2D D_8 LGT

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1 Gauss' Laws

We write two commuting and independent Gauss' laws:

$$\prod_i \theta^L_{a^2x}(x,i)\theta^{R\dagger}_{a^2x}(x-e_i,i)\left|\psi\right\rangle = e^{i\pi\eta_0^\dagger(x)\eta_0(x)+s(x)}\left|\psi\right\rangle$$
$$\prod_i \theta^L_{x}(x,i)\theta^{R\dagger}_{x}(x-e_i,i)\left|\psi\right\rangle = e^{i\pi\eta_1^\dagger(x)\eta_1(x)+s(x)}\left|\psi\right\rangle$$

form now on I'll call $a^2x = 0$ and $x = 1$. Moreover note that: $\theta^{R\dagger} = \theta^R$ and that $s(x)$ come from the staggering. We can rewrite the Gauss' laws in an easier way working on the RHS:

$$\begin{aligned}
e^{i\pi\eta_0^\dagger(x)\eta_0(x)+s(x)} &= (-1)^{s(x)}e^{i\pi\eta_0^\dagger(x)\eta_0(x)} = (-1)^{s(x)}\sum_{k=0}\frac{(i\pi n_0)^k}{k!} \\
&= (-1)^{s(x)}\left[1+\sum_{k=1}\frac{(i\pi)^k}{k!}n_0\right] \qquad \text{as } n_0 = n_0^k \\
&= (-1)^{s(x)}\left[1+\sum_{k=0}\frac{(i\pi)^k}{k!}n_0 - n_0\right] \\
&= (-1)^{s(x)}\left[1+e^{i\pi}n_0 - n_0\right] = (-1)^{s(x)}(1-2n_0)
\end{aligned}$$

Hence defining $G = \prod_i \theta^L_{a^2x}(x,i)\theta^{R\dagger}_{a^2x}(x-e_i,i)$ we have that:

$$G = (-1)^{s(x)}(1-2n_0) \tag{1}$$
$$(-1)^{s(x)}n_0 = \frac{(-1)^{s(x)}-G}{2} \tag{2}$$

and this is useful for the realization of the transformation and also for rewriting the matter term as we would do later.
Similarly to what Guy did in the 1D case we can extend some definitions :

$$\begin{aligned}
&\hat{\Theta}_m(x) = \theta^L_{m,h}(x)\theta^L_{m,v}(x)\theta^R_{m,h}(x-e_1)\theta^R_{m,v}(x-e_2) \\
&P^{\pm}_m = \frac{1}{2}\left[1\mp\epsilon(x)\hat{\Theta}_m(x)\right] \\
&P^+_m - P^-_m = -\epsilon(x)\hat{\Theta}_m(x) \\
&\hat{\Theta}_1(x)\hat{\Theta}_0(x) = \hat{\Theta}_0(x)\hat{\Theta}_1(x) = \Pi_v(x)\Pi_h(x)\Pi_h(x-e_1)\Pi_v(x-e_2) \\
&\Pi^2_r(x) = 1 \qquad \Pi_r(x)\theta^{L/R}_{0,r}(x) = \theta^{L/R}_{1,r}(x) \qquad r = h,v
\end{aligned}$$

2 Hamiltonian under transformation: hard-core bosons + JW

I generalize the interesting things for 2D from Guy's note.
Considering 1 we now that the epsilon terms are given by (using the "pi" notation).

$$\begin{aligned}\xi_h &= \Pi_v(x)\Pi_h(x-e_1)\Pi_v(x-e_2)\Pi_v(x+e_1-e_2)\\ \xi_v &= \Pi_h(x-e_1)\Pi_v(x-e_2)\end{aligned}$$

Hence, after transformation of [1] + local Jordan-Wigner, the parts of the hamiltonian are almost the same of the ones that Guy found. Now now instead of only one "pi" we will have 4 or 2 depending on the horizontal or vertical case.

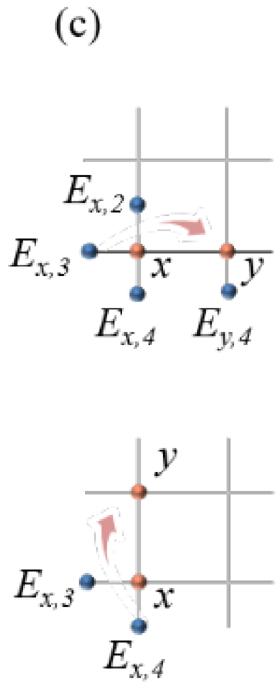


Figure 1: Convention from *Zohar-Cirac 2018*

2.1 Horizontal

$$\begin{aligned}H^h_{00} &= \xi_h\tau_0^+(x)U_{00}(x,h)\tau_0^-(x+e_1)\\ H^h_{01} &= -\xi_h\tau_0^+(x)U_{01}(x,h)\tau_0^z(x+e_1)\tau_1^-(x+e_1)\\ H^h_{10} &= -\xi_h\tau_1^+(x)\tau_0^z(x)U_{10}(x,h)\tau_0^-(x+e_1)\\ H^h_{11} &= \xi_h\tau_1^+(x)\tau_0^z(x)U_{11}(x,h)\tau_0^z(x+e_1)\tau_1^-(x+e_1)\end{aligned}$$

2.2 Vertical

$$\begin{aligned}H^v_{00} &= \xi_v\tau_0^+(x)U_{00}(x,v)\tau_0^-(x+e_2)\\ H^v_{01} &= -\xi_v\tau_0^+(x)U_{01}(x,v)\tau_0^z(x+e_2)\tau_1^-(x+e_2)\\ H^v_{10} &= -\xi_v\tau_1^+(x)\tau_0^z(x)U_{10}(x,v)\tau_0^-(x+e_2)\\ H^v_{11} &= \xi_v\tau_1^+(x)\tau_0^z(x)U_{11}(x,v)\tau_0^z(x+e_2)\tau_1^-(x+e_2)\end{aligned}$$

Now we apply the transformation of [2] to completely remove the matter. The transformation for the link operator (U) and the ones for the matter (sigma and tau matrices) are the same of what guy did in the 1D case.

Recall that in terms of gates we have:

$$U_{00} = 1 \cdot Z_{02} \qquad U_{01} = -Z \cdot Z_{13} \qquad U_{10} = 1 \cdot Z_{13} \qquad U_{11} = Z \cdot Z_{02} \qquad \Pi = 1 \cdot (X_{02} + X_{13}) \tag{3}$$

3 Rewriting terms with projectors

3.1 Horizontal

$$\begin{aligned}
\langle out| H^h_{00} |out\rangle &= \xi_h(x) P_0^+(x) U_{00}(x,h) P_0^+(x+e_1) \\
\langle out| H^h_{01} |out\rangle &= -\xi_h(x) P_0^+(x) U_{01}(x,h) \Bigg[P_0^+(x+e_1) - P_0^-(x+e_1) \Bigg] P_1^+(x+e_1) \\
\langle out| H^h_{10} |out\rangle &= -\xi_h(x) P_1^+(x) \Bigg[P_0^+(x) - P_0^-(x) \Bigg] U_{10}(x,h) P_0^+(x+e_1) \\
\langle out| H^h_{10} |out\rangle &= -\xi_h(x) P_1^+(x) \Bigg[P_0^+(x) - P_0^-(x) \Bigg] U_{11}(x,h) \Bigg[P_0^+(x+e_1) - P_0^-(x+e_1) \Bigg] P_1^+(x+e_1)
\end{aligned}$$

3.2 Vertical

$$\begin{aligned}
\langle out| H^v_{00} |out\rangle &= \xi_v(x) P_0^+(x) U_{00}(x,v) P_0^+(x+e_2) \\
\langle out| H^v_{01} |out\rangle &= -\xi_v(x) P_0^+(x) U_{01}(x,v) \Bigg[P_0^+(x+e_2) - P_0^-(x+e_2) \Bigg] P_1^+(x+e_2) \\
\langle out| H^v_{10} |out\rangle &= -\xi_v(x) P_1^+(x) \Bigg[P_0^+(x) - P_0^-(x) \Bigg] U_{10}(x,v) P_0^+(x+e_2) \\
\langle out| H^h_{10} |out\rangle &= -\xi_v(x) P_1^+(x) \Bigg[P_0^+(x) - P_0^-(x) \Bigg] U_{11}(x,v) \Bigg[P_0^+(x+e_2) - P_0^-(x+e_2) \Bigg] P_1^+(x+e_2)
\end{aligned}$$

4 Expliciting the terms

4.1 Horizontal

- $$\begin{aligned}
H^h_{00} &= \frac{1}{4} \Bigg[\xi_h(x) - \epsilon(x)\xi_h(x)\hat{\Theta}_0(x) \Bigg] U_{00}(x,h) \Bigg[1 - \epsilon(x+e_1)\hat{\Theta}_0(x+e_1) \Bigg] \\
&= \frac{1}{4} \Bigg[\xi_h(x) - \epsilon(x)\Pi_v(x+e_1-e_2)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2) \Bigg] U_{00}(x,h) \Bigg[1 - \epsilon(x+e_1)\hat{\Theta}_0(x+e_1) \Bigg]
\end{aligned}$$

  where we used:

  $$\begin{aligned}
  &\xi_h(x)\hat{\Theta}_0(x) = \Pi_v(x)\Pi_h(x)\Pi_h(x-e_2)\Pi_v(x+e_1-e_2) \cdot \theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{0,h}(x-e_1)\theta^R_{0,v}(x-e_2) \\
  &= \Pi_v(x+e_1+e_2)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)
  \end{aligned}$$

- $$\begin{aligned}
H^h_{01} = \frac{1}{4} \Bigg[\xi_h(x) - \epsilon(x)\Pi_v(x+e_1+e_2)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2) \Bigg] U_{01}(x,h) \cdot \\
\cdot \Bigg[\epsilon(x+e_1)\hat{\Theta}_0(x+e_1) - \Pi_h(x+e_1)\Pi_v(x+e_1)\Pi_h(x)\Pi_v(x+e_1-e_2) \Bigg]
\end{aligned}$$

- $$H^h_{10} = \frac{1}{4}\left[\epsilon(x)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2) - \Pi_h(x)\right]\Pi_v(x+e_1-e_2)U_{10}(x,h)\left[1-\epsilon(x+e_1)\hat{\Theta}_0(x+e_1)\right]$$

- $$H^h_{11} = \frac{1}{4}\left[\epsilon(x)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2) - \Pi_h(x)\right]\Pi_v(x+e_1-e_2)U_{11}(x,h)\cdot$$
$$\cdot\left[\epsilon(x+e_1)\hat{\Theta}_0(x+e_1) - \Pi_h(x+e_1)\Pi_v(x+e_1)\Pi_h(x)\Pi_v(x+e_1-e_2)\right]$$

4.2 Vertical

- $$H^v_{00} = \frac{1}{4}\left[\xi_v(x) - \epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\right]U_{00}(x,v)\left[1-\epsilon(x+e_2)\hat{\Theta}_0(x+e_2)\right]$$

- $$H^v_{01} = \frac{1}{4}\left[\xi_v(x) - \epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\right]U_{01}(x,v)\cdot$$
$$\cdot\left[\epsilon(x+e_2)\hat{\Theta}_0(x+e_2) - \Pi_h(x+e_2)\Pi_v(x+e_2)\Pi_h(x+e_2-e_1)\Pi_v(x)\right]$$

- $$H^v_{10}- = \frac{1}{4}\left[\epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2) - \Pi_h(x)\Pi_v(x)\right]U_{10}(x,v)\left[1-\epsilon(x+e_2)\hat{\Theta}_0(x+e_2)\right]$$

- $$H^v_{11} = \frac{1}{4}\left[\epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2) - \Pi_h(x)\Pi_v(x)\right]U_{10}(x,v)\cdot$$
$$\cdot\left[\epsilon(x+e_2)\hat{\Theta}_0(x+e_2) - \Pi_h(x+e_2)\Pi_v(x+e_2)\Pi_h(x+e_2-e_1)\Pi_v(x)\right]$$

5 Anti-hermitian terms

As Guy did for the 1d case, we want to consider only the terms that are anti-hermitian and that survives thanks to the presence of ’i”.

5.1 Method

:

- All the theta matrices (”pi” is also a theta matrix) used in the computation are hermitian (Note: not all the theta matrices are hermitian!).
- All the $U_{mn}(x)$ are hermitian matrix, so if we don’t act on it (and change it somehow) we can forget about that term.
- Hence, we have only to take care of the terms on site x where I have both U and theta matrices and see if I get an anti-hermitian matrix; also the index ”h/v” must be the same.
- I did this check using Matlab as a brute-force method + check with Guy later.

5.2 Horizontal

$$H^h_{00} = -i[\xi_h(x)U_{00}(x,h)\epsilon(x+e_1)\hat{\Theta}_0(x+e_1) + \epsilon(x)\Pi_v(x+e_1-e_2)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)U_{00}(x,h)]$$

$$H^h_{01} = i[-\xi_h(x)U_{01}(x,h)\Pi_h(x+e_1)\Pi_v(x+e_1)\Pi_h(x)\Pi_v(x+e_1-e_2)+$$
$$+(-)\epsilon(x)\Pi_v(x+e_1-e_2)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)U_{01}(x,h)\epsilon(x+e_1)\hat{\Theta}_0(x+e_1)]$$

$$H^h_{10} = i[-\Pi_h(x)\Pi_v(x+e_1-e_2)U_{10}(x,h)+$$
$$+(-)\epsilon(x)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\Pi_v(x+e_1-e_2)U_{10}(x,h)\epsilon(x+e_1)\hat{\Theta}_0(x+e_1)]$$

$$H^h_{11} = i[-\epsilon(x)\theta^L_{0,h}(x)\theta^L_{1,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)U_{11}(x,h)\Pi_h(x+e_1)\Pi_v(x+e_1)\Pi_h(x)+$$
$$+(-)\Pi_h(x)\Pi_v(x+e_1-e_2)U_{11}(x,h)\epsilon(x+e_1)\hat{\Theta}_0(x+e_1)]$$

Reordering:

$$(-1)^x H^h_{00} = -i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{0,h}(x)U_{00}(x,h)\theta^L_{1,v}(x)\Pi_v(x+e_1-e_2)+$$
$$i\Pi_h(x-e_1)\Pi_v(x-e_2)\Pi_v(x)U_{00}(x,h)\theta^R_{0,h}(x)\theta^L_{0,h}(x+e_1)\theta^L_{0,v}(x+e_1)\theta^R_{1,v}(x+e_1-e_2)$$

$$H^h_{01} = i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{1,v}(x)\theta^L_{0,h}(x)U_{01}(x,h)\theta^R_{0,h}(x)\theta^L_{0,h}(x+e_1)\theta^L_{0,v}(x+e_1)\theta^R_{1,v}(x+e_1-e_2)$$
$$-i\Pi_h(x-e_1)\Pi_v(x-e_2)\Pi_v(x)U_{01}(x,h)\Pi_h(x)\Pi_h(x+e_1)\Pi_v(x+e_1)$$

$$H^h_{10} = i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{1,v}(x)\theta^L_{0,h}(x)U_{10}(x,h)\theta^R_{0,h}(x)\theta^L_{0,v}(x+e_1)\theta^L_{0,h}(x+e_1)\theta^R_{1,v}(x+e_1-e_2)$$
$$-i\Pi_h(x)U_{10}(x,h)\Pi_v(x+e_1-e_2)$$

$$(-1)^x H^h_{11} = i\Pi_h(x)U_{11}(x,h)\theta^R_{0,h}(x)\theta^L_{0,v}(x+e_1)\theta^L_{0,h}(x+e_1)\theta^R_{1,v}(x+e_1-e_2)$$
$$-i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{1,v}(x)\theta^L_{0,h}(x)U_{11}(x,h)\Pi_h(x)\Pi_h(x+e_1)\Pi_v(x+e_1)$$

Comment: we now have many-bodies interaction and, for instance, the first term of the H^h_{00} is a five-link-interaction while the second one is a seven-link interaction term. Remember that, in our formulation, every link is built out of a quibit and a qudit. So 7-link interaction doesn't imply 7-body interaction; it could be more than 7.

