Numerical Simulations of 3-level Λ system

Guy Levinberg

January 2023

1 Theory

1.1 Rotating Wave Hamiltonian

In order to numerically simulate a 3-level Λ system, and exhibit CPT phenomena the following model was processed:
For an unperturbated 3-level system, the Hamiltonian takes the following term:

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hbar\omega_2 & 0 \\ 0 & 0 & \hbar\omega_3 \end{pmatrix} \tag{1}$$

Where our basis is $\{|1\rangle, |2\rangle, |3\rangle\}$ with $|1\rangle$ having the lowest energy.
Note that here the first ground state has zero energy so that ω_2 and ω_3 are the frequencies of the transition energies from the first ground state.
The system is illuminated by a bi-chromatic radiation field that is (in a semi-classical model) described as:

$$E = \mathcal{E}_1 e^{i(k_1\cdot r-\omega_{L1}t)} + \mathcal{E}_1^* e^{-i(k_1\cdot r-\omega_{L1}t)} + \mathcal{E}_2 e^{i(k_2\cdot r-\omega_{L2}t)} + \mathcal{E}_2^* e^{-i(k_2\cdot r-\omega_{L2}t)} \tag{2}$$

Where the laser has components with frequencies $\omega_{L1} \sim \omega_3$ and $\omega_{L2} \sim (\omega_3 - \omega_2)$
This electric field interacts with the atomic dipole, given by an operator $\hat{d} = -e\hat{r}$. which may be represented in terms of the atomic eigenstates and the dipole matrix elements:

$$\hat{d} = \sum_{n,n'} |n'\rangle \langle n| \mu_{n',n} = |1\rangle \langle 3| \mu_{1,3} + |2\rangle \langle 3| \mu_{2,3} + h.c \tag{3}$$

Where $\mu_{n',n} = \langle n'| \hat{d} |n\rangle = \mu_{n,n'}^*$ is the dipole matrix element of the $|n'\rangle \to |n\rangle$ transition, and with the assumption that there is no dipole transition from $|1\rangle$ to $|2\rangle$. Our interaction Hamiltonian is then:

$$\begin{aligned} H_I = &-(\mathcal{E}_1 e^{-i\omega_{L1}t} + \mathcal{E}_1^* e^{i\omega_{L1}t})(|1\rangle \langle 3| \mu_{1,3} + h.c) + \\ &-(\mathcal{E}_2 e^{-i\omega_{L2}t} + \mathcal{E}_2^* e^{i\omega_{L2}t})(|2\rangle \langle 3| \mu_{2,3} + h.c) \end{aligned} \tag{4}$$

With the total Hamiltonian being

$$H = H_0 + H_I \tag{5}$$

A general state can be written as: $|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$ Moving to a rotated frame we can write:

$$|\psi\rangle = c_1|1\rangle + \tilde{c_2}e^{i(\omega_{L1}-\omega_{L2})t}|2\rangle + \tilde{c_3}e^{i\omega_{L1}t}|3\rangle \tag{6}$$

We will now use Schrodinger equation:

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle \tag{7}$$

With multiplying both sides with $\langle g1|, \langle g2|, \langle e|$ from left we can get the EOMs for the amplitudes $c_1, \tilde{c_2}\tilde{c_3}$ and using the Hamiltonian from (5).

$$\langle 1|\, i\hbar\frac{d}{dt}|\psi\rangle = i\hbar\dot{\tilde{c_1}} = (\mathcal{E}_1 e^{-i\omega_{L1}t} + \mathcal{E}_1^* e^{i\omega_{L1}t})e^{i\omega_{L1}t}\tilde{c_3}\mu_{1,3} \tag{8}$$

cancelling the counter-rotating term $e^{2i\omega_{L1}t}$ we get:

$$i\hbar\dot{\tilde{c_1}} = \mathcal{E}_1^*\mu_{1,3}\tilde{c_3} \equiv \Omega_1\hbar\tilde{c_3} \tag{9}$$

