3-10-23

2 NUMERICAL SOLUTION OF SYSTEMS OF 2 HYPERBOLIC EQUATIONS

ENGINEERING EXAMPLES:

- TRANSMISSION LINES, CABLES — V, I
- ELASTIC WAVES IN A BAR — σ, v
- TRANSIENT PIPE FLOW — p, v OR H, Q

2.1 CHARACTERISTIC FORM OF A SYSTEM OF 2 HYPERBOLIC EQUATIONS

PROPAGATION OF A SIGNAL ALONG A CABLE IS DESCRIBED BY THE FOLLOWING EQUATIONS:

$$\frac{\partial V}{\partial t} + \frac{1}{C'}\frac{\partial I}{\partial x} = 0 \quad (1)$$

$$\frac{\partial I}{\partial t} + \frac{1}{L'}\frac{\partial V}{\partial x} = 0 \quad (2)$$

V - VOLTAGE [V]
I - CURRENT [A]
C' - UNIT CAPACITANCE [F/m]
L' - UNIT INDUCTANCE [H/m]
CABLE RESISTANCE NEGLECTED!

(1) + α (2) →

$$\frac{\partial V}{\partial t} + \frac{1}{C'}\frac{\partial I}{\partial x} + \alpha\frac{\partial I}{\partial t} + \frac{\alpha}{L'}\frac{\partial V}{\partial x} = 0$$

$$\underbrace{\frac{\partial V}{\partial t} + \frac{\alpha}{L'}\frac{\partial V}{\partial x}}_{=\frac{dV}{dt} \text{ FOR } \frac{\alpha}{L'} = \frac{dx}{dt}} + \alpha\underbrace{\left(\frac{\partial I}{\partial t} + \frac{1}{\alpha C'}\frac{\partial I}{\partial x}\right)}_{=\frac{dI}{dt} \text{ FOR } \frac{1}{\alpha C'} = \frac{dx}{dt}} = 0 \quad (3)$$

$$\frac{dx}{dt} = \frac{\alpha}{L'} = \frac{1}{\alpha C'} \rightarrow \alpha^2 = \frac{L'}{C'} \qquad \boxed{\alpha = \pm\sqrt{\frac{L'}{C'}}}$$

$$\frac{dx}{dt} = \pm\frac{1}{L'}\sqrt{\frac{L'}{C'}} = \pm\frac{1}{\sqrt{L'C'}} = \pm c$$

