Communication Systems - Assignment 5

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Spatial Multiplexing

Given a 2×2 MIMO system, suppose the channel matrix H is known. We use its Singular Value Decomposition:

$$H = U \Sigma V^h$$

where h denotes the Hermitian (conjugate transpose), U and V are unitary matrices and Σ is a diagonal matrix

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$$

where λ_1 and λ_2 are the eigenvalues of HH^h.

Precoding

At the transmitted side, we generate two independent QAM symbols X_1, X_2 and we collect them into a vector:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

We apply this precoder operation:

$$X' = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = VX$$

and we transmit the two complex symbols X_1' and X_2'. Note that:

- For the unitary matrix properties, X and X' have the same energy.
- X_1' and X_2' are not QAM symbols, but generic complex symbols.

Channel

We receive the vector

$$Y = HX' + N$$

where

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

is the noise vector, made by two independent samples of Gaussian complex random variables with zero mean. (In our assignment, we will not consider the noise and we will focus on the TX and RX symbols only.)

Postcoding

At the receiver side, given Y we apply the postcoding operation to obtain

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = U^h Y$$

This way, it is easy to show that we have

$$Z = \Sigma X + N'$$

where $N' = \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} = U^h N$ has the same energy of N.
Then, the two received symbols are equal to

$$Z_1 = \sqrt{\lambda_1} X_1 + N_1'$$
$$Z_2 = \sqrt{\lambda_2} X_2 + N_2'$$

and we can recover X_1 and X_2 by using their respective QAM decision regions.

Exercise 5.1

1. Generate two 4-QAM symbols

$$X_1, X_2 \in \{\pm 1 \pm j\}$$

and collect them into a vector X.

2. Generate a channel matrix

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

where every element is a sample of an independent Raylegh pdf with $\sigma = 1$:

$$\mathcal{CN}\left(0, \sigma^2 = 1\right)$$

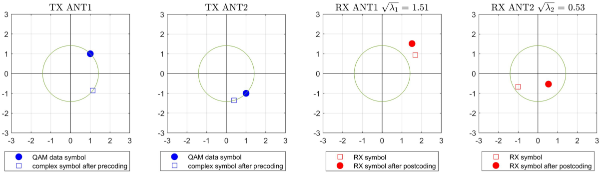
(Then the real and imaginary parts are samples of independent Gaussian pdfs $\mathcal{N}\left(0, \sigma^2 = 1/2\right)$.)

Exercise 5.1

3. Compute the SVD of $H = U\Sigma V^h$. (Note: the Matlab svd function automatically orders the eigenvalues of Σ in descending order.)
4. Apply the precoding $X' = VX$.
5. Plot the original QAM symbols and the new complex symbols after precoding: X and X'.

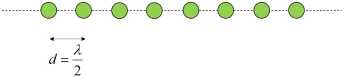
Exercise 5.1

6. Compute the received vector $Y = HX'$.
7. Apply the postcoding $Z = U^h Y$.
8. Plot the received symbols and the symbols after postcoding: Y and Z.
9. Compute the square roots of the (sorted) eigenvalues of HH^h: $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$.
10. Write them above the figure and verify that $Z_1 = \sqrt{\lambda_1} X_1$ and $Z_2 = \sqrt{\lambda_2} X_2$.


TX ANT1
QAM data symbol
complex symbol after precoding
TX ANT2
QAM data symbol
complex symbol after precoding
RX ANT1 $\sqrt{\lambda_1} = 1.51$
RX symbol
RX symbol after postcoding
RX ANT2 $\sqrt{\lambda_2} = 0.53$
RX symbol
RX symbol after postcoding

Uniform Linear Array

Consider a Uniform Linear Array (ULA) with M antenna elements at distance $d = \lambda$. Fix $f = 3.6GHz$.



