Oscillatory Solution Regime in Multi-Strain Epidemiological Model

Liam B. Mannor, Guy A. Melamed
Technion – Israel Institute of Technology
Course: Mathematical Models of Infectious Diseases

January 15, 2026

Abstract

In the world of infectious diseases, we see many diseases with more than one strain, where some interact through the human immunological response. In our project, we shall look at the cases where one strain develops a cross-immunity while the other develops a full immunity. Using numerical and analytical tools we can reach a very good estimate of the conditions of stability and instability given the characteristics of the two strains ($\beta_i, \gamma_i, \sigma, \mu$). Each type of stability is vastly different, we will show how we can reach self-sustaining oscillation even though the system is highly asymmetric.

Contents

List of Figures

1 Introduction

We study an SIR model where a disease has 2 different strains A and B such that contracting strain A gives one full immunity to B and contracting B gives partial immunity.

A real life example for such a system is the Orthopoxvirus family. Famously Edward Jenner (1749-1823) discovered in 1798 that hosts previously infected with Cowpox (CPXV) gain full immunity to Smallpox. The reverse is not true; hosts that were previously infected with Smallpox can still be infected by Cowpox.

In our project, we use analysis tools to determine different equilibrium points and whether they are stable or not.
In our SIR model, we may have 4 types of equilibrium points: DFE (Disease Free Equilibrium) and CE (Coexistence Equilibrium), and two single strain equilibria - SE_i (single existence Equilibrium of strain i). We will investigate their existence, stability, and behavior, with the goal of finding stable equilibria. Our motivation will be to show that the system will converge to a point, and thus we will be able to determine what will happen to a population with this type of disease.
We will use tools such as pertubation theorem to make our system symbolically solavble, dynamical systems analysis to understand the behaivour of our disease and numerical graphs to confirm our findings.

2 Model Formulation

This is the flowchart of our model.

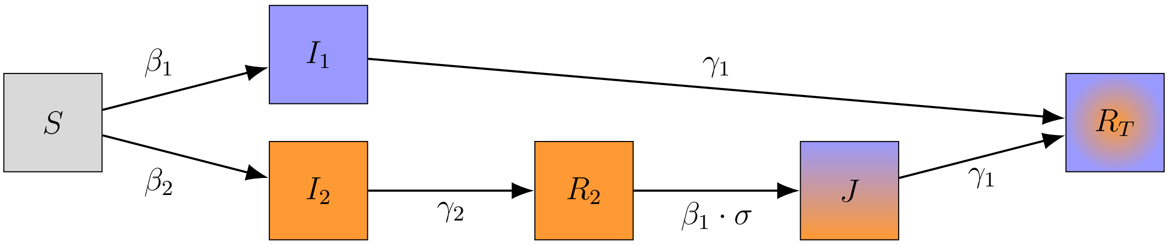

Figure 1: Model Flow Chart

- β_i is the transmission rate for strain i.
- γ_i is the recovery rate for strain i.
- σ is the relative susceptibility to strain 2 for an individual previously infected with and recovered from strain 1. Where:

2.1 Compartments and Variables

When we look at our specific SIR model we define each cohort as such

- S - General susceptible population, before have not contracted any strain yet.
- I_i - Individuals who are currently sick with strain i.
- R_2 - Individuals who have recovered from strain 2 but are still susceptible to strain 1.
- J - Individuals who are currently sick with strain 1 and already recovered from strain 2.
- R_T - Individuals who gained full immunity, either from getting strain 2 and then 1 or just strain 1.

2.2 ODE System

This is the normalized ODE system derive from the flowchart.

$$\frac{dS}{dt} = \mu - (\beta_1(I_1 + J) + \beta_2 I_2)S - \mu S, \tag{1}$$
$$\frac{dI_1}{dt} = \beta_1 S(I_1 + J) - (\mu + \gamma_1)I_1, \tag{2}$$
$$\frac{dI_2}{dt} = \beta_2 S(I_2) - (\mu + \gamma_2)I_2, \tag{3}$$
$$\frac{R_2}{dt} = \gamma_2 I_2 - \beta_1\sigma(I_1 + J)R_2 - \mu R_2, \tag{4}$$
$$\frac{dJ}{dt} = \beta_1\sigma(I_1 + J)R_2 - (\mu + \gamma_1)J, \tag{5}$$
$$\frac{dR_T}{dt} = \gamma_1 I_1 + \gamma_2 J - \mu R_T.$$

Where μ is the natural birth and mortality rate of our population.
We consider the initial conditions which satisfy :

$$\begin{aligned} &S(0), I_i(0), J(0), R_2(0), R_T(0) \geq 0 \quad i = 1, 2, \\ &S(0) + \sum_{i=1}^{2} I_i(0) + J + R_2(0) + R_T(0) = 1. \end{aligned} \tag{6}$$

and thus $\forall t > 0$

$$\begin{aligned} &S(t), I_i(t), J(t), R_2(t), R_T(t) \geq 0 \quad i = 1, 2, \\ &S(t) + \sum_{i=1}^{2} I_i(t) + J + R_2(t) + R_T(t) \equiv 1. \end{aligned} \tag{7}$$

6 and 7 follow as we have conservation in the system through the natural birth and death, and Thus in what follows we may express R_T as $1 - \left(\sum_{i=1}^{2} I_i + J + R_2 +\right)$ and then we can consider the model 1–5

2.3 Assumption on model parameters

We consider the model fig. 1 in the case where $\sigma \in [0, 1]$ and with $\mu > 0$ as we assume death and birth ensue and $\gamma_i > 0$ as recovery and $\beta_i > 0$ as the diseases spreads.

$$\sigma \in [0, 1], \quad \mu > 0, \quad \beta_i > 0 \quad \gamma_i > 0, \quad i = 1, 2 \tag{8}$$

Where $\sigma = 0$ means full immunity, $\sigma = 1$ means no immunity and $\sigma \in (0, 1)$ offers partial cross-immunity.

2.4 Basic Reproduction Number

Definition 1. F - Transmission matrix (people going into the infected cohorts)

Definition 2. V - Transition matrix (people leaving the infected cohorts)

In our model F and V are respectively :

$$\begin{bmatrix} \beta_1 S(t) & 0 \\ (\beta_2 + \sigma\beta_1)S(t) & \beta_2 S(t) \end{bmatrix} \quad \begin{bmatrix} \gamma_1 + \mu & 0 \\ 0 & \gamma_2 + \mu \end{bmatrix}$$

Proposition 3. *Calculation of the basic reproduction number.*

To calculate our R_0 we need to find the spectral radius of our NGM (Next Generation Matrix) which is defined by $NGM = FV^{-1}$.

$$FV^{-1}|_{\text{DFE}} = \begin{bmatrix} \frac{\beta_1 S(t)}{\gamma_1+\mu} & 0 \\ \frac{(\beta_2+\sigma\beta_1)S(t)}{\gamma_1+\mu} & \frac{\beta_2 S(t)}{\gamma_2+\mu} \end{bmatrix}$$

Insert the values for normalized DFE, which by definition is (S,I) = (1,0):

$$\begin{bmatrix} \frac{\beta_1}{\gamma_1+\mu} & 0 \\ \frac{\beta_2+\sigma\beta_1}{\gamma_1+\mu} & \frac{\beta_2}{\gamma_2+\mu} \end{bmatrix}$$

Now we need to find the eigenvalues and find the spectral radius :

$$\begin{aligned} &(\frac{\beta_1}{\gamma_1+\mu} - \lambda)(\frac{\beta_2}{\gamma_2+\mu} - \lambda) = 0, \\ &\mathcal{R}_0 = \max(\frac{\beta_1}{\gamma_1+\mu}, \frac{\beta_2}{\gamma_2+\mu}). \end{aligned}$$

Thus the basic reproduction number for our model fig. 1 is

$$\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\}, \tag{9}$$

where R_1, R_2 are the reproduction number of each strain which is

$$\mathcal{R}_i := \frac{\beta_i}{\gamma_i + \mu}.$$

3 Equilibrium Points

Given our system of equations, we want to find the different types of equilibrium points, their conditions, and their classifications.

