67731 | Convex Optimization and Applications | Ex 2

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1.
Recall that LP is written as:

$$\underset{x}{min}\ x^\intercal x$$

$$\text{s.t}\, Gx \preceq h,\ Ax = b$$

And the linear conic programming formulation:

$$K = \mathbb{R}^d$$

$$f_1(x) = h - Gx$$

$$h_1(x) = b - Ax$$

Notice that $\mathbb{R}^d$ is a cone, and that both f_1, h_1 are affine, and so:

$$Ax = b \Leftrightarrow h_1(x) = 0 \quad , \quad Gx \preceq h \Leftrightarrow f_1(x) \in K$$

As for SOCP:

$$c = f$$

$$h_1(x) = Cx - d$$

$$f_i(x) = \begin{bmatrix} A_i \\ c^\intercal \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix}.$$

As we can see both f_i, h_1 are affine and so:

$$\|A_i x + b_i\| \leq c_i^\intercal x + d_i \Leftrightarrow$$

$$\begin{bmatrix} A_i x + b_i \\ c_i^\intercal x + d_i \end{bmatrix} \in K \Leftrightarrow$$

$$\begin{bmatrix} A_i \\ c^\intercal \end{bmatrix} x + \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in K$$

2.
1.
Lets look at each row of the problem:

$$\underset{x}{min}\, c^\intercal x$$

$$s.t\ \ A_i x \leq b_i\, \forall A \in \mathcal{A}$$

And for each scalar we get:

$$\underset{x}{min}\ c^{\intercal}x$$

$$\underset{A\in\mathcal{A}}{max} A_i x \leq b_i$$

Notice that :

$$A_{i,j} \leq \hat{A_{i,j}} x_j + V_{i,j} x_j$$

and that inequality holds for each coordinate.
In fact we get that :

$$\underset{A\in\mathcal{A}}{max} A_i x = \hat{A}_i + V_i |x|$$

Lets denote $z = |x|$ and write the problem as:

$$\hat{A}x + Vz \leq b$$

$$s.t\ \ -z \leq x \leq z$$

2.
We can say that

$$P_i = U D_i^1 U^{\intercal}$$

$$G = U D_i^2 U^{\intercal}$$

And so:

$$\sum_i x_i U D_i^1 U^{\intercal} + U D_i^2 U^{\intercal} \preceq 0$$

And getting $U, U^{\intercal}$ to the sides we get:

$$U \sum (x_i D_i^1 + D^2) U^{\intercal} \leq 0 \rightarrow \sum (x_i D_i^1 + D^2) \leq 0$$

Since D^1, D^2 are both diagonal we can fit their values in 1d vectors
and so we get that we can rearrange the problem:

$$\hat{D}^1 x \leq -D^2$$

Which is LP.
3.
set $z = \begin{bmatrix} R(x) \\ I(x) \end{bmatrix}$
for $p = 1$:

$$\underset{x}{min} \|x\|_1\ \ s.t\ \ Ax = b$$

$$= \underset{z\in\mathbb{R}^{2n}}{min} \sum_{i=1}^{n} \|z_i ... z_{i+n}\|_2$$

$$s.t : \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix}, z = \begin{bmatrix} R(b) \\ I(b) \end{bmatrix}$$

$$= \underset{z\in\mathbb{R}^{2n}, t\in\mathbb{R}}{min}\ 1^{\intercal}t$$

$$s.t : \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix}, z = \begin{bmatrix} R(b) \\ I(b) \end{bmatrix}, \|z_i ... z_{i+n}\|_2 \leq t$$

for $p = 2$:

$$\underset{x}{min}\|x\| \ \ s.t \ \ Ax = b$$

$$= \underset{z\in\mathbb{R}^{2n}}{min}\|z\|_2$$

$$s.t : \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix}, z = \begin{bmatrix} R(b) \\ I(b) \end{bmatrix}$$

$$= \underset{z\in\mathbb{R}^{2n}, t\in\mathbb{R}}{min} t$$

$$s.t : \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix}, z = \begin{bmatrix} R(b) \\ I(b) \end{bmatrix}, \|z\|_2 \leq t$$

for $p = \infty$:

$$\underset{x}{min}\|x\|_\infty \ \ s.t \ \ Ax = b$$

$$= \underset{z\in\mathbb{R}^{2n}}{min}\|z\|_2$$

$$= \underset{z\in\mathbb{R}^{2n}}{min} max\|z_i...z_{i+n}\|_2$$

$$s.t : \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix}, z = \begin{bmatrix} R(b) \\ I(b) \end{bmatrix}$$

$$= \underset{z\in\mathbb{R}^{2n}, t\in\mathbb{R}}{min} t$$

$$s.t : \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix}, z = \begin{bmatrix} R(b) \\ I(b) \end{bmatrix}, \|z_i...z_{i+n}\|_2 \leq t$$

4.

1.

$$\|A^*q\| = \|\frac{uq^\intercal}{\|u\| \cdot \|q\|}q\| = \|q\|$$

also:

$$\|Aq\| \leq \|A\|_F \cdot \|q\| \leq \|q\| \rightarrow \underset{\|A\|_F\leq 1}{max}\|Aq\| \leq \|q\|$$

And together with $\|A^*q\| = \|q\|$ we get that $\underset{\|A\|_F\leq 1}{max}\|Aq\| = \|q\|$

2.