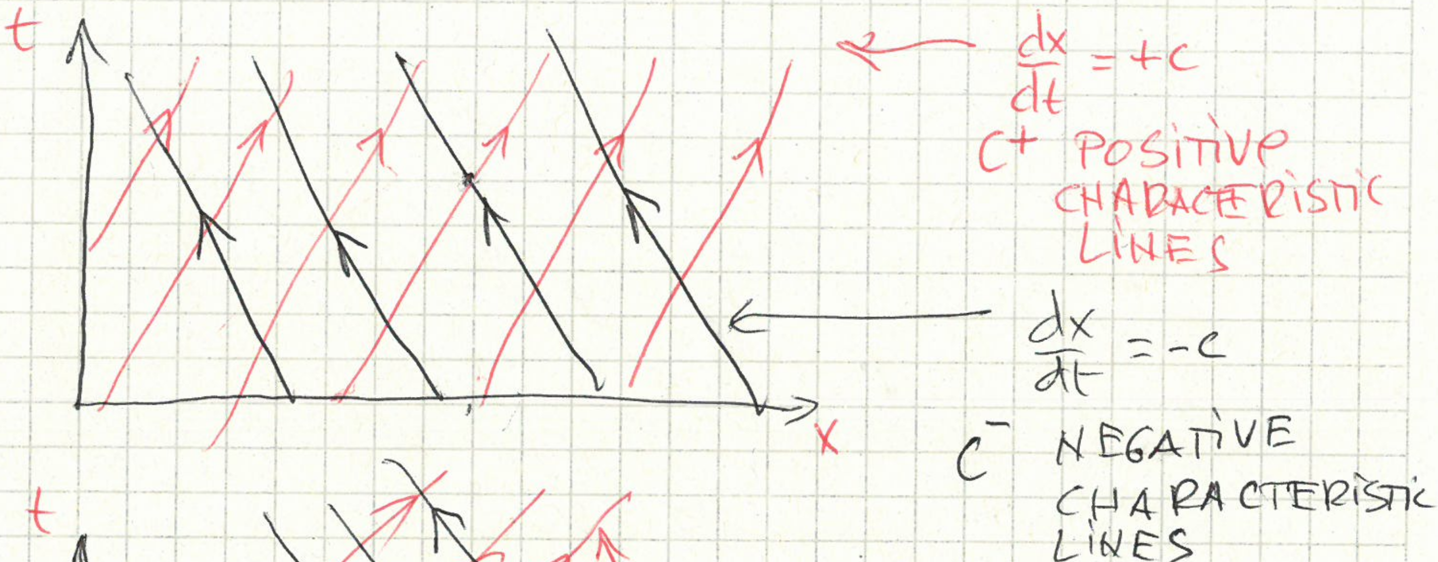
c - SPEED OF ELECTROMAGNETIC WAVE

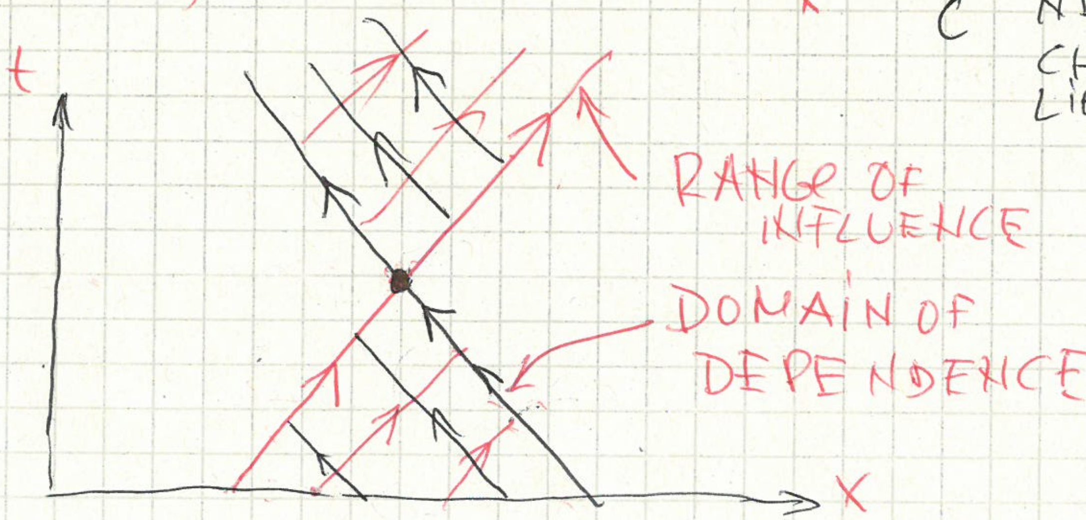
(3) $$\frac{dV}{dt} \pm \sqrt{\frac{L'}{C'}}\frac{dI}{dt} = 0 \quad \text{FOR} \quad \frac{dx}{dt} = \pm\frac{1}{\sqrt{L'C'}}$$

$$dV \pm \underbrace{\sqrt{\frac{L'}{C'}}}_{Z_0}\, dI = 0 \qquad \text{FOR} \quad \frac{dx}{dt} = \pm \frac{1}{\sqrt{L'C'}}$$

