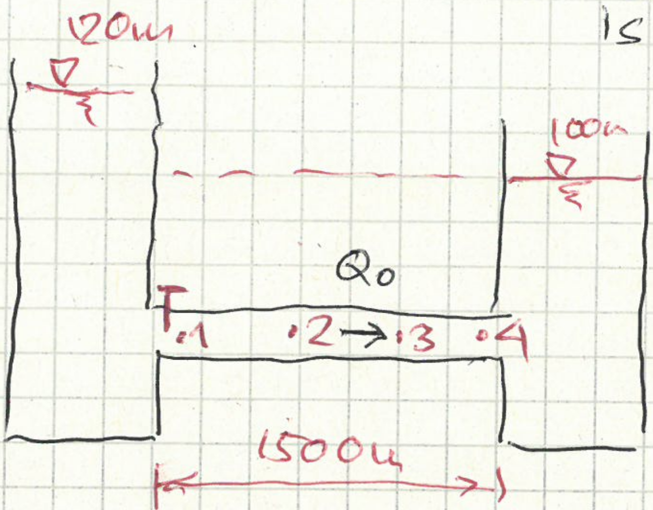
24-11-23

EXAMPLE PIPE5 - SIMILAR TO PIPE 2

EXCEPT THAT THE VALVE CLOSURE IS GRADUAL



$c = 1000\,m/s$

$g \cong 10\,m/s^2$

$A = 0.01\,m^2$

$A_{V0} = \frac{1}{4} A$

$C_Q = 0.125$

$Z_0 = \frac{c}{gA} = 10^4\,s/m^2$

$M_0 = \frac{1}{2g\,C_Q^2\,A_{V0}^2} = 512\,000\,\frac{s^2}{m^5}$

IC - STEADY STATE

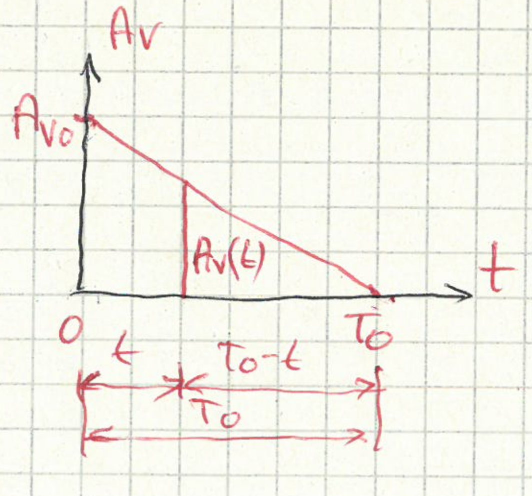
$$M_0 Q_0^2 = 120m - 100m = 20m$$

$$Q_0 = 6.25 \times 10^{-3}\,m^3/s$$

L.B.C GRADUAL VALVE CLOSURE DURING TIME T_0

$T_0 = 3s\,;\,6s$

- FOR $0 \le t < T_0$ ← IMPORTANT TO APPLY FOR $t < T_0$, NOT $t = T_0$



WE ASSUME THAT A_V CHANGES LINEARLY WITH TIME.

$$\frac{A_v(t)}{A_{V0}} = \frac{T_0 - t}{T_0}$$

$$\boxed{A_v(t) = A_{V0}\left(1 - \frac{t}{T_0}\right)}$$

~~...~~ THE L.B.C. IS THE VALVE:

$$M^{n+1} Q_1^{n+1} |Q_1^{n+1}| = H_T - H_1^{n+1} \qquad (1)$$

COMBINING (1) WITH THE C^- BETWEEN $(n,2)$ & $(n+1,1)$ WE GET:

$$M^{n+1} Q_1^{n+1} |Q_1^{n+1}| + Z_0 Q_1^{n+1} = N^n \qquad N^n = 120_u - H_2^n + Z_0 Q_2^n$$