Example:

$$i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{0,h}(x)U_{00}(x,h)\theta^L_{1,v}(x)\Pi_v(x+e_1-e_2)$$

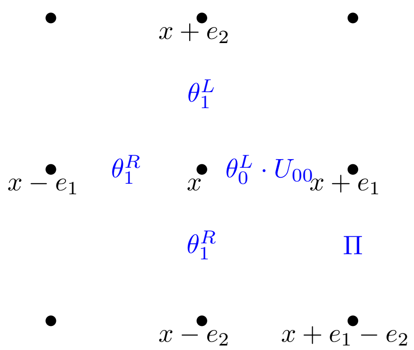


Figure 2: First interaction term of H^h_{00}

And we can do the same for the second:

$$-i\Pi_h(x-e_1)\Pi_v(x-e_2)\Pi_v(x)U_{00}(x,h)\theta^R_{0,h}(x)\theta^L_{0,h}(x+e_1)\theta^L_{0,v}(x+e_1)\theta^R_{1,v}(x+e_1-e_2)$$

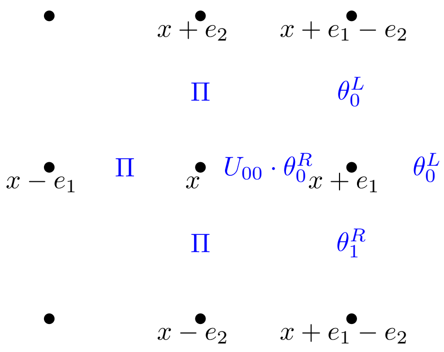


Figure 3: Second interaction term of H^h_{00}

5.3 Horizontal in terms of gates

Now we try to translate it in terms of qudits and qubits so that we can understand how many bodies are involved in the interaction. The first operator acts on the qubit while the second one on the qudit:

$$(-1)^x H^h_{00} = -\Big[X\cdot 1\Big]_{h,x-e_1}\times\Big[X\cdot 1\Big]_{v,x-e_2}\times\Big[X\cdot Y_{02}\Big]_{h,x}\times\Big[X\cdot(P_{02}+X_{13})\Big]_{v,x}\times\Big[1\cdot(X_{02}+X_{13})\Big]_{v,x+e_1-e_2}-$$
$$\Big[1\cdot(X_{02}+X_{13})\Big]_{h,x-e_1}\times\Big[1\cdot(X_{02}+X_{13})\Big]_{v,x-e_2}\times\Big[1\cdot(X_{02}+X_{13})\Big]_{v,x}\times\Big[X\cdot Y_{02}\Big]_{h,x}\times\Big[X\cdot(X_{02}+P_{13})\Big]_{h,x+e_1}\times$$
$$\times\Big[X\cdot(X_{02}+P_{13})\Big]_{v,x+e_1}\times\Big[X\cdot 1\Big]_{v,x+e_1-e_2}$$

Here we can immediately notice that the first term involves 4 quibits and 3 qudits while the second term 4 qubits and 6 qudits. So in total we have a 7-body interaction and a 10 body-interaction.

$$H^h_{01} = -\Big[X\cdot 1\Big]_{h,x-e_1}\times\Big[X\cdot 1\Big]_{v,x-e_2}\times\Big[X\cdot(P_{02}+X_{13})\Big]_{v,x}\times\Big[Z\cdot Y_{13}\Big]_{h,x}\times\Big[X\cdot(X_{02}+P_{13})\Big]_{h,x+e_1}\times$$
$$\times\Big[X\cdot(X_{02}+P_{13})\Big]_{v,x+e_1}\times\Big[X\cdot 1\Big]_{v,x+e_1-e_2}-\Big[1\cdot(X_{02}+X_{13})\Big]_{h,x-e_1}\times\Big[1\cdot(X_{02}+X_{13})\Big]_{v,x-e_2}$$
$$\times\Big[1\cdot(X_{02}+X_{13})\Big]_{v,x}\times\Big[Z\cdot Y_{13}\Big]_{h,x}\times\Big[1\cdot(X_{02}+X_{13})\Big]_{h,x+e_1}\times\Big[1\cdot(X_{02}+X_{13})\Big]_{v,x+e_1}$$

For the first term, we have 7 quibits and 4 qudits so it's a 11-body interaction. For the second term, we have 1 qubit and 6 qudits so it's a 7-body interaction term.

$$H^h_{10} = -\Big[X\cdot 1\Big]_{h,x-e_1}\times\Big[X\cdot 1\Big]_{v,x-e_2}\times\Big[X\cdot(P_{02}+X_{13})\Big]_{v,x}\times\Big[1\cdot Y_{13}\Big]_{h,x}\times\Big[X\cdot(X_{02}+P_{13})\Big]_{v,x+e_1}\times$$
$$\times\Big[X\cdot(X_{02}+P_{13})\Big]_{h,x+e_1}\times\Big[X\cdot 1\Big]_{v,x+e_1-e_2}-\Big[1\cdot Y_{13}\Big]_{h,x}\times\Big[1\cdot(X_{02}+X_{13})\Big]_{v,x+e_1-e_2}$$

Here we have that the first term involves 6 qubits and 4 qudits so is a 10-body interaction. The second term involves only 2 qudits, hence it's a 2-body interaction.

$$(-1)^x H^h_{11} = \Big[Y \cdot Z_{02}\Big]_{h,x} \times \Big[X \cdot (X_{02} + P_{13})\Big]_{v,x+e_1} \times \Big[X \cdot (X_{02} + P_{13})\Big]_{h,x+e_1} \times \Big[X \cdot 1\Big]_{v,x+e_1-e_2} +$$
$$\Big[X \cdot 1\Big]_{h,x-e_1} \times \Big[X \cdot 1\Big]_{v,x-e_2} \times \Big[X \cdot (P_{02} + X_{13})\Big]_{v,x} \times \Big[Y \cdot Z_{02}\Big]_{h,x} \times \Big[1 \cdot (X_{02} + X_{13})\Big]_{h,x+e_1} \times \Big[1 \cdot (X_{02} + X_{13})\Big]_{v,x+e_1}$$

Here we have that the first term involves 4 qubits and 3 qudits so is a 7-body interaction. On the other hand the second term involves 4 qubits and 4 qudits, hence it's 8-body interaction.

5.4 Vertical

$$H^v_{00} = -i\epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)U_{00}(x,v) - i\xi_v(x)U_{00}(x,v)\epsilon(x+e_2)\hat{\Theta}_0(x+e_2)$$

$$H^v_{01} = -i\xi_v(x)U_{01}(x,v)\Pi_h(x+e_2)\Pi_v(x+e_2)\Pi_v(x)\Pi_h(x+e_2-e_1)+$$
$$+(-)i\epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)U_{01}(x,v)\epsilon(x+e_2)\hat{\Theta}_0(x+e_2)$$

$$H^v_{10} = i\Pi_h(x)\Pi_v(x)U_{10}(x,v) - i\epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)U_{10}(x,v)\epsilon(x+e_2)\hat{\Theta}_0(x+e_2)$$

$$H^v_{11} = -i\epsilon(x)\theta^L_{0,h}(x)\theta^L_{0,v}(x)\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)U_{11}(x,v)\Pi_h(x+e_2)\Pi_v(x+e_2)\Pi_h(x+e_2-e_1)\Pi_v(x)+$$
$$+(-)i\Pi_h(x)\Pi_v(x)\epsilon(x+e_2)\hat{\Theta}_0(x+e_2)$$

Reordering:

$$(-1)^x H^v_{00} = -i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{0,h}(x)\theta^L_{0,v}(x)U_{00}(x,v)+$$
$$+i\Pi_h(x-e_1)\Pi_v(x-e_2)U_{00}(x,v)\theta^R_{0,v}(x)\theta^R_{0,h}(x-e_1+e_2)\theta^L_{0,h}(x+e_2)\theta^L_{0,v}(x+e_2)$$

$$H^v_{01} = -i\Pi_h(x-e_1)\Pi_v(x-e_2)U_{01}(x,v)\Pi_v(x)\Pi_h(x+e_2)\Pi_v(x+e_2)\Pi_h(x+e_2-e_1) + i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{0,h}(x)\cdot$$
$$\cdot\,\theta^L_{0,v}(x)U_{01}(x,v)\theta^R_{0,v}(x)\theta^L_{0,h}(x+e_2)\theta^L_{0,v}(x+e_2)\theta^R_{0,h}(x+e_2-e_1)$$

$$H^v_{10} = -i\Pi_h(x)\Pi_v(x)U_{10}(x,v) + i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{0,h}(x)\theta^L_{0,v}(x)U_{10}(x,v)\theta^R_{0,v}(x)\theta^L_{0,h}(x+e_2)\theta^L_{0,v}(x+e_2)\theta^R_{0,h}(x+e_2-e$$

$$(-1)^x H^v_{11} = -i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{0,h}(x)\theta^L_{0,v}(x)U_{11}(x,v)\Pi_v(x)\Pi_h(x+e_2)\Pi_v(x+e_2)\Pi_h(x+e_2-e_1)+$$
$$+i\Pi_h(x)\Pi_v(x)U_{11}(x,v)\theta^R_{0,v}(x)\theta^L_{0,h}(x+e_2)\theta^L_{0,v}(x+e_2)\theta^R_{0,h}(x+e_2-e_1)$$

Pictorial representation of the interaction terms of H^v_{00}:

$$-i\theta^R_{1,h}(x-e_1)\theta^R_{1,v}(x-e_2)\theta^L_{0,h}(x)\theta^L_{0,v}(x)U_{00}(x,v)+$$

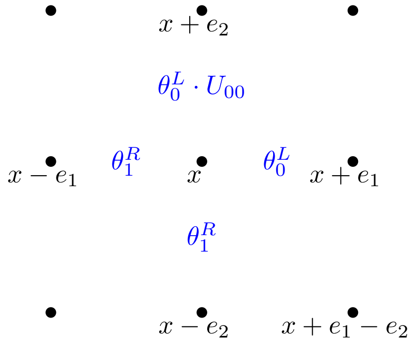


Figure 4: First interaction term of H_{00}^v

Second term:

$$+i\Pi_h(x-e_1)\Pi_v(x-e_2)U_{00}(x,v)\theta_{0,v}^R(x)\theta_{0,h}^R(x-e_1+e_2)\theta_{0,h}^L(x+e_2)\theta_{0,v}^L(x+e_2)$$

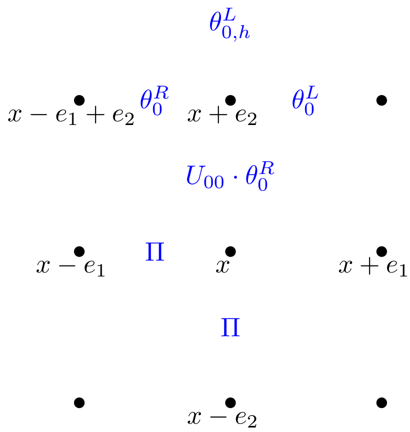


Figure 5: Second interaction term of H_{00}^v

5.5 Vertical in terms of gates

And in terms of gates we have:

$$(-1)^x H_{00}^v = -\Big[X \cdot 1\Big]_{h,x-e_1} \times \Big[X \cdot 1\Big]_{v,x-e_2} \times \Big[X \cdot (X_{02}+P_{13})\Big]_{h,x} \times \Big[X \cdot Y_{02}\Big]_{v,x} - \Big[1 \cdot (X_{02}+X_{13})\Big]_{h,x-e_1} \times$$
$$\Big[1 \cdot (X_{02}+X_{13})\Big]_{v,x-e_2} \times \Big[X \cdot Y_{02}\Big]_{v,x} \times \Big[X \cdot (X_{02}+X_{13})\Big]_{h,x-e_1+e_2} \times \Big[X \cdot (X_{02}+P_{13})\Big]_{h,x+e_2} \times \Big[X \cdot (X_{02}+P_{13})\Big]_{v,x+e_2}$$

We can see that here we have a 4 qubits + 2 qudits so a 6-body term and for the second one we need 4 qubits and 6 qudits so a total of 10-body interaction term.

$$H_{01}^v = -\Big[1 \cdot X_{02}+X_{13}\Big]_{h,x-e_1} \times \Big[1 \cdot X_{02}+X_{13}\Big]_{v,x-e_2} \times \Big[Z \cdot Y_{13}\Big]_{v,x} \times \Big[1 \cdot X_{02}+X_{13}\Big]_{h,x+e_2} \times \Big[1 \cdot X_{02}+X_{13}\Big]_{v,x+e_2} \times$$
$$\Big[1 \cdot X_{02}+X_{13}\Big]_{h,x+e_2-e_1} - \Big[X \cdot 1\Big]_{h,x-e_1} \times \Big[X \cdot 1\Big]_{v,x-e_2} \times \Big[X \cdot (X_{02}+P_{13})\Big]_{h,x} \times \Big[Z \cdot Y_{13}\Big]_{v,x} \times \Big[X \cdot (X_{02}+P_{13})\Big]_{h,x+e_2} \times$$
$$\times \Big[X \cdot (X_{02}+P_{13})\Big]_{v,x+e_2} \times \Big[X \cdot (X_{02}+X_{13})\Big]_{h,x+e_2-e_1}$$

Here we have a 1 qubits + 6 qudits so a 7-body term and for the second one we need 7 qubits and 5 qudits so it's 12-body interaction term.

$$H^v_{10} = -\Bigg[1\cdot X_{02}+X_{13}\Bigg]_{h,x}\times\Bigg[1\cdot Y_{13}\Bigg]_{v,x} - \Bigg[X\cdot 1\Bigg]_{h,x-e_1}\times\Bigg[X\cdot 1\Bigg]_{v,x-e_2}\times\Bigg[X\cdot(X_{02}+P_{13})\Bigg]_{h,x}\times\Bigg[1\cdot Y_{13}\Bigg]_{v,x}\times$$
$$\times\Bigg[X\cdot(X_{02}+P_{13})\Bigg]_{h,x+e_2}\times\Bigg[X\cdot(X_{02}+P_{13})\Bigg]_{v,x+e_2}\times\Bigg[X\cdot(X_{02}+X_{13})\Bigg]_{h,x+e_2-e_1}$$

Here we have 2 qudits so a 2-body term and, for the second one, we need 6 qubits and 5 qudits so it's 11-body interaction term.

$$(-1)^x H^v_{11} = \Bigg[X\cdot 1\Bigg]_{h,x-e_1}\times\Bigg[X\cdot 1\Bigg]_{v,x-e_2}\times\Bigg[X\cdot(X_{02}+P_{13})\Bigg]_{h,x}\times\Bigg[Y\cdot Z_{02}\Bigg]_{v,x}\times\Bigg[1\cdot(X_{02}+X_{13})\Bigg]_{h,x+e_2}\times$$
$$\times\Bigg[1\cdot(X_{02}+X_{13})\Bigg]_{v,x+e_2}\times\Bigg[1\cdot(X_{02}+X_{13})\Bigg]_{h,x+e_2-e_1}+\Bigg[1\cdot(X_{02}+X_{13})\Bigg]_{h,x}\times\Bigg[Y\cdot Z_{02}\Bigg]_{v,x}\times\Bigg[X\cdot(X_{02}+P_{13})\Bigg]_{h,x+e_2}\times$$
$$\times\Bigg[X\cdot(X_{02}+P_{13})\Bigg]_{v,x+e_2}\times\Bigg[X\cdot(X_{02}+X_{13})\Bigg]_{h,x+e_2-e_1}$$

Here we have 4 qubits and 5 qudits for both terms, so, they are 9-body interaction terms.

