Homework 02

Vision-Aided Navigation

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1.

To show the correspondence between the Gaussian distribution in its standard form and its information form, we start with the Gaussian distribution given by

$$x \sim \mathcal{N}(\mu, \Sigma), \tag{1.1}$$

which has the probability density function (PDF)

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right) \tag{1.2}$$

and we aim to express it in terms of the information form

$$x \sim \mathcal{N}^{-1}(\eta, \Lambda). \tag{1.3}$$

where $\Lambda = \Sigma^{-1}$ is the information, or precision matrix (inverse of the covariance matrix) and $\eta = \Lambda\mu$ is the information vector. The PDF in the information form can be written as

$$\mathcal{N}^{-1}(x; \eta, \Lambda) = \frac{e^{-\frac{1}{2}\eta^\top \Lambda^{-1}\eta}}{\sqrt{\det(2\pi\Lambda^{-1})}} e^{-\frac{1}{2}x^\top \Lambda x + \eta^\top x}. \tag{1.4}$$

To show this, we start by expressing the exponent of the Gaussian PDF in terms of η and Λ:

$$\begin{aligned} -\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu) &= -\frac{1}{2}x^\top \Lambda x + x^\top \Lambda\mu - \frac{1}{2}\mu^\top \Lambda\mu \\ &\overset{\eta=\Lambda\mu}{=} -\frac{1}{2}x^\top \Lambda x + \eta^\top x - \frac{1}{2}\eta^\top \Lambda^{-1}\eta \end{aligned}$$

The normalization factor also needs to be expressed in terms of Λ. Since $\Lambda = \Sigma^{-1}$, we have:

$$\sqrt{(2\pi)^n \det(\Sigma)} = \sqrt{\det(2\pi\Lambda^{-1})}.$$

Putting it all together, the PDF in the information form is:

$$\mathcal{N}^{-1}(x; \eta, \Lambda) = \frac{e^{-\frac{1}{2}\eta^\top \Lambda^{-1}\eta}}{\sqrt{\det(2\pi\Lambda^{-1})}} e^{-\frac{1}{2}x^\top \Lambda x + \eta^\top x}$$

2.

Given the standard observation model involving a random variable $x \in \mathbb{R}^n$:

$$z = h(x) + v \tag{2.1}$$

where $v \sim \mathcal{N}(0, \Sigma_v)$, and assuming the initial belief regarding the state x is Gaussian with mean $\hat{x}_0$ and covariance Σ_0.

(a)

We can write the expressions for the prior $p(x)$ and the measurement likelihood $p(z|x)$ as follows:

The prior distribution of x, $p(x)$, is given by the initial belief about the state, which is a Gaussian distribution

$$
\begin{aligned}
p(x) &= \mathrm{N}(x; \hat{x}_0, \Sigma_0) \\
&= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma_0)}} \exp\left(-\frac{1}{2}(x - \hat{x}_0)^\top \Sigma_0^{-1} (x - \hat{x}_0)\right)
\end{aligned}
$$

The measurement likelihood $p(z|x)$ describes the probability of observing z given the state x, taking into account the noise v. Since v is Gaussian distributed, and $z = h(x) + v$, the likelihood is also Gaussian, centered around $h(x)$ with covariance Σ_v:

$$
\begin{aligned}
p(z|x) &= \mathrm{N}(z; h(x), \Sigma_v) \\
&= \frac{1}{\sqrt{(2\pi)^m \det(\Sigma_v)}} \exp\left(-\frac{1}{2}(z - h(x))^\top \Sigma_v^{-1} (z - h(x))\right)
\end{aligned}
$$

Here, m is the dimension of z, and $h(x)$ represents the nonlinear transformation applied to x before adding the measurement noise v. This formulation allows for incorporating both the prior knowledge about the state x and the information gained from observing z in a Bayesian framework.

(b)

Given a measurement z_1 acquired through the measurement model $z = h(x) + v$, where v is the measurement noise, and assuming $v \sim \mathrm{N}(0, \Sigma_v)$, the posterior probability $p(x|z_1)$ can be derived using Bayes' theorem. Bayes' theorem allows us to update our prior belief $p(x)$ about the state x in light of the new evidence provided by the measurement z_1. The posterior probability $p(x|z_1)$ is given by:

$$
\begin{aligned}
p(x|z_1) &= \frac{p(z_1|x) \cdot p(x)}{p(z_1)} \\
&= \frac{p(z_1|x) \cdot p(x)}{\int_{-\infty}^{\infty} p(z_1|\tilde{x}) p(\tilde{x}) d\tilde{x}}
\end{aligned} \tag{2.2}
$$

Here, $p(z_1|x)$ is the measurement likelihood, which is the probability of observing z_1 given the state x, and $p(x)$ is the prior probability of the state x before observing z_1. The denominator $p(z_1)$ serves as a normalization factor, ensuring that the posterior probabilities sum to 1.

