Calculus Exams 2

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1 Question 1: 2pt

Compute the limit of the following sequence

$$s_n = \frac{7n^{10} + 2n^6 + 2n + 1/n}{2n^{10} + 3n^6 + 9} \tag{1}$$

1.1 Solution

$$\frac{7n^{10} + 2n^6 + 2n + 1/n}{2n^{10} + 3n^6 + 9} \sim \frac{7n^{10}}{2n^{10}} = \frac{7}{2}$$

2 Question 1: 1pt

Show that these two functions are asymptotic to each other for $x \to 0$.

$$f(x) = x^3 + 2x^4 + x^5 e^x \tag{2}$$

$$g(x) = x^2(e^x - 1) \tag{3}$$

2.1 Solution

$$\frac{x^2(e^x - 1)}{x^3 + 2x^4 + x^5 e^x} \sim \frac{x^2 \times x}{x^3}$$

$$\lim_{x \to 0} \frac{x^3}{x^3} = 1$$

3 Question 3: 1pt

Write a Python script to approximate the following integral wuth respect to a differential using $N = 20$ rectangles:

$$\int_{-1}^{1} \cos(x^3) \mathrm{d}x^2 \tag{4}$$

3.1 Solution

```
f = lambda x: np.cos(x**3)
    a = -1
    b = 1
    N = 20
    dx = (b - a)/N
    x_range = np.linspace(a,b,N)
    result = sum([f(x)*((x+dx)**2 - x**2) for x in x_range])
```

4 Question 4: 2pt

Compute derivative of the following function

$$f(x) = e^{-\alpha(x)} \sin(\omega(x)x + \phi_0) \tag{5}$$

where $\omega(x)$ and $\alpha(x)$ are arbitrary functions and ϕ_0 is a real number.

4.1 Solution

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = e^{-\alpha(x)} \left(-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x} \sin(\omega(x)x + \phi_0) + \cos(\omega(x)x + \phi_0) \left(\frac{\mathrm{d}\omega(x)}{\mathrm{d}x} x + \omega(x) \right) \right)$$

5 Question 4: 2pt

Compute the following integral:

$$\int_0^{\pi/2} \cos(x) \sin(x) e^{\sin(x)} \mathrm{d}x \tag{6}$$

Hint: You need to use both substitution and integration by parts.

5.1 Solution

You start with the substitution $u(x) = \sin(x)$, $du(x) = \cos(x)\mathrm{d}x$.

$$\int_0^{\pi/2} \cos(x) \sin(x) e^{\sin(x)} \mathrm{d}x = \int_0^{\pi/2} u(x) e^{u(x)} \mathrm{d}u(x) \tag{7}$$

you can now use the integration by parts formula:

$$\int_0^{\pi/2} u(x) e^{u(x)} \mathrm{d}u(x) + \int_0^{\pi/2} e^{u(x)} \mathrm{d}u(x) = u(1) e^{u(1)} - u(0) e^{u(0)} \tag{8}$$

Hence:

$$\int_0^{\pi/2} u(x) e^{u(x)} \mathrm{d}u(x) = -e^{\sin(\pi/2)} + e^{\sin(0)} + \sin(\pi/2) e^{\sin(\pi/2)} - \sin(0) e^{\sin(0)} \tag{9}$$

$$= 1 \tag{10}$$

6 Question 5: 2pt

Prove that the derivative of e^{2x} is $2e^{2x}$ using the definition of derivative.

6.1 Solution

$$\frac{e^{2(x+h)} - e^{2x}}{h} = \frac{e^{2x}e^{2h} - e^{2x}}{h} = e^{2x}\frac{e^{2h} - 1}{h} \sim e^{2x}\frac{2h}{h} = 2e^{2x}$$

Therefore:

$$\frac{\mathrm{d}e^{2x}}{\mathrm{d}x} = \lim_{h\to 0}\frac{e^{2(x+h)} - e^{2x}}{h} = 2e^{2x}$$