The $M \times 1$ steering vector is given by

$$S = \begin{bmatrix} 1 \\ \vdots \\ e^{-jm\gamma} \\ \vdots \\ e^{-j(M-1)\gamma} \end{bmatrix} \quad 0 \leq m \leq (M-1)$$

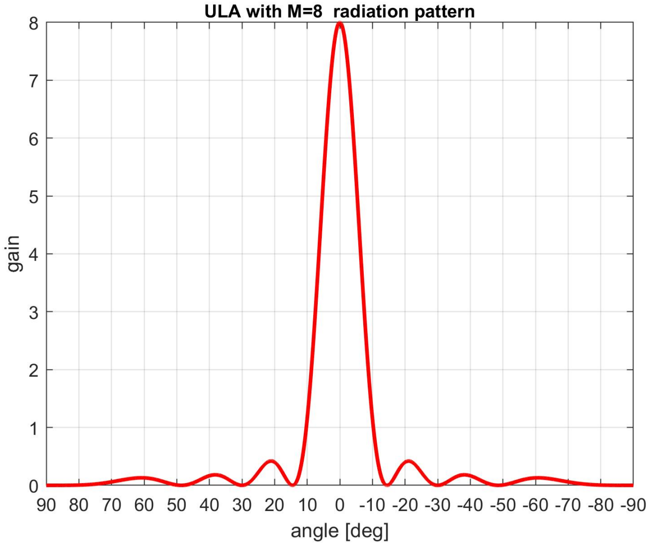
where

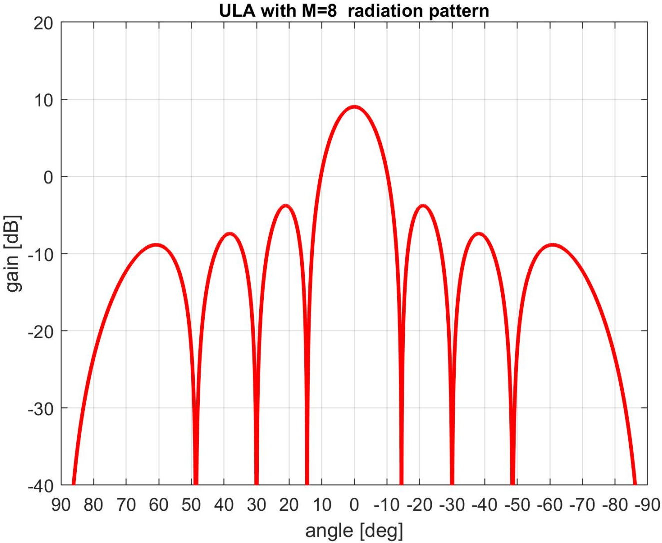
$$\gamma = \frac{2\pi d \sin(\Theta)}{\lambda} = \pi \sin(\Theta)$$

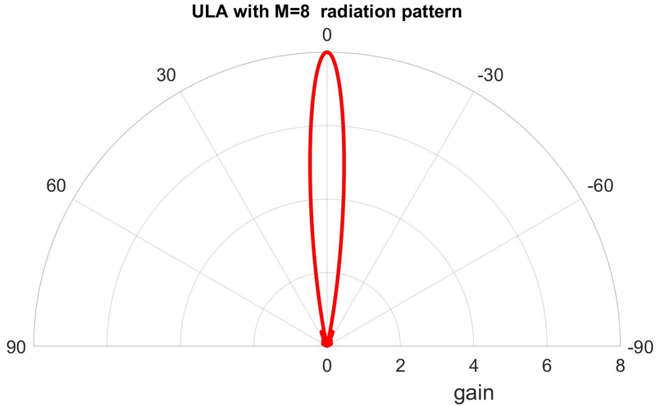
Exercise 5.2

Write a Matlab program that for any M:

1. Generates a $1 \times M$ beamformer vector
$$B = \frac{1}{\sqrt{M}} [1, \cdots, 1]$$
2. Computes the gain
$$g(\Theta) = |BS|^2$$
3. Plots g vs. linear Θ
4. Plots g [dB] vs. linear Θ
5. Plots g vs. polar Θ


ULA with M=8 radiation pattern
gain
angle [deg]
8
7
6
5
4
3
2
1
0
90 80 70 60 50 40 30 20 10 0 -10 -20 -30 -40 -50 -60 -70 -80 -90


