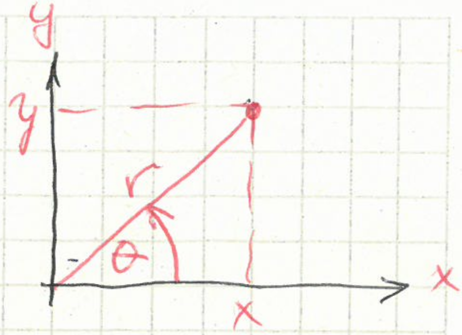
26-9-23

1.2 STABILITY ANALYSIS

COMPLEX NUMBERS



$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$i = \sqrt{-1}$$

$$|z| = r = \sqrt{x^2 + y^2}$$

VON NEUMANN STABILITY ANALYSIS – CHECK IF A NUMERICAL SOLUTION AMPLIFIES AT EVERY TIME STEP

$$\phi^{n+1} = \xi\, \phi^n$$

ϕ^n – SOLUTION OF A PDE

ξ – AMPLIFICATION FACTOR

$$|\xi| = \begin{cases} < 1 & \text{STABLE, DUMPED} \\ = 1 & \text{NEUTRALLY STABLE} \\ > 1 & \text{UNSTABLE} \end{cases}$$

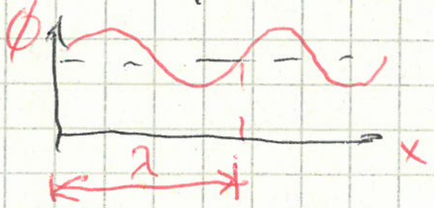
WE WILL PRESENT ϕ^n AS FOURIER SERIES:

$$\phi^n = \sum_{1}^{\infty} A_k^n e^{ikx}$$

k – WAVE NUMBER

$k = 2\pi/\lambda$

λ – WAVE LENGTH



BECAUSE WE STUDY LINEAR SCHEMES WE CAN FOCUS ON A SINGLE FOURIER TERM.

$$\phi^n = \xi^{n} e^{ikx}$$ EXPONENT!

$n=0 \quad \phi^0 = e^{ikx} = \phi_0$

$n=1 \quad \phi^1 = \xi\phi_0$

$n=2 \quad \phi^2 = \xi\phi^1 = \xi^2\phi_0$

$n \quad \phi^n = \xi\phi^{n-1} = \xi^n\phi_0$

FOR UNIFORM GRID

$$x_j = j\Delta x$$

$$\boxed{\phi_j^n = \xi^{n} e^{ikj\Delta x}}$$

EXPONENT!

FTCS: $$c_j^{n+1} = c_j^n - \frac{\overbrace{u\Delta t}^{C_r}}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right)$$

$$C_r = \frac{u\Delta t}{\Delta x}$$

COURANT NUMBER

PLUG IN $$c_j^n = \xi^n e^{ikj\Delta x}$$

$$\xi^{n+1} e^{ikj\Delta x} = \xi^n e^{ikj\Delta x} - \frac{C_r}{2}\left[\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}\right]$$

$$\times \frac{1}{\xi^n}$$

$$\xi e^{ikj\Delta x} = e^{ikj\Delta x} - \frac{C_r}{2}\left[e^{ikj\Delta x} e^{ik\Delta x} - e^{ikj\Delta x} e^{-ik\Delta x}\right]$$

$$\times \frac{1}{e^{ikj\Delta x}}$$

$$\xi = 1 - \frac{C_r}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$\xi = 1 - \frac{C_r}{2}\left[\cancel{\cos(k\Delta x)} + i\sin(k\Delta x) - \underbrace{\cancel{\cos(-k\Delta x)}}_{=\cos(k\Delta x)} - i\underbrace{\sin(-k\Delta x)}_{-\sin(k\Delta x)}\right]$$

$$\xi = 1 - \frac{C_r}{2}\, 2i\sin(k\Delta x)$$

$$\boxed{\xi = 1 - i C_r \sin(k\Delta x)}$$

AMPLIFICATION FACTOR FOR FTCS SCHEME

$$|\xi| = \sqrt{1 + C_r^2 \sin^2(k\Delta x)} > 1$$

$|\xi| > 1 \longrightarrow$ FTCS IS UNCONDITIONALLY UNSTABLE

FOR THE LAX SCHEME THE AMPLIFICATION FACTOR IS:

$$\boxed{\xi = \cos(k\Delta x) - i C_r \sin(k\Delta x)}$$

$$|\xi| = \sqrt{\cos^2(k\Delta x) + C_r^2 \sin^2(k\Delta x)}$$

$$|\xi| \begin{cases} > 1 & C_r > 1 \\ \leq & \boxed{C_r \leq 1} \end{cases}$$

→ UNSTABLE

→ LAX SCHEME IS CONDITIONALLY STABLE

$C_r = \frac{u\Delta t}{\Delta x}$

CFL STABILITY CRITERION

COURANT - FRIEDRICH - LEVY

WHY IS THE LAX SCHEME STABLE?

$$c_j^{n+1} = \frac{1}{2} c_{j+1}^n + \frac{1}{2} c_{j-1}^n - \frac{u\Delta t}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right) \qquad \times \frac{1}{\Delta t}$$

$$\frac{c_j^{n+1}}{\Delta t} - \underline{\frac{1}{2\Delta t} c_{j+1}^n} - \underline{\frac{1}{2\Delta t} c_{j-1}^n} \underline{- \frac{c_j^n}{\Delta t} + \frac{c_j^n}{\Delta t}} = -\frac{u}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right)$$

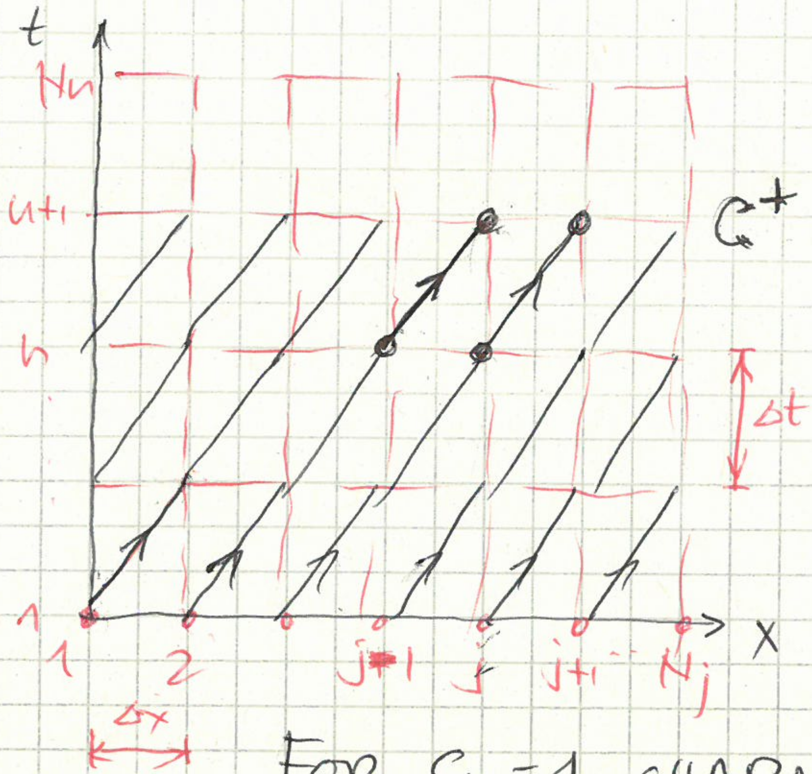
$$\frac{c_j^{n+1} - c_j^n}{\Delta t} - \frac{c_{j+1}^n + c_{j-1}^n - 2c_j^n}{2\Delta t} = -\frac{u}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right)$$

$$\frac{c_j^{n+1} - c_j^n}{\Delta t} + \frac{u}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right) = \underbrace{\frac{\Delta x^2}{2\Delta t}}_{D} \frac{c_{j+1}^n + c_{j-1}^n - 2c_j^n}{\Delta x^2}$$

$$\underbrace{\frac{c_j^{n+1} - c_j^n}{\Delta t}}_{\simeq \frac{\partial c}{\partial t}} + \underbrace{\frac{u}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right)}_{\simeq u \frac{\partial c}{\partial x}} = D \underbrace{\frac{\frac{c_{j+1}^n - c_j^n}{\Delta x} - \frac{c_j^n - c_{j-1}^n}{\Delta x}}{\Delta x}}_{\simeq D \frac{\partial^2 c}{\partial x^2}}$$

