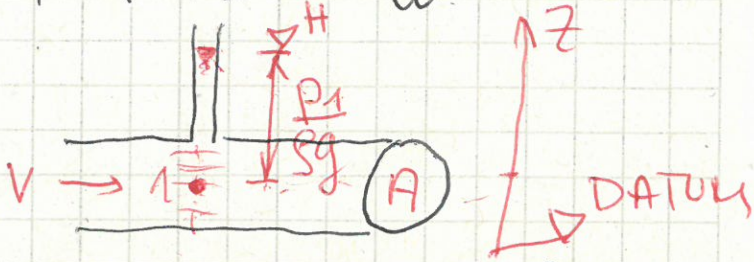
13-10-23

2.4 MoC FOR TRANSIENT PIPE FLOW

EQUATIONS:

$$\frac{\partial H}{\partial t} + \frac{c^2}{gA}\frac{\partial Q}{\partial x} = 0 \quad (1)$$

$$\frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} = 0 \quad (2)$$



Q - DISCHARGE $[m^3/s]$

$Q = VA$

H - HYDRAULIC HEAD $[m]$

$$H = \underbrace{\frac{p}{\rho g}}_{\text{PRESSURE HEAD}} + z$$

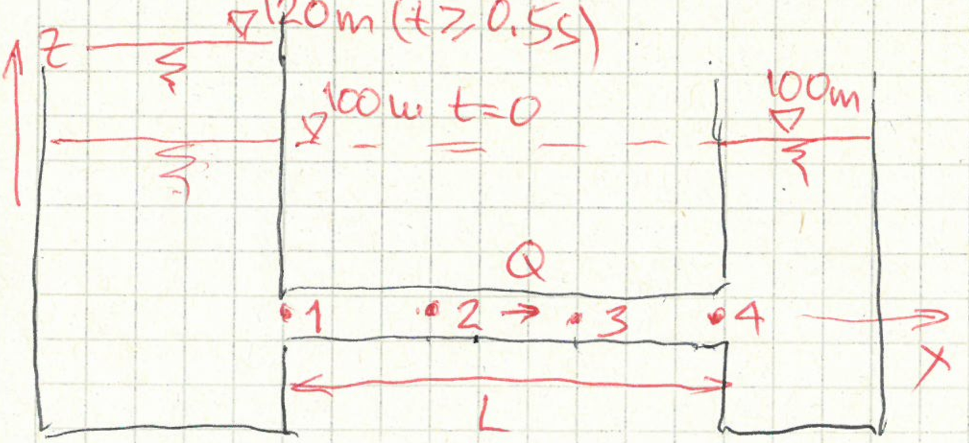
$$H_1 = \frac{p_1}{\rho g} + z_1$$

We can combine (1) & (2) to derive the characteristic form:

$$\boxed{\begin{array}{ll} dH + z_0\, dQ = 0 & C^+: \frac{dx}{dt} = c \\ dH - z_0\, dQ = 0 & C^-: \frac{dx}{dt} = -c \end{array}}$$

$$z_0 = \frac{c}{gA}$$

EXAMPLE PIPE 1 - SUDDEN SURGE OF PRESSURE (H)



$g \approx 10\ m/s^2$

$L = 1500\ m$

$D = 113\ mm$

$A = 0.01\ m^2$

$c = 1000\ m/s$

$\Delta x = 500\ m$

$\Delta t = \frac{\Delta x}{c} = 0.5\ s$

$z_0 = \frac{c}{gA} = 10^4\ s/m^2$

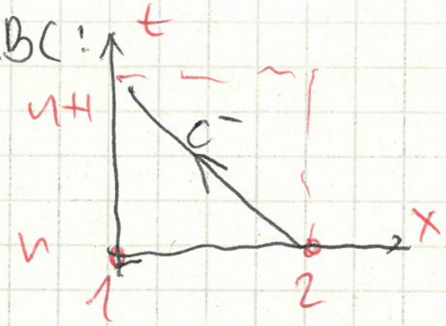
IC: $H(0 \leq x \leq L, t=0) = 100\ m$

$Q(0 \leq x \leq L, t=0) = 0$

LBC: $H(x=0, t \geq 0.5s) = 120\ m$

RBC: $H(x=L, t) = 100\ m$

LBC:



