Introduction to computer vision EX1 Guy Lutsker:

1. Prove: You can choose whether to prove for the continuous or the discrete case (1D is sufficient).

   (a) The Convolution Theorem: $\mathfrak{F}\{f \star g\} = F \cdot G$ (where $F = \mathfrak{F}\{f\}$ and $G = \mathfrak{F}\{g\}$). (in the discrete case you may have a scale factor corresponding to the size of the image).

   (b) The convolution is commutative: $f \star g = g \star f$.

   (c) Show that: $f \star (\alpha g + \beta h) = \alpha (f \star g) + \beta (f \star h)$, where α and β are scalars.

   (d) The convolution is shift invariant: $f(x) \star g(x-d) = (f \star g)(x-d)$.

$$Solution:$$

$$(a)$$

$$Let\ us\ recall\ the\ definition\ of\ fourier\ transform:$$

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) \cdot e^{-\frac{2\pi iux}{N}}$$

$$and\ the\ convolution:$$

$$(f \star g)(x) = \sum_{n=0}^{N-1} f(n) \cdot g(x-n)$$

$$and\ so, \mathfrak{F}(f \star g)(u) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{n=0}^{N-1} f(n) \cdot g(x-n) \cdot e^{-\frac{2\pi iux}{N}} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{x=0}^{N-1} f(n) \cdot g(x-n) \cdot e^{-\frac{2\pi iux}{N}} =$$

$$substitute\ \{y = x - n\} \rightarrow x = y + n:$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{y=-n}^{N-1-(y-x)} f(n) \cdot g(y+n-n) \cdot e^{-\frac{2\pi iu(y+n)}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{y=-n}^{N-1-(y-x)} f(n) \cdot g(y) \cdot e^{-\frac{2\pi iuy}{N}} \cdot e^{-\frac{2\pi iun}{N}}$$

$$but\ we\ have\ cyclic\ assumption\ about\ our\ descrete\ signals\ and\ so:$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{y=0}^{N-1} f(n) \cdot g(y) \cdot e^{-\frac{2\pi iuy}{N}} \cdot e^{-\frac{2\pi iun}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} f(n) \cdot e^{-\frac{2\pi iun}{N}} \cdot \sum_{y=0}^{N-1} g(y) \cdot e^{-\frac{2\pi iuy}{N}} = F \cdot G$$

$$(b)$$

$$(f \star g)(x) = \sum_{n=0}^{N-1} f(n) \cdot g(x-n) =^{y=x-n} \sum_{x=y}^{x-N+1} f(x-y) \cdot g(y) =^{flip\ summation} \sum_{x=0}^{N-1} f(x-y) \cdot g(y)$$

$$(c)$$

$$f \star (ag + bh) = \sum_{n=0}^{N-1} f(n) \cdot (ag+bh)(x-n) =^{function\ distributive}$$

$$\sum_{n=0}^{N-1} f(n) \cdot \big(ag(x-n) + bh(x-n)\big) =^{function\ associativity} \sum_{n=0}^{N-1} f(n) \cdot ag(x-n) + f(n) \cdot bh(x-n)$$

$$= a \sum_{n=0}^{N-1} f(n) \cdot g(x-n) + b \sum_{n=0}^{N-1} f(n) \cdot h(x-n) = a(f \star g) + b(f \star h)$$

$$(d)$$

$$f(x) \star g(x-d) = \sum_{n=0}^{N-1} f(n) \cdot g(x-d-n) = (f \star g)(x-d)$$

2. What is $\frac{d}{dx}\{f(x) \star g(x)\}$?

Hint: use the convolution theorem.

