10-10-23

## 2.3 MoC FOR 1D ELASTIC WAVES IN A SOLID MATERIAL

EQUATIONS:

$$\frac{\partial \sigma}{\partial t} - E\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} - \frac{1}{\rho}\frac{\partial \sigma}{\partial x} = 0$$

$\sigma$ - NORMALL STRESS [Pa]

V - SPEED OF PARTICLES [m/s]

E - MODULUS OF ELASTICITY [Pa]

$\rho$ - DENSITY [kg/m3]

TO BE CONSISTENT WITH OTHER ENG. APPLICATIONS WE WILL USE PRESSURE, P, INSTEAD OF $\sigma$

$$p = -\sigma$$

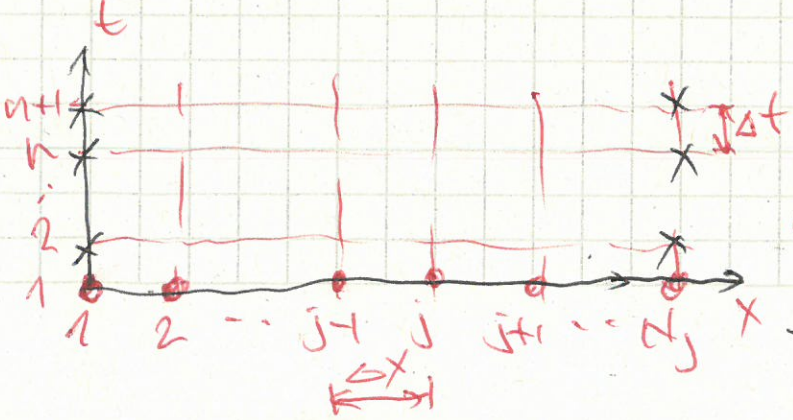
$$\frac{\partial p}{\partial t} + E\frac{\partial v}{\partial x} = 0 \quad (1)$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad (2)$$

BY MAKING A LINEAR COMBINATION OF (1) & (2) WE DERIVE THE CHARACTERISTIC FORM OF THE SYSTEM (1), (2).

$$\boxed{\begin{array}{lll} dp + z\,dV = 0 & \text{ALONG: } C^+: \frac{dx}{dt} = c & c = \sqrt{\frac{E}{\rho}} \\ dp - z\,dV = 0 & \text{ALONG } C^-: \frac{dx}{dt} = -c & z_0 = \sqrt{E\rho} \end{array}}$$

MoC IS APPLIED IN THE SAME WAY AS BEFORE




$$\frac{\Delta x}{\Delta t} = c \qquad C_r = \frac{c\Delta t}{\Delta x} = 1$$

• IC. KNOWN p & V AT t=0

x LBC KNOWN p OR V OR p(V) AT x=0

x RBC KNOWN p OR V OR p(V) AT x=L

# EXAMPLE ROD 1


$E = 200$ GPa
$\rho = 8000$ kg/m$^3$
$A = 10^{-3}$ m$^2$

WAVE SPEED:

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{200 \times 10^9}{8000}} = 5000 \frac{m}{s}$$

IMPEDANCE

$$z_0 = \sqrt{E\rho} = 40 \times 10^6 \frac{Pa\,s}{m} = \frac{MPa\,s}{m}$$

$$p_{max} = \frac{F_{max}}{A} = \frac{10^3 N}{10^{-3} m^2} = 1 MPa$$

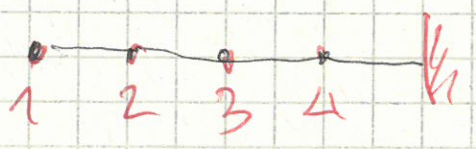
IC: $p(0 \leq x \leq L, t=0) = 0$
$v(0 \leq x \leq L, t=0) = 0$