Proposition 4. *In our system we get 4 different equilibria.*

depending on the values of our coefficients, each with its own epidemiological meaning:

1. DFE (Disease Free Equilibrium) - Our disease has not yet affected anyone.

$$\phi_{DFE} = (1, 0, 0, 0, 0, 0) \tag{10}$$

2. SE1 (Single-existence Equilibrium) - strain 2 dies out, leaving only strain 1. $\mathcal{R}_1 > 1$ & $\mathcal{R}_2 < 1$

$$\phi_{SE1} = (\frac{\gamma_1 + \mu}{\beta_1}, \frac{\mu(\beta_1 - \gamma_1 - \mu)}{\beta_1(\gamma_1 + \mu)}, 0, 0, 0, \frac{\gamma_1(\beta_1 - \gamma_1 - \mu)}{\beta_1(\gamma_1 + \mu)}). \tag{11}$$

3. SE2 (Single-existence Equilibrium) - Strain 1 dies out, leaving only strain 2, $\mathcal{R}_1 < 1$ & $\mathcal{R}_2 > 1$

$$\phi_{SE2} = (\frac{\gamma_2 + \mu}{\beta_2}, 0, \frac{\mu(\beta_2 - \gamma_2 - \mu)}{\beta_2(\gamma_2 + \mu)}, \frac{\gamma_2(\beta_2 - \mu - \gamma_2)}{\beta_2(\mu + \gamma_2)}, 0, 0). \tag{12}$$

4. CE (Co-existence Equilibrium) - Our strains co-exist, meaning $I_i > 0$ for $i \in 1, 2$, $\mathcal{R}_2 > \mathcal{R}_1 > 1$. ϕ_{CE} will be shown in section 5.

Proof. To find our equilibria we define the steady-state ϕ_E consists of six variables $(S, I_1, I_2, J, R_2, R_T)$ and we need to find $(S^*, I_1^*, I_2^*, R_2^*, J^*, R_T^*)$ where :

$$(\frac{dS}{dt}, \frac{dI_1}{dt}, \frac{dI_2}{dt}, \frac{dJ}{dt}, \frac{dR_2}{dt}, \frac{dR_T}{dt})|_{(S*, I_1^*, I_2^*, J^*, R_2^*, R_T^*)} = (0, 0, 0, 0, 0, 0)$$

We get this system of equations

$$0 = \mu - (\beta_1(I_1 + J) + \beta_2 I_2)S - \mu S, \tag{13}$$
$$0 = \beta_1 S(I_1 + J) - (\mu + \gamma_1)I_1, \tag{14}$$
$$0 = \beta_2 S(I_2) - (\mu + \gamma_2)I_2, \tag{15}$$
$$0 = \gamma_2 I_2 - \beta_1 \sigma(I_1 + J)R_2 - \mu R_2, \tag{16}$$
$$0 = \beta_1 \sigma(I_1 + J)R_2 - (\mu + \gamma_1)J, \tag{17}$$
$$0 = \gamma_1 I_1 + \gamma_2 J - \mu R_T. \tag{18}$$

- Case (1) - DFE : By definition $\phi_{DFE} = (1, 0, 0, 0, 0, 0)$ Which all holds in our equations.

- Case (2) - SE1: $I_2 = 0, J = 0, R_2 = 0$ from equation 14 :

$$0 = \beta_1 I_1 S - (\mu + \gamma_1)I_1, \quad I_1 > 0 \Rightarrow \quad S = \frac{\mu + \gamma_1}{\beta_1} \tag{19}$$

Now we use that into 13 :

$$0 = \mu - \beta_1 I_2 S - \mu S, \qquad 0 = \mu - (\mu + \gamma_1) I_1 - \mu \frac{\mu + \gamma_!}{\beta_1}, \tag{20}$$

$$I_1 = \frac{\mu(\beta_1 - \mu - \gamma_1)}{\beta_1(\mu + \gamma_1)}. \tag{21}$$

Plug I_1 into 18:

$$0 = \gamma_1 I_1 - \mu R_T, \qquad R_T = \frac{\gamma_1 I_1}{\mu}, \tag{22}$$

$$R_T = \frac{\gamma_1(\beta_1 - \mu - \gamma_1)}{\beta_1(\mu + \gamma_1)}. \tag{23}$$

Thus proving

$$\phi_{SE1} = (\frac{\gamma_1 + \mu}{\beta_1}, \frac{\mu(\beta_1 - \gamma_1 - \mu)}{\beta_1(\gamma_1 + \mu)}, 0, 0, 0, \frac{\gamma_1(\beta_1 - \gamma_1 - \mu)}{\beta_1(\gamma_1 + \mu)}). \tag{24}$$

- Case (3) - SE2 : $I_1 = 0, J = 0$
  We can derive this exactly the same way as in Case (2) to get :

$$\phi_{SE2} = (\frac{\gamma_2 + \mu}{\beta_2}, 0, \frac{\mu(\beta_2 - \gamma_2 - \mu)}{\beta_2(\gamma_2 + \mu)}, \frac{\gamma_2(\beta_2 - \mu - \gamma_2)}{\beta_2(\mu + \gamma_2)}, 0, 0). \tag{25}$$

- ϕ_{CE} Will be shown and proved in section 5.

□

Theorem 5. *Existence and uniqueness of the equilibrium points DFE, SE1, SE2*

Proof. Let $\mu, \beta_i, \gamma_i, \sigma$ parameters that satisfy 8 as DFE, SE1 and SE2 are direct result of a linear system of equations, they exists and are unique if the coordinates are positive as we cannot have a negative amount of infected and recovered.

- the DFE exists and is unique under the parameter space.
- the SE1 exists and is unique if $\beta_1 > \gamma_1 + \mu$
- the SE2 exists and is unique if $\beta_2 > \gamma_2 + \mu$

□

4 Single Strain Steady States

In This chapter we will analyze the stability of ϕ_{SE1} and ϕ_{SE2}.

Proposition 6. *ϕ_{SE1} and ϕ_{SE2} are both stable given $\gamma_1, \gamma_2 \gg \mu$.*

Proof. Denote J to be the Jacobian of our system. $J_1 = J|_{\phi_{SE1}}$, $J_2 = J|_{\phi_{SE2}}$.

$$J_1 = \begin{pmatrix} -\frac{\beta_1\mu}{\gamma_1+\mu} & -\gamma_1-\mu & -\frac{\beta_2(\gamma_1+\mu)}{\beta_1} & 0 & -\gamma_1-\mu \\ -\frac{\mu(\gamma_1-\beta_1+\mu)}{\gamma_1+\mu} & 0 & 0 & 0 & \gamma_1+\mu \\ 0 & 0 & \frac{\beta_2(\gamma_1+\mu)}{\beta_1}-\mu-\gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_2 & \frac{\mu\sigma(\gamma_1-\beta_1+\mu)}{\gamma_1+\mu}-\mu & 0 \\ 0 & 0 & 0 & -\frac{\mu\sigma(\gamma_1-\beta_1+\mu)}{\gamma_1+\mu} & -\gamma_1-\mu \end{pmatrix}$$

$$\begin{aligned}
\lambda_1^{(1)} &= -(\gamma_1+\mu), \\
\lambda_2^{(1)} &= -\frac{\beta_1\mu - \sqrt{\mu\,(\beta_1^2\mu - 4\beta_1\gamma_1^2 - 8\beta_1\gamma_1\mu - 4\beta_1\mu^2 + 4\gamma_1^3 + 12\gamma_1^2\mu + 12\gamma_1\mu^2 + 4\mu^3)}}{2(\gamma_1+\mu)}, \\
\lambda_3^{(1)} &= -\frac{\beta_1\mu + \sqrt{\mu\,(\beta_1^2\mu - 4\beta_1\gamma_1^2 - 8\beta_1\gamma_1\mu - 4\beta_1\mu^2 + 4\gamma_1^3 + 12\gamma_1^2\mu + 12\gamma_1\mu^2 + 4\mu^3)}}{2(\gamma_1+\mu)}, \\
\lambda_4^{(1)} &= -\frac{\gamma_1\mu - \mu^2\sigma + \mu^2 + \beta_1\mu\sigma - \gamma_1\mu\sigma}{\gamma_1+\mu}, \\
\lambda_5^{(1)} &= -\gamma_2 - \mu - \frac{\beta_2(\gamma_1-\mu)}{\beta_1}.
\end{aligned}$$

We get that $\lambda_1^{(1)}, \lambda_4^{(1)}, \lambda_5^{(1)}$ all have $\Re(\lambda_i^{(1)}) < 0$, so we just need the conditions on the 2 other eigenvalues. Essentially we just need to prove

$$\begin{aligned}
\beta_1^2\mu^2 >& \sqrt{\mu\,(\beta_1^2\mu - 4\beta_1\gamma_1^2 - 8\beta_1\gamma_1\mu - 4\beta_1\mu^2 + 4\gamma_1^3 + 12\gamma_1^2\mu + 12\gamma_1\mu^2 + 4\mu^3)}, \\
0 >& \gamma_1^3 + 3\gamma_1^2\mu + 3\gamma_1\mu^2 + \mu^3 - \beta_1\gamma_1^2 - 2\beta_1\gamma_1\mu - \beta_1\mu^2, \\
0 >& (\gamma_1+\mu)^3 - \beta_1(\gamma_1+\mu)^2, \\
\frac{\beta_1}{\gamma_1+\mu} =& \mathcal{R}_1 > 0.
\end{aligned}$$

Which holds under our assumptions, meaning all our eigenvalues have negative real parts, and thus ϕ_{SE1} is stable.

