Robust ORCA: Ellipsoidal Uncertainty

Zhaoli Wang Guxin Du Yiran Yang

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Multi-Agent Path Planning (MAPP)

Problem setting:

- We consider N agents moving in a shared 2D workspace.
- Each agent chooses a velocity and then moves in a **straight line** at constant speed.
- After a fixed time step, each agent updates its velocity again.
- The key requirement: **agents must avoid collisions** while moving toward their goals.

Goal:

- Enable all agents to reach their destinations efficiently **without colliding**.

Key Challenges in MAPP

1. **Multi-stage decision making**
   - Agents repeatedly select velocities and update their positions.
   - Decisions across time steps are tightly coupled.
2. **Centralized coupling**
   - A centralized solver must choose velocities for all agents jointly.
   - The number of pairwise collision constraints scales as $O(N^2)$.
3. **Non-convex feasible region**
   - Collision avoidance requires

$$\|\boldsymbol{p}_i + t\,\boldsymbol{v}_i - (\boldsymbol{p}_j + t\,\boldsymbol{v}_j)\|_2 \;\geq\; r_i + r_j, \quad \forall\, t \in [t_{cur}, t_{cur} + \tau].$$

   - The feasible set of $\boldsymbol{v}_i$ is the complement of a disk in velocity space $\Rightarrow$ non-convex.
4. **Uncertainty in real systems**
   - Sensing and actuation noise make future positions uncertain.

ORCA: Advantages and Applications

Key advantages

- **Fully decentralized**: each agent selects its own velocity using only local neighbor information.
- **One-step decision making**: at every time step ORCA chooses a velocity for the next horizon, without planning a full trajectory.
- **Real-time performance**: scales to thousands of agents at interactive frame rates.

Application domains

- Video games and animation (e.g., crowd scenes in *Warhammer 40,000: Space Marine* and other AAA titles).
- Pre-training / guiding RL policies for crowd and robot navigation.
- Large-scale, high-density scenarios

ORCA: Basic Idea

- At each decision step and for each neighbor j, define a velocity obstacle

$$\mathrm{VO}_{i|j}^{\tau} := \{\, v_i - v_j \in \mathbb{R}^2 \mid \text{ leads to collision within horizon } \tau \}.$$

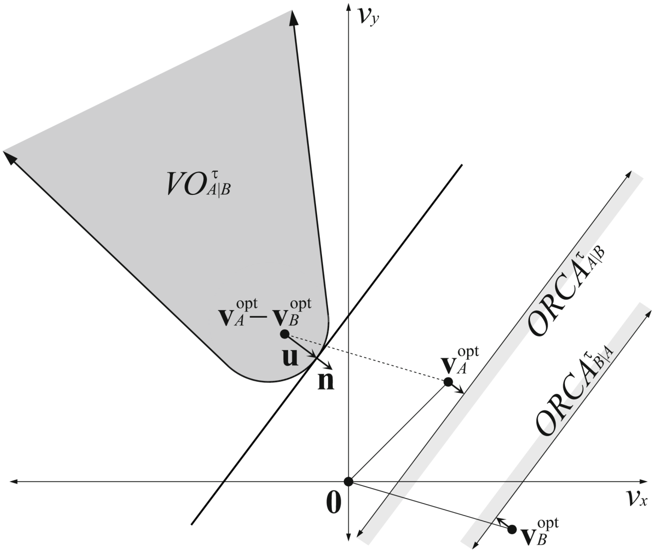
- Relax $\mathrm{VO}_{i|j}^{\tau}$ into a separating half-space, and then, by the reciprocal protocol, assign half of this avoidance responsibility to agent i as $\mathrm{ORCA}_{i|j}^{\tau}$.
- The feasible velocity region for agent i is the intersection

$$\mathcal{V}_i^{\tau} = \bigcap_{j \in \mathcal{N}_i} \mathrm{ORCA}_{i|j}^{\tau}.$$

- Update velocity by projecting the preferred velocity onto this feasible set:

$$v_i^{\text{new}} = \arg\min_{v \in \mathcal{V}_i^{\tau}} \|v - v_i^{\text{pref}}\|_2^2.$$

ORCA: Feasible Region Relaxation



ORCA: Optimization Model

Optimization formulation for agent i:

$$\begin{aligned} \min_{\boldsymbol{v}\in\mathbb{R}^2} \quad & \|\boldsymbol{v} - \boldsymbol{v}_i^{\mathrm{pref}}\|_2^2 \\ \text{s.t.} \quad & \left(\boldsymbol{v} - (\boldsymbol{v}_i + \tfrac{1}{2}\,\boldsymbol{u}_{i|j})\right)^\top \boldsymbol{n}_{i|j} \;\geq\; 0, \qquad \forall j \in \mathcal{N}_i, \\ & \|\boldsymbol{v}\|_2 \leq v_i^{\max}. \end{aligned}$$