RIGHT BC:

$$v(x=L, t) = 0$$

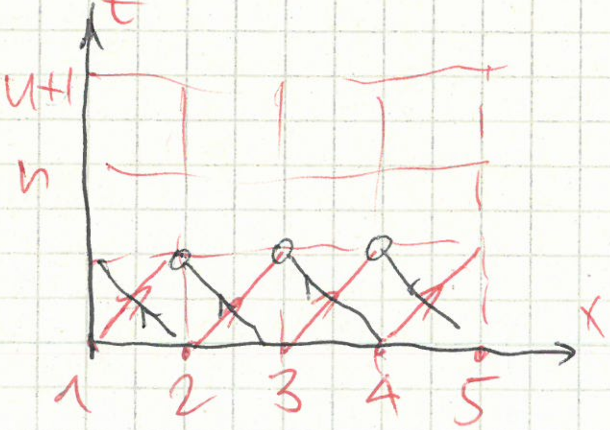
LEFT BC – KNOWN LOADING FOR 2 CASES:

- QUICK CHAGE OF F TO:
  $F_0 = 1$ kN IN 0.2 ms
- PERIODIC CHANGE OF F
  $F_{max} = 1$ kN, $f = 1$ kHz




$\Delta x = 1$ m

$$\Delta t = \frac{\Delta x}{c} = \frac{1}{5000 s} = 0.2 ms$$

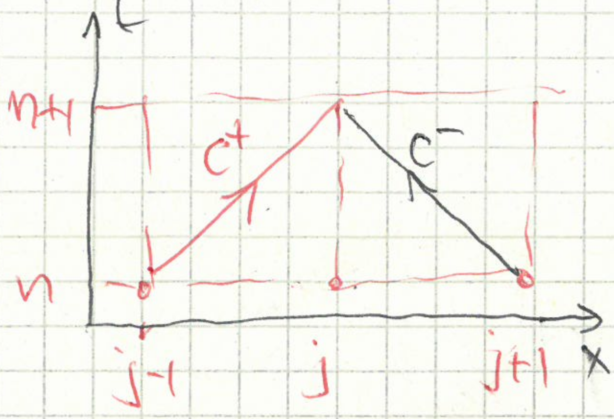



IC: $p_j^1 = 0$   $j = 1, \dots 5$
$v_j^n = 0$

LBC: KNOWN $p_1^n$   $n = 1, \dots N_n$

RBC: KNOWN $v_5^n = 0$

MIDDLE POINTS $j=2,3,4$




$C^+: dp + z_0\,dv = 0$

$C^-: dp - z_0\,dv = 0$

INTEGRATED BETWEEN:

$C^+: (n, j-1)$ & $(n+1, j)$

$C^-: (n, j+1)$ & $(n+1, j)$

THE EQUATIONS BECOME

$$P_j^{n+1} - P_{j-1}^{n} + z_0 \left( V_j^{n+1} - V_{j-1}^{n} \right) = 0 \qquad (3)$$

$$P_j^{n+1} - P_{j+1}^{n} - z_0 \left( V_j^{n+1} - V_{j+1}^{n} \right) = 0 \qquad (4)$$

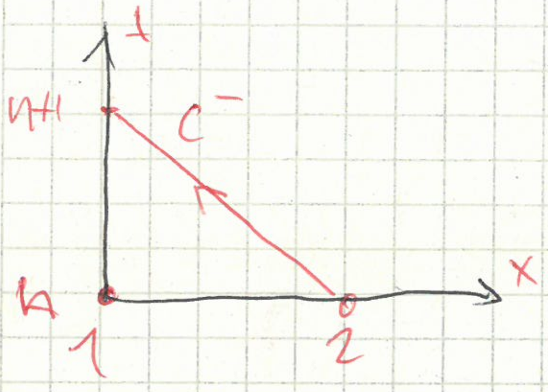
(3)+(4) $$2P_j^{n+1} - P_{j-1}^{n} - P_{j+1}^{n} - z_0 V_{j-1}^{n} + z_0 V_{j+1}^{n} = 0$$

$$\boxed{P_j^{n+1} = \frac{1}{2}\left(P_{j-1}^{n} + P_{j+1}^{n}\right) + \frac{z_0}{2}\left(V_{j-1}^{n} - V_{j+1}^{n}\right)}$$