For ϕ_{SE2} we use the same reasoning and get eigenvalues of J_2 are:

$$\lambda_1^{(2)} = -\mu,$$
$$\lambda_2^{(2)} = -\gamma_1 - \mu,$$
$$\lambda_3^{(2)} = -\frac{\beta_2\mu^2 - \beta_1\mu^2 - \beta_1\gamma_2^2 + \beta_2\gamma_1\gamma_2 - 2\beta_1\gamma_2\mu + \beta_2\gamma_1\mu + \beta_2\gamma_2\mu + \beta_1\gamma_2^2\sigma + \beta_1\gamma_2\mu\sigma - \beta_1\beta_2\gamma_2\sigma}{\beta_2(\gamma_2+\mu)},$$
$$\lambda_4^{(2)} = -\frac{\sqrt{\mu}\left(\sqrt{\beta_2^2\mu - 4\beta_2\gamma_2^2 - 8\beta_2\gamma_2\mu - 4\beta_2\mu^2 + 4\gamma_2^3 + 12\gamma_2^2\mu + 12\gamma_2\mu^2 + 4\mu^3} + \beta_2\sqrt{\mu}\right)}{2(\gamma_2+\mu)},$$
$$\lambda_5^{(2)} = \frac{\sqrt{\mu}\left(\sqrt{\beta_2^2\mu - 4\beta_2\gamma_2^2 - 8\beta_2\gamma_2\mu - 4\beta_2\mu^2 + 4\gamma_2^3 + 12\gamma_2^2\mu + 12\gamma_2\mu^2 + 4\mu^3} - \beta_2\sqrt{\mu}\right)}{2(\gamma_2+\mu)}.$$

$\lambda_1^{(2)}, \lambda_2^{(2)}, \lambda_4^{(2)}$ are all negative. For $\lambda_3^{(2)}$ to be negative, we need $\frac{\beta_2}{\gamma_2} > \frac{\beta_1}{\gamma_1}$ which holds due to $\mathcal{R}_2 > 1 > \mathcal{R}_1$. For $\lambda_5^{(2)}$:

$$\beta_2\sqrt{\mu} > \sqrt{\beta_2^2\mu - 4\beta_2\gamma_2^2 - 8\beta_2\gamma_2\mu - 4\beta_2\mu^2 + 4\gamma_2^3 + 12\gamma_2^2\mu + 12\gamma_2\mu^2 + 4\mu^3},$$
$$\frac{\beta_2}{\gamma_2+\mu} = \mathcal{R}_2 > 1.$$

Which holds under our assumptions, meaning all our eigenvalues have negative real parts, and thus ϕ_{SE2} is stable.
Proving that ϕ_{SE1}, ϕ_{SE2} are both stable. $\square$

5 Coexistence (Interior) Steady-State

We wish to find when the system 2.2 gives rise to a steady-state coexistence ϕ_{CE} in which $I_i > 0$ and to find the compartment sizes in the equilibrium.

Proposition 7. *Steady-state co-existence and compartment sizes*

in our steady state $\phi_{\mathrm{CE}} = (S^*, I_1^*, I_2^*, R_2^*, J^*, R_T^*)$ We have these compartment sizes.

$$S^* = \frac{\mu + \gamma_2}{\beta_2}$$

$$R_2^* = \frac{\beta_2(\mu + \gamma_1) - \beta_1(\mu + \gamma_2)}{\beta_1(\mu + \gamma_2)}$$

$$I_1^* = -\frac{\mu(\gamma_2 + \mu)\Big(\beta_2\Big(\gamma_1(\beta_1\mu\sigma + \gamma_2) - \beta_1\gamma_2\sigma + \beta_1\mu^2\sigma + \gamma_2\mu\Big) + \beta_1\gamma_2(\sigma - 1)(\gamma_2 + \mu)\Big)}{\beta_2\sigma\Big(\beta_1\Big(\beta_2\gamma_1^2\mu\sigma + \gamma_1(2\beta_2\mu^2\sigma + \gamma_2^2 + \gamma_2\mu) + \beta_2\mu^3\sigma - \gamma_2^3 - \gamma_2^2\mu\Big) + \beta_2\gamma_2(\gamma_1 + \mu)(\gamma_2 + \mu)\Big)}$$

$$I_2^* = -\frac{\mu(\gamma_1 + \mu)\Big(\beta_2(\gamma_1 + \mu)(-\beta_1\sigma + \gamma_1 - \gamma_2) + \beta_1(\gamma_2 + \mu)(\gamma_1(\sigma - 2) + \gamma_2 + \mu(\sigma - 1))\Big)}{\beta_1\Big(\beta_2\gamma_1^2\mu\sigma + \gamma_1(2\beta_2\mu^2\sigma + \gamma_2^2 + \gamma_2\mu) + \beta_2\mu^3\sigma - \gamma_2^3 - \gamma_2^2\mu\Big) + \beta_2\gamma_2(\gamma_1 + \mu)(\gamma_2 + \mu)}$$

$$R_T^* = -\frac{\Big(\beta_2\gamma_2(\gamma_1 + \mu) + \beta_1(\gamma_1 - \gamma_2)(\gamma_2 + \mu)\Big)\Big(\beta_1\gamma_2(\gamma_2 + \mu)(\sigma - 1) + \beta_2\Big(\beta_1\sigma\mu^2 + \gamma_2\mu - \beta_1\gamma_2\sigma + \gamma_1(\gamma_2 + \beta_1\mu\sigma)\Big)\Big)}{\beta_1\beta_2\sigma\Big(\beta_2\gamma_2(\gamma_1 + \mu)(\gamma_2 + \mu) + \beta_1\Big(- \gamma_2^3 - \mu\gamma_2^2 + \beta_2\mu^3\sigma + \beta_2\gamma_1^2\mu\sigma + \gamma_1(\gamma_2^2 + \mu\gamma_2 + 2\beta_2\mu^2\sigma)\Big)\Big)}$$

$$J^* = \frac{\mu\Big(\beta_1(\gamma_2 + \mu) - \beta_2(\gamma_1 + \mu)\Big)\Big(\beta_1\gamma_2(\gamma_2 + \mu)(\sigma - 1) + \beta_2\Big(\beta_1\sigma\mu^2 + \gamma_2\mu - \beta_1\gamma_2\sigma + \gamma_1(\gamma_2 + \beta_1\mu\sigma)\Big)\Big)}{\beta_1\beta_2\sigma\Big(\beta_2\gamma_2(\gamma_1 + \mu)(\gamma_2 + \mu) + \beta_1\Big(- \gamma_2^3 - \mu\gamma_2^2 + \beta_2\mu^3\sigma + \beta_2\gamma_1^2\mu\sigma + \gamma_1(\gamma_2^2 + \mu\gamma_2 + 2\beta_2\mu^2\sigma)\Big)\Big)}$$

Proof. So using equation 15 we define $\frac{dI_2}{dt} = 0$ getting:

$$0 = \beta_2 S I_2 - (\mu + \gamma)I_2 = I_2(\beta_2 S - (\mu + \gamma)) \tag{26}$$