$$Solution:$$

$$\mathfrak{F}(f) = F$$

$$we\ learned\ in\ class\ that\ \mathfrak{F}\left(\frac{d}{dx} f(x)\right)(u) = 2\pi i u F(u)$$

$$and\ so\ we\ know\ from\ the\ convolution\ thm\ that:$$

$$\mathfrak{F}(f \star g) = F \cdot G$$

$$\mathfrak{F}\left(\frac{d}{dx} f \star g\right)(u) = \overbrace{2\pi i u F(u)}^{\mathfrak{F}\left(\frac{d}{dx} f(x)\right)(u)} \cdot \overbrace{G(u)}^{\mathfrak{F}(g(x))(u)}$$

$$and\ so\ we\ can\ do\ the\ convolution\ thm\ in\ the\ other\ direction\ and\ get:$$

$$\frac{d}{dx}(f(x) \star g(x)) = (\frac{d}{dx} f(x)) \star g(x)$$

3. Prove: You can choose whether to prove for the continuous or the discrete case (1D is sufficient).

   (a) If f is a symmetric function, then its Fourier transform $F = \mathfrak{F}\{f\}$ is also symmetric.

   (b) If f is symmetric and real, then F is also symmetric and real.

Solution:

(a)

$$let\ f\ be\ a\ function\ such\ that\ f(x) = f(-x)$$

$$\mathfrak{F}(f)(u) = \frac{1}{N}\int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi iux}dx =^{symmetry} \frac{1}{N}\int_{-\infty}^{\infty} f(-x)\cdot e^{-2\pi iux}dx = \{y=-x, dy=-dx\}$$

$$= \frac{1}{N}\int_{\infty}^{-\infty} -f(y)\cdot e^{2\pi iut}dx = \frac{1}{N}\int_{-\infty}^{\infty} f(y)\cdot e^{2\pi iut}dx = \frac{1}{N}\int_{-\infty}^{\infty} f(y)\cdot e^{-2\pi i(-u)t}dx = \mathfrak{F}(f)(-u)$$

(b)

$$let\ us\ decompose\ \mathfrak{F}(f)\ its\ imaginary\ and\ real\ parts:$$

$$\mathfrak{F}(f)(u) = \frac{1}{N}\int_{-\infty}^{\infty} f(x)\cdot e^{-2\pi iux}dx = \frac{1}{N}\int_{-\infty}^{\infty} f(x)\cdot (\cos(-2\pi ux) + i\cdot \sin\,(-2\pi ux)dx$$

$$the\ imaginary\ part\ of\ thisexpression\ is:$$

$$\frac{1}{N}\int_{-\infty}^{\infty} f(x)\cdot i\cdot \sin(-2\pi ux)\,dx\,, \quad so\ we\ will\ show\ that\ \int_{-\infty}^{\infty} f(x)\cdot\sin(-2\pi ux)\,dx = 0$$

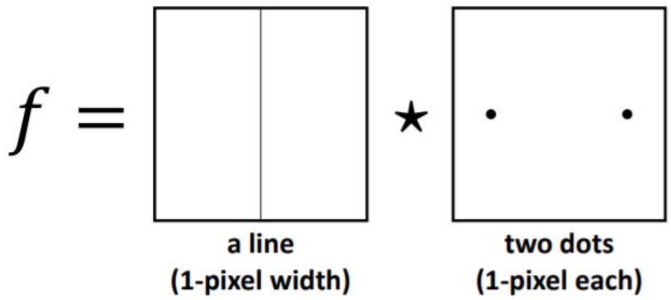
$$\int_{-\infty}^{\infty} f(x)\cdot\sin(-2\pi ux)\,dx = \overbrace{\int_{-\infty}^{0} f(x)\cdot\sin(-2\pi ux)\,dx}^{C} + \int_{0}^{\infty} f(x)\cdot\sin(-2\pi ux)\,dx$$

$$= C + \int_{0}^{\infty} f(x)\cdot\sin(-2\pi ux)\,dx = \{y=-x, dy=-dx\}$$

$$= C + \int_{0}^{-\infty} -f(-t)\cdot\sin(2\pi ut)\,dt =^{f\ symmetric} C + \int_{-\infty}^{0} f(t)\cdot\sin(2\pi ut)\,dt$$

$$=^{\sin x = -\sin\,(-x)} \overbrace{C}^{\int_{-\infty}^{0} f(x)\cdot\sin(-2\pi ux)dx} - \int_{-\infty}^{0} f(t)\cdot\sin(-2\pi ut)\,dt = 0$$

4. Consider the image f defines as:



(a) What does the image f look like?