$$H_1^{n+1} = 120\,m$$

$$C^-: \; dH - Z_0\, dQ = 0$$

INTEGRATED BETWEEN $(n,2)$ & $(n+1,1)$

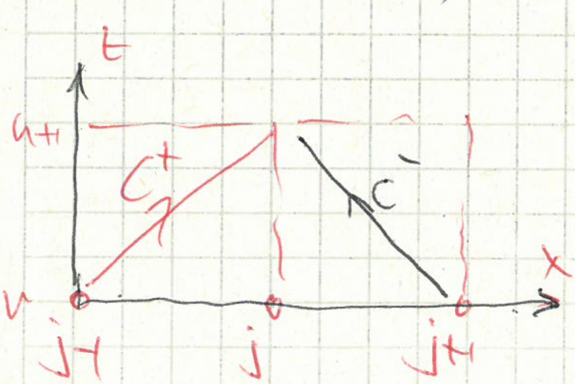
$$H_1^{n+1} - H_2^n - Z_0\left(Q_1^{n+1} - Q_2^n\right) = 0 \quad (3)$$

H_1^{n+1} IS KNOWN SO WE SOLVE (3) FOR Q_1^{n+1}:

$$H_1^{n+1} - H_2^n - Z_0\, Q_1^{n+1} + Z_0\, Q_2^n = 0$$

$$\boxed{Q_1 = Q_2^n + \frac{1}{Z_0}\left(H_1^{n+1} - H_2^n\right)}$$

MIDDLE POINTS



$$C^+: \; dH + Z_0\, dQ = 0$$

$$C^-: \; dH - Z_0\, dQ = 0$$

INTEGRATED BETWEEN $C^+: (n, j-1)$ & $(n+1, j)$
$C^-\,(n, j+1)$ & $(n+1, j)$

$$H_j^{n+1} - H_{j-1}^n + Z_0\left(Q_j - Q_{j-1}^n\right) = 0 \quad (4)$$

$$H_j^{n+1} - H_{j+1}^n - Z_0\left(Q_j^{n+1} - Q_{j+1}^n\right) = 0 \quad (5)$$

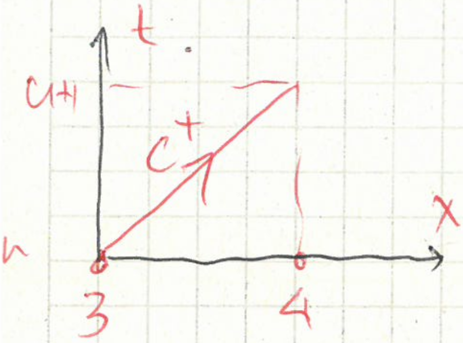
(4) + (5)

$$H_j^{n+1} = \frac{1}{2}\left(H_{j-1}^n + H_{j+1}^n\right) + \frac{Z_0}{2}\left(Q_{j-1}^n - Q_{j+1}^n\right)$$

(4) − (5)

$$Q_j^{n+1} = \frac{1}{2}\left(Q_{j-1}^n + Q_{j+1}^n\right) + \frac{1}{2Z_0}\left(H_{j-1}^n - H_{j+1}^n\right)$$

RBC



$$H_4^{n+1} = 100\,m$$

$$c^+: \; dH + z_0\, dQ = 0$$

INTEGRATED BETWEEN H(n,3) & (n+1,4).

$$H_4^{n+1} - H_3^{n} + z_0\left(Q_4^{n+1} - Q_3^{n}\right) = 0$$

H_4^{n+1} IS KNOWN,

$$H_4^{n+1} - H_3^{n} + z_0 Q_4^{n+1} - z_0 Q_3^{n} = 0$$

$$\boxed{Q_4^{n+1} = Q_3^{n} + \frac{1}{z_0}\left(H_3^{n} - H_4^{n+1}\right)}$$

ALL EQUATIONS

$$H_1^{n+1} = 120\,m \qquad Q_1^{n+1} = Q_1^{n} + 10^{-4}\left(H_1^{n+1} - H_2^{n}\right)$$