Since we defined $I_i > 0$ we get:

$$S^* = \frac{\mu + \gamma_2}{\beta_2} \tag{27}$$

Where we can ignore the 3rd equation since we derived S^* from it. We get a five-equation nonlinear system with five variables. We rarely can get a numerical solution for the general case , but we can try to reduce it into a system defined by a variable that we can work with. Looking at 16 we have:

$$I_1 = \frac{\beta_1 S^*}{\mu+\gamma_1}(I_1 + J) \Rightarrow I_1(1 - \frac{\beta_1 S^*}{\mu+\gamma_1}) = J \tag{28}$$

$$I_1 = \frac{\beta_1 S^*}{\mu+\gamma_1 - \beta_1 S^*} J \tag{29}$$

For ease of use lets define : $\alpha = \frac{\beta_1 S^*}{\mu+\gamma_1-\beta_1 S^*}$. So $I_1 = \alpha J$ .Now if we were to look at 16 :

$$\gamma_2 I_2 = \beta_1 \sigma (I_1 + J) R_2 + \mu R_2 \tag{30}$$

$$R_2 = \frac{\gamma_2 I_2}{\beta_1 \sigma (\alpha+1) J + \mu} \tag{31}$$

From 17:

$$(\mu+\gamma_1) J = \beta_1 \sigma (\alpha+1) J R_2 \tag{32}$$

$$R_2 = \frac{\mu+\gamma_1}{\beta_1 \sigma (\alpha+1)} \tag{33}$$

using 27 we get R_2^*
So from 29 & 31 we get:

$$\frac{\gamma_2 I_2}{\beta_1 \sigma (\alpha+1) J + \mu} = \frac{\mu+\gamma_1}{\beta_1 \sigma (\alpha+1)}$$

$$\implies I_2 = \frac{(\mu+\gamma_1)(\beta_1 \sigma (\alpha+1) J + \mu)}{\gamma_2} \tag{34}$$

So now we have I_1^*, I_2^*, as a function of J from 18 we get

$$\mu R_T = \gamma_1 I_1 + \gamma_2 J \tag{35}$$

and from 7 we get another relation to R_T by comparing them we get that

$$S^* + I_1 + I_2 + R_2^* + J + \frac{\gamma_1 I_1 + \gamma_2 J}{\mu} = 1 \tag{36}$$

from 34, 33 and 29 we get

$$J^* = \frac{\mu\Big(\beta_1(\gamma_2+\mu) - \beta_2(\gamma_1+\mu)\Big)\Big(\beta_1\gamma_2(\gamma_2+\mu)(\sigma-1) + \beta_2\Big(\beta_1\sigma\mu^2 + \gamma_2\mu - \beta_1\gamma_2\sigma + \gamma_1(\gamma_2+\beta_1\mu\sigma)\Big)\Big)}{\beta_1\beta_2\sigma\Big(\beta_2\gamma_2(\gamma_1+\mu)(\gamma_2+\mu) + \beta_1\Big(-\gamma_2^3 - \mu\gamma_2^2 + \beta_2\mu^3\sigma + \beta_2\gamma_1^2\mu\sigma + \gamma_1(\gamma_2^2 + \mu\gamma_2 + 2\beta_2\mu^2\sigma)\Big)\Big)}$$

then we get I_1^*, I_2^*, R_T^* from34, 29 and 35 □

proposition 7 gives us a symbolic solution to our steady state, and the following theorems are built upon that finding to show that this solution exists and is unique under our parameter space.

5.1 Existence and Uniqueness of Co-existence Equilibrium

The relation $\mu \ll \gamma_i$ is equivalent to $\frac{1}{\mu} \gg \frac{1}{\gamma_i}$ which we can interpret as the average lifespan is much larger than the average recovery time, This case happens in many diseases, for example, influenza have an average recovery time of 5 days yet the average lifespan is 70-80 years, as this case happen in almost any disease it is worth talking about and exploring.

Theorem 8. *Existence and uniqueness of ϕ_{CE}*

we have an interior point if $\mathcal{R}_2 > \mathcal{R}_1 > 1$ and $\mu \ll \gamma_i$

Proof. Let $\Omega^\mu := \mathbf{Span}\{\beta_i, \gamma_i, \mu\},\ \mu << \gamma_i,\ \beta_i, \gamma_i, \sigma, \mu$ parameters that satisfy 8. We have existence and uniqueness as a direct result of a system of linear equations if all of our coordinates are positive, this is a very complicated task yet we may look at the asymptotic region where $\mu \ll \gamma_i$ to simplify it.
we may look at our coordinates up to $O(\mu^2)$, if we were to look up to $O(\mu)$ when looking at our leading order coefficient we shall get

$$S^* = \frac{\gamma_2}{\beta_2} \tag{37}$$

$$R_2^* = \frac{\gamma_1(\gamma_1 - \beta_1\frac{\gamma_2}{\beta_2})}{\beta_1\sigma\gamma_1} \tag{38}$$

$$I_1^* = 0 \tag{39}$$

$$I_2^* = 0 \tag{40}$$

$$R_T^* = 1 - \frac{\gamma_2}{\beta_2} + \frac{1}{\sigma}\left(\frac{\gamma_2}{\beta_2} - \frac{\gamma_1}{\beta_1}\right) \tag{41}$$

$$J^* = 0 \tag{42}$$

which is a boundary equilibrium, we may only develop I_i^*, J^* as the conditions for the other coordinates can be found from the leading order alone thus we may develop up to $O(\mu^2)$ to get an interior point, now for $O(\mu)$

$$I_1^* = \frac{-\beta_1\gamma_2(\sigma-1) + \beta_2(\beta_1\sigma - \gamma_1)}{\beta_2\sigma\beta_1(\gamma_1-\gamma_2) + \beta_2\gamma_1} \tag{43}$$

$$I_2^* = \frac{-\gamma_1(\beta_1\gamma_2(\gamma_1(\sigma-2)+\gamma_2) - \beta_2\gamma_1(\beta_1\gamma_1(\beta_1\sigma - \gamma_1 + \gamma_2))}{\gamma_2^2(\beta_1(\gamma_1-\gamma_2) + \beta_2\gamma_1)} \tag{44}$$

$$J^* = \frac{(\beta_1\gamma_2 - \beta_2\gamma_1)(\beta_1\gamma_2(\sigma-1) - \beta_2(\beta_1\sigma-\gamma_1))}{\beta_1\beta_2\gamma_2\sigma(\beta_1(\gamma_1-\gamma_2) + \beta_2\gamma_1} \tag{45}$$

as we want our coordinates to be positive we shall get these conditions:
from 38 we get $\mathcal{R}_1 < \mathcal{R}_2$ as $\sigma \in [0,1]$ we get $43, 44$ is positive if $\mathcal{R}_1 > 1, \mathcal{R}_2 > 1$ which apply for 45as well and to 41 □

6 Analyzing Our Coexistence Equilibrium

In order to find the type of equilibrium, we need to look at how it behaves near ϕ_{CE}. To do so, we need to find the Jacobian at that point, find the eigenvalues, and check the behavior. To ease our equations, instead of looking at a 6x6 Jacobian matrix, we can look at a 5x5 Jacobian. Since $R_T = 1 - S - I_1 - I_2 - R_2 - J$, R_T is linearly dependent on the other cohorts, we know the rest, and we know R_T
$J(S, I_1, I_2, R_2, J) =$

$$J = \begin{bmatrix} -\mu - \beta_2 I_2 - \beta_1(I_1 + J) & -\beta_1 S & -\beta_2 S & 0 & -\beta_1 S \\ \beta_1(I_1 + J) & \beta_1 S - \mu - \gamma_1 & 0 & 0 & \beta_1 S \\ \beta_2 I_2 & 0 & \beta_2 S - \mu - \gamma_2 & 0 & 0 \\ 0 & -\beta_1 \sigma R_2 & \gamma_2 & -\mu - \beta_1 \sigma (I_1 + J) & -\beta_1 \sigma R_2 \\ 0 & \beta_1 \sigma R_2 & 0 & \beta_1 \sigma (I_1 + J) & \beta_1 \sigma R_2 - \mu - \gamma_1 \end{bmatrix}$$