THE EQUATION THAT THE LAX SCHEME APPROXIMATES IS:

METHOD OF CHARACTERISTICS FOR SOLVING ADVECTION EQUATION



IF WE CHOOSE Δt SO THAT

$$\frac{\Delta x}{\Delta t} = u$$

$$\Delta t = \frac{1}{u}\Delta x$$

$$\boxed{\frac{u\Delta t}{\Delta x} = C_r = 1}$$

FOR $C_r = 1$ CHARACTERISTIC LINES ALWAYS GO FROM A COMPUTATIONAL NODE TO ANOTHER ONE, SHIFTED BY $(\cancel{\Delta x}, \Delta t)$

$$\boxed{C_j^{n+1} = C_{j-1}^{n}} \quad \text{or} \quad C_{j+1}^{n+1} = C_j^{n}$$

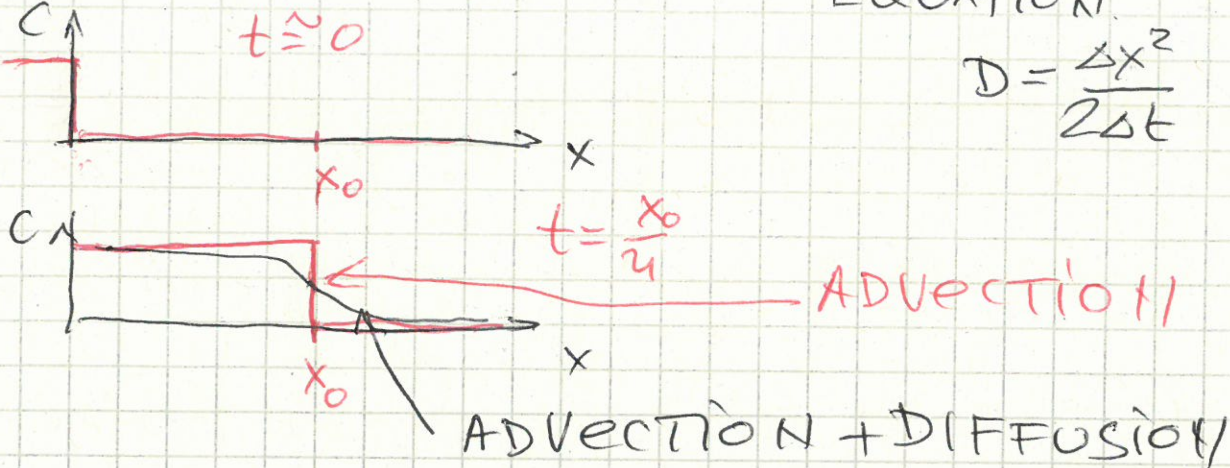
MoC FOR SOLVING ADVECTION EQUATION

NUMERICAL DIFFUSION COEFF.

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2}$$

ADVECTION - DIFUSION EQUATION

$$D = \frac{\Delta x^2}{2\Delta t}$$



1,3 CHARACTERISTIC FORM OF ADVECTION EQUATION

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0$$

$C(x,t)$

$$dC = \frac{\partial c}{\partial t}dt + \frac{\partial c}{\partial x}dx$$

FOR $u = \frac{dx}{dt}$ THE AE SIMPLIFIES TO:

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{dx}{dt}\frac{\partial c}{\partial x}$$

$\frac{dc}{dt} = 0$	FOR $\frac{dx}{dt} = u$
$dc = 0$	FOR $\frac{dx}{dt} = u$
$c = \text{const}$	FOR $\frac{dx}{dt} = u$

CHARACTERISTIC FORM OF ADVECTION EQUATION

(WE SWITCHED FROM PDE TO ODE)