(b) What does its Fourier transform F look like? (The magnitude $|F|$ only) Explain why.

Solution:

(a)

Two vertical lines as in the left image, at the location of the dots in the right image, because of the duplication property of convolving with single points (delta function).

(b)

The magnitude of F will look like the multiplication of the Fourier's of the images (as per the convolution theorem).

The Fourier of the left image is a horizontal line. This is because all the vertical frequencies are zero, and the horizontal frequency always gets a single delta function.

The Fourier of the right image is cosine. This is something we saw in class and is easy to show that when we get two horizontal deltas in the Fourier domain, we get a cosine (the imaginary part holding the sine coefficient zeroes out)

multiplying these (The Fourier's we just derived) together will result in a horizontal cosine across the x axis, which would look like a dotted line across the horizontal center of the image.

5. Let $F(u, v)$ be the Fourier transform of an $M \times N$ image $f(x, y)$. Let $g(x, y)$ be an image of dimensions $(2M) \times (2N)$ whose Fourier transform $G(u, v)$ is defined as follows:

$$G(u,v) = \begin{cases} F(u,v) & \text{if } 0 \le u < M \text{and } 0 \le v < N \\ 0 & \text{otherwise} \end{cases}$$

What does the image $g(x, y)$ look like in terms of $f(x, y)$? Show mathematically, and explain the result (one sentence, it suffices to explain the values at even coordinates).

$$Solution:$$

$$we\ get\ a\ clear\ defenition\ of\ G, then\ lets\ look\ at\ g:$$

$$\mathfrak{F}^{-1}\big(G(u,v)\big)(x,y) = \sum_{u=0}^{2N-1}\sum_{v=0}^{2M-1} G(u,v)\cdot e^{2\pi i\left(\frac{ux}{2N}+\frac{vy}{2M}\right)} =^{G(\ge N,\ge M)=0} \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} G(u,v)\cdot e^{2\pi i\left(\frac{ux}{2N}+\frac{vy}{2M}\right)}$$

$$which\ will\ result\ in\ f\ in\ the\ even\ coordinates\ since:$$

$$g(2x,2y) = \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} G(u,v)\cdot e^{2\pi i\left(\frac{u2x}{2N}+\frac{v2y}{2M}\right)} = \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} G(u,v)\cdot e^{2\pi i\left(\frac{ux}{N}+\frac{vy}{M}\right)} = f(x,y)$$

$$and\ consiquently, 0\ everywhere\ else.$$

Meaning f might seem like some image in $N \times M$ space, and g will be f in $2N \times 2M$ space, spread out, padded with zeros in the uneven coordinates.

6. Let $f(x, y)$ be an $M \times N$ image, and let $F(u, v)$ be its Fourier transform. Let $g(x, y)$ be an image of dimensions $(2M) \times (2N)$, whose Fourier transform $G(u, v)$ is generated from $F(u, v)$ by inserting a row of Zeros (0's) between every two rows of F, and a column of Zeros (0's) between every two columns of F. What does the image $g(x, y)$ look like in terms of $f(x, y)$? Show mathematically and explain the result (one sentence). Remember, $g(x, y)$ is of size $(2M) \times (2N)$.