Similarly, we can get

$$i\hbar\dot{\tilde{c_2}} = \hbar\Omega_2\tilde{c_3} + \hbar\Delta_1\tilde{c_2} \tag{10}$$

$$i\hbar\dot{\tilde{c_3}} = \hbar\Omega_2^*\tilde{c_2} + \hbar\Omega_1^*\tilde{c_1} + \hbar\Delta_2\tilde{c_3} \tag{11}$$

Where $\Delta_1 \equiv \omega_2 + \omega_{L1} - \omega_{L2}$ and $\Delta_2 \equiv \omega_3 + \omega_{L1}$, and $\Omega_{1/2} = \frac{\mu_{1/2,3}\mathcal{E}_{1/2}^*}{\hbar}$ are the Rabi frequencies.
The EOMs (9) (10) (11) could be derived from a rotated frame Hamiltonian:

$$H_{RF} = \hbar\begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & \Delta_1 & \Omega_2 \\ \Omega_1^* & \Omega_2^* & \Delta_2 \end{pmatrix} \tag{12}$$

1.2 Master Equation

In order to take into account the openness of the system, i.e the loss of energy to the environment through stimulated and spontaneous emissions, as well as decoherence, a density matrix formalism is used. The density matrix is defined as:

$$\hat{\rho} = \sum_n P_n|\psi_n\rangle\langle\psi_n| \tag{13}$$

Where $\{|\psi_n\rangle\}$ is a complete set of the Hilbert space and P_n are the probabilities of a single particle to be in a $|\psi_n\rangle$.
This definitions enables a distinction between a statistical mixture of states and a superposition of states.
The density matrix evolves with time through the Von-Neumann equation:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[H, \hat{\rho}] \tag{14}$$

Where the Hamiltonian consists of the Hamiltonian of the system in question, the environment and the interactions between the two.

$$H_{tot} = H_{sys} + H_{env} + H_{int} \tag{15}$$

Since we're only interested in the system, we can perform a partial trace over the environmental degrees of freedom and obtain a master equation for the evolution of the system which doesn't conserve energy as the environment can lead to dissipation (and therefore, the equation is not equivalent to Schrodinger equation). The most general form of the master equation which preservers the trace of the density matrix is the Lindblad-master equation:

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_n \frac{1}{2}[2C_n\rho(t)C_n^\dagger - \rho(t)C_n^\dagger C_n - C_n^\dagger C_n\rho(t)] \tag{16}$$

Where C_n are called "jump operators" and describe some decay rate from some level to another level. The assumptions upon which the Lindblad-master equation is derived are:
-The environment is large and unchanging (the Born approximation)
-The environment has 'no memory', i.e very small correlation time (Markov approximation)
Applying the Lindblad-master approximation to n-level atoms yields a set of differential equations for the density matrix components also known as Optical Bloch Equations.
In our case for the λ system, the jump operators are:

$$C_1 = \sqrt{\gamma_{31}}\,|1\rangle\langle 3| \tag{17}$$

$$C_2 = \sqrt{\gamma_{32}}\,|2\rangle\langle 3| \tag{18}$$

These operators describe a decay process from the excited level to the two ground-states with rates γ_{31}, γ_{32}. There's no assumed transitions between the two ground-states.

2 Numerical simulations with Qutip - quantum toolbox in python

2.1 Steady state solutions

Assuming that the detuning of Laser1 is exactly the detuning of Laser2 with a minus sign, (which is to say that the Carrier laser frequency is exactly between the resonances) is equal to $\frac{\Delta}{2}$ (Δ is the two-photon detuning). Scanning Δ from some negative initial value to a final positive value and for each one calculating the populations $\rho_{11}, \rho_{22}, \rho_{33}$ of the system under steady state conditions: $\dot{\rho_{11}} = \dot{\rho_{22}} = \dot{\rho_{33}} = 0$ and with preserving the trace $\rho_{11} + \rho_{22} + \rho_{33} = 1$.
Using the Optical Bloch equations with the Hamiltonian from (12) and the two

jump operators (17) (18) . With the initial state being a statistical mixture of 50-50 between the two ground states.