ULA with M=8 radiation pattern
gain [dB]
20
10
0
-10
-20
-30
-40
90 80 70 60 50 40 30 20 10 0 -10 -20 -30 -40 -50 -60 -70 -80 -90
angle [deg]


ULA with M=8 radiation pattern
0
30
-30
60
-60
90
-90
0
2
4
6
8
gain

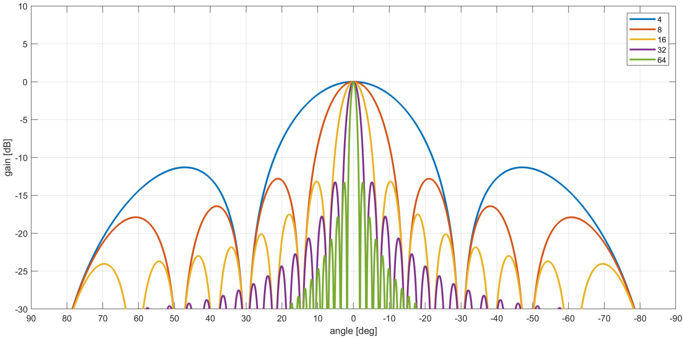
Polar plot

Matlab code

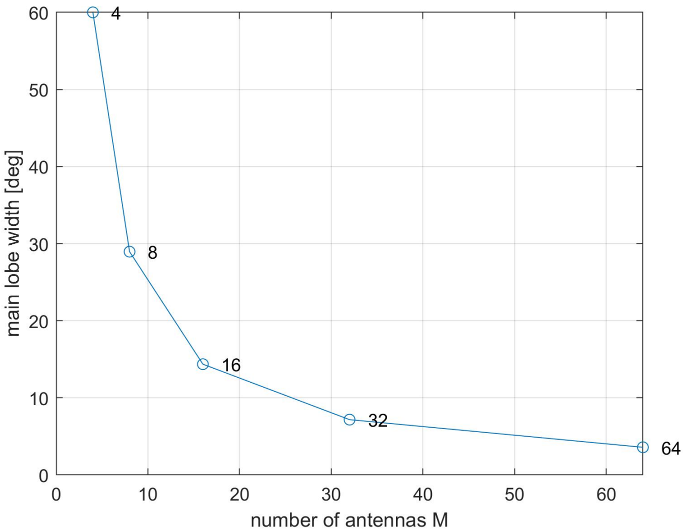
```
polarplot(theta, gain, 'r', 'LineWidth', 2);
ax = gca;
ax.ThetaZeroLocation = 'top';
ax.ThetaDir = 'counterclockwise';
ax.ThetaLim = [-90 90];
```

Exercise 5.2

1. Plot the **normalized** gains $g/\max g$ [dB] for $4 \leq M \leq 64$.
2. Plot the main lobe (null-to-null) width for $4 \leq M \leq 64$.
3. In the first graph, we note that the first secondary lobe peaks have almost the same amplitudes, prove why by using the analytic expression of g.



gain [dB]
angle [deg]
10
5
0
-5
-10
-15
-20
-25
-30
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
4
8
16
32
64



60
50
40
30
20
10
0
0
10
20
30
40
50
60
main lobe width [deg]
number of antennas M
4
8
16
32
64

Conventional Beamforming

Write a Matlab program that for any M, given an angle Θ_0:

1. Generates a $1 \times M$ beamformer vector

$$B = \frac{1}{\sqrt{M}} \left[1, \cdots, e^{jm\gamma_0}, \cdots, e^{j(M-1)\gamma_0} \right]$$