Uncertainty model:

- The observed positions and velocities depend on a **joint noise vector**

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_{|\mathcal{N}_i|}).$$

- The ORCA displacement and normal depend on this joint noise:

$$\boldsymbol{u}_{i|j} = \boldsymbol{u}_{i|j}(\boldsymbol{\xi}), \qquad \boldsymbol{n}_{i|j} = \boldsymbol{n}_{i|j}(\boldsymbol{\xi}).$$

Stochastic Model 1: Minimizing the Expected Loss

We model the ORCA update as a stochastic program:

$$z^* = \min_{\boldsymbol{v} \in X} \; \mathbb{E}_{\boldsymbol{\xi}}\left[\, f(\boldsymbol{v}, \boldsymbol{\xi}) \,\right].$$

Random loss function:

$$f(\boldsymbol{v}, \boldsymbol{\xi}) = \begin{cases} \|\boldsymbol{v} - \boldsymbol{v}_i^{\mathrm{pref}}\|_2^2, & \text{if ORCA constraints hold for all neighbors } j, \\ +\infty, & \text{otherwise.} \end{cases}$$

Feasible set:

$$X = \{\, \boldsymbol{v} \in \mathbb{R}^2 : \|\boldsymbol{v}\|_2 \leq v_i^{\max} \,\}.$$

Stochastic Model 2: Robust Optimization

Uncertainty set:

$$\boldsymbol{\xi} \in \Xi.$$

Robust ORCA formulation for agent i:

$$\begin{aligned}
\min_{\boldsymbol{v} \in \mathbb{R}^2} \quad & \|\boldsymbol{v} - \boldsymbol{v}_i^{\text{pref}}\|_2^2 \\
\text{s.t.} \quad & \left(\boldsymbol{v} - (\boldsymbol{v}_i + \tfrac{1}{2}\,\boldsymbol{u}_{i|j}(\boldsymbol{\xi}))\right)^\top \boldsymbol{n}_{i|j}(\boldsymbol{\xi}) \;\geq\; 0, \\
& \qquad\qquad\qquad \forall\, j \in \mathcal{N}_i,\ \forall\, \boldsymbol{\xi} \in \Xi, \\
& \|\boldsymbol{v}\|_2 \leq v_i^{\max}.
\end{aligned}$$

Interpretation:

- The chosen velocity must remain collision-free **for all** noise realizations in the uncertainty set Ξ.
- Guarantees worst-case safety.

Approximation Method: Sample Average Approximation

Given N sampled noise realizations $\{\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(N)}\}$,

SAA formulation:

$$\min_{\boldsymbol{v}\in\mathbb{R}^2} \frac{1}{N}\sum_{k=1}^{N} f\big(\boldsymbol{v}, \boldsymbol{\xi}^{(k)}\big)$$

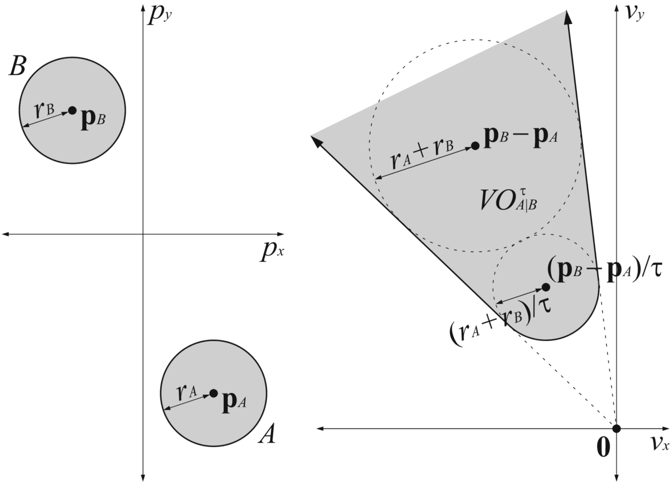
Explicit SAA for ORCA:

$$\begin{aligned}
\min_{\boldsymbol{v}\in\mathbb{R}^2} \quad & \frac{1}{N}\sum_{k=1}^{N} \|\boldsymbol{v} - \boldsymbol{v}_i^{\text{pref}}\|_2^2 \\
\text{s.t.} \quad & \big(\boldsymbol{v} - (\boldsymbol{v}_i + \tfrac{1}{2}\,\boldsymbol{u}_{i|j}(\boldsymbol{\xi}^{(k)}))\big)^{\top}\boldsymbol{n}_{i|j}(\boldsymbol{\xi}^{(k)}) \ \geq \ 0, \\
& \qquad\qquad \forall j \in \mathcal{N}_i,\ \forall k = 1, \dots, N, \\
& \|\boldsymbol{v}\|_2 \leq v_i^{\max}.
\end{aligned}$$

