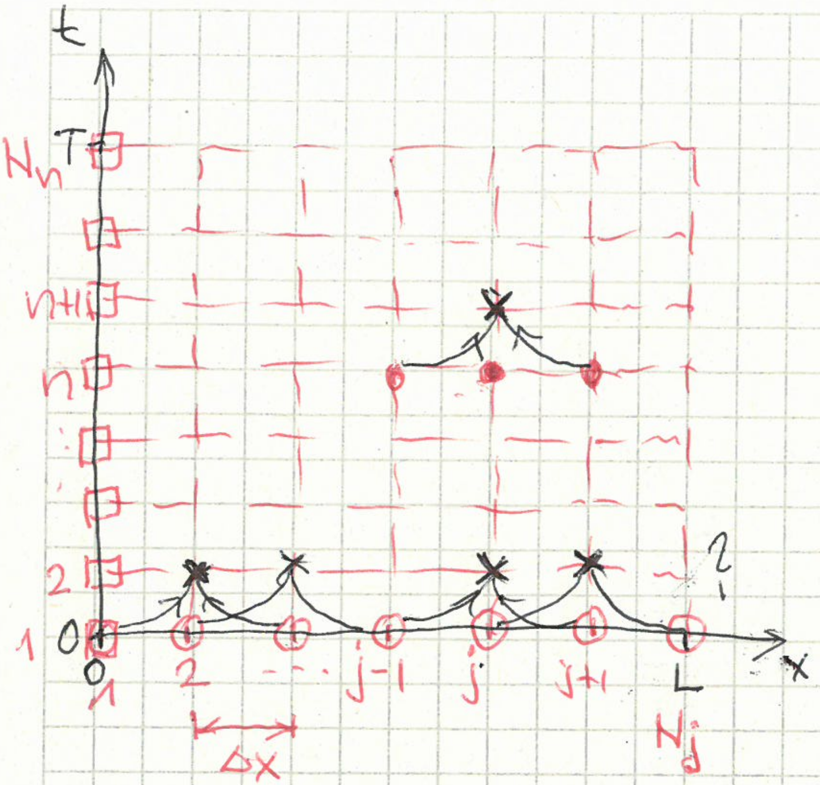
22-9-23



Δx - SPATIAL STEP

Δt - TIME STEP

COUNTERS:

$j \;\ldots\; 1, 2, \ldots N_j$

$n \;\ldots\; 1, 2, \ldots N_n$

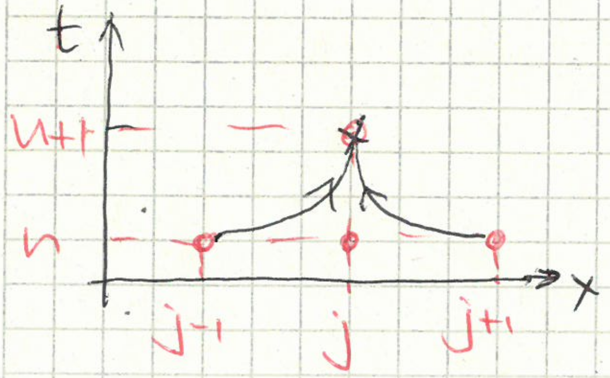
$x = (j-1)\,\Delta x$

$t = (n-1)\,\Delta t$

$C_j^n \leftarrow$ NOT AN EXPONENT!

