Vision-Aided Navigation (086761)
Homework #1

January 15, 2024

Submission in pairs. Please email your submission (a single file in the format ID1-ID2.zip) to andreyz@campus.technion.ac.il by 28 January 2024, 23:59.

Basic probability

1. Consider a random vector x with a Gaussian distribution:

$$x \sim \mathcal{N}(\mu_x, \Sigma_x).$$

   (a) Write an explicit expression for $p(x)$.

   (b) Consider a linear transformation $y = Ax + b$. Assuming A is invertible, show y has a Gaussian distribution, $y \sim \mathcal{N}(\mu_y, \Sigma_y)$, and find expressions of μ_y and Σ_y in terms of μ_x and Σ_x. Note the above is true for singular A as well. Hint: one way to show that y is Gaussian is using the random vector transformation formula.

   (c) (10pts bonus) Show the above also holds for singular A. That is, $y \sim \mathcal{N}(\mu_y, \Sigma_y)$ with mean and covariance as in the previous clause even when $\det(A) = 0$.

2. Let $p(x) = \mathcal{N}(\hat{x}_0, \Sigma_0)$ be a prior distribution over $x \in \mathbb{R}^n$ with known mean $\hat{x}_0 \in \mathbb{R}^n$ and covariance $\Sigma_0 \in \mathbb{R}^{n \times n}$. Consider a given measurement $z \in \mathbb{R}^m$ with a corresponding linear measurement model $z = Hx + v$, where H is a measurement matrix and v is Gaussian noise $v \sim \mathcal{N}(0, R)$ with covariance R. The matrices $H \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{m \times m}$ are known.

   (a) Write an expression for the a posteriori probability function (pdf) over x, $p(x|z)$, in terms of solely the prior $p(x)$ and measurement likelihood $p(z|x)$.

   (b) Derive analytically an expression for the maximum a posteriori (MAP) estimate $x^\star$ and the associated covariance Σ or information matrix $I = \Sigma^{-1}$ such that $p(x|z) = \mathcal{N}(x^\star, \Sigma)$. Useful relation: $\|a\|_\Sigma^2 = \|\Sigma^{-1/2} a\|^2$ where $\Sigma^{-1} = \Sigma^{-T/2}\Sigma^{-1/2}$.

Hands-on Exercises - Please print and submit your code.

1. Rotations. Implement transformation from rotation matrix to Euler angles and vice versa

   (a) Implement a function that receives as input Euler angles (roll angle ϕ, pitch angle θ, and yaw angle ψ) and calculates the corresponding rotation matrix assuming roll pitch-yaw order from Body to Global: $R = R_Z(\psi) R_Y(\theta) R_X(\phi)$.

   (b) What is the rotation matrix from Body to Global for $\psi = \pi/7$, $\theta = \pi/5$, and $\phi = \pi/4$?

   (c) Implement a function that receives as input a rotation matrix and calculates the corresponding Euler angles assuming roll-pitch-yaw order.

   (d) What are the Euler angles in degrees for the following rotation matrix (Body to Global, assuming roll-pitch-yaw order):

$$R_B^G = \begin{pmatrix} 0.813797681 & -0.440969611 & 0.378522306 \\ 0.46984631 & 0.882564119 & 0.0180283112 \\ -0.342020143 & 0.163175911 & 0.925416578 \end{pmatrix}$$

2. 3D rigid transformation. The coordinates of a 3D point in a global frame are

$$l^G = (450, 400, 50)^T$$

This 3D point is observed by a camera whose pose is described by the following rotation and translation with respect to the global frame:

$$R_G^C = \begin{pmatrix} 0.5363 & -0.8440 & 0 \\ 0.8440 & 0.5363 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t_{C\to G}^G = (-451.2459, 257.0322, 400)^T$$

Calculate the 3D point coordinates in a camera frame (l^C =?). Write an explicit expression for the appropriate 3D transformation (4×4 matrix) in terms of R_G^C and $t_{C\to G}^G$

3. Pose composition. An autonomous ground vehicle (robot) is commanded to move forward by 1 meter each time step. Due to imperfect control system, the robot instead moves forward by 1.01 meter and also rotates by 1 degree. Remark: In this exercise we consider a 2D scenario, where pose is defined in terms of x-y coordinates and an orientation (heading) angle.

   (a) Write expressions for the corresponding commanded and actual transformations - note these are relative to the robot frame. Guidance: calculate the rotation R_k^{k+1} and translation $t_{k+1\to k}^k$ relating robot frames at consecutive times k, $k+1$ (commanded and actual). Use these to express commanded and actual transformations T_k^{k+1}.

   (b) Assuming robot starts moving from the origin, calculate evolution of robot pose (in terms of x-y position and orientation angle) for 10 steps using pose composition. Draw the commanded and actual robot pose for 10 steps. What is the dead reckoning error at the end? (write down x_{error}, y_{error} in meters, heading error in rad).