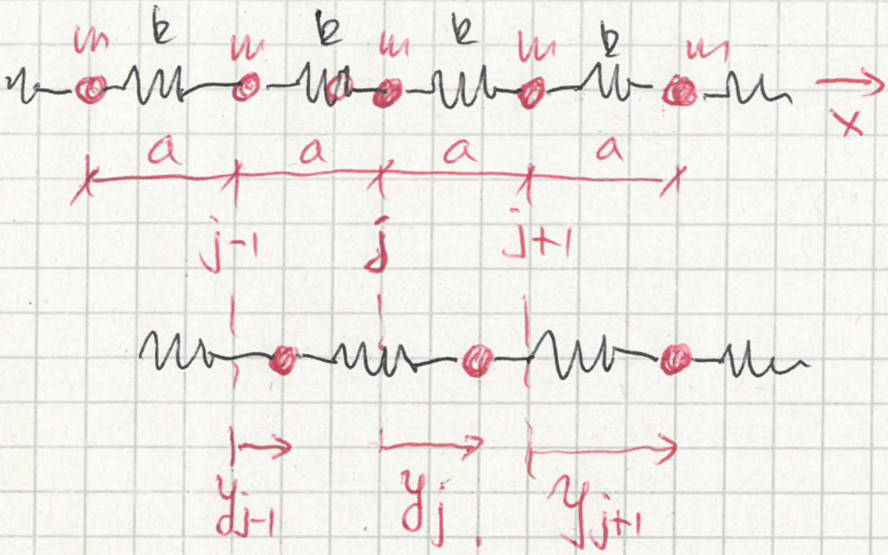
31-10-23

3 TRAVELLING WAVES

LARGE OR INFINITE NUMBER OF COUPLED OSCILLATORS

3.1 COUPLED MASSES AND SPRINGS

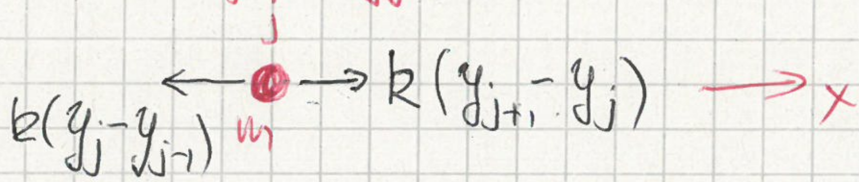


EQUILIBRIUM

y - DISPLACEMENT

$V = \frac{dy}{dt}$ VELOCITY

$\frac{d^2y}{dt^2}$ ACCELERATION



MOTION EQUATION FOR MASS j:

$$m\frac{d^2y}{dt^2} = k(y_{j+1} - y_j) - k(y_j - y_{j-1})$$

$$\frac{d^2y}{dt^2} = \frac{k}{m}(y_{j+1} - y_j) - \frac{k}{m}(y_j - y_{j-1}) \qquad (1)$$

ASSUMING THAT y IS A SMOOTH FUNCTION OF X WE CAN APPLY TAYLOR SERIES

$$y_{j+1} - y_j = \frac{\partial y}{\partial x}a + \frac{\partial^2 y}{\partial x^2}\frac{a^2}{2} + \dots$$

$$y_{j-1} - y_j = -\frac{\partial y}{\partial x}a + \frac{\partial^2 y}{\partial x^2}\frac{a^2}{2} + \dots$$

PLUGGING THESE INTO (1) AND REPLACING d^2/dt^2 WITH $\partial^2/\partial t^2$ WE GET

$$\frac{\partial^2 y}{\partial t^2} = \frac{k}{m}\left[\cancel{\frac{\partial y}{\partial x}a} + \frac{\partial^2 y}{\partial x^2}\frac{a^2}{2}\right] + \frac{k}{m}\left[\cancel{-\frac{\partial y}{\partial x}a} + \frac{\partial^2 y}{\partial x^2}\frac{a^2}{2}\right]$$

$$(2)\quad \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{k}{m}a^2\frac{\partial^2 y}{\partial x^2}}$$