6 Electric part

In 2D we have:

$$\begin{aligned} H_E &= \lambda_E\Bigg[\sum_{h\ links}|2,mn\rangle\langle 2,mn| + \sum_{v\ links}|2,mn\rangle\langle 2,mn|\Bigg] \\ &\sum_{h\ links}\frac{\lambda_E}{2}\Bigg(1-\Pi_h\Bigg)+\sum_{v\ links}\frac{\lambda_E}{2}\Bigg(1-\Pi_v\Bigg) \qquad \text{or in terms of x} \\ &\sum_x\frac{\lambda_E}{2}\Bigg(2-\Pi_h(x)-\Pi_v(x)\Bigg) \qquad \text{we don't care about constants} \\ &\sum_x -\frac{\lambda_E}{2}\Bigg(\Pi_h(x)+\Pi_v(x)\Bigg) \end{aligned}$$

6.1 Translation into gates

$$H_E = -\frac{\lambda_E}{2}\sum_x\Bigg[1\cdot(X_{02}+X_{13})\Bigg]_{h,x}+\Bigg[1\cdot(X_{02}+X_{13})\Bigg]_{v,x}$$

7 Magnetic part

Again, we use the convention of the paper "Zohar-Cirac 2018". Nothing change under the transformation of 2019 so the magnetic part remains:

$$H_B = \lambda_b\sum_p \xi_p Tr\Bigg[U_aU_bU_c^\dagger U_d^\dagger\Bigg]+h.c. \qquad \text{I rewrite it in terms of the sites}$$
$$= \lambda_b\sum_x \Pi_h(x)\Pi_v(x+e_1)\Pi_v(x+e_2)\Pi_h(x-e_1+e_2)Tr\Bigg[U(x,h)U(x+e_1,v)U^\dagger(x+e_2,h)U^\dagger(x,v)\Bigg]+h.c.$$

Among the 32 terms that compose the magnetic term we would like to find what is the maximum number of bodies involved in the realizations of the terms. We are doing this because in case of a digital simulation this would be fundamental and very important. I recall that the the "pi" operators act only on the qudit, so we need 2 qudits for the realization of the "last" two "pi" operators in the phase factor. Now, considering the definition 3 of $U_{00}, U_{01}, U_{10}, U_{11}$ we consider the worst possibile scenario and therefore either U_{01} or U_{11} where both the qubit and the qudit are involved. Hence, in the worst case we have 4 qubits and 6 qudits and a total of 10-body interaction term.

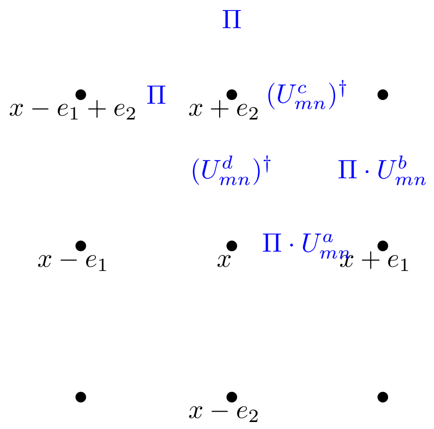


Figure 6: Magnetic term of a single plaquette

8 Conclusion

We have that the interaction terms involves up to 12 bodies and therefore is probably very hard to implement with the current hardware. We hope that in the near-term future we will have a technological advance such that the 2D simulation will be implemented.

	Max # qubits	Max # qudits	Max # bodies	structure of the hardest term
Electric term	0	1	1	1 qudits
Magnetic term	4	6	10	4 qubits + 6 qudits
Vertical int.	7	6	12	7 qubits + 5 qudits
Horizontal int.	7	6	11	7 qubits + 4 qudits

Table 1: Features of the Hamiltonian parts.

Note: The fact that for the Vertical interaction (same reasoning is valid for the horizontal interaction) we have Max # qubits = 7 and Max # qudits = 6 doesn't mean that we have up to 13-body interactions. Indeed, the Max # belongs to different terms.

Note: With "hardest term" we mean the one with the highest number of bodies involved. From an experimental point of view the concept of "hardness" could be different as qubits and qudits surely have different difficulties in the implementation.

9 Single Plaquette: pure gauge terms

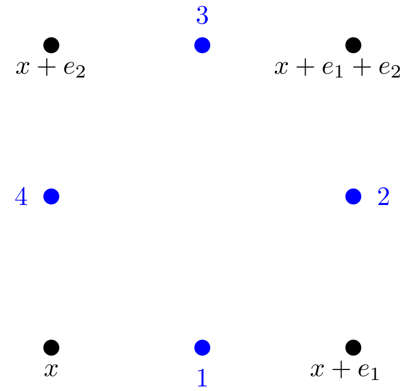


Figure 7: Single plaquette, in blue the gauge field while in black the matter before being removed. x and $x+e_1+e_2$ are taken as the even sites while $x+e_1$ and $x+e_2$ as the odd ones. Per each site we have at most two fermions and they correspond to two different flavours; we are therefore using the fundamental representation for the matter.

9.1 Electric part

$$\begin{aligned}
H_E &= \lambda_E \Bigg[\sum_{h\ links} |2, mn\rangle \langle 2, mn| + \sum_{v\ links} |2, mn\rangle \langle 2, mn| \Bigg] \\
&= -\frac{\lambda_E}{2} \sum_{h\ links} \Pi_h - \frac{\lambda_E}{2} \sum_{v\ links} \Pi_v \\
&= -\frac{\lambda_E}{2} \Bigg(\Pi_1 + \Pi_3 + \Pi_2 + \Pi_4 \Bigg) \\
&= -\frac{\lambda_E}{2} \sum_{i=1}^{4} \Bigg[1 \cdot (X_{02} + X_{13}) \Bigg]_i
\end{aligned}$$

I defined the new Hilbert space as the product space of the links (1, 2, 3 and 4). In this way the notation is easier and I don't have anymore $x, h/v$. All the electric terms are 1-body term interaction and the terms are commuting one with each other as defined on different Hilbert spaces.

9.2 Magnetic part

$$H_B = \lambda_b \xi_p Tr \Bigg[U^1 U^2 U^{3\dagger} U^{4\dagger} \Bigg] + h.c. \qquad \xi_p = \Pi_h(x)\Pi_v(x+e_1)\Pi_v(x+e_2)\Pi_h(x-e_1+e_2)$$

"Z" form: $U_{00} = 1 \cdot Z_{02}$ $\quad U_{01} = -Z \cdot Z_{13}$ $\quad U_{10} = 1 \cdot Z_{13}$ $\quad U_{11} = Z \cdot Z_{02}$ $\quad \Pi = 1 \cdot (X_{02} + X_{13})$

"Ya" form: $\Pi U_{00} = -i1 \cdot Y_{02}$ $\quad \Pi U_{01} = iZ \cdot Y_{13}$ $\quad \Pi U_{10} = -i1 \cdot Y_{13}$ $\quad \Pi U_{11} = -iZ \cdot Y_{02}$

"Yb" form: $U_{00}\Pi = i1 \cdot Y_{02}$ $\quad U_{01}\Pi = -iZ \cdot Y_{13}$ $\quad U_{10}\Pi = i1 \cdot Y_{13}$ $\quad U_{11}\Pi = iZ \cdot Y_{02}$

Considering the definition of the magnetic Hamiltonian we have to consider only the first two "pi" factors.

Moreover, analyzing the U_{mn} terms we see that these are all hermitian (and real) as well as the Πs. Hence:

$$
\begin{aligned}
H_B &\equiv \sum_{m,n,n',m'=0,1} H_B(m,m',n,n') = \lambda_b \xi_p Tr\left[U^1 U^2 U^{3\dagger} U^{4\dagger}\right] + h.c. \\
&= \lambda_b \Pi_1 \Pi_2 Tr\left[U^1 U^2 U^{3\dagger} U^{4\dagger}\right] + \bar{\lambda_b}\left\{\Pi_1 \Pi_2 Tr\left[U^1 U^2 U^{3\dagger} U^{4\dagger}\right]\right\}^\dagger \\
&= \lambda_b Tr\left[(\Pi^1 U^1)(\Pi^2 U^2) U^{3\dagger} U^{4\dagger}\right] + \bar{\lambda_b}\left\{Tr\left[(\Pi^1 U^1)(\Pi^2 U^2) U^{3\dagger} U^{4\dagger}\right]\right\}^\dagger \\
&= \sum_{m,n,n',m'=0,1} \lambda_b \Pi^1 U^1_{mn} \Pi^2 U^2_{nn'} U^3_{m'n'} U^4_{mm'} + \bar{\lambda_b}(\Pi^1 U^1_{mn})^\dagger (\Pi^2 U^2_{nn'})^\dagger U^3_{m'n'} U^4_{mm'} \\
&= \sum_{m,n,n',m'=0,1} \lambda_b \Pi^1 U^1_{mn} \Pi^2 U^2_{nn'} U^3_{m'n'} U^4_{mm'} + \bar{\lambda_b}({U^1_{mn}}^\dagger \Pi^1)({U^2_{nn'}}^\dagger \Pi^2) U^3_{m'n'} U^4_{mm'} \qquad \text{using: } \Pi U^1_{mn} = -U^1_{mn}\Pi \\
&= \sum_{m,n,n',m'=0,1} \lambda_b \Pi^1 U^1_{mn} \Pi^2 U^2_{nn'} U^3_{m'n'} U^4_{mm'} + \bar{\lambda_b} \Pi^1 U^1_{mn} \Pi^2 U^2_{nn'} U^3_{m'n'} U^4_{mm'} \\
&= 2Re[\lambda_b] \sum_{m,n,n',m'=0,1} \Pi^1 U^1_{mn} \Pi^2 U^2_{nn'} U^3_{m'n'} U^4_{mm'}
\end{aligned}
$$

and in total we have 16 different terms. **Note:** operators acting on the first and second link are in the "Ya" and "Yb" form as they are coming from ΠU_{mn} and $U_{mn}\Pi$ while the operators on the third and fourth link are just the usual U_{mn} and so in the "Z" form. The "Ya" and "Yb" change only for a sign.
Note: this formulation is to be taken as correct. Indeed, the resulting Hamiltonian commutes with the transformed Gauss' laws as expected. We did not show, as for the electric, mass and interaction term, the correctness of the magnetic term. Nevertheless, the commutativity check already gives us a flavour that we are right.

#	m	n	n'	m	$U^1_{mn}U^2_{nn'}U^3_{m'n'}U^4_{mm'}$	#	m	n	n'	m	$U^1_{mn}U^2_{nn'}U^3_{m'n'}U^4_{mm'}$
1	0	0	0	0	$U^1_{00}U^2_{00}U^3_{00}U^4_{00}$	9	1	0	0	0	$U^1_{10}U^2_{00}U^3_{00}U^4_{10}$
2	0	0	0	1	$U^1_{00}U^2_{00}U^3_{10}U^4_{01}$	10	1	0	0	1	$U^1_{10}U^2_{00}U^3_{10}U^4_{11}$
3	0	0	1	0	$U^1_{00}U^2_{01}U^3_{01}U^4_{00}$	11	1	0	1	0	$U^1_{10}U^2_{01}U^3_{01}U^4_{10}$
4	0	0	1	1	$U^1_{00}U^2_{01}U^3_{11}U^4_{01}$	12	1	0	1	1	$U^1_{10}U^2_{01}U^3_{11}U^4_{11}$
5	0	1	0	0	$U^1_{01}U^2_{10}U^3_{00}U^4_{00}$	13	1	1	0	0	$U^1_{11}U^2_{10}U^3_{00}U^4_{10}$
6	0	1	0	1	$U^1_{01}U^2_{10}U^3_{10}U^4_{01}$	14	1	1	0	1	$U^1_{11}U^2_{10}U^3_{10}U^4_{11}$
7	0	1	1	0	$U^1_{01}U^2_{11}U^3_{01}U^4_{00}$	15	1	1	1	0	$U^1_{11}U^2_{11}U^3_{01}U^4_{10}$
8	0	1	1	1	$U^1_{01}U^2_{11}U^3_{11}U^4_{01}$	16	1	1	1	1	$U^1_{11}U^2_{11}U^3_{11}U^4_{11}$

Table 2: Terms involved in the magnetic Hamiltonian for the LGT with symmetry group D_8

9.3 Mass part

Recall the equations we developed before in 2; we have that:

$$
\begin{aligned}
H_M &= M\sum_{x,i}(-1)^{s(x)} n_i(x) \qquad \text{in our case we have only one plaquette, moreover fundamental representation} \implies i = 1,2 \\
&= M\sum_{p,i}\frac{(-1)^{s(x)} - G_i(x)}{2} \qquad (-1)^{s(x)}\text{cancels out for the single plaquette supposing that } (-1)^{s(x)} = -(-1)^{s(x+e_i)} \\
&= -\frac{M}{2}\sum_p G_0(x) + G_1(x) \\
&= -\frac{M}{2}\Big[\theta^L_{0,v}(x)\theta^L_{0,h}(x) + \theta^L_{0,v}(x+e_1)\theta^R_{0,h}(x) + \theta^R_{0,h}(x+e_2)\theta^R_{0,v}(x+e_1) + \theta^L_{0,h}(x+e_2)\theta^R_{0,v}(x) + \\
&\quad \theta^L_{1,v}(x)\theta^L_{1,h}(x) + \theta^L_{1,v}(x+e_1)\theta^R_{1,h}(x) + \theta^R_{1,h}(x+e_2)\theta^R_{1,v}(x+e_1) + \theta^L_{1,h}(x+e_2)\theta^R_{1,v}(x)\Big]
\end{aligned}
$$

and this could be translated into gates:

$$H_M = -\frac{M}{2}\left\{\Big[X\cdot(X_{02}+P_{13})\Big]_4 \times \Big[X\cdot(X_{02}+P_{13})\Big]_1 + \Big[X\cdot(X_{02}+P_{13})\Big]_2 \times \Big[X\cdot(X_{02}+X_{13})\Big]_1 + \right.$$
$$\left.\Big[X\cdot(X_{02}+X_{13})\Big]_3 \times \Big[X\cdot(X_{02}+X_{13})\Big]_2 + \Big[X\cdot(X_{02}+P_{13})\Big]_3 \times \Big[X\cdot(X_{02}+X_{13})\Big]_4\right\} +$$
$$-\frac{M}{2}\left\{\Big[X\cdot(P_{02}+X_{13})\Big]_4 \times \Big[X\cdot(P_{02}+X_{13})\Big]_1 + \Big[X\cdot(P_{02}+X_{13})\Big]_2 \times \Big[X\cdot 1\Big]_1 + \right.$$
$$\left.\Big[X\cdot 1\Big]_3 \times \Big[X\cdot 1\Big]_2 + \Big[X\cdot(P_{02}+X_{13})\Big]_3 \times \Big[X\cdot 1\Big]_4\right\} \equiv -\frac{M}{2}\sum_{i=1}^{8} H_M(i)$$

10 Single Plaquette: interaction terms

$$H_{int} = \epsilon\Big[\psi^\dagger(x)U^1\psi(x+e_1) + \psi^\dagger(x+e_1)U^2\psi(x+e_1+e_2) + \psi^\dagger(x+e_2)U^3\psi(x+e_1+e_2) + \psi^\dagger(x)U^4\psi(x+e_2) + h.c.\Big]$$
$$= H_1 + H_2 + H_3 + H_4 + h.c. \qquad \text{alternatively using the "site" notation}$$
$$= H_h(x) + H_v(x+e_1) + H_h(x+e_2) + H_v(x) + h.c.$$

10.1 Horizontal interaction

$H_h = H_1 + H_3 + h.c.$

$H_1 = H_h(x) = \epsilon\psi^\dagger(x)U^1\psi(x+e_1) + h.c. = \epsilon\psi^\dagger_m(x)U^1_{mn}\psi_n(x+e_1) + h.c.$ where the Einstein summation is implied

$$= \epsilon\Big[\psi^\dagger_0(x)U^1_{00}\psi_0(x+e_1) + \psi^\dagger_1(x)U^1_{10}\psi_0(x+e_1) + \psi^\dagger_0(x)U^1_{01}\psi_1(x+e_1) + \psi^\dagger_1(x)U^1_{11}\psi_1(x+e_1) + h.c.\Big]$$

and after both the transformations we have:

$$= \frac{\epsilon}{2}(\tilde{H}^1_{00} + \tilde{H}^1_{01} + \tilde{H}^1_{10} + \tilde{H}^1_{11})$$

