Calculus 1- Exercise 4

All exercises should be submitted by November 29th by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand.
These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

<u>Questions:</u>

1. Define each of the following terms. Give a full and complete mathematical definition, without using any negation symbol, except for maybe $\neq$.
   If your definition includes a secondary term that was defined in class, you must define it as well! You do not have to define any term that appears prior to the words "Define the term".
   Let f be a function defined on $(-\infty, b]$ for some $b \in \mathbb{R}$.
   (a) Define the term: 5 is not a limit of f as $x \to -\infty$.
   (b) Define the term: the limit $\lim\limits_{x \to -\infty} f(x)$ does not exist.
2. Prove each of the following claims:
   (a) For every $x \in \mathbb{R}$, we have $\left|-3x+3\right| \geq \left|\left|x+2\right| - \left|5-2x\right|\right|$.
   (b) For every $x, y \in \mathbb{R}$, we have $\min\{x, y\} = \frac{1}{2}(x + y - |x-y|)$.
   (c) For every $x \in \mathbb{R}$ such that $\left|x-7\right| < \frac{1}{2}$, we have $\left|\frac{x^2-4x-21}{x-6}\right| < \frac{21}{2}$.

(d) For every $L, y \in \mathbb{R}$ and for every $r \geq 0$, we have
$\mid y - L \mid \leq r$ if and only if $L - r \leq y \leq L + r$.

(e) For every $a, b, x \in \mathbb{R}$ such that $a \leq x \leq b$, we have $\mid x \mid \leq \max \{\mid a \mid, \mid b \mid\}$.

Hint: denote $r = \max \{\mid a \mid, \mid b \mid\}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and suppose that $\lim\limits_{x \to \infty} f(x) = 4$.
Prove that there exists an $M > 0$ such that for every $x > M$, we have $15.5 < [f(x)]^2 < 16.5$.

4. Prove each statement by using the definition of the corresponding limit. In items (a) and (b), find a value of $M > 0$ (or $M < 0$ when $x \to -\infty$), such that for every $x > M$ (or $x < M$) , $|f(x) - L| < \frac{1}{100}$.

(a) $\lim\limits_{x \to \infty} \frac{2x+5}{-x+2} = -2$

(b) $\lim\limits_{x \to -\infty} \frac{2x+5}{-x+2} = -2$

(c) $\lim\limits_{x \to -\infty} \left(\sqrt{x^2 - 4x + 3} + x\right) = 2$

(d) $\lim\limits_{x \to -\infty} \frac{5}{\lfloor x \rfloor} = 0$

Hint: for every $x \in \mathbb{R}$, we have $x - 1 \leq \lfloor x \rfloor \leq x$.

5. Let f be a function that is defined on $[a, \infty)$ for some $a \in \mathbb{R}$ and let $L \in \mathbb{R}$. Prove or disprove each of the following claims:

(a) $\lim\limits_{x \to \infty} f(x) = L$ if and only if there exists an $M > 0$ such that for every $\epsilon > 0$ and every $x > M$, we have $|f(x) - L| < \epsilon$.

(b) $\lim\limits_{x \to \infty} f(x) = L$ if and only if for every $0 < \epsilon \leq 0.04$ there exists an $M \geq 70$ such that for every $x > M$, we have $|f(x) - L| < \epsilon$.