WAVE EQUATION $\qquad c^2 = \frac{ka^2}{m}$

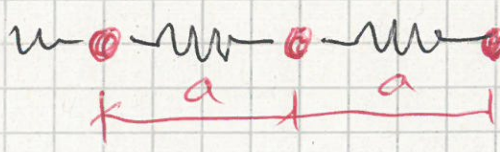
ALTERNATIVELY: $\quad V = \frac{\partial y}{\partial t} \qquad S = \frac{\partial y}{\partial x}$

$$(2) \rightarrow \quad \boxed{\begin{aligned} \frac{\partial V}{\partial t} &= \frac{k}{m}a^2\frac{\partial S}{\partial x} \\ \frac{\partial S}{\partial t} &= \frac{\partial V}{\partial x} \end{aligned}}$$

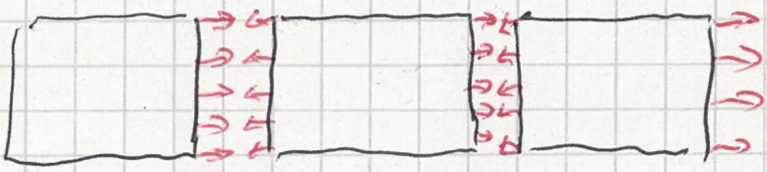
$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial V}{\partial x}$$

WAVE EQUATION $\qquad \frac{ka^2}{m} = c^2$

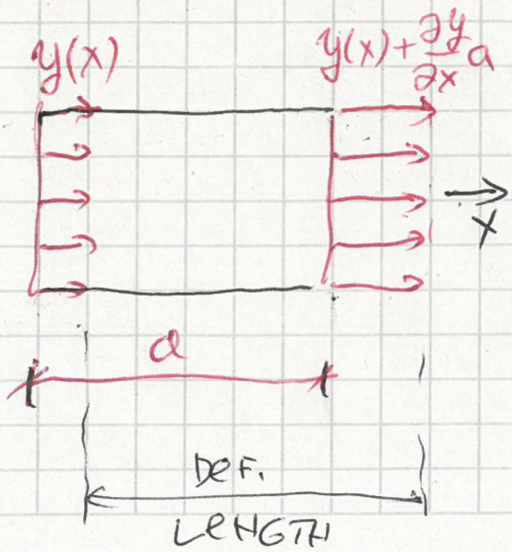
3.2 ELASTIC WAVES IN A BAR



$\rightarrow x \qquad F = ky$



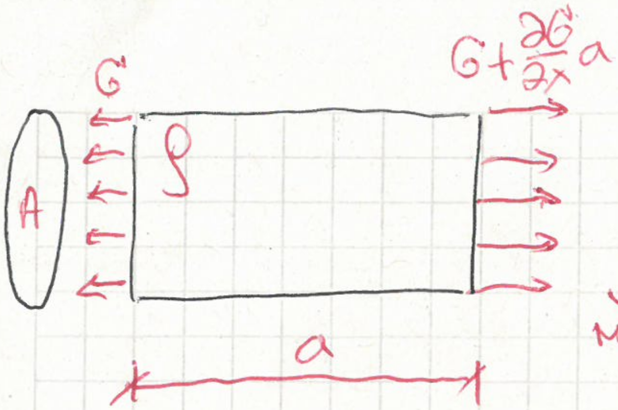
$\rightarrow x \qquad \sigma = E\varepsilon$



$$\text{STRAIN} = \frac{\text{DEFORMED LENGTH} - \text{ORIGINAL LENGTH}}{\text{ORIGINAL LENGTH}}$$

$$\text{DEFORMED LENGTH} = x + a + y + \frac{\partial y}{\partial x}a - (x+y) = a + \frac{\partial y}{\partial x}a$$

$$\varepsilon = \frac{\cancel{a} + \frac{\partial y}{\partial x}\cancel{a} - \cancel{a}}{\cancel{a}} = \frac{\partial y}{\partial x}$$



MOTION EQUATION

$$\underbrace{\rho A a}_{\text{MASS}} \frac{\partial^2 y}{\partial t^2} = \left(\sigma + \frac{\partial \sigma}{\partial x} a\right) A - \sigma A$$

$$\rho \cancel{A a} \frac{\partial^2 y}{\partial t^2} = \frac{\partial \sigma}{\partial x} \cancel{a A}$$

