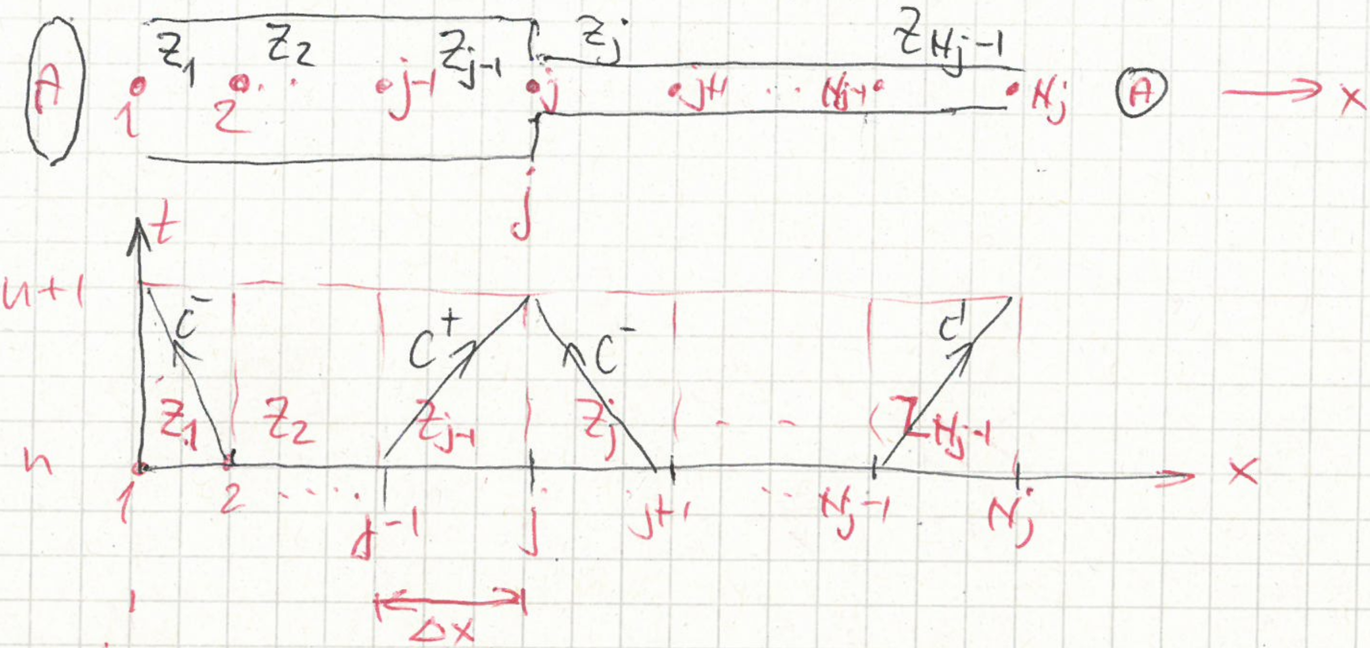
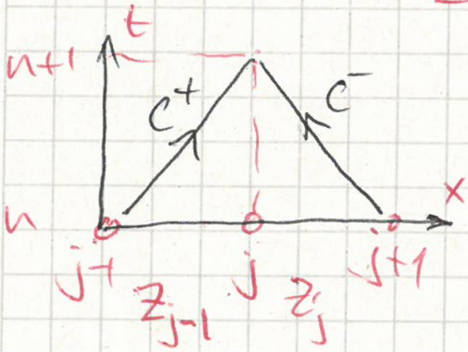
17-10-23

2.5 Variable impedance

Parameter Z may vary in space, $Z = Z(x)$





$$C^+: \; dH + Z_{j-1}\, dQ = 0$$
$$C^-: \; dH - Z_j\, dQ = 0$$

Integrated between $(n, j-1)$ & $(n+1, j)$
$(n, j+1)$ & $(n+1, j)$

$$H_j^{n+1} - H_{j-1}^n + Z_{j-1}\left(Q_j^{n+1} - Q_{j-1}^n\right) = 0 \qquad (1)$$
$$H_j^{n+1} - H_{j+1}^n - Z_j\left(Q_j^{n+1} - Q_{j+1}^n\right) = 0 \qquad (2)$$