SOLUTION

$$Q_1^{n+1} = \frac{N^n}{q} \qquad q = \frac{1}{2}\left[Z_0 + \sqrt{Z_0 + 4 M^{n+1} |N^n|}\right.$$

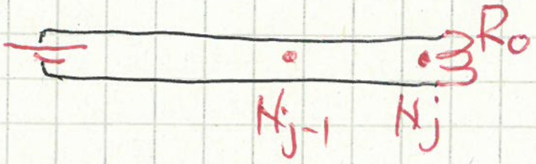
$$M^{n+1} = \frac{1}{2g\, C_Q^2 \, (A_v(t)^{n+1})^2}$$

$$(1) \rightarrow \quad H_1^{n+1} = H_T - M^{n+1} Q_1^{n+1} |Q_1^{n+1}|$$

- FOR $T_0 \leqslant t$ THE L.B.C $Q_1^{n+1} = 0$

FROM C^- WE GET $\quad H_1^{n+1} = H_2^n - Z_0 Q_2^n$

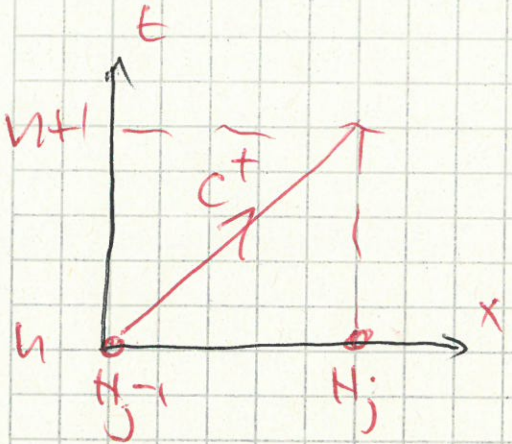
2.5.2 LINEAR BOUNDARY CONDITION



$$V_{N_j} = R_0 I_{N_j}$$

VALID AT ANY TIME SO:

$$V_{N_j}^{n+1} = R_0 I_{N_j}^{n+1}$$



C^+: $dV + Z_0 dI = 0$

INTEGRATED BETWEEN (n, N_{j-1}) & $(n+1, N_j)$

$$V_{N_j}^{n+1} - V_{N_{j-1}}^{n} + Z_0 \left(I_{N_j}^{n+1} - I_{N_{j-1}}^{n} \right) = 0$$

$$\underline{R_0 I_{N_j}^{n+1}} - V_{N_{j-1}}^{n} + Z_0 \underline{I_{N_j}^{n+1}} - Z_0 I_{N_{j-1}}^{n} = 0$$

$$\boxed{\begin{aligned} I_{N_j}^{n+1} &= I_{N_{j-1}}^{n} \frac{Z_0}{Z_0 + R_0} + V_{N_{j-1}} \frac{1}{Z_0 + R_0} \\ V_{N_j}^{n+1} &= R_0 I_{N_j}^{n+1} \end{aligned}}$$

2.6 LAX SCHEME FOR SYSTEMS OF TWO EQUATIONS

$$\frac{\partial p}{\partial t} + E\frac{\partial v}{\partial x} = 0 \quad (1)$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad (2)$$

