FinKont2: Hand-In 3

Gustav Dyhr / msh606

April 2025

I have created a full GitHub of my code used for this Hand-In at `https://github.com/guxel/HandIn-3`. The main code is in HandIn3.Py, while the rest are supporting objects. Additionally, short code snippets have been inserted where relevant.

The Hand-In references Tomas Björk's book *Arbitrage Theory in Continuous Time* (4th edition), simply refereed to as Björk.

1 Portfolio Dynamics

We must inspect the dynamics of a portfolio that invests a fixed fraction α in the stock S. We call the value process of the portfolio A_t^α and seek to prove using dA_t^α that the process is a Geometric Brownian Motion (GBM) under both the measure P and Q. We must also determine the drifts and the volatility of the process. Finally we must show that the value process can be written as:

$$A_t^\alpha = g(t)\,(S_t)^\alpha \tag{1.1}$$

We use the Black Scholes model for the asset price, which means that it follows the following GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P \tag{1.2}$$

I.e. we have a log-normally distributed asset with returns that have a constant drift μ and constant volatility σ. We can now rewrite the asset price from the real world measure P to the bankbook martingale measure Q for replication and pricing. We do this using Girsanovs Theorem, from Björk's theorem 12.3 and the Black-Scholes Girsanov kernel from Björk's lemma 13.1:

$$dW_t^P = \rho_t dt + dW_t^Q = \frac{r-\mu}{\sigma} dt + dW_t^Q \tag{1.3}$$

Inserting this in our real-world asset price process, yields the asset price process under Q:

$$dS_t = \mu S_t dt + \sigma S_t \left(\frac{r-\mu}{\sigma} dt + dW_t^Q\right) = \left(\mu + \sigma\frac{r-\mu}{\sigma}\right) S_t dt + \sigma S_t dW_t^Q = rS_t dt + \sigma S_t dW_t^Q \tag{1.4}$$

We see that the two asset processes are nearly identical, except for the drift, we can therefore easily shift from one to the other by exchanging μ and r (and P and Q). As we have a constant risk free rate r, then we know that the bankbook follows the following process:

$$dB_t = rB_t dt \tag{1.5}$$

We are now ready to determine dA_t^α, for this we use Björks lemma 6.12 for self-financing portfolios. We know that we invest a fixed fraction into the stock, so $w_S = \alpha$. For our portfolio weights to be valid, they must sum to 1. This allows for us to derive the weight for our only other asset, $w_B = 1 - \alpha$. This definition is very need as it does not require $0 \leq \alpha \leq 1$, instead it allows for both short and leveraged positions. Especially leverage is surprisingly attractive for some stylized pension funds. We can now insert in the lemma:

$$
\begin{aligned}
dA_t^\alpha =& A_t^\alpha \left(w_S \frac{dS_t + dD_t^S}{S_t} + w_B \frac{dB_t + dD_t^B}{B_t} \right) - c_t dt = A_t^\alpha \left(\alpha \frac{dS_t}{S_t} + (1-\alpha) \frac{dB_t}{B_t} \right) \\
=& A_t^\alpha \left(\alpha \frac{\mu S_t dt + \sigma S_t dW_t^P}{S_t} + (1-\alpha) \frac{r B_t dt}{B_t} \right) \\
=& A_t^\alpha \left(\alpha \left(\mu dt + \sigma dW_t^P \right) + (1-\alpha) r dt \right) = \left(\alpha\mu + (1-\alpha) r \right) A_t^\alpha dt + \alpha\sigma A_t^\alpha dW_t^P \\
=& r A_t^\alpha dt + \alpha\sigma A_t^\alpha dW_t^Q
\end{aligned}
\tag{1.6}
$$

Where we in the last step have reapplied Girsanovs theorem to go from the real world measure to the martingale measure. In the second step we used that our assets do not pay dividends and that we have no consumption. We note that under both measures the portfolio value process follows a classic GBM. We see that under the real world measure, the portfolio return has constant drift $(\alpha\mu + (1-\alpha)r)$, an average between the return of the bank account and the stock, weighted by the portfolio weights. Under the bankbook martingale measure, the portfolio simply has drift r, making the discounted portfolio value process a martingale as expected. Both versions of the dynamics share volatility parameter $\alpha\sigma$, a constant volatility reflecting the volatility of the asset scaled with the asset weight.

Now we must prove (1.1). As we know that A_t^α follows a GBM, we can use the solutions for GBM from Björk's proposition 5.2:

$$
dX_t = \alpha X_t dt + \sigma X_t dW_t \Rightarrow X_t = X_0 e^{(a - \sigma^2/2)t + \sigma W_t} \tag{1.7}
$$

Applying to our portfolio under the risk-neutral measure (choice of measure does not matter here) we gain:

$$
A_t^\alpha = A_0^\alpha e^{(r - (\alpha\sigma)^2/2)t + \alpha\sigma W_t^Q} \tag{1.8}
$$

In addition, we also know that the stock has a similar solution as it also follows a GBM:

$$
\frac{S_t}{S_0} = e^{(r - \sigma^2/2)t + \sigma W_t^Q} \tag{1.9}
$$

We now wish to rewrite (1.8) to something where we can insert (1.9).

$$
\begin{aligned}
A_t^\alpha =& A_0^\alpha e^{(r-(\alpha\sigma)^2/2)t + \alpha\sigma W_t^Q} = A_0^\alpha \left(e^{(r-(\alpha\sigma)^2/2)t/\alpha + \alpha\sigma W_t^Q/\alpha} \right)^\alpha \\
=& A_0^\alpha \left(e^{(r-(\alpha\sigma)^2/2)t/\alpha} e^{\sigma W_t^Q} \right)^\alpha = A_0^\alpha \left(e^{(r-(\alpha\sigma)^2/2)t/\alpha - (r-\sigma^2/2)t} e^{(r-\sigma^2/2)t + \sigma W_t^Q} \right)^\alpha \\
=& A_0^\alpha e^{(r-(\alpha\sigma)^2/2)t/\alpha \cdot \alpha - (r-\sigma^2/2)t\alpha} \left(\frac{S_t}{S_0} \right)^\alpha \\
=& \frac{A_0^\alpha}{(S_0)^\alpha} e^{(r-(\alpha\sigma)^2/2)t - (r-\sigma^2/2)t\alpha} \left(S_t \right)^\alpha = \frac{A_0^\alpha}{(S_0)^\alpha} e^{(r+\alpha\sigma^2/2)(1-\alpha)t} \left(S_t \right)^\alpha = g(t) \left(S_t \right)^\alpha
\end{aligned}
$$

We have hereby shown that we can define the portfolio value process as a function of the stock with the power α and a deterministic function of time $g(t)$ defined as:

$$
g(t) = \frac{A_0^\alpha}{(S_0)^\alpha} e^{(r+\alpha\sigma^2/2)(1-\alpha)t} \tag{1.10}
$$

2 Portfolio Insurance Contract

This question revolves around a revolutionary insurance product has the payoff function:

$$\pi_T = (K - A_T^{\alpha})^+ \tag{2.1}$$

I.e. a put option on the portfolio value. We have to determine a function for the arbitrage-free price of the product and investigate the possibility of discretely hedging it. To determine a pricing function, we use the risk-neutral valuation formula (with a constant interest rate), from Björk's theorem 7.11:

$$\begin{aligned}\pi_t =& e^{-r(T-t)} E_t^Q [\pi_T] = e^{-r(T-t)} E_t^Q \left[(K - A_T^{\alpha})^+\right] \\ =& e^{-r(T-t)} E_t^Q \left[\left(K - A_t^{\alpha} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma W_{T-t}^Q} \right) \mathbf{1}_{K - A_t^{\alpha} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma W_{T-t}^Q} > 0} \right]\end{aligned} \tag{2.2}$$

We now start by simplifying our term in the indicator function:

$$\begin{aligned} K - A_t^{\alpha} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma W_{T-t}^Q} &> 0 \\ K &> A_t^{\alpha} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma W_{T-t}^Q} \\ \frac{K}{A_t^{\alpha}} e^{-(r-(\alpha\sigma)^2/2)(T-t)} &> e^{\alpha\sigma\sqrt{T-t}Z} \\ \ln\left(\frac{K}{A_t^{\alpha}}\right) - (r-(\alpha\sigma)^2/2)(T-t) &> \alpha\sigma\sqrt{T-t}Z \\ \frac{\ln\left(\frac{K}{A_t^{\alpha}}\right) - (r-(\alpha\sigma)^2/2)(T-t)}{\alpha\sigma\sqrt{T-t}} &> Z \\ -d_2 &> Z \end{aligned} \tag{2.3}$$

Where Z is a standard normal random variable and d_2 is a variable we made up for simplicity. We insert this and rewrite the equation:

$$\begin{aligned}\pi_t =& e^{-r(T-t)} E_t^Q \left[\left(K - A_t^{\alpha} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma W_{T-t}^Q} \right) \mathbf{1}_{-d_2 A_t^{\alpha} > Z} \right] \\ =& e^{-r(T-t)} E_t^Q \left[K\mathbf{1}_{-d_2>Z} - A_t^{\alpha} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma W_{T-t}^Q} \mathbf{1}_{-d_2>Z} \right] \\ =& e^{-r(T-t)} \left(KP^Q[-d_2 > Z] - A_t^{\alpha} E_t^Q \left[e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma\sqrt{T-t}Z} \mathbf{1}_{-d_2>Z} \right] \right) \\ =& e^{-r(T-t)} \left(K\Phi(-d_2) - A_t^{\alpha} \int_{-\infty}^{\infty} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma\sqrt{T-t}Z} \mathbf{1}_{-d_2>Z} \phi(Z) dZ \right) \\ =& e^{-r(T-t)} \left(K\Phi(-d_2) - A_t^{\alpha} \int_{-\infty}^{\infty} e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma\sqrt{T-t}Z} \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} \mathbf{1}_{-d_2>Z} dZ \right) \\ =& e^{-r(T-t)} \left(K\Phi(-d_2) - A_t^{\alpha} \int_{-\infty}^{\infty} e^{(r-(\alpha\sigma)^2/2)(T-t)} \frac{1}{\sqrt{2\pi}} e^{\alpha\sigma\sqrt{T-t}Z - Z^2/2} \mathbf{1}_{-d_2>Z} dZ \right)\end{aligned}$$

Where $\phi(x)$ is the probability density function of the standard normal distribution and $\Phi(x)$ is the cumulative density function. For a pretty solution, we must rewrite the integral to be on the form of $\Phi(x)$. To accomplish this, we change the integrand by defining a new variable such that:

$$
\begin{aligned}
Y =& Z - \alpha\sigma\sqrt{T-t} \Rightarrow \\
Z =& Y + \alpha\sigma\sqrt{T-t} \Rightarrow \\
\alpha\sigma\sqrt{T-t}Z - Z^2/2 =& \alpha\sigma\sqrt{T-t}(Y + \alpha\sigma\sqrt{T-t}) - (Y + \alpha\sigma\sqrt{T-t})^2/2 \\
=& \alpha\sigma\sqrt{T-t}Y + (\alpha\sigma\sqrt{T-t})^2 - Y^2/2 - (\alpha\sigma\sqrt{T-t})^2/2 - 2\alpha\sigma\sqrt{T-t}Y/2 \\
=& (\alpha\sigma\sqrt{T-t})^2/2 - Y^2/2
\end{aligned} \tag{2.4}
$$

We can now substitute with this new varaible:

$$
\begin{aligned}
\pi_t =& e^{-r(T-t)}\left(K\Phi(-d_2) - A_t^\alpha \int_{-\infty}^{\infty} e^{(r-(\alpha\sigma)^2/2)(T-t)} \frac{1}{\sqrt{2\pi}} e^{(\alpha\sigma\sqrt{T-t})^2/2 - Y^2/2} \mathbf{1}_{-d_2 > Y + \alpha\sigma\sqrt{T-t}} dY\right) \\
=& e^{-r(T-t)}(K\Phi(-d_2) - A_t^\alpha \int_{-\infty}^{\infty} e^{(r-(\alpha\sigma)^2/2)(T-t) + (\alpha\sigma\sqrt{T-t})^2/2} \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} \mathbf{1}_{-d_2 - \alpha\sigma\sqrt{T-t} > Y} dY \\
=& e^{-r(T-t)}(K\Phi(-d_2) - A_t^\alpha e^{(r-(\alpha\sigma)^2/2)(T-t) + (\alpha\sigma)^2/2(T-t)} \int_{-\infty}^{-d_1} \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} dY \\
=& e^{-r(T-t)}\left(K\Phi(-d_2) - A_t^\alpha e^{(r-(\alpha\sigma)^2/2 + (\alpha\sigma)^2/2)(T-t)} \Phi(-d_1)\right) \\
=& e^{-r(T-t)}\left(K\Phi(-d_2) - A_t^\alpha e^{r(T-t)} \Phi(-d_1)\right) \\
=& e^{-r(T-t)} K\Phi(-d_2) - A_t^\alpha \Phi(-d_1)
\end{aligned}
$$

Where:

$$
\begin{aligned}
d_2 =& \frac{\ln\left(\frac{A_t^\alpha}{K}\right) + (r - (\alpha\sigma)^2/2)(T-t)}{\alpha\sigma\sqrt{T-t}} \\
d_1 =& d_2 + \alpha\sigma\sqrt{T-t} = \frac{\ln\left(\frac{A_t^\alpha}{K}\right) + (r + (\alpha\sigma)^2/2)(T-t)}{\alpha\sigma\sqrt{T-t}}
\end{aligned} \tag{2.5}
$$

We have hereby derived a formula for pricing the portfolio. We see that it is identical to the standard Black-Scholes formula for pricing put options, except that we now have a scale parameter α multiplied on our σ to reflect our exposure.

Using our parameters $r = 2\%$, $\sigma = 20\%$, $A_0^\alpha = S_0 = 1$,$T = 30$,$K = e^{rT}$ and $\alpha = 50\%$ we get the price:

$$
\pi_0 = e^{-0.02(30-0)} K\Phi(-d_2) - 1\Phi(-d_1) = 0.2158 \tag{2.6}
$$

We now have to investigate the possibilities of discretely hedging the option by trading in the underlying stock. To do this we need to construct a portfolio of the stuck that fully hedges the stochastic part of π_t. I.e. we need to delta hedge the option. To accomplish this, we must first determine the delta of the option:

$$
\begin{aligned}
\Delta =& \frac{\partial \pi_t}{\partial S_t} = e^{-r(T-t)} K \frac{\partial}{\partial S_t}\Phi(-d_2) - \frac{\partial}{\partial S_t}\left(A_t^\alpha \Phi(-d_1)\right) = e^{-r(T-t)} K \frac{\partial}{\partial S_t}\Phi(-d_2) - \frac{\partial}{\partial S_t}\left(g(t)\left(S_t\right)^\alpha \Phi(-d_1)\right) \\
=& e^{-r(T-t)} K \frac{\partial}{\partial S_t}\Phi(-d_2) - \alpha g(t)\left(S_t\right)^{\alpha-1}\Phi(-d_1) - g(t)\left(S_t\right)^\alpha \frac{\partial}{\partial S_t}\Phi(-d_1)
\end{aligned} \tag{2.7}
$$

We used the product rule in the last step. We must now determine the partial derivatives with regards to the CDF functions. For this we use the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial S_t}\Phi(-d_2) =& \frac{\partial}{\partial S_t}\Phi\left(-\frac{\ln\left(\frac{A_t^\alpha}{K}\right)+(r-(\alpha\sigma)^2/2)(T-t)}{\alpha\sigma\sqrt{T-t}}\right)\\
=& \frac{\partial}{\partial S_t}\Phi\left(\frac{\ln\left(\frac{K}{g(t)(S_t)^\alpha}\right)-(r-(\alpha\sigma)^2/2)(T-t)}{\alpha\sigma\sqrt{T-t}}\right)\\
=& \phi(-d_2)\left(-\frac{1}{S_t\sigma\sqrt{T-t}}\right)\\
=& -\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}
\end{aligned} \tag{2.8}$$

Where we used that the derivative of the CDF is the PDF. We now do the same for $\frac{\partial}{\partial S_t}\Phi(-d_1)$:

$$\begin{aligned}
\frac{\partial}{\partial S_t}\Phi(-d_1) =& \frac{\partial}{\partial S_t}\Phi\left(\frac{\ln\left(\frac{K}{g(t)(S_t)^\alpha}\right)-(r+(\alpha\sigma)^2/2)(T-t)}{\alpha\sigma\sqrt{T-t}}\right)\\
=& \phi(-d_1)\left(-\frac{1}{S_t\sigma\sqrt{T-t}}\right) = -\frac{\frac{1}{\sqrt{2\pi}}e^{-(-d_1)^2/2}}{S_t\sigma\sqrt{T-t}}\\
=& -\frac{\frac{1}{\sqrt{2\pi}}e^{-(-d_2-\alpha\sigma\sqrt{T-t})^2/2}}{S_t\sigma\sqrt{T-t}} = -\frac{\frac{1}{\sqrt{2\pi}}e^{-(d_2^2+(\alpha\sigma\sqrt{T-t})^2+2\alpha\sigma\sqrt{T-t}d_2)/2}}{S_t\sigma\sqrt{T-t}}\\
=& -\frac{e^{-((\alpha\sigma\sqrt{T-t})^2+2\alpha\sigma\sqrt{T-t}d_2)/2}\frac{1}{\sqrt{2\pi}}e^{-d_2^2/2}}{S_t\sigma\sqrt{T-t}}\\
=& -e^{-\alpha\sigma\sqrt{T-t}\frac{\ln\left(\frac{A_t^\alpha}{K}\right)+(r-(\alpha\sigma)^2/2)(T-t)}{\alpha\sigma\sqrt{T-t}}-(\alpha\sigma)^2/2(T-t)}\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}\\
=& -e^{\ln\left(\frac{K}{g(t)(S_t)^\alpha}\right)-(r-(\alpha\sigma)^2/2)(T-t)-(\alpha\sigma)^2/2(T-t)}\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}\\
=& -\frac{K}{g(t)(S_t)^\alpha}e^{-r(T-t)}\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}
\end{aligned} \tag{2.9}$$