Here we have an even site so $(-1)^x = 1$.

$$\tilde{H}^1_{00} = -\left\{\Big[X\cdot Y_{02}\Big]_1 \times \Big[X\cdot(P_{02}+X_{13})\Big]_4 + \Big[X\cdot Y_{02}\Big]_1 \times \Big[1\cdot(X_{02}+X_{13})\Big]_4 \times \Big[X\cdot(X_{02}+P_{13})\Big]_2\right\}$$

$$\tilde{H}^1_{01} = -\left\{\Big[X\cdot(P_{02}+X_{13})\Big]_4 \times \Big[Z\cdot Y_{13}\Big]_1 \times \Big[X\cdot(X_{02}+P_{13})\Big]_2 + \Big[1\cdot(X_{02}+X_{13})\Big]_4 \times \Big[Z\cdot Y_{13}\Big]_1 \times \Big[1\cdot(X_{02}+X_{13})\Big]_2\right\}$$

$$\tilde{H}^1_{10} = -\left\{\Big[X\cdot(P_{02}+X_{13})\Big]_4 \times \Big[1\cdot Y_{13}\Big]_1 \times \Big[X\cdot(X_{02}+P_{13})\Big]_2 + \Big[1\cdot Y_{13}\Big]_1\right\}$$

$$\tilde{H}^1_{11} = \Big[Y\cdot Z_{02}\Big]_1 \times \Big[X\cdot(X_{02}+P_{13})\Big]_2 + \Big[X\cdot(P_{02}+X_{13})\Big]_4 \times \Big[Y\cdot Z_{02}\Big]_1 \times \Big[1\cdot(X_{02}+X_{13})\Big]_2$$

Similarly for $H_3 = H_h(x+e_2) = \frac{\epsilon}{2}(\tilde{H}^3_{00} + \tilde{H}^3_{01} + \tilde{H}^3_{10} + \tilde{H}^3_{11})$, now we have an odd site so $-(1)^x = -1$:

$$\tilde{H}^3_{00} = \Big[X \cdot 1\Big]_4 \times \Big[X \cdot Y_{02}\Big]_3 \times \Big[1 \cdot (X_{02} + X_{13})\Big]_2 + \Big[1 \cdot (X_{02} + X_{13})\Big]_4 \times \Big[X \cdot Y_{02}\Big]_3 \times \Big[X \cdot 1\Big]_2$$

$$\tilde{H}^3_{01} = -\Big\{\Big[X \cdot 1\Big]_4 \times \Big[Z \cdot Y_{13}\Big]_3 \times \Big[X \cdot 1\Big]_2 + \Big[1 \cdot (X_{02} + X_{13})\Big]_4 \times \Big[Z \cdot Y_{13}\Big]_3\Big\}$$

$$\tilde{H}^3_{10} = -\Big\{\Big[X \cdot 1\Big]_4 \times \Big[1 \cdot Y_{13}\Big]_3 \times \Big[X \cdot 1\Big]_2 + \Big[1 \cdot Y_{13}\Big]_3 \times \Big[1 \cdot (X_{02} + X_{13})\Big]_4\Big\}$$

$$\tilde{H}^3_{11} = -\Big\{\Big[Y \cdot Z_{02}\Big]_3 \times \Big[X \cdot 1\Big]_2 + \Big[X \cdot 1\Big]_4 \times \Big[Y \cdot Z_{02}\Big]_3\Big\}$$

10.2 Vertical interaction

We want to compute $H_v = H_2 + H_4 + h.c.$
For for $H_2 = H_v(x + e_1) = \frac{\epsilon}{2}(\tilde{H}^2_{00} + \tilde{H}^2_{01} + \tilde{H}^2_{10} + \tilde{H}^2_{11})$, we have an odd site so $(-1)^x = -1$:

$$\tilde{H}^2_{00} = \Big[X \cdot 1\Big]_1 \times \Big[X \cdot Y_{02}\Big]_2 + \Big[1 \cdot (X_{02} + X_{13})\Big]_1 \times \Big[X \cdot Y_{02}\Big]_2 \times \Big[X \cdot (X_{02} + X_{13})\Big]_3$$

$$\tilde{H}^2_{01} = -\Big\{\Big[1 \cdot X_{02} + X_{13}\Big]_1 \times \Big[Z \cdot Y_{13}\Big]_2 \times \Big[1 \cdot X_{02} + X_{13}\Big]_3 + \Big[X \cdot 1\Big]_1 \times \Big[Z \cdot Y_{13}\Big]_2 \times \Big[X \cdot (X_{02} + X_{13})\Big]_3\Big\}$$

$$\tilde{H}^2_{10} = -\Big\{\Big[1 \cdot Y_{13}\Big]_2 + \Big[X \cdot 1\Big]_1 \times \Big[1 \cdot Y_{13}\Big]_2 \times \Big[X \cdot (X_{02} + X_{13})\Big]_3\Big\}$$

$$\tilde{H}^2_{11} = -\Big\{\Big[X \cdot 1\Big]_1 \times \Big[Y \cdot Z_{02}\Big]_2 \times \Big[1 \cdot (X_{02} + X_{13})\Big]_3 + \Big[Y \cdot Z_{02}\Big]_2 \times \Big[X \cdot (X_{02} + X_{13})\Big]_3\Big\}$$

Similarly to what we did before, after the transformation we have for $H_4 = H_v(x) = \frac{\epsilon}{2}(\tilde{H}^4_{00} + \tilde{H}^4_{01} + \tilde{H}^4_{10} + \tilde{H}^4_{11})$, now $(-1)^x = 1$:

$$\tilde{H}^4_{00} = -\Big\{\Big[X \cdot (X_{02} + P_{13})\Big]_1 \times \Big[X \cdot Y_{02}\Big]_4 + \Big[X \cdot Y_{02}\Big]_4 \times \Big[X \cdot (X_{02} + P_{13})\Big]_3\Big\}$$

$$\tilde{H}^4_{01} = -\Big\{\Big[Z \cdot Y_{13}\Big]_4 \times \Big[1 \cdot (X_{02} + X_{13})\Big]_3 + \Big[X \cdot (X_{02} + P_{13})\Big]_1 \times \Big[Z \cdot Y_{13}\Big]_4 \times \Big[X \cdot (X_{02} + P_{13})\Big]_3\Big\}$$

$$\tilde{H}^4_{10} = -\Big\{\Big[1 \cdot X_{02} + X_{13}\Big]_1 \times \Big[1 \cdot Y_{13}\Big]_4 + \Big[X \cdot (X_{02} + P_{13})\Big]_1 \times \Big[1 \cdot Y_{13}\Big]_4 \times \Big[X \cdot (X_{02} + P_{13})\Big]_3\Big\}$$

$$\tilde{H}^4_{11} = \Big[X \cdot (X_{02} + P_{13})\Big]_1 \times \Big[Y \cdot Z_{02}\Big]_4 \times \Big[1 \cdot (X_{02} + X_{13})\Big]_3 + \Big[1 \cdot (X_{02} + X_{13})\Big]_1 \times \Big[Y \cdot Z_{02}\Big]_4 \times \Big[X \cdot (X_{02} + P_{13})\Big]_3$$

11 Comments

Why should we trust the math I did for the interaction part?

- If we consider the horizontal cases we see that there is a similar pattern: H^1 very similar to H^3 as expected and the same holds for the vertical case.
- it is consistent with the 1D case that Guy did.
- For H^1 and H^2 I did all the math again (but this time considering only the Πs in the plaquette for the ξ_v and ξ_h.) and I arrived to the same result.
- the pseudo-2D case is correct and it comes from the same model so it means that also this is probably correct.

	Max # qubits	Max # qudits	Max # bodies	structure of the hardest term
Electric term	0	1	1	1 qudits
Magnetic term	4	4	8	4 qubits + 4 qudits
Vertical int.	3	3	6	3 qubits + 3 qudits
Horizontal int.	3	3	6	3 qubits + 3 qudits

Table 3: Features of the different parts of the single-plaquette Hamiltonian.

Even though the complexity of the total Hamiltonian is smaller compared to the total 2D model, it's still hard to simulate, either digital or in an analogue way, with the current technology. Nevertheless, we could try to realize an analysis through VQE algorithms. With this goal we analyze the hamiltonian:

$$H_{plaquette} = H_M + H_E + H_B + H_{INT} \tag{4}$$

and we want to find groups of terms that are commuting among themselves so that we can later run a variational quantum algorithm on it.

12 Preparation for the VQE implementation

12.1 Consideration before building the groups

We have 44 different terms: 8 coming from the mass part, 4 from the electric one, 16 from the magnetic and 16 from the interaction term. General features:

- $[X_{mn}, P_{mn}] = 0$ for every qudit.
- All the 16 terms of the magnetic part commutes among themselves. Indeed: $[U_{ab}, U_{cd}] = 0 \ \forall \ a, b, c, d$ as well as $[\Pi U_{ab}, \Pi U_{cd}] = 0 \ \forall \ a, b, c, d$ **Note:** we don't consider $[\Pi U_{mn}, U_{m'n'}]$ as we don't have this case.
- As expected (we built the theory such that this is respected) we have that the "gate version" of the theta matrices commute. The same we have between theta matrices and "pi". Indeed, the elements $\{a^2x, x, a^2\}$ form an Abelian sub-group; this tell us also that we could extend the "elimination process" to also a "spin 1" representation of the matter so where we have three components and not just two as in the fundamental one.
  Hence, we have to look at the links where we have the U_{mn}; these could give raise to non commuting parts.
- The 8 terms of the mass hamiltonian commutes among themselves as they are all built out of $X, 1, P$ and we have: $[X, 1] = [X, P] = [P, 1] = 0$. Moreover we remember that all the theta matrices (involved) commute among themselves as they are representations of commuting elements.
- It works similarly for the electric part; all the 4 elements commute among themselves.
- Each H^r_{mn} where $m, n = 0, 1$ and $r = v, h$ has 2 commuting terms.
- I can try to see if it's possible to unify some of the 16 H^r_{mn} groups such that I will get something easier.

13 First attempt (worst)

Analyzing further the terms of the interaction part we see that we can simplify further; indeed, whatever link you consider we have that $\{H_{00}, H_{10}, H_{11}\}$ commutes among themselves; unfortunately this is not valid for the element H_{01}. An alternative would be to have $\{H_{00}, H_{11}\}$ and $\{H_{01}H_{10},\}$ so that the division is symmetric; nevertheless this division doesn't seem to improve anything later in the computation. We call such groups in this way:

$$\begin{aligned}
A^1 &\equiv \{H^1_{00}, H^1_{10}, H^1_{11}\} \\
A^2 &\equiv \{H^2_{00}, H^2_{10}, H^2_{11}\} \\
A^3 &\equiv \{H^3_{00}, H^3_{10}, H^3_{11}\} \\
A^4 &\equiv \{H^4_{00}, H^4_{10}, H^4_{11}\}
\end{aligned}$$

We see that we can distribute the terms of the electric part in the groups of the interaction part; more specifically:

$$\begin{aligned}
\Pi_3 &\longrightarrow A^1 \\
\Pi_1 &\longrightarrow A^3 \\
\Pi_2 &\longrightarrow A^4 \\
\Pi_4 &\longrightarrow A^2
\end{aligned}$$

Let's now consider the other terms of the interaction Hamiltonian:

$$H^1_{01} = -\left\{\Big[X\cdot(P_{02}+X_{13})\Big]_4 \times \Big[Z\cdot Y_{13}\Big]_1 \times \Big[X\cdot(X_{02}+X_{13})\Big]_2 + \Big[1\cdot(X_{02}+X_{13})\Big]_4 \times \Big[Z\cdot Y_{13}\Big]_1 \times \Big[1\cdot(X_{02}+X_{13})\Big]_2\right\}$$

$$H^3_{01} = -\left\{\Big[X\cdot 1\Big]_4 \times \Big[Z\cdot Y_{13}\Big]_3 \times \Big[X\cdot 1\Big]_2 + \Big[1\cdot(X_{02}+X_{13})\Big]_4 \times \Big[Z\cdot Y_{13}\Big]_3\right\}$$

$$H^2_{01} = -\left\{\Big[1\cdot X_{02}+X_{13}\Big]_1 \times \Big[Z\cdot Y_{13}\Big]_2 \times \Big[1\cdot X_{02}+X_{13}\Big]_3 + \Big[X\cdot 1\Big]_1 \times \Big[Z\cdot Y_{13}\Big]_2 \times \Big[X\cdot(X_{02}+X_{13})\Big]_3\right\}$$

$$H^4_{01} = -\left\{\Big[Z\cdot Y_{13}\Big]_4 \times \Big[1\cdot X_{02}+X_{13}\Big]_3 + \Big[X\cdot(X_{02}+P_{13})\Big]_1 \times \Big[Z\cdot Y_{13}\Big]_4 \times \Big[X\cdot(X_{02}+P_{13})\Big]_3\right\}$$

We can see that, here, we can have two more grups of commuting variables:

$$\begin{aligned}
A^5 &\equiv \{H^1_{01}, H^3_{01}\} \\
A^6 &\equiv \{H^2_{01}, H^4_{01}\}
\end{aligned}$$

Indeed, we have that $\left[H^1_{01}, H^2_{01}\right] \neq 0$; this is evident considering the first terms in both expressions.
Now, we would like to divide the elements of the mass Hamiltonian in the existing group such that we can get a smaller set of non-commuting clusters:

$$\begin{aligned}
H_M(3), H_M(4) &\longrightarrow A^1 \\
H_M(5), H_M(8) &\longrightarrow A^2 \\
H_M(1), H_M(2) &\longrightarrow A^3 \\
H_M(6), H_M(7) &\longrightarrow A^4
\end{aligned} \tag{45}$$

How did I choose the elements?
I recall that the elements $\{a^2x, x, a^2\}$ commutes among themselves and therefore the associated matrices. The

mass term in entirely built out of these terms. Now, whatever link I take, the Hamiltonian related to that link is a multi-link Hamiltonian, but on the other links acts only through matrices associated to $\{a^2x, x, a^2\}$. So if I consider the mass terms that don't act on the link the Hamiltonian terms are characterized (ex. link 1 for H^1_{mn}) then the terms will commute. This, as explained before, follows from the commutation of the aforementioned elements.
Note that what we did in 45 is not the only possible way of dividing the terms in the groups. Indeed, for example I could have put 4 terms in A^1 and the other four in A^2, I preferred to follow a symmetric way (as it usually preferable in physics).

Can I reduce further the number of sets following this track? Probably not, indeed the only remaining doubt would be to see if we are able to divide the terms of the magnetic part into the existing sets. Unfortunately this is not possible, indeed we found that there is at least one element that doesn't commute with at least one term of each set.
Example: consider the magnetic term characterized by all the indices 1.

$$H_B(1,1,1,1) = -2\lambda_B \Big[Z \cdot Y_{02}\Big]_1 \times \Big[Z \cdot Y_{02}\Big]_2 \times \Big[Z \cdot Z_{02}\Big]_3 \times \Big[Z \cdot Z_{02}\Big]_4$$

Group 1: doesn't commute with the first element of H^1_{10}

Group 2: doesn't commute with the second element of H^2_{10}

Group 3: doesn't commute with the first element of H^3_{10}

Group 4: doesn't commute with the first element of H^4_{10}

Group 5: doesn't commute with the second element of H^1_{01}

Group 6: doesn't commute with the first element of H^2_{01}

hence we must have one more group that we will call A^7.