(c)

By incorporating the explicit expressions for $p(x)$ and $p(z_1|x)$ as defined within Gaussian distribution frameworks, we arrive at:

- The prior $p(x) = \mathrm{N}(x; \hat{x}_0, \Sigma_0)$
- The likelihood $p(z_1|x) = \mathrm{N}(z_1; h(x), \Sigma_v)$

Under the assumptions of minor errors and a linear correlation, it can be deduced that:

$$
\begin{aligned}
x^\star &= \widehat{x} + \widetilde{x} \\
h\left(x^\star\right) &= h(\widehat{x}_0) + H\widetilde{x}
\end{aligned}
$$

where $H = \left.\dfrac{dh}{dx}\right|_{x=\widehat{x}_0}$. When applied to $p(x|z)$:

$$
p(x|z) = \exp\left[-\frac{1}{2}\left(\|x - \widehat{x}_0\|^2_{\Sigma_0} + \left\|z - h(\widehat{x}_0) - H\widetilde{x}\right\|^2_{\Sigma_v}\right)\right]. \tag{2.3}
$$

To minimize and derive the optimal solution, we utilize the relation $\|a\|^2_\Sigma = \left\|\Sigma^{-1/2}a\right\|^2$

$$
\|x - \widehat{x}_0\|^2_{\Sigma_0} + \left\|z - h(\widehat{x}_0) - H\widetilde{x}\right\|^2_{\Sigma_v} = \left\|\Sigma_0^{-1/2}x\right\|^2 + \left\|\Sigma_v^{-1/2}\left(z - h\left(\widehat{x}_0 - H\widetilde{x}\right)\right)\right\|^2 \tag{2.4}
$$

This leads to a linear least squares problem, with a well-established solution:

$$
\widetilde{x} = \left(A^\top A\right)^{-1} A^\top B
$$

Here,

$$
A = \begin{bmatrix} \Sigma_0^{-1/2} \\ \Sigma_v^{-1/2} H \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ \Sigma_v^{-1/2} h(\widehat{x}_0 - z) \end{bmatrix}
$$

Yielding:

$$
\widetilde{x} = \left(\Sigma_0^{-1} + H^\top \Sigma_v^{-1} H\right)^{-1} H^\top \Sigma_v^{-1} h(\widehat{x}_0 - z) \tag{2.5}
$$

Therefore, we have:

$$
\begin{aligned}
\widehat{x}_1 &= \widehat{x}_0 + \left(\Sigma_0^{-1} + H^\top \Sigma_v^{-1} H\right)^{-1} H^\top \Sigma_v^{-1} h(\widehat{x}_0 - z) \\
\Sigma_1 &= \left(\Sigma_0^{-1} + H^\top \Sigma_v^{-1} H\right)^{-1}
\end{aligned}
$$

(d)

Under markov assumption

$$
p(x|z_1, z_2) = \frac{p(z_2|x)p(x|z_1)}{p(z_2)}
$$

Hence

$$
\begin{aligned}
\widehat{x}_2 &= \widehat{x}_1 + \left(\Sigma_1^{-1} + H^\top \Sigma_v^{-1} H\right)^{-1} H^\top \Sigma_v^{-1} (h(\widehat{x}_1 - z)) \\
\Sigma_2 &= \left(\Sigma_1^{-1} + H^\top \Sigma_v^{-1} H\right)^{-1}
\end{aligned}
$$

3.

