Calculus 1-Exercise 10

All exercises should be submitted by January 10th by 23:00.
Delays won't be accepted aside for special cases which will be approved beforehand. These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Define each of the following terms. Give a full and complete mathematical definition, without using any negation symbol, except for maybe $\neq$.
   If your definition includes a secondary term that was defined in class, you must define it as well! You do not have to define any term that appears prior to the words "Define the term".
   (a) Let $f : [a, b] \to \mathbb{R}$, and let $x_0 \in [a, b]$. Define the term: x_0 is not an extreme point of f in $[a, b]$.
   (b) Let $f : (a, b) \to \mathbb{R}$, and let $x_0 \in (a, b)$. Define the term: f is not differentiable at x_0.
2. Let f be a function that is defined on a neighbourhood of $x_0 \in \mathbb{R}$. For each of the following cases, determine whether f is differentiable at x_0, and compute the value of $f'(x_0)$, or prove that it doesn't exist.
   (a) $f(x) = \left|x^2 - 3x - 4\right|$, $x_0 = 4$.

(b) $f(x) = \sqrt[3]{x^2 - |\, x \,|}$, $x_0 = 0$.

(c) $f(x) = \begin{cases} \frac{\sin(x^2)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x_0 = 0$.

3. Let f be a function that is continuous on $[a, b]$ for some $a < b$. Suppose that $f(a) = f(b) = 0$, and let $M = \sup\{f(x) : x \in [a, b]\}$.

   (a) Explain why M is well-defined.

   (b) Suppose that $M > 0$, and let $0 \leq r < M$. Prove that the set $\{x \in [a, b] \; : \; f(x) = r\}$ contains at least two distinct points.

4. Prove or disprove each of the following statements:

   (a) Let f be function that is continuous on $[0, 1]$. Suppose that $f(x) > 0$ for every $x \in [0, 1]$.
   Then there exists an $\epsilon > 0$ such that $f(x) > \epsilon$ for every $x \in [0, 1]$.

   (b) Let f be a function defined on a neighbourhood of 1, and assume that f is differentiable at 1.
   If $f(1) = 2$ and $f'(1) = 3$, then $\lim\limits_{x \to 1} \frac{f^2(x) - 2f(x)}{x - 1} = 6$.

   (c) Let f, g be two functions that are defined on a neighborhood of 0 such that $f(x) = x \cdot g(x)$ for every x in that neighborhood.
   Then f is differentiable at 0 if and only if $\lim\limits_{x \to 0} g(x)$ exists.

   (d) Let f be a function that is continuous on (a, b) for some $a < b$.
   Suppose that a and b are removable discontinuities of f.
   Then f is bounded in (a, b).

   (e) Let f be a function that is defined on a neighbourhood of 0.
   Suppose that $\lim\limits_{x \to 0} \frac{f(x)}{|x|} = 1$. Then f is not differentiable at 0.

5. Let $L_1, L_2 \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x \to \infty} f(x) = L_1$ and $\lim\limits_{x \to -\infty} f(x) = L_2$.

   (a) Give an example of a function satisfying the above conditions, and such that f doesn't have any extreme point in $\mathbb{R}$ (neither a maximum nor a minimum).

   (b) Assume that $L_1 = L_2$. Prove that f has an extreme point (maximum or minimum) in $\mathbb{R}$.