○ IC (INITIAL CONDITION): $C(0 \le x \le L,\ t=0)$

□ BC (BOUNDARY CONDITION): $C(x=0,\ 0 \le t \le T)$



$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0$$

$\uparrow\ \partial \neq d$

TO FIND APPROXIMATE FORMULAS FOR $\partial c/\partial t$ AND $\partial c/\partial x$ WE USE TAYLOR SERIES

• IN TIME

$$C_j^{n+1} = C_j^n + \frac{\Delta t}{1!}\left(\frac{\partial c}{\partial t}\right)_j^n + \frac{\Delta t^2}{2!}\left(\frac{\partial^2 c}{\partial t^2}\right)_j^n + \frac{\Delta t^3}{3!}\left(\frac{\partial^3 c}{\partial t^3}\right)_j^n + \ldots$$

$$C_j^{n+1} - C_j^n = \Delta t\left(\frac{\partial c}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 c}{\partial t^2}\right)_j^n + \frac{\Delta t^3}{3!}\left(\frac{\partial^3 c}{\partial t^3}\right)_j^n + \ldots \quad \Big/ \frac{1}{\Delta t}$$

$$\frac{c_j^{n+1}-c_j^n}{\Delta t} = \left(\frac{\partial c}{\partial t}\right)_j^n + \underbrace{\frac{\Delta t}{2}\left(\frac{\partial^2 c}{\partial t^2}\right)_j^n + \frac{\Delta t^2}{3!}\left(\frac{\partial^3 c}{\partial t^3}\right)_j^n + \dots}_{O(\Delta t^1)}$$

SMALL QUANTITY OF FIRST ORDER

$$\left.\frac{\partial c}{\partial t}\right|_j^n = \frac{c_j^{n+1}-c_j^n}{\Delta t} + O(\Delta t)$$

$$\boxed{\frac{\partial c}{\partial t} \simeq \frac{c_j^{n+1}-c_j^n}{\Delta t}}$$

APPROXIMATE FIRST ORDER ACCURATE FORMULA FOR $\partial c/\partial t$

- IN SPACE

(1) $$c_{j+1}^n = c_j^n + \frac{\Delta x}{1!}\left(\frac{\partial c}{\partial x}\right)_j^n + \frac{\Delta x^2}{2!}\left(\frac{\partial^2 c}{\partial x^2}\right)_j^n + \frac{\Delta x^3}{3!}\left(\frac{\partial^3 c}{\partial x^3}\right) + \dots$$

(2) $$c_{j-1}^n = c_j^n + \frac{\overbrace{x_{j-1}-x_j}^{-\Delta x}}{1!}\left(\frac{\partial c}{\partial x}\right)_j^n + \frac{(x_{j-1}-x_j)^2}{2!}\left(\frac{\partial^2 c}{\partial x^2}\right)_j^n + \frac{(x_{j-1}-x_j)^3}{3!}\frac{\partial^3 c}{\partial x^3} + \dots$$

(1) − (2)

$$c_{j+1}^n - c_{j-1}^n = 2\Delta x\left(\frac{\partial c}{\partial x}\right)_j^n + 2\frac{\Delta x^3}{3!}\left(\frac{\partial^3 c}{\partial x^3}\right)_j^n + \dots \Big/ \frac{1}{\Delta x}$$

$$\frac{c_{j+1}^n - c_{j-1}^n}{\Delta x} = 2\left(\frac{\partial c}{\partial x}\right)_j^n + \underbrace{2\frac{\Delta x^2}{3!}\left(\frac{\partial^3 c}{\partial x^3}\right)_j^n + \dots}_{O(\Delta x^2)}$$

$$\left(\frac{\partial c}{\partial x}\right)_j^n = \frac{c_{j+1}^n - c_{j-1}^n}{2\Delta x} + O(\Delta x^2)$$

$$\boxed{\frac{\partial c}{\partial x} \simeq \frac{c_{j+1}^n - c_{j-1}^n}{2\Delta x}}$$

SECOND ORDER ACCURATE APPROXIMATE FORMULA FOR $\frac{\partial c}{\partial x}$

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = 0$$

PLUGGING IN APPROXIMATE FORMULAS FOR $\frac{\partial c}{\partial t}, \frac{\partial c}{\partial x}$ YIELDS:

$$\frac{c_j^{n+1} - c_j^n}{\Delta t} + u \frac{c_{j+1}^n - c_{j-1}^n}{2\Delta x} = 0$$

$$\boxed{c_j^{n+1} = c_j^n - \frac{u\Delta t}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right)}$$