$$Solution:$$

$$This\ time\ we\ get\ that:$$

$$G(u,v) = F\left(\frac{u}{2}, \frac{v}{2}\right) \cdot \mathbf{1}\{\boldsymbol{u_{mod2}}, \boldsymbol{v_{mod2}} = \mathbf{0}\}$$

$$and\ we\ can\ derive\ g\ directly\ from\ G:$$

$$\mathfrak{F}^{-1}\big(G(u,v)\big)(x,y) = g(x,y) = \sum_{u=0}^{2N-1} \sum_{v=0}^{2M-1} G(u,v) \cdot e^{2\pi i\left(\frac{ux}{2N}+\frac{vy}{2M}\right)} =^{odd\ indecies\ equal\ 0} =$$

$$(*) \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} G(2u,2v) \cdot e^{2\pi i\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

$$notice\ that\ this\ is\ exactly\ \mathfrak{F}^{-1}(F) = f$$

but recall that we substituted variables for their double in equation $(*)$*, and so since* g *is twice the size* f *in each dimension, we get that* g *have* f *embeded in* g *in the left upper corner.*

in addition in descrete finite fourrier analysis we have a cyclic assumption over our data and so in the rest*of* g*, we get* f *again. such that* g *can be devided into 4 parts with* f *duplicated in them.*

7. (a) Is it possible to discretely sample the function $f_1(x) = \sin(\alpha x)$ without losing any information? (i.e., can we sample $f_1(x)$ discretely, and be able to reconstruct the continuous $f_1(x)$ back from its discrete samples?) If the answer is Yes - then what is the maximal allowed distance between the samples? If the answer is No - explain why.

(b) Same question as above, but for

$$f_2(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Same question as above, now for

$$f_3(x) = f_1(x) \star f_2(x)$$

$$Solution:$$

$$(a)$$

$As\ Nyquist - Shannon\ sampling\ theorem\ states - we\ can\ recover\ any\ band\ limmited\ signal.$

$the\ sine\ wave\ has\ only\ two\ frequances\ (oposite\ one\ another\ on\ the\ imagenary\ axis)\ and\ so\ is\ band\ limited$
$\rightarrow can\ be\ recovered.$

$$choosing\ \Omega = \frac{\alpha}{2\pi}, we\ get\ that\ F(>\Omega) = 0.$$

$$\Delta x\ is\ chosen\ such\ that\ it\ is\ less\ than \frac{1}{2\Omega} = \frac{\pi}{\alpha}, \qquad which\ is\ the\ maximal\ allowed\ distance.$$

$$(b)$$

$$The\ answer\ is\ no. This\ is\ because\ this\ signal\ is\ not\ band\ limited.$$

$$lets\ try\ to\ find\ the\ roots\ of\ the\ signal\ in\ the\ frequancy\ domain:$$

$$\mathfrak{F}(f_2) =^{f_2\ is\ a\ rect\ function}\ 2 \cdot 1 \cdot 1 sinc(2\pi u) = \frac{\sin(2\pi u)}{\pi u}$$

$$\frac{\sin(2\pi u)}{\pi u} = 0 \rightarrow \sin(2\pi u) = 0 \rightarrow u = \frac{t}{2}\ \forall t \in \mathbb{N}$$

$$and\ so\ for\ any\ other\ value\ u\ \ \mathfrak{F}(f_2) \neq 0 \rightarrow f_2\ is\ not\ band\ limited, and\ so\ cannot\ be\ recovered.$$

$$(c)$$

$$We\ know\ from\ the\ convolution\ theorem\ that\ \mathfrak{F}(f_1 \star f_2) = F_1 \cdot F_2, so\ lets\ look\ at\ them.$$

$$F_1(u) = (from\ the\ lecture) = \frac{1}{2i}\delta\left(u - \frac{\alpha}{2\pi}\right) - \frac{1}{2i}\delta(u + \frac{\alpha}{2\pi})$$

$$F_2(u) = (also\ from\ lecture) = \frac{\sin(2\pi u)}{\pi u}$$

$$and\ so\ F_3 = \left(\frac{1}{2i}\delta\left(u - \frac{\alpha}{2\pi}\right) - \frac{1}{2i}\delta\left(u + \frac{\alpha}{2\pi}\right)\right) \cdot \frac{\sin(2\pi u)}{\pi u}$$

which is only two two points in the complex plain, and so is of course band limitted, and is recoverable

$$explicitly, F_3 = \begin{cases} \frac{1}{2i} \cdot C\ , u = \frac{\alpha}{2\pi} \\ -\frac{1}{2i} \cdot C\ , u = \frac{-\alpha}{2\pi} \end{cases}$$