13.1 Summary of the groups

$$\begin{aligned}
A^1 &\equiv \{H^1_{00}, H^1_{10}, H^1_{11}, \Pi_3, H_M(3), H_M(4)\}\\
A^2 &\equiv \{H^2_{00}, H^2_{10}, H^2_{11}, \Pi_4, H_M(5), H_M(8)\}\\
A^3 &\equiv \{H^3_{00}, H^3_{10}, H^3_{11}, \Pi_1, H_M(1), H_M(2)\}\\
A^4 &\equiv \{H^4_{00}, H^4_{10}, H^4_{11}, \Pi_2, H_M(6), H_M(7)\}\\
A^5 &\equiv \{H^1_{01}, H^3_{01}\}\\
A^6 &\equiv \{H^2_{01}, H^4_{01}\}\\
A^7 &\equiv \{\sum_{m,m',n,n'} H_B(m,m',n,n')\}
\end{aligned}$$

where all the 44 terms have been distributed.
We notice that considering or not considering the mass term doesn't increase the complexity of the simulation in terms of # of sets of commuting elements.

14 Second attempt (best one)

- We notice that H^2 (H^4) doesn't have any term acting on the fourth (second) link; moreover, except for the 2 (4) link all the other parts are build through θ or Π matrices that commute among themselves. So we can put H^2 and H^4 in the same group. Following the same reasoning we get together H^1 and H^3 in the same group.
- The electric part is built out of Π operators so it commutes with all the other θ and Π. Hence, we can put the horizontal Π in the set where we have the vertical interaction term. Indeed, in this case they would act on links where there is not any U_{mn} and so they will commute. The same holds for the horizontal case.

$$
\begin{aligned}
&A^1 \equiv \{H^1_{00}, H^1_{10}, H^1_{11}, H^3_{00}, H^3_{10}, H^3_{11}, \Pi_2, \Pi_4\} \\
&A^2 \equiv \{H^2_{00}, H^2_{10}, H^2_{11}, H^4_{00}, H^4_{10}, H^4_{11}, \Pi_3, \Pi_1\} \\
&A^3 \equiv \{H^1_{01}, H^3_{01}\} \\
&A^4 \equiv \{H^2_{01}, H^4_{01}\} \\
&A^5 \equiv \{\sum_{m,m',n,n'} H_B(m, m', n, n')\}
\end{aligned}
$$

In case we want to include also the mass we have:

$$A^6 \equiv \{H_M\}$$

Indeed, with this group formulation we can't use the technique we developed before to distribute the mass terms.
Example: consider $H_M(1) = -\frac{M}{2}\Big[X \cdot (X_{02} + P_{13})\Big]_4 \times \Big[X \cdot (X_{02} + P_{13})\Big]_1$

- Group 1 and 2: $\big[H^1_{00}, H_M(1)\big] \neq 0 \neq \big[H^4_{00}, H_M(1)\big]$ and this is evident if you take the first element of these parts as we have anticommutation relations.
- Group 3 and 4: $\big[H^1_{01}, H_M(1)\big] \neq 0 \neq \big[H^4_{01}, H_M(1)\big]$ and this is evident if we take the first element of these parts as we have anticommutation relations.
- Group 5: consider $H_B(1,1,1,0) = 2\lambda_B \Big[Z \cdot Y_{02}\Big]_1 \times \Big[Z \cdot Y_{02}\Big]_2 \times \Big[1 \cdot Z_{13}\Big]_3 \times \Big[Z \cdot Z_{13}\Big]_4$ and it's clear that $H_B(1,1,1,0)$ and $H_M(1)$ anticommute.

Hence we need one more group.
We can conclude that for a massless VQE implementation it would be enough to consider 5 sets of commuting terms while, if we want to consider also the mass we have 6 sets.
The second attempt should be better; nevertheless it's interesting to have another option in case we have some experimental/algorithmic constraints.

14.1 Gauge states to be simulated

When running the VQE algorithm, for real time evolution is interesting to see how states of a particular sectors evolve in time. As an example we consider the easiest gauge invariant state we found and one with an excitation.

$$
\begin{aligned}
\left|\tilde{\psi_1}\right\rangle &= \frac{1}{2}\Big[\,|200\rangle + |210\rangle - |201\rangle - |211\rangle\,\Big]\,|0\rangle\,|0\rangle\,|0\rangle \\
\left|\tilde{\psi_0}\right\rangle &= |0\rangle\,|0\rangle\,|0\rangle\,|0\rangle\,,
\end{aligned}
$$

but, as the Hamiltonian has been written in the group element basis, we need to rewrite them. Let's do this by firstly considering only the one link state:

- $$|0\rangle = \sum_{g\in\mathbb{D}_8} |g\rangle\,\langle g|0\rangle = \sum_{g\in\mathbb{D}_8} \sqrt{\frac{dim(0)}{|G|}}\,D^0(g)\,|g\rangle = \sum_{g\in\mathbb{D}_8} \frac{1}{2\sqrt{2}}\,|g\rangle$$

- $$
\begin{aligned}
\frac{1}{2}\Big[\,|200\rangle + |210\rangle - |201\rangle - |211\rangle\,\Big] &= \frac{1}{2}\Big[\sum_{g\in\mathbb{D}_8} |g\rangle\,\sqrt{\frac{dim(2)}{|G|}}\Big(D^2_{00}(g) + D^2_{10}(g) - D^2_{01}(g) - D^2_{11}(g)\Big) \\
&= \frac{1}{2}\Big[\,|a\rangle - \left|a^3\right\rangle + |x\rangle - \left|a^2x\right\rangle\,\Big]
\end{aligned}
$$

Consequently:

$$\left|\tilde{\psi_1}\right\rangle = \frac{1}{2}\Bigg[\,|200\rangle + |210\rangle - |201\rangle - |211\rangle\Bigg]\,|0\rangle\,|0\rangle \tag{5}$$

$$= \frac{1}{32\sqrt{2}}\Bigg(|a\rangle - \left|a^3\right\rangle + |x\rangle - \left|a^2 x\right\rangle\Bigg) \otimes \Bigg(\sum_{g\in\mathbb{D}_8} |g\rangle\Bigg) \otimes \Bigg(\sum_{g\in\mathbb{D}_8} |g\rangle\Bigg) \otimes \Bigg(\sum_{g\in\mathbb{D}_8} |g\rangle\Bigg) \tag{6}$$

$$\left|\tilde{\psi_0}\right\rangle = |0\rangle\,|0\rangle\,|0\rangle\,|0\rangle = \frac{1}{64}\Bigg(\sum_{g\in\mathbb{D}_8} |g\rangle\Bigg) \otimes \Bigg(\sum_{g\in\mathbb{D}_8} |g\rangle\Bigg) \otimes \Bigg(\sum_{g\in\mathbb{D}_8} |g\rangle\Bigg) \otimes \Bigg(\sum_{g\in\mathbb{D}_8} |g\rangle\Bigg) \tag{7}$$

15 Pseudo-2D model with matter

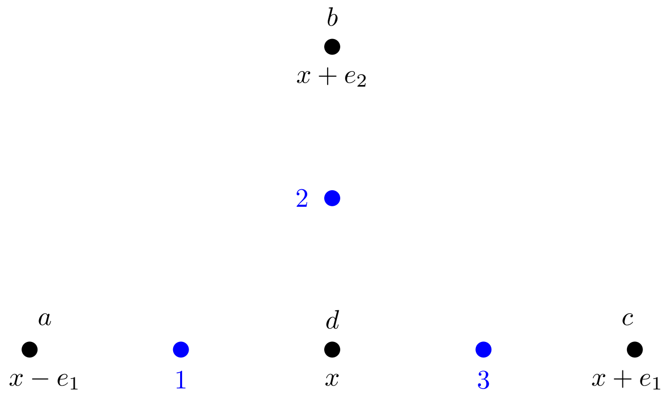


Figure 8: Pseudo-2D model, in blue the gauge field while in black the matter before being removed. We take a, b and c as the even sites while d as the odd one. Per each site we have at most two fermions and they correspond to two different flovours; we are therefore using the fundamental representation for the matter.

15.1 Introduction

We have a model where gauge-invariant states are constrained by the four Gauss' laws that we have in each vertex. They are respectively:

$$\begin{aligned}
&\theta^L_g(1)\left|\Psi\right\rangle = \theta^m_g(x-e_1)\left|\Psi\right\rangle && \forall g \in D_8\\
&\theta^{R\dagger}_g(2)\left|\Psi\right\rangle = \theta^m_g(x-e_2)\left|\Psi\right\rangle && \forall g \in D_8\\
&\theta^{R\dagger}_g(3)\left|\Psi\right\rangle = \theta^m_g(x+e_1)\left|\Psi\right\rangle && \forall g \in D_8\\
&\theta^{R\dagger}_g(1)\theta^L_g(2)\theta^L_g(3)\left|\Psi\right\rangle = \theta^m_g(x)\left|\Psi\right\rangle && \forall g \in D_8.
\end{aligned}$$

where $\theta^{L/R}$ acts on the link space while θ^m on the vertices. Following the paper of Zohar and Burello [3], the operator acting on the matter field is defined as following (the index j stands for the representation and $det(g^{-1})^N$ comes from the staggering):

$$\begin{aligned}
&\theta^{m,j}_g(x) = det(g^{-1})^N e^{i\Psi^\dagger_a q_{ab}(g)\Psi_b} \qquad \text{where}\\
&q^j(g) = -ilog(D^j(g)).
\end{aligned}$$

For the group element operators we use the representation basis as explained in [4]:

$$\begin{aligned}
\left\langle j'm'n'\right|\theta^L_g\left|jmn\right\rangle &= \delta_{jj'}\delta_{nn'}\bar{D}^j_{m'm}(g^{-1})\\
\left\langle j'm'n'\right|\theta^R_g\left|jmn\right\rangle &= \delta_{jj'}\delta_{m'm}\bar{D}^{j'}_{nn'}(g^{-1})
\end{aligned}$$

	e	a	a^2	a^3	x	ax	a^2x	a^3x
g^{-1}	e	a^3	a^2	a	x	ax	a^2x	a^3x

Table 4: Inverse elements for the D_8 group.

15.2 Enforcing gauge-invariance

We now want to impose gauge invariance and find the states following these constraints. In analogy with the Wigner-Eckart theorem we want that, for each vertex, the right sum of angular momenta is equal to the left sum. We can enforce it using the Clebsch-Gordan Coefficients and, then, checking that the state respects the gauge-invariance. Indeed, recall that for compact groups we have:

$$\sum_i L_i - R_i = Q_m.$$

and we can extend it to finite group using C-G coefficients. The most general states is given by:

$$|\Psi\rangle = \sum_{J_1,J_2,J_3,M_1,M_2,M_3,N_1,N_2,N_3} \alpha^{J_1 J_2 J_3 j_a j_b j_c j_d}_{M_1 M_2 M_3 N_1 N_2 N_3} |J_1 M_1 N_1\rangle |J_2 M_2 N_2\rangle |J_3 M_3 N_3\rangle \times \\ \times (a_0^\dagger)^{n_{a_0}} (a_1^\dagger)^{n_{a_1}} (b_0^\dagger)^{n_{b_0}} (b_1^\dagger)^{n_{b_1}} (c_0^\dagger)^{n_{c_0}} (c_1^\dagger)^{n_{c_1}} (d_0^\dagger)^{n_{d_0}} (d_1^\dagger)^{n_{d_1}} |\Omega\rangle$$

where $|\Omega\rangle$ stands for the *Fock vacuum.* Nevertheless, we can rewrite the general state enforcing gauge-invariance and half-filling for the fermions:

$$|\Psi\rangle = \sum_{J_1,J_2,J_3} \alpha^{J_1 J_2 J_3 j_a j_b j_c j_d} \langle J_1 M_1 | j_a m_a\rangle \langle 0 | J_2 N_2 j_b m_b\rangle \langle 0 | J_3 N_3 j_c m_c\rangle \sum_J \langle J_2 M_2 J_3 M_3 | JM\rangle \langle JM | J_1 N_1 j_d m_d\rangle \times \tag{8}$$

$$\times |J_1 M_1 N_1\rangle |J_2 M_2 N_2\rangle |J_3 M_3 N_3\rangle \delta_{\sum n,4} (a_0^\dagger)^{n_{a_0}} (a_1^\dagger)^{n_{a_1}} (b_0^\dagger)^{n_{b_0}} (b_1^\dagger)^{n_{b_1}} (c_0^\dagger)^{n_{c_0}} (c_1^\dagger)^{n_{c_1}} (d_0^\dagger)^{n_{d_0}} d_1^\dagger)^{n_{d_1}} |\Omega\rangle \tag{9}$$

The next step is to see what is the relation between the occupation number and the fermions and the associated representation. For this we need to divide the even sites from the odd sites; indeed they transform differently under the local unitary transformations θ_g^m.

	$\lvert\Omega\rangle$	$a_j^\dagger \lvert\Omega\rangle$	$a_0^\dagger a_1^\dagger \lvert\Omega\rangle$
even	$\lvert 0\rangle$	$\lvert 2j\rangle$	$\lvert\bar{0}\rangle$
odd	$\lvert\bar{0}\rangle$	$-\varepsilon_{ij} \lvert 2i\rangle$	$\lvert 0\rangle$

Table 5: Summary of the matter representation

Now, to find all the states, we consider the only two possible cases we have to place the matter in the sites: we can either have one site full and all the other empty or have two sites with only one fermion.

coefficients				value
$\langle 0\|00\rangle$,	$\langle 0\|\bar{0}\bar{0}\rangle$,	$\langle 0\|11\rangle$,	$\langle 0\|\bar{1}\bar{1}\rangle$	1
$\langle \bar{0}\|\bar{0}0\rangle$,	$\langle \bar{0}\|0\bar{0}\rangle$,	$\langle \bar{0}\|1\bar{1}\rangle$,	$\langle \bar{0}\|\bar{1}1\rangle$	1
$\langle 1\|10\rangle$,	$\langle 1\|01\rangle$,	$\langle 1\|\bar{0}\bar{1}\rangle$,	$\langle 1\|\bar{1}\bar{0}\rangle$	1
$\langle \bar{1}\|\bar{1}0\rangle$,	$\langle \bar{1}\|0\bar{1}\rangle$,	$\langle \bar{1}\|\bar{0}1\rangle$,	$\langle \bar{1}\|1\bar{0}\rangle$	1
	$\langle 0, 2m\|2m'\rangle$,	$\langle 2m, 0\|2m'\rangle$		$\delta_{m,m'}$
	$\langle \bar{0}, 2m\|2m'\rangle$,	$\langle 2m, \bar{0}\|2m'\rangle$		$\varepsilon_{m,m'}$
	$\langle 1, 2m\|2m'\rangle$,	$\langle 2m, 1\|2m'\rangle$		$(\sigma_z)_{m,m'}$
	$\langle \bar{1}, 2m\|2m'\rangle$,	$\langle 2m, \bar{1}\|2m'\rangle$		$(\sigma_x)_{m,m'}$

coefficients	$(m,m') =$ $(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$\langle 0\|2m,2m'\rangle = Q^0_{m,m'}$	$\frac{1}{\sqrt{2}}(1$	0	0	$1)$
$\langle \bar{0}\|2m,2m'\rangle = Q^{\bar{0}}_{m,m'}$	$\frac{1}{\sqrt{2}}(0$	1	-1	$0)$
$\langle 1\|2m,2m'\rangle = Q^1_{m,m'}$	$\frac{1}{\sqrt{2}}(1$	0	0	$-1)$
$\langle \bar{1}\|2m,2m'\rangle = Q^{\bar{1}}_{m,m'}$	$\frac{1}{\sqrt{2}}(0$	1	1	$0)$

Table 6: Clebsch-Gordan Coefficients, $m, m' \in \{0, 1\}$.

15.3 Finding the states

First option: one site full and the others empty:

- **Matter only in $x - e_1$:** $|matter\rangle = a_0^\dagger a_1^\dagger |\Omega\rangle$

  And using the definition of 9 it is easy to see that the only possibility is: $\boxed{|\Psi_1\rangle = |\bar{0}, 0, 0\rangle a_0^\dagger a_1^\dagger |\Omega\rangle}$.