$\sigma = E\varepsilon$
$\varepsilon = \partial y / \partial x$

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho}\frac{\partial \sigma}{\partial x} = \frac{E}{\rho}\frac{\partial \varepsilon}{\partial x} = \frac{E}{\rho}\frac{\partial^2 y}{\partial x^2}$$

WAVE EQUATION

$$\boxed{\frac{\partial^2 y}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 y}{\partial x^2}}$$

$$c^2 = \frac{E}{\rho}$$

ALTERNATIVELY

$$v = \frac{\partial y}{\partial t}$$

$$\frac{\partial v}{\partial t} = \frac{E}{\rho}\frac{\partial \varepsilon}{\partial x} = \frac{1}{\rho}\frac{\partial \sigma}{\partial x}$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial y}{\partial t} = \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} = \frac{1}{\rho}\frac{\partial \sigma}{\partial x}$$

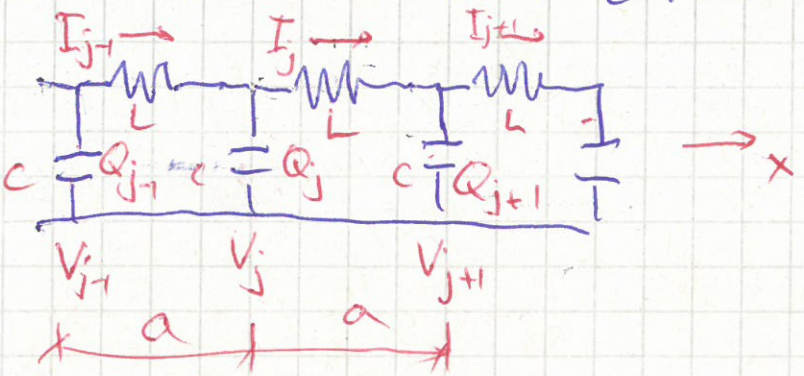
$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial x} \quad / \times E$$

$$\boxed{\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{\rho}\frac{\partial \sigma}{\partial x} \\ \frac{\partial \sigma}{\partial t} &= E\frac{\partial v}{\partial x} \end{aligned}}$$

WAVE EQUATIONS

$$c^2 = \frac{E}{\rho}$$

3.3 TRANSMISSION LINES



Q – CHARGE [C]

I – CURRENT [A]

V – VOLTAGE [V]

CHANGE OF VOLTAGE GENERATES CURRENT $\qquad V_j - V_{j+1} = L \frac{dI_j}{dt}$

CONSERVATION OF CHARGE FOR j $\qquad I_{j-1} - I_j = \frac{dQ_j}{dt} = C \frac{dV_j}{dt}$

$$V_{j+1} - V_j = \frac{\partial V}{\partial x} a \quad \rightarrow \quad -\frac{\partial V}{\partial x} a = L \frac{\partial I}{\partial t}$$

$$I_j - I_{j-1} = \frac{\partial I}{\partial x} a \qquad -\frac{\partial I}{\partial x} a = C \frac{\partial V}{\partial t}$$

$$\frac{\partial I}{\partial t} = -\frac{a}{L} \frac{\partial V}{\partial x}$$

$$\frac{\partial V}{\partial t} = -\frac{a}{C} \frac{\partial I}{\partial x}$$

$C' = \frac{C}{a}$ UNIT CAPACITANCE $\left[\frac{F}{m}\right]$

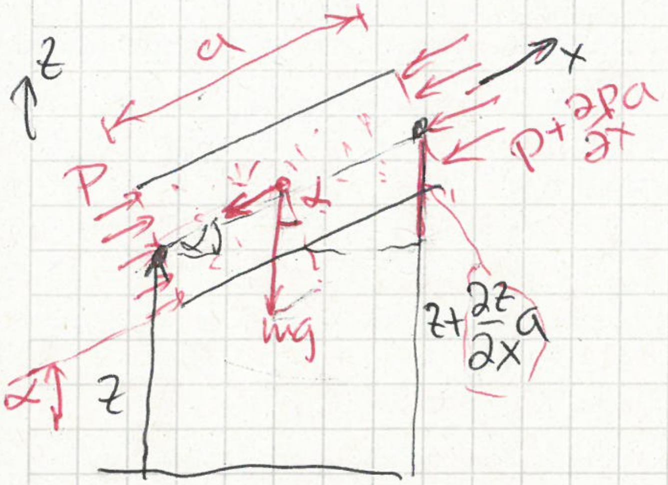
$L' = \frac{L}{a}$ UNIT INDUCTANCE $\left[\frac{H}{m}\right]$

$$\boxed{\begin{aligned} \frac{\partial I}{\partial t} &= -\frac{1}{L'} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial t} &= -\frac{1}{C'} \frac{\partial I}{\partial x} \end{aligned}}$$

