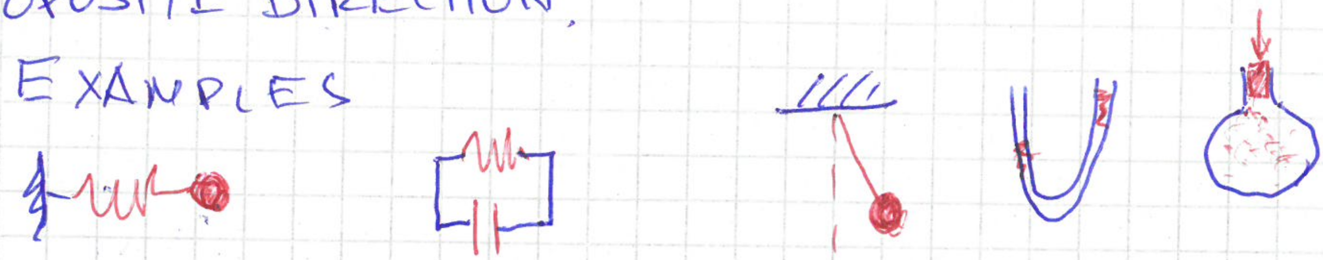
15-11-22

PART B FUNDAMENTAL EQUATIONS GOVERNING WAVES

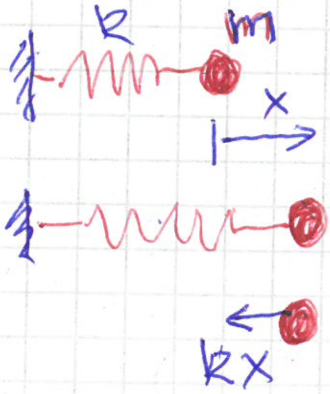
1 FREE OSCILLATIONS OF A SINGLE SYSTEM

MANY SYSTEMS, WHEN DISPLACED FROM EQUILIBRIUM BY A DISPLACEMENT x, REACT BY GENERATING A RESTORING FORCE PROPORTIONAL TO x, WHICH ACTS IN THE OPOSITE DIRECTION.

EXAMPLES



1.1 MASLESS SPRING WITH A MASS (NO ENERGY DISSIPATION)



MOTION EQUATION: $\dot{x} = \frac{dx}{dt} = v$

$$\boxed{m\ddot{x} = -kx} \quad (1)$$

$\ddot{x} = \frac{d^2x}{dt^2}$ = ACCELERATION

WE CAN TRY A SIMPLE HARMONIC FUNCTION FOR x:

$$x = a\cos\omega t + b\sin\omega t \qquad \omega = \frac{2\pi}{T}$$

$$\dot{x} = -\omega a \sin\omega t + b\omega\cos\omega t$$

$$\ddot{x} = -\omega^2 a\cos\omega t - b\omega^2 \sin\omega t =$$

$$= -\omega^2 \underbrace{(\cos\omega t + \sin\omega t)}_{x} = -\omega^2 x$$

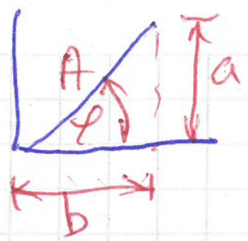
$\ddot{x} = -\omega^2 x$ PLUGGING THIS INTO (1):

$$-m\omega^2 \cancel{x} = -k\cancel{x}$$

$$\boxed{\omega^2 = k/m} \qquad \boxed{\omega = \sqrt{\frac{k}{m}}} \text{ NATURAL FREQUENCY}$$

ALTERNATIVE WAY TO EXPRESS X

$$x = \underbrace{A\sin\varphi}_{a}\cos\omega t + \underbrace{A\cos\varphi}_{b}\sin\omega t$$