Where we used the definition of the PDF and that it is symmetric. We can now insert these formulas:

$$\begin{aligned}
\Delta =& e^{-r(T-t)}K\left(-\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}\right)-\alpha g(t)\left(S_t\right)^{\alpha-1}\Phi(-d_1)-g(t)\left(S_t\right)^\alpha\left(-\frac{K}{g(t)(S_t)^\alpha}e^{-r(T-t)}\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}\right)\\
=& -e^{-r(T-t)}K\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}-\alpha g(t)\left(S_t\right)^{\alpha-1}\Phi(-d_1)+e^{-r(T-t)}K\frac{\phi(-d_2)}{S_t\sigma\sqrt{T-t}}\\
=& -\alpha g(t)\left(S_t\right)^{\alpha-1}\Phi(-d_1) = -\alpha\frac{A_t^\alpha}{S_t}\Phi(-d_1)
\end{aligned} \tag{2.10}$$

We have now calculated delta, the partial derivative of the insurance contract with regards to the underlying stock. We thereby know how many units of stock that we need to hold to replicate the option. We note that the delta is equal to the overall sensitivity to changes in the stock price $-\Phi(-d_1)$, scaled with the weight of the stock α and the relative price of the portfolio compared to the stock $\frac{A_t^\alpha}{S_t}$. Intuitively then we will need a lot of units if we wish to hedge a \$300m. portfolio of penny stocks with a large weight. The next step is implementing this in code for a delta hedging experiment, so that we can verify if it works. For this we simply use $A_t^\alpha = g(t)(S_t)^\alpha$ to evaluate A_t. We could also just have used the second last step and skipped evaluating it. We choose not to do that, to allow for using either A_t^α or t, A_0^t, S_0 as arguments. We start by coding the insurance contract:

```
class insurance(claim):

    def payoff(self,**v):
        if not "A" in v:
            g = v['A0'] / np.power(v['S0'],v['a']) * np.exp((v['r']+v['a']*(v['sigma']**2)/2)*(1-v['a'])*v['t'])
            v['A'] = g * np.power(v['S'],v['a'])
        return np.maximum(v['K']-v['A'],0)

    def price(self,**v):
        with np.errstate(divide='ignore', invalid='ignore'): #divide by 0 mute
            if not "A" in v:
                g = v['A0'] / np.power(v['S0'],v['a']) * np.exp((v['r']+v['a']*(v['sigma']**2)/2)*(1-v['a'])*v['t'])
                v['A'] = g * np.power(v['S'],v['a'])
            d1 = (np.log(v['A'] / v['K']) + (v['r'] + 0.5 * (v['sigma']*v['a']) ** 2) * v['T']) / (v['a']*v['sigma'] * np.sqrt(v['T']))
            d2 = d1 - v['a']*v['sigma'] * np.sqrt(v['T'])

            opt = - v['A'] * scipy.stats.norm.cdf(-d1) + v['K'] * np.exp(-v['r'] *(v['T'])) * scipy.stats.norm.cdf(-d2)
            return  np.where(v['T']==0,self.payoff(**v),opt)

    def delta(self,**v):
        with np.errstate(divide='ignore', invalid='ignore'): #divide by 0 mute
            if not "A" in v:
                g = v['A0'] / np.power(v['S0'],v['a']) * np.exp((v['r']+v['a']*(v['sigma']**2)/2)*(1-v['a'])*v['t'])
                v['A'] = g * np.power(v['S'],v['a'])
            d1 = (np.log(v['A'] / v['K']) + (v['r'] + 0.5 * (v['sigma']*v['a']) ** 2) * v['T']) / (v['a']*v['sigma'] * np.sqrt(v['T']))
            delta = - v['a'] * v['A'] / v['S'] * scipy.stats.norm.cdf(-d1)
            return  np.where(v['T']==0,0,delta)
```

Note that $v['T']$ denotes time till maturity from t and therefore rather $T-t$ than actually T. Slightly inconvenvinient use of notation. For implementing the discrete hedge we use a method extremely similar to in Hand-In 1. We First simulate asset prices with the Black-Scholes model under Q. We do this for 5000 paths and a number of time points. For each time point in each path we then compute the delta of the insurance contract using the current simulated asset price and factors like the remaining time to maturity. Finally we compute the performance of portfolios that hold these delta hedges and then discretely re-adjusts after a time shift of size Δt. We implement this strategy in the below dynamic code:

```
def SimulateSimpleDeltaHedge(stockSim : np.ndarray, option : claim,T,r,sigma, **v):
    m, n = stockSim.shape
    matTime = T - np.arange(0,m).reshape(-1, 1)*T/(m-1)
    pfV = np.zeros((m, n))
    pfS = np.zeros((m, n))
    pfB = np.zeros((m, n))
    pfV[0] = option.price( S=stockSim[0],T=T,r=r,sigma=sigma, t=0, **v)
    pfS = option.delta(S=stockSim,T=matTime,r=r,sigma=sigma, t=T-matTime, **v)

    interest = np.exp(r*matTime[-2])
```

```
    for i in range(1,m):
        pfB[i-1] = pfV[i-1]-pfS[i-1]*stockSim[i-1]
        pfV[i] = pfS[i-1]*stockSim[i] + pfB[i-1]* interest

    pf0 = option.payoff( S=stockSim[-1],t=T,r=r,sigma=sigma,**v)
    discountedHedgeError = (pfV[-1]-pf0) * np.exp(-r*T)
    return {'discounted hedge error' : discountedHedgeError,
            'option payoff' : pf0,
            'hedging pf value' : pfV[-1],
            'hedging pf path' : pfV,
            'hedging pf S' : pfS,
            'hedging pf B' : pfB}
```

We are now ready to test the hedging-strategy, as we have done in Figure 1. We note that the strategy appears to hedge the payoff of the option well, especially for a high rebalance rate. Additionally, the payoff function has a very interesting curvature shape, caused by the non-linear relationship between the asset price S_t and the portfolio price A_t^α used in the payoff function. The payoff function is naturally still perfectly horizontal for $A_T^\alpha = g(t)(S_T)^\alpha > K$ as while $K - A_T^\alpha = K - g(t)(S_T)^\alpha$ is not linear in S_T, 0 definitely is.

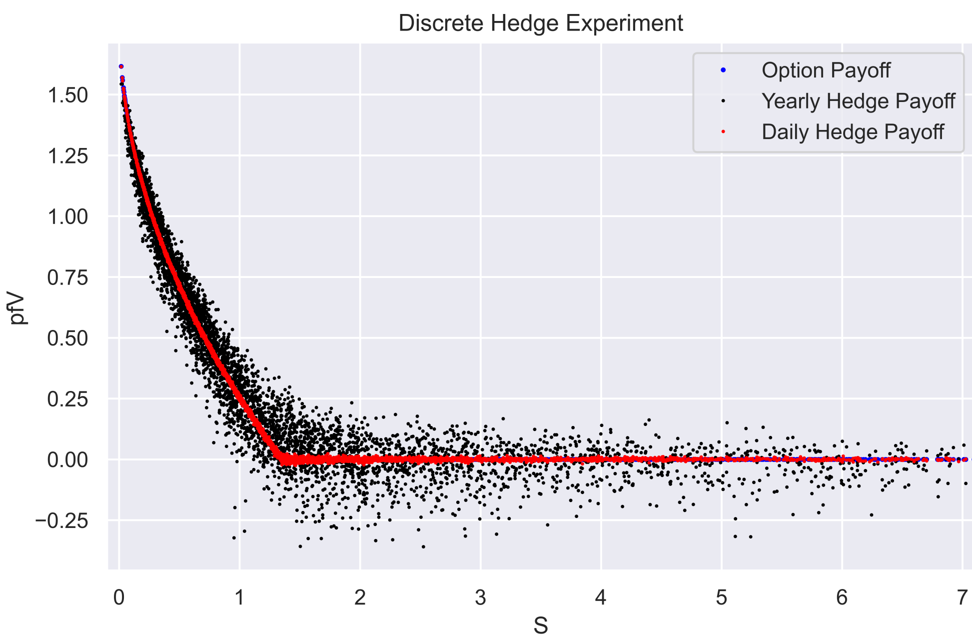

Figure 1

There are however still some errors, so we need to convince ourselves that the errors are caused by the discretisation of what should be a continuously rebalancing hedge. We do this as we did in Hand-In 1, by examining the relationship between the errors and our amount of hedge points. First we definite the discounted hedge error:

$$\varepsilon = e^{-rT}(V_{T,pf} - (K - A_T^\alpha)^+) \tag{2.11}$$

Where $V_{T,pf}$ denotes the terminal value of our hedging portfolio. Assuming that our errors have mean 0 (which they approximately do), we can define the relative size of the hedging errors as the standard deviation of the hedging errors:

$$\sigma_\varepsilon = \sqrt{Var[\varepsilon]} = \sqrt{E[\varepsilon^2] - E[\varepsilon]^2} = \sqrt{E[\varepsilon^2] - 0^2} = \sqrt{E[\varepsilon^2]} \tag{2.12}$$

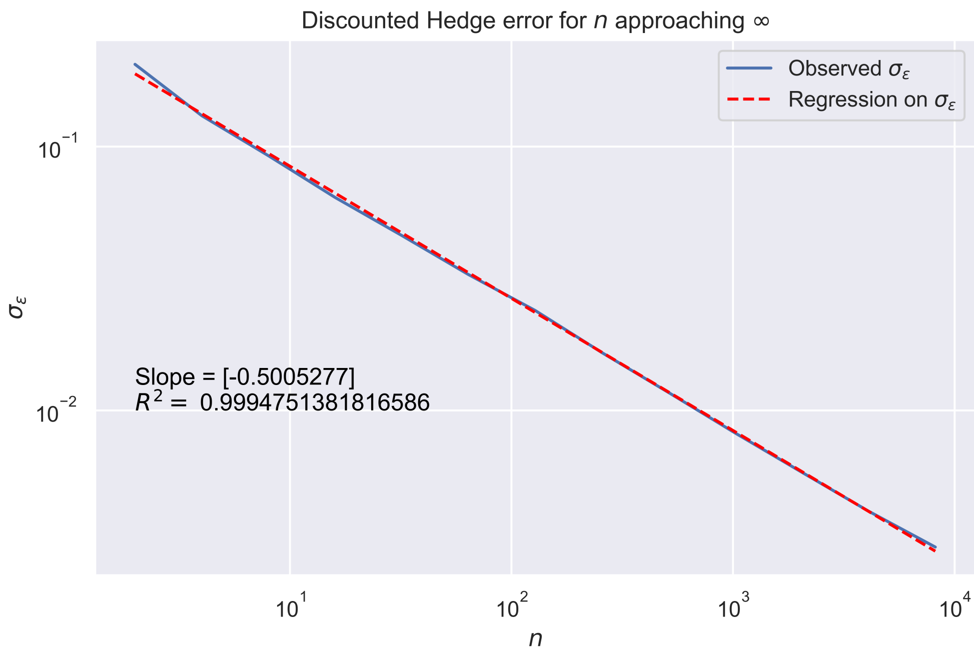

Figure 2: Standard deviation of simulated hedging errors for 1000 simulations of delta hedging the insurance contract with different amounts of rebalance dates.

Empirically we have:

$$\hat{\sigma_\varepsilon} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} \varepsilon_i^2 - \bar{\varepsilon}} \tag{2.13}$$

$$\bar{\varepsilon} = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \approx 0 \tag{2.14}$$

We compute this for simulations with varying amounts of rebalance time points n. As we wish to understand the behaviour of $\lim_{n\to\infty} \sigma_\varepsilon$ then we plot the relationship between the log transformed standard deviation and the log transformed n. We also fit a linear regression between the two. The log-log relationship is interesting because it indicates if the two factors share the relationship:

$$\sigma_\varepsilon = c \cdot n^\beta \tag{2.15}$$

Where β is the coefficient of the linear regression. We plot the results in Figure 2. We see that the regression has an extremely high R^2, which indicates that the variables in fact do share the relationship in equation (2.15). Additionally we see that this relationship holds for even extremely large n and small σ_ε, this is nice as for bad hedging strategies we would see the errors converge on a non-zero level. We note that the regression coefficient is $\beta = -0.5$, which means that the hedge errors converge on zero with order $O(n^{-0.5}) = O(1/\sqrt{n})$. This is the same as that of the classical Black-Scholes Call/Put and our Quanto Put from Hand-In 1. We therefore argue that, by the same logic as in Hand-In 1, if the errors of hedging a Black-Scholes Call option converge on zero for $\Delta t \to 0$ then so does the errors for hedging our insurance. It is therefore possible to replicated the insurance contract by trading dynamically in the stock and delta hedging the exposure.

3 Pricing Other Options

In this section we have to derive the pricing formulas for the below equations:

1. $e^{-rT}E^P[(K-A_T^\alpha)^+]$ (Insurance contract under P)
2. $e^{-rT}E^Q[(K-(S_T)^\alpha)^+]$ (Suboptimal insurance contract)
3. $\alpha e^{-rT}E^Q[(K-S_T)^+]$ (α put options)

We start with the first equations, which is the real world expected payoff of the insurance contract. We start by taking one of our equations from question 2:

$$e^{-r(T-t)}E_t^Q[(K-A_T^\alpha)^+] = e^{-r(T-t)}\left(KP^Q[-d_2>Z] - A_t^\alpha E_t^Q\left[e^{(r-(\alpha\sigma)^2/2)(T-t)+\alpha\sigma\sqrt{T-t}Z}\mathbf{1}_{-d_2>Z}\right]\right) \tag{3.1}$$

From this, we can conclude that the following naturally holds with $t=0$: (we could also easily show this in a couple of steps)

$$e^{-rT}E^P[(K-A_T^\alpha)^+] = e^{-rT}\left(KP[-d_2>Z] - A_0^\alpha E^P\left[e^{(r-(\alpha\sigma)^2/2)T+\alpha\sigma\sqrt{T}Z}\mathbf{1}_{-d_2>Z}\right]\right) \tag{3.2}$$

We defined $Z=\dfrac{W_T^Q}{\sqrt{T}}$ and we know from equation (1.3) that $dW_T^Q = dW_T^P - \dfrac{r-\mu}{\sigma}dt \Rightarrow W_T^Q = W_T^P - \dfrac{r-\mu}{\sigma}T$ so we can define another standard normal variable X (now under P) such that:

$$Z=\frac{W_T^Q}{\sqrt{T}} = \frac{dW_T^P - \frac{r-\mu}{\sigma}T}{\sqrt{T}} = \frac{X\sqrt{T}-\frac{r-\mu}{\sigma}T}{\sqrt{T}} = X - \frac{r-\mu}{\sigma}\sqrt{T} \tag{3.3}$$

We do this as Z is only a standard normally distributed variable under Q, not under P. The opposite is the for X.

$$\begin{aligned}
e^{-rT}E^P[(K-A_T^\alpha)^+] =& e^{-rT}\left(KP[-d_2>X-\frac{r-\mu}{\sigma}\sqrt{T}] - A_0^\alpha E^P\left[e^{(r-(\alpha\sigma)^2/2)T+\alpha\sigma\sqrt{T}\left(X-\frac{r-\mu}{\sigma}\sqrt{T}\right)}\mathbf{1}_{-d_2>X-\frac{r-\mu}{\sigma}\sqrt{T}}\right]\right)\\
=& e^{-rT}\left(KP\left[-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X\right] - A_0^\alpha E^P\left[e^{(r-(\alpha\sigma)^2/2)T-\alpha(r-\mu)T+\alpha\sigma\sqrt{T}X}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X}\right]\right)\\
=& e^{-rT}\left(K\Phi\left(-d_2+\frac{r-\mu}{\sigma}\sqrt{T}\right) - A_0^\alpha E^P\left[e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T+\alpha\sigma\sqrt{T}X}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X}\right]\right)
\end{aligned}$$

We now rewrite the expectation by writing it on integral form and rewriting to a standard normal CDF. We define the variable Æ similar to Y before.

$$\begin{aligned}
Æ =& X-\alpha\sigma\sqrt{T}\Rightarrow\\
X =& Æ+\alpha\sigma\sqrt{T}\Rightarrow\\
\alpha\sigma\sqrt{T}X - X^2/2 =& \alpha\sigma\sqrt{T}(Æ+\alpha\sigma\sqrt{T}) - (Æ+\alpha\sigma\sqrt{T})^2/2\\
=& \alpha\sigma\sqrt{T}Æ + (\alpha\sigma\sqrt{T})^2 - Æ^2/2 - (\alpha\sigma\sqrt{T})^2/2 - 2\alpha\sigma\sqrt{T}Æ/2\\
=& T(\alpha\sigma)^2/2 - Æ^2/2
\end{aligned} \tag{3.4}$$