$Z_j\,(1) + Z_{j-1}\,(2) \rightarrow$

$$Z_j H_j^{n+1} + Z_{j-1} H_j^{n+1} - Z_j H_{j-1}^n - Z_{j-1} H_{j+1}^n - Z_j Z_{j-1} Q_{j-1}^n + Z_j Z_{j-1} Q_{j+1}^n = 0$$

$$\boxed{H_j^{n+1} = \frac{Z_j H_{j-1}^n + Z_{j-1} H_{j+1}^n}{Z_j + Z_{j-1}} + \frac{Z_j Z_{j-1}}{Z_j + Z_{j-1}}\left(Q_{j-1}^n - Q_{j+1}^n\right)}$$

For $Z_j = Z_{j-1} = Z_0$

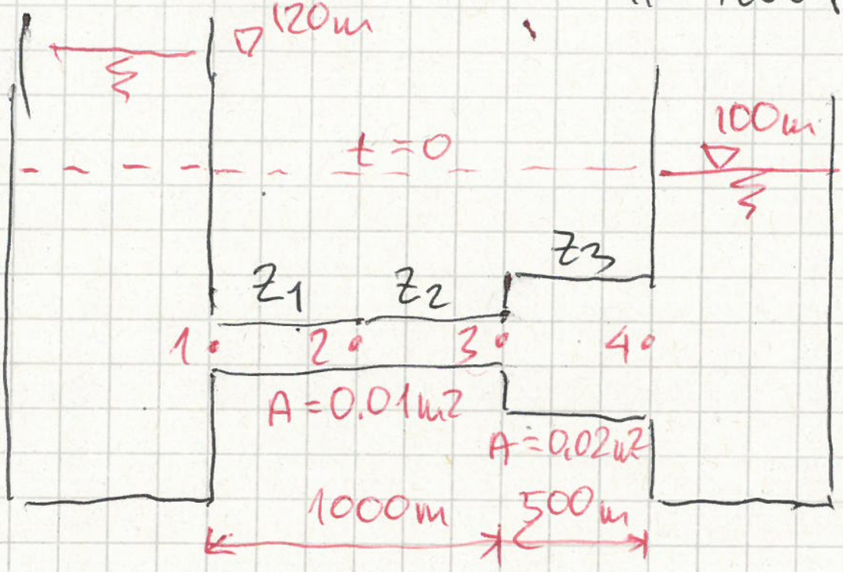
$$\frac{H_{j-1}^n + H_{j+1}^n}{2} + \frac{Z_0}{2}\left(Q_{j-1}^n - Q_{j+1}^n\right)$$

(1)-(2) →

$$H_{j+1}^{n} - H_{j-1}^{n} + z_{j-1} Q_j^{n+1} + z_j Q_j^{n+1} - z_{j-1} Q_{j-1}^{n} - z_j Q_{j+1}^{n} = 0$$

$$\boxed{Q_j^{n+1} = \frac{z_{j-1} Q_{j-1}^{n} + z_j Q_{j+1}^{n}}{z_{j-1} + z_j} + \frac{1}{z_{j-1} + z_j}\left(H_{j-1}^{n} - H_{j+1}^{n}\right)}$$

EXAMPLE PIPE 3 - SUDDEN SURGE (PIPE 1) IN TWO PIPES



$z_1 = 10^4 \ s/m^2$

$z_2 = 10^4 \ s/m^2$

$z_3 = 5\,000 \ s/m^2$

$\Delta x = 500m$

$c = 1000 m/s$

$\Delta t = 0.5s$

EQUATIONS:

LB: $H_1^{n+1} = 120m$ $\qquad Q_1^{n+1} = Q_2^{n} + \frac{1}{z_1}\left(H_1^{n+1} - H_2^{n}\right)$