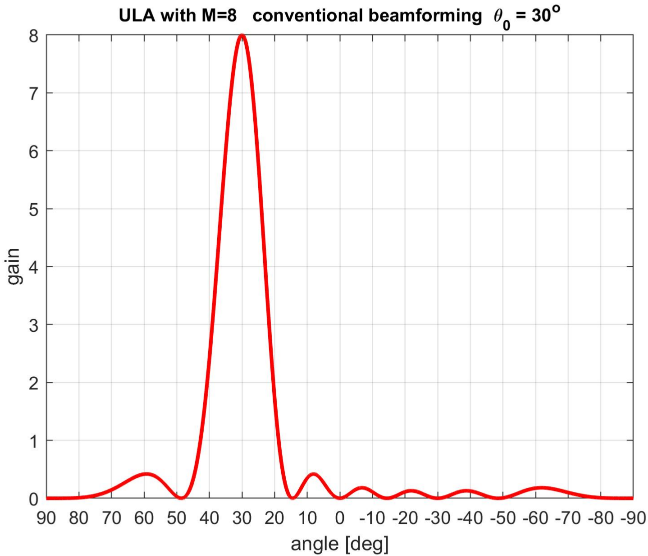
   where

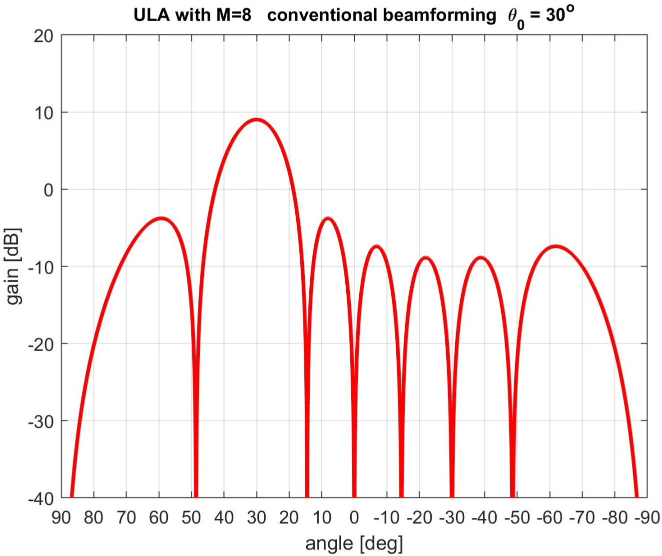
$$\gamma_0 = \frac{2\pi d \sin\left(\Theta_0\right)}{\lambda} = \pi \sin\left(\Theta_0\right)$$

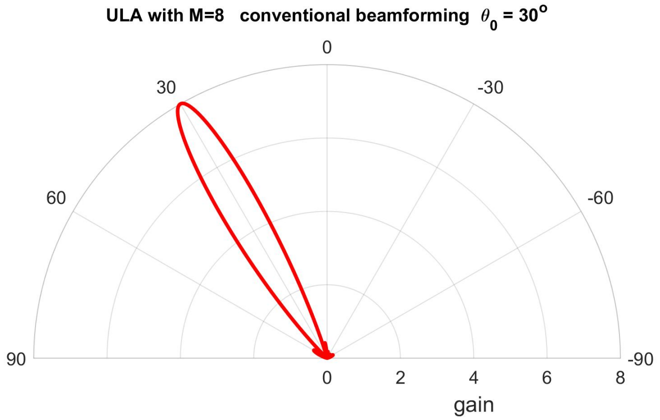
2. Computes the gain

$$g\left(\Theta\right) = \left|BS\right|^2$$

3. Plots g vs. linear Θ
4. Plots g [dB] vs. linear Θ
5. Plots g vs. polar Θ


ULA with M=8 conventional beamforming θ_0 = 30°
gain
angle [deg]
0
1
2
3
4
5
6
7
8
90 80 70 60 50 40 30 20 10 0 -10 -20 -30 -40 -50 -60 -70 -80 -90


ULA with M=8 conventional beamforming θ_0 = 30°
gain [dB]
angle [deg]
20
10
0
-10
-20
-30
-40
90 80 70 60 50 40 30 20 10 0 -10 -20 -30 -40 -50 -60 -70 -80 -90



ULA with M=8 conventional beamforming θ_0 = 30°
0
30
-30
60
-60
90
-90
0
2
4
6
8
gain