We now compute the expectation using our new variable for substitution:

$$
\begin{aligned}
E^P[...] =& E^P\left[e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T+\alpha\sigma\sqrt{T}X}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X}\right] \\
=& \int_{-\infty}^{\infty} e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T+\alpha\sigma\sqrt{T}X}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X}\phi(X)dX \\
=& \int_{-\infty}^{\infty} e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T+\alpha\sigma\sqrt{T}X}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X}\frac{1}{\sqrt{2\pi}}e^{-X^2/2}dX \\
=& e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T}\int_{-\infty}^{\infty}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X}\frac{1}{\sqrt{2\pi}}e^{\alpha\sigma\sqrt{T}X-X^2/2}dX \\
=& e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T}\int_{-\infty}^{\infty}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>\text{Æ}+\alpha\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{T(\alpha\sigma)^2/2-\text{Æ}^2/2}d\text{Æ} \\
=& e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T+T(\alpha\sigma)^2/2}\int_{-\infty}^{\infty}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}-\alpha\sigma\sqrt{T}>\text{Æ}}\frac{1}{\sqrt{2\pi}}e^{-\text{Æ}^2/2}d\text{Æ} \\
=& e^{(r(1-\alpha)+\mu\alpha)T}\int_{-\infty}^{\infty}\mathbf{1}_{-d_1+\frac{r-\mu}{\sigma}\sqrt{T}>\text{Æ}}\phi(\text{Æ})d\text{Æ} \\
=& e^{(r(1-\alpha)+\mu\alpha)T}\int_{-\infty}^{-d_1+\frac{r-\mu}{\sigma}\sqrt{T}}\phi(\text{Æ})d\text{Æ} \\
=& e^{(r(1-\alpha)+\mu\alpha)T}\Phi\left(-d_1+\frac{r-\mu}{\sigma}\sqrt{T}\right)
\end{aligned}
$$

We can now finally insert this expectation back in our equation:

$$
\begin{aligned}
e^{-rT}E^P[(K-A_T^\alpha)^+] =& e^{-rT}\left(K\Phi\left(-d_2+\frac{r-\mu}{\sigma}\sqrt{T}\right)-A_0^\alpha E^P\left[e^{(r(1-\alpha)+\mu\alpha-(\alpha\sigma)^2/2)T+\alpha\sigma\sqrt{T}X}\mathbf{1}_{-d_2+\frac{r-\mu}{\sigma}\sqrt{T}>X}\right]\right) \\
=& e^{-rT}\left(K\Phi\left(-d_2+\frac{r-\mu}{\sigma}\sqrt{T}\right)-A_0^\alpha e^{(r(1-\alpha)+\mu\alpha)T}\Phi\left(-d_1+\frac{r-\mu}{\sigma}\sqrt{T}\right)\right) \\
=& e^{-rT}K\Phi\left(-d_2+\frac{r-\mu}{\sigma}\sqrt{T}\right)-A_0^\alpha e^{-rT+(r(1-\alpha)+\mu\alpha)T}\Phi\left(-d_1+\frac{r-\mu}{\sigma}\sqrt{T}\right) \\
=& e^{-rT}K\Phi\left(-d_2+\frac{r-\mu}{\sigma}\sqrt{T}\right)-e^{\alpha(\mu-r)T}A_0^\alpha\Phi\left(-d_1+\frac{r-\mu}{\sigma}\sqrt{T}\right)
\end{aligned}
\tag{3.5}
$$

We have hereby determined the formula for the expected discounted payoff of the insurance contract under P. We see that the option now also depends on the real drift of the stock μ. We will interpret the formula deeper after deriving the others.

We now look at equation 2, which is similar to a bad hedge where we are missing the factor $g(T)$. We have:

$$
\begin{aligned}
e^{-rT}E^Q[(K-(S_T)^\alpha)^+] =& e^{-rT}E^Q[(K-\frac{1}{g(T)}A_T^\alpha)^+] \\
=& e^{-rT}E^Q\left[(K-\frac{1}{g(T)}A_T^\alpha)\mathbf{1}_{K>\frac{1}{g(T)}A_T^\alpha}\right] \\
=& e^{-rT}\left(E^Q\left[K\mathbf{1}_{K>\frac{1}{g(T)}A_T^\alpha}\right]-E^Q\left[\frac{1}{g(T)}A_T^\alpha\mathbf{1}_{K>\frac{1}{g(T)}A_T^\alpha}\right]\right) \\
=& e^{-rT}\left(KP^Q\left[Kg(T)>A_T^\alpha\right]-\frac{1}{g(T)}E^Q\left[A_T^\alpha\mathbf{1}_{Kg(T)>A_T^\alpha}\right]\right)
\end{aligned}
$$

We now see some clear parallels to our formula in question two. We note that the only difference is the $g(T)$ multiplier on K in our indicator functions and the $g(T)^{-1}$ multiplier on our second term. While we could redo all our rewriting, integrals and substitutions, we will simply insert these obvious differences. This yields:

$$
e^{-rT}E^Q[(K-(S_T)^\alpha)^+] = e^{-rT}K\Phi(-d_2^*)-\frac{A_0^\alpha}{g(T)}\Phi(-d_1^*) \tag{3.6}
$$

Where:

$$
\begin{aligned}
d_2^* =& \frac{\ln\left(\frac{A_0^\alpha}{Kg(T)}\right)+(r-(\alpha\sigma)^2/2)T}{\alpha\sigma\sqrt{T}} \\
d_1^* =& d_2^*+\alpha\sigma\sqrt{T}=\frac{\ln\left(\frac{A_0^\alpha}{Kg(T)}\right)+(r+(\alpha\sigma)^2/2)T}{\alpha\sigma\sqrt{T}}
\end{aligned} \tag{3.7}
$$

We again wait with interpreting the result and instead focus on the last equation. We notice that the equation is simply a standard put option on the stock, multiplied with the fraction α. Using this we can isolate a put option, we can then derive the put options price using Björk's proposition 10.2 (the put-call parity) and 7.13 (Black-Scholes Call price formula):

$$
\begin{aligned}
\alpha e^{-rT}E^Q[(K-S_T)^+] =& \alpha p(0,S_0)=\alpha\left(e^{-rT}K+c(0,S_0)-S_0\right) \\
=& \alpha\left(e^{-rT}K+S_0\Phi(d_1)-e^{-rT}K\Phi(d_2)-S_0\right) \\
=& \alpha\left(e^{-rT}K+S_0\Phi(d_1)-e^{-rT}K\Phi(d_2)-S_0\right) \\
=& \alpha\left(e^{-rT}K+S_0(1-\Phi(-d_1))-e^{-rT}K(1-\Phi(-d_2))-S_0\right) \\
=& \alpha\left(-S_0\Phi(-d_1)+e^{-rT}K\Phi(-d_2)\right) \\
=& \alpha\left(e^{-rT}K\Phi(-d_2)-S_0\Phi(-d_1)\right)
\end{aligned} \tag{3.8}
$$

Here we used that $\Phi(x)=1-\Phi(-x)$. The equation naturally yields α multiplied with the Black-Scholes put option price. We can now try computing the price of the options using our numbers from before and $\mu=7\%$:

$$
e^{-rT}E^Q[(K-A_T^\alpha)^+] = 0.2158 \tag{3.9}
$$

$$
e^{-rT}E^P[(K-A_T^\alpha)^+] = 0.0304 \tag{3.10}
$$

$$
e^{-rT}E^Q[(K-(S_T)^\alpha)^+] = 0.4119 \tag{3.11}
$$

$$\alpha e^{-rT}E^{Q}[(K-S_T)^+] = 0.2081 \tag{3.12}$$

We see that the discounted expected payoff of the option under P is substantially smaller than under Q. This is because $\mu > r$, as the stock has a risk premium, so its value is expected to grow at a higher rate than the bank account. The discounted expected payoff under Q is however still the correct arbitrage-free price for the contract, as while the payoff under P is what a contract holder might expect, it is not the price of replicating the option. Additionally, the payoffs of this contract will be large in paths where money is "worth" more (has higher utility). The exact relationship between Q and P is however a greater topic encompassing facets such as risk appetite, optimal consumption and state-price utility. We can however conclude that the real world expected value of the contract is lower than the price, as it pays a risk premium by indirectly shorting the stock.

The second equation we were given $e^{-rT}E^{Q}[(K-(S_T)^{\alpha})^+]$ gave a price substantially higher than the original option. Examining its definition and the pricing formula we derived in equation (3.6), it is clear that this is as **the equation is equal to the insurance contract, but with the price of the asset at expiry scaled with the deterministic** $g(T)^{-1}$. We can also yield a more intuitive formula be rewriting $(K-(S_T)^{\alpha})^+ = (K-g(T)^{-1}A_T^{\alpha})^+ = g(T)^{-1}(g(T)K-A_T^{\alpha})^+ \forall g(T) > 0$. A more intuitive explanation of equation two is therefore the arbitrate price of $g(T)^{-1}$ insurance contracts on the portfolio with the strike $g(T)K$. As $g(T) = 1.57 > 1$ then we have fewer contracts, but with a lower strike.

Usually $0 < \dfrac{\partial}{\partial K}\pi_t < 1$, which might make one think that this should lead to an overall smaller price, but we have a multiplicative relationship and not a additive one. Since our strike is larger than the put price then the effects are not applied evenly. Therefore we do not mind having fewer contracts, as the price difference for each one outweighs the lower amount. To examine the relationship we can attempt to calculate the derivative:

$$\begin{aligned}
\frac{\partial}{\partial g(T)}e^{-rT}g(T)^{-1}E^{Q}[(g(T)K-A_T^{\alpha})^+] =& \frac{\partial}{\partial g(T)}g(T)^{-1}\pi_0\left(g(T)K,...\right)\\
=& -g(T)^{-2}\pi_0\left(g(T)K,...\right) + g(T)^{-1}\frac{\partial}{\partial g(T)}\pi_0\left(g(T)K,...\right)
\end{aligned} \tag{3.13}$$

We calculate the derivative of the insurance contract with regards to scaling K by $g(T)$:

$$\begin{aligned}
\frac{\partial}{\partial g(T)}\pi_0\left(g(T)K,...\right) =& e^{-rT}\frac{\partial}{\partial g(T)}\left(g(T)K\Phi(-d_2^*)\right) - A_0^{\alpha}\frac{\partial}{\partial g(T)}\left(\Phi(-d_1^*)\right)\\
=& e^{-rT}K\Phi(-d_2^*) + e^{-rT}g(T)K\frac{\partial}{\partial g(T)}\left(\Phi(-d_2^*)\right) - A_0^{\alpha}\frac{\partial}{\partial g(T)}\left(\Phi(-d_1^*)\right)\\
=& e^{-rT}K\Phi(-d_2^*) + e^{-rT}g(T)K\frac{\phi(-d_2^*)}{g(T)\alpha\sigma\sqrt{T}} - A_0^{\alpha}\frac{\phi(-d_1^*)}{g(T)\alpha\sigma\sqrt{T}}\\
=& e^{-rT}K\Phi(-d_2^*) + e^{-rT}g(T)K\frac{\phi(-d_2^*)}{g(T)\alpha\sigma\sqrt{T}} - A_0^{\alpha}\frac{g(T)K}{g(0)(S_0)^{\alpha}}e^{-rT}\frac{\phi(-d_2^*)}{g(T)\alpha\sigma\sqrt{T}}\\
=& e^{-rT}K\Phi(-d_2^*) + e^{-rT}K\frac{\phi(-d_2^*)}{\alpha\sigma\sqrt{T}} - e^{-rT}K\frac{\phi(-d_2^*)}{\alpha\sigma\sqrt{T}}\\
=& e^{-rT}K\Phi(-d_2^*)
\end{aligned} \tag{3.14}$$

We see that this is the usual put derivative with regards to K, but scaled with K as we are looking at the derivative of something multiplied to K. We can now insert this:

$$\begin{aligned}
\frac{\partial}{\partial g(T)}e^{-rT}g(T)^{-1}E^{Q}[(g(T)K-A_T^{\alpha})^+] =& -g(T)^{-2}\pi_0\left(g(T)K,...\right) + g(T)^{-1}\frac{\partial}{\partial g(T)}\pi_0\left(g(T)K,...\right)\\
=& -g(T)^{-2}\left(e^{-rT}g(T)K\Phi(-d_2^*) - A_0^{\alpha}\Phi(-d_1^*)\right) + g(T)^{-1}e^{-rT}K\Phi(-d_2^*)\\
=& g(T)^{-2}A_0^{\alpha}\Phi(-d_1^*)
\end{aligned} \tag{3.15}$$

$$g(T)^{-2}A_0^\alpha \Phi(-d_1^*)\big|_{g(T)=1} = 0.3920 \tag{3.16}$$

$$g(T)^{-2}A_0^\alpha \Phi(-d_1^*)\big|_{g(T)=1.5683} = 0.2879 \tag{3.17}$$

$$(1.5683-1)\frac{0.3920+0.2879}{2} = 0.1932 \approx 0.1961 = e^{-rT}g(T)^{-1}E^Q[(g(T)K - A_T^\alpha)^+] - e^{-rT}E^Q[(K - A_T^\alpha)^+] \tag{3.18}$$

We see that this is always positive and the sizes are inline with what we would expect from our price changes. Therefore, scaling the strike of the payoff function up while scaling the amount of options down with the same factor always increases the value of the position. This is not a specific result for our insurance contract, but rather a general result for put options. The effect is diminishing in $g(T)$ as the option becomes more and more ITM.

It is however NOT possible to recreate the original insurance contract using this new one, as the effective strike now is $g(T)K > K$.

We now have the final equation, which is the arbitrage free price of simply holding α put options on the stock. We note that the price is similar to the price of the insurance contract, but not equal. This is as $A_0^\alpha = S_0$ then the exposure of the simple put strategy perfectly matches that of the insurance contract at time $t = 0$. This is visible from the delta:

$$\Delta_0 = -\alpha\frac{A_0^\alpha}{S_0}\Phi(-d_1) = -\alpha\Phi(-d_1) = -\alpha\frac{\partial}{\partial S_t}\text{put} \tag{3.19}$$

It is however also visible that for $t > 0$ then the two portfolios will surely diverge as their deltas differ by a factor of $\frac{A_t^\alpha}{S_t} = g(t)(S_t)^{\alpha-1}$. This is why the prices appear similar, but are in fact different.

4 Proving the spanning formula

We must prove the spanning formula from *Carr & Madan* (2001). The equation details how to theoretically span any twice continuously differentiable payoff using a continuous set of OTM put/call options over all strikes, a risk free asset and the underlying. Let $f(S_T)$ be the payoff function that we wish to replicate, then the spanning formula is:

$$f(S_T) = f(S_0) - f'(S_0)S_0 + f'(S_0)S_T + \int_0^{S_0} f''(K)(K - S_T)^+ dK + \int_{S_0}^{\infty} f''(K)(S_T - K)^+ dK \tag{4.1}$$

Which equates to a replicating strategy holding $f'(S_0)$ units of stock, $f(S_0) - f'(S_0)S_0+$ dollars (or the local currency) in the bankbook and $f''(K)dK$ OTM options at strike K.