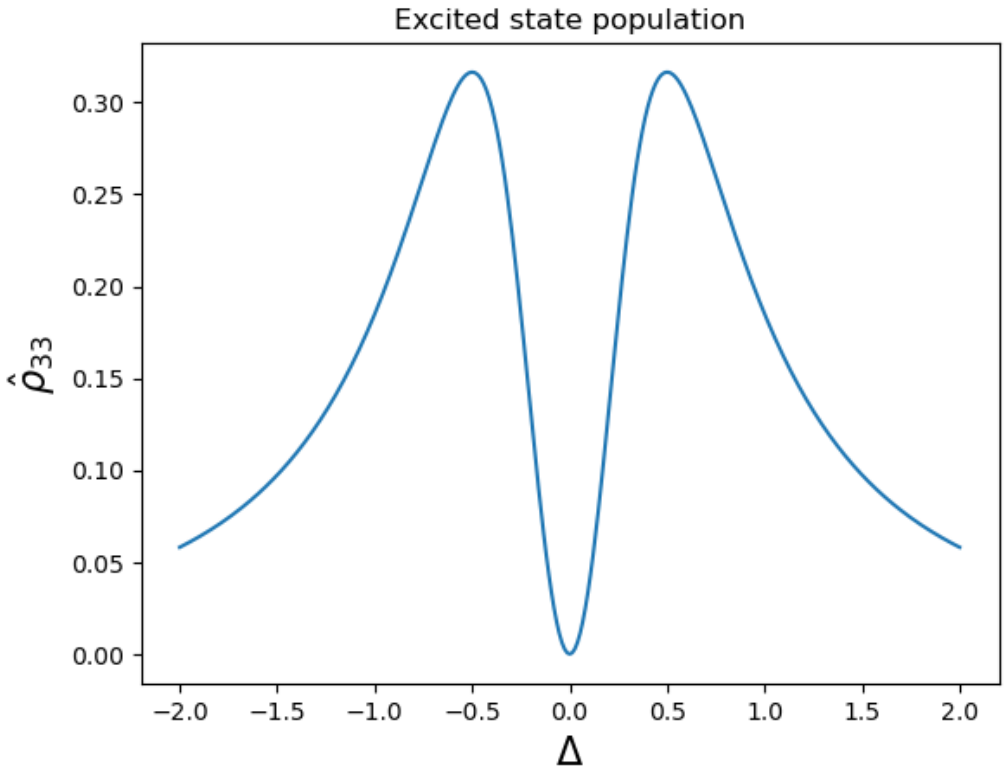

Figure 1: Population of the excited state vs 2-photon detuning. Close to resonance, the population is greatly depleted.

The susceptibility satisfies:

$$\chi \propto \rho_{31} \tag{19}$$

So the refractive index n and the absorption are proportional to the real part and imaginary part of χ respectively.