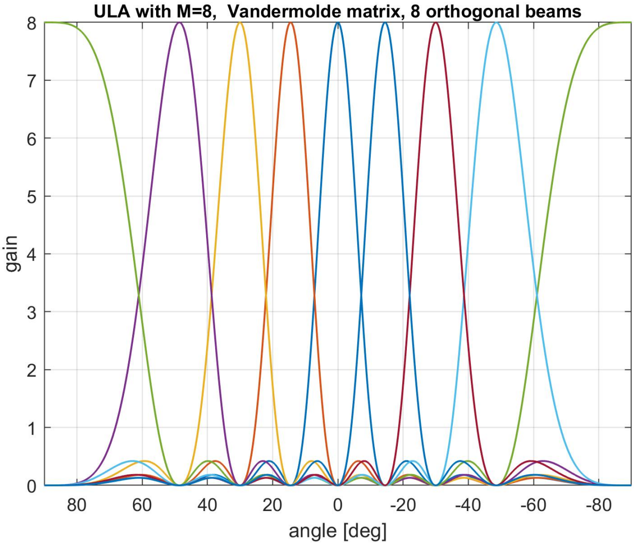
Vandermolde DFT Matrix

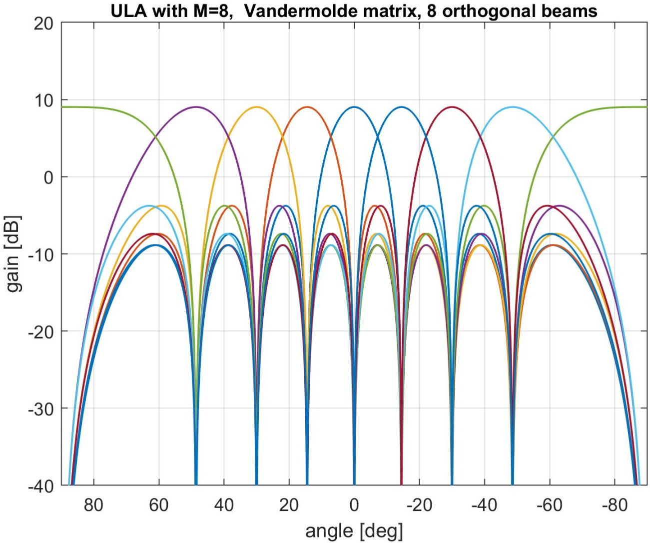
Write a Matlab program that for any M:

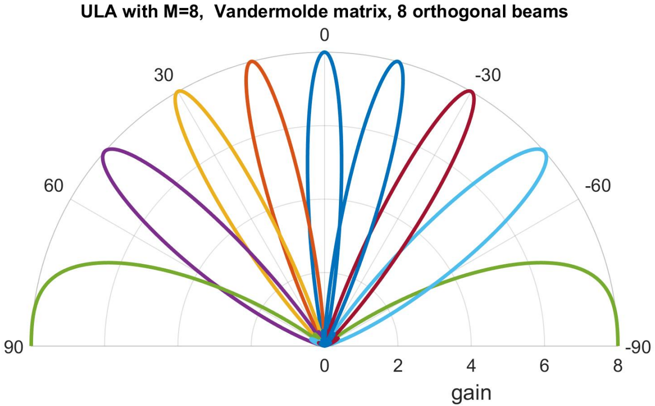
1. Generates the $M \times M$ Vandermolde matrix

$$\left[e^{j\frac{2\pi}{M}\alpha\beta} \right] \qquad 0 \leq \alpha \leq (M-1) \qquad 0 \leq \beta\,(M-1)$$

2. Uses the M rows as beamformer vectors to generate M orthogonal beams
3. Plots g vs. linear Θ
4. Plots g [dB] vs. linear Θ
5. Plots g vs. polar Θ