FTCS - FORWARD TIME, CENTERED SPACE

UNCONDITIONALLY UNSTABLE

LAX SCHEME $c_j^n \to \frac{1}{2}\left(c_{j+1}^n + c_{j-1}^n\right)$

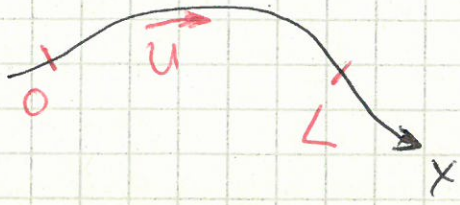
$$\boxed{c_j^{n+1} = \frac{1}{2}\left(c_{j+1}^n - c_{j-1}^n\right) - \frac{u\Delta t}{2\Delta x}\left(c_{j+1}^n - c_{j-1}^n\right)}$$

UPWIND SCHEME

$$\boxed{c_j^{n+1} = c_j^n - \frac{u\Delta t}{\Delta x}\left(c_j^n - c_{j-1}^n\right)}$$

FIRST ORDER ACCURATE IN SPACE

EXAMPLE RIVER



$L = 2000\,m$

$u = 0.5\,m/s$

IC: $c(0 \leq x \leq L, t = 0) = 0$

LEFT BC: $c(x = 0, 0 \leq t \leq 2000s) = 4\ mg/l$

$c(x = 0, 2000s < t) = 0$

SOLUTION

$\Delta x = 500\,m$

FOR STABILITY:

$$C_r = \frac{u \Delta t}{\Delta x} \leq 1$$

$$\Delta t \leq \frac{\Delta x}{u} = \frac{500}{0.5}$$

$\Delta t \leq 1000s$

LAX SCHEME:

- $\Delta t = 1000s$ $\quad \frac{u \Delta t}{\Delta x} = 1$

$$c_j^{n+1} = \frac{1}{2}\left(c_{j-1}^n + c_{j+1}^n\right) - \underbrace{\frac{u \Delta t}{2 \Delta x}}_{=1}\left(c_{j+1}^n - c_{j-1}^n\right)$$

$$c_j^{n+1} = \frac{1}{2} c_{j-1}^n + \cancel{\frac{1}{2} c_{j+1}^n} - \cancel{\frac{1}{2} c_{j+1}^n} + \frac{1}{2} c_{j-1}^n$$

$$\boxed{c_j^{n+1} = c_{j-1}^n}$$

(We would get the same formula for $C_r = 1$ for upwind scheme)

C [mg/l]

x(m) / t(s)	0	500	1000	1500	2000
	1	2	3	4	5
0	4	0	0	0	0
1000	4	4	0	0	0
2000	4	4	4	0	0
3000	0	4	4	4	0
4000	0	0	4	4	4

← IC

- $\Delta t = 500_s$ $\quad C_r = u\Delta t / \Delta x = 1/2$

$$C_j^{n+1} = \frac{1}{2}\left(C_{j-1}^n + C_{j+1}^n\right) - \underbrace{\frac{u\Delta t}{2\Delta x}}_{=1/2}\left(C_{j+1}^n - C_{j-1}^n\right)$$

$$C_j^{n+1} = \frac{1}{2} C_{j-1}^n + \frac{1}{2} C_{j+1}^n - \frac{1}{4} C_{j+1}^n + \frac{1}{4} C_{j-1}^n$$

$$\boxed{C_j^{n+1} = \frac{3}{4} C_{j-1}^n + \frac{1}{4} C_{j+1}^n}$$

TRANSMISSIVE BOUNDARY "GHOST CELL"

C [mg/l]

x(m) / t(s)	0	500	1000	1500	2000	2500
	1	2	3	4	5	6
IC 0	4	0	0	0	0	0
500	4	3	0	0	0	0
1000	4	3	9/4	0	0	0
1500	4	3+ 9/16	9/4	27/16	0	0
2000	4					
2500	0					