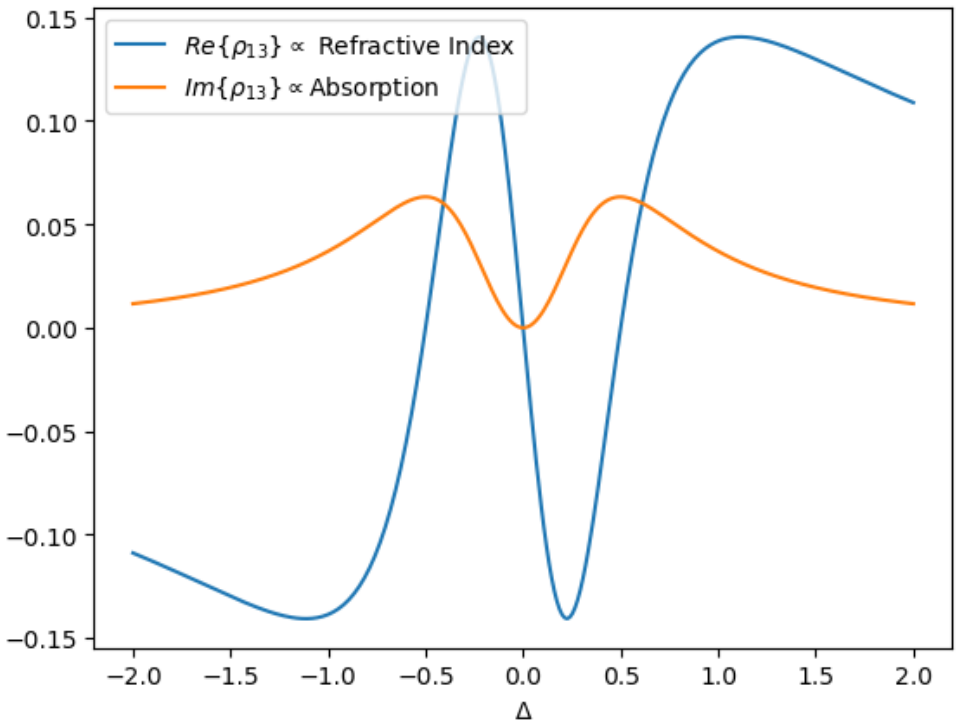

Figure 2: The real and imaginary part of ρ_{31}. The absorption is suppressed at resonance. The refractive index is greatly changed around the resonance

2.2 Transient solutions

The optical Bloch Equations could also be solved dynamically with a slow varying Hamiltonian. The Hamiltonian from (12) is now written as:

$$H_{RF} = \hbar \begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1^* & \Omega_2^* & 0 \end{pmatrix} + \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta(t) & 0 \\ 0 & 0 & \frac{\Delta(t)}{2} \end{pmatrix} \tag{20}$$

The first term being time-independent and second is the slowly changing detuning parameter that behaves as:

$$\Delta(t) = \Delta(0) + S \cdot t \tag{21}$$

Where $\Delta(0)$ is some (negative) initial detuning value and S is a sweep rate. This simulates a ramp sweep. Here are the results for various sweep rates:

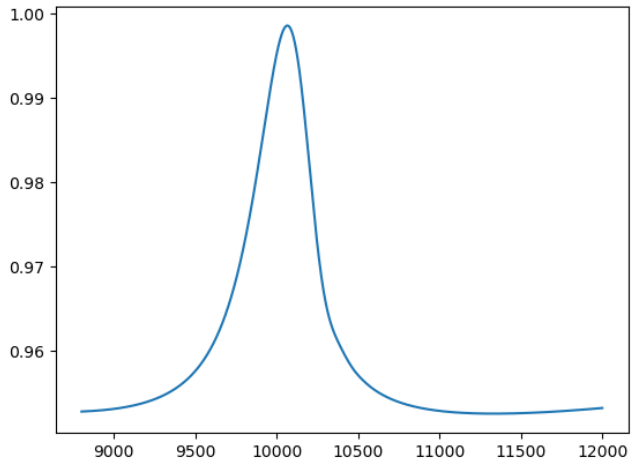

Figure 3: Transmission vs time for sweep rate $S = 1 \cdot 10^{-3}\Gamma$

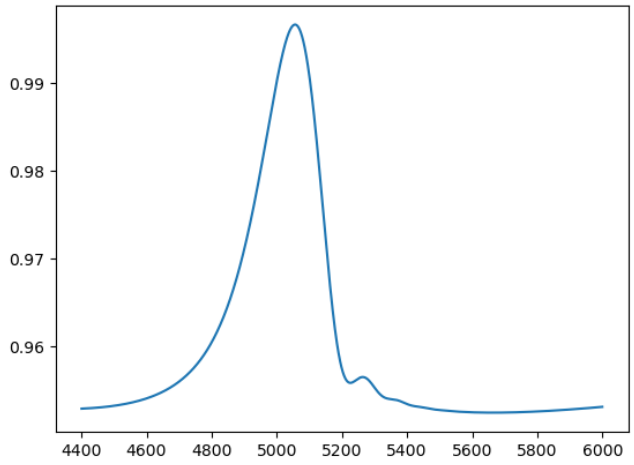

Figure 4: Transmission vs time for sweep rate $S = 2 \cdot 10^{-3}\Gamma$

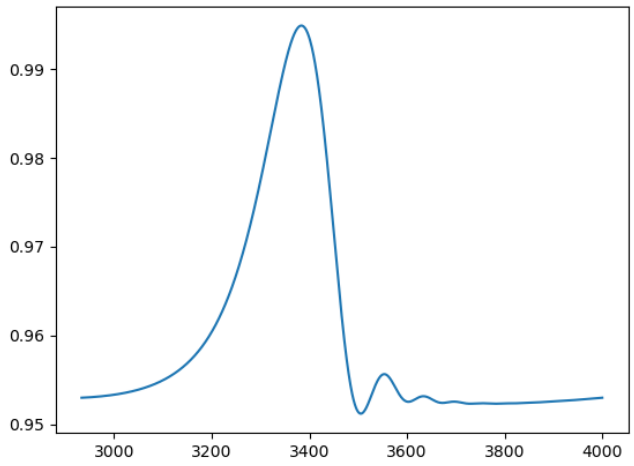

Figure 5: Transmission vs time for sweep rate $S = 3 \cdot 10^{-3}\Gamma$

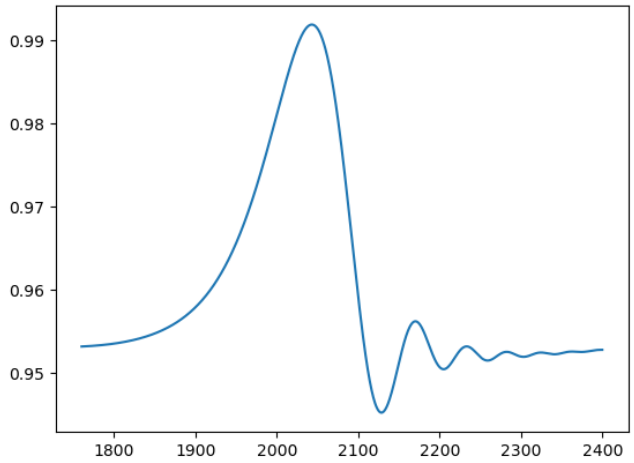

Figure 6: Transmission vs time for sweep rate $S = 1 \cdot 10^{-5}\Gamma$

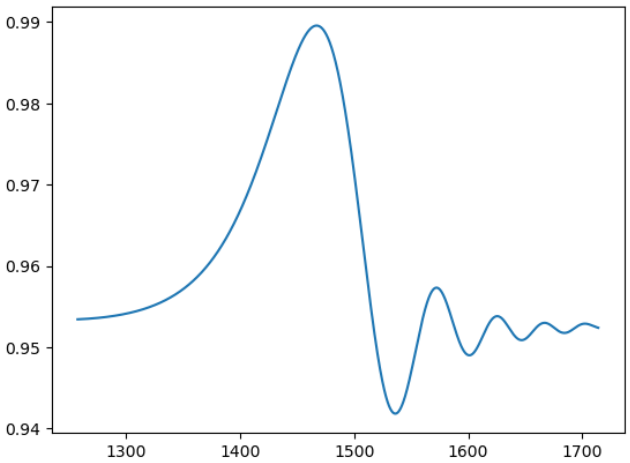

Figure 7: Transmission vs time for sweep rate $S = 7 \cdot 10^{-3}\Gamma$

Here are some CPT lineshape measurement for an 8mm cylindrical cell with buffer gas pressure of 20torr , laser power of $100\mu W$, RF power of 7dBm , magnetic field of $\sim 200mG$ and temperature of $84°C$.