To prove this formula we start with the fundamental theorem of calculus. The theorem in its most basic state states that:

$$f(S_T) - f(S_0) = \int_{S_0}^{S_T} f'(S)ds \tag{4.2}$$

I.e. the change over an interval is equal to the integral over infinitesimal changes on the interval, provided that $f'(S)$ exists over the interval $[S_0, S_T]$. We can rewrite this a bit:

$$f(S_T) = f(S_0) + \int_{S_0}^{S_T} f'(S)ds \tag{4.3}$$

We know that an integral where the upper limit is below the lower limit ($S_T < S_0$) is equal to the negative of the integral with the limits reversed. Using this to expand yields:

$$f(S_T) = f(S_0) + \mathbf{1}_{S_T>S_0} \int_{S_0}^{S_T} f'(S)ds - \mathbf{1}_{S_T<S_0} \int_{S_T}^{S_0} f'(S)ds \tag{4.4}$$

We can now re-apply the fundamental theorem of calculus inside itself, this means that:

$$f'(s) = f'(S_0) + \int_{S_0}^{s} f''(u)du = f'(S_0) - \int_{s}^{S_0} f''(u)du \tag{4.5}$$

$$\begin{aligned}
f(S_T) =& f(S_0) + \mathbf{1}_{S_T>S_0} \int_{S_0}^{S_T} \left(f'(S_0) + \int_{S_0}^{s} f''(u)du \right) ds - \mathbf{1}_{S_T<S_0} \int_{S_T}^{S_0} \left(f'(S_0) - \int_{s}^{S_0} f''(u)du \right) ds \\
=& f(S_0) + \mathbf{1}_{S_T>S_0} \left(f'(S_0)(S_T - S_0) + \int_{S_0}^{S_T} \left(\int_{S_0}^{s} f''(u)du \right) ds \right) - \mathbf{1}_{S_T<S_0} \left(f'(S_0)(S_0 - S_T) - \int_{S_T}^{S_0} \left(\int_{s}^{S_0} f''(u)du \right) ds \right) \\
=& f(S_0) + f'(S_0)(S_T - S_0) + \mathbf{1}_{S_T>S_0} \int_{S_0}^{S_T} \left(\int_{S_0}^{s} f''(u)du \right) ds + \mathbf{1}_{S_T<S_0} \int_{S_T}^{S_0} \left(\int_{s}^{S_0} f''(u)du \right) ds
\end{aligned} \tag{4.6}$$

Here we used that $f'(S_0)$ does not depend on s to move it outside the integrals. Additionally, we used that $\mathbf{1}_{x>0}x + \mathbf{1}_{x<0}x = x$. Next *Carr & Madan* use Fubini's theorem. The goal is to switch the order of the integrals, such that we can simplify our expression further. The issue with this is that the limit of our inner integral depends directly on the outer integral. This gives us triangular integration regions that can be described as:

$$R_1 = \{(s, u), S_0 \leq s \leq S_T, S_0 \leq u \leq s\} \tag{4.7}$$

$$R_2 = \{(s, u), S_T \leq s \leq S_0, s \leq u \leq S_0\} \tag{4.8}$$

The idea of Fubini's theorem is that we can collect our iterated integrals into collected Lebesgue double integrals over these regions, before splitting them again. We are allowed to do this if f is twice continuously differentiable, which therefore is a requirement for the spanning formula. In addition both R_1 and R_2 are finite (because of the hard limits S_0, S_T). This yields the new double integrals:

$$f(S_T) = f(S_0) + f'(S_0)(S_T - S_0) + \mathbf{1}_{S_T > S_0} \iint_{R_1} f''(u)duds + \mathbf{1}_{S_T < S_0} \iint_{R_2} f''(u)duds \tag{4.9}$$

We now wish to split these double integrals into iterated integrals with the opposite order. To accomplish this we first gave to rewrite the regions such that u is defined first without a constraint from s. This is relatively simple given our simple triangular regions.

$$R_1 = \{(s,u), S_0 \leq s \leq S_T, S_0 \leq u \leq s\} = \{(u,s), S_0 \leq u \leq S_T, u \leq s \leq S_T\} \tag{4.10}$$

$$R_2 = \{(s,u), S_T \leq s \leq S_0, s \leq u \leq S_0\} = \{(u,s), S_T \leq u \leq S_0, S_T \leq s \leq u\} \tag{4.11}$$

It is clear that these are in fact the same regions with the same boundaries. We can now split our integrals up again:

$$f(S_T) = f(S_0) + f'(S_0)(S_T - S_0) + \mathbf{1}_{S_T > S_0} \int_{S_0}^{S_T} \left(\int_u^{S_T} f''(u)ds \right) du + \mathbf{1}_{S_T < S_0} \int_{S_T}^{S_0} \left(\int_{S_T}^{u} f''(u)ds \right) du \tag{4.12}$$

We can now evaluate the inner integrals, as they do not depend on s.

$$\begin{aligned} f(S_T) &= f(S_0) + f'(S_0)(S_T - S_0) + \mathbf{1}_{S_T > S_0} \int_{S_0}^{S_T} f''(u)(S_T - u)du + \mathbf{1}_{S_T < S_0} \int_{S_T}^{S_0} f''(u)(u - S_T)du \\ &= f(S_0) + f'(S_0)(S_T - S_0) + \int_{S_0}^{S_T} f''(u)(S_T - u)\mathbf{1}_{S_T > S_0} du + \int_{S_T}^{S_0} f''(u)(u - S_T)\mathbf{1}_{S_T < S_0} du \\ &= f(S_0) + f'(S_0)(S_T - S_0) + \int_{S_0}^{S_T} f''(u)(S_T - u)^+ du + \int_{S_T}^{S_0} f''(u)(u - S_T)^+ du \\ &= f(S_0) + f'(S_0)(S_T - S_0) + \int_{S_0}^{\infty} f''(u)(S_T - u)^+ du + \int_{-\infty}^{S_0} f''(u)(u - S_T)^+ du \\ &= f(S_0) + f'(S_0)(S_T - S_0) + \int_{S_0}^{\infty} f''(u)(S_T - u)^+ du + \int_{0}^{S_0} f''(u)(u - S_T)^+ du \end{aligned} \tag{4.13}$$

We now have something that resembles options nicely. In the third last step we have used that if $S_T \leq S_0 \Leftrightarrow (u - S_T) \leq 0 \forall u \in [S_0, S_T]$, thereby $(u - S_T)^+$ encapsulates the effect of $\mathbf{1}_{S_T < S_0}$. Additionally, then as $u \leq S_T$ then $(u - S_T)^+$ does not remove any other values than those. The reasoning is similar but opposite for the second integral. In the second last step we let the values run free as any new values above/below the old limit now are truncated by the new positivity requirements. Finally we limit the last integral at 0, with the idea that (traditional) assets prices never are negative, and neither are strikes. To fully accentuate the relationship between our formula and call/put options, we rename our integration variable from u to K:

$$\begin{aligned} f(S_T) &= f(S_0) + f'(S_0)(S_T - S_0) + \int_{S_0}^{\infty} f''(K)(S_T - K)^+ dK + \int_0^{S_0} f''(K)(K - S_T)^+ dK \\ &= f(S_0) - f'(S_0)S_0 + f'(S_0)S_T + \int_0^{S_0} f''(K)(K - S_T)^+ dK + \int_{S_0}^{\infty} f''(K)(S_T - K)^+ dK \end{aligned} \tag{4.14}$$

And we have hereby proved the Spanning formula in equation (4.1), by from the fundamental theorem of calculus.

5 Spanning the Portfolio Insurance Contract

We now wish to use the spanning formula to price our insurance contract from question 2, i.e:

$$f(S_T) = (K - A_T^\alpha)^+ = (K - g(T)(S_T)^\alpha)^+ \tag{5.1}$$

We note that the question mentions a portfolio of put options only. With the spanning formula then if the money starts OTM such that $f(S_0) = f'(S_0) = f''(S_0) = 0$ then we would hold none of the risk free asset, none of the stock and no call options. This is however **not** the case with our numeric example as $K > g(T)(S_0)^\alpha$. Therefore, according to the spanning formula we should also hold these other assets. We will therefore follow the spanning formula and expand our portfolio to these additional assets. We will later investigate the possibility of replacing our call options with put options, and if we then still require the bond and stock positions.

To implement the spanning formula, we must first evaluate the derivatives. We have that for ITM options:

$$K > g(T)x^\alpha \Leftrightarrow x < \frac{K}{g(T)}^{1/\alpha} = S_{ATM} \Rightarrow f'(x) = -\alpha g(T) x^{\alpha-1} \tag{5.2}$$

$$x > S_{ATM} \Rightarrow f'(x) = 0 \tag{5.3}$$

$$f'(S_0) = -\alpha g(T) S_0^{\alpha-1} \tag{5.4}$$

$$x < S_{ATM} \Rightarrow f''(x) = (\alpha - \alpha^2) g(T) x^{\alpha-2} \tag{5.5}$$

$$x > S_{ATM} \Rightarrow f''(x) = 0 \tag{5.6}$$

The last case we need to evaluate is the edge case where the option is ATM. Here we can either steal the result from Rolf Poulsen's *Things I Learned This Semester the Fourth.* If we wish to calculate it ourselves then we can use Björk's *The Pedestrian's Guide to Local Time*, as we will do now. The paper describes the Heavyside function as:

$$H_y(x) = \begin{cases} 0, & x < y \\ 1, & x \geq y \end{cases} \tag{5.7}$$

Which is simply a special form of the indicator function. It is however very usefull for options as (with some constraints) $\frac{d}{dx}(x-K)^+ = H_K(x) = 1_{x \geq K}$. For put options we naturally have:

$$\frac{d}{dx}(K - x)^+ = 1 - H_K(x) = 1_{x \leq K} \tag{5.8}$$

In addition, one can show that in a (quote) distributional sense, the derivative of the heavy side function is the Dirac delta:

$$\frac{d}{dx} H_K(x) = \delta_K(x) \tag{5.9}$$

We are now armed to attempt to extract the full derivatives for $f(x)$. We begin by using the chain rule:

$$\begin{aligned} f'(x) =& \frac{\partial}{\partial x}(K - g(T)(x)^\alpha)^+ = -(1 - H_K(g(T)(x)^\alpha))\alpha g(T) x^{\alpha-1} = -(1 - H_{S_{ATM}}(x))\alpha g(T) x^{\alpha-1} \\ =& (H_{S_{ATM}}(x) - 1)\alpha g(T) x^{\alpha-1} \end{aligned} \tag{5.10}$$

We can now compute the derivative of this using both the product rule:

$$f''(x) = (H_{S_{ATM}}(x) - 1)(\alpha^2 - \alpha) g(T) x^{\alpha-2} + \delta_{S_{ATM}}(x)\, \alpha g(T) x^{\alpha-1} \tag{5.11}$$

We can insert in our spanning formula now:

$$
\begin{aligned}
f(S_T) =& f(S_0) - f'(S_0)S_0 + f'(S_0)S_T + \int_0^{S_0} f''(K)(K-S_T)^+ dK + \int_{S_0}^{\infty} f''(K)(S_T-K)^+ dK \\
=& (K - g(T)(S_0)^\alpha)^+ + (H_{S_{ATM}}(S_0) - 1)\alpha g(T) S_0^{\alpha-1}(S_T - S_0) \\
&+ \int_0^{S_0} \left((H_{S_{ATM}}(K) - 1)(\alpha^2-\alpha) g(T) K^{\alpha-2} + \delta_{S_{ATM}}(K)\, \alpha g(T) K^{\alpha-1}\right)(K-S_T)^+ dK \\
&+ \int_{S_0}^{\infty} \left((H_{S_{ATM}}(K) - 1)(\alpha^2-\alpha) g(T) K^{\alpha-2} + \delta_{S_{ATM}}(K)\, \alpha g(T) K^{\alpha-1}\right)(S_T-K)^+ dK \\
=& K - g(T)(S_0)^\alpha + (0-1)\alpha g(T) S_0^{\alpha-1}(S_T - S_0) \\
&+ \int_0^{S_0} \left((0-1)(\alpha^2-\alpha) g(T) K^{\alpha-2}\right)(K-S_T)^+ dK \\
&+ \int_{S_0}^{S_{ATM}} \left((0-1)(\alpha^2-\alpha) g(T) K^{\alpha-2}\right)(S_T-K)^+ dK \\
&+ 1\alpha g(T) S_{ATM}^{\alpha-1}(S_T - S_{ATM})^+ \\
=& K - (1-\alpha) g(T)(S_0)^\alpha - \alpha g(T) S_0^{\alpha-1} S_T \\
&+ \int_0^{S_0} g(T) K^{\alpha-2}(\alpha - \alpha^2)(K-S_T)^+ dK \\
&+ \int_{S_0}^{S_{ATM}} g(T) K^{\alpha-2}(\alpha - \alpha^2)(S_T-K)^+ dK \\
&+ \alpha g(T) S_{ATM}^{\alpha-1}(S_T - S_{ATM})^+
\end{aligned}
\tag{5.12}
$$

Here we have used abilities and definitions for the Heavyside and Diaric delta functions. We have hereby inserted in the spanning formula and we have the portfolio that we need to replicate the option. We see that we need to hold $f(S_0) - f'(S_0)S_0 = K - (1-\alpha)g(T)(S_0)^\alpha = 1.0379$ units of zero coupon bonds and $f'(S_0) = -\alpha g(T) S_0^{\alpha-1} = -0.7841$ units of stock. In addition we also have to hold some call options. This all deviates from Rolf's article, but that is just as we have a setup where $S_0 < S_{ATM}$. If that was not the case, we would not be holding any bonds, stock or call options. With regards to call options, we hold $f''(x)dK = g(T)K^{\alpha-2}(\alpha - \alpha^2)dK = 0.3920K^{-1.5}dK$ units for each strike $K \in [S_0, S_{ATM}]$. For the ATM option we hold an **additional** $\alpha g(T) S_{ATM}^{\alpha-1} = 0.6749$. Finally, for put options, we hold the same as call options $f''(x)dK = g(T)K^{\alpha-2}(\alpha - \alpha^2)dK = 0.3920K^{-1.5}dK$.

We now have to perfect static hedge, that we can simply buy and hold EXCEPT that we sadly are unable to actually buy tiny fractions of options with infinity many strikes. If we wish to somehow implement this spanning hedge, then we must discretize our integrals to simply being the sum of n options. This is what we will do next.

We first decide to purchase as many stock units and ATM calls as in our continuous spanning formula, we also insert the same amount of cash in the bankbook. We additionally have the interval that we wish to purchase options in $[0, S_{ATM}]$. We could split this interval using different methods, but we will simply split it in $n+1$ evenly sized intervals, where n is the amount of additional options with different strikes. In other words, we approximate the integral using a Riemann sum. This yields the new spanning approximation:

$$
\begin{aligned}
f(S_T) \approx& K - (1-\alpha) g(T)(S_0)^\alpha - \alpha g(T) S_0^{\alpha-1} S_T \\
&+ \sum_{i=1}^{n} g(T) K_i^{\alpha-2}(\alpha-\alpha^2)((S_T-K)^+ H_{S_0}(K_i) + (K-S_T)^+(1 - H_{S_0}(K_i))) \frac{S_{ATM} - 0}{n} \\
&+ \alpha g(T) S_{ATM}^{\alpha-1}(S_T - S_{ATM})^+
\end{aligned}
\tag{5.13}
$$

Which works for $n \to \infty$. K_i denotes the i'th partition point of the interval. We have written how this discrete hedging looks for the $n = 5$ case in Table 1. We note that the time 0 price of the portfolio is 0.2121, not exactly the arbitrage-free price of the insurance contract, but it is relatively close for a static hedge of only 6 options.

Product	Bank	Stock	Put 1	Put 2	Put 3	Put 4	Call	ATM Call
Strike (K_i)			0.2250	0.4500	0.6749	0.8999	1.1249	1.3499
Position	1.0379	-0.7841	0.9920	0.3507	0.1909	0.1240	0.0887	0.6749
Price	0.5696	-0.7841	0.0037	0.0082	0.0115	0.0138	0.0493	0.3400

Table 1: Discrete spanning portfolio with extra options $n = 5$.

We should however examine what happens as $n \to \infty$, and if the price of the dynamic hedge portfolio actually converges on the arbitrage-free price of the insurance contract. In addition, there is the even more important issue of whether the portfolio actually replicates the contract. To examine how the spanning formula works in practice, we develop the below code:

```
def span(option,n, area = None, **coef):
    result = {}
    putOpt = Put.put()
    callOpt = Call.call()

    coefEnd = coef.copy()
    coefEnd['t'] = coef['T']
    coefEnd['T'] = 0
    coef['t'] = 0

    dBonds = option.payoff(**coefEnd,S=coef['S0']) - option.payoffD(**coefEnd,S=coef['S0']) * coef
        ['S0']
    dStock = option.payoffD(**coefEnd,S=coef['S0'])

    Ks = np.linspace(0, option.atm(**coefEnd), n+2)[1:]
    dOpt = option.payoffDD(**coefEnd,S=Ks)
    dOpt[:-1] *= option.atm(**coefEnd)/n #not diarec delta

    coefOpts = coef.copy()
    coefOpts['K'] = Ks

    pOpt = np.where(Ks<coef['S0'],putOpt.price(**coefOpts,S=coef['S0']),callOpt.price(**coefOpts,S
        =coef['S0']))

    result['Positions'] = np.concatenate((np.array([dBonds, dStock]), dOpt))
    result['Prices'] = np.concatenate((np.array([dBonds*np.exp(-coef['T']*coef['r']), dStock]),
        dOpt * pOpt))
    result['Price'] = np.sum(result['Prices'])
    if not area is None:
        areaTiled = np.tile(area.reshape(-1, 1),n+1)
        result['Fit'] = dBonds + dStock * area + np.dot(np.where(Ks<coef['S0'],putOpt.payoff(**
            coefOpts,S=areaTiled),callOpt.payoff(**coefOpts,S=areaTiled)),dOpt)
    return result
```

Using this code we first examine our approximation of $f(S_T)$ with different n. We plot some of these in figure 3. We note that the payoff of the spanning portfolio appears to very quickly converge on the payoff of the insurance contract. This is a great sign, but we should also examine the convergence of the errors.