CONTINUOUS FORM OF THE MATHEMATICAL MODEL

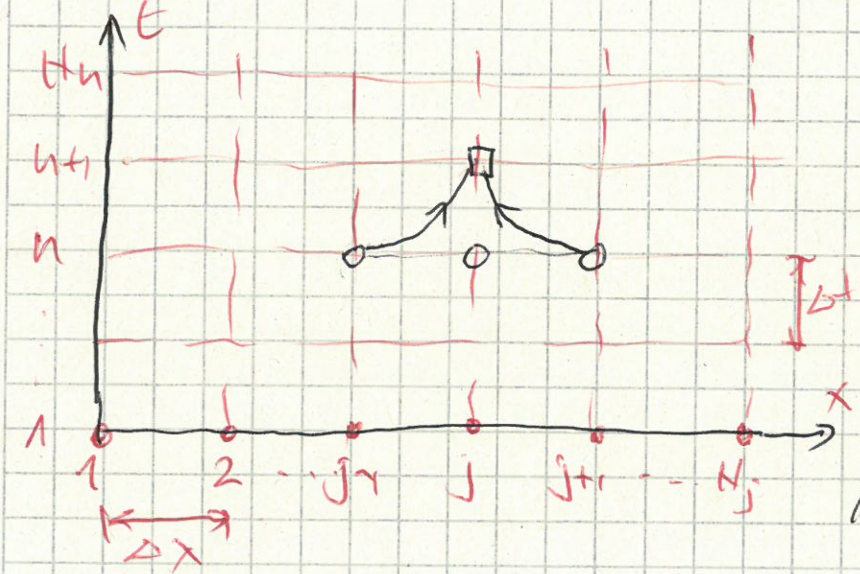
CHARACTERISTIC FORM:

$dp + z_0\, dV = 0$ ALONG $\frac{dx}{dt} = c$

$dp - z_0\, dV = 0$ ALONG $\frac{dx}{dt} = -c$

$z_0 = \sqrt{E\rho} \qquad c = \sqrt{\frac{E}{\rho}}$

DISCRETE FORM OF THE SYSTEM (1), (2)



TO APPLY THE LAX SCHEME

$$\left(\frac{\partial p}{\partial t}\right)_j \simeq \frac{1}{\Delta t}\left[p_j^{n+1} - \frac{1}{2}\left(p_{j-1}^n + p_{j+1}^n\right)\right]$$

$$\left(\frac{\partial p}{\partial x}\right)_j \simeq \frac{1}{2\Delta x}\left(p_{j+1}^n - p_{j-1}^n\right)$$

APPROXIMATE EQUATIONS FOR $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$ ARE ANALOGOUS

MIDDLE POINTS

$$\frac{1}{\Delta t}\left[p_j^{n+1} - \frac{1}{2}\left(p_{j-1}^n + p_{j+1}^n\right)\right] + \frac{E}{2\Delta x}\left(v_{j+1}^n - v_{j-1}^n\right) = 0 \rightarrow p_j^{n+1}$$

$$\frac{1}{\Delta t}\left[v_j^{n+1} - \frac{1}{2}\left(v_{j-1}^n + v_{j+1}^n\right)\right] + \frac{1}{\rho\, 2\Delta x}\left(p_{j+1}^n - p_{j-1}^n\right) = 0 \rightarrow v_j^{n+1}$$

$$\boxed{\begin{aligned} p_j^{n+1} &= \frac{1}{2}\left(p_{j-1}^n + p_{j+1}^n\right) + \frac{E\Delta t}{2\Delta x}\left(v_{j-1}^n - v_{j+1}^n\right) \\ v_j^{n+1} &= \frac{1}{2}\left(v_{j-1}^n + v_{j+1}^n\right) + \frac{\Delta t}{2\rho\Delta x}\left(p_{j-1}^n - p_{j+1}^n\right) \end{aligned}}$$

$$\frac{E\,\Delta t}{\Delta x}\frac{c}{c} = E\, C_r \frac{\sqrt{\rho}}{\sqrt{E}} = C_r \underbrace{\sqrt{E\rho}}_{Z_0} = Z_0 C_r \qquad c = \sqrt{\frac{E}{\rho}}$$