- **Matter only in $x+e_2$:** $|matter\rangle = b_0^\dagger b_1^\dagger |\Omega\rangle$
  Similarly, the only choice is : $\boxed{|\Psi_2\rangle = |0,\bar{0},0\rangle\, b_0^\dagger b_1^\dagger |\Omega\rangle}$.

- **Matter only in $x+e_1$:** $|matter\rangle = c_0^\dagger c_1^\dagger |\Omega\rangle$
  Similarly, the only choice is : $\boxed{|\Psi_3\rangle = |0,0,\bar{0}\rangle\, c_0^\dagger c_1^\dagger |\Omega\rangle}$.

- **Matter only in x:** $|matter\rangle = d_0^\dagger d_1^\dagger |\Omega\rangle$
  Similarly, the only choice is : $\boxed{|\Psi_4\rangle = |0,0,0\rangle\, d_0^\dagger d_1^\dagger |\Omega\rangle}$.

Second option: two even site with one fermion:

- **Matter in $x-e_1$ and $x+e_2$:** $|matter\rangle = a_{m_a}^\dagger b_{m_b}^\dagger |\Omega\rangle$. We trivially get: $J_1, M_1 = 2, m_a$; $J_2 = 2$ and $J_3 = 0$.

$$|\Psi_5\rangle = Q^0_{N_2,m_b}\varepsilon_{N_1,M_2}\,|2m_aN_1\rangle\,|2M_2N_2\rangle\,|0\rangle\,a_{m_a}^\dagger b_{m_b}^\dagger|\Omega\rangle$$
$$= ...$$
$$|\Psi_5\rangle = \frac{1}{2\sqrt{2}}\Bigg[|200\rangle\,|210\rangle\,|0\rangle\,a_0^\dagger b_0^\dagger|\Omega\rangle + |210\rangle\,|210\rangle\,|0\rangle\,a_1^\dagger b_0^\dagger|\Omega\rangle - |201\rangle\,|200\rangle\,|0\rangle\,a_0^\dagger b_0^\dagger|\Omega\rangle - |211\rangle\,|200\rangle\,|0\rangle\,a_1^\dagger b_0^\dagger|\Omega\rangle$$
$$+ |200\rangle\,|211\rangle\,|0\rangle\,a_0^\dagger b_1^\dagger|\Omega\rangle + |210\rangle\,|211\rangle\,|0\rangle\,a_1^\dagger b_1^\dagger|\Omega\rangle - |201\rangle\,|201\rangle\,|0\rangle\,a_0^\dagger b_1^\dagger|\Omega\rangle - |211\rangle\,|201\rangle\,|0\rangle\,a_1^\dagger b_1^\dagger|\Omega\rangle\Bigg]$$

- **Matter in $x-e_1$ and $x+e_1$:** $|matter\rangle = a_{m_a}^\dagger c_{m_c}^\dagger |\Omega\rangle$. We trivially get: $J_1, M_1 = 2, m_a$; $J_2 = 2$ and $J_3 = 0$.

$$|\Psi_6\rangle = Q^0_{N_3,m_c}\varepsilon_{N_1,M_3}\,|2m_aN_1\rangle\,|0\rangle\,|2M_3N_3\rangle\,|0\rangle\,a_{m_a}^\dagger c_{m_c}^\dagger|\Omega\rangle$$
$$= ...$$
$$|\Psi_6\rangle = \frac{1}{2\sqrt{2}}\Bigg[|200\rangle\,|0\rangle\,|210\rangle\,a_0^\dagger c_0^\dagger|\Omega\rangle + |210\rangle\,|0\rangle\,|210\rangle\,a_1^\dagger c_0^\dagger|\Omega\rangle - |201\rangle\,|0\rangle\,|200\rangle\,a_0^\dagger c_0^\dagger|\Omega\rangle - |211\rangle\,|0\rangle\,|200\rangle\,a_1^\dagger c_0^\dagger|\Omega\rangle$$
$$+ |200\rangle\,|0\rangle\,|211\rangle\,a_0^\dagger c_1^\dagger|\Omega\rangle + |210\rangle\,|0\rangle\,|211\rangle\,a_1^\dagger c_1^\dagger|\Omega\rangle - |201\rangle\,|0\rangle\,|201\rangle\,a_0^\dagger c_1^\dagger|\Omega\rangle - |211\rangle\,|0\rangle\,|201\rangle\,a_1^\dagger c_1^\dagger|\Omega\rangle\Bigg]$$

- **Matter in $x+e_2$ and $x+e_1$:** $|matter\rangle = b_{m_b}^\dagger c_{m_c}^\dagger |\Omega\rangle$. We trivially get: $J_1 = 0$; $J_2 = 2$ and $J_3 = 0$.

$$|\Psi_7\rangle = Q^{\bar{0}_{M_2,M_3}Q^0_{N_2,m_b}Q^0_{N_3,m_c}|0\rangle|2M_2N_2\rangle|0\rangle|2M_3N_3\rangle b_{m_b}^\dagger c_{m_c}^\dagger|\Omega\rangle}$$
$$= ...$$
$$|\Psi_7\rangle = \frac{1}{2\sqrt{2}}\Bigg[|0\rangle\,|200\rangle\,|210\rangle\,b_0^\dagger c_0^\dagger|\Omega\rangle + |0\rangle\,|200\rangle\,|211\rangle\,b_0^\dagger c_1^\dagger|\Omega\rangle + |0\rangle\,|201\rangle\,|210\rangle\,b_1^\dagger c_0^\dagger|\Omega\rangle + |0\rangle\,|201\rangle\,|211\rangle\,b_1^\dagger c_1^\dagger|\Omega\rangle$$
$$- |0\rangle\,|210\rangle\,|200\rangle\,b_0^\dagger c_0^\dagger|\Omega\rangle - |0\rangle\,|210\rangle\,|201\rangle\,b_0^\dagger c_1^\dagger|\Omega\rangle - |0\rangle\,|211\rangle\,|200\rangle\,b_1^\dagger c_0^\dagger|\Omega\rangle - |0\rangle\,|211\rangle\,|201\rangle\,b_1^\dagger c_1^\dagger|\Omega\rangle\Bigg]$$

Third option: one even site and odd site with one fermion:

- **Matter in $x-e_1$ and x:** $|matter\rangle = a_{m_a}^\dagger d_{m_d}^\dagger |\Omega\rangle$. We trivially get: $J_1, M_1 = 2, m_a$; $J_2 = 0$ and $J_3 = 0$.

$$|\Psi_8\rangle = \varepsilon_{n_d,m_d}Q^0_{N_1,n_d}\,|2m_aN_1\rangle\,|0\rangle\,|0\rangle\,a_{m_a}^\dagger d_{m_d}^\dagger|\Omega\rangle$$
$$\boxed{|\Psi_8\rangle = \frac{1}{2}\Bigg[|200\rangle\,|0\rangle\,|0\rangle\,a_0^\dagger d_1^\dagger|\Omega\rangle + |210\rangle\,|0\rangle\,|0\rangle\,a_1^\dagger d_1^\dagger|\Omega\rangle - |201\rangle\,|0\rangle\,|0\rangle\,a_0^\dagger d_0^\dagger|\Omega\rangle - |211\rangle\,|0\rangle\,|0\rangle\,a_1^\dagger d_0^\dagger|\Omega\rangle\Bigg]}$$

- **Matter in $x+e_1$ and x:** $|matter\rangle = c^\dagger_{m_c} d^\dagger_{m_d} |\Omega\rangle$. We trivially get: $J_1 = 0$; $J_2 = 0$ and $J_3, M_3 = 2, n_d$.

$$|\Psi_9\rangle = \varepsilon_{n_d,m_d} Q^0_{N_3,m_c} |0\rangle |0\rangle |2n_d N_3\rangle c^\dagger_{m_c} d^\dagger_{m_d} |\Omega\rangle$$

$$\boxed{|\Psi_9\rangle = \frac{1}{2}\Bigg[|0\rangle |0\rangle |200\rangle c^\dagger_0 d^\dagger_1 |\Omega\rangle + |0\rangle |0\rangle |201\rangle c^\dagger_1 d^\dagger_1 |\Omega\rangle - |0\rangle |0\rangle |210\rangle c^\dagger_0 d^\dagger_0 |\Omega\rangle - |0\rangle |0\rangle |211\rangle c^\dagger_1 d^\dagger_0 |\Omega\rangle \Bigg]}$$

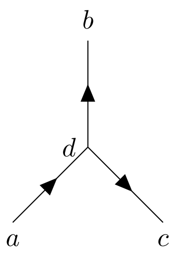
- **Matter in $x+e_2$ and x:** $|matter\rangle = b^\dagger_{m_b} d^\dagger_{m_d} |\Omega\rangle$. We trivially get: $J_1 = 0$; $J_2, M_2 = 2, n_d$ and $J_3 = 0$.

$$|\Psi_{10}\rangle = \varepsilon_{n_d,m_d} Q^0_{N_2,m_b} |0\rangle |2n_d N_2\rangle |0\rangle b^\dagger_{m_b} d^\dagger_{m_d} |\Omega\rangle$$

$$\boxed{|\Psi_{10}\rangle = \frac{1}{2}\Bigg[|0\rangle |200\rangle |0\rangle b^\dagger_0 d^\dagger_1 |\Omega\rangle + |0\rangle |201\rangle |0\rangle b^\dagger_1 d^\dagger_1 |\Omega\rangle - |0\rangle |210\rangle |0\rangle b^\dagger_0 d^\dagger_0 |\Omega\rangle - |0\rangle |211\rangle |0\rangle b^\dagger_1 d^\dagger_0 |\Omega\rangle \Bigg]}$$

15.4 Checking that the states are rotational invariant

We checked manually that the ten states are respecting the different Gauss' laws, nevertheless we would like to check also their rotational invariance (not of the states but of set as a whole) just to be sure that we are not forgetting anything. The system could be visualized as the following:



The direction of the arrow is the key-point of the non-abelian property of this model. The matter transformation under clockwise rotations of 120 degrees is:

$$\begin{cases} d^\dagger_m \longrightarrow d^\dagger_m \\ a^\dagger_m \longrightarrow b^\dagger_m \longrightarrow c^\dagger_m \longrightarrow a^\dagger_m \end{cases}$$

While for the gauge connection we have (recall that $(U^\dagger)_{mn} = U^\dagger_{nm}$):

$$\begin{cases} U^{(1)}_{mn} \longrightarrow U^{(2)\dagger}_{nm} \\ U^{(2)}_{mn} \longrightarrow U^{(3)}_{mn} \\ U^{(3)}_{mn} \longrightarrow U^{(1)\dagger}_{nm} \end{cases}$$

Hence, recall that $|JMN\rangle = \sqrt{dim(j)} U^{j\dagger}_{nm} |0\rangle = \sqrt{dim(j)} U^j_{nm} |0\rangle$ where we use the fact that, for D_8, $U = U^\dagger$. With this and knowing that $|0\rangle$ is the singlet and therefore invariant under all transformation, we can find how the states given by the 2D-irreducible representation change under these rotations:

$$\begin{cases} |JMN\rangle_1 \longrightarrow |JNM\rangle_2 \\ |JMN\rangle_2 \longrightarrow |JMN\rangle_3 \\ |JMN\rangle_3 \longrightarrow |JNM\rangle_1 \end{cases}$$

so that at the end we have: $|JMN\rangle_1 \longrightarrow |JNM\rangle_2 \longrightarrow |JNM\rangle_3 \longrightarrow |JMN\rangle_1$.
Note: for the states given by 1D irreducible representation we don't have any change as they are not sensible to the left/right algebra.
To show rotational invariance, we have to check that the Hamiltonian is rotation invariant and that the states we found before are somehow related by rotations. For the first, the mass and electric terms are trivially invariant while we must pay more attention on the interaction term:

$$H_{GM} = -J\Bigg[a^\dagger_m U^{(1)}_{mn} d_n + b^\dagger_m (U^{(2)})^\dagger_{mn} d_n + c^\dagger_m (U^{(3)})^\dagger_{mn} d_n + h.c. \Bigg]$$

but using the rules we stated before we can see that each of this term is mapped in its $h.c.$ under rotations. For the ladder, it is easy to see (again following the rules above) that:

$$\begin{cases} |\Psi_1\rangle \longrightarrow |\Psi_2\rangle \longrightarrow |\Psi_3\rangle \longrightarrow |\Psi_1\rangle \\ |\Psi_4\rangle \longrightarrow |\Psi_4\rangle \\ |\Psi_5\rangle \longrightarrow |\Psi_6\rangle \longrightarrow |\Psi_7\rangle \longrightarrow |\Psi_5\rangle \\ |\Psi_8\rangle \longrightarrow |\Psi_9\rangle \longrightarrow |\Psi_{10}\rangle \longrightarrow |\Psi_8\rangle \end{cases}$$

hence everything seems to work smoothly.

15.5 Hamiltonian over gauge invariant states

We remind that the Hamiltonin for this model is made out of three terms: the mass term, the electric term and the gauge-matter interaction term.

$$H = M\sum_{x,i}(-1)^{s(x)}n_i(x) + \lambda_E \sum_{links} |2, mn\rangle \langle 2, mn| - J\left[a^\dagger_m U^{(1)}_{mn} d_n + d^\dagger_m U^{(2)}_{mn} b_n + d^\dagger_m U^{(3)}_{mn} c_n + h.c.\right]$$

From now on we will use as a basis $\{|\Psi_i\rangle\}_{i=1,2,\ldots 10}$ and the following matrices are evaluated over it.

Mass term:

It is trivial to see that $\boxed{H_M = 2M diag(1, 1, 1, -1, 1, 1, 1, 0, 0, 0)}$.

Electric term:

Here we need to count the number of links that are in the 2D-irreps.
Again, it is trivial to see that $\boxed{H_E = \lambda_E diag(0, 0, 0, 0, 2, 2, 2, 1, 1, 1)}$.

