67731 | Convex Optimization and Applications | Ex 1

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Convex Sets

Which of the following is convex sets? prove or give a counter example:

1. $\{x \in \mathbb{R}^d | \alpha_i \leq x_i \leq \beta_i . i \in [d]\}$

The set is convex. Proof:

We can describe the aforementioned as two convex sets we have covered:

Denote: $A = \{x \in \mathbb{R}^d | \alpha_i \leq x_i\}, B = \{x \in \mathbb{R}^d | x_i \leq \beta_i\}$

Notice that both A, B are halfspaces, and so are convex.

We have learned in class that an intersection of convex sets, is convex it self, and so: B

$$\{x \in \mathbb{R}^d | \alpha_i \leq x_i \leq \beta_i . i \in [d]\} = \overbrace{A}^{\text{Convex}} \cap \overbrace{B}^{\text{Convex}} \rightarrow \overbrace{A \cap B}^{\text{Convex}} = \overbrace{\{x \in \mathbb{R}^d | \alpha_i \leq x_i \leq \beta_i . i \in [d]\}}^{\text{Convex}}$$

2. $\{x \in \mathbb{R}^d | \; \|x\|_0 \leq k\}$

The set is **not** convex. Counter example:

$$\text{Let } k = 1 \text{ and let } \mathbb{R}^d \ni v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbb{R}^d \ni v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

Its clear that both v_1, v_2 hold that $\forall i \in [2] : \|v_i\|_0 \leq k$ and so both are in the aforementioned set

Let us look at a convex combination of v_1, v_2: Let us choose $\theta = 0.5$

$$\theta \cdot v_1 + (1-\theta) \cdot v_2 = \begin{bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \| \begin{bmatrix} 0.5 \\ 0.5 \\ \vdots \\ 0 \end{bmatrix} \|_0 = 2 \neq 1 = k \rightarrow \theta \cdot v_1 + (1-\theta) \cdot v_2 \notin$$

The set$\rightarrow$ The set is not convex.

3. $\{x \in \mathbb{R}^d | \; \|x - x_0\|_2 \leq \|x - y\|_2 \; \forall y \in Y, x_0 \in \mathbb{R}^d\}$

The set is convex. Proof:

Notice that for each point $y \in Y$ what we are essentially creating here is a halfspace.

(To me this problem sounds alot like SVM, even though I might be exaggerating.

And the point in Y which are closect to x_0 are the "support vectors" which help us

define the halfspace - our convex set here is the one that contains x_0)

(you can tell me how off I am, but for now lets get back to reality)

I will prove that for a point $y \in Y$ we get a convex set, and the convexity of the set will follow,

since it will be an intersection of convex sets.

Let $y \in Y$ and lets look at $\{x \in \mathbb{R}^d | \; \|x - x_0\|_2 \leq \|x - y\|_2\}$, and 2 points v, w in the set.

Let $0 \leq \theta \leq 1$.

Denote our candidate $c = \theta \cdot v + (1 - \theta) \cdot w$

Lets check: $\|c - x_0\|_2 \overset{?}{\leq} \|c - y\|_2$

4. $\{X \in \mathbb{R}^{d \times d} | rank(X) < d\}$

The set is **not** convex. Counter example:

Let us take $\delta_{i,j} = \begin{cases} 0 & i \neq j \\ f(i,j) & i = j \end{cases}$ where f is some function such that $f : \mathbb{N}^2 \to \{0, 1\}$

Also f holds that its image has both 0 and 1, meaning $\{0, 1\} \in Im(f)$

Let us construct $A \in \mathbb{R}^{d \times d}$ such that $A_{i,j} = \delta_{i,j}$.

Let us also construct $\eta_{i,j} = \begin{cases} 0 & i \neq j \\ g(i,j) & i = j \end{cases}$ where g is some function such that

$f : \mathbb{N}^2 \to \{0, 1\}$ and says the oposite of f such that:

$$g(i,j) = \begin{cases} 0 & f(i,j) = 1 \\ 1 & f(i,j) = 0 \end{cases}$$

And g also holds that $\{0, 1\} \in Im(g)$

Let us then construct $B \in \mathbb{R}^{d \times d}$ such that $B_{i,j} = \eta_{i,j}$.

Notice that by the way of construction, both A, B have a rank between 1 and $d-1$.