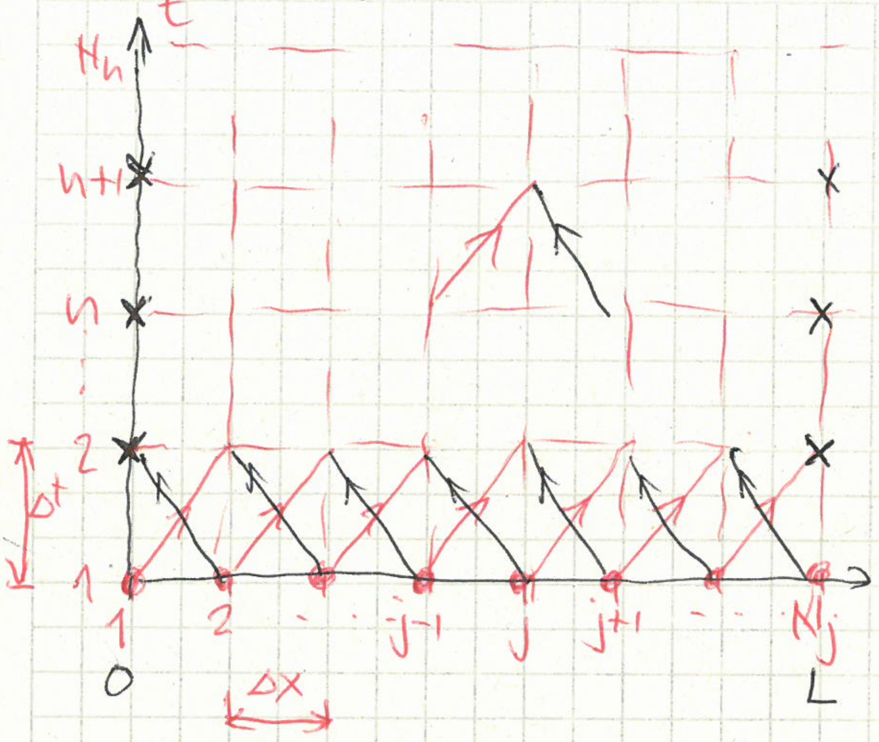
$$\boxed{\begin{array}{lll} dV + Z_0\, dI = 0 & \text{FOR} \quad \dfrac{dx}{dt} = +c & Z_0 = \sqrt{\dfrac{L'}{C'}} \\ dV - Z_0\, dI = 0 & \text{FOR} \quad \dfrac{dx}{dt} = -c & c = \dfrac{1}{\sqrt{L'C'}} \end{array}}$$

CHARACTERISTIC FORM OF THE SYSTEM (1), (2)





2.2. METHOD OF CHARACTERISTICS (MoC)



FOR THE MoC

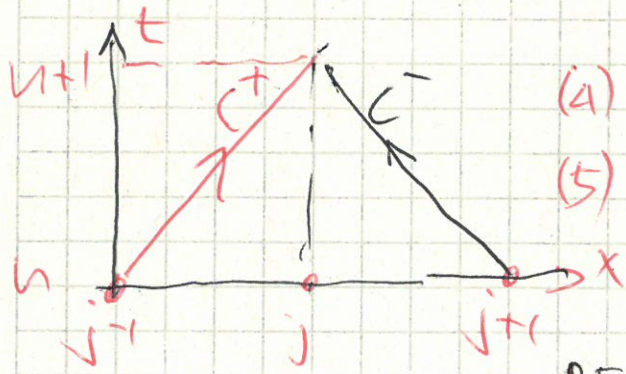
$$\boxed{\begin{aligned} &\frac{\Delta x}{\Delta t} = c = \frac{1}{\sqrt{L'C'}} \\ &C_r = \frac{c\Delta t}{\Delta x} = 1 \end{aligned}}$$

IC: V, I AT $t=0$

LBC AT $X=0$

RBC AT $X=L$

MIDDLE POINTS: $j = 2, 3 \dots N_j - 1$



(4) C^+: $dV + Z_0\, dI = 0$ ALONG $\frac{dx}{dt} = c$

(5) C^-: $dV - Z_0\, dI = 0$ ALONG $\frac{dx}{dt} = -c$

We INTEGRATE (4) AND (5) BETWEEN (4): $(n, j-1)$ & $(n+1, j)$
(5): $(n, j+1)$ & $(n+1, j)$