MIDDLE POINTS $j = 2,3$

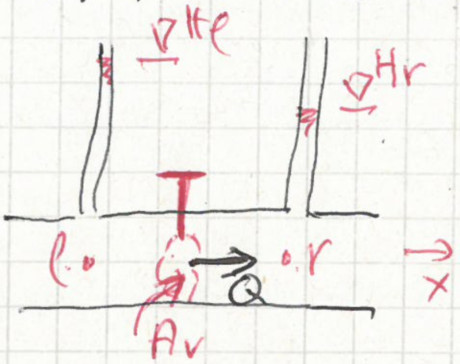
$$H_j^{n+1} = \frac{z_j H_{j-1}^{n} + z_{j-1} H_{j+1}^{n}}{z_j + z_{j-1}} + \frac{z_j z_{j-1}}{z_j + z_{j-1}}\left(Q_{j-1}^{n} - Q_{j+1}^{n}\right)$$

$$Q_j^{n+1} = \frac{z_{j-1} Q_{j-1}^{n} + z_j Q_{j+1}^{n}}{z_j + z_{j-1}} + \frac{1}{z_j + z_{j-1}}\left(H_{j-1}^{n} - H_{j+1}^{n}\right)$$

RB: $H_4^{n+1} = 100m$ $\qquad Q_4^{n+1} = Q_3^{n} + \frac{1}{z_3}\left(H_3^{n} - H_4^{n+1}\right)$

2.6 SPECIAL KINDS OF BOUNDARY CONDITIONS

2.6.1 NON-LINEAR BOUNDARY CONDITION



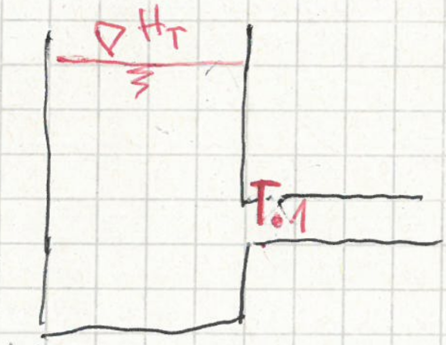
IF $H_l > H_r$ $\quad Q = C_Q A_v \sqrt{2g(H_l - H_r)}$

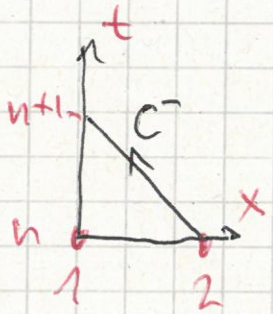
IF $H_l < H_r$ $\quad Q = -C_Q A_v \sqrt{2g(H_r - H_l)}$