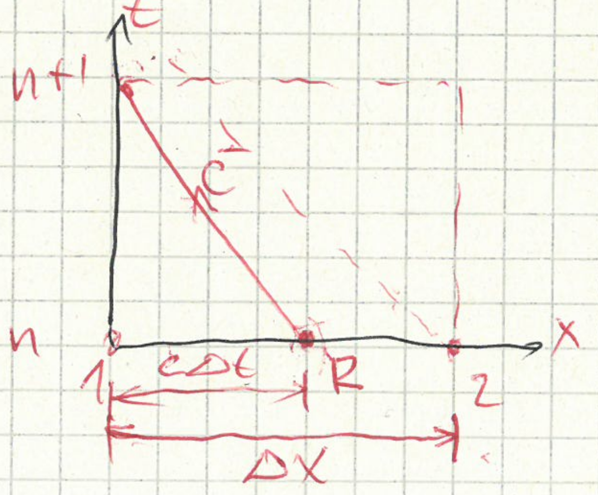
$$\frac{\Delta t}{\rho\,\Delta x}\frac{c}{c} = \frac{1}{\rho} C_r \frac{\sqrt{\rho}}{\sqrt{E}} = \frac{C_r}{\sqrt{E\rho}} = \frac{C_r}{Z_0}$$

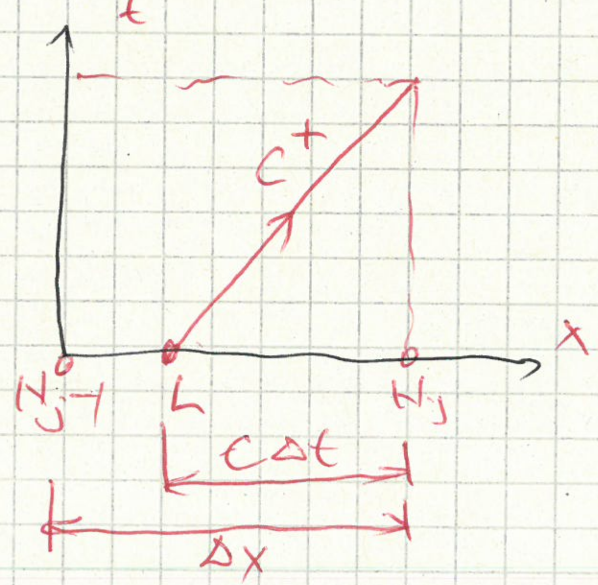
$$\boxed{\begin{aligned} P_j^{n+1} &= \frac{1}{2}\left(P_{j-1}^n + P_{j+1}^n\right) + \frac{C_r Z_0}{2}\left(V_{j-1}^n - V_{j+1}^n\right) \\ V_j^{n+1} &= \frac{1}{2}\left(V_{j-1}^n + V_{j+1}^n\right) + \frac{C_r}{2 Z_0}\left(P_{j-1}^n - P_{j+1}^n\right) \end{aligned}}$$