Using a methodology similar to in question 2, we use the square root of our quadratic errors to measure our errors:

$$\sqrt{\sum \varepsilon^2} = \sqrt{\sum_{i=1}^{m} \left(f(S_{T,i}) - \text{Span payoff}(S_{T,i})\right)^2} \tag{5.14}$$

This time we do not assume that our average error is 0 as figure 3 clearly shows that that is not the case. We calculate this measure for 10,000 points in the span of $S_{T,i} \in]0, 3[$. At the same time we also look at the deviations between the price of the span portfolio and the price of the insurance contract. We plot the results in Figure 4. We see that both errors behave similarly, they have a relatively linear log-log relationship with n, but with some

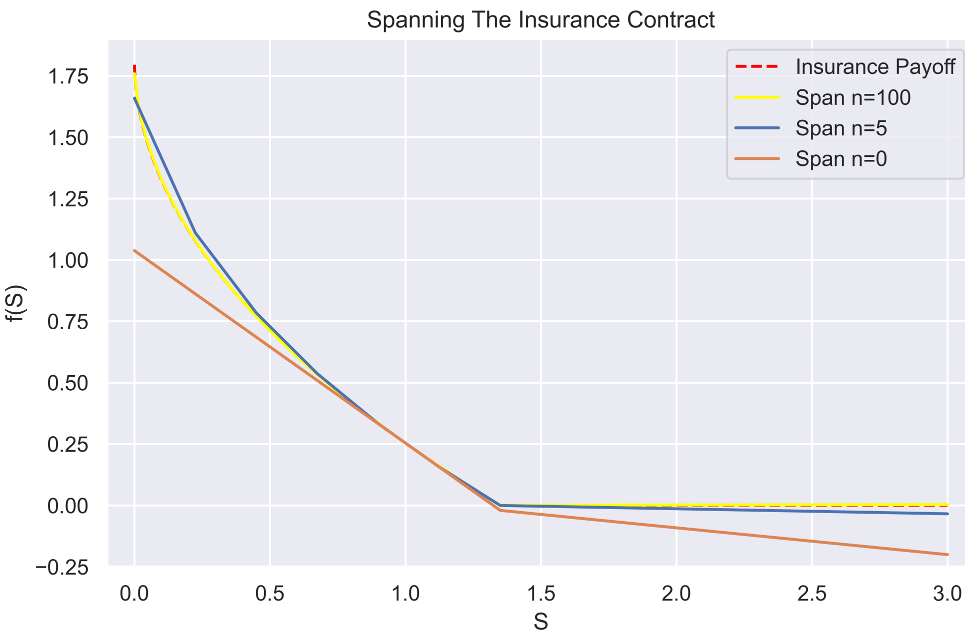

Figure 3

noise/random variation. The R^2 coefficients are however extremely large for both and nothing indicates structural deviance from the linear log-log relationship, such as convergence to some level or divergence. We additionally see that the slopes both are (numerically) large at $\beta = -0.9382$ which indicates that both of these errors converge even faster than that of the delta hedge portfolio, with convergence order close to $O\left(n^{-1}\right)$. We therefore conclude that it is possible to create a static hedge that uses a discrete version of the spanning formula to buy an initial portfolio of bonds, stock and options that fully hedges our insurance contract. We note that the price of this hedge portfolio converges on the insurance contract price, showing absence of arbitrage.

The question does however ask for a portfolio of only put options, so we are not fully satisfied with our answer. We can however use the put-call pariety to transform all our OTM Call options to ITM put options. We do so:

$$\begin{aligned}
f(S_T) \approx & K - (1-\alpha)g(T)(S_0)^\alpha - \alpha g(T) S_0^{\alpha-1} S_T \\
& + \sum_{i=1}^{n} g(T) K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}\left((K-S_T)^+ - Ke^{-r0} + S_T\right) \\
& + \alpha g(T) S_{ATM}^{\alpha-1}\left((S_{ATM}-S_T)^+ - S_{ATM}e^{-r0} + S_T\right) \\
= & K - \alpha g(T)\left(S_{ATM}^\alpha + (1-\alpha)\frac{S_{ATM}}{n}\sum_i K_i^{\alpha-1}\mathbf{1}_{K_i \geq S_0}\right) - (1-\alpha)g(T)(S_0)^\alpha \\
& \quad + (S_{ATM}^{\alpha-1} - S_0^{\alpha-1} + (1-\alpha)\frac{S_{ATM}}{n}\sum_i K_i^{\alpha-2}\mathbf{1}_{K_i \geq S_0})\alpha g(T) S_T \\
& + \sum_{i=1}^{n} g(T) K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}(K-S_T)^+ \\
& + \alpha g(T) S_{ATM}^{\alpha-1}(S_{ATM}-S_T)^+
\end{aligned} \tag{5.15}$$

We now have a portfolio with no call options, but we have instead gained a more complicated amount of bonds and stocks. To simplify these terms we will look closer at our sums $\frac{S_{ATM}}{n}\sum_i K_i^{\alpha-1}\mathbf{1}_{K_i \geq S_0}$ and $\frac{S_{ATM}}{n}\sum_i K_i^{\alpha-2}\mathbf{1}_{K_i \geq S_0}$.

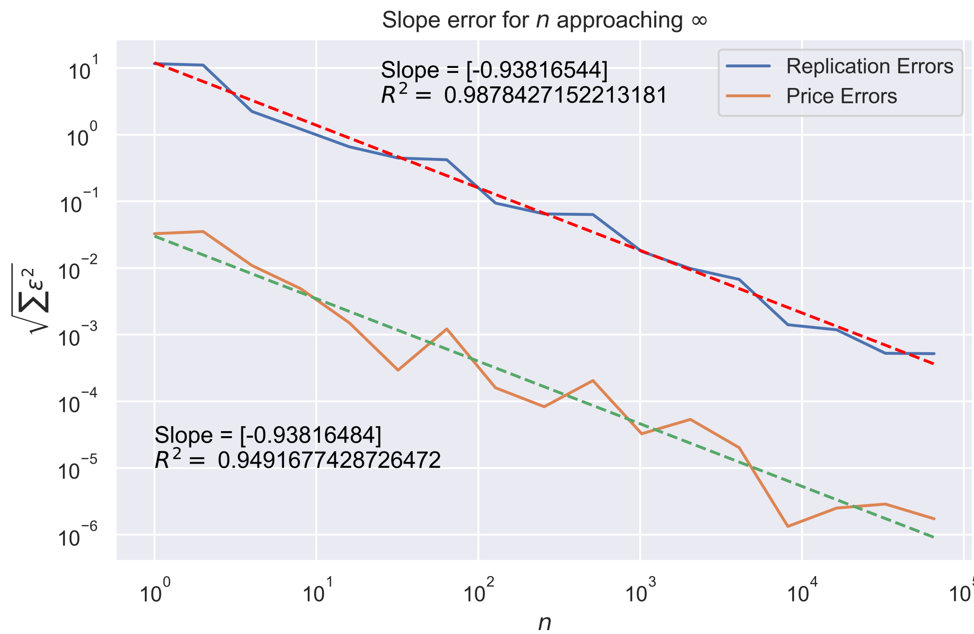

Figure 4

Our idea is that the holdings of bonds and stocks might vanish in the limit, so we set $n \to \infty$ and rewrite the sums as integrals:

$$\lim_{n\to\infty} \frac{S_{ATM}}{n} \sum_i K_i^{\alpha-1} \mathbf{1}_{K_i \geq S_0} = \int_{S_0}^{S_{ATM}} K^{\alpha-1} dK = \frac{S_{ATM}^{\alpha} - S_0^{\alpha}}{\alpha} \tag{5.16}$$

$$\lim_{n\to\infty} \frac{S_{ATM}}{n} \sum_i K_i^{\alpha-2} \mathbf{1}_{K_i \geq S_0} = \int_{S_0}^{S_{ATM}} K^{\alpha-2} dK = \frac{S_{ATM}^{\alpha-1} - S_0^{\alpha-1}}{\alpha - 1} = -\frac{S_{ATM}^{\alpha-1} - S_0^{\alpha-1}}{1-\alpha} \tag{5.17}$$

Where we assume $\alpha \neq 1$ and $\alpha \neq 0$, in which case the problem would simplify to something much simpler anyway (just hold the correct put option, or nothing). This is simply reversing the discretisation we did earlier. We also stress that the above formula only holds for $n \to \infty$. We can now insert:

$$
\begin{aligned}
\lim_{n\to\infty} f(S_T) \approx & K - \alpha g(T)\left(S_{ATM}^{\alpha} + (1-\alpha)\frac{S_{ATM}^{\alpha} - S_0^{\alpha}}{\alpha}\right) - (1-\alpha)g(T)(S_0)^{\alpha} \\
& + (S_{ATM}^{\alpha-1} - S_0^{\alpha-1} - (1-\alpha)\frac{S_{ATM}^{\alpha-1} - S_0^{\alpha-1}}{1-\alpha})\alpha g(T)S_T \\
& + \sum_{i=1}^{n} g(T)K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}(K-S_T)^{+} \\
& + \alpha g(T)S_{ATM}^{\alpha-1}(S_{ATM}-S_T)^{+} \\
= & K - \alpha g(T)\left(\frac{S_{ATM}^{\alpha} - S_0^{\alpha}}{\alpha} + S_0^{\alpha}\right) - (1-\alpha)g(T)(S_0)^{\alpha} \\
& + (0)\alpha g(T)S_T \\
& + \sum_{i=1}^{n} g(T)K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}(K-S_T)^{+} \\
& + \alpha g(T)S_{ATM}^{\alpha-1}(S_{ATM}-S_T)^{+} \\
= & K - g(T)S_{ATM}^{\alpha} \\
& + \sum_{i=1}^{n} g(T)K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}(K-S_T)^{+} \\
& + \alpha g(T)S_{ATM}^{\alpha-1}(S_{ATM}-S_T)^{+} \\
= & K - g(T)\left(\frac{K}{g(T)}^{1/\alpha}\right)^{\alpha} \\
& + \sum_{i=1}^{n} g(T)K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}(K-S_T)^{+} \\
& + \alpha g(T)S_{ATM}^{\alpha-1}(S_{ATM}-S_T)^{+} \\
= & 0 \\
& + \sum_{i=1}^{n} g(T)K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}(K-S_T)^{+} \\
& + \alpha g(T)S_{ATM}^{\alpha-1}(S_{ATM}-S_T)^{+}
\end{aligned} \tag{5.18}
$$

We can now conclude that for $n \to \infty$ the effects from the put-call parity perfectly matches our initial holdings of bonds and stock. We can therefore statically hedge the with only put options and no other assets, as long as we purchase enough of them. For each put with strike in the interval $K \in]0, S_{ATM}]$ we purchase exactly $f''(K)\frac{S_{ATM}}{n} = g(T)K_i^{\alpha-2}(\alpha-\alpha^2)\frac{S_{ATM}}{n}$ units, except for the option with strike S_{ATM}, where we purchase an additional $-f'(S_{ATM}) = \alpha g(T)S_{ATM}^{\alpha-1}$. We have made some slight modifications to our code and plotted the errors of this strategy in Figure 5. We see that the price of the portfolio of put options also converges on the arbitrage-free insurance contract price:

$$
\lim_{n\to\infty} \pi_0^{\text{SPAN-Puts}} = 0.2158 = \lim_{n\to\infty} \pi_0^{\text{SPAN-Puts+Calls+Stock+Bankbook}} = \pi_0^{\text{Risk-Neutral Valuation Formula}} \tag{5.19}
$$

The replication errors also converge on zero. We additionally note that the put only portfolio converges faster and with less noise than the mixed assets span. The errors are also generally of smaller size. This simpler portfolio of only put options thereby actually does a better job of spanning the portfolio. This possibly because of its simplicity, but it might also be that it is easier to replicate a "put-like" contract using puts rather than calls.

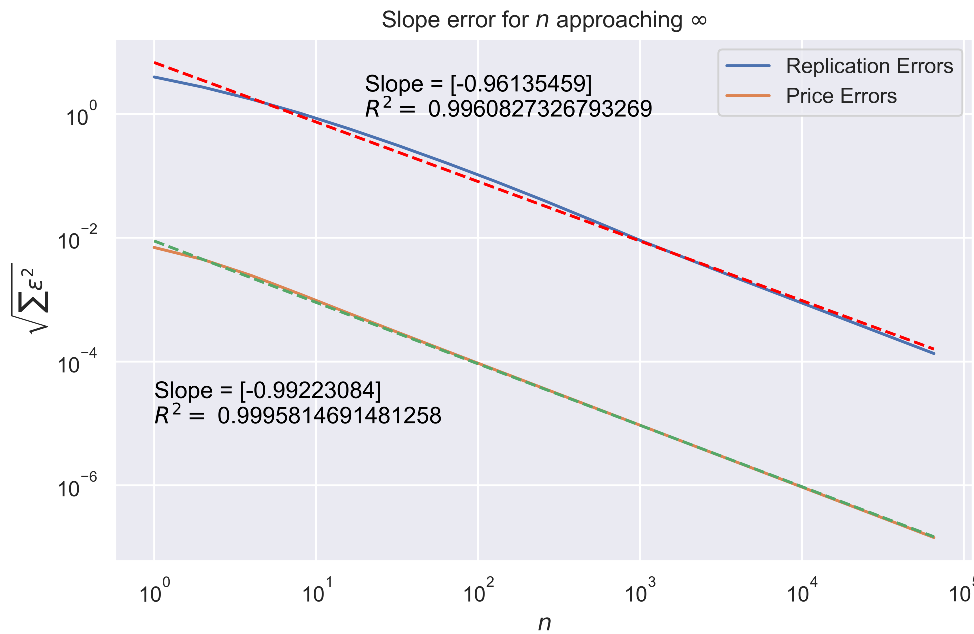

Figure 5

6 Heston Price Formula

For the rest of the questions we will move from the Black-Scholes GBM model to the heston model. That means that we will assume that the stock price follows the dynamics under Q:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1,t}^Q \tag{6.1}$$

$$dV_t = \kappa(\theta - V_t)dt + \epsilon\sqrt{V_t} dW_{2,t}^Q \tag{6.2}$$

$$dW_{1,t}^Q dW_{2,t}^Q = \rho \tag{6.3}$$

We must now determine the new dynamics of the portfolio dA_t^α and explain how $e^{-rT}E^Q[(K - A_t^\alpha)^+]$ can be calculated using the Heston formula.

Our Björk's lemma 6.12 for weighted portfolios (from question 1) still holds, we use it:

$$\begin{aligned} dA_t^\alpha =& A_t^\alpha \left(w_S \frac{dS_t + dD_t^S}{S_t} + w_B \frac{dB_t + dD_t^B}{B_t} \right) - c_t dt = A_t^\alpha \left(\alpha \frac{dS_t}{S_t} + (1-\alpha)\frac{dB_t}{B_t} \right) \\ =& A_t^\alpha \left(\alpha \frac{rS_t dt + \sqrt{V_t} S_t dW_{1,t}^Q}{S_t} + (1-\alpha) \frac{rB_t dt}{B_t} \right) \\ =& A_t^\alpha \left(\alpha \left(rdt + \sqrt{V_t} dW_{1,t}^Q \right) + (1-\alpha) rdt \right) \\ =& A_t^\alpha \left(rdt + \alpha\sqrt{V_t} dW_{1,t}^Q \right) \end{aligned} \tag{6.4}$$

We see that similar to in question 1, our portfolio value process is equal to the process stock, except with a α multiplier to the stochastic term. The new dynamics are however not a GBM, as we have a non-constant and stochastic $\sigma = \alpha\sqrt{V_t}$. As the portfolio dynamics are not on the standard Heston form (because of α), we will have trouble incorporating the Heston formula. Therefore we define a new random process ν_t such that $\alpha\sqrt{V_t} = \sqrt{\alpha^2 V_t} = \sqrt{\nu_t}$. This new scaled volatility process then has the dynamics:

$$
\begin{aligned}
d\nu_t = \alpha^2 dV_t =& \alpha^2\kappa(\theta - V_t)dt + \alpha^2\epsilon\sqrt{V_t}dW^Q_{2,t} \\
=& \kappa(\alpha^2\theta - \alpha^2 V_t)dt + \alpha\epsilon\sqrt{\alpha^2 V_t}dW^Q_{2,t} \\
=& \kappa(\alpha^2\theta - \nu_t)dt + \alpha\epsilon\sqrt{\nu_t}dW^Q_{2,t} \\
=& \kappa(\Theta - \nu_t)dt + \varepsilon\sqrt{\nu_t}dW^Q_{2,t}
\end{aligned}
\tag{6.5}
$$

We note that this volatility proccess follows the standard Heston volatility process form, except the small adjustment to three coefficients:

$$\Theta = \alpha^2\theta \tag{6.6}$$

$$\varepsilon = \alpha\epsilon \tag{6.7}$$

$$\nu_t = \alpha^2 V_t \tag{6.8}$$

We now have something that we almost can use for the Heston formula. We note that the equation that we are trying to price is the arbitrage-free price of a put option on A^α_t, with the Risk-Neutral Valuation formula. We therefore wish to price:

$$Put_0 = e^{-rT}E^Q[(K - A^\alpha_T)^+] = e^{-rT}E^Q[Put_T] \tag{6.9}$$

The Heston formula from Heston's *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options* (1993) says that for a call option on a process S following the Heston dynamics we have the arbitrage-free price:

$$Call_0 = S_0P_1 - KP(0,T)P_2 = S_0P_1 - e^{-rT}KP_2 \tag{6.10}$$

Where we have a ton of supporting formulas:

$$f_j(x, v, t; \phi) = e^{C_j(T-t,\phi)+D_j(T-t,\phi)v+i\phi x} \tag{6.11}$$

$$C_j(\tau;\phi) = r\phi i\tau + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\phi i + d_j)\,\tau - 2\ln\left(\frac{1 - ge^{d_j\tau}}{1-g}\right)\right\} \tag{6.12}$$

$$D_j(\tau;\phi) = \frac{b_j - \rho\sigma\phi i + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - ge^{d_j\tau}}\right] \tag{6.13}$$

$$g_j = \frac{b_j - \rho\sigma\phi i + d_j}{b_j - \rho\sigma\phi i - d_j} \tag{6.14}$$

$$d_j = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)} \tag{6.15}$$

$$P_j(x, v, T, K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K)}f_j(x, v, T;\phi)}{i\phi}\right]d\phi \tag{6.16}$$

For $j \in \{1, 2\}$.