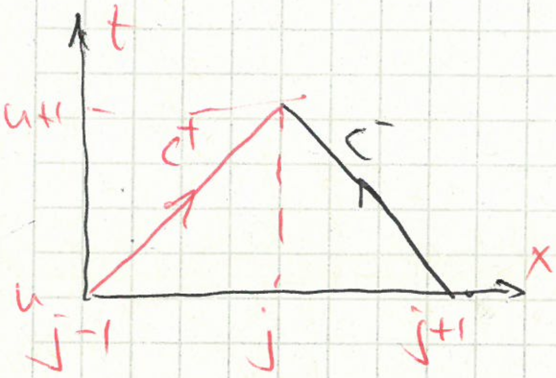
$$(4) \rightarrow \int_{n,j-1}^{n+1,j} dV + Z_0 \int_{n,j-1}^{n+1,j} dI = 0$$

$$(5) \rightarrow \int_{n,j+1}^{n+1,j} dV - Z_0 \int_{n,j+1}^{n+1,j} dI = 0$$

$$V_j^{n+1} - V_{j-1}^{n} + Z_0\left(I_j^{n+1} - I_{j-1}^{n}\right) = 0 \qquad (6)$$

$$V_j^{n+1} - V_{j+1}^{n} - Z_0\left(I_j^{n+1} - I_{j+1}^{n}\right) = 0 \qquad (7)$$

MIDDLE POINTS: $j = 2, 3, 4, 5$



$$c^+ : dV + Z_0\, dI = 0$$
$$c^- : dV - Z_0\, dI = 0$$

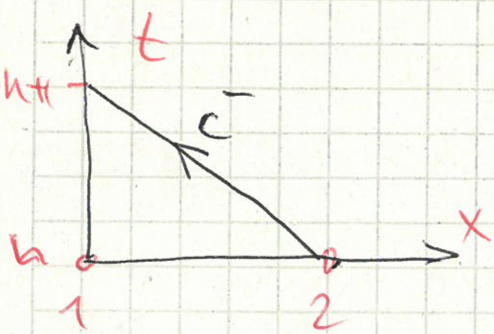
INTEGRATED BETWEEN $(j-1, n)$ & $(j, n+1)$
$(j+1, n)$ & $(j, n+1)$

THEY BECOME:

$$V_j^{n+1} - V_{j-1}^{n} + Z_0 \left(I_j^{n+1} - I_{j-1}^{n} \right) = 0 \quad (8)$$
$$V_j^{n+1} - V_{j+1}^{n} - Z_0 \left(I_j^{n+1} - I_{j+1}^{n} \right) = 0 \quad (9)$$

(8)+(9) $\rightarrow$
$$\boxed{V_j^{n+1} = \frac{1}{2}\left(V_{j-1}^{n} + V_{j+1}^{n} \right) + \frac{Z_0}{2}\left(I_{j-1}^{n} - I_{j+1}^{n} \right)}$$
(8)−(9) $\rightarrow$
$$\boxed{I_j^{n+1} = \frac{1}{2}\left(I_{j-1}^{n} + I_{j+1}^{n} \right) + \frac{1}{2Z_0}\left(V_{j-1}^{n} - V_{j+1}^{n} \right)}$$

LEFT BC



$$c^- : dV - Z_0\, dI = 0$$

INTEGRATED BETWEEN $(2, n)$ & $(1, n+1)$

$$V_1^{n+1} - V_2^{n} - Z_0 \left(I_1^{n+1} - I_2^{n} \right) = 0 \quad (10)$$

V_1^{n+1} IS KNOWN SO WE SOLVE (10) FOR I_1^{n+1}

$$V_1^{n+1} - V_2^{n} - Z_0 I_1^{n+1} + Z_0 I_2^{n} = 0$$

$$\boxed{I_1^{n+1} = I_2^{n} + \frac{1}{Z_0}\left(V_1^{n+1} - V_2^{n} \right)}$$

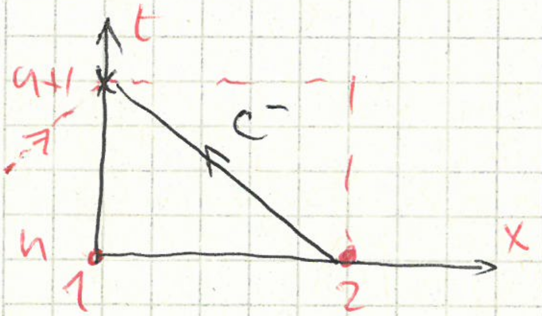
$(6)+(7) \rightarrow \quad 2V_j^{n+1} - V_{j-1}^n - V_{j+1}^n - Z_0 I_{j-1}^n + Z_0 I_{j+1}^n = 0$