j = 2, 3

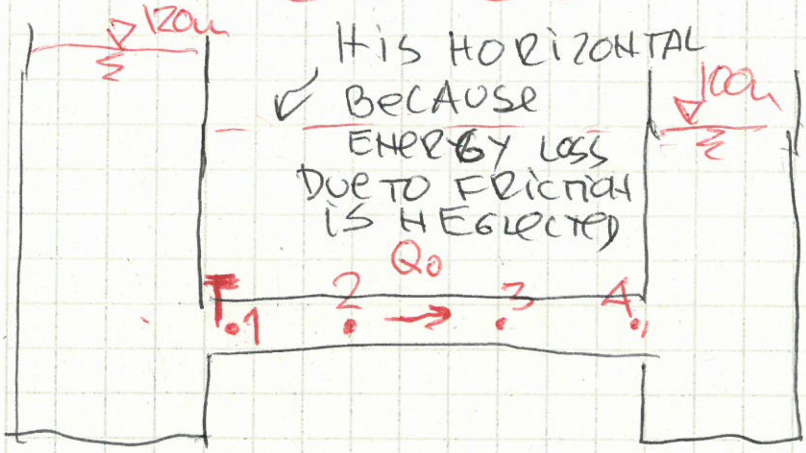
$$H_j^{n+1} = \frac{1}{2}\left(H_{j-1}^{n} + H_{j+1}^{n}\right) + \frac{10^4}{2}\left(Q_{j-1}^{n} - Q_{j+1}^{n}\right)$$

$$Q_j^{n+1} = \frac{1}{2}\left(Q_{j-1}^{n} + Q_{j+1}^{n}\right) + \frac{1}{2\times 10^4}\left(H_{j-1}^{n} - H_{j+1}^{n}\right)$$

$$H_4^{n+1} = 100\,m \qquad Q_4^{n+1} = Q_3^{n} + 10^{-4}\left(H_3^{n} - H_4^{n+1}\right)$$

x(m)	0		500		1000		1500		
t(s)	H (m)	Q (m^3/s)	H	Q	H	Q	H	Q	
0	100	0	100	0	100	0	100	0	← IC
0.5	120	20×10^{-4}	100	0	100	0	100	0	
1.0	120	20×10^{-4}	120	20×10^{-4}	100	0	100	0	
1.5	120	20×10^{-4}	120	20×10^{-4}	120	20×10^{-4}	100	0	
	120						100	40×10^{-4}	
	LBC						RBC		