φ - PHASE

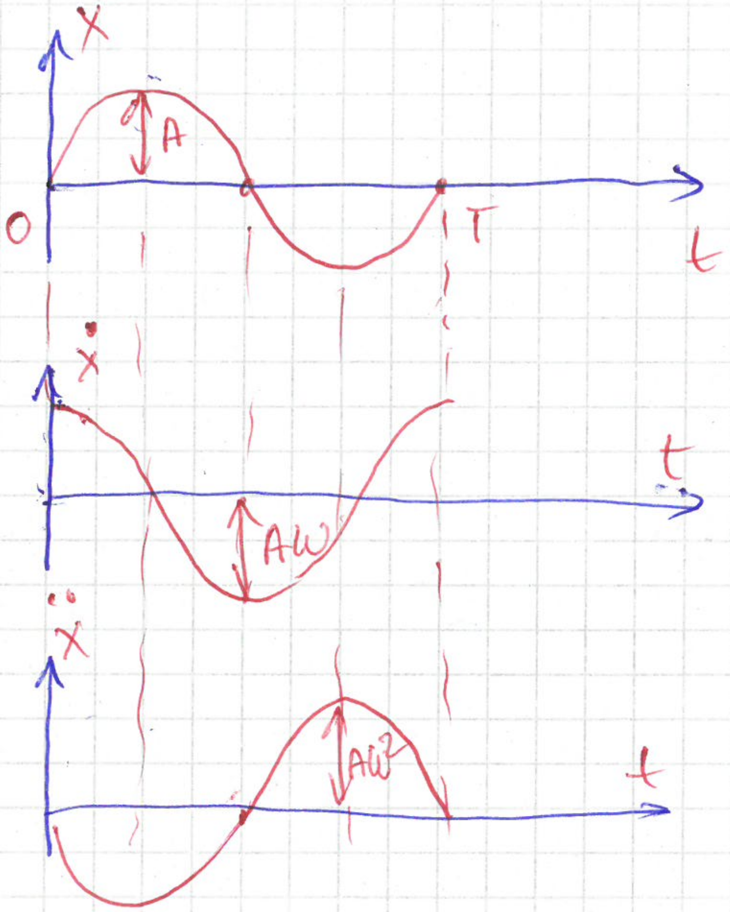
$\omega = \frac{2\pi}{T}$

T - PERID

$$x = A\sin(\omega t + \varphi)$$

$$\dot{x} = A\omega\cos(\omega t + \varphi)$$

$$\ddot{x} = -A\omega^2\sin(\omega t + \varphi)$$



X LAGS BEHIND $\dot{X}$ BY $T/4$ $\left(\frac{\pi}{2}\right)$

$\ddot{X}$ IS OUT OF PHASE WITH X BY $\frac{T}{2}$ (π)

ENERGY $\quad m\ddot{x} + kx = 0$

$$E_k = \int m\ddot{x}\,dx = m\int \frac{d\dot{x}}{dt}\,dx = m\int \dot{x}\,d\dot{x} = \frac{1}{2}m\dot{x}^2$$

$$E_p = \int kx\,dx = \frac{1}{2}kx^2$$

$$\text{"}E_k\text{"} = \int L\ddot{Q}\,dQ = L\int \frac{d\dot{Q}}{dt}\,dQ = L\int \dot{Q}\,d\dot{Q} = \frac{1}{2}L\dot{Q}^2$$

$$\dot{Q}$$

$$\text{"}E_p\text{"} = \int \frac{1}{C}Q\,dQ = \frac{1}{2}\frac{1}{C}Q^2$$

GENERALISED "MOTION EQUATION"

$$M\ddot{X} + KX = 0$$

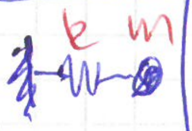
$$E_k = \frac{1}{2}M\dot{X}^2$$

$$E_p = \frac{1}{2}KX^2$$

X - GENERALISED DISPLACEMENT

M - "MASS"

K - "STIFFNESS"

SYSTEM	X	M	K
	x	m	k
	Q	L	$\frac{1}{C}$

TOTAL ENERGY

$x = A \sin \omega t$

$\dot{x} = A\omega \cos \omega t$

$$E = E_k + E_p = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 =$$

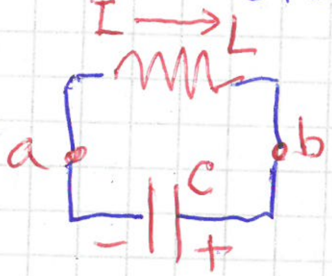
$$= \frac{1}{2} m A^2 \underbrace{\omega^2}_{k/m} \cos^2 \omega t + \frac{1}{2} k A^2 \sin^2 \omega t =$$

$$= \frac{1}{2} A^2 k \cos^2 \omega t + \frac{1}{2} k A^2 \sin^2 \omega t =$$

$$= \frac{1}{2} A^2 k \, (\underbrace{\cos^2 \omega t + \sin^2 \omega t}_{=1})$$

$$\boxed{E = \frac{1}{2} A^2 k = \frac{1}{2} A^2 \omega^2 \cdot m = const}$$

1.2. LC CIRCUIT



L - INDUCTANCE (HENRY)