(a)

$$p(x_k|x_{k-1}, u_{k-1}) \;=\; \frac{1}{\sqrt{|2\pi\Sigma_w|}} exp\left(-\frac{1}{2}||x_k - f(x_{k-1}, u_{k-1})||^2_{\Sigma_w}\right)$$

(b)

using Bayes rule and Markov:

$$\underbrace{p(x_1|z_1, u_0)}_{post} \;=\; \frac{p(z_1|x_1, u_0)p(x_1, u_0)}{p(z_1, u_0)} \propto \eta p(z_1|x_1, u_0)p(x_1, u_0) = \eta p(z_1|x_1)p(x_1, u_0)$$

$$= \eta p(z_1|x_1)\int p(x_1, x_0|u_0)\, dx_0 \;= \eta p(z_1|x_1)\int p(x_1|x_0, u_0)\, p(x_0, u_0)\, dx_0$$

$$= \eta p(z_1|x_1)\int \underbrace{p(x_1|x_0, u_0)}_{prior}\, p(x_0)\, dx_0$$

also used motion and observation models.

(c)

also using the Bayes Theorem:

$$p(x_0, x_1|z_1, u_0) \;=\; \frac{p(z_1|x_1, x_0, u_0)p(x_1, x_0, u_0)}{p(z_1, u_0)} \propto \eta p(z_1|x_1, x_0, u_0)p(x_1, x_0, u_0)$$

$$= \eta p(z_1|x_1)p(x_1, x_0|u_0)$$

$$= \eta p(z_1|x_1)p(x_1|x_0, u_0)p(x_0|u_0)$$

$$= \eta p(z_1|x_1)p(x_1|x_0, u_0)p(x_0)$$

That is,

$$p(x_0, x_1|z_1, u_0) \;= \eta\frac{1}{\sqrt{|2\pi\Sigma_v|}} exp\left(-\frac{1}{2}||z_1 - h(x_1)||^2_{\Sigma_v}\right)$$

$$\cdot\, \eta\frac{1}{\sqrt{|2\pi\Sigma_w|}} exp\left(-\frac{1}{2}||x_1 - f(x_0, u_0)||^2_{\Sigma_w}\right)$$

$$\cdot\, \eta\frac{1}{\sqrt{|2\pi\Sigma_0|}} exp\left(-\frac{1}{2}||x_0 - \widehat{x}_0||^2_{\Sigma_w}\right)$$

MAP for x is:

$$x^* \;= arg_x max\{\eta\frac{1}{\sqrt{|2\pi\Sigma_v|}} exp\left(-\frac{1}{2}||z_1 - h(x_1)||^2_{\Sigma_v}\right)$$

$$\cdot \eta \frac{1}{\sqrt{|2\pi\Sigma_w|}} exp\left(-\frac{1}{2}||x_1 - f(x_0, u_0)||^2_{\Sigma_w}\right)$$

$$\cdot \eta \frac{1}{\sqrt{|2\pi\Sigma_0|}} exp\left(-\frac{1}{2}||x_0 - \hat{x}_0||^2_{\Sigma_w}\right)\}$$

log func of this term, won't change the solution.

$$x^* = arg_x max\{log\left[\eta \frac{1}{\sqrt{|2\pi\Sigma_v|}} \cdot \eta \frac{1}{\sqrt{|2\pi\Sigma_w|}} \cdot \eta \frac{1}{\sqrt{|2\pi\Sigma_0|}}\right]$$

$$\cdot \left(-\frac{1}{2}||z_1 - h(x_1)||^2_{\Sigma_v}\right) \cdot \left(-\frac{1}{2}||x_1 - f(x_0, u_0)||^2_{\Sigma_w}\right) \cdot \left(-\frac{1}{2}||x_0 - \hat{x}_0||^2_{\Sigma_w}\right)\}$$

extracting the $-\frac{1}{2}$ ⇨ will change the argument to min:

$$x^* = arg_x min\{log\left[\eta \frac{1}{\sqrt{|2\pi\Sigma_v|}} \cdot \eta \frac{1}{\sqrt{|2\pi\Sigma_w|}} \cdot \eta \frac{1}{\sqrt{|2\pi\Sigma_0|}}\right]$$

$$\cdot \left(||z_1 - h(x_1)||^2_{\Sigma_v}\right) \cdot \left(||x_1 - f(x_0, u_0)||^2_{\Sigma_w}\right) \cdot \left(||x_0 - \hat{x}_0||^2_{\Sigma_w}\right)\}$$

taking out all the determinants in the log function since they are independent of x

$$x^* = arg_x min\left\{\left(||z_1 - h(x_1)||^2_{\Sigma_v}\right) \cdot \left(||x_1 - f(x_0, u_0)||^2_{\Sigma_w}\right) \cdot \left(||x_0 - \hat{x}_0||^2_{\Sigma_w}\right)\right\}$$

now using this relation $||a||^2_{\Sigma} = ||\Sigma^{-\frac{1}{2}}a||$:

$$x^* = arg_x min\left\{\left(||\Sigma_v^{-\frac{1}{2}}(z_1 - h(x_1))||\right) \cdot \left(||\Sigma_w^{-\frac{1}{2}}(x_1 - f(x_0, u_0))||\right) \cdot \left(||\Sigma_w^{-\frac{1}{2}}(x_0 - \hat{x}_0)||\right)\right\}$$