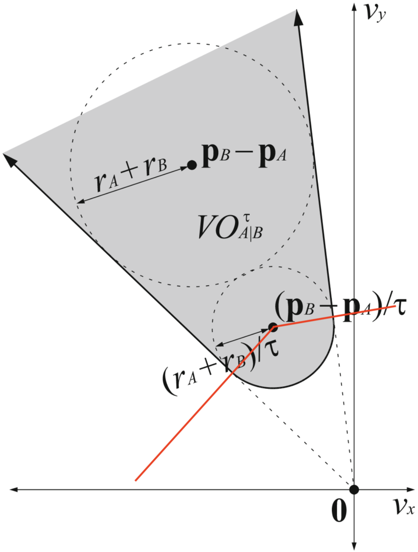
Notes:

- Converts the stochastic problem into a deterministic but larger QP.
- Constraints scale linearly with sample size N.

A Closer Look at the VO



A Closer Look at the VO



Robust Optimization under Ellipsoidal Velocity Noise

Uncertainty model:

$$\tilde{\boldsymbol{v}}_j = \bar{\boldsymbol{v}}_j + \boldsymbol{Q}_j \xi_j, \qquad \|\xi_j\|_2 \le \Gamma_j.$$

When the closest point on the VO boundary lies on the extreme ray $\boldsymbol{r}_1$ (rather than on the lower-right circular cap), the robust ORCA constraint for agent i w.r.t. neighbor j is:

$$\left(\boldsymbol{v} - (\boldsymbol{v}_i + \tfrac{1}{2}\,\boldsymbol{u}_{i|j}(\xi_j))\right)^\top \boldsymbol{n}_{i|j} \;\ge\; 0, \qquad \forall\, \|\xi_j\|_2 \le \Gamma_j.$$

This can be written in deterministic closed form as:

$$\boldsymbol{n}_{i|j}^\top \boldsymbol{v} \;\ge\; \boldsymbol{n}_{i|j}^\top \boldsymbol{v}_i + \tfrac{1}{2}\boldsymbol{n}_{i|j}^\top (\boldsymbol{r}_1 \boldsymbol{r}_1^\top - I)(\boldsymbol{v}_i - \bar{\boldsymbol{v}}_j) + \tfrac{1}{2}\,\Gamma_j \left\|\boldsymbol{Q}_j^\top (I - \boldsymbol{r}_1 \boldsymbol{r}_1^\top)\, \boldsymbol{n}_{i|j}\right\|_2.$$

Preliminaries: Affine Form of $\boldsymbol{u}_{i|j}(\xi_j)$

Velocity noise model.

$$\tilde{\boldsymbol{v}}_j = \bar{\boldsymbol{v}}_j + \boldsymbol{Q}_j \xi_j, \qquad \|\xi_j\|_2 \le \Gamma_j.$$

$$\tilde{\boldsymbol{v}}_{i|j} = \boldsymbol{v}_i - \tilde{\boldsymbol{v}}_j = (\boldsymbol{v}_i - \bar{\boldsymbol{v}}_j) - \boldsymbol{Q}_j \xi_j = \bar{\boldsymbol{v}}_{i|j} - \boldsymbol{Q}_j \xi_j.$$

Projection onto the extreme ray $\boldsymbol{r}_1$. Assume the closest point on the VO boundary lies on the ray $\boldsymbol{r}_1$ with $\|\boldsymbol{r}_1\|_2 = 1$:

$$\mathrm{Proj}_{\mathrm{span}\{\boldsymbol{r}_1\}}(\tilde{\boldsymbol{v}}_{i|j}) = (\tilde{\boldsymbol{v}}_{i|j}^\top \boldsymbol{r}_1)\, \boldsymbol{r}_1.$$

ORCA displacement definition.

