Calculus 1 - Exercise 9

December 25, 2020

All exercises should be submitted by January 3rd by 23:00.
Delays won't be accepted aside for special cases which will be approved beforehand. These are the submission regulations (also available on the Moodle):

1. Exercises are personal and cannot be submitted in groups.
2. Write your name, ID and tutorial group in the header of the exercise.
3. They should be written clearly on A4 pages. Hard-to-read exercises will not be graded.
4. Serious effort has to be shown by the student. Unreadable or extremely partial answers will be disregarded.
5. Exercises submitted late without the TA's approval will not be accepted.

Questions:

1. Define each of the following terms. Give a full and complete mathematical definition, without using any negation symbol, except for maybe $\neq$.
   If your definition includes a secondary term that was defined in class, you must define it as well! You do not have to define any term that appears prior to the words "Define the term".
   (a) Let $f : \mathbb{R} \to \mathbb{R}$. Define the term: f is not continuous on (a, b).
   (b) Let f be a function defined on a deleted neighbourhood of $x_0 \in \mathbb{R}$.
   Define the term: x_0 is an essential discontinuity.

2.
   (a) Let $2 < r < 4$. Prove that there exists an $1 < x < 4$ for which $3x - x^{3/2} = r$.
   (b) Let $0 < \alpha < 2$. Prove that there exists an $0 < x < 1$ for which $2^x = \frac{\alpha}{x}$.
   (c) Let $f : [0, 1] \to [0, 1]$ continuous. Assume that $f(0) = f(1)$.
   Prove that there exists an $0 \leq x \leq \frac{1}{2}$ such that $f(x) = f\left(x + \frac{1}{2}\right)$.

3. Prove or disprove each of the following statements:
   (a) Let f be a function that is continuous continuous at $x_0 \in \mathbb{R}$. Then $g = f \cdot D$ is continuous at x_0 if and only if $f(x_0) = 0$, where $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.
   (b) There exists an $0.33 < x < 0.34$ for which $\ln(x)$ is irrational.
   (c) Let $f : (0, \infty) \to \mathbb{R}$ be a function for which $f(x) \cdot f\left(\frac{1}{x}\right) < 0$ for every $0 < x \neq 1$.
   Suppose that f is continuous at $x_0 = 1$, then $f(1) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function that is not constant, and let $M = \sup Im(f)$, $m = \inf Im(f)$. Prove that $(m, M) \subseteq Im(f) \subseteq [m, M]$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $\mathbb{Z}$ and discontinuous on $\mathbb{R}\backslash\mathbb{Z}$ (that is, continuous at x_0 if and only if $x_0 \in \mathbb{Z}$).

   (a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the above requirements.

   (b) Student A has proven that there is no such function in the following way:
   Suppose, by way of contradiction, that such a function exists. Let $n \in \mathbb{Z}$.
   As f is continuous at n, then there exists a $\delta_1 > 0$ such that for every $|x - n| < \delta_1$ we have

$$|f(x) - f(n)| < \boxed{\frac{\epsilon_0}{2}}$$

   As $\mathbb{R}\backslash\mathbb{Z}$ is dense, there exists an $x_1 \in \mathbb{R}\backslash\mathbb{Z}$ such that $|x_1 - n| < \delta_1$ which implies that

$$|f(x_1) - f(n)| < \boxed{\frac{\epsilon_0}{2}}$$

   As f is not continuous at x_1, then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $|x - x_1| < \delta$ but $|f(x) - f(x_1)| \geq \epsilon_0$.

   Choose $\delta > 0$ to be small enough such that $n - \delta_1 < x_1 - \delta$ and $x_1 + \delta < n + \delta_1$ (surely, there exists such a δ). Then there exists an $x_2 \in \mathbb{R}$ such that $|x_2 - x_1| < \delta$ but $|f(x_2) - f(x_1)| \geq \epsilon_0$.

   Note that $n - \delta_1 < x_1 - \delta < x_2 < x_1 + \delta < n + \delta_1$ and therefore $|x_2 - n| < \delta_1$, which implies that

$$|f(x_2) - f(n)| < \boxed{\frac{\epsilon_0}{2}}$$

   By the 'trick' we obtain

$$|f(x_2) - f(x_1)| \leq |f(x_2) - f(n)| + |f(x_1) - f(n)| < \boxed{\frac{\epsilon_0}{2}} + \boxed{\frac{\epsilon_0}{2}} = \epsilon_0$$

   Contradiction!

   Find and explain student A's mistake.