C - CAPACITANCE (FARADAY)

$$V_a - V_b = L \frac{dI}{dt}$$

$$V_b - V_a = \frac{1}{C} Q$$

$$V_a - V_b = L \frac{d\dot{Q}}{dt} = -\frac{1}{C} Q$$

Q - CHARGE (COULOMB)

I - CURRENT

$$I = \frac{dQ}{dt} = \dot{Q}$$

$$L \ddot{Q} = -\frac{1}{C} Q$$

$$L \ddot{Q} + \frac{1}{C} Q = 0$$

$$\boxed{\ddot{Q} + \frac{1}{LC} Q = 0}$$

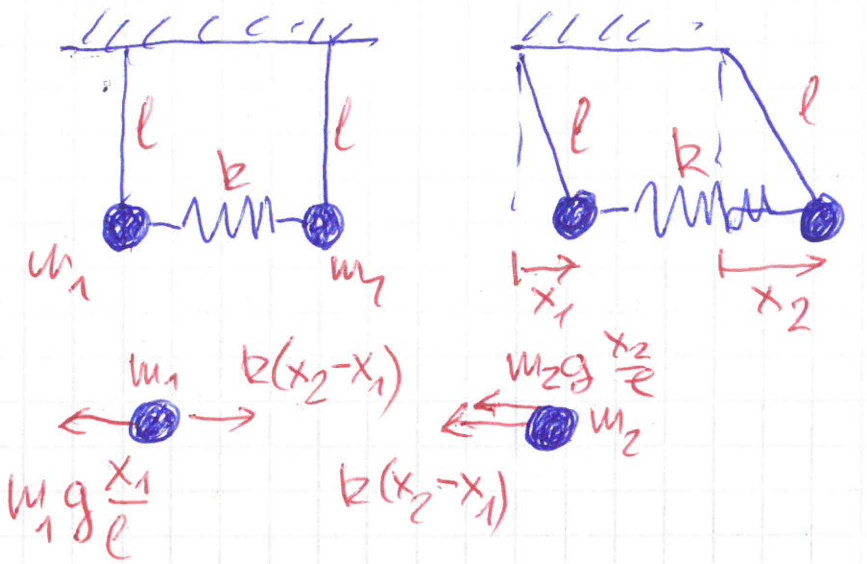
"MOTION EQUATION"

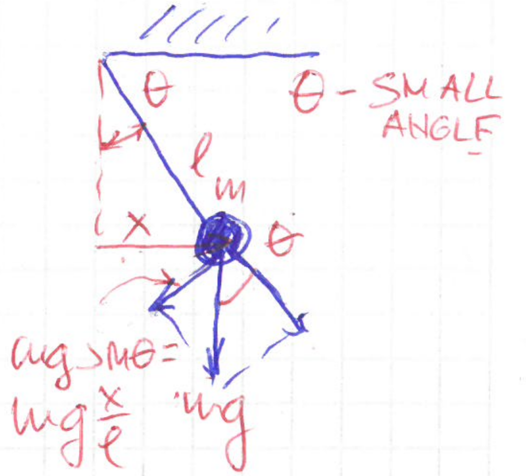
$$\omega^2 = \frac{1}{LC}$$

NATURAL FREQUENCY

2 TWO COUPLED SIMPLE HARMONIC OSCILLATORS

2.1 TWO COUPLED MASSES WITH A SPRING





MOTION EQUATION

$$\cancel{m}\ddot{x} = -\cancel{m}g\frac{x}{l}$$

$$\ddot{x} + \underbrace{\left(\frac{g}{l}\right)}_{\omega_0^2} x = 0$$

MOTION EQUATIONS:

$$m_1\ddot{x}_1 = -m_1\frac{g}{l}x_1 + k(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -k(x_2 - x_1) - m_2\frac{g}{l}x_2$$

$m_1 = m_2 = m$ $\qquad g/l = \omega_0^2$

$$m\ddot{x}_1 = -m\omega_0^2 x_1 + kx_2 - kx_1 \qquad / \times \frac{1}{m}$$

$$m\ddot{x}_2 = -kx_2 + kx_1 - m\omega_0^2 x_2 \qquad / \times \frac{1}{m}$$

$$\ddot{x}_1 + \left(\omega_0^2 + \frac{k}{m}\right)x_1 - \frac{k}{m}x_2 = 0$$

$$\ddot{x}_2 - \frac{k}{m}x_1 + \left(\omega_0^2 + \frac{k}{m}\right)x_2 = 0$$

ASSUME:

$x_1 = A_1\cos\omega t$

$x_2 = A_2\cos\omega t$

$$-A_1\omega^2\cancel{\cos\omega t} + \left(\omega_0^2 + \frac{k}{m}\right)A_1\cancel{\cos\omega t} - \frac{k}{m}A_2\cancel{\cos\omega t} = 0$$

$$-A_2\omega^2\cancel{\cos\omega t} - \frac{k}{m}A_1\cancel{\cos\omega t} + \left(\omega_0^2 + \frac{k}{m}\right)A_2\cancel{\cos\omega t} = 0$$

$$\left(\omega_0^2+\frac{k}{m}-\omega^2\right)A_1-\frac{k}{m}A_2=0 \quad (1) \qquad \omega_0^2=\frac{g}{\ell}$$

$$-\frac{k}{m}A_1+\left(\omega_0^2+\frac{k}{m}-\omega^2\right)A_2=0 \quad (2)$$

UNKNOWNS: A_1, A_2, ω

THE SYSTEM OF ALGEBRAIC EQUATIONS HAS A NON-TRIVIAL SOLUTION FOR A_1 & A_2 IF:

$$\det\begin{bmatrix}\omega_0^2+\frac{k}{m}-\omega^2 & -\frac{k}{m}\\ -k/m & \omega_0^2+\frac{k}{m}-\omega^2\end{bmatrix}=0$$

$$\left(\omega_0^2+\frac{k}{m}-\omega^2\right)^2-\left(\frac{k}{m}\right)^2=0 \qquad \left[a^2-b^2=(a+b)(a-b)\right.$$

$$\underbrace{\left(\omega_0^2+\cancel{\frac{k}{m}}-\omega^2-\cancel{\frac{k}{m}}\right)}_{=0 \text{ FOR } \omega^2=\omega_0^2}\underbrace{\left(\omega_0^2+\frac{k}{m}-\omega^2+\frac{k}{m}\right)}_{=0 \text{ FOR } \omega^2=\omega_0^2+2\frac{k}{m}}=0$$

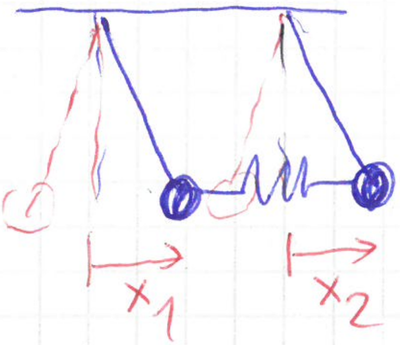
THE TWO VALUES OF ω ARE THE TWO NORMAL FREQUENCIES OF THE SYSTEM AND THE TWO KINDS OF MOTION ARE CALLED TWO NORMAL MODES

$$(1)\rightarrow \quad A_1/A_2=\frac{k/m}{\omega_0^2+\frac{k}{m}-\omega^2}=\begin{cases}1 & \text{FOR } \omega^2=\omega_0^2\\ -1 & \text{FOR } \omega^2=\omega_0^2+2\frac{k}{m}\end{cases}$$

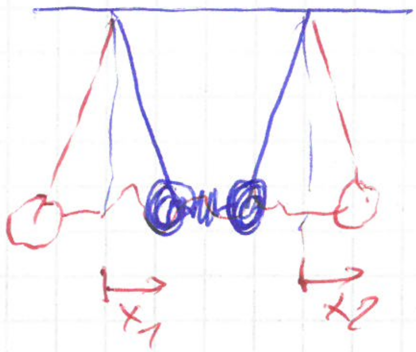
FOR $\omega^2=\omega_0^2$

$$x_1 = a\cos\omega t$$
$$x_2 = a\cos\omega t$$

$$\omega^2 = \omega_0^2 + 2\frac{k}{m}$$
$$x_1 = a\cos\omega t$$
$$x_2 = -a\cos\omega t$$