FOR $C_r = 1$ THE LAX SCHEME GIVES THE SAME DISCRETISED EQUATIONS AS THE MoC

$C_r \leq 1$ HAS TO BE SATISFIED FOR STABILITY

FOR BOUNDARY CONDITIONS IT IS STILL BETTER TO USE THE MoC

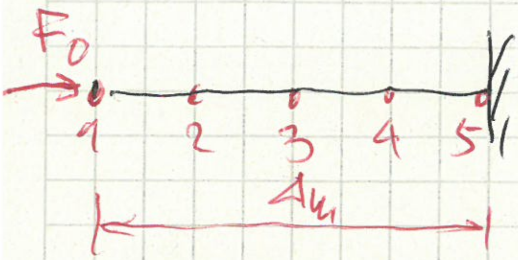




$$C_r = \frac{c\,\Delta t}{\Delta x}$$

$$\Delta x \overset{c}{=} c\,\Delta t$$

EXAMPLE ROD 2 - SAME AS ROD 1 EXCEPT THAT WE USE THE LAX SCHEME
$C_r = 0.8$



$E = 200\,GPa$ $\qquad c = \sqrt{\frac{E}{\rho}} = 5000\,\frac{m}{s}$

$\rho = 8000\,kg/m^3$ $\qquad Z_0 = \sqrt{E\rho} = 40\,\frac{MPa}{s}$

$A = 10^{-3}\,m^2$

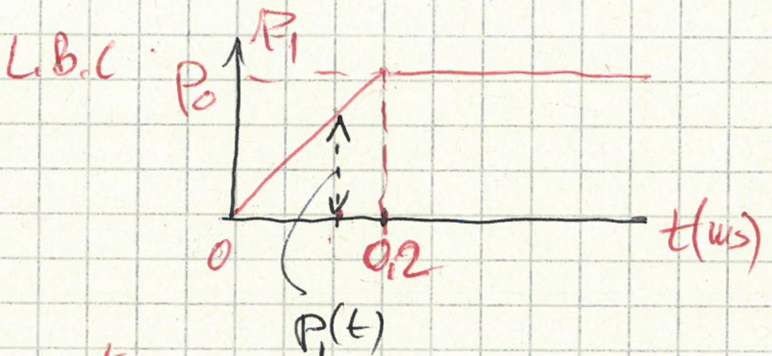
$P_0 = \frac{F_0}{A_0} = 1\,MPa$

$\Delta x = 1\,m$

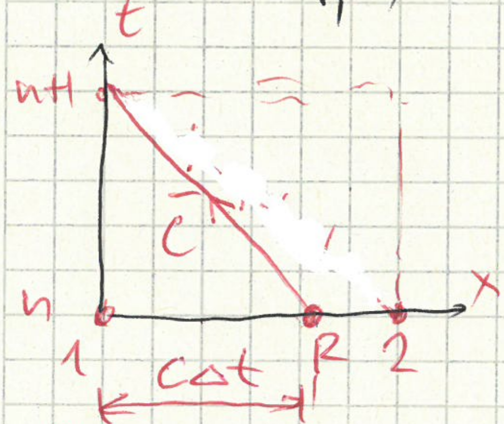
$C_r = 0.8$ $\qquad \frac{c\Delta t}{\Delta x} = 0.8 \rightarrow$

$\Delta t = 0.16\,ms$

L.B.C:



$$p = \begin{cases} t \leq 0.2\,ms & P_1(t) = P_0 \frac{t}{0.2} \\ t > 0.2\,ms & P_1 = P_0 \end{cases}$$