(3)-(4) $$P_{j+1}^{n} - P_{j-1}^{n} + 2z_0 V_j^{n+1} - z_0 V_{j-1}^{n} - z_0 V_{j+1}^{n} = 0$$

$$\boxed{V_j^{n+1} = \frac{1}{2}\left(V_{j-1}^{n} + V_{j+1}^{n}\right) + \frac{1}{2z_0}\left(P_{j-1}^{n} - P_{j+1}^{n}\right)}$$

## LEFT BC




$C^-: dp - Z_0\, dV = 0$

INTEGRATE BETWEEN $(n,2)$ & $(n+1,1)$

$$P_1^{n+1} - P_2^{n} - Z_0\left(V_1^{n+1} - V_2^{n}\right) = 0 \qquad (5)$$

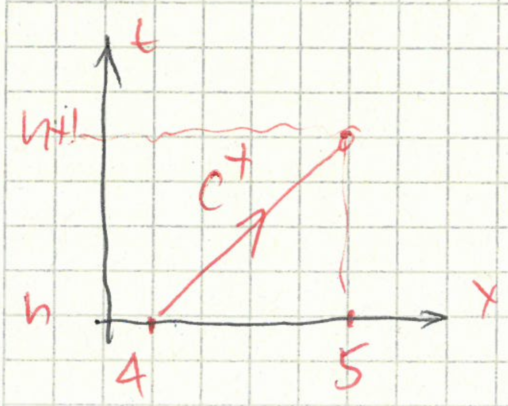
WE KNOW $P_1^{n+1}$ FROM THE LBC

SO WE SOLVE (5) FOR $V_1^{n+1}$

$$P_1^{n+1} - P_2^{n} - Z_0 V_1^{n+1} + Z_0 V_2^{n} = 0$$

$$\boxed{V_1^{n+1} = V_2^{n} + \frac{1}{Z_0}\left(P_1^{n+1} - P_2^{n}\right)}$$

## RIGHT BC




$C^+: dp + Z_0\, dV = 0$

INTEGRATED BETWEEN
$(n,4)$ & $(n+1,5)$

$$P_5^{n+1} - P_4^{n} + Z_0\left(\underbrace{V_5^{n+1}}_{=0} - V_4^{n}\right) = 0 \qquad (6)$$

$V_5^{n+1}$ IS KNOWN SO WE SOLVE
(6) FOR $P_5^{n+1}$

$$\boxed{P_5^{n+1} = P_4^{n} + Z_0 V_4^{n}}$$

## ALL EQUATIONS

LB: $j=1$

$P_1^{n+1}$ is KNOWN

$$V_1^{n+1} = V_2^n + \frac{1}{40}\left(P_1^{n+1} - P_2^n\right)$$

MIDDLE $j=2,3,4$

$$P_j^{n+1} = \frac{1}{2}\left(P_{j-1}^n + P_{j+1}^n\right) + 20\left(V_{j-1}^n - V_{j+1}^n\right)$$

$$V_j^{n+1} = \frac{1}{2}\left(V_{j-1}^n + V_{j+1}^n\right) + \frac{1}{80}\left(P_{j-1}^n - P_{j+1}^n\right)$$

RB $j=5$

$$V_5^{n+1} = 0$$

$$P_5^{n+1} = P_4^n + 40 \cdot V_4^n$$

## SUDDEN CHANGE OF LOADING

| j = | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| x(m) | 0 | | 1 | | 2 | | 3 | | 4 | |
| t(ms) | P(MPa) | V(mm/s) | P | V | P | V | P | V | P | V |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.2 | 1 | 25 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.4 | 1 | 25 | 1 | 25 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.6 | 1 | 25 | 1 | 25 | 1 | 25 | 0 | 0 | 0 | 0 |

← IC

↑ LBC

↑ RBC