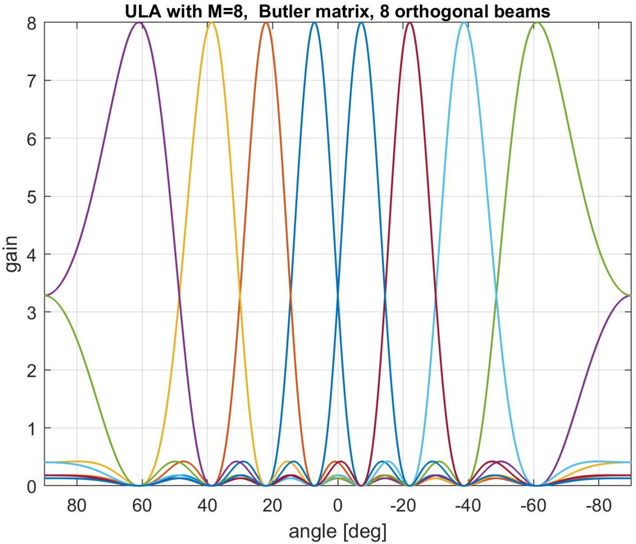

ULA with M=8, Vandermolde matrix, 8 orthogonal beams
gain
8
7
6
5
4
3
2
1
0
80
60
40
20
0
-20
-40
-60
-80
angle [deg]

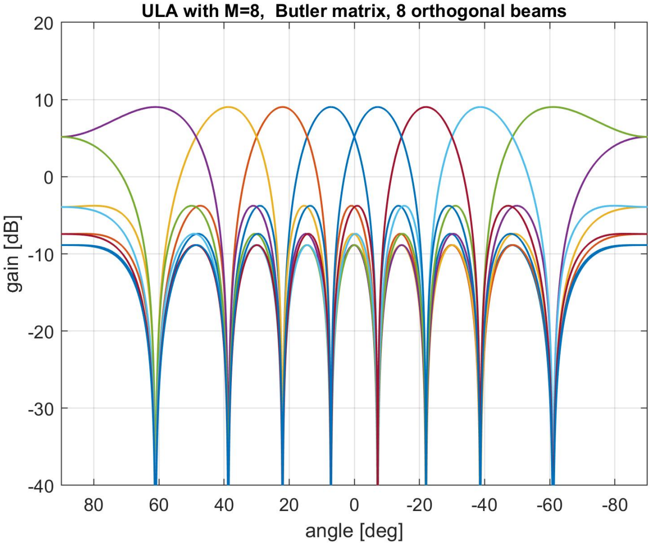

ULA with M=8, Vandermolde matrix, 8 orthogonal beams
gain [dB]
20
10
0
-10
-20
-30
-40
80
60
40
20
0
-20
-40
-60
-80
angle [deg]

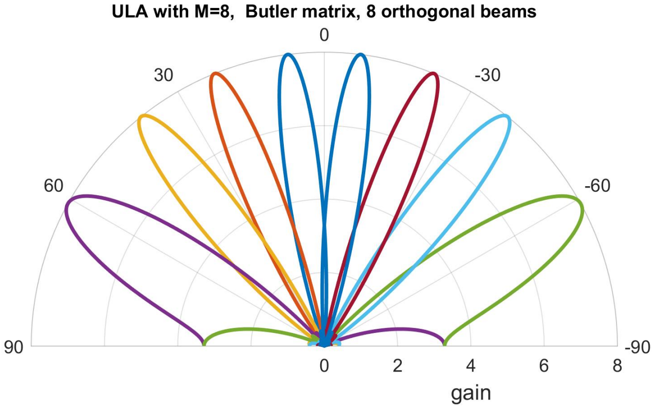

ULA with M=8, Vandermolde matrix, 8 orthogonal beams
0
30
-30
60
-60
90
-90
0
2
4
6
8
gain

Butler Matrix

1. Change the previous program to generate M orthogonal beams symmetric with respect to 0 degrees.
2. Plots g vs. linear Θ
3. Plots g [dB] vs. linear Θ
4. Plots g vs. polar Θ


ULA with M=8, Butler matrix, 8 orthogonal beams
gain
angle [deg]
8
7
6
5
4
3
2
1
0
80
60
40
20
0
-20
-40
-60
-80


ULA with M=8, Butler matrix, 8 orthogonal beams
gain [dB]
20
10
0
-10
-20
-30
-40
80
60
40
20
0
-20
-40
-60
-80
angle [deg]



ULA with M=8, Butler matrix, 8 orthogonal beams
0
30
-30
60
-60
90
-90
0
2
4
6
8
gain