We will not explain these formulas, but they allow for a closed form solution for pricing call options in the Heston model. The idea is that since we are unable to create a closed form solution for our cumalative density functions, then we can instead attempt to calculate them through the characteristic functions $f_1, f_2 = \phi^F, \phi$, which we are able to write solutions for. The probabilities $P_1, P_2 = Q^F(S_T > K), Q(S_T > K) = E^Q[S_T/S_0\mathbf{1}_{S_T>K}], E^Q[\mathbf{1}_{S_T>K}]$ are simply $\Phi(d_1), \Phi(d_2)$ in the Black-Scholes model. The formulas are currently not useable as they use another factorisation of the model, so we insert our own parameters and some other facts:

$$e^x = S_t = A^\alpha_t \tag{6.17}$$

$$a = \kappa\Theta = \kappa\alpha^2\theta \tag{6.18}$$
$$u_1 = 1/2 \tag{6.19}$$
$$u_2 = -1/2 \tag{6.20}$$
$$\sigma = \varepsilon = \alpha\epsilon \tag{6.21}$$
$$\lambda = 0 \tag{6.22}$$
$$b_1 = \kappa + \lambda - \rho\sigma = \kappa + 0 - \rho\alpha\epsilon = \kappa - \rho\alpha\epsilon \tag{6.23}$$
$$b_2 = \kappa + \lambda = \kappa + 0 = \kappa \tag{6.24}$$
$$v = \nu_0 = \alpha^2 v_0 \tag{6.25}$$

We now have all the tools we need to price a put option on A_t^α. We however wish to price a put option, so we can now simply use the Put-Call parity as before:

$$\begin{aligned}
e^{-rT}E^Q[(K - A_T^\alpha)^+] =& Put_0 = K^{-rT} + Call_0 - A_0^\alpha \\
=& K^{-rT} + A_0^\alpha P_1 - e^{-rT}KP_2 - A_0^\alpha \\
=& K^{-rT}(1 - P_2) - A_0^\alpha(1 - P_1)
\end{aligned} \tag{6.26}$$

We again see a clear relationship to the Black-Scholes model where $\Phi(-d_1) = 1 - \Phi(d_1)$. We now have a solution for $e^{-rT}E^Q[(K - A_T^\alpha)^+]$ and as this is the risk-neutral valuation formula for a put option on A_T^α, then we know from Björk's Theorem 7.11 that we have the arbitrage-free pricing formula for the insurance contract. Our use of the put-call parity does not ruin this fact, as the parity is based of a replication argument and it thereby makes no assumptions about the distributions of the underlying. We do however note that as this is a model with two brownian motions driving the value of the option (the stock and the volatility) then we are unable to hedge the option using only the stock. This means that unless we have options we can trade in, we have an incomplete market and the arbitrage-free price is not unique or enforceable. We are now ready to implement this formula, which we will do in the next part of the hand-in.

Another thing what we will need later is the relationship:

$$A_t^\alpha = f(t, S_t) = g(t)(S_t)^\alpha \tag{6.27}$$

But does this relationship also hold in the Heston model? We can examine this using Itô's lemma:

$$g'(t) = \left(r + \alpha\sigma^2/2\right)(1 - \alpha)g(t)$$

$$\begin{aligned}
f_t = g'(t)(S_t)^\alpha = \left(r + \alpha\sigma^2/2\right)(1-\alpha)g(t)(S_t)^\alpha &= \left(r + \alpha\sigma^2/2\right)(1-\alpha)A_t^\alpha \\
f_s &= g(t)\alpha S^{\alpha-1} \\
f_{ss} &= g(t)(\alpha^2 - \alpha)S^{\alpha-2} \\
\mu_t &= rS_t \\
\sigma_t &= \sqrt{V_t}S_t
\end{aligned}$$

$$\begin{aligned}
dA_t = d(g(t)(S_t)^\alpha) =& (g'(t)(S_t)^\alpha + rS_t g(t)\alpha S^{\alpha-1} + \frac{1}{2}(\sqrt{V_t}S_t)^2 g(t)(\alpha^2 - \alpha)S_t^{\alpha-2})dt + \sqrt{V_t}S_t g(t)\alpha S_t^{\alpha-1}dW_{1,t}^Q \\
=& (\left(r + \alpha\sigma^2/2\right)(1-\alpha)A_t^\alpha + r\alpha A_t^\alpha + \frac{1}{2}V_t(\alpha^2 - \alpha)A_t^\alpha)dt + \alpha\sqrt{V_t}A_t^\alpha dW_{1,t}^Q \\
=& (rA_t^\alpha - \frac{1}{2}\sigma^2(\alpha^2 - \alpha)A_t^\alpha + \frac{1}{2}V_t(\alpha^2 - \alpha)A_t^\alpha)dt + \alpha\sqrt{V_t}A_t^\alpha dW_{1,t}^Q \\
=& (rA_t^\alpha + \frac{1}{2}(V_t - \sigma^2)(\alpha^2 - \alpha)A_t^\alpha)dt + \alpha\sqrt{V_t}A_t^\alpha dW_{1,t}^Q
\end{aligned} \tag{6.28}$$

We note that for this to hold with our old formula, we most replace the old σ with $\sqrt{V_t}$. We therefore now have:

$$g(t) = \frac{A_0^\alpha}{(S_0)^\alpha} e^{(r+\alpha\sigma^2/2)(1-\alpha)t} = \frac{A_0^\alpha}{(S_0)^\alpha} e^{(r+\alpha V_t/2)(1-\alpha)t} \tag{6.29}$$

We note two things, firstly $\sigma = \sqrt{V_0}$ in our example, so it does not change anything at exactly time 0. Secondly, our old deterministic formula is now stochastic and changes based of movements in the volatility level $\sqrt{V_t}$. If we assume that $\alpha = 1$ we circumvent this problem, but else we will be unable to determine A_t^α using a deterministic formula and S_t. This will turn out to be an issue in the next question.

7 Heston Price Computation

We must calculate the arbitrage-free price of the insurance contract with our formula from question 6 and our numeric example. We have also gained the additional new coefficients $V_0 = \theta = 0.2^2, \kappa = 2, \epsilon = 1, \rho = -0.5$.

We derived the formula that we must solve in the last question, so our challenge now is to find a good way to solve the integral in equation (6.16) numerically. While there are other specifications of the Heston model that are much better to work with, we will start with this one and use a naive method to solve the integrals numerically.

To solve the integrals numerically, we will approximate them through discretisation as we did in question five. That means that we will use a Riemann sum:

$$P_j(x, v, T, K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K)} f_j(x, v, T; \phi)}{i\phi}\right] d\phi \approx \frac{1}{2} + \frac{\phi_{Lim}}{n\pi}\sum_{k=1}^{n} \mathrm{Re}\left[\frac{e^{-i\phi_k \ln(K)} f_j(x, v, T; \phi_k)}{i\phi_k}\right] \tag{7.1}$$

Where ϕ_k are the partition points that split the space $[0, \phi_{Lim}]$ into $n+1$ even sized intervals and ϕ_{Lim} is some upper limit we set in hopes that the sum will have converged at that point, and that any greater ϕ_{Lim} would add nothing of substance to the sum. This should happen because of the terms $1/\phi$ and $e^{-\phi}$, but we might need a very large ϕ. We implement this Naive Heston model in the below code. We implement it with the standard parametrisation, and not our special one (ε, not $\alpha\epsilon$...). Instead, we will simply adjust the parameters when calling the model. We have:

```python
class naiveHestonPut(claim):

    def payoff(self,**v):
        return np.maximum(v['K']-v['S'],0)

    def _us(self):
        return np.array([1/2,-1/2])

    def _bs(self,**v):
        return np.array([v['kappa'] + v['lambd'] - v['rho'] * v['epsilon'],v['kappa'] + v['lambd']])

    def _ds(self,phi,bs,us,**v):
        return np.sqrt(np.power(v['rho'] * v['epsilon']*phi*1j-bs,2) - v['epsilon']**2 *(2*us*phi*1j-phi**2) )

    def _gs(self,phi,ds,bs,**v):
        return (bs- v['rho'] * v['epsilon']*phi*1j + ds)/(bs- v['rho'] * v['epsilon']*phi*1j - ds)

    def _Ds(self,phi,gs,ds,bs,tau,**v):
        expon = np.exp(ds*tau)
        return ((bs-v['rho'] * v['epsilon']*phi*1j + ds)/(v['epsilon']**2)) * ((1-expon)/(1-gs*expon))

    def _Cs(self,phi,gs,ds,bs,tau,**v):
        expon = np.exp(ds*tau)
        return v['r']*phi*1j*tau+(v['kappa']*v['theta']/(v['epsilon']**2))*((bs-v['rho']*v['epsilon']*phi*1j+ds)*tau-2*np.log((1-gs*expon)/(1-gs)))

    def characteristic(self,phi,bs=None,**v):
        if bs is None:
            bs = self._bs(**v)
        ds = self._ds(phi, bs, self._us(), **v)
        gs = self._gs(phi, ds, bs, **v)
        return np.exp(self._Cs(phi, gs, ds, bs, v['T'], **v)+self._Ds(phi, gs, ds, bs, v['T'], **v)*v['V']+1j*phi*v['S'])


    def Ps(self,phiLim,n,**v):
        phis =  np.tile(np.linspace(0, phiLim,n+2)[1:-1].reshape(-1,1),2)
```

```
        logK =  np.log(v['K'])
        bs = self._bs(**v)
        return 1/2 + phiLim/(np.pi*n) * np.sum(np.real(np.exp(-1j*phis*logK)*self.characteristic(
            phis,bs,**v)/1j*phis))

    def price(self,**v):
        with np.errstate(divide='ignore', invalid='ignore'): #divide by 0 mute
            Ps = self.Ps(**v)
            return  np.where(v['T']==0,self.payoff(**v),v['K']*np.exp(-v['r']*v['T'])*(1-Ps[1])-v[
                'S']*(1-Ps[0]))
```

Before we go straight to pricing our option, we should start by examining the behaviour of Re $\left[\frac{e^{-i\phi_k \ln(K)} f_j(x,v,T;\phi_k)}{i\phi_k}\right]$ for different ϕ_k. Then we can determine how granular we need to be (n) and how far we need to go (ϕ). We plot this for $j \in \{1,2\}$ with $n = 1,000,000$ in Figure 6 with our numeric parameters. We see that the characteristic functions appear to be poorly behaved, but they do converge on 0 and while they do jump a little, they do not appear to oscillate too much. We therefore continue pricing the option with the same parameters and we gain the price -0.0052... This price is naturally completely wrong, and after a couple of hours of research we figure out why in Martin Bech Rasmussen's Master's thesis *Heston modellen*.

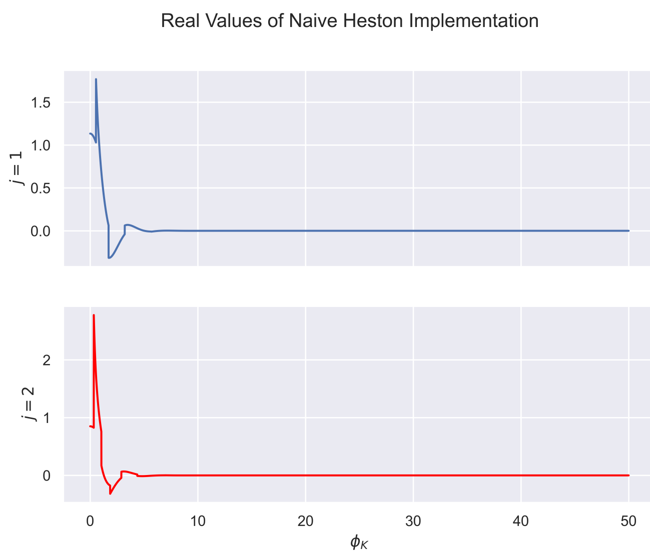

Figure 6

He notes on page 37 that the complex root d is able to take on two values, the one derived by Heston and one that is exactly the same except with the opposite sign. Additional, the original root d is shown the be unstable for some parameter values... We therefore attempt flipping the sign of d and we then again calculate our characteristic functions in figure 7. We see that they are now much more well-behaved and smooth, and they still both converge on 0 fast.

Armed with this strengthened method we again try to price the insurance contract, and we now gain a much more reasonable price:

$$Put_0^{Naive} = e^{-rT} E_0^Q[(K - A_T^\alpha)^+] = 0.2039 \tag{7.2}$$

This result is much more reasonable and is likely correct. We note that the price is a bit below the Black-Scholes price 0.2158. This is partly as we have an ATMF (At-The-Money Forward) option, so the volatility skew, which usually drives the price difference, has a minimal influence.

There are still some issues with the traditional Heston formula, also mentioned in *Heston....Again?!* by Leif Andersen and Mark Lake. Issues such as having to evaluate two integrals, possibilities of rapid oscillations (jumps

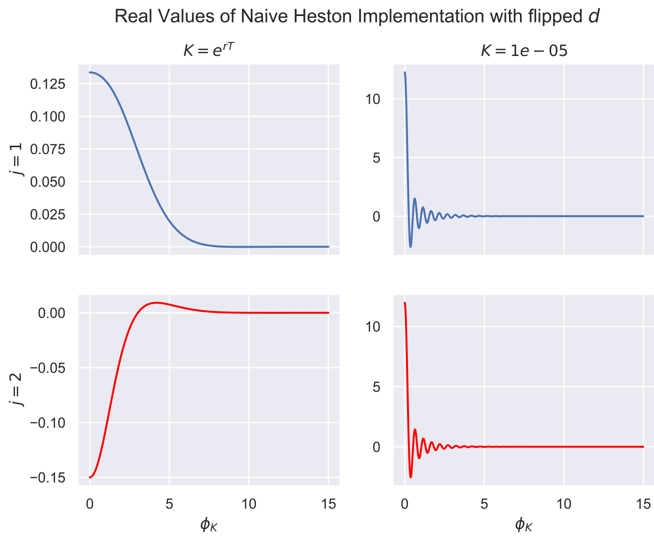

Figure 7

up and down), truncation errors and a low dampening effect. We note that, as visible in our graph, our characteristic functions are so well behaved, that none of these issues are of any real concern for the $K = e^{-rT}$ case. These issues are however very apparent for the $K = 1e - 10$ case, which is very relevant for the next question.

We will therefore implement the higher efficiency model explained by Andersen & Lake in their paper. We use the call option formula from equation 16.

$$Call_0 = e^{-rT}\left(R - \frac{e^{rT}S_0}{\pi}I\right) \tag{7.3}$$

Where I is an integral and R is a constant defined as:

$$R = e^{rT}S_0\mathbf{1}_{\alpha\leq 0} - K\mathbf{1}_{\alpha\leq -1} - \frac{1}{2}\left(e^{rT}S_0\mathbf{1}_{\alpha=0} - K\mathbf{1}_{\alpha=-1}\right) \tag{7.4}$$

Which is a parametrisation of the classic Heston formula that allows for collecting the two integrals in one. Thereby we one have one complicated integral to evaluate. We have additionally included a discounting not in the original equation. For the integral itself we take equation 24.

$$I = e^{\alpha\omega}\int_0^\infty \mathrm{Re}\left\{e^{-x\tan(\varphi)\omega}e^{ix\omega}Q_H(h(x))(1 + i\tan(\varphi))\right\}dx \tag{7.5}$$

Where:

$$h(x) = -i\alpha + x(1 + i\tan(\varphi)) \tag{7.6}$$

$$Q_H(x) = \frac{\phi(x-i)}{x(x-i)} \tag{7.7}$$

$$\omega = \ln(e^{rT}S_0/K) \tag{7.8}$$

Here they changed the original integral to an angle line contour integral, thereby managing to gain the same result but with higher computation efficiency and nice dampening effects. We extract their characteristic function algorithm from appendix A1 of $X = \ln(S_T/K)$ where:

$$\phi(x) = e^{A(x)+B(x)V_0} \tag{7.9}$$

$$A(x) = \frac{\kappa\theta}{\epsilon^2}((\beta - D)T - 2\ln\left(\frac{1 - Ge^{-DT}}{1 - G}\right)) \tag{7.10}$$

$$B(x) = \frac{\beta - D}{\epsilon^2}\left(\frac{1 - e^{-DT}}{1 - Ge^{-DT}}\right) \tag{7.11}$$

$$\beta = \kappa - i\epsilon\rho x \tag{7.12}$$

$$D = \sqrt{\beta^2 + \epsilon^2 x(x + i)} \tag{7.13}$$

$$G = \frac{\beta - D}{\beta + D} \tag{7.14}$$

The appendix describes how to implement this characteristic function effectively while reducing cancellation errors.

For approximating the integral we nap their Fixed Tanh-Sinh rule method from section 4.3. We use section 3.1 for determining φ, but for determining the dampening factor α we steal the implementation by tcPedersen. We however optimise it for working with array inputs. We then use the Put-call parity to transform the call option price to a put.