$$\boxed{V_j^{n+1} = \frac{1}{2}\left(V_{j-1}^n + V_{j+1}^n\right) + \frac{Z_0}{2}\left(I_{j-1}^n - I_{j+1}^n\right)}$$

$(6)-(7) \quad V_{j+1}^n - V_{j-1}^n + 2Z_0 I_j^{n+1} - Z_0 I_{j-1}^n - Z_0 I_{j+1}^n = 0$

$$\boxed{I_j^{n+1} = \frac{1}{2}\left(I_{j-1}^n + I_{j+1}^n\right) + \frac{1}{2Z_0}\left(V_{j-1}^n - V_{j+1}^n\right)}$$

LEFT BOUNDARY CONDITION:



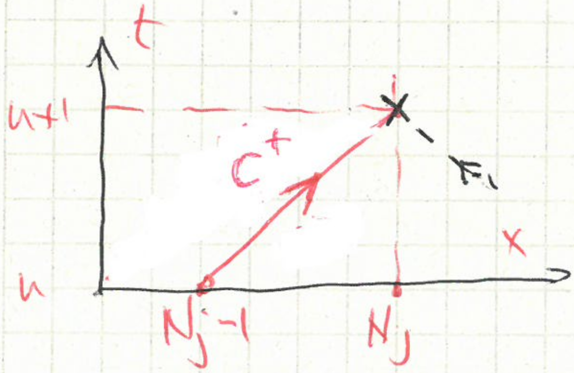
$C^-: \ dV - Z_0 \, dI = 0$

INTEGRATED BETWEEN $(n,2)$ & $(n+1,1)$

$V_2^{n+1} - V_1^n - Z_0\left(I_1^{n+1} - I_2^n\right) = 0$

THIS IS AN EQUATION WITH 2 UNKNOWNS SO WE HAVE TO ADD: V_1^{n+1} OR I_1^{n+1} OR $V_1^{n+1}(I_1^{n+1})$

RIGHT BOUNDARY CONDITION



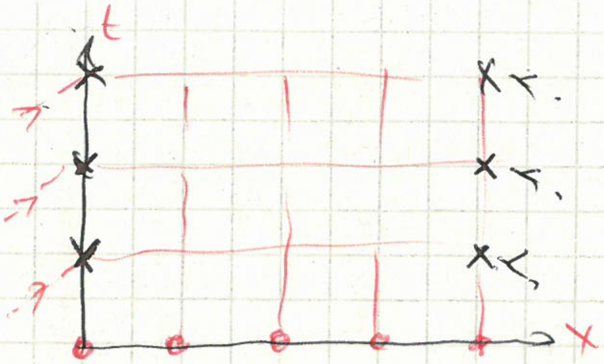
$C^+: \ dV + Z_0 \, dI = 0$

INTEGRATED BETWEEN $(n, N_j - 1)$ & $(n+1, N_j)$

$V_{N_j}^{n+1} - V_{N_j-1}^n + Z_0\left(I_{N_j}^{n+1} - I_{N_j-1}^n\right) = 0$

WE HAVE TO PRESCRIBE THE RBC: $V_{N_j}^{n+1}$ OR $I_{N_j}^{n+1}$ OR $V_{N_j}^{n+1}(I_{N_j}^{n+1})$

GENERAL RULE: THE NUMBER OF BOUNDARY CONDITIONS THAT HAVE TO BE PRESCRIBED AT A BOUNDARY IS EQUAL TO THE NUMBER OF CHARACTERISTIC LINES WHICH ENTER THE SIMULATION DOMAIN THROUGH THIS BOUNDARY



