Calculus 1- Exercise 12

All exercises should be submitted by January 22th by 23:00. Delays won't be accepted aside for special cases which will be approved beforehand. These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

<u>Questions:</u>

1.

(a) Prove that the equation

$$x \cdot \left(1 + \sqrt{x^2 + 1}\right)^3 = \frac{1}{2}$$

has a unique solution in $\mathbb{R}$.

(b) Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ such that $a_1, a_2, a_3 > 0$ and $b_1 < b_2 < b_3$. Prove that the equation

$$\frac{a_1}{x - b_1} + \frac{a_2}{x - b_2} + \frac{a_3}{x - b_3} = 0$$

has exactly two distinct solutions in $\mathbb{R}$.

2. Compute each one of the following limits or prove that it doesn't exist:

   (a) $\lim\limits_{x\to 0} \frac{x^2 - x\sin x}{(e^x - 1)\cdot\ln(1+x)}$

   (b) $\lim\limits_{x\to 0^+} \left(x^2 \cdot \ln x\right)$

   (c) $\lim\limits_{x\to 0^-} \left(\frac{1}{x} \cdot e^{\frac{1}{x}}\right)$

   (d) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{\tan x}}$

3. Let $L \in \mathbb{R}$ and let f be a function that is differentiable on a deleted neighbourhood of $x_0 \in \mathbb{R}$ such that $\lim\limits_{x\to x_0} f'(x) = L$.

   (a) Give an example of a function satisfying the above conditions, and such that f is not differentiable at x_0.

   (b) Assume in addition that f is continuous at x_0 and prove that f is differentiable at x_0 and $f'(x_0) = L$.

4. Let $f : [0,\infty) \to \mathbb{R}$ be a differentiable function.
   Assume that f' is increasing and $f(0) = 0$.

   (a) Prove that for every $x > 0$ we have $\frac{f(x)}{x} \le f'(x)$.

   (b) Let $g(x) = \frac{f(x)}{x}$ for every $x > 0$. Prove that g is increasing on $(0,\infty)$.

5. Prove or disprove each of the following statements:

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If the equation $f'(x) = 0$ has exactly one solution, then the equation $f(x) = 0$ has at least two solutions.

   (b) Let $a \in \mathbb{R}$ and let $f, g : (-\infty, a] \to \mathbb{R}$ be two differentiable functions. Suppose that $f(a) \le g(a)$ and $f'(x) > g'(x)$ for every $x \in (-\infty, a]$. Then $f(x) < g(x)$ for every $x < a$.

   (c) For every $x > 0$, $\arctan(x) < x$.

   (d) The function

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

   is differentiable.

(e) We have $1.984^{2.020} < 2.020^{1.984}$.

Hint: Consider the function $f(x) = x^{\frac{1}{x}}$ for every $x > 0$.

(f) Let $f : [a, b) \to \mathbb{R}$ be a differentiable function such that $\lim\limits_{x \to b^-} f(x)$ does not exist in the extended sense.
Then there exists an $x_0 \in [a, b)$ for which $f'(x_0) = 0$.