Calculating $J(\phi_{CE})$ we get:

$$\begin{bmatrix} \dfrac{-\mu \cdot H_1}{\sigma \cdot H_4} & -\frac{\beta_1(\gamma_2+\mu)}{\beta_2} & -(\gamma_2 + \mu) & 0 & -\frac{\beta_1(\gamma_2+\mu)}{\beta_2} \\ -\dfrac{\mu(\gamma_1 + \mu) \cdot H_2}{\sigma \cdot H_4} & \frac{\beta_1(\gamma_2+\mu)}{\beta_2} - \mu - \gamma_1 & 0 & 0 & \frac{\beta_1(\gamma_2+\mu)}{\beta_2} \\ -\dfrac{\beta_2 \mu(\gamma_1 + \mu) \cdot H_3}{H_4} & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_1\gamma_2 - \beta_2\gamma_1 + \beta_1\mu - \beta_2\mu}{\beta_2} & \gamma_2 & \frac{\gamma_2 \mu \cdot H_3}{H_4} & \frac{\beta_1\gamma_2 - \beta_2\gamma_1 + \beta_1\mu - \beta_2\mu}{\beta_2} \\ 0 & -\frac{\beta_1\gamma_2 - \beta_2\gamma_1 + \beta_1\mu - \beta_2\mu}{\beta_2} & 0 & -\frac{\mu(\gamma_1+\mu) \cdot H_2}{H_4} & -\frac{\beta_1(\gamma_2+\mu)}{\beta_2} \end{bmatrix}$$

$$\begin{aligned} H_1 = {} & \beta_1\gamma_1\gamma_2^2 - \beta_2^2\gamma_1^3\sigma - \beta_2\gamma_1^2\gamma_2 + \beta_1\gamma_2\mu^2 + \beta_1\gamma_2^2\mu - \beta_2\gamma_2\mu^2 \\ & - \beta_1\gamma_2^3\sigma + \beta_1\beta_2^2\gamma_1^2\sigma^2 + \beta_1\beta_2^2\mu^2\sigma^2 + \beta_2\gamma_1\gamma_2^2\sigma - \beta_1\gamma_2\mu^2\sigma - 2\beta_1\gamma_2^2\mu\sigma \\ & + \beta_2\gamma_2\mu^2\sigma + \beta_2\gamma_2^2\mu\sigma + \beta_2^2\gamma_1^2\gamma_2\sigma - \beta_2^2\gamma_1\mu^2\sigma - 2\beta_2^2\gamma_1^2\mu\sigma + \beta_2^2\gamma_2\mu^2\sigma \\ & + \beta_1\gamma_1\gamma_2\mu - 2\beta_2\gamma_1\gamma_2\mu - \beta_1\beta_2\gamma_1^2\gamma_2\sigma^2 + 2\beta_1\beta_2^2\gamma_1\mu\sigma^2 - \beta_1\beta_2\gamma_2\mu^2\sigma^2 \\ & + \beta_1\beta_2\gamma_1\gamma_2\sigma + \beta_1\beta_2\gamma_2\mu\sigma + \beta_2\gamma_1\gamma_2\mu\sigma - \beta_1\beta_2\gamma_1\gamma_2^2\sigma + 2\beta_1\beta_2\gamma_1^2\gamma_2\sigma \\ & + \beta_1\beta_2\gamma_1\mu^2\sigma + \beta_1\beta_2\gamma_1^2\mu\sigma - \beta_1\beta_2\gamma_2^2\mu\sigma + 2\beta_2^2\gamma_1\gamma_2\mu\sigma \\ & - 2\beta_1\beta_2\gamma_1\gamma_2\mu\sigma^2 + 2\beta_1\beta_2\gamma_1\gamma_2\mu\sigma, \end{aligned}$$

$$H_2 = \beta_2\gamma_1\gamma_2 - \beta_1\gamma_2^2 - \beta_1\gamma_2\mu + \beta_2\gamma_2\mu + \beta_1\gamma_2^2\sigma + \beta_1\gamma_2\mu\sigma + \beta_1\beta_2\mu^2\sigma - \beta_1\beta_2\gamma_2\sigma + \beta_1\beta_2\gamma_1\mu\sigma,$$

$$\begin{aligned} H_3 = {} & \beta_1\gamma_2^2 + \beta_2\gamma_1^2 - \beta_1\mu^2 + \beta_1\mu^2\sigma - 2\beta_1\gamma_1\gamma_2 - \beta_2\gamma_1\gamma_2 - 2\beta_1\gamma_1\mu + \beta_2\gamma_1\mu - \beta_2\gamma_2\mu + \beta_1\gamma_1\gamma_2\sigma \\ & - \beta_1\beta_2\mu\sigma + \beta_1\gamma_1\mu\sigma + \beta_1\gamma_2\mu\sigma - \beta_1\beta_2\gamma_1\sigma, \end{aligned}$$

$$\begin{aligned} H_4 = {} & \beta_1\gamma_1\gamma_2^2 - \beta_1\gamma_2^3 + \beta_2\gamma_1\gamma_2^2 - \beta_1\gamma_2^2\mu + \beta_2\gamma_2\mu^2 + \beta_2\gamma_2^2\mu + \beta_1\beta_2\mu^3\sigma + \beta_1\gamma_1\gamma_2\mu + \beta_2\gamma_1\gamma_2\mu \\ & + 2\beta_1\beta_2\gamma_1\mu^2\sigma + \beta_1\beta_2\gamma_1^2\mu\sigma. \end{aligned}$$

The characteristic polynomial, is a convoluted and complex quintic equation, from the Abel-Rufffini theorem we know that there is no simple solution to find all 5 roots. Using perturbation theory we can get a close estimation to our roots.

Theorem 9. *Our eigenvalues of the Jacobian at ϕ_{CE} are :*

$$\lambda^{(1)} = \sqrt{\mu}\sqrt{\frac{-(A+B)+\sqrt{(A+B)^2-4\gamma_1(AB*f)}}{\gamma_1}}) + q_1\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{46}$$

$$\lambda^{(2)} = -\sqrt{\mu}\sqrt{\frac{-(A+B)+\sqrt{(A+B)^2-4\gamma_1(AB*f)}}{\gamma_1}}) + q_2\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{47}$$

$$\lambda^{(3)} = \sqrt{\mu}\sqrt{\frac{-(A+B)-\sqrt{(A+B)^2-4\gamma_1(AB*f)}}{\gamma_1}}) + q_3\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{48}$$

$$\lambda^{(4)} = -\sqrt{\mu}\sqrt{\frac{-(A+B)-\sqrt{(A+B)^2-4\gamma_1(AB*f)}}{\gamma_1}}) + q_4\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{49}$$

$$\lambda^{(5)} = -\gamma_1 + \mu + \mathcal{O}(\mu^2). \tag{50}$$

Proof. From section 5.1 we develop a power series with respect to μ to analyze the equilibrium.

$$P(\lambda,\mu) = \sum_{k=0}^{n}\sum_{m=0}^{5} \lambda^m P_{m,k}\mu^k + \mathcal{O}(\mu^{n+1}), \tag{51}$$

$$\lambda^{(i)}(\mu) = \sum_{k=0}^{n} \lambda_k^{(i)}\mu^k + \mathcal{O}(\mu^{n+1}) \qquad i \in 1,2,3,4,5. \tag{52}$$

We know $\lambda_0^{(i)}$ is the solution to our char. poly. when $\mu = 0$ and get :

$$0 = \left(\frac{\sigma\beta_1^2\beta_2\gamma_1^3\gamma_2^4 - 2\sigma\beta_1^2\beta_2\gamma_1^2\gamma_2^5 + \sigma\beta_1^2\beta_2\gamma_1\gamma_2^6 + 2\sigma\beta_1\beta_2^2\gamma_1^3\gamma_2^4 - 2\sigma\beta_1\beta_2^2\gamma_1^2\gamma_2^5 + \sigma\beta_2^3\gamma_1^3\gamma_2^4}{\beta_2\sigma\left(\beta_1\gamma_1\gamma_2^2 - \beta_1\gamma_2^3 + \beta_2\gamma_1\gamma_2^2\right)^2}\right)\left(\lambda^5\gamma_1 + \lambda^4\right).$$

Thus getting :

$$\lambda_0^{(1)} = \lambda_0^{(2)} = \lambda_0^{(3)} = \lambda_0^{(4)} = 0, \quad \lambda_0^{(5)} = -\gamma_1$$

So if we take a 3 term expansion of our system :

$$P(\lambda,\mu) = \lambda^4(\lambda+\gamma_1) + \mu\sum_{i=0}^{4} P_{1,i}\lambda^i + \mu^2\sum_{i=0}^{4} P_{2,i}\lambda^i + \mu^3\sum_{i=0}^{4} P_{3,i}\lambda^i + \mathcal{O}(\mu^4), \tag{53}$$

$$\lambda^{(i)}(\mu) = 0 + \mathcal{O}(\mu) \qquad i \in 1,2,3,4, \tag{54}$$

$$\lambda^{(5)}(\mu) = -\gamma_1 + \mu\lambda_1^{(5)} + \mathcal{O}(\mu^2). \tag{55}$$