We find a new price that is identical to the naive price down the the fifth decimal ($Put_0^{Naive} - Put_0^{Lake} = 4e{-}05$). But, we now have a stronger model that we can use for the next question. We leave our code below:

```
from .base import claim
import numpy as np
import scipy.special as sp
from .StolenAlpha import calc_alpha2 #https://github.com/tcpedersen/anderson-lake-python/blob/
    master/pricers.py

class lakeHestonPut(claim):

    def payoff(self,**v):
        return np.maximum(v['K']-v['S'],0)

    def _beta(self,u,**v):
        return v['kappa']-1j*v['epsilon']*v['rho']*u

    def _D(self,u,beta,**v):
        return np.sqrt(beta**2 + (v['epsilon']**2)*u*(u+1j))

    def characteristic(self,u,**v):
        beta = self._beta(u,**v)
        D = self._D(u,beta,**v)

        r = np.where((np.real(beta)*np.real(D)+np.imag(beta)*np.imag(D))>0,
                     -v['epsilon'] ** 2 * u * (u+1j)/(beta+D),
                     beta-D)

        y = np.where(D!=0,
                     np.expm1(-D*v['T'])/(2*D),
                     -v['T']/2)

        A = v['kappa']*v['theta']/(v['epsilon']**2)*(r*v['T']-2*np.log1p(-r*y))
        B = u*(u+1j)*y/(1-r*y)

        return np.exp(A+B*v['V'])

    def Int(self,alpha,N=500,**v):
        ns = np.tile(np.arange(-N,N+1).reshape((-1,1)),len(np.atleast_1d(v['K'])))
        h = sp.lambertw(2*np.pi*N)/N
        qs = np.exp(-np.pi*np.sinh(ns*h))
        ys = 2 * qs / (1 +qs )

```

```
        ws = ys / (1+qs) * np.pi * np.cosh(ns*h)
        xs = 1 - ys

        omega = np.log(np.exp(v['T']*v['r'])*v['S']/v['K'])
        phi = np.pi / 12 * np.sign(omega)

        phi = np.where(v['rho'] - v['epsilon'] * omega /( v['V'] + v['kappa'] * v['theta'] * v['T'] )*omega< 0,phi,0)

        u = lambda f,x: (2/np.power(1-x,2))*f((1+x)/(1-x))
        Q = lambda z: self.characteristic(z-1j,**v)/(z*(z-1j))
        hfunc = lambda x: -1j*alpha + x*(1+1j*np.tan(phi))
        inner = lambda x:  np.real(
                            np.exp(-x*np.tan(phi)*omega)*np.exp(1j*x*omega)*Q(hfunc(x))*(1+1j*np.tan(phi))
                            )
        I = h * np.sum(np.nan_to_num(ws * u(inner,xs),0),axis=0)

        return np.exp(alpha*omega) * I

    def _R(self,forward,alpha,**v):
        return np.where(alpha<=0,forward,0)-np.where(alpha<=-1,v['K'],0)-(np.where(alpha==0,forward,0)-np.where(alpha==-1,v['K'],0))/2

    def price(self,**v):
        with np.errstate(divide='ignore', invalid='ignore'):
            forward = np.exp(v['T']*v['r'])*v['S']
            alpha = calc_alpha2(**v)
            R = self._R(forward,alpha,**v)
            call = np.real(np.exp(-v['r']*v['T'])*(R-forward/np.pi *self.Int(alpha,**v)))
            return  np.where(v['T']==0,self.payoff(**v),v['K']*np.exp(-v['r'] * v['T'])-v['S']+call)
```

8 Heston Static Hedge

We must now consider a static hedge strategy as in question 5, where we replicate the payoff of the original insurance contract $(g(T)S_T^{\alpha})$ by holding a portfolio of other put options with same maturity date.

We first note that as the payoff function $f(x)$ does not directly depend on the process of the underlying, then all our results with regards to the spanning formula, $f'(x)$ and $f''(x)$ still hold. Additionally, then as we are usign the same static hedge portfolio, then we still have the same $g(T)$. The only difference is now that the options that we use to hedge have a different price. We therefore know for certain that as we could replicate the payoff function in the Black-Scholes model, we can do the same with the same portfolio in the Heston Model. It will simply cost something else.

We calculate these prices using the Heston model with our default parameters θ, ϵ, V_t and not our modified $\alpha^2\theta, \alpha\epsilon, \alpha^2 V_t$. This is as we are pricing the options on the underlying strike which we wish to use for hedging, and not portfolio insurance contracts at different strikes. We use our Anderson-Lake implementation from question 7. We plot the result in Figure 8.

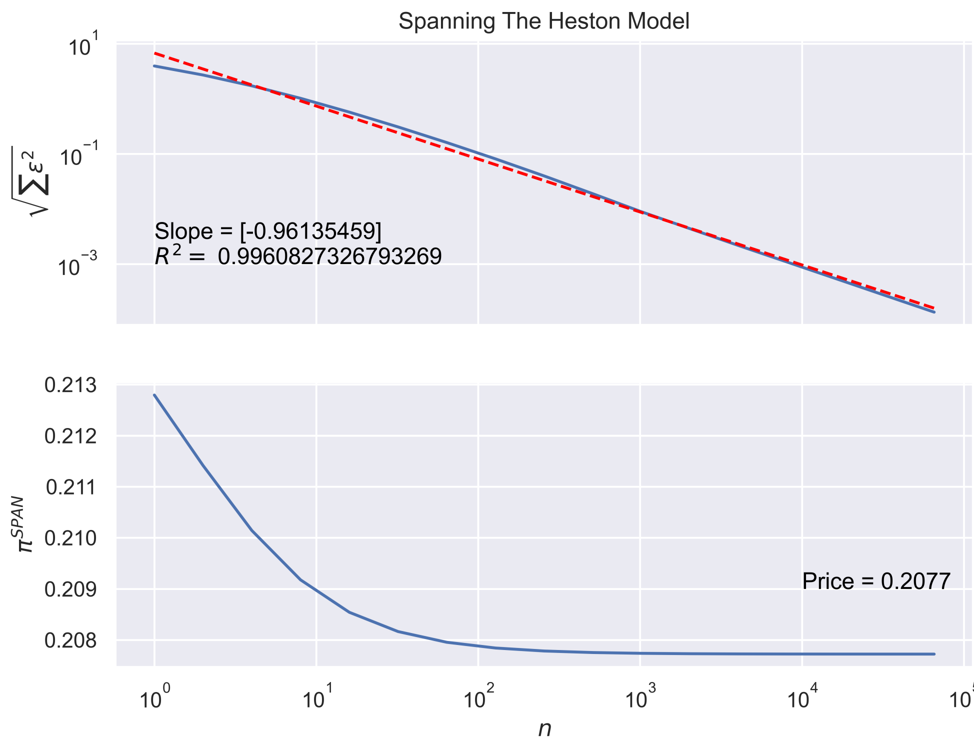

Figure 8

We note that as expected, the payout of the spanning hedge converges on payout of the insurance contract as fast as in the Black-Scholes model. We also see the the price of the hedge converges stably on a level. The price is 0.2077, which is slightly greater than what we found in question 7. There are multiple possible explanations for this, one is that while the Anderson-Lake model is significantly more precise, the is still some level of errors. This is especially true for far ITM and ATM options, which we need to complete our hedge. The Anderson-Lake model is however one of the most effective computation methods available, so this should not be the case.

Another possible factor is that the Heston model is incomplete, this means that we are unable to replicate the option perfectly by using simply trading in the stock. This has no effect in this case, as we are simply replicating the terminal payoff. We have already assumed factors caused by the incompleteness (such as the market price of volatility risk $\lambda = 0$) and these factors are identical for all the options. The incompleteness therefore does not explain the price deviations.

The third factor is issues with our weights from the spanning formula are incorrect under the new model. As explained above, the spanning formula only relies on the terminal function $f(x)$. We do however assume that A_T^α is still equal to $g(T)S_t^\alpha$, and as we showed in the end of question 7, that assumption is no longer true. The prices therefore deviate as the payoff that we are now spanning with the original portfolio of put options is not the same as the payoff of our portfolio insurance contract. As the two payoffs are not the same, it is also natural that the two prices for the option are not the same.

We now look at the special case $\alpha = 1$. In this case our portfolio degenerates to simply a holding of the stock and the return of the portfolio is exactly equal to that of the stocks. Thereby, the insurance contracts becomes simply becomes a position in a put option on the stock. The starting relative size of the portfolio to the stock A_0^α/S_0 does complicate this a bit as we must also scale the strike and put position with this relationship. If $A_0^\alpha = 2 = 2S_0$ then to accomplish an insurance contract on the portfolio with $K = 1$ then we need the payoff $(1 - A_0)^+ = (1 - 2S_0)^+ = 2(0.5 - S_0)^+$. This can also be seen by inserting $g(t)$:

$$(K - A_t^\alpha)^+ = (K - g(t)S_t^\alpha)^+ = (K - \frac{A_0^\alpha}{S_0}S_t)^+ \tag{8.1}$$

And then using that a call price is homogeneous in S and K:

$$Call_0(\frac{A_0^\alpha}{S_0}S, K) = \frac{A_0^\alpha}{S_0}Call_0(S, \frac{S_0}{A_0^\alpha}K) \tag{8.2}$$

Given the put-call parity, this naturally extends to put options.

$$\begin{aligned} Insurance_0^{\alpha=1}(S,K) = Put_0(\frac{A_0^\alpha}{S_0}S,K) =& e^{-rT}K - \frac{A_0^\alpha}{S_0}S + Call_0(\frac{A_0^\alpha}{S_0}S,K) = \frac{A_0^\alpha}{S_0}\left(\frac{S_0}{A_0^\alpha}e^{-rT}K - S + Call_0(S, \frac{S_0}{A_0^\alpha}K)\right) \\ =& \frac{A_0^\alpha}{S_0}Put_0(S, \frac{S_0}{A_0^\alpha}K) \end{aligned} \tag{8.3}$$

We therefore see that our dependence on $\sqrt{V_t}$ which made $g(t)$ stochastic no longer exists. We are therefore now able to get payoffs that actually match the payoff of the insurance contract. In our numeric case we simply have $A_0^\alpha = S_0$, so this is not needed. Pricing with Anderson-Lake we get:

$$Put_0^{Lake,\alpha=1} = 0.3841 \tag{8.4}$$

Using the Spanning formula we get the same price:

$$Put_0^{Span,\alpha=1} = 0.3841 \tag{8.5}$$

Where the spanning formula naturally simply holds one put option with strike K and for all other strikes, it holds exactly 0.

9 Heston Delta Hedge

We must examine the possibilities of delta hedging the insurance contract when $\alpha = 0$. We have from question 8 that in this case the insurance contract degenerates to simply being A_0^α / S_0 put options with strike $S_0/A_0^\alpha K$. We therefore must determine the delta of such a portfolio, which is equal to simply A_0^α/S_0 multiplied with the delta of one put option with strike $K^* = S_0/A_0^\alpha K$.

Sadly Anderson & Lake's article contains no information on how to calculate the greeks with their parametrisation. We can however attempt to simply take the derivative of their formula to get the Call Delta:

$$
\begin{aligned}
\Delta_{call} =& \frac{\partial}{\partial S_0} Call_0 = e^{-rT}\left(\frac{\partial}{\partial S_0} R - \frac{\partial}{\partial S_0}\left(\frac{e^{rT} S_0}{\pi} I\right)\right)\\
=& e^{-rT}\left(e^{rT}\mathbf{1}_{\alpha\le 0} - e^{rT}\frac{1}{2}\mathbf{1}_{\alpha=0} - \frac{e^{rT}}{\pi} I - \frac{e^{rT}S_0}{\pi}\frac{\partial}{\partial S_0}\left(e^{\alpha\omega}\int_0^\infty \mathrm{Re}\left\{e^{-x\tan(\varphi)\omega} e^{ix\omega} Q_H(h(x))(1+i\tan(\varphi))\right\} dx\right)\right)\\
=& \mathbf{1}_{\alpha\le 0} - \frac{1}{2}\mathbf{1}_{\alpha=0} - \frac{1}{\pi} I + \frac{S_0}{\pi}\left(\int_0^\infty \frac{\partial}{\partial S_0}\mathrm{Re}\left\{e^{\alpha\omega} e^{-x\tan(\varphi)\omega} e^{ix\omega} Q_H(h(x))(1+i\tan(\varphi))\right\} dx\right)\\
=& \mathbf{1}_{\alpha\le 0} - \frac{1}{2}\mathbf{1}_{\alpha=0} - \frac{1}{\pi} I - \frac{S_0}{\pi}\left(\int_0^\infty \mathrm{Re}\left\{\frac{\partial}{\partial S_0} e^{\alpha\omega} e^{-x\tan(\varphi)\omega} e^{ix\omega} Q_H(h(x))(1+i\tan(\varphi))\right\} dx\right)\\
=& \mathbf{1}_{\alpha\le 0} - \frac{1}{2}\mathbf{1}_{\alpha=0} - \frac{1}{\pi} I - \frac{S_0}{\pi}\left(\int_0^\infty \mathrm{Re}\left\{\frac{\alpha + x(i-\tan(\varphi))}{S_0} e^{\alpha\omega} e^{-x\tan(\varphi)\omega} e^{ix\omega} Q_H(h(x))(1+i\tan(\varphi))\right\} dx\right)\\
=& \mathbf{1}_{\alpha\le 0} - \frac{1}{2}\mathbf{1}_{\alpha=0} - \frac{1}{\pi} I - \frac{1}{\pi}\left(e^{\alpha\omega}\int_0^\infty \mathrm{Re}\left\{(\alpha + x(i-\tan(\varphi))) e^{-x\tan(\varphi)\omega} e^{ix\omega} Q_H(h(x))(1+i\tan(\varphi))\right\} dx\right)\\
=& \mathbf{1}_{\alpha\le 0} - \frac{1}{2}\mathbf{1}_{\alpha=0} - \frac{1}{\pi}\left(I + I_2\right)
\end{aligned} \tag{9.1}
$$

Here we have assumed that our choices of α and φ stay constant if we change S_0 marginally. We note that this likely is not the most efficient method for calculation as it requires evaluating two seperate integrals, but it is satisfactory for us. We can now use the put-call parity to convert this to a put delta:

$$
\Delta_{put} = \frac{\partial}{\partial S_0} Put_0 = \frac{\partial}{\partial S_0}(Ke^{-rT} - S + Call_0) = -1 + \Delta_{call} \tag{9.2}
$$

We implement this in the following code as part of the lakeHestonPut class:

```
    def Int2(self,alpha,N=500,**v):
        ns = np.tile(np.arange(-N,N+1).reshape((-1,1)),len(np.atleast_1d(v['K'])))
        h = np.real(sp.lambertw(2*np.pi*N)/N)
        qs = np.exp(-np.pi*np.sinh(ns*h))
        ys = 2 * qs / (1 +qs )

        ws = ys / (1+qs) * np.pi * np.cosh(ns*h)
        xs = 1 - ys

        forward = np.exp(v['T']*v['r'])*v['S']
        omega = np.log(forward/v['K'])
        phi = np.pi / 12 * np.sign(omega)

        phi = np.where(v['rho'] - v['epsilon'] * omega /( v['V'] + v['kappa'] * v['theta'] * v['T'] )*omega< 0,phi,0)

        u = lambda f,x: (2/np.power(1-x,2))*f((1+x)/(1-x))
        Q = lambda z: self.characteristic(z-1j,**v)/(z*(z-1j))
        hfunc = lambda x: -1j*alpha + x*(1+1j*np.tan(phi))
        inner = lambda x:  np.real(
                            (alpha+x*(1j-np.tan(phi)))*np.exp(-x*np.tan(phi)*omega)*np.exp(1j*x*omega)*Q(hfunc(x))*(1+1j*np.tan(phi))
```

```
                            )
        I2 = h * np.sum(np.nan_to_num(ws * u(inner,xs),0),axis=0)
        return np.exp(alpha*omega) * I2

    def _RdS(self,forward,alpha,**v):
        return np.where(alpha<=0,1,0)-np.where(alpha==0,1/2,0)

    def delta(self,**v):
        with np.errstate(divide='ignore', invalid='ignore'):
            forward = np.exp(v['T']*v['r'])*v['S']
            alpha = calc_alpha2(**v)
            Reffect = self._RdS(forward,alpha,**v)
            IntEffect = - (self.Int(alpha, **v)+self.Int2(alpha, **v))/np.pi
            return  np.real(np.where(v['T']==0,0,-1+Reffect+IntEffect))
```

To test the method we calculate the delta of our option at the start time:

$$\Delta_0 = -0.3585 \tag{9.3}$$

We note that this result matches other implementations.

We can now compute the delta exposure of any Heston put, plus our insurance contract by using the trick from question 8. Using that relationship, we actually see that:

$$\Delta^{\alpha=1}_{Insurance} = \frac{A^\alpha_0}{S_0}\Delta_{Put,K=\frac{S_0}{A^\alpha_0}K} \tag{9.4}$$

As we have $S_0 = A^\alpha_0$, then this degenerates to $\Delta^{\alpha=1,S_0=A^\alpha_0}_{Insurance} = \Delta_{Put}$.