Interaction term:

We divide this term into three parts as it will be easier to evaluate them:

- $-Ja^\dagger_m U^{(1)}_{mn} d_n = H_1$ Many matrix elements are just zero as we don't have any fermion (of that specif species) to annihilate. Hence the only non-vanishing terms are:

$$\langle\Psi_8| H_1 |\Psi_4\rangle = \langle\Psi_1| H_1 |\Psi_8\rangle = -J\sqrt{2}$$
$$\langle\Psi_6| H_1 |\Psi_9\rangle = \langle\Psi_5| H_1 |\Psi_{10}\rangle = -J$$

- $-Jd^\dagger_m U^{(1)}_{mn} b_n = H_2$ Following the same reasoning:

$$\langle\Psi_4| H_2 |\Psi_{10}\rangle = \langle\Psi_{10}| H_2 |\Psi_2\rangle = -J\sqrt{2}$$
$$\langle\Psi_9| H_2 |\Psi_7\rangle = \langle\Psi_8| H_2 |\Psi_5\rangle = -J$$

- $-Jd^\dagger_m U^{(1)}_{mn} c_n = H_3$ Following the same reasoning:

$$\langle\Psi_9| H_3 |\Psi_3\rangle = \langle\Psi_4| H_3 |\Psi_9\rangle = -J\sqrt{2}$$
$$\langle\Psi_{10}| H_3 |\Psi_7\rangle = \langle\Psi_8| H_3 |\Psi_6\rangle = -J$$

We can therefore finally compute the spectrum $H = H_M + H_E + \underbrace{H_1 + H_2 + H_3}_{H_{INT}}$: :

$$H = \begin{bmatrix} 2M & 0 & 0 & 0 & 0 & 0 & 0 & -J\sqrt{2} & 0 & 0 \\ 0 & 2M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -J\sqrt{2} \\ 0 & 0 & 2M & 0 & 0 & 0 & 0 & 0 & -J\sqrt{2} & 0 \\ 0 & 0 & 0 & -2M & 0 & 0 & 0 & -J\sqrt{2} & -J\sqrt{2} & -J\sqrt{2} \\ 0 & 0 & 0 & 0 & M+2\lambda_E & 0 & 0 & -J & 0 & -J \\ 0 & 0 & 0 & 0 & 0 & M+2\lambda_E & 0 & -J & -J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M+2\lambda_E & 0 & -J & -J \\ -J\sqrt{2} & 0 & 0 & -J\sqrt{2} & -J & -J & 0 & \lambda_E & 0 & 0 \\ 0 & 0 & -J\sqrt{2} & -J\sqrt{2} & 0 & -J & -J & 0 & \lambda_E & 0 \\ 0 & -J\sqrt{2} & 0 & -J\sqrt{2} & -J & 0 & -J & 0 & 0 & \lambda_E \end{bmatrix}$$

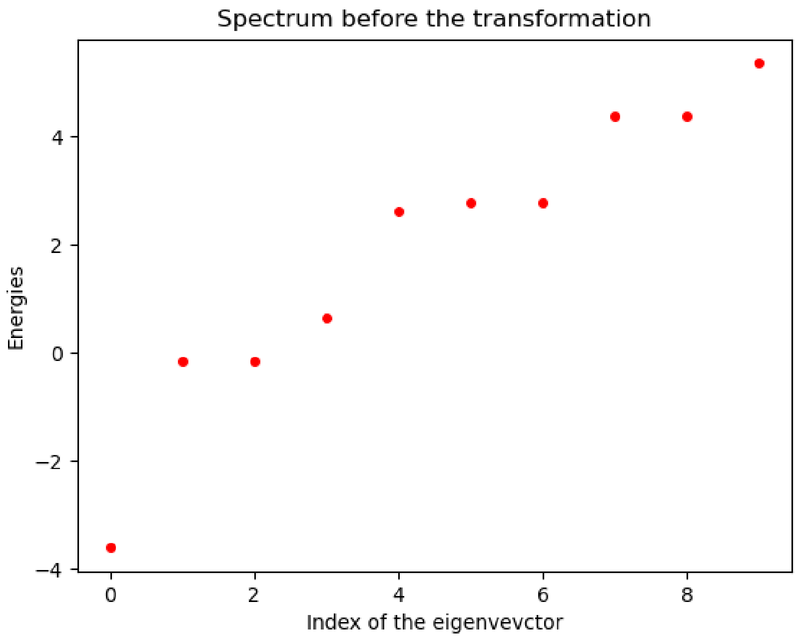


Figure 9: Spectrum evaluated with the following values for the parameters: $M = 1$, $\lambda_E = 1$ and $J = 1$.

16 Pseudo-2D model without matter

16.1 Hamiltonian under the transformation:

Note: The gate version was done using the group element basis and not the representation basis as we did for the calculations with pen and paper. Recall that $0 = a^2x$ and $1 = x$

16.2 Mass Hamiltonian

Recall that for the $0, 1$ indeces as well as for Π we have that these are hermitian operators and therefore we can drop the †.

$$\begin{aligned}
H_M &= M\sum_{x,i}(-1)^{s(x)}n_i(x) && \text{in our case we chose the fundamental representation} \implies i = 1, 2 \\
&= M\sum_{x,i}\frac{(-1)^{s(x)} - G_i(x)}{2} && \sum_{x,i}(-1)^{s(x)} = 4 \\
&= 2M - \frac{M}{2}\sum_x G_0(x) + G_1(x) && \text{as always, we don't care about constant and so we can drop } 2M. \\
&= -\frac{M}{2}\Bigg[\theta_0^R(1)\theta_0^L(2)\theta_0^L(3) + \theta_1^R(1)\theta_1^L(2)\theta_1^L(3) + \theta_0^L(1) + \theta_1^L(1) + \theta_0^R(2) + \theta_1^R(2) + \theta_0^R(3) + \theta_1^R(3)\Bigg]
\end{aligned}$$

and the gate formulation becomes:

$$\begin{aligned}
H_M = -\frac{M}{2}\Bigg\{&\Big[X\cdot(X_{02}+X_{13})\Big]_1 \times \Big[X\cdot(X_{02}+P_{13})\Big]_2 \times \Big[X\cdot(X_{02}+P_{13})\Big]_3 + \\
&+ \Big[X\cdot 1\Big]_1 \times \Big[X\cdot(P_{02}+X_{13})\Big]_2 \times \Big[X\cdot(P_{02}+X_{13})\Big]_3 + \Big[X\cdot(X_{02}+P_{13})\Big]_1 + \Big[X\cdot(P_{02}+X_{13})\Big]_1 + \Big[X\cdot(X_{02}+X_{13})\Big]_2 \\
&+ \Big[X\cdot 1\Big]_2 + \Big[X\cdot(X_{02}+X_{13})\Big]_3 + \Big[X\cdot 1\Big]_3\Bigg\} \equiv -\frac{M}{2}\sum_{i=1}^{8}H_M(i)
\end{aligned}$$

16.3 Electric part

$$\begin{aligned}
H_E &= \lambda_E\sum_{links}|2,mn\rangle\langle 2,mn| \\
&= -\frac{\lambda_E}{2}\Bigg(\Pi_1 + \Pi_2 + \Pi_3\Bigg) \\
&= -\frac{\lambda_E}{2}\sum_{i=1}^{3}\Bigg[1\cdot(X_{02}+X_{13})\Bigg]_i
\end{aligned}$$

16.4 Interaction parts

16.4.1 First link

Note: here we have $(-1)^x = 1$

$$\begin{aligned}
H^1 &= H^1_{00} + H^1_{01} + H^1_{10} + H^1_{11} \\
H^1_{00} &= iU_{00}(1)\theta_0^R(1)\theta_0^L(2)\theta_0^L(3) - i\theta_0^L(1)U_{00}(1) \\
H^1_{01} &= i\theta_0^L(1)U_{01}(1)\theta_0^R(1)\theta_0^L(2)\theta_0^L(3) - iU_{01}(1)\Pi(1)\Pi(2)\Pi(3) \\
H^1_{10} &= i\theta_0^L(1)U_{10}(1)\theta_0^R(1)\theta_0^L(2)\theta_0^L(3) - i\Pi(1)U_{10}(1) \\
H^1_{11} &= i\Pi(1)U_{11}(1)\theta_0^R(1)\theta_0^L(2)\theta_0^L(3) - i\theta_0^L(1)U_{11}(1)\Pi(1)\Pi(2)\Pi(3)
\end{aligned}$$

And now we write them in terms of gates:

$$H^1_{00} = -\left\{ \Big[X \cdot Y_{02} \Big]_1 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[X \cdot Y_{02} \Big]_1 \right\}$$

$$H^1_{01} = -\left\{ \Big[Z \cdot Y_{13} \Big]_1 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[Z \cdot Y_{13} \Big]_1 \times \Big[1 \cdot (X_{02} + X_{13}) \Big]_2 \times \Big[1 \cdot (X_{02} + X_{13}) \Big]_3 \right\}$$

$$H^1_{10} = -\left\{ \Big[1 \cdot Y_{13} \Big]_1 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[1 \cdot Y_{13} \Big]_1 \right.$$

$$H^1_{11} = \Big[Y \cdot Z_{02} \Big]_1 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[Y \cdot Z_{02} \Big]_1 \times \Big[1 \cdot (X_{02} + X_{13}) \Big]_2 \times \Big[1 \cdot (X_{02} + X_{13}) \Big]_3$$

16.4.2 Second link

Note: here we have $(-1)^x = -1$

$$\begin{aligned}
H^2 &= H^2_{00} + H^2_{01} + H^2_{10} + H^2_{11} \\
H^2_{00} &= i\theta^R_1(1)\theta^L_0(2)U_{00}(2)\theta^L_0(3) - i\Pi(1)U_{00}(2)\theta^R_0(2) \\
H^2_{01} &= i\theta^R_1(1)\theta^L_0(2)U_{01}(2)\theta^R_0(2)\theta^L_0(3) - i\Pi(1)U_{01}(2)\Pi(2) \\
H^2_{10} &= i\theta^R_1(1)\theta^L_0(2)U_{10}(2)\theta^R_0(2)\theta^L_0(3) - i\Pi(2)U_{10}(2)\Pi(3) \\
H^2_{11} &= i\theta^R_1(1)\theta^L_0(2)U_{11}(2)\Pi(2)\theta^L_0(3) - i\Pi(2)U_{11}(2)\theta^R_0(2)\Pi(3)
\end{aligned}$$

And now we write them in terms of gates:

$$H^2_{00} = \Big[X \cdot 1 \Big]_1 \times \Big[X \cdot Y_{02} \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[1 \cdot (X_{02} + X_{13}) \Big]_1 \times \Big[X \cdot Y_{02} \Big]_2$$

$$H^2_{01} = -\left\{ \Big[X \cdot 1 \Big]_1 \times \Big[Z \cdot Y_{13} \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[1 \cdot (X_{02} + X_{13}) \Big]_1 \times \Big[Z \cdot Y_{13} \Big]_2 \right\}$$

$$H^2_{10} = -\left\{ \Big[X \cdot 1 \Big]_1 \times \Big[1 \cdot Y_{13} \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[1 \cdot Y_{13} \Big]_2 \times \Big[1 \cdot (X_{02} + X_{13}) \Big]_3 \right\}$$

$$H^2_{11} = -\left\{ \Big[X \cdot 1 \Big]_1 \times \Big[Y \cdot Z_{02} \Big]_2 \times \Big[X \cdot (X_{02} + P_{13}) \Big]_3 + \Big[Y \cdot Z_{02} \Big]_2 \times \Big[1 \cdot (X_{02} + X_{13}) \Big]_3 \right\}$$

16.4.3 Third link

Note: here we have $(-1)^x = -1$

$$\begin{aligned}
H^3 &= H^3_{00} + H^3_{01} + H^3_{10} + H^3_{11} \\
H^3_{00} &= i\theta^R_1(1)\theta^L_1(2)\theta^L_0(3)U_{00}(3) - i\Pi(1)\Pi(2)U_{00}(3)\theta^R_0(3) \\
H^3_{01} &= i\theta^R_1(1)\theta^L_1(2)\theta^L_0(3)U_{01}(3)\theta^R_0(3) - i\Pi(1)\Pi(2)U_{01}(3)\Pi(3) \\
H^3_{10} &= i\theta^R_1(1)\theta^L_1(2)\theta^L_0(3)U_{10}(3)\theta^R_0(3) - i\Pi(3)U_{10}(3) \\
H^3_{11} &= i\theta^R_1(1)\theta^L_1(2)\theta^L_0(3)U_{11}(3)\Pi(3) - i\Pi(3)U_{11}(3)\theta^R_0(3)
\end{aligned}$$

And now we write them in terms of gates:

$$H^3_{00} = \Big[X \cdot 1\Big]_1 \times \Big[X \cdot (P_{02} + X_{13})\Big]_2 \times \Big[X \cdot Y_{02}\Big]_3 + \Big[1 \cdot (X_{02} + X_{13})\Big]_1 \times \Big[1 \cdot (X_{02} + X_{13})\Big]_2 \times \Big[X \cdot Y_{02}\Big]_3$$

$$H^3_{01} = -\Big\{\Big[X \cdot 1\Big]_1 \times \Big[X \cdot (P_{02} + X_{13})\Big]_2 \times \Big[Z \cdot Y_{13}\Big]_3 + \Big[1 \cdot (X_{02} + X_{13})\Big]_1 \times \Big[1 \cdot (X_{02} + X_{13})\Big]_2 \times \Big[Z \cdot Y_{13}\Big]_3\Big\}$$

$$H^3_{10} = -\Big\{\Big[X \cdot 1\Big]_1 \times \Big[X \cdot (P_{02} + X_{13})\Big]_2 \times \Big[1 \cdot Y_{13}\Big]_3 + \Big[1 \cdot Y_{13}\Big]_3\Big\}$$

$$H^3_{11} = -\Big\{\Big[X \cdot 1\Big]_1 \times \Big[X \cdot (P_{02} + X_{13})\Big]_2 \times \Big[Y \cdot Z_{02}\Big]_3 + \Big[Y \cdot Z_{02}\Big]_3\Big\}$$

	Max # qubits	Max # qudits	Max # bodies	structure of the hardest term
Electric term	0	1	1	1 qudits
Mass term	3	3	6	3 qubits + 3 qudits
Vertical int.	3	2	5	3 qubits + 2 qudits
Horizontal int.	3	3	6	3 qubits + 3 qudits

Table 7: Features of the different parts of the pseudo-2D Hamiltonian.

16.5 Gauge-invariant states under transformation

In the previous section we found ten gauge-invariant states, to benchmark the hamiltonian of the pseudo-2D model without the matter we would like to compute H over the new transformed states which we can obtain applying the transformation $\mathcal{U}$. Recall that:

$$\mathcal{U} = \prod_x \mathcal{U}_1(x)\mathcal{U}_0(x) = \prod_x \Big[P_1^+ \otimes (\eta_1 + \eta_1^\dagger) + P_1^- \otimes 1\Big]\Big[P_0^+ \otimes (\eta_0 + \eta_0^\dagger) + P_0^- \otimes 1\Big] = \mathcal{U}_a\mathcal{U}_b\mathcal{U}_c\mathcal{U}_d$$

$$= \Big[P_1^+ \otimes (a_1 + a_1^\dagger) + P_1^- \otimes 1\Big]\Big[P_0^+ \otimes (a_0 + a_0^\dagger) + P_0^- \otimes 1\Big]\Big[P_1^+ \otimes (b_1 + b_1^\dagger) + P_1^- \otimes 1\Big]\Big[P_0^+ \otimes (b_0 + b_0^\dagger) + P_0^- \otimes 1\Big]\times$$

$$\times \Big[P_1^+ \otimes (c_1 + c_1^\dagger) + P_1^- \otimes 1\Big]\Big[P_0^+ \otimes (c_0 + c_0^\dagger) + P_0^- \otimes 1\Big]\Big[P_1^+ \otimes (d_1 + d_1^\dagger) + P_1^- \otimes 1\Big]\Big[P_0^+ \otimes (d_0 + d_0^\dagger) + P_0^- \otimes 1\Big].$$

We can now apply it to $\{|\psi_i\rangle\}_{i=1,2\ldots10}$:

First option: one site full and the others empty:

$$\begin{aligned}
\mathcal{U}|\psi_1\rangle &= \mathcal{U}_a\mathcal{U}_b\mathcal{U}_c\mathcal{U}_d a_0^\dagger a_1^\dagger |\Omega\rangle |\bar{0},0,0\rangle && \mathcal{U}_b\mathcal{U}_c\mathcal{U}_d|\psi_1\rangle = |\psi_1\rangle \\
&= \mathcal{U}_a a_0^\dagger a_1^\dagger |\Omega\rangle |\bar{0},0,0\rangle \\
&= P_1^+(1)P_0^-(1)(a_1 + a_1^\dagger)(a_0 + a_0^\dagger)a_0^\dagger a_1^\dagger |\Omega\rangle |\bar{0},0,0\rangle && (a_m^\dagger)^2 = 0 \\
P_1^+(1)P_0^-(1)a_1 a_0 a_0^\dagger a_1^\dagger |\Omega\rangle |\bar{0},0,0\rangle &= P_1^+(1)P_0^-(1)|\Omega\rangle |\bar{0},0,0\rangle \\
&= \frac{1}{4}|\Omega\rangle \Big(1 - \theta_x^L(1) - \theta_{a^2x}^L(1) + \theta_{a^2}^L(1)\Big)|\bar{0},0,0\rangle && \text{and now we perform the projection}
\end{aligned}$$

$$\boxed{\left|\tilde{\psi}_1\right\rangle = |\bar{0},0,0\rangle}$$

Following the same technique we can similarly find that:

$$\boxed{\left|\tilde{\psi}_2\right\rangle = |0,\bar{0},0\rangle}$$

$$\boxed{\left|\tilde{\psi_3}\right\rangle = |0,0,\bar{0}\rangle}$$

$$\boxed{\left|\tilde{\psi_4}\right\rangle = |0,0,0\rangle}$$

1

Second option: two even site with one fermion:

$$\begin{aligned}
&\mathcal{U}\,|\psi_5\rangle = \mathcal{U}_a\mathcal{U}_b\mathcal{U}_c\mathcal{U}_d\,|\psi_5\rangle \\
&= \ldots \\
&= \frac{1}{2\sqrt{2}}[P_0^+(1)P_0^+(2)\,|200\rangle\,|210\rangle\,|0\rangle + P_1^+(1)P_0^+(2)\,|210\rangle\,|210\rangle\,|0\rangle - P_0^+(1)P_0^+(2)\,|201\rangle\,|200\rangle\,|0\rangle - P_1^+(1)P_0^+(2)\,|211\rangle\,|200\rangle\,|0\rangle \\
&+ P_0^+(1)P_1^+(2)\,|200\rangle\,|211\rangle\,|0\rangle + P_1^+(1)P_1^+(2)\,|210\rangle\,|211\rangle\,|0\rangle - P_0^+(1)P_1^+(2)\,|210\rangle\,|210\rangle\,|0\rangle - P_1^+(1)P_1^+(2)\,|211\rangle\,|210\rangle\,|0\rangle]
\end{aligned}$$

$$\boxed{\begin{aligned}\left|\tilde{\psi_5}\right\rangle = \frac{1}{2\sqrt{2}}\Bigg[&|200\rangle\,|210\rangle\,|0\rangle + |210\rangle\,|210\rangle\,|0\rangle - |201\rangle\,|200\rangle\,|0\rangle - |211\rangle\,|200\rangle\,|0\rangle + \\ &|200\rangle\,|211\rangle\,|0\rangle + |210\rangle\,|211\rangle\,|0\rangle - |210\rangle\,|210\rangle\,|0\rangle - |211\rangle\,|210\rangle\,|0\rangle\Bigg]\end{aligned}}.$$

Similarely:

$$\boxed{\begin{aligned}\left|\tilde{\psi_6}\right\rangle = \frac{1}{2\sqrt{2}}\Bigg[&|200\rangle\,|0\rangle\,|210\rangle + |210\rangle\,|0\rangle\,|210\rangle - |201\rangle\,|0\rangle\,|200\rangle - |211\rangle\,|0\rangle\,|200\rangle \\ &+ |200\rangle\,|0\rangle\,|211\rangle + |210\rangle\,|0\rangle\,|211\rangle - |201\rangle\,|0\rangle\,|201\rangle - |211\rangle\,|0\rangle\,|201\rangle\Bigg]\end{aligned}}.$$

$$\boxed{\begin{aligned}\left|\tilde{\psi_7}\right\rangle = \frac{1}{2\sqrt{2}}\Bigg[&|0\rangle\,|200\rangle\,|210\rangle + |0\rangle\,|200\rangle\,|211\rangle + |0\rangle\,|201\rangle\,|210\rangle + |0\rangle\,|201\rangle\,|211\rangle \\ &- |0\rangle\,|210\rangle\,|200\rangle - |0\rangle\,|210\rangle\,|201\rangle - |0\rangle\,|211\rangle\,|200\rangle - |0\rangle\,|211\rangle\,|201\rangle\Bigg]\end{aligned}}.$$

Third option: one even site and odd site with one fermion:

$$\boxed{\left|\tilde{\psi_8}\right\rangle = \frac{1}{2}\Bigg[|200\rangle\,|0\rangle\,|0\rangle + |210\rangle\,|0\rangle\,|0\rangle - |201\rangle\,|0\rangle\,|0\rangle - |211\rangle\,|0\rangle\,|0\rangle\Bigg]}.$$

$$\boxed{\left|\tilde{\psi_9}\right\rangle = \frac{1}{2}\Bigg[|0\rangle\,|0\rangle\,|200\rangle + |0\rangle\,|0\rangle\,|201\rangle - |0\rangle\,|0\rangle\,|210\rangle - |0\rangle\,|0\rangle\,|211\rangle\Bigg]}.$$

$$\boxed{\left|\tilde{\psi_{10}}\right\rangle = \frac{1}{2}\Bigg[|0\rangle\,|200\rangle\,|0\rangle + |0\rangle\,|201\rangle\,|0\rangle - |0\rangle\,|210\rangle\,|0\rangle - |0\rangle\,|211\rangle\,|0\rangle\Bigg]}.$$

Note that the rotational invariance is preserved. Indeed, while with the transformation breaks the 90° invariance of the Hamiltonian, the states remain invariant under rotation of 90°.

[1] In the group element basis we have: $|0\rangle = \sum_g |g\rangle\langle g|0\rangle = \sum_g \sqrt{\frac{dim(j)}{|G|}} D^j_{mn}(g)\,|g\rangle = \sum_g \frac{1}{2\sqrt{2}}\,|g\rangle$

16.6 Hamiltonian over the transformed states

Mass term:

It is trivial to see that $\boxed{H_M = 2Mdiag(1,1,1,-1,1,1,1,0,0,0)}$.

Electric term:

Here we need to count the number of links that are in the 2D-irreps.
Again, it is trivial to see that $\boxed{H_E = \lambda_E diag(0,0,0,0,2,2,2,1,1,1)}$.

Note: we can see that the electric and mass terms are exactly the same of what we found in 15.5.

Interaction term:

We divide this term into three parts as it will be easier to evaluate them (we exactly evaluate them using Python):

- Many matrix elements are just zero as we don't have any fermion (of that specif species) to annihilate. Hence the only non-vanishing terms are:

$$\left\langle \tilde{\psi_4} \right| H_1 \left| \tilde{\psi_8} \right\rangle = \left\langle \tilde{\psi_1} \right| H_1 \left| \tilde{\psi_8} \right\rangle = -iJ\sqrt{2} = -\left\langle \tilde{\psi_8} \right| H_1 \left| \tilde{\psi_4} \right\rangle = -\left\langle \tilde{\psi_8} \right| H_1 \left| \tilde{\psi_1} \right\rangle$$
$$\left\langle \tilde{\psi_6} \right| H_1 \left| \tilde{\psi_9} \right\rangle = \left\langle \tilde{\psi_5} \right| H_1 \left| \tilde{\psi_{10}} \right\rangle = -iJ = -\left\langle \tilde{\psi_9} \right| H_1 \left| \tilde{\psi_6} \right\rangle = -\left\langle \tilde{\psi_{10}} \right| H_1 \left| \tilde{\psi_5} \right\rangle$$

- Similarly:

$$\left\langle \tilde{\psi_2} \right| H_2 \left| \tilde{\psi_{10}} \right\rangle = \left\langle \tilde{\psi_4} \right| H_2 \left| \tilde{\psi_{10}} \right\rangle = iJ\sqrt{2} = -\left\langle \tilde{\psi_{10}} \right| H_2 \left| \tilde{\psi_2} \right\rangle = -\left\langle \tilde{\psi_{10}} \right| H_2 \left| \tilde{\psi_4} \right\rangle =$$
$$\left\langle \tilde{\psi_5} \right| H_2 \left| \tilde{\psi_8} \right\rangle = \left\langle \tilde{\psi_7} \right| H_2 \left| \tilde{\psi_9} \right\rangle = iJ = -\left\langle \tilde{\psi_8} \right| H_2 \left| \tilde{\psi_5} \right\rangle = -\left\langle \tilde{\psi_9} \right| H_2 \left| \tilde{\psi_7} \right\rangle$$

- Following the same reasoning:

$$\left\langle \tilde{\psi_3} \right| H_3 \left| \tilde{\psi_9} \right\rangle = \left\langle \tilde{\psi_4} \right| H_3 \left| \tilde{\psi_9} \right\rangle = iJ\sqrt{2} = -\left\langle \tilde{\psi_9} \right| H_3 \left| \tilde{\psi_3} \right\rangle = -\left\langle \tilde{\psi_9} \right| H_3 \left| \tilde{\psi_4} \right\rangle$$
$$\left\langle \tilde{\psi_6} \right| H_3 \left| \tilde{\psi_8} \right\rangle = \left\langle \tilde{\psi_7} \right| H_3 \left| \tilde{\psi_{10}} \right\rangle = iJ = -\left\langle \tilde{\psi_8} \right| H_3 \left| \tilde{\psi_6} \right\rangle = -\left\langle \tilde{\psi_{10}} \right| H_3 \left| \tilde{\psi_7} \right\rangle$$

We can therefore finally compute the spectrum of $H' = H_M + H_E + \underbrace{H_1 + H_2 + H_3}_{H_{INT}}$:

$$H' = \begin{bmatrix} 2M & 0 & 0 & 0 & 0 & 0 & 0 & -iJ\sqrt{2} & 0 & 0 \\ 0 & 2M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & iJ\sqrt{2} \\ 0 & 0 & 2M & 0 & 0 & 0 & 0 & 0 & iJ\sqrt{2} & 0 \\ 0 & 0 & 0 & -2M & 0 & 0 & 0 & -iJ\sqrt{2} & iJ\sqrt{2} & iJ\sqrt{2} \\ 0 & 0 & 0 & 0 & M+2\lambda_E & 0 & 0 & iJ & 0 & -iJ \\ 0 & 0 & 0 & 0 & 0 & M+2\lambda_E & 0 & iJ & -iJ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M+2\lambda_E & 0 & iJ & iJ \\ iJ\sqrt{2} & 0 & 0 & iJ\sqrt{2} & -iJ & -iJ & 0 & \lambda_E & 0 & 0 \\ 0 & 0 & -iJ\sqrt{2} & -iJ\sqrt{2} & 0 & iJ & -iJ & 0 & \lambda_E & 0 \\ 0 & -iJ\sqrt{2} & 0 & -iJ\sqrt{2} & iJ & 0 & -iJ & 0 & 0 & \lambda_E \end{bmatrix}$$

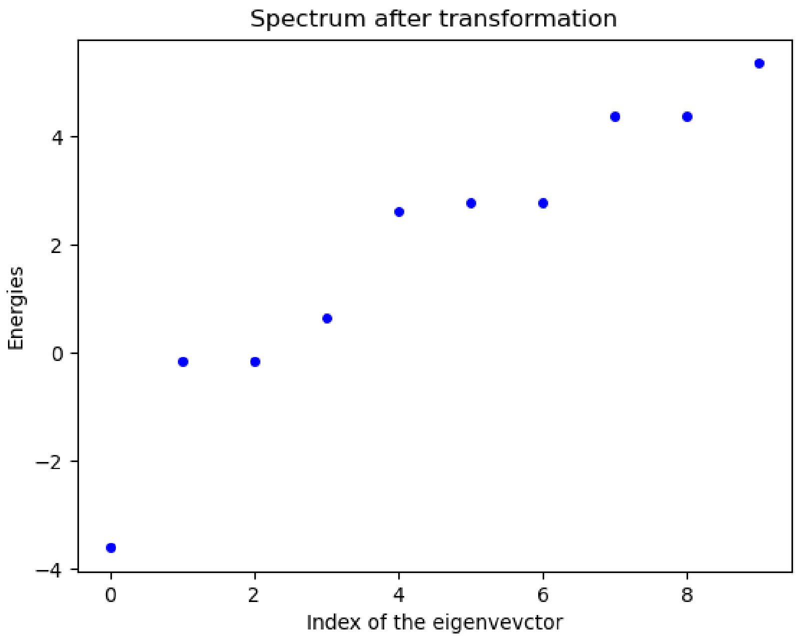


Figure 10: Spectrum evaluated with the following values for the parameters: $M = 1$, $\lambda_E = 1$ and $J = 1$. As we can easily see, it is exactly the same of what we obtained in 9. Moreover, to close the loop, we found a unitary matrix D such that $H' = DHD^\dagger$ as expected and we also checked that $DD^\dagger = 1$. We can find the explicit form of H and H' in 15.5 and 16.6.

16.7 VQE preparation of the pseudo-2D model

We reuse again the rules that were written in 12.1 and we try to build the smallest number of sets:

$$
\begin{aligned}
A^1 &\equiv \{H^1_{00}, H^1_{10}, H^1_{11}, H^3_{00}, H^3_{10}, H^3_{11}, \Pi_2\} \\
A^2 &\equiv \{H^2_{00}, H^2_{10}, H^2_{11}, \Pi_3, \Pi_1\} \\
A^3 &\equiv \{H^1_{01}, H^3_{01}, H_M(6)\} \\
A^4 &\equiv \{H^2_{01}, H_M(3), H_M(4), H_M(5), H_M(7), H_M(8)\} \\
A^5 &\equiv \{H_M(1), H_M(2)\}.
\end{aligned}
$$

Unfortunately, we are not able to reduce utterly the mass term into other sets and we are left with 5 sets; a valid alternative is also:

$$
\begin{aligned}
A^1 &\equiv \{H^1_{00}, H^1_{10}, H^1_{11}, H^3_{00}, H^3_{10}, H^3_{11}, \Pi_2\} \\
A^2 &\equiv \{H^2_{00}, H^2_{10}, H^2_{11}, \Pi_3, \Pi_1\} \\
A^3 &\equiv \{H^1_{01}, H^3_{01}\} \\
A^4 &\equiv \{H^2_{01}\} \\
A^5 &\equiv \{H_M\}.
\end{aligned}
$$

where we have all the mass terms together.
When running the VQE, for real time evolution is interesting to see how states of a particular sectors evolve in time. As an example we consider the easiest gauge invariant state we found and one with an excitation.

$$
\begin{aligned}
\left|\tilde{\psi}_8\right\rangle &= \frac{1}{2}\Bigg[\,|200\rangle + |210\rangle - |201\rangle - |211\rangle\,\Bigg]\,|0\rangle\,|0\rangle \\
\left|\tilde{\psi}_4\right\rangle &= |0,0,0\rangle\,,
\end{aligned}
$$

but, as the Hamiltonian has been written in the group element basis, we need to rewrite them. Let's do this by firstly considering only the one link state:

- $$|0\rangle = \sum_{g\in\mathbb{D}_8} |g\rangle \langle g|0\rangle = \sum_{g\in\mathbb{D}_8} \sqrt{\frac{dim(0)}{|G|}} D^0(g) |g\rangle = \sum_{g\in\mathbb{D}_8} \frac{1}{2\sqrt{2}} |g\rangle$$

- $$\begin{aligned}\frac{1}{2}\Bigg[|200\rangle + |210\rangle - |201\rangle - |211\rangle\Bigg] &= \frac{1}{2}\Bigg[\sum_{g\in\mathbb{D}_8} |g\rangle \sqrt{\frac{dim(2)}{|G|}}\Big(D^2_{00}(g) + D^2_{10}(g) - D^2_{01}(g) - D^2_{11}(g)\Big)\\ &= \frac{1}{2}\Bigg[|a\rangle - \left|a^3\right\rangle + |x\rangle - \left|a^2x\right\rangle\Bigg]\end{aligned}$$

Consequently:

$$\left|\tilde{\psi_8}\right\rangle = \frac{1}{2}\Bigg[|200\rangle + |210\rangle - |201\rangle - |211\rangle\Bigg] |0\rangle |0\rangle \tag{10}$$

$$= \frac{1}{16}\Big(|a\rangle - \left|a^3\right\rangle + |x\rangle - \left|a^2x\right\rangle\Big) \otimes \Big(\sum_{g\in\mathbb{D}_8} |g\rangle\Big) \otimes \Big(\sum_{g\in\mathbb{D}_8} |g\rangle\Big) \tag{11}$$

$$\left|\tilde{\psi_4}\right\rangle = |0,0,0\rangle = \frac{1}{16\sqrt{2}}\Big(\sum_{g\in\mathbb{D}_8} |g\rangle\Big) \otimes \Big(\sum_{g\in\mathbb{D}_8} |g\rangle\Big) \otimes \Big(\sum_{g\in\mathbb{D}_8} |g\rangle\Big) \tag{12}$$

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