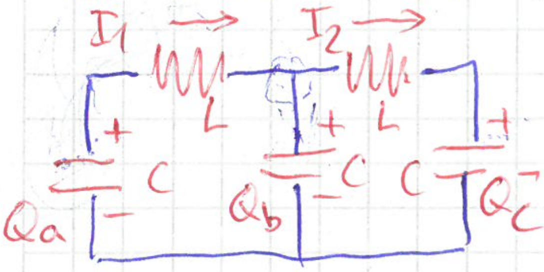


$$\omega = \omega_0$$



$$\omega = \sqrt{\omega_0^2 + 2\frac{k}{m}}$$

2.2 TWO COUPLED LC CIRCUITS



"MOTION" EQUATION FOR THE TWO INDUCTORS:

$$\frac{d}{dt}\Big/ \quad L\frac{dI_1}{dt} = V_a - V_b = \frac{1}{C}Q_a - \frac{1}{C}Q_b$$

$$\frac{d}{dt}\Big/ \quad L\frac{dI_2}{dt} = V_b - V_c = \frac{1}{C}Q_b - \frac{1}{C}Q_c$$

CONSERVATION EQUATIONS FOR CHARGE

$$I_1 = -\frac{dQ_a}{dt} = -\dot{Q}_a$$

$$I_1 = I_2 + \frac{dQ_b}{dt} = I_2 + \dot{Q}_b$$

$$I_2 = \dot{Q}_c$$

$$L\ddot{I}_1 = \frac{1}{C}\dot{Q}_a - \frac{1}{C}\dot{Q}_b$$

$$L\ddot{I}_2 = \frac{1}{C}\dot{Q}_b - \frac{1}{C}\dot{Q}_c$$

$$L\ddot{I}_1 = -\frac{1}{C}I_1 - \frac{1}{C}(I_1 - I_2)$$

$$L\ddot{I}_2 = \frac{1}{C}(I_1 - I_2) - \frac{1}{C}I_2$$

$$L\ddot{I}_1 = -\frac{2}{C} I_1 + \frac{1}{C} I_2$$

$$L\ddot{I}_2 = \frac{1}{C} I_1 - \frac{2}{C} I_2$$

WE CAN ASSUME $I_1 = A_1 \cos\omega t$
$I_2 = A_2 \cos\omega t$

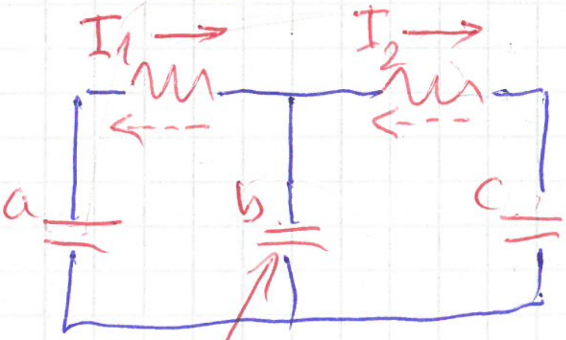
AND USE THE SAME PROCEDURE AS FOR THE 2 MASSES TO WORK OUT THE NORMAL FREQUENCIES

$$\omega = \frac{1}{\sqrt{LC}} \qquad \omega = \frac{\sqrt{3}}{\sqrt{LC}}$$

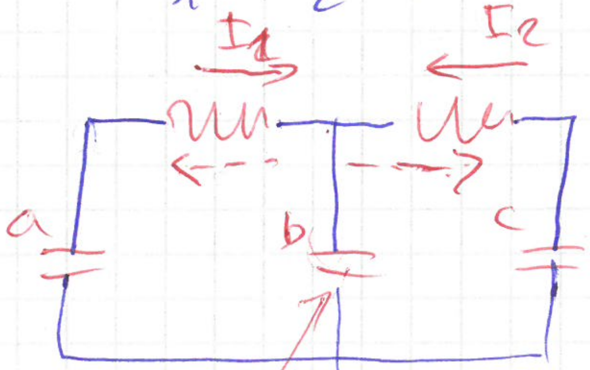
AND FIND THE CORRESPONDING NORMAL MODES:

$$\omega = \frac{1}{\sqrt{LC}} \qquad\qquad \omega = \frac{\sqrt{3}}{\sqrt{LC}}$$

$$I_1 = I_2 = a\cos\omega t \qquad\qquad I_1 = -I_2 = a\cos\omega t$$



CAPACITOR b NEVER CHARGED



MAXIMUM CHARGING/DEPLETING OF CAPACITOR b