Numerical Simulations of 13-level Λ system

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1 Theory

1.1 Zeeman Splitting

Expanding our discussion of 3-level systems, we introduce Zeeman splitting and selection rules.
With the laser fields having a σ^+ polarization, due to selection rules, only transitions with $\Delta m_F = +1$ are possible, as depicted in figure1

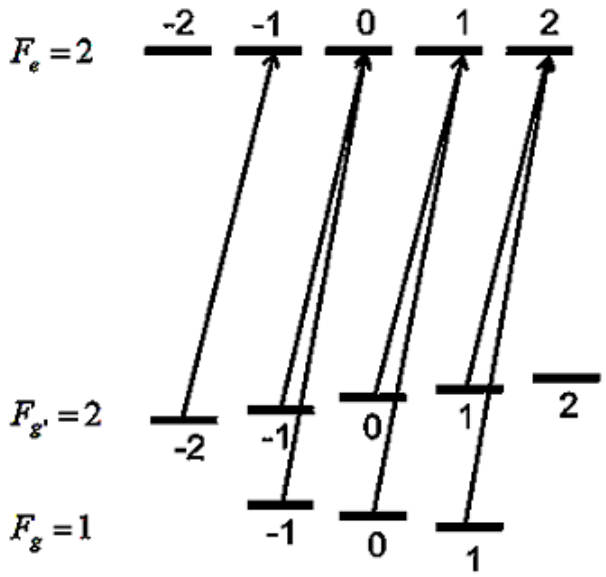

Figure 1: The Zeeman sub-levels manifold for Rb87 D1 transition

The energy Splitting in the weak field regime is:

$$\Delta E = \mu_B g_F m_F B_z \tag{1}$$

Where μ_B is the Bohr magneton, g_F is the Lande-g factor m_f is the projection of the angular momentum in the $\hat{z}$ direction and B_z is the magnetic field in the $\hat{z}$ direction.
Plugging the appropriate constants, the energy shift for each quanta of angular momentum m_F is $-0.7 \times B\frac{[MHz]}{[G]}$ for the bottom ground state, $+0.7 \times B\frac{[MHz]}{[G]}$ for the second ground state and $+0.23 \times B\frac{[MHz]}{[G]}$ for the excited state. We can now write a 13×13 atom-Hamiltonian, with the diagonal terms being:

$$H_0 = \hbar \cdot \begin{pmatrix} 0 \\ A \\ 2A \\ \Delta E_{hf} - A \\ \Delta E_{hf} \\ \Delta E_{hf} + A \\ \Delta E_{hf} + 2A \\ \Delta E_{hf} + 3A \\ \omega_1 + \frac{A}{3} \\ \omega_1 + \frac{2A}{3} \\ \omega_1 + A \\ \omega_1 + \frac{4A}{3} \\ \omega_1 + \frac{5A}{3} \end{pmatrix} \tag{2}$$

Where $A = (0.7 \times 10^6 B[G])$, is the Zeeman splitting energy.
$\Delta E_{hf} = 2\pi \times 6.834 \cdot 10^9$ is the hyperfine splitting frequency and $\omega_1 = 2\pi \times 377.107568 \cdot 10^{12}$ is the optical transition frequency for the D1 transition: $|F = 1\rangle \rightarrow |F' = 2\rangle$.
Since σ^+ polarization is applied, the interaction terms will contain only transitions with $\Delta m_F = +1$ the off-diagonal terms, in the Rotating Wave approximation are:

$$\begin{aligned} d_1 H_{1,13} = d_2 H_{2,12} = d_3 H_{3,11} = \hbar\Omega_1 e^{\imath\omega_{L1} t} \\ d_4 H_{4,10} = d_5 2H_{5,11} = d_6 H_{6,12} = d_7 H_{7,13} = \hbar\Omega_2 e^{\imath\omega_{L2} t} \end{aligned} \tag{3}$$

Where $\Omega_{1/2}$ are the Rabi frequencies of the transitions and $\omega_{L_{1/2}}$ are the laser frequencies and the factors d_i are additional factor of the reduced matrix element and are given by [1]:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ -2 \\ -\sqrt{12} \\ \sqrt{6} \\ 2 \\ 2 \\ \sqrt{6} \end{pmatrix} \tag{4}$$

The conjugate terms obey: $H_{i,j} = H^*_{j,i}$

1.2 Interaction Picture

In order to eliminate the time dependencies in the Hamiltonian, we can use a unitary transformation U and move to a co-rotating frame. The Hamiltonian in the new frame will be:

$$H_{RF} = UHU^{+} + i\hbar\frac{dU}{dt}U^{+} \tag{5}$$

And the unitary transformation will take the form:

$$U = \begin{pmatrix} e^{i\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{i\lambda_2 t} & \dots & \\ \vdots & 0 & \ddots & \\ 0 & \vdots & 0 & e^{i\lambda_{13} t} \end{pmatrix} \tag{6}$$

Plugging equations (4) and (5) into the terms in equation (3)

$$H^{RF}_{i,j} = \hbar\Omega_{1/2} e^{i(\lambda_i + \omega_{L_{1/2}} - \lambda_j)t} \tag{7}$$