This problem, which involves minimizing the squared differences where the function $h(x)$ is non-linear, can be effectively tackled using optimization techniques introduced in our classes, such as the Gauss-Newton or Levenberg-Marquardt algorithms.

(d)

seems to be like:

The marginal covariance $\Sigma_1 \ for \ x_1$ can be extracted directly from the appropriate block of the joint covariance matrix since the off-diagonal blocks do not influence it after marginalization.

$$\Sigma_1 = \Sigma_{11}$$

The marginal information matrix $I_1 \ for \ x_1$ requires subtracting the influence of x_0 from the total information about x_1 which is done by the right term.

$$I_1 = I_{11} - I_{01}^T I_{00}^{-1} I_{10}$$

Hands-on Exercises

4.

*this Q in matlab code.

(a)

Camera projection matrix is:

$$P = \begin{bmatrix} f_X & s & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [R \mid t]$$

$[R \mid t]$ is the camera pose, s is the skew.

will write down 3D trasformation matrix:

$$T_G^C = \begin{bmatrix} 0.5363 & -0.8440 & 0 & -451.2459 \\ 0.8440 & 0.5363 & 0 & 257.0322 \\ 0 & 0 & 1 & 400 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

given (u_0, v_0) and also $f_X = f_y = 400$.

assume $s = 0$.

$$P = \begin{bmatrix} 480 & 0 & 320 \\ 0 & 480 & 270 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5363 & -0.8440 & 0 & -451.2459 \\ 0.8440 & 0.5363 & 0 & 257.0322 \\ 0 & 0 & 1 & 400 \end{bmatrix}$$

$$P = \begin{bmatrix} 257.424 & -405.12 & 320 & -8.8598e4 \\ 405.12 & 257.424 & 270 & 2.318e5 \\ 0 & 0 & 1 & 400 \end{bmatrix}$$

(b)

need to calculate:

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = P \cdot X^G$$

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} 9.158e4 \\ 2.994e5 \\ 365 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \cdot \frac{1}{\tilde{w}} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} 250.905 \\ 820.168 \end{pmatrix}$$

(c)

Re-projection error will calculated by:

$$Err = z - \pi(x, l) = \begin{pmatrix} u \\ v \end{pmatrix}_{pix} - \begin{pmatrix} 250.905 \\ 820.168 \end{pmatrix} = \begin{pmatrix} -9.45 \\ -651.17 \end{pmatrix}$$

the total different in pixels is: 651.236

*seems quite big

5.

*this Q in matlab code.

(a)

done.

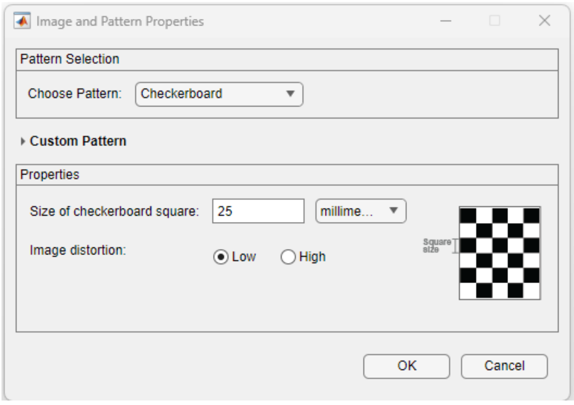
(b)