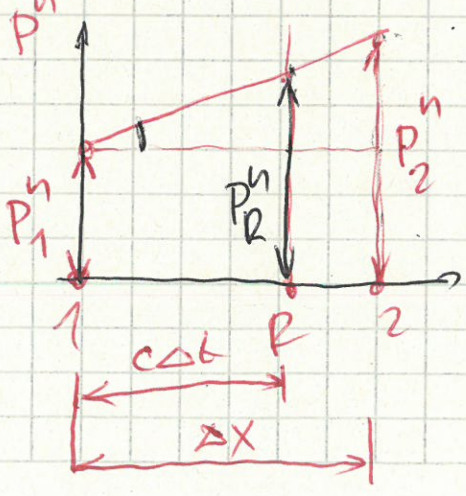
C^-: $dp - Z_0\,dv = 0$

INTEGRATED BETWEEN (n, R) & $(n+1, 1)$

$$P_1^{n+1} - P_R^n - Z_0\left(V_1^{n+1} - V_R^n\right) = 0$$

$$P_1^{n+1} - P_R^n - Z_0 V_1^{n+1} + Z_0 V_R^n = 0$$

$$\boxed{V_1^{n+1} = V_R^n + \frac{1}{Z_0}\left(P_1^{n+1} - P_R^n\right)}$$



WE FIND P_R^n BY INTERPOLATION

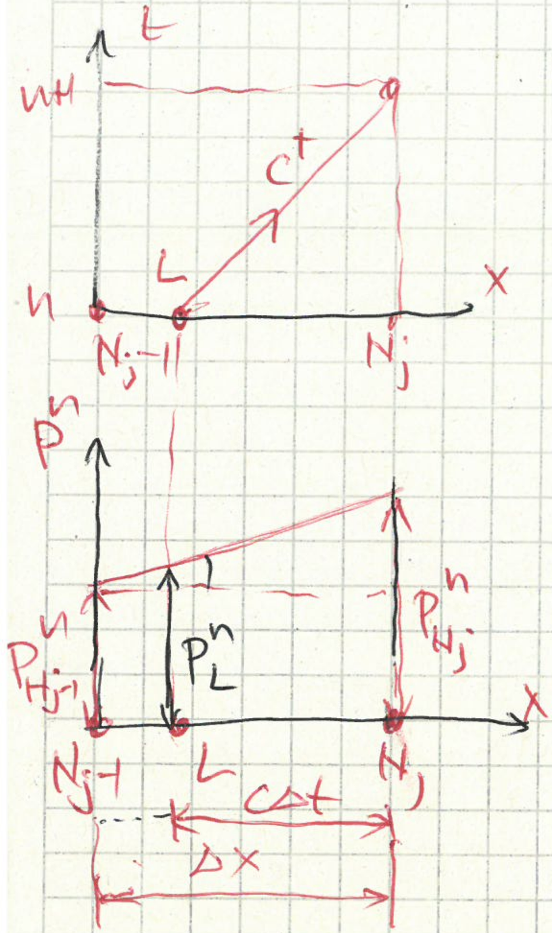
$$P_R^n = P_1^n + \frac{P_2^n - P_1^n}{\Delta x} c\Delta t$$

C_r

$$P_R^n = P_1^n + C_r\left(P_2^n - P_1^n\right)$$

$$\boxed{P_R^n = P_1^n(1 - C_r) + P_2^n C_r}$$

RIGHT BC



C^+: $dp + Z_0 dV = 0$

INTEGRATED BETWEEN (n, L) & $(n+1, N_j)$

$$P_{N_j}^{n+1} - P_L^n + Z_0 \left(\underbrace{V_{N_j}^{n+1}}_{=0} - V_L^n\right) = 0$$

$$\boxed{P_{N_j}^{n+1} = P_L^n + Z_0 V_L^n}$$

$$P_L^n = P_{N_{j-1}}^n + \frac{P_{N_j}^n - P_{N_{j-1}}^n}{\Delta x}(\Delta x - c\Delta t)$$

$$P_L^n = P_{N_{j-1}}^n + (P_{N_j}^n - P_{N_{j-1}}^n)(1 - C_r)$$

$$P_L^n = \cancel{P_{N_{j-1}}^n} + P_{N_j}^n - \cancel{P_{N_{j-1}}^n} - C_r P_{N_j}^n + C_r P_{N_{j-1}}^n$$

$$\boxed{\begin{aligned} P_L^n &= P_{N_j}^n (1 - C_r) + P_{N_{j-1}}^n C_r \\ V_L^n &= V_{N_j}^n (1 - C_r) + V_{N_{j-1}}^n C_r \end{aligned}}$$

RESULTS

x(m)	0		0.8		1		2		3		3.2		4		
j	1		R		2		3		4		L		5		
t (ms)	P MPa	V m/s	P MPa	V m/s	P	V	P	V	P	V	P	V	P	V	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	IC
0.16	0.8	20	0.16	4.0	0	0	0	0	0	0	0	0	0	0	
0.32	1.0	25	0.77	19.4	0.72	18	0	0	0	0	0	0	0	0	
0.48	1.0	25	0.92	23.0	0.90	22.5	0.65	16.2	0	0	0	0	0	0	
0.60	1.0	25	0.97	24.3	0.96	24.1	0.81	20.2	0.58	14.6	0.46	11.7	0	0	
0.72	1.0	25	0.98	24.6	0.98	24.5	0.93	23.2	0.73	18.2	0.77	14.6	0.93	0	
0.88	1.0	25	0.99	24.9	0.99	24.8	0.96	23.9	1.30	11.5	1.31	9.2	1.35	0	

LBC

RBC