Where C is the value of the delta function at the point it is not 0 *− depends of convention − could be* 1 *or* ∞*, but here it only matters that it is not equal* 0.

$$and\ so\ \Omega = \frac{2\alpha}{2\pi} = \frac{\alpha}{\pi} \rightarrow \Delta x = \frac{\pi}{\alpha}$$

8. Exploring the importance of magnitude vs. phase: We took pairs of images and mixed their magnitude and phase in the following manner:

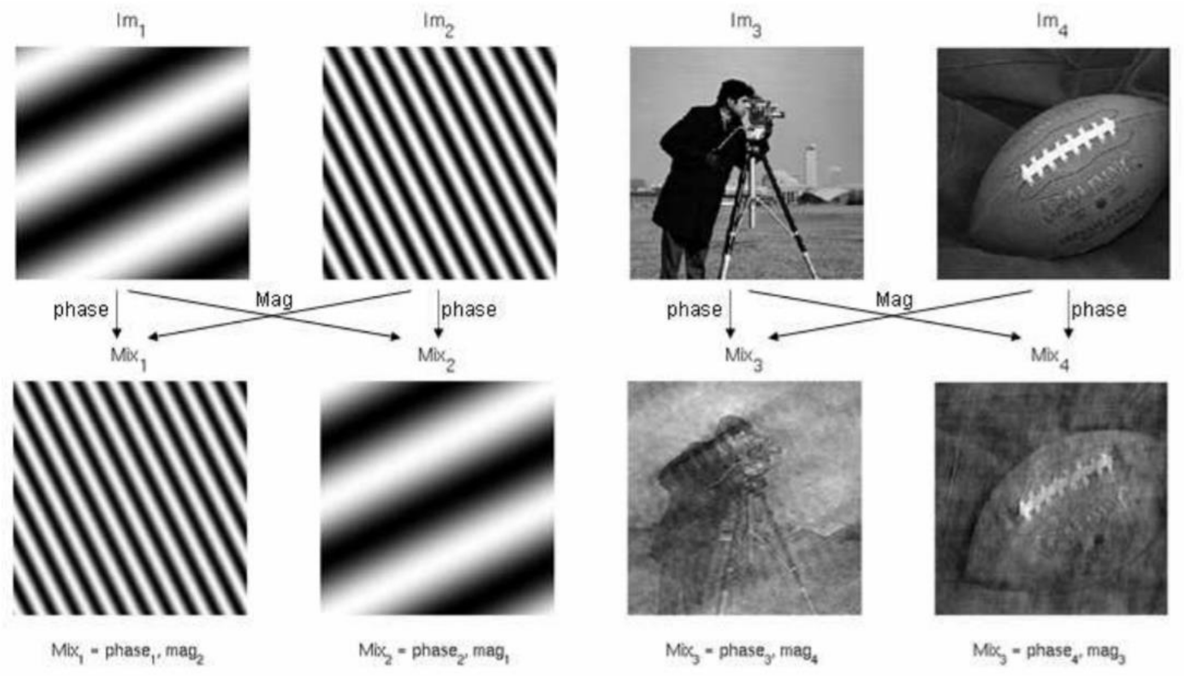
$$\mathrm{Mix}_1 = \mathfrak{F}^{-1}\left\{\left|\mathfrak{F}\left\{\mathrm{Im}_2\right\}\right| e^{i\cdot\measuredangle(\mathfrak{F}\{\mathrm{Im}_1\})}\right\}$$

$$\mathrm{Mix}_2 = \mathfrak{F}^{-1}\left\{\left|\mathfrak{F}\left\{\mathrm{Im}_1\right\}\right| e^{i\cdot\measuredangle(\mathfrak{F}\{\mathrm{Im}_2\})}\right\}$$