EXAMPLE PIPE 2



$g \cong 10\, m/s^2$

$L = 1500\, m$

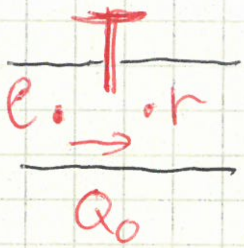
$A = 0.01\, m/s$

$Z_0 = 10^4\, s/m^2$

VALVE

$C_Q = 0.125$

$A_v = \frac{1}{4} A$



IC: $H(0 \le x \le L, t=0) = 100\, m$

$Q(0 \le x \le L, t=0) = Q_0$

$Q_0 = C_Q A_v \sqrt{2g(H_l - H_r)} =$

$Q_0 = 0.125 \frac{1}{4} 0.01 \sqrt{20 \times (120-100)}$

$Q_0 = 6.25 \times 10^{-3}\, m^3/s$

LBC: $Q(x=0, t \ge 0.5s) = 0$

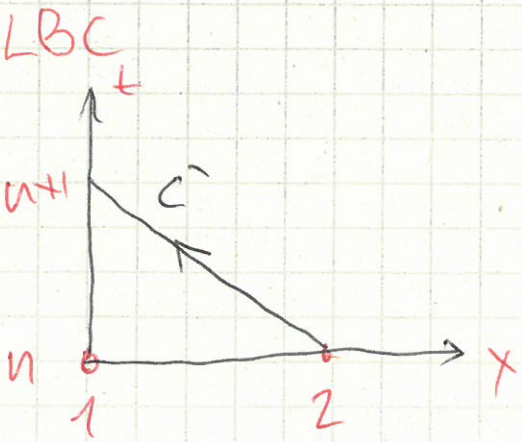
RBC: $H(x=L, t) = 100\, m$

$\Delta x = 500\, m$

$\Delta t = 0.5\, s$

SOLUTION

LBC



$Q_1^{n+1} = 0$

C^-: $dH - Z_0\, dQ = 0$

INTEGRATED BETWEEN $(n,2)$ & $(n+1,1)$

$H_1^{n+1} - H_2^n - Z_0 (Q_1^{n+1} - Q_2^n) = 0$

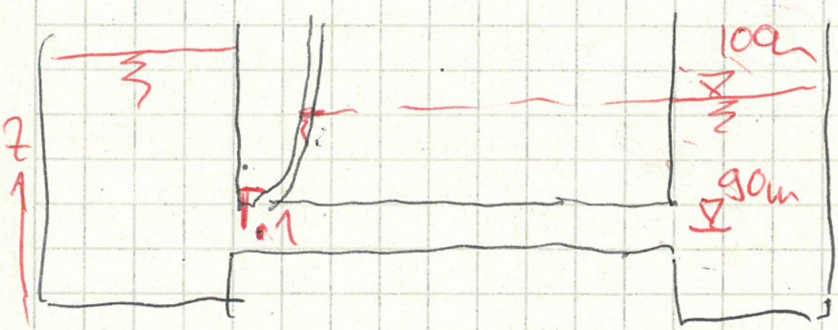
Q_1^{n+1} IS KNOWN $\rightarrow$ $H_1^{n+1} = H_2^n + Z_0 (\underbrace{Q_1^{n+1}}_{0} - Q_2^n)$

$$\boxed{H_1^{n+1} = H_2^n - Z_0 Q_2^n}$$

ALL OTHER EQUATIONS ARE THE SAME AS IN PIPE1

X(m)	0		500		1000		1500		
t (s)	H(m)	Q($10^{-3}\frac{m^3}{s}$)	H	Q	H	Q	H	Q	
0	100	6.25	100	6.25	100	6.25	100	6.25	IC
0.5	37.5	0	100	6.25	100	6.25	100	6.25	
1.0	37.5	0	37.5	0	100	6.25	100	6.25	
1.5	37.5	0	37.5	0	37.5	0	100	6.25	
2.0	37.5	0	37.5	0	37.5	0	100	-6.25	

CHECK IF THE SIMULATION RESULT IS PHYSICALLY POSSIBLE



$t = 0$

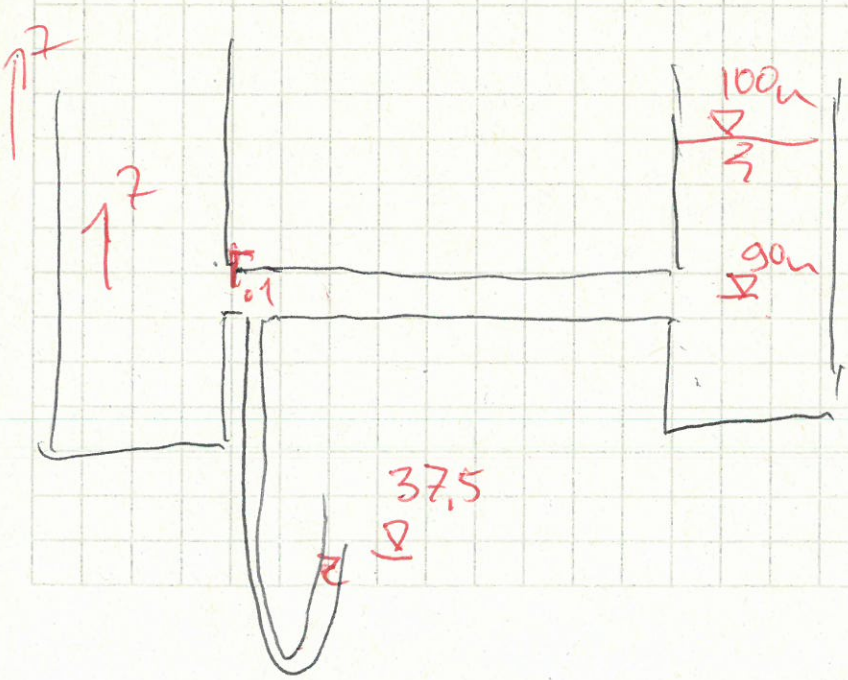
$H_1 = 100m$

$z_1 = 90m$

$$\frac{P_1}{\rho g} = H_1 - z_1 = 10m$$

$P_1 \sim 1 bar$

GAUGE PRESSURE



$t = 0.5s$

$$\frac{P_1}{\rho g} = H_1 - z_1 = 37.5 - 90m = -52.5m$$

$P_1 \sim -5 bar$

NOT POSSIBLE BECAUSE WATER WOULD EVAPORATE AT VERY LOW BUT POSITIVE P