WAVE EQUATIONS

3.4 TRANSIENT PIPE FLOW

(a) CHANGE OF PRESSURE GENERATES MOTION



$$\sin\alpha = \frac{\frac{\partial z}{\partial x}a}{a} = \frac{\partial z}{\partial x}$$

MOTION EQUATION

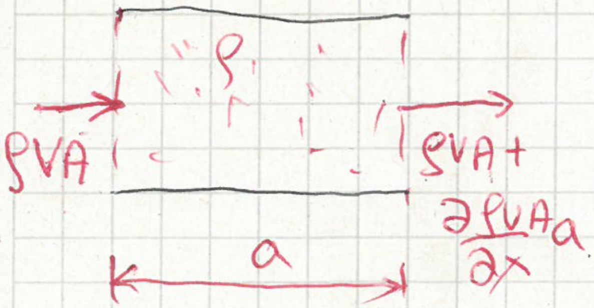
$$\rho a A \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} a A - \underbrace{\rho a A}_{\text{MASS}} g \sin\alpha$$

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} - \rho g \frac{\partial z}{\partial x} \Big/ \frac{1}{\rho g}$$

$$\frac{1}{g}\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x}\Big(\underbrace{\frac{p}{\rho g} + z}_{=H}\Big)$$

$$\boxed{\frac{\partial v}{\partial t} = -g\frac{\partial H}{\partial x}} \quad (1)$$

(b) FLUID MOTION GENERATES CHANGE OF DENSITY



MASS CONSERVATION

$\underbrace{\text{MASS IN} - \text{MASS OUT}}_{\text{PER UNIT TIME}}$ = CHANGE OF MASS

$$\rho v A - \Big(\rho v A + \frac{\partial \rho v A}{\partial x} a\Big) = \frac{\partial \rho a A}{\partial t}$$

$$-\frac{\partial \rho v}{\partial x} \cancel{Aa} = \cancel{aA}\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} = -\rho\frac{\partial v}{\partial x} - \underbrace{v\frac{\partial \rho}{\partial x}}_{\ll \rho\frac{\partial v}{\partial x}}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\rho\frac{\partial v}{\partial x}} \quad (2)$$

(c) CHANGE OF DENSITY GENERATES CHANGE OF PRESSURE

$$dp = -B\frac{dV}{V}$$

V – VOLUME [m^3]
B – BULK MODULUS [Pa]

$$dp = -B\frac{d(\cancel{m}/\rho)}{\cancel{m}/\rho} = -B\rho\, d\left(\frac{1}{\rho}\right) = -B\rho\left(-\frac{1}{\rho^2}\right)d\rho = \frac{B}{\rho}d\rho$$

$$\frac{dp}{d\rho} = \frac{B}{\rho}$$

$$\frac{\partial\rho}{\partial t} = -\rho\frac{\partial v}{\partial x}$$

$$\frac{d\rho}{dp}\frac{\partial p}{\partial t} = -\rho\frac{\partial v}{\partial x}$$

$$\frac{\cancel{\rho}}{B}\frac{\partial p}{\partial t} = -\cancel{\rho}\frac{\partial v}{\partial x}$$

$$\frac{\partial p}{\partial t} = -B\frac{\partial v}{\partial x} \quad \Big/ \frac{1}{\rho g}$$

$$\frac{\partial z}{\partial t} = 0$$

$$\frac{\partial\overbrace{\left(\frac{p}{\rho g} + z\right)}^{H}}{\partial t} = -\frac{B}{\rho g}\frac{\partial v}{\partial x} \qquad c^2 = \frac{B}{\rho}$$

$$\boxed{\begin{aligned}\frac{\partial H}{\partial t} &= -\frac{c^2}{g}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial t} &= -g\frac{\partial H}{\partial x}\end{aligned}}$$

WAVE EQUATIONS