$$\frac{Q|Q|}{C_Q^2 A_v^2 2g} = H_l - H_r$$

$$\boxed{M Q|Q| = H_l - H_r \qquad M = \frac{1}{C_Q^2 A_v^2 2g}}$$

VALVE AT THE BEGINNING OF A PIPE CONNECTED TO A TANK WITH A CONSTANT LEVEL





C^-: $dH - z_0\, dQ = 0$

INTEGRATED BETWEEN $(n, 2)$ & $(n+1, j)$

$$H_1^{n+1} - H_2^n - z_0 \left(Q_1^{n+1} - Q_2^n\right) = 0$$

LBC: $\quad M Q_1 |Q_1| = H_T - H_1$

VALID AT ANY TIME

$$M^{n+1} Q_1^{n+1} |Q_1^{n+1}| = H_T - H_1^{n+1}$$

$$\boxed{H_1^{n+1} = H_T - M^{n+1} Q_1^{n+1} |Q_1^{n+1}|}$$

LBC

$$H_T - M^{n+1} Q_1^{n+1} |Q_1^{n+1}| - H_2^n - z_0 Q_1^{n+1} + z_0 Q_2^n = 0$$

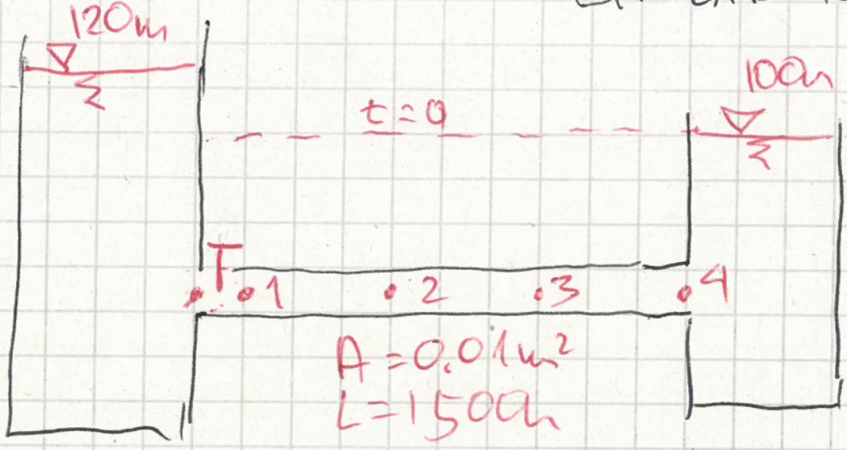
$$M^{n+1} Q_1^{n+1} |Q_1^{n+1}| + z_0 Q_1^{n+1} = \underbrace{H_T - H_2^n + z_0 Q_2^n}_{H^n}$$

$$M^{n+1} Q_1^{n+1} |Q_1^{n+1}| + z_0 Q_1^{n+1} = H^n$$

$$q = \frac{1}{2}\left[z_0 + \sqrt{z_0^2 + 4 M^{n+1} |N^n|}\right] \qquad M^{n+1} = \frac{1}{2g\, C_Q^2 (A_V^{n+1})^2}$$

$$Q_1^{n+1} = \frac{N^n}{q} \qquad N^n = H_T - H_2^n + z_0 Q_2^n$$

EXAMPLE PIPE <1: SUDDEN VALVE OPENING (SIMILAR TO PIPE 1)



$c = 1000\ m/s$

$g \cong 10\ m/s^2$

$z_0 = 10^4\ s/m^2$

IC $Q(0 \le x \le L, t=0) = 0$

$H(0 \le x \le L, t=0) = 100m$

LBC: $t \geq 0.5s$ THE VALVE OPENS

$A_V = \frac{1}{4} A$

$C_Q = 0.125$

EQUATIONS:

LB. $$Q_1^{n+1} = \frac{N^n}{q} \qquad q = \frac{1}{2}\left(z_0 + \sqrt{z_0^2 + 4M^{n+1}|N^n|}\right)$$

$$N^n = 120m - H_2^n + z_0 Q_2^n$$

$$H_1^{n+1} = 120m - M^{n+1} Q_1^{n+1} |Q_1^{n+1}|$$

$j=2,3$ $$H_j^{n+1} = \frac{H_{j-1}^n + H_{j+1}^n}{2} + \frac{z_0}{2}\left(Q_{j-1}^n - Q_{j+1}^n\right)$$

$$Q_j^{n+1} = \frac{Q_{j-1}^n + Q_{j+1}^n}{2} + \frac{1}{2z_0}\left(H_{j-1}^n - H_{j+1}^n\right)$$

RB $$H_4^{n+1} = 100m \qquad Q_4^{n+1} = Q_3^n + \frac{1}{z_0}\left(H_3^n - H_4^{n+1}\right)$$

1

t(s)	x(m) / N(m)	0 H(m)	0 Q(l/s)	500 H	500 Q	1000 H	1000 Q	1500 H	1500 Q
0	20	100	0	100	0	100	0	100	0
0.5	20	118.3	1.8	100	0	100	0	100	0
1.0	~~2.7~~	118.3	1.8	118.3	1.8	100	0	100	0
1.5	1.7								

← IC

↑ RBC