We now need a method for simulating stock movements under the Heston model, for our discrete hedging experiment. Sadly Heston does not have a closed form solution like GBM, as the volatility changes simultaneously with the stock prices. Therefore, we will have to simulate the dynamics. This sadly introduces discritisation erros, and potential for negative stock prices and volatility. To avoid this we use a full truncation rule to replace all negative values with 0. This is shown to be generally the best method in *A comparison of biased simulation schemes for stochastic volatility models* by Roger Lord, Remmert Koekkoek and Dick Van Dijk. We also use a smaller *dt* for simulating that we will use for hedging. We implement this in the below code:

```
def simulateHestonDynamics(T,dt,S0,mu,V0,theta,kappa,epsilon,rho,n=1):
    """
    Uses dynamics for simulation, not accurate for large dt
    """
    m = int(T/dt)
    path = np.zeros([m+1,n,2])
    path[0,:,0] = S0
    path[0,:,1] = V0

    stockBrownian = np.sqrt(dt)*np.random.normal(0,1,[m,n])
    volMovements = epsilon*(rho*stockBrownian +np.sqrt(1-rho**2)*np.sqrt(dt)*np.random.normal
        (0,1,[m,n]))

    driftStock = mu*dt
    kapdt = kappa * dt
    driftVol = theta*kapdt


    for i in range(1,m+1):
        vol = np.sqrt(path[i-1,:,1])
        path[i,:,0] = np.exp(
                        np.log(path[i-1,:,0])
                        +driftStock
                        -0.5*path[i-1,:,1]*dt
                        +vol*stockBrownian[i-1]
                        )
```

```
        path[i,:,1] = np.maximum(
                                path[i-1,:,1]
                                +driftVol-path[i-1,:,1]*kapdt
                                +vol*volMovements[i-1]
                                ,0)
    return path
```

Note that this code truncates the volatility when it is calculated, and not when it is used. This means that we avoid later attempting to hedge negative volatility. We simulate $m = 5,000$ paths with $dt = 1/(252 \cdot 24)$, this is not a small enough dt to accurately simulate the dynamics, but as we have $m \cdot dt^{-1} \cdot T = 907,200,000$ observations of both stock prices and volatilities to handle, then this is already pushing our computational limits. We implement our delta hedging method as in question 2, but now with our new delta function. We also now have varying volatility inputs for our delta, dependent on the time and the path, instead of a fixed σ. We plot the result of a yearly and a daily hedging strategy in Figure 9. We see that our hedges perform significantly worse than in question 2, where we used the same hedging time horisons. We additionally see with our Moving Average that the hedge actually systematically deviates. For low S_T it under-performs and for high S_T it over-performs compared to the payoff. This is possibly caused by the correlation between the stock price and the volatility.

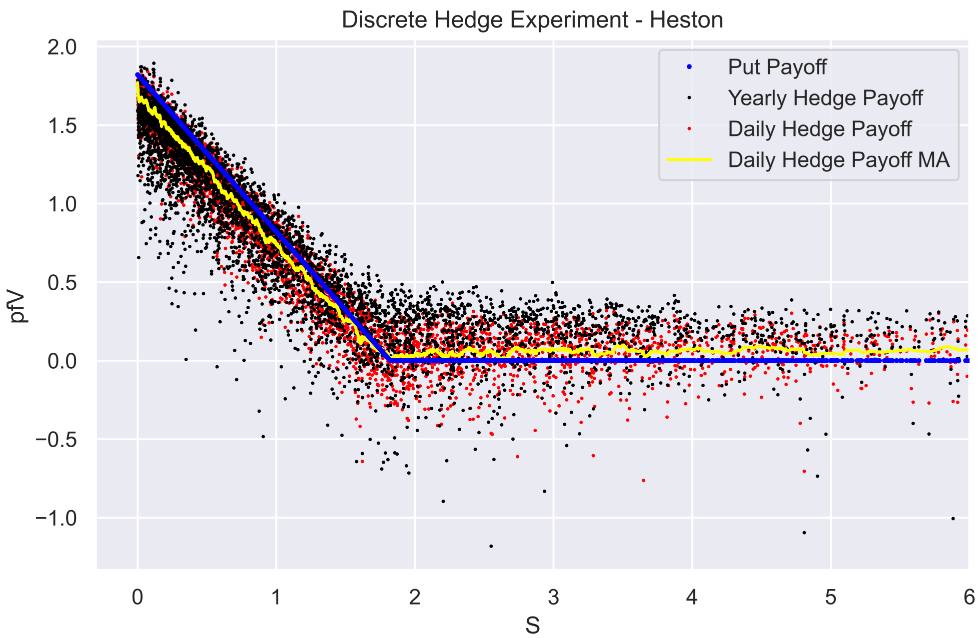

Figure 9

We should however for good measure examine the limit where the amount of hedge rebalances $n \to \infty$, as we did in earlier questions. We do this in Figure 10, but with fewer n for computational reasons. We use our total quadratic variation measure as we saw before that the errors had bias. We note that while the regression has a fine R^2, it has a very low slope coefficient. The error level also appears to stabilise for large n. We have hereby demonstrated by simulation that a self-financing delta hedging strategy does not perfectly hedge the put option.

This mismatch is caused by the unhedged stochastic volatility. If the volatility of an asset increases while the underlying is constant in price then we will lose money on our short put position, while we gain no payments from the stock position. The only way to solve this is by also trading in an asset with exposure to the volatility process, such as another option. This is exactly what we will do in the next question.

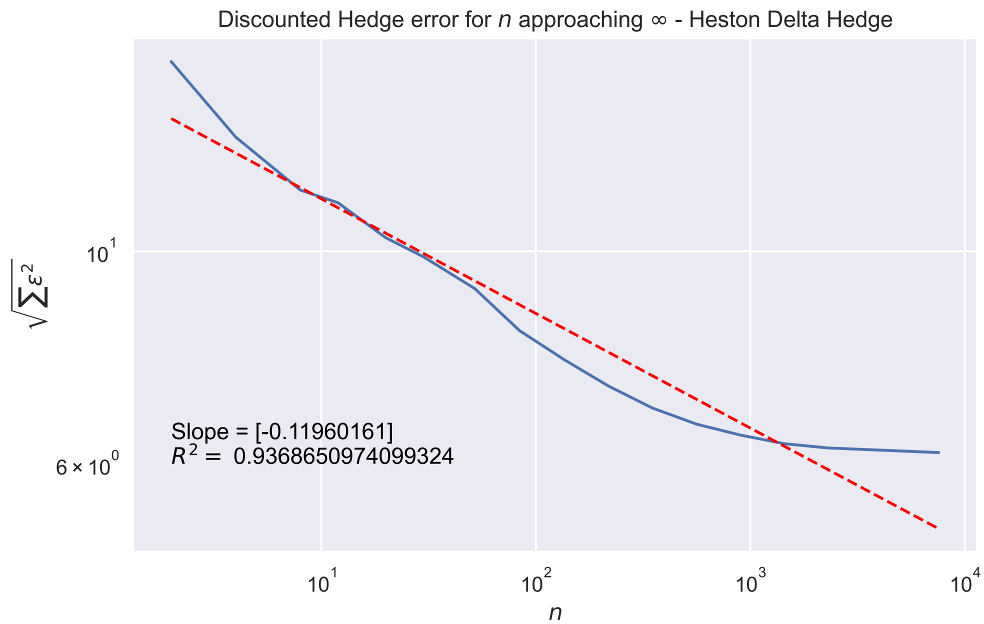

Figure 10

10 Volatility Hedging

We must now consider if we can accurately replicate the option by trading dynamically. We saw in the last question that simply trading in the risk free asset and the stock is not enough, so we will now include another put option, that also has expiry T but strike K_0.

We will investigate this strategy mathematically, following the methodology in Gatheral's book page 5. We start by considering a strategy which holds one short position in the strike K_1 put option that we wish to replicate. Additionally it holds h_s units of the stock and h_0 units of the other put option. This yields the portfolio:

$$V_t^{pf} = -put_t^1 + h_s S_t + h_0 put_t^0 \tag{10.1}$$

This is a self-financing trading strategy if h_s and h_0 are constant, with time 0 price:

$$V_0^{pf} = -put_0^1 + h_s S_0 + h_0 put_0^0 \tag{10.2}$$

We will however expect to rebalance our portfolio continuously as the exposure of our original put changes. To insure that our portfolio stays self-financing we can include a position in the risk-free bankbook B_t, which offsets all the other positions. This also allows for the portfolio to have time 0 value $V_0^{pf} = 0$. If there is absence of arbitrage then the portfolio will have value 0 for every t. This can be thought of as borrowing or investing all cash-flow generated by the net of the other positions. this yields the full portfolio:

$$V_t^{pf} = -put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t \tag{10.3}$$

$$h_b = -\frac{-put_t^1 + h_s S_t + h_0 put_t^0}{B_t} = \frac{put_t^1 - h_s S_t - h_0 put_t^0}{B_t} \tag{10.4}$$

We can now write up the dynamics of our portfolio:

$$
\begin{aligned}
dV_t^{pf} =& - dput_t^1 + h_s dS_t + h_0 dput_t^0 + h_b dB_t \\
=& \frac{\partial}{\partial t}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right) dt \\
&+ \frac{\partial}{\partial S_t}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right) dS_t \\
&+ \frac{\partial}{\partial V_t}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right) dV_t \\
&+ \frac{1}{2}\frac{\partial^2}{(\partial S_t)^2}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right)(dS_t)^2 \\
&+ \frac{1}{2}\frac{\partial^2}{(\partial V_t)^2}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right)(dV_t)^2 \\
&+ \frac{\partial^2}{\partial V_t \partial S_t}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right) dV_t dS_t
\end{aligned}
\tag{10.5}
$$

Where we have used a multi-dimensional variant of Itô's lemma. We have that:

$$
(dS_t)^2 = (S_t(rdt + \sqrt{V_t} dW_{1,t}^Q))^2 = S_t^2 V_t dt \tag{10.6}
$$

$$
(dV_t)^2 = (\kappa(\theta - V_t)dt + \epsilon\sqrt{V_t} dW_{2,t}^Q)^2 = \varepsilon^2 V_t dt \tag{10.7}
$$

$$
dS_t dV_t = S_t(rdt + \sqrt{V_t} dW_{1,t}^Q)(\kappa(\theta - V_t)dt + \epsilon\sqrt{V_t} dW_{2,t}^Q) = S_t V_t \epsilon dt \tag{10.8}
$$

Using some standard rules for stochastic calculus. We can however simplify our expression as we do not care about the drift parts. This yields:

$$
\begin{aligned}
dV_t^{pf} =& (\ldots)\, dt \\
&+ \frac{\partial}{\partial S_t}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right) dS_t \\
&+ \frac{\partial}{\partial V_t}\left(-put_t^1 + h_s S_t + h_0 put_t^0 + h_b B_t\right) dV_t \\
=& (\ldots)\, dt \\
&+ \left(-\frac{\partial}{\partial S_t} put_t^1 + h_s + h_0 \frac{\partial}{\partial S_t} put_t^0 + 0\right) dS_t \\
&+ \left(-\frac{\partial}{\partial V_t} put_t^1 + h_0 \frac{\partial}{\partial V_t} put_t^0 + 0\right) dV_t
\end{aligned}
\tag{10.9}
$$

We see that this portfolio has exposure to three processes. Changes in time dt, changes in the asset price dS_t and changes in the volatility (variance) dV_t. Therefore a portfolio with no risk, such that put_1 is perfectly replicated, must have the following:

$$
\begin{aligned}
-\frac{\partial}{\partial V_t} put_t^1 + h_0 \frac{\partial}{\partial V_t} put_t^0 = 0 \Leftrightarrow \\
h_0 = \frac{\frac{\partial}{\partial V_t} put_t^1}{\frac{\partial}{\partial V_t} put_t^0} = \frac{\nu_1}{\nu_0}
\end{aligned}
\tag{10.10}
$$

$$
\begin{aligned}
-\frac{\partial}{\partial S_t}put_t^1 + h_s + h_0\frac{\partial}{\partial S_t}put_t^0 = 0 \Leftrightarrow \\
h_s = \frac{\partial}{\partial S_t}put_t^1 - h_0\frac{\partial}{\partial S_t}put_t^0 = \Delta_1 - \frac{\nu_1}{\nu_0}\Delta_0
\end{aligned}
\tag{10.11}
$$

We see that we must still hedge the delta exposure of the option, but we must also hedge the delta of our new option. We can however now also hedge away the volatility (vega) exposure of the put by holding another put with size equal to their proportional Vega exposures. We will not actually compute and insert the formulas for computing the Vega $\frac{\partial}{\partial V_t}put_t$, but it can be done as we did for delta in question 9. We also see that our hedge error stemmed from us having $h_0 = 0$. One would be able to update the discrete hedge to hold h_0, h_s and h_b according to the formulas above, yielding a perfectly replicating strategy. As the Anderson-Lake call formula only depends on $\sqrt{V_0}$ through the characteristic function $\phi(x)$ then we have:

$$
\begin{aligned}
\nu_{Call} =& \frac{\partial}{\partial\sqrt{V_0}}Call_0 = e^{-rT}\left(0 + \frac{e^{rT}S_0}{\pi}\frac{\partial}{\partial\sqrt{V_0}}I\right) \\
=& \frac{S_0}{\pi}e^{\alpha\omega}\int_0^\infty \mathrm{Re}\left\{e^{-x\tan(\varphi)\omega}e^{ix\omega}2\sqrt{V_0}B(h(x-i))Q_H(h(x))(1+i\tan(\varphi))\right\}dx \\
=& 2\sqrt{V_0}\frac{S_0}{\pi}e^{\alpha\omega}\int_0^\infty \mathrm{Re}\left\{e^{-x\tan(\varphi)\omega}e^{ix\omega}B(h(x-i))Q_H(h(x))(1+i\tan(\varphi))\right\}dx
\end{aligned}
\tag{10.12}
$$

From the definition of $Call_0, Q_H(x), h(x), \phi(x)$. We then have the put Vega from the Put-Call Parity:

$$
\begin{aligned}
\nu_{put} =& \frac{\partial}{\partial\sqrt{V_0}}put_0 = 0 + \frac{\partial}{\partial\sqrt{V_0}}Call_0 - 0 \\
=& 2\sqrt{V_0}\frac{S_0}{\pi}e^{\alpha\omega}\int_0^\infty \mathrm{Re}\left\{e^{-x\tan(\varphi)\omega}e^{ix\omega}B(h(x-i))Q_H(h(x))(1+i\tan(\varphi))\right\}dx
\end{aligned}
\tag{10.13}
$$

If we where to replace the put put_t^0 with a variance swap contract, then our dynamics would simplify further. This is as the variance swap is a contract with Vega exposure, but no delta exposure. The variance swap is a bet on changes in the variance of the underlying, so it does not matter if it goes up or down, only that it moves. If we only consider the naive theoretical aspects of a volatility swap, then we should have that:

$$\Delta_{VS} = 0 \tag{10.14}$$

$$\nu_{VS} = 1 \tag{10.15}$$

From the definition that a volatility swap is a contract with a payout matching the underlying's volatility. In this case the replicating portfolio would be:

$$h_{VS} = \frac{\partial}{\partial V_t}put_t^1 = \nu_1 \tag{10.16}$$

$$h_s = \frac{\partial}{\partial S_t}put_t^1 = \Delta_1 \tag{10.17}$$

This does however not fully match the real world. As mentioned in *Equity Variance Swap Greeks* by Richard White, the derivatives naturally depend on the model assumptions. $\Delta_{VS} = 0$ holds in the Black-Scholes model, but not in models where the volatility $\sqrt{V_t}$ depends directly on S_T. In the Heston model, there is a correlation between the Brownian motion driving the asset price and the Brownian motion driving the volatility (ρ). This is however not enough as it is not a direct dependence, so we still have $\Delta_{VS} = 0$. An example of a direct dependence is local volatility models. *A Note on Variance Swap Greeks* by Justin Kirkby shows mathematically that $\Delta_{VS} \to 0$ for stochastic volatility models of generic form (including but not limited to Heston).

Equity Variance Swap Greeks also shows that while $\text{Vega}_{VS} = 1$ in the Black-Scholes model, other elements such as dividends and assumptions about the shape of the volatility surface can have an effect on the exact Vega. This intuitively makes sense, as lump-sum dividends cause predictable price changes that count as volatility. It also makes sense that factors like θ and κ can affect Vega as an early large volatility shift is worth more if it is less likely to mean revert quickly. Therefore, shifts in the instantaneous volatility $\sqrt{V_0}$ are unlikely to have a $1:1$ effect on the realised volatility $\sqrt{RV}$.

In our case we have no dividends, but a non-flat volatility surface (skewed volatility smile). Therefore the proper amount of volatility swap contracts are:

$$h_{VS} = \frac{\frac{\partial}{\partial V_t} put_t^1}{\frac{\partial}{\partial V_t} VS} = \frac{\nu_1}{\nu_{VS}} \tag{10.18}$$

Determining Vega is easier said than done, as the Variance Swap has no closed form solution. Instead it is usually priced by splitting the contracts into two parts. A log contract and a position in the underlying that Delta hedges the (direct) exposure to the underlying. As the Delta hedge is Vega neutral, we would then simply have to determine the Vega of the log contract. The log contract is usually spanned, making determining Vega more complicated. It can be however be approximated by simply bumping $\sqrt{V_0}$ and calculating the relatice price change:

$$\lim_{\delta \downarrow 0} \nu_{VS} \approx \frac{VS(\sqrt{V_0} + \delta) - VS(\sqrt{V_0})}{\delta} \tag{10.19}$$

And while it might not be exactly 1, it should be close.