Getting:
$\sum P_{1,i}, \quad i \in 1,2,3$ by definition is all the items in the char. poly. that are the coefficients

of μ^i:

$$0 = \sum_{i=0}^{5} P_{1,i} = P_{1,4}\lambda^4 + P_{1,3}\lambda^3 + P_{1,2}\lambda^2 = \lambda^4 \frac{b_1}{\beta_2 \sigma} + \lambda^3 \frac{b_2}{b_4} - \lambda^2 \frac{b_3}{b_4}, \qquad \mathbb{R} \ni b_i \neq 0. \tag{56}$$

$$0 = \sum_{i=0}^{5} P_{2,i} = P_{2,4}\lambda^4 + P_{2,3}\lambda^3 + P_{2,2}\lambda^2 + P_{2,1}\lambda + P_{2,0}, \qquad \mathbb{R} \ni P_{2,i} \neq 0. \tag{57}$$

$$0 = \sum_{i=0}^{5} P_{3,i} = P_{3,4}\lambda^4 + P_{3,3}\lambda^3 + P_{3,2}\lambda^2 + P_{3,1}\lambda, \qquad \mathbb{R} \ni P_{3,i} \neq 0. \tag{58}$$

$$\tag{59}$$

So plugging 56 in 53 and we get :

$$0 = P(\lambda, \mu) = \lambda^4(\lambda + \gamma_1) + \mu\, \lambda^2(\lambda^2 P_{1,4} + \lambda P_{1,3} + P_{1,2}) + \mathcal{O}(\mu^2) \tag{60}$$

Now we need to find $\lambda_1^{(5)}$. To do so, we will substitute $P(\lambda, \mu) \to P(-\gamma_1 + \lambda_1^{(5)}\mu,\, \mu) + \mathcal{O}(\mu^2)$ Since λ depends on μ, we expand $P(\lambda, \mu)$ as a Taylor series around $\lambda = -\gamma_1, \mu = 0$:

$$0 = P(\lambda, \mu) = P(-\gamma_1, 0) + \mu \left(\lambda_1^{(5)} \frac{\partial P}{\partial \lambda}\bigg|_{(-\gamma_1,0)} + \frac{\partial P}{\partial \mu}\bigg|_{(-\gamma_1,0)} \right) + \mathcal{O}(\mu^2),$$

We already know that $-\gamma_1$ is a root at $\mu = 0$ so $P(-\gamma_1, 0) = 0$, thus getting:

$$0 = \mu \left(\lambda_1^{(5)} \frac{\partial P}{\partial \lambda}\bigg|_{(-\gamma_1,0)} + \frac{\partial P}{\partial \mu}\bigg|_{(-\gamma_1,0)} \right) + \mathcal{O}(\mu^2), \qquad \mu > 0 \tag{61}$$

$$\lambda_1^{(5)} = - \left. \frac{\frac{\partial P}{\partial \mu}}{\frac{\partial P}{\partial \lambda}} \right|_{(-\gamma_1,0)}. \tag{62}$$

We can calculate both $\frac{\partial P}{\partial \lambda}$ and $\frac{\partial P}{\partial \mu}$ and the evaluate at $(-\gamma_1, 0)$ and we get:

$$\frac{\partial P}{\partial \lambda}\bigg|_{(-\gamma_1,0)} = \frac{\partial P}{\partial \mu}\bigg|_{(-\gamma_1,0)} = \gamma_1^4. \tag{63}$$

So putting 63 into 62 and then into 55 :

$$\lambda^{(5)} = -\gamma_1 + \mu + \mathcal{O}(\mu^2). \tag{64}$$

In order to find $\lambda_1^{(i)}, i \in 1,2,3,4$, we use dominant balance (as shown in [1]) to derive the orders of the remaining eigenvalues. Assume $\lambda = A\mu^{\alpha}$, using 60 :

$$A^4\gamma_1\mu^{4\alpha} \sim A^2 P_{1,2}\mu^{1+2\alpha}, \qquad \alpha = \frac{1}{2}$$

Meaning, $\lambda_1^{(i)} = \mathcal{O}(\sqrt{\mu}), \qquad i \in 1,2,3,4$. Assume $\lambda = a_1\mu^{\frac{1}{2}} + a_2\mu + a_3\mu^{1.5} + a_4\mu^2 + a_5\mu^{2.5} + a_6\mu^3 + \mathcal{O}(\mu^{3.5})$, substitute into 60

$$0 = P_{1,0} = P_{1,1} = P_{3,0} \Rightarrow \tag{65}$$

$$0 = \mu^2(\gamma_1 a_1^4 + P_{1,2}a_1^2 + P_{2,0}) + \mathcal{O}(\mu^{1.5}), \tag{66}$$

$$a_1 = \pm\sqrt{\frac{-P_{1,2} + \sqrt{P_{1,2}^2 - 4\gamma_1 P_{2,0}}}{\gamma_1}}, \pm\sqrt{\frac{-P_{1,2} - \sqrt{P_{1,2}^2 - 4\gamma_1 P_{2,0}}}{\gamma_1}}. \tag{67}$$

Thus from 67 and 64 we get all our 5 eigenvalues for our Jacobian at ϕ_{CE} are :

$$\begin{aligned}
\lambda^{(1)} &= \sqrt{\mu}\sqrt{\frac{-P_{1,2} + \sqrt{P_{1,2}^2 - 4\gamma_1 P_{2,0}}}{\gamma_1}} + \mathcal{O}(\mu), \\
\lambda^{(2)} &= -\sqrt{\mu}\sqrt{\frac{-P_{1,2} + \sqrt{P_{1,2}^2 - 4\gamma_1 P_{2,0}}}{\gamma_1}} + \mathcal{O}(\mu), \\
\lambda^{(3)} &= \sqrt{\mu}\sqrt{\frac{-P_{1,2} - \sqrt{P_{1,2}^2 - 4\gamma_1 P_{2,0}}}{\gamma_1}} + \mathcal{O}(\mu), \\
\lambda^{(4)} &= -\sqrt{\mu}\sqrt{\frac{-P_{1,2} - \sqrt{P_{1,2}^2 - 4\gamma_1 P_{2,0}}}{\gamma_1}} + \mathcal{O}(\mu), \\
\lambda^{(5)} &= -\gamma_1 + \mu + \mathcal{O}(\mu^2).
\end{aligned}$$

We simplify further by defining A, B, f all functions of $\beta_1, \beta_2, \gamma_1, \gamma_2, \sigma$ such that:

$$P_{1,2} = A + B, \tag{68}$$

$$P_{2,0} = ABf. \tag{69}$$

Getting:

$$\lambda^{(1)} = \sqrt{\mu}\sqrt{\frac{-(A+B) + \sqrt{(A+B)^2 - 4\gamma_1(AB*f)}}{\gamma_1}}) + q_1\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{70}$$

$$\lambda^{(2)} = -\sqrt{\mu}\sqrt{\frac{-(A+B) + \sqrt{(A+B)^2 - 4\gamma_1(AB*f)}}{\gamma_1}}) + q_2\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{71}$$

$$\lambda^{(3)} = \sqrt{\mu}\sqrt{\frac{-(A+B) - \sqrt{(A+B)^2 - 4\gamma_1(AB*f)}}{\gamma_1}}) + q_3\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{72}$$

$$\lambda^{(4)} = -\sqrt{\mu}\sqrt{\frac{-(A+B) - \sqrt{(A+B)^2 - 4\gamma_1(AB*f)}}{\gamma_1}}) + q_4\mu + \mathcal{O}(\mu\sqrt{\mu}), \tag{73}$$

$$\lambda^{(5)} = -\gamma_1 + \mu + \mathcal{O}(\mu^2). \tag{74}$$

Where :

$$\begin{aligned}
A &= \frac{\beta_1\gamma_1^2\gamma_2(\sigma-1)(\beta_1\gamma_1\gamma_2-\beta_1\gamma_2^2+\beta_2\gamma_1\gamma_2)}{b_2},\\
B &= \frac{\beta_2\gamma_1\gamma_2^2(\sigma-1)(\beta_1\gamma_1\gamma_2-\beta_1\gamma_2^2+\beta_2\gamma_1\gamma_2)}{b_2},\\
f &= \frac{1}{\gamma_1\gamma_2}\left[1-\frac{(1-\sigma)^2\beta_1\beta_2\gamma_1\gamma_2 b_2}{(\sigma\beta_1\gamma_1+(1-\sigma)\beta_2\gamma_2)(\sigma\beta_2\gamma_2+(1-\sigma)\beta_1\gamma_1)}\right],\\
b_2 &= \beta_2\,\sigma\left(\beta_1\,\gamma_1\,{\gamma_2}^2-\beta_1\,{\gamma_2}^3+\beta_2\,\gamma_1\,{\gamma_2}^2\right)^2.
\end{aligned}$$

Using $S^*, I_1^*, I_2^*, R_2^*, J^*, R_T^* \geq 0$ in 7, and $\mathcal{R}_1, \mathcal{R}_2 > 1$ we can prove $A, B \geq 0$. q_i will be defined in the next chapter. $\square$

6.1 Analysis of Stability and Bifurcation

From theorem 8 we have an interior equilibrium, we will show that we have an oscillatory steady-state within $\mathcal{O}(\mu\sqrt{\mu})$.

