Calculus 1 - Exercise 7

All exercises should be submitted by December 20th by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand.
These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Define each of the following terms. Give a full and complete mathematical definition, without using any negation symbol, except for maybe $\neq$.
   If your definition includes a secondary term that was defined in class, you must define it as well! You do not have to define any term that appears prior to the words "Define the term".
   Let $f : \mathbb{R} \to \mathbb{R}$.
   (a) Define the term: f is not bounded.
   (b) Define the term: $\lim\limits_{x \to -\infty} f(x)$ is not $-\infty$.

2. Let $f : (0, \infty) \to \mathbb{R}$ be the function given by
$$f(x) = \frac{1}{x^2} + \frac{2}{x^2} + \ldots + \frac{\lfloor x \rfloor}{x^2}, \quad \forall x > 0$$
   (a) Prove, by using the formula $1 + 2 + ... + n = \frac{n(n+1)}{2}$, $\forall n \in \mathbb{N}$ and the squeeze rule, that $\lim\limits_{x \to \infty} f(x) = \frac{1}{2}$.

(b) Student "A" (from HW 6) proved that $\lim_{x\to\infty} f(x) = 0$ in the following way:
"For every $x \geq 1$, and for every $1 \leq k \leq \lfloor x \rfloor$ we have

$$0 \leq \frac{k}{x^2} \leq \frac{\lfloor x \rfloor}{x^2} \leq \frac{1}{x}$$

and therefore, by the squeeze rule, $\lim_{x\to\infty} \frac{k}{x^2} = 0$ for every $k \in \mathbb{N}$. Therefore, by the algebra of limits, we have

$$\lim_{x\to\infty} f(x) = 0 + 0 + ... + 0 = 0$$

"

Explain student A's mistake.

3. Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be an increasing function.

(a) Prove that there exists an $\underline{s} \in \mathbb{R}$ such that $\underline{s} = \inf \{f(x) \mid a < x \leq b\}$.

(b) Prove that $\lim_{x\to a^+} f(x)$ exists, and is equal to $\underline{s}$.

(c) Give an example where $\lim_{x\to a^+} f(x) \neq \inf \{f(x) \mid a \leq x \leq b\}$.

4. (The Inverse squeeze rule) Let f, g, h be three functions defined on a deleted neighbourhood of $x_0 \in \mathbb{R}$ and $L \in \mathbb{R}$. Suppose that:

(a) $f(x) \leq h(x) \leq g(x)$ for every x in that deleted neighbourhood.

(b) $\lim_{x\to x_0} h(x) = L$.

(c) $\lim_{x\to x_0} (g(x) - f(x)) = 0$.

Prove that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist and that

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = L$$

5. Prove or disprove each of the following arguments:

(a) There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\infty} \frac{f(x)}{x} = 3$ and $\lim_{x\to\infty} \frac{f(x)\cdot\sin x}{x^2} = 6$.

(b) Let f, g be two functions that are defined on a deleted neighbourhood of zero. Suppose that $xf(x) \leq g(x) \leq x$ for every x in the that neighbourhood. Then $\lim_{x\to 0} g(x) = 0$.

(c) Let f, g be two functions defined on a deleted neighbourhood of $x_0 \in \mathbb{R}$. If $g(x) \neq 0$ for every x on the deleted neighbourhood and $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$, then $\lim_{x \to x_0} (f(x) - g(x)) = 0$.

(d) Let f, g be two functions defined on $[a, \infty)$ for some $a \in \mathbb{R}$.
If $\lim_{x \to \infty} [f(x) \cdot g(x)] = \infty$, then $\lim_{x \to \infty} |f(x)| = \infty$ or $\lim_{x \to \infty} |g(x)| = \infty$.