Calculus 1- Exercise 3

All exercises should be submitted by November 22th by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand.
These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Define each of the following terms. Give a full and complete mathematical definition, without using any negation symbol, except for maybe $\neq$.
   If your definition includes a secondary term that was defined in class, you must define it as well! You do not have to define any term that appears prior to the words "Define the term". Let $\phi \neq A \subseteq \mathbb{R}$.
   (a) Define the term: A has no minimal value.
   (b) Let $M \in \mathbb{R}$ be an upper bound of A. Define the term: M is not the least upper bound of A.
   (c) Define the term: A is not dense in $\mathbb{R}$.

2. For each of the following sets, compute $\inf A$, $\sup A$, $\min A$, $\max A$ or prove that it does not exist.
   (a) $A = \left\{ \frac{2n+(-1)^n}{n+2} \middle| n \in \mathbb{N} \right\}$
   (b) $A = \left\{ \frac{n}{nm+1} \middle| n, m \in \mathbb{N} \right\}$

3. Let $\phi \neq A, B \subseteq \mathbb{R}$ be two bounded sets.
Suppose that $\forall a \in A \exists b \in B \,:\, a \leq b$, and $\forall b \in B \exists a \in A \,:\, b \leq a$.

   (a) Explain why $\sup(A)$, $\sup(B)$ exist (i.e., are real numbers) and prove that $\sup(A) = \sup(B)$.

   (b) Find an example for $\phi \neq A, B \subseteq \mathbb{R}$ satisfying the above conditions, and such that $\inf(A) \neq \inf(B)$.

4. Let $\phi \neq A, B \subseteq (0, \infty)$ be two sets that are bounded from above.
We define
$$\frac{A}{B} = \left\{ \frac{a}{b} \middle| \, a \in A, \, b \in B \right\}$$

   (a) Give an example for such sets A and B for which the set $\frac{A}{B}$ is not bounded.

   (b) Prove that $\frac{A}{B}$ is bounded if and only if $\inf B > 0$.

   (c) Assume in addition that B has a minimum.
   Prove that $\sup\left(\frac{A}{B}\right) = \frac{\sup A}{\min B}$.

5. Let $\phi \neq A, B \subseteq \mathbb{R}$. We say that A *is dense in* B, if $A \subseteq B$, and, in addition, if for every $x, y \in B$ such that $x < y$ there exists an $a \in A$ such that $x < a < y$.
Prove or disprove each of the following statements:

   (a) If A is dense in B and B is dense in $\mathbb{R}$, then A is dense in $\mathbb{R}$.

   (b) If A is dense in $(0, 1)$, then A is dense in $[0, 1]$.

   (c) If A is dense in $\mathbb{R}$, then $A \cap \mathbb{Q}$ is dense in $\mathbb{R}$ or $A \cap (\mathbb{R} \setminus \mathbb{Q})$ is dense in $\mathbb{R}$.

   (d) The set
   $$A = \left\{ \frac{n}{m^2} \middle| \, n \in \mathbb{Z}, m \in \mathbb{N} \right\}$$
   is dense in $\mathbb{R}$.