Calculus 1 - Exercise 13

All exercises should be submitted by January 31th by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand. These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

<u>Questions:</u>

1. Let $0 < \alpha < \beta < \frac{\pi}{2}$. Prove that

$$\frac{\beta - \alpha}{\cos^2 \alpha} < \tan \beta - \tan \alpha < \frac{\beta - \alpha}{\cos^2 \beta}$$

2. Let f, g be two functions that defined on $[a, b]$ for $a < b$.
   Assume that f, g are continuous on $[a, b]$ and differentiable on (a, b) such that $g'(x) \neq 0$ for every $x \in (a, b)$.
   Prove that there exists an $a < c < b$ for which

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

   Hint: Consider the function $F(x) = (f(x) - f(a)) \cdot (g(b) - g(x))$ for every $x \in [a, b]$.

3. Compute each one of the following limits or prove that it doesn't exist:

   (a) $\lim\limits_{x \to 0} \frac{\ln(\cos(3x))}{\ln(\cos(5x))}$

   (b) $\lim\limits_{x \to 1^+} [\ln x \cdot \ln(\ln x)]$

   (c) $\lim\limits_{x \to \infty} \left(e^{2x} - x + 1\right)^{\frac{1}{x}}$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function.
   Assume that $f''(x) > 0$ for every $x \in \mathbb{R}$.
   Prove that for every $x \in \mathbb{R}$ we have

$$\frac{f(x+1) + f(x-1)}{2} > f(x)$$

5. Prove or disprove each of the following statements:

   (a) For every $a, b \in [1, 2]$ such that $a \leq b$, we have

$$2(\ln b - \ln a) \leq b^2 - a^2$$

   (b) Let $f(x) = \frac{3^x - 2^x - 4^x}{3^x + 2^x + 4^x}$ for every $x \in \mathbb{R}$.
   Then there exists a point $c \in \mathbb{R}$ such that $f'(c) = 0$.

   (c) Let f be a function defined on $[a, b]$. Assume that f is continuous on $[a, b]$ and differentiable on (a, b).
   Let $c \in (a, b)$. Then there exist two points x_1, x_2 such that
   $a \leq x_1 < x_2 \leq b$ and $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

   (d) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $L \in \mathbb{R}$.
   If $\lim\limits_{x \to \infty} \frac{f(x) + x \cdot f'(x)}{x} = L$, then $\lim\limits_{x \to \infty} \frac{f(x)}{x} = \frac{L}{2}$.

   Hint: Consider the function $g(x) = x \cdot f(x)$ for every $x \in \mathbb{R}$.

Optional:
Let $R(t)$ denote the basic reproduction number of an infectious disease at time t (i.e., the expected number of cases directly generated by one case in a population where all individuals are susceptible to infection).
Let $N(t)$ denote the number of diagnosed infections in a population.
Suppose that there exist numbers $n_0 > 0$ and $r > 0$ such that for every $t \geq 0$ we have

$$N(t) = n_0 [R(t)]^{\frac{t}{r}}$$

Suppose also that $R(t) \geq 1$ for every $t \geq 0$ and let $\underline{s} = \inf\{R(t) : t \geq 0\}$.

1. Prove that if $\underline{s} > 1$ then $\lim_{t \to \infty} N(t) = \infty$.
2. Prove that if N is bounded from above, then $\lim_{t \to \infty} R(t) = 1$.