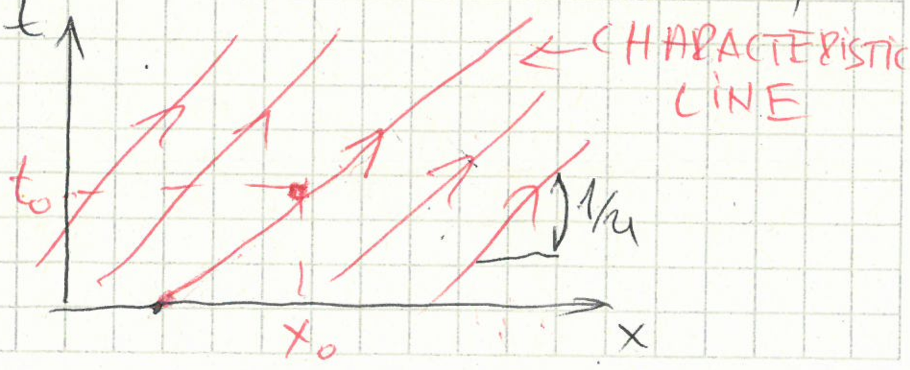
$$\frac{dx}{dt} = u$$

$$dx = u\,dt$$

$$\int_{t_0}^{t} dt = \frac{1}{u}\int_{x_0}^{x} dx$$

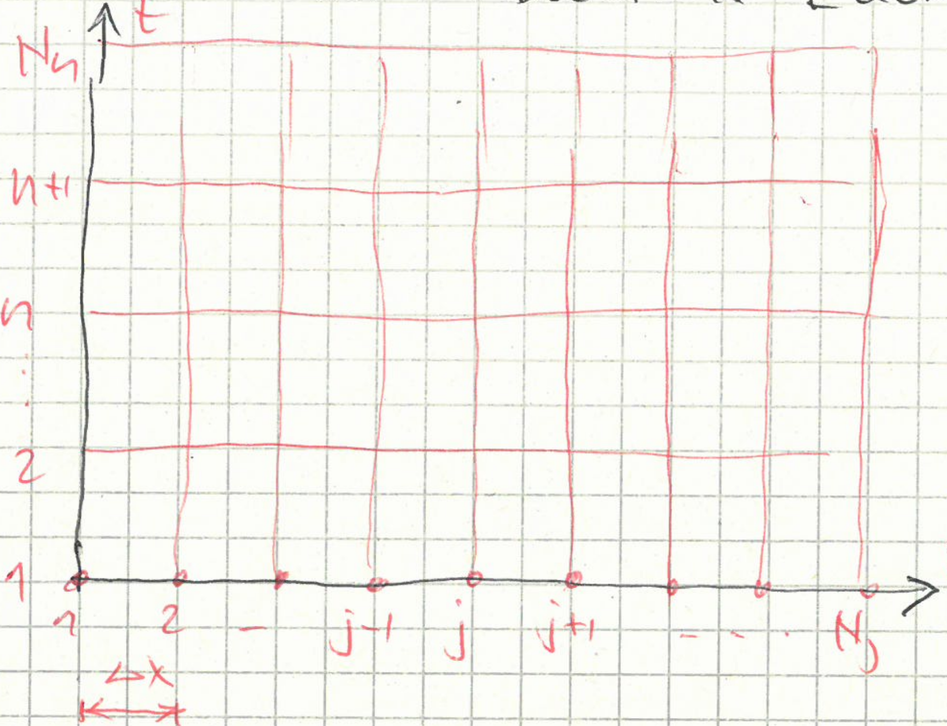
$$(t - t_0) = \frac{1}{u}(x - x_0)$$

$$\boxed{t = \overbrace{\frac{1}{u}}^{a}x \underbrace{- \frac{x_0}{u} + t_0}_{b}}$$



$t = ax + b$

SUMMARY OF NUMERICAL SCHEMES FOR SOLVING ADVECTION EQUATION



$$C_r = \frac{u \Delta t}{\Delta x}$$

- FTCS $\qquad \dfrac{C_j^{n+1} - C_j^n}{\Delta t} = -u \dfrac{C_{j+1}^n - C_{j-1}^n}{2\Delta x}$

  UNCONDITIONALLY UNSTABLE

- LAX $\qquad \dfrac{C_j^{n+1} - 0{,}5\left(C_{j+1}^n + C_{j-1}^n\right)}{\Delta t} = -u \dfrac{C_{j+1}^n - C_{j-1}^n}{2\Delta x}$

  STABLE FOR $C_r \leq 1$

- UPWIND $\qquad \dfrac{C_j^{n+1} - C_j^n}{\Delta t} = -u \dfrac{C_j^n - C_{j-1}^n}{\Delta x}$

  STABLE FOR $C_r \leq 1$

- MoC $\qquad C_j^{n+1} = C_{j-1}^n$

  $C_r = 1$