Theorem 10. *ϕ_{CE} is a stable fixed point for a large amounf of cases within Ω^μ.*

Proof. In order to show that the point is stable, we look at the eigenvalues from theorem 9 and prove they have negative real part.
We begin with $\lambda^{(5)}$ and as $\gamma_1 \gg \mu$ and $\gamma_1 > 0$ we get that $\lambda^{(5)}$ has a negative real part. As for the other four eigenvalues as our polynomial has real coefficient we may say that we have complex conjugates in our roots for the polynomial, we look the pairs $\lambda^{(1)}, \lambda^{(2)}$ and $\lambda^{(3)}, \lambda^{(4)}$, each pair have the same real part so we need to show that the real part of $\lambda^{(2)}, \lambda^{(4)}$ is negative .
Let $\Delta \coloneqq (A+B)^2 - 4\gamma_1(ABf)$ (the nested root of our eigenvalues).$\Delta = (A+B)^2 - 4\gamma_1(ABf) \Rightarrow \Delta \geq 0$ from 8 & $(\mathcal{R}_1, \mathcal{R}_2) \notin \Gamma^*$, so $\frac{-(A+B)\pm\sqrt{\Delta}}{\gamma_1} < 0$.
Implementing into 70,71,72 and 73, all 4 eigenvalues are purely imaginary.
We define $\{r_i\}_{i=1}^4$ to be the coefficient of $\sqrt{\mu}$ of $\lambda^{(i)}$.

$$\lambda^{(i)} = \sqrt{\mu} r_i + \mu q_i + \mathcal{O}(\mu\sqrt{\mu}) \tag{75}$$

We substitute into $P(\lambda, \mu) = 0$, (53), and we get:

$$0 = 4\gamma_1 r_i^3 q_i + 2P_{1,2} r_i q_i + r_i^5 + P_{1,4} r_i^4 + P_{1,3} r_i^3, \tag{76}$$

$$q_i = -\frac{r_i^4 + P_{1,4} r_i^3 + P_{1,3} r_i^2}{4\gamma_1 r_i^2 + 2P_{1,2}}, \tag{77}$$

$$q_i = -\frac{\omega_\pm^2 - i\omega_\pm^{3/2} P_{1,4} - \omega_\pm P_{1,3}}{-4\gamma_1\omega_\pm + 2P_{1,2}}, \tag{78}$$

$$\Re(q_i) = -\frac{\omega_\pm(\omega_\pm - P_{1,3})}{5(A+B) \pm 4\Delta}, \qquad i = 1, 2, 3, 4. \tag{79}$$

Where:

$$P_{1,4} = \frac{H_1 + H_4\,\sigma - H_3\,\gamma_2\,\sigma}{H_4\,\sigma},$$

$$P_{1,3} = -\frac{\gamma_1\,\left(H_2\,\beta_1\,\gamma_2 - H_1\,\beta_2 + H_3\,{\beta_2}^2\,\gamma_2\,\sigma - H_2\,\beta_1\,\gamma_2\,\sigma + H_2\,\beta_2\,\gamma_1\,\sigma + H_3\,\beta_2\,\gamma_2\,\sigma\right)}{H_4\,\beta_2\,\sigma},$$

$$P_{1,2} = 68,$$

$$H_i = 6,$$

$$\omega_\pm = \frac{(A+B) \pm \sqrt{\Delta}}{\gamma_1}$$

Our system is stable iff $\omega_\pm - P_{1,3} > 0$ (since denominator is ≥ 0) thus ensuring $\Re(q_i) < 0$. We cannot prove this symbolically under our basic assumptions, unless we assume specific cases. If we compute numerically we get that $q_i < 0$ for a large amount of cases within Ω^μ / Γ^*. $\square$

Proposition 11. *For any parameters in* Ω^{μ}, $\quad \exists \sigma_c \in [0,1]$ *such that a bifuraction occurs at* σ_c

Proof. Since $\{r_i\}_{i=1}^4$ are all purely imaginary, our bifurcation occurs in $\mathcal{O}(\mu)$, looking at $\{q_i\}_{i=1}^4$, we see that $q_1 = q_2$, $q_3 = q_4$. To check when the bifurcation occurs we need to find when the sign of $\Re(\lambda^{(i)})$ changes signs. Since $\omega_{\pm} > 0$, the sign changes as we cross $\omega_{\pm} = P_{1,3}$. Define both $\omega_{\pm} = P_{1,3}$ as a function of σ and we look to find when $\omega_{\pm}(\sigma) = P_{1,3}(\sigma)$

$$\gamma_1 P_{1,3}(\sigma) = A(\sigma) + B(\sigma) \pm \sqrt{\Delta(\sigma)}, \tag{80}$$

$$\pm\sqrt{\Delta(\sigma)} = \gamma_1 P_{1,3}(\sigma) - A(\sigma) - B(\sigma), \tag{81}$$

$$\Delta(\sigma) = (\gamma_1 P_{1,3}(\sigma) - A(\sigma) - B(\sigma))^2, \tag{82}$$

$$0 = (\gamma_1 P_{1,3}(\sigma) - A(\sigma) - B(\sigma))^2 - \Delta(\sigma). \tag{83}$$

This is a septic poly, there are 7 roots that are extremely cumbersome to prove if one lays within $[0,1]$. Instead we wrote a code that takes 10^5 random values of $\beta_i, \gamma_i \in$ $(0,1]$, $(0,1]$, $(0,5]$, $(0,10] \cap \Omega^{\mu}$, $\quad i \in \{1,2\}$. and a code which does the same but in $\Omega^{\mu=0}$.

- Case $\mu = 0$: In this system we get that for all parameters in $\Omega^{\mu=0}$ the roots of 83 come in pairs within $[0,1]$ meaning we get Hopf-windows, small sections of instability, as the real part of 2 of our eigenvalues cross the $\Re = 0$ axis.

- Case $\mu \neq 0$: In the pertubed system for any $\mu > 0$ we get that our system changes drasticlly by now having only an odd amount of roots for 83. Meaning if $n = 1$ (n being the amount of roots within $[0,1]$) we obtain a regular Hopf-Bifurcation. If $n = 3, 5, 7$ (less than 1% of the cases) we get a Hopf-window and then another bifurcation.

□

Lemma 1. $\exists \Gamma^* \subset \Omega^{\mu=0}$*, where for* $\forall (\mathcal{R}_1, \mathcal{R}_2) \in \Gamma^*$ *our system is unstable.*

Proof. From 11, we know that exists at least 1 Hopf-window in Ω^0 where we gain instability. Setting σ to be a constant we define at $\Gamma^*(\mathcal{R}_1, \mathcal{R}_2) \coloneqq \Delta(\mathcal{R}_1, \mathcal{R}_2) = 0$. From the definition of our eigenvalues in 9 we see that by $\Delta = 0$ our Γ^* is a curve of degeneracy since we get repeated eigenvalues. As shown by Gavish in [2], chapter 5, along Γ^* by using a differemt pertubation of our eigenvalues we get that the coeff. of $\mu^{\frac{3}{4}}$ are all positive and real, and thus we have instability along Γ^*. □

7 Numerical results

We may look at some numerical results to compare with previous analysis, we shall mainly focus on the coexistence equilibrium.