Where the Rabi frequncy $\Omega_{1/2}$ and the laser frequency $\omega_{L_{1/2}}$ are with accordance to i, j in equation (3). Taking the expression in exponents to be zero, we get a set of equations for the transformation parameters λ_i:

$$\begin{aligned} \lambda_1 + \omega_{L_1} - \lambda_{13} &= 0 \\ \lambda_2 + \omega_{L_1} - \lambda_{12} &= 0 \\ \lambda_3 + \omega_{L_1} - \lambda_{11} &= 0 \\ \lambda_4 + \omega_{L_2} - \lambda_{10} &= 0 \\ \lambda_5 + \omega_{L_2} - \lambda_{11} &= 0 \\ \lambda_6 + \omega_{L_2} - \lambda_{12} &= 0 \\ \lambda_7 + \omega_{L_2} - \lambda_{13} &= 0 \end{aligned} \tag{8}$$

Since we have only 7 equations and 13 parameters, we can choose:

$$\begin{aligned} \lambda_1 = \lambda_2 = \lambda_3 &= 0 \\ \lambda_4 = \lambda_5 = \lambda_8 &\\ \lambda_9 = \lambda_{10} & \end{aligned} \tag{9}$$

Which makes the solution to the set of equation (7) to be trivial:

$$\begin{aligned} \lambda_{13} = \lambda_{12} = \lambda_{11} = \lambda_{10} = \lambda_9 &= \omega_{L_1} \\ \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 &= \omega_{L_1} - \omega_{L_2} \end{aligned} \tag{10}$$

And our time independent, rotated Hamiltonian is according to (4):

$$\begin{aligned} H_{RF} = \sum_{i=1}^{13}(E_i - \lambda_i)\left|i\right\rangle\left\langle i\right| + \Omega_1(\left|13\right\rangle\left\langle 1\right|\frac{-1}{\sqrt{2}} + \left|12\right\rangle\left\langle 2\right|\frac{-1}{2} + \left|11\right\rangle\left\langle 3\right|\frac{-1}{\sqrt{12}} + h.c) + \\ \Omega_2(\left|10\right\rangle\left\langle 4\right|\frac{1}{\sqrt{6}} + \left|11\right\rangle\left\langle 5\right|\frac{1}{2} + \left|12\right\rangle\left\langle 6\right|\frac{1}{2} + \left|13\right\rangle\left\langle 7\right|\frac{1}{\sqrt{6}} + h.c) \end{aligned} \tag{11}$$

Where the energies E_i are the bare energies found on (2). Assuming the carrier in our setup is exactly in between the transitions:$|F=1,m_F=0\rangle \rightarrow |F'=2,m_F=1\rangle$ and $|F=2,m_F=0\rangle \rightarrow |F'=2,m_F=1\rangle$, we can denote:

$$\begin{aligned} \omega_{L_1} &= \omega_1 + \frac{\Delta}{2} + \frac{A}{3} \\ \omega_{L_2} &= \omega_1 - \Delta E_{hf} - \frac{2A}{3} - \frac{\Delta}{2} \\ \omega_{L_1} - \omega_{L_2} &= \Delta E_{hf} + \Delta + A \end{aligned} \tag{12}$$

Where Δ is our two-photon detuning, we would expect a resonant Λ system when $\Delta = 0$

1.3 Master Equation

In order to take into account the openness of the system, i.e the loss of energy to the environment through stimulated and spontaneous emissions, as well as decoherence through other processes, a density matrix formalism is used. The density matrix is defined as:

$$\hat{\rho} = \sum_n P_n \left|\psi_n\right\rangle \left\langle\psi_n\right| \tag{13}$$

Where $\{|\psi_n\rangle\}$ is a complete set of the Hilbert space and P_n are the probabilities of a single particle to be in a $|\psi_n\rangle$.
This definitions enables a distinction between a statistical mixture of states and a superposition of states.
The density matrix evolves with time through the Von-Neumann equation:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[H, \hat{\rho}] \tag{14}$$

Where the Hamiltonian consists of the Hamiltonian of the system in question, the environment and the interactions between the two.

$$H_{tot} = H_{sys} + H_{env} + H_{int} \tag{15}$$

Since we're only interested in the system, we can perform a partial trace over the environmental degrees of freedom and obtain a master equation for the evolution of the system which doesn't conserve energy as the environment can lead to dissipation (and therefore, the equation is not equivalent to Schrodinger equation). The most general form of the master equation which preservers the trace of the density matrix is the Lindblad-master equation:

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_n \frac{1}{2}[2C_n\rho(t)C_n^\dagger - \rho(t)C_n^\dagger C_n - C_n^\dagger C_n \rho(t)] \tag{16}$$

Where C_n are called "jump operators" (also called "collapse operators") and describe some decay rate from some level to another level. The assumptions

upon which the Lindblad-master equation is derived are:
-The environment is large and unchanging (the Born approximation)
-The environment has 'no memory', i.e very small correlation time (Markov approximation)
Applying the Lindblad-master approximation to n-level atoms yields a set of differential equations for the density matrix components also known as Optical Bloch Equations.
The jump operators describing spontaneous emission from state i to j are:

$$C_{i\to j} = \sqrt{\gamma_{i\to j}}\,|i\rangle\,\langle j| \tag{17}$$

Where $\gamma_{i\to j}$ is the spontaneous decay rate of the transition. From selection rules considerations we will only take account of decays transitions that obey $\Delta m_F = \pm 1$ (corresponding to $\sigma^{\pm}$ transitions) and $\Delta m_F = 0$ (for π transitions). The same sort of jump operators from eq. (16) can be used to describe ground state relaxation process where the $|i\rangle$ is an $F = 2$ ground state and $\langle j|$ is a $F = 1$ ground state. Additional decoherence processes can be described with collapse operators of the sort:

$$C_i = \sqrt{\gamma_i}\,|i\rangle\,\langle i| \tag{18}$$

References

[1] Daniel A Steck. Rubidium 87 d line data (revision 2.1. 5, 13 january 2015), 2015.