Calculus 1 - Exercise 2

All exercises should be submitted by November 15th by 23:00.
Delays won't be accepted aside for special cases which will be approved beforehand.These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Let $p > 1$ be a prime number. Prove that $\sqrt[3]{p}$ is irrational.

   Remark: You don't need to prove that $\sqrt[3]{p} \in \mathbb{R}$.

2. Let $\phi \neq A \subseteq \mathbb{R}$.

   (a) Prove the greatest lower bound theorem: Suppose that A is bounded from below. Then there exists a unique greatest lower bound of A, $\underline{s} \in \mathbb{R}$.

   (b) Prove the ϵ property of the greatest lower bound:
   Suppose that A is bounded from below and let $\underline{s} \in \mathbb{R}$ be a lower bound of A . Then $\underline{s} = \inf(A)$ if and only if for every $\epsilon > 0$ there exists an $a \in A$ such that $a < \underline{s} + \epsilon$.

   (c) Suppose that A is bounded from below.
   Prove that $\inf A \in A$ if and only if there exists a minimum in A.

3. For each of the following sets compute $\sup(A)$, $\inf(A)$, $\min(A)$, $\max(A)$ or prove that they don't exist.

   (a) $A = \left\{ \frac{2x-3}{3x+5} \mid x \geq 0 \right\}$

   (b) $A = \left\{ \frac{1}{n} + (-1)^n \mid n \in \mathbb{N} \right\}$

   (c) $A = \left\{ a + \frac{1}{a} \mid 0 < a \in \mathbb{R} \right\}$
   Hint: for every $0 < a \in \mathbb{R}$,$\left(\sqrt{a} - \frac{1}{\sqrt{a}} \right)^2 \geq 0$.

4. Let $\phi \neq A, B \subseteq \mathbb{R}$.

   (a) Prove that if $A \leq B$, then $\sup(A) \leq \inf(B)$.

   (b) Prove that if $\sup(A) = \inf(B)$, then $A \cap B$ contains at most one element.

5. Let $\phi \neq A, B \subseteq \mathbb{R}$.
   Prove or disprove each of the following statements:

   (a) If A is bounded from above and $B \subseteq A$, then B is bounded from above and $\sup(B) \leq \sup(A)$.

   (b) Let $0 < \alpha \in \mathbb{R}$.
   We define $\alpha \cdot A = \{\alpha \cdot a \mid a \in A\}$.
   If A is bounded from above, then $\sup(\alpha \cdot A) = \alpha \cdot \sup(A)$.

   (c) We define $-A = \{-a \mid a \in A\}$.
   If A is bounded from above, then $-A$ is bounded from below and $\inf(-A) = -\sup(A)$.

   (d) If A is bounded from above and $\sup(A) \notin A$. Then there exists an $\epsilon > 0$ such that the set $C = \{x \in A \mid x \leq \sup(A) - \epsilon\}$ has a maximum.