Calculus 1 - Exercise 11

All exercises should be submitted by January 17th by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand.
These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Consider the function $f(x) = \arccos(x)$ for every $x \in [-1, 1]$.
   (a) Prove that f is differentiable on $(-1, 1)$ and that $f'(x) = -\frac{1}{\sqrt{1-x^2}}$ for every $-1 < x < 1$.
   (b) Prove that f is not differentiable at any of the points $x_0 = -1, 1$.

2. Prove that the function $f(x) = (x \cdot \ln x)^{\ln x}$ is differentiable for every $x > 1$ and compute the value of $f'(x)$ for every $x > 1$.

3. (a) Let $L \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be an odd function, such that $\lim_{x\to\infty} g(x) = L$. Prove that $\lim_{x\to-\infty} g(x) = -L$.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.
   Consider the function defined by $g(x) = f(x) - f(-x)$ for every $x \in \mathbb{R}$. Suppose that $\lim_{x\to\infty} g(x) = 0$.
   Conclude, using exercise 10 question 5b, that there exists an $x_0 \in \mathbb{R}$ for which $g'(x_0) = 0$.

(c) Prove that there exists an $x_1 \in \mathbb{R}$ for which $f'(x_1) = 0$.

4. Prove or disprove each of the following statements:

   (a) Let $f : [a, b] \to \mathbb{R}$. Suppose that b is a minimum in $[a, b]$ of f and that f is differentiable at b. Then $f'(b) \leq 0$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} x^{\frac{3}{2}} \sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$.

   Then f' is continuous.

   (c) There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ for which

   $$f'(x) = \begin{cases} \frac{\cos x - 1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

   (d) Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function for which $f'(x) \cdot f'\left(\frac{1}{x}\right) < 0$ for every $0 < x \neq 1$.
   Then $f'(1) = 0$.

   (e) Let $x_0 \in \mathbb{R}$ and let f be a differentiable function on (x_0, b) for some $b > x_0$. If $\lim\limits_{x \to x_0^+} f(x) = \infty$, then $\lim\limits_{x \to x_0^+} f'(x) = -\infty$.

5. Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic, if there exists a $0 < T \in \mathbb{R}$ such that $f(x + T) = f(x)$ for every $x \in \mathbb{R}$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic, differentiable function.

   (a) Prove that $f'(0) = f'(T)$.

   (b) Prove that there exist two distinct points $x_1, x_2 \in [0, T]$ for which $f'(x_1) = f'(x_2) = 0$.