• IC INITIAL CONDITION

$V(0 \leq x \leq L, t=0)$
$I(0 \leq x \leq L, t=0)$

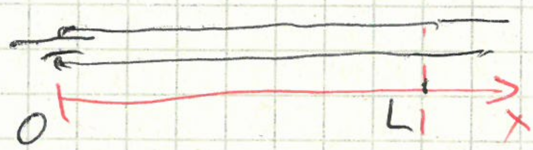
× LBC

$V(x=0,t)$ OR $I(x=0,t)$ OR $V(I)(x=0,t)$

× RBC

$V(x=L,t)$ OR $I(x=L,t)$ OR $V(I)(x=L,t)$

EXAMPLE CABLE 1



$L = 1000\,m$

$C' = 94.8 \times 10^{-12}\ F/m$

$L' = 0{,}236 \times 10^{-6}\ H/m$

IC: $V(x, t=0) = 0$
$I(x, t=0) = 0$

LBC: $V(x=0, t \geq 1.2\mu s) = 4V$

RBC: TRANSMISSIVE

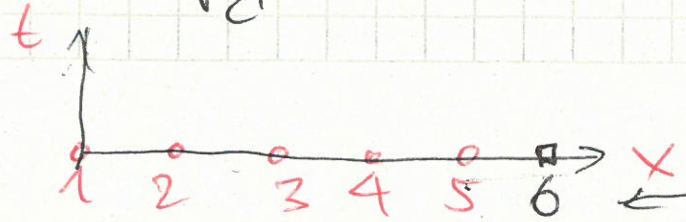
SOLUTION:

$c = \frac{1}{\sqrt{L'C'}} \simeq 211 \times 10^{6}\ \frac{m}{s}$

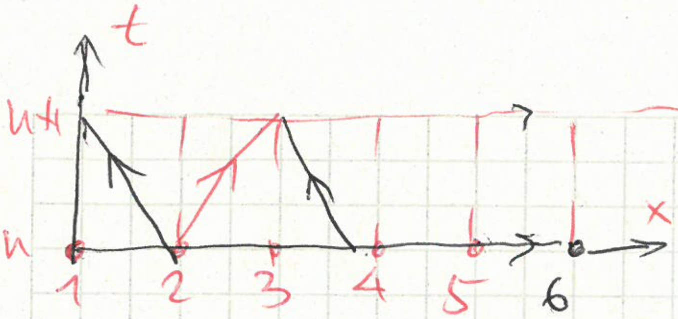
$\Delta x = 250\,m$

$Z_0 = \sqrt{\frac{L'}{C'}} \simeq 50\,\Omega$

$\Delta t = \frac{\Delta x}{c} = \frac{250}{211 \times 10^{6}} \simeq 1.2\,\mu s$



'GHOST' NODE



ALL EQUATIONS:

LBC $V_1^{n+1} = 4V$

$$I_1^{n+1} = I_2^n + \frac{1}{50}\left(V_1^{n+1} - V_2^n\right)$$

MIDDLE POINTS $j = 2,3,4,5$

$$V_j^{n+1} = \frac{1}{2}\left(V_{j-1}^n + V_{j+1}^n\right) + 25\left(I_{j-1}^n - I_{j+1}^n\right)$$

$$I_j^{n+1} = \frac{1}{2}\left(I_{j-1}^n + I_{j+1}^n\right) + \frac{1}{100}\left(V_{j-1}^n - V_{j+1}^n\right)$$

x(m)	0		250		500		750		1000		GHOST 1250	
j	1		2		3		4		5		6	
t (ms)	V(V)	I(A)	V	I	V	I	V	I	V	I	V	I
0	0	0	0	0	0	0	0	0	0	0 →	0	0
1.2	4	4/50	0	0	0	0	0	0	0	0 →	0	0
2.4	4	4/50	4	4/50	0	0	0	0	0	0 →	0	0
3.6	4	4/50	4	4/50	4	4/50	0	0	0	0 →	0	0
4.8	4											