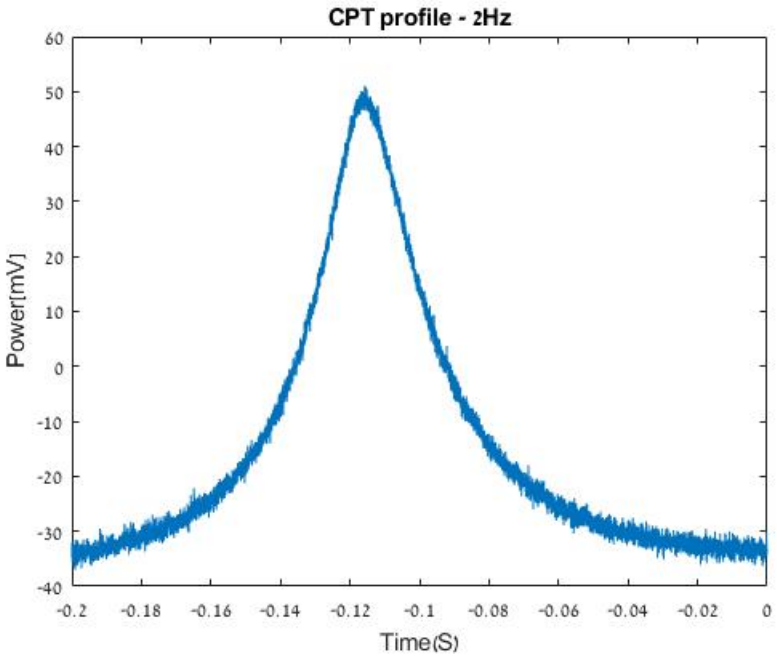

Figure 8

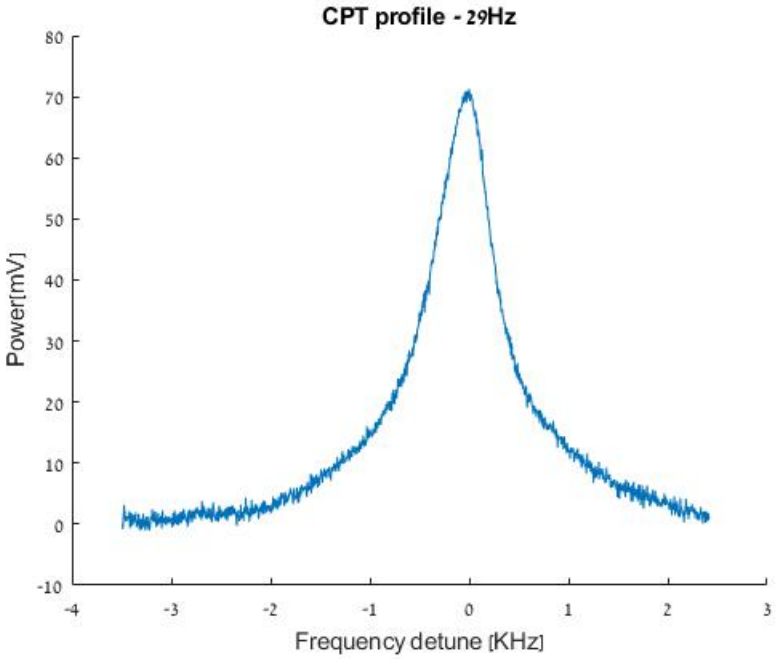
CPT profile - 29Hz
Power[mV]
Frequency detune [KHz]
80
70
60
50
40
30
20
10
0
-10
-4
-3
-2
-1
0
1
2
3

