Calculus 1 – Exercise 5

All exercises should be submitted by December 6 by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand. These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Prove each statement by only using the corresponding definition of the limit:
   (a) $\lim\limits_{x\to 3}\left|2-\frac{1}{x}\right|=\frac{5}{3}$
   (b) $\lim\limits_{x\to -3}\frac{x^2-5x}{x-3}=-4$
2. Compute each of the following limits, or prove that it doesn't exist:
   (a) $\lim\limits_{x\to 2}\frac{\sqrt{2x^2+8}-4}{(x-2)\cdot(x^2+2x+4)}$
   (b) $\lim\limits_{x\to 1}\left(\frac{2}{1-x^2}-\frac{3}{1-x^3}\right)$
   (c) $\lim\limits_{x\to -\infty}\left(\sqrt{4x^2-6x}-\sqrt{4x^2-8x}\right)$

3. Let $A \subseteq \mathbb{R}$ and let $a, b \in \mathbb{R}$. Suppose that A and $\mathbb{R} \setminus A$ are dense in $\mathbb{R}$ and that $a \neq b$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a & x \in A \\ b & x \notin A \end{cases}$$

Prove that the limit $\lim\limits_{x \to \infty} f(x)$ does not exist.

4. Let f, g be two functions defined on a deleted neighbourhood of $x_0 \in \mathbb{R}$ and let $L \in \mathbb{R}$. Prove or disprove each of the following claims:

   (a) $\lim\limits_{x \to x_0} f(x) = L$ if and only if for every $0 \neq \alpha \in \mathbb{R}$ $\lim\limits_{x \to x_0} [\alpha \cdot f(x)] = \alpha \cdot L$.

   (b) If $\lim\limits_{x \to x_0} (f(x) + g(x)) = L$, then $\lim\limits_{x \to x_0} f(x)$, $\lim\limits_{x \to x_0} g(x)$ both exist, or $\lim\limits_{x \to x_0} f(x)$, $\lim\limits_{x \to x_0} g(x)$ both do not exist.

   (c) Let $h : \mathbb{R} \to \mathbb{R}$. If $\lim\limits_{x \to a} h(x) = L$ for every $a \in \mathbb{R}$, then $h(x) = L$ for every $x \in \mathbb{R}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ and let $m \in \mathbb{R}$. Consider the set

$$A = \{x \in [a, b) \mid f(x) > m\}$$

   (a) Suppose that $A \neq \phi$. Prove that $\underline{s} = \inf(A)$ exists and that $a \leq \underline{s} < b$.

   (b) Let $L \in \mathbb{R}$, and suppose that $\lim\limits_{x \to \underline{s}} f(x) = L$. Suppose in addition that $\underline{s} \notin A$. Prove that $L \geq m$.