$$\begin{aligned}
\boldsymbol{u}_{i|j}(\xi_j) &= \mathrm{Proj}_{\mathrm{span}\{\boldsymbol{r}_1\}}(\tilde{\boldsymbol{v}}_{i|j}) \; - \; \tilde{\boldsymbol{v}}_{i|j} \\
&= \left(\bar{\boldsymbol{v}}_{i|j}^\top \boldsymbol{r}_1\right)\boldsymbol{r}_1 \; - \; \bar{\boldsymbol{v}}_{i|j} \; + \; (I - \boldsymbol{r}_1 \boldsymbol{r}_1^\top)\boldsymbol{Q}_j \xi_j \\
&= (\boldsymbol{r}_1 \boldsymbol{r}_1^\top - I)(\boldsymbol{v}_i - \bar{\boldsymbol{v}}_j) \; + \; (I - \boldsymbol{r}_1 \boldsymbol{r}_1^\top)\boldsymbol{Q}_j \xi_j.
\end{aligned}$$

Affine form:

$$\boldsymbol{u}_{i|j}(\xi_j) = \underbrace{(\boldsymbol{r}_1 \boldsymbol{r}_1^\top - I)(\boldsymbol{v}_i - \bar{\boldsymbol{v}}_j)}_{\bar{\boldsymbol{u}}_j} + \underbrace{(I - \boldsymbol{r}_1 \boldsymbol{r}_1^\top)\boldsymbol{Q}_j}_{\boldsymbol{U}_j} \xi_j.$$

Ben-Tal–Nemirovski Ellipsoidal Form

the robust ORCA constraint for agent i w.r.t. neighbor j is

$$\left(\boldsymbol{v} - (\boldsymbol{v}_i + \tfrac{1}{2}\,\boldsymbol{u}_{i|j}(\xi_j))\right)^\top \boldsymbol{n}_{i|j} \;\geq\; 0, \qquad \forall\, \|\xi_j\|_2 \leq \Gamma_j.$$

Define the augmented decision vector

$$\boldsymbol{x} := \begin{bmatrix} \boldsymbol{v} \\ 1 \end{bmatrix}.$$

The uncertain coefficient vector is

$$\tilde{\boldsymbol{a}}_j(\xi_j) = \begin{bmatrix} -\,\boldsymbol{n}_{i|j} \\ \boldsymbol{n}_{i|j}^\top \boldsymbol{v}_i + \tfrac{1}{2}\,\boldsymbol{n}_{i|j}^\top \bar{\boldsymbol{u}}_j \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\tfrac{1}{2}\,\boldsymbol{U}_j^\top \boldsymbol{n}_{i|j} \end{bmatrix} \xi_j.$$

Thus the robust constraint becomes

$$\tilde{\boldsymbol{a}}_j(\xi_j)^\top \boldsymbol{x} \;\leq\; 0, \qquad \forall \|\xi_j\|_2 \leq \Gamma_j.$$

This matches the standard Ben-Tal–Nemirovski ellipsoidal form:

$$\tilde{\boldsymbol{a}}_j(\xi_j) = \bar{\boldsymbol{a}}_j + \boldsymbol{B}_j \xi_j, \qquad \|\xi_j\|_2 \leq \Gamma_j.$$

the Other Case

When the closest point on the VO boundary does **not** lie on either extreme rays, i.e., on the circular arc, the same robust-optimization framework still applies.

Key idea: geometric relaxation.

- Select a **supporting hyperplane** of the VO boundary together with its associated half-space that excludes the VO.
- Instead of projecting onto $\mathrm{span}\{\boldsymbol{r}_1\}$ (extreme-ray case), project the noisy relative velocity onto this supporting hyperplane.
- This again yields a **Ben-Tal–Nemirovski ellipsoidal form** and the same type of deterministic robust constraint.

How to find the supporting hyperplane.

- Use the **nominal relative velocity** $\bar{v}_{i|j}$ to identify the closest point on the VO boundary.
- The tangent at that boundary point defines the supporting hyperplane and thus the ORCA half-space.

Effect of Position Uncertainty on the VO

Position uncertainty model.

$$\tilde{\boldsymbol{p}}_j = \bar{\boldsymbol{p}}_j + E_j\,\zeta_j, \qquad \|\zeta_j\|_2 \le \Gamma_j.$$

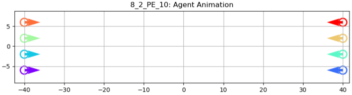
Impact on the VO geometry.

- Each noise realization ζ_j perturbs the relative position and thus produces a **different VO**.
- The uncertain VO is the **union** of these perturbed cones.
- Because position noise only shifts or slightly rotates the VO apex and boundary rays, the union remains a **truncated cone**.
- Therefore, given the noise between the observation position and velocity are independent, the robust ORCA still works!

