Calculus 1 - Exercise 6

All exercises should be submitted by December 13th by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand.
These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Two students solved the following exercise in different ways. Who was right and who was wrong? Explain.
   Student A:
   $$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(x\sqrt{1+\frac{1}{x}}-x\right)\underset{\lim\limits_{x\to\infty}\sqrt{1+\frac{1}{x}}=1}{=}\lim_{x\to\infty}(x-x)=0$$
   Student B:
   $$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}\right)=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}$$
2. Prove each of the following statements:
   (a) $\lim\limits_{x\to\infty}\sqrt[3]{x^2-1}=\infty$.
   (b) $\lim\limits_{x\to 1^+}\sqrt[3]{x^2-1}=0$.

3. Let $L \in \mathbb{R}$ and let f, g be two functions defined on a deleted neighborhood of $x_0 \in \mathbb{R}$. Prove each of the following statements:

   (a) If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = \infty$, then $\lim_{x \to x_0} (f(x) + g(x)) = \infty$.

   (b) If $\lim_{x \to x_0} f(x) = L > 0$ and $\lim_{x \to x_0} g(x) = \infty$, then $\lim_{x \to x_0} (f(x) \cdot g(x)) = \infty$.

4. Let f be a function that is defined on a deleted neighbourhood of $x_0 \in \mathbb{R}$, such that $f(x) \neq 0$ for every x on that deleted neighbourhood. Suppose that $\lim_{x \to x_0} \left(f(x) + \frac{1}{|f(x)|} \right) = 0$.

   (a) Prove that there exists a $\delta > 0$ such that for every $0 < |x - x_0| < \delta$, we have $f(x) < 0$.
   (Hint: prove that for every $y > 0$, $y + \frac{1}{y} \geq 2$)

   (b) Set $g(x) = f(x) + \frac{1}{|f(x)|}$ for every $0 < |x - x_0| < \delta$. Find an expression for $f(x)$ as a function of $g(x)$ for every $0 < |x - x_0| < \delta$.

   (c) Conclude that $\lim_{x \to x_0} f(x) = -1$.

5. Prove or disprove each of the following claims:

   (a) There exists a function f that is defined on a deleted neighbourhood of $x_0 \in \mathbb{R}$, such that $f(x) \neq 0$ for every x on that deleted neighbourhood and $\lim_{x \to x_0} \left(f(x) + \frac{1}{f(x)} \right) = 0$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| = |x - y|$. Then $\lim_{x \to \infty} |f(x)| = \infty$.