$$\mathrm{Mix}_3 = \mathfrak{F}^{-1}\left\{\left|\mathfrak{F}\left\{\mathrm{Im}_4\right\}\right| e^{i\cdot\measuredangle(\mathfrak{F}\{\mathrm{Im}_3\})}\right\}$$

$$\mathrm{Mix}_4 = \mathfrak{F}^{-1}\left\{\left|\mathfrak{F}\left\{\mathrm{Im}_3\right\}\right| e^{i\cdot\measuredangle(\mathfrak{F}\{\mathrm{Im}_4\})}\right\}$$

where $|\mathfrak{F}\{\text{Im}\}|$ is the magnitude of the Fourier transform of Im, and $\measuredangle(\mathfrak{F}\{\text{Im}\})$ is its phase.



(a) In the first mixed image pair (Mix_1 and Mix_2) which input image are they more similar to - the one from which the phase was taken, or the one from which the magnitude was taken? In what way is the output image different form the most similar input image? Explain why.

(b) Same question as the above, but for the second mixed image pair (Mix_3 and Mix_4). Explain why the importance of phase vs. magnitude in this case is reversed?

Solution:

(a)

In both Mix_1 *as well as* Mix_2 *we get that The result is a little move (translation?) magnitude of the images*

Both of the input images had only one frequancy in them:

some singe combination of $\sin$ *& cos functions – resulted in this diagonal wave.*

This means that in the fourier domain we only had 1 *value with non* 0 *magnitude.*

And so both Mix_1, Mix_2 had only one frequancy in their magnitudes.

What is differant is the phase − this term contributed to the tiny translation we get.

And so we get that the single magnitude we colected contributed the shape of the resulted image

and the phase from the other image contributed to the shift of the resulting image.

(b)

This, as apposed to the previous example, is a natural image.

And what is important to remember in answering this question is that

in general the magnitude of the fourier transform of natural images

is quite simmilar across all natural images.

This is in the sence that its much differant than the edge case of the single wave as in (a)

in natural images we get that almost all frequancies dont get exactly 0 magnitude.

and so here when we change the magnitude of two images, we get a noised version of the image

because the weight for each frequancy is mixed up, but the phase stay's the same.

and so the frequencies dont zero out in the mixing disappear

This all means that in this mixing question, the phase will be

play a bigger role because it affects the shift of the frequencies.

This is, as already mentioned, very differant than the previous example

because then we had a sort of an edge case where we had only one frequancy with non

zero magnitude. and so we got a reverse of importance.

9. Let $g(x)$ be a scaled version of $f(x)$, i.e.:

$$g(x) = f(sx)$$

where s is a positive scalar. What does the Fourier transform G of g look like (in terms of the Fourier transform F of f)?

$$Solution:$$

$$we\ know\ from\ the\ scaling\ property\ that\ \mathfrak{F}\big(f(ax)\big) = \frac{1}{|\alpha|}F(\frac{u}{\alpha})$$

$$But\ I\ guess\ we\ can\ quickly\ prove\ it\ :)$$

$$\mathfrak{F}\big(g(x)\big)(u) = \int_{-\infty}^{\infty} g(x) \cdot e^{-2\pi iux} dx =^{set\ g(x)=f(sx)\ as\ stated\ in\ the\ question}$$

$$= \int_{-\infty}^{\infty} f(sx) \cdot e^{-2\pi iux} dx = change\ variables\ \{c = sx, dc = c \cdot dx\}$$

$$= \int_{-\infty}^{\infty} f(c) \cdot \frac{e^{-\frac{2\pi iuc}{s}} dc}{s} = \frac{1}{s}\int_{-\infty}^{\infty} f(c) \cdot e^{-2\pi ic(\frac{u}{s})} dc = \frac{1}{s}\mathfrak{F}(f)(\frac{u}{s})$$