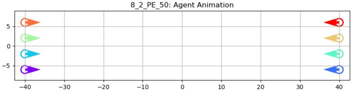
Position Uncertainty and Complexity Preservation

- Each neighbor j contributes **one linear half-space**, regardless of which VO face is active or how uncertainty enters.
- The robust counterpart remains the **same convex QP** as in deterministic ORCA.
- Per-agent computation stays $O(|\mathcal{N}_i|)$, enabling real-time performance.

More samples can lead to worse performance for (PE)

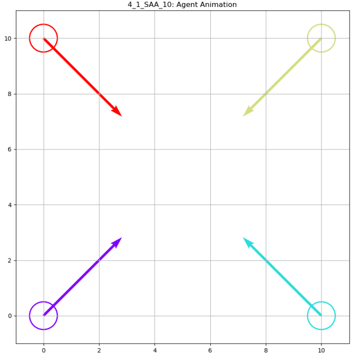


10 samples

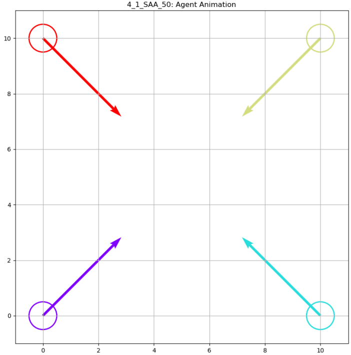


50 samples

More samples can lead to worse performance for (SAA)

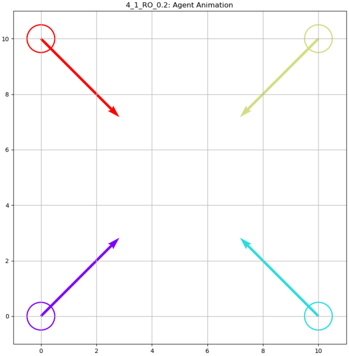


10 samples

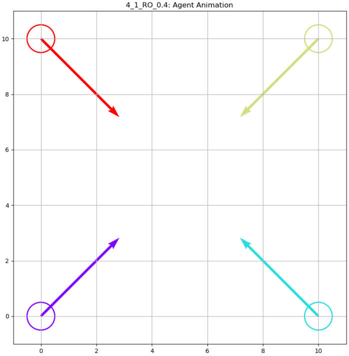


50 samples

Larger budget leads to more conservative behaviour (RO)

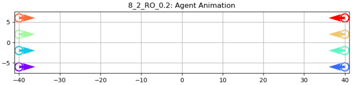


$\Gamma = 0.2$

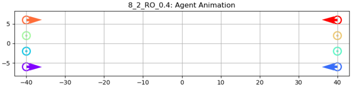


$\Gamma = 0.4$

Larger budget leads to more conservative behaviour (RO)



$\Gamma = 0.2$



$\Gamma = 0.4$

Results Summary

Table: Steps needed to achieve the targets, number of collisions and number of relaxations for different methods on various instances. The best results for each instance and measure are highlighted in bold.

Instance	Measures	Methods						
		PE_1	PE_10	PE_50	SAA_10	SAA_50	RO_0.2	RO_0.4
4_1	Steps	**13**	14	**13**	17	17	14	14
	Collisions	**0**	3	**0**	3	3	**0**	**0**
	Relaxations	**1**	4	**1**	15	16	4	3
4_2	Steps	**42**	43	**42**	**42**	44	**42**	**42**
	Collisions	6	14	10	4	2	**0**	**0**
	Relaxations	**0**	**0**	**0**	**0**	1	**0**	**0**
8_1	Steps	**45**	50	**45**	49	48	7475	50
	Collisions	21	14	17	5	17	**0**	2
	Relaxations	32	26	33	29	50	**10**	19
8_2	Steps	≥10000	66	70	≥10000	54	**47**	69
	Collisions	2	2	2	**0**	**0**	**0**	**0**
	Relaxations	4	5	**3**	8	14	12	12
15_1	Steps	**50**	51	55	52	52	56	≥10000
	Collisions	44	30	53	53	34	24	**0**
	Relaxations	78	56	75	114	108	68	**0**

Conclusion

- We proposed a robust ORCA framework that accounts for both position and velocity uncertainty.
- The robust ORCA constraints can be expressed in closed-form deterministic equivalents, preserving the computational efficiency of ORCA.
- Experimental results demonstrate that the robust ORCA effectively reduces collisions under uncertainty while maintaining reasonable path efficiency.

Thank you!