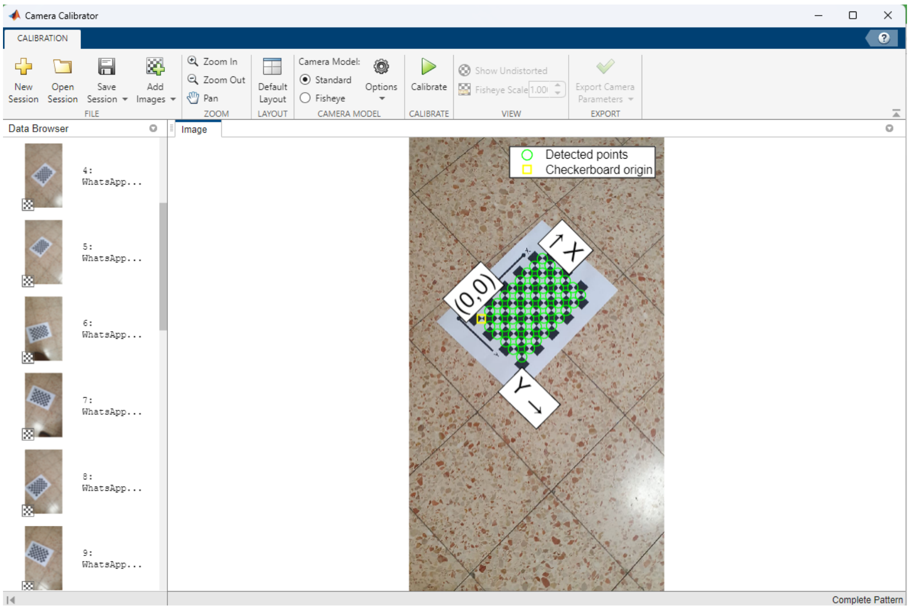
In total, 21 images of a checkerboard pattern

using Samsung Galaxy s10+ camera.

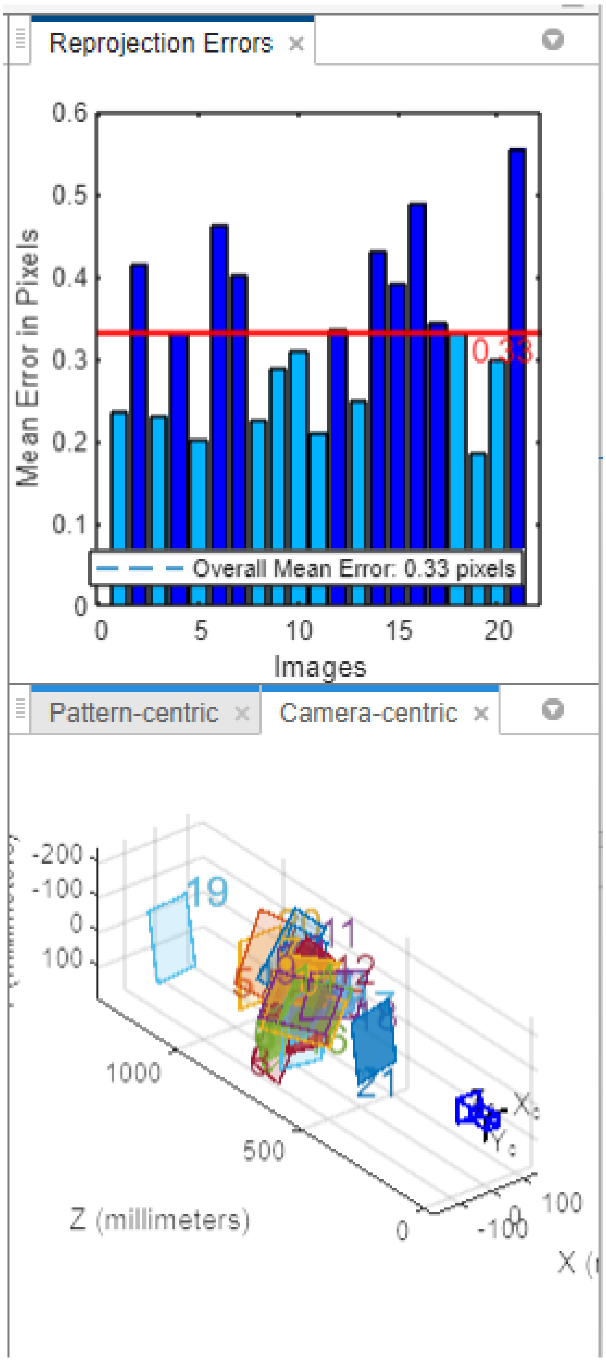
(c)

First uploading all the 21 images into Matlab Calibration GUI, verified that each rib is $25mm$.:

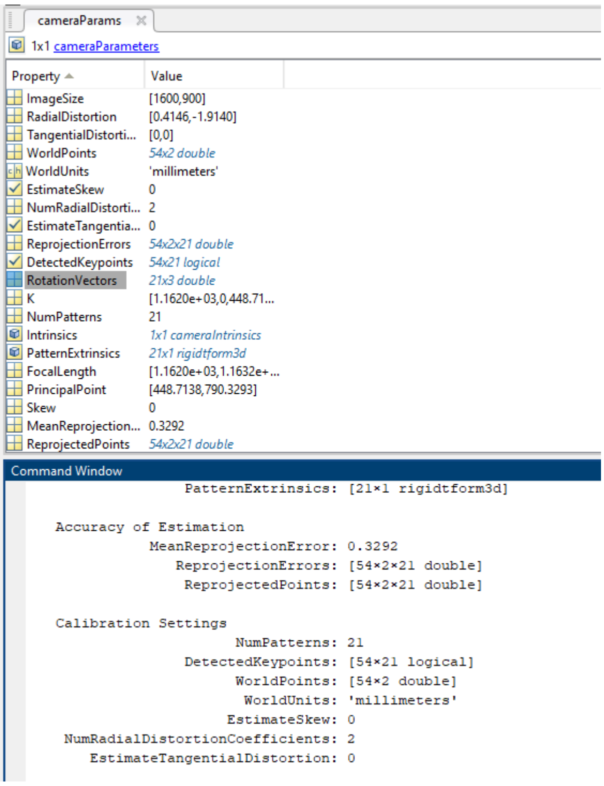




Pressing Calibration button and derived all data:



the data derived:



(d)

As we saw before, the definition for K matrix:

$$K = \begin{bmatrix} f_X & s & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.162e3 & 0 & 448.7138 \\ 0 & 1.163e3 & 790.3293 \\ 0 & 0 & 1 \end{bmatrix}$$

The following is the principal point, in each axis:

$$\begin{bmatrix} 448.7138 \\ 790.3293 \end{bmatrix}$$

The following is the focal length, in each axis:

$$\begin{bmatrix} 1.62e3 \\ 1.163e3 \end{bmatrix}$$

6.

*this Q in Python code.

(a)

done

(b)

done

(c)

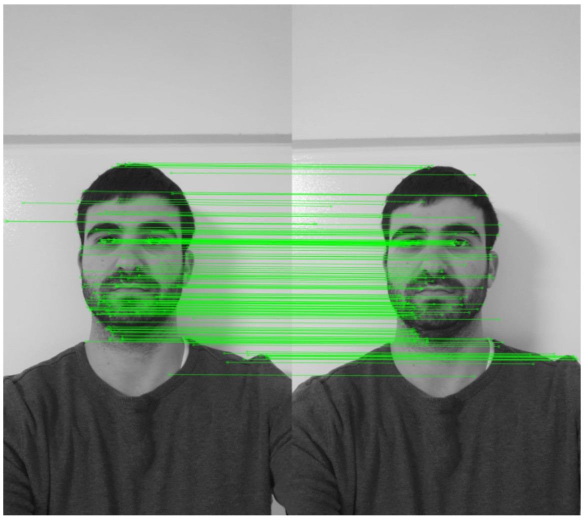
SIFT features were extracted in both images

$$Scale = 1.94134, \quad Orientation = 3.718\ rad$$

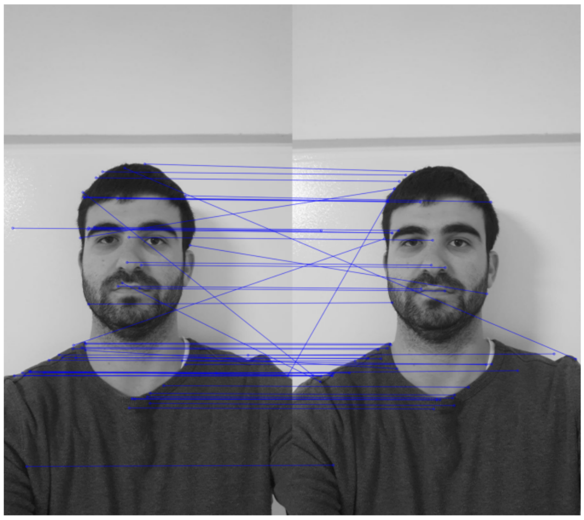
*for the first keypoint that measured.

(d)

Inliers (Good Matches)



Outliers (Bad Matches)



* with Threshold of 5.