Figure 9

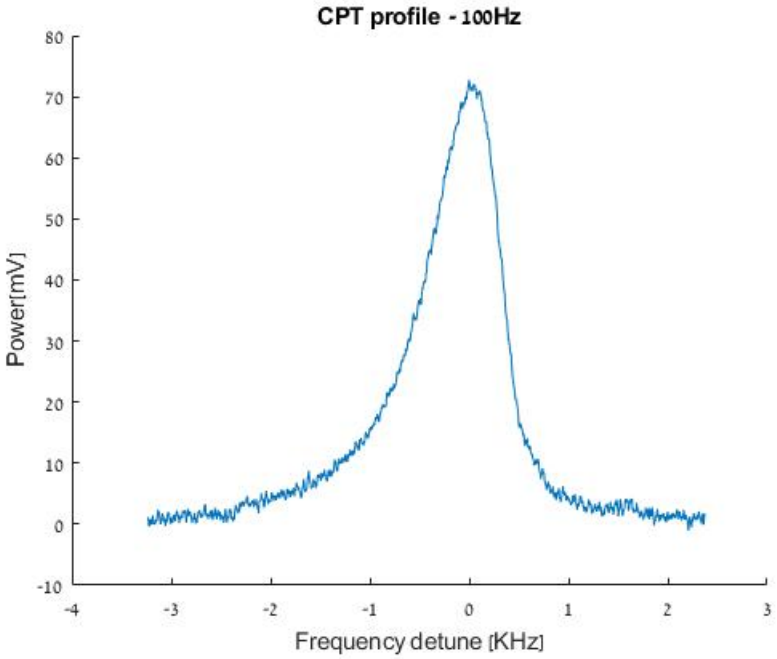
CPT profile - 100Hz
Power[mV]
Frequency detune [KHz]
80
70
60
50
40
30
20
10
0
-10
-4
-3
-2
-1
0
1
2
3

Figure 10

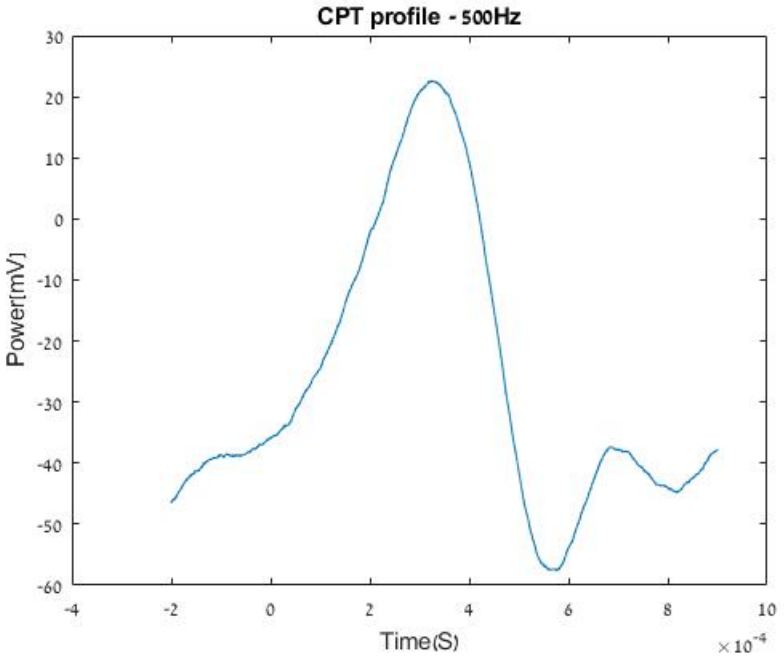

Figure 11

This indicates that the asymmetry and ringing in the CPT lineshape is closely related to the time-dependant nature of the sweep and is a feature of even the simplest model of the atoms, one that doesn't take into account additional Zeeman sub levels and other broadening effects.