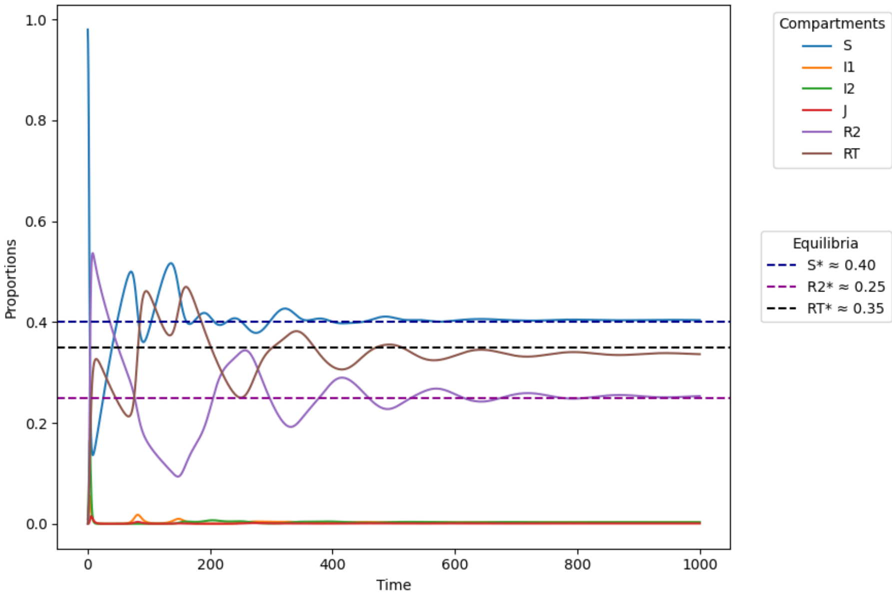

Figure 2: Compartment Size Plotting

The parameters are $\gamma_i = 1, \beta_1 = 1.5, \beta_2 = 2, \sigma = 0.4, \mu = 0.01$, our starting conditions are $S = 0.98, I_1 = 0.01, I_2 = 0.01$, the compartment approximation comes from the leading order; here $\mathcal{R}_2 > \mathcal{R}_1 > 1, \mu \ll \gamma_i$ then from theorem 8 we have an oscillatory stable interior fixed point.

We may now observe a bifurcation scan under the same conditions, and we may iterate over σ to find σ_{crit}.

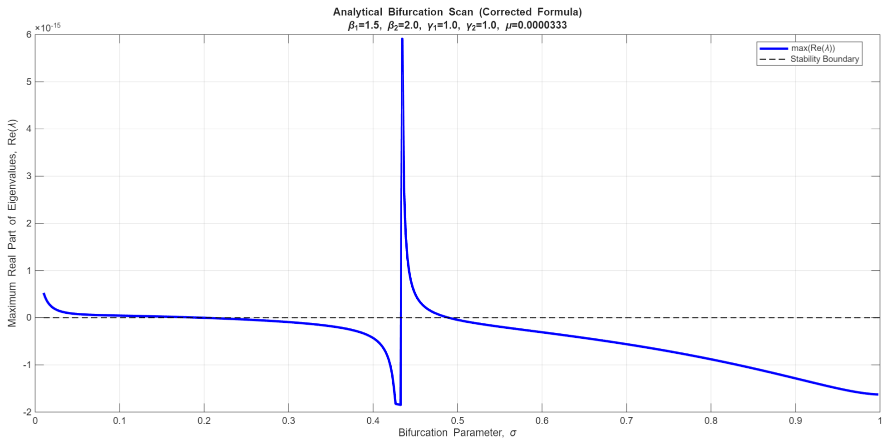

Figure 3: Bifurcation Scan

Under the conditions in the title we can clearly see the 3 bifucation points and even the Hopf instability window.

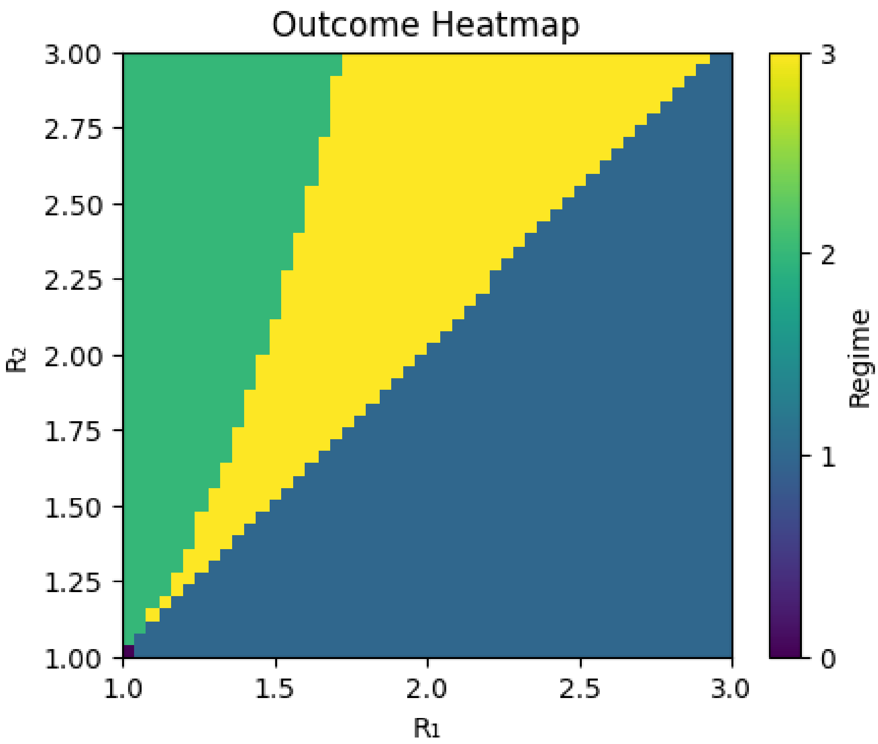

Figure 4: Outcome Heatmap

Under the same conditions as in fig. 2 with changing β_i, where the regime is where the system converges, where green is SE2, blue is SE1, purple is DFE and yellow is CE. We note that even though the interior point exists, we may still converge into SE2.

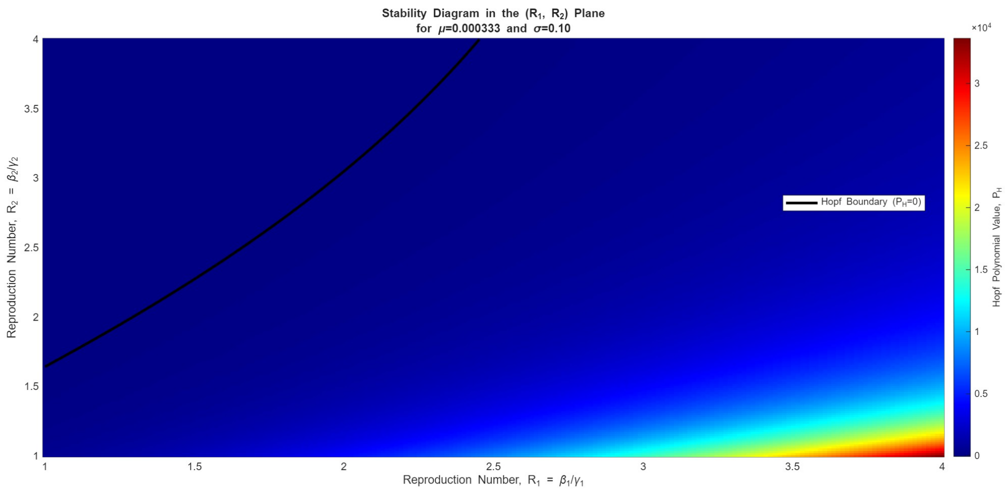

Figure 5: Stability Heatmap

Here the colours signify the negativity of our roots, (more red = more negative, black = non-negative). We can clearly see the Γ^* curve of instability. For any $(\mathcal{R}_1, \mathcal{R}_2) \notin \Gamma^*$ we see that our Hopf polynomial is > 0 meaning we get a degeneracy on all values of the black curve , and due to the further analysis done by Gavish in [2], we know that the degeneracy causes instability.

References

[1] M.H. Holmes. *Introduction to Perturbation Methods*. Vol. 20. New York: Springer, 2012.

[2] Nir Gavish. *A new oscillatory regime in two-strain epidemic models with partial cross-immunity*. Technion, Haifa: arXiv, 2024.