This is true since both f, g hold that they are diagonal with a number of 1's between 1,$d-1$.

Not only that, but they have "completing" ranks, s.t $rank(A) + rank(B) = d$.

This is again true because of the construction of f, g.

Notice that since $rank(A), rank(B) < d$ they are both is the set.

But for any convex combination of them (exluding $\theta \in \{0, 1\}$) the result will be a full rank matrix

Since it will be a diagonal matrix, and so will not be in the set.

And so the set is not convex.

5. $\{x \in \mathbb{R}^d | d(x, S) \leq d(x, T), s.t\ d(x, S) = \underset{y \in S}{min} \|x - y\|\}$

The set is **not** convex. Counter example:

Let $d = 1$

Let $S = \{1, -1\}, T = \{0\}$

Let $a = 1, b = -1$ be in the set since $\|1 - (1)\| = 0 \leq 0 = \| -1 - (-1)\|$.

But taking $\theta = 0.5$ gives us 0 which is closer to $T \rightarrow \underset{y \in S}{min} \|0 - y\| = 1 \nleq 0 = \|0 - 0\| = \underset{y \in T}{min} \|x - y\|$

And so the set is not convex.

6. $\{X \in \mathbb{S}^d_+ | Tr(X) \leq r\}$

As the hint suggests, if with will show that $Tr(X)$ is a linear function we will be done since the set

will be a halfspace and so will be convex.

Notice that $Tr(A + B) = \sum_i a_{i,i} + b_{i,i} = \sum_i a_{i,i} + \sum_i b_{i,i} = Tr(A) + Tr(B)$

And that: $Tr(cA) = \sum_i ca_{i,i} = c \sum_i a_{i,i} = c \cdot Tr(A)$.

And so, trace is a linear function, and the set is a halfspace, which is convex.

7. Prove that C is convex (according to the definition from the lecture) iff it contains every convex combination of its points.

$$Solution:$$

We shall prove on the convex set C by induction over the length of the convex combination n.

Base: Let $n = 1$,then for any point $x_1 \in C$ a convex combination must require $\theta = 1$ and we immidiatly get the derivation.

Step: Assume the statement is true for any $n - 1$ and we shall prove for n

Let $x_1 - x_n \in C$.

We want to proove that $\sum_i \theta_i x_i \in C$ where $\sum_i \theta_i = 1$

From I.H we know that for $n - 1$ the statement holds, and so we know that we can take

$\sum_i^{n-1} \theta_i = 1 \to$ we can "renormalize" for the new expression θ_n thusly:

$$\theta_i^{'} = \frac{\theta_i}{1 - \theta_n}$$

And now:

$$\sum_i^n \theta_i x_i = \overbrace{(1 - \theta_n) \sum_{i=1}^{n-1} \frac{\theta_i \cdot x_i}{1 - \theta_n}}^{\in C \text{ via the I.H}} + \theta_n \overbrace{\cdot x_n}^{\text{Chosen from } C} \overbrace{\in}^{\sum_{i=1}^n \theta_i = 1} C$$

8. Prove that a convex hull is equivalent to $convS$: the smallest convex set that contains S

$$Solution$$

Let S be a set, and H be its convex hull, and K be the smallest convex set that contains S

$H \subset K$:

From Q7 we know that a convex set must contain all of the convex combinations in it.

Given that H is simply some convex combination of k points in S (which are also in H by definition)

K must contain this set otherwise its not convex.

$K \subset H$:

H is a set of convex combinations, and so is also convex (Q7 is iff)

and so if K portrays to be the smallest convex set that contains S

Then H is some convex combinations, and must be by definition larger or equal to the smallest convex set contaning S,

And so the statement is derived.

9. a

10. a

11. Let $f(\begin{bmatrix} x \\ t \end{bmatrix}) = \frac{Ax+b}{ct+d}$,such that $dom(f) = \{\begin{bmatrix} x \\ t \end{bmatrix} \mid ct + d > 0\}$

$$Solution:$$

Let $S \subset \mathbb{R}^d$ be a convex set, show that $\{f(x) \mid x \in S \cap dom(f)\}$ is convex.

Lets take the hints suggestion and look at $S_1 = \{\begin{bmatrix} A & 0 \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} \mid \begin{bmatrix} x \\ t \end{bmatrix} \in S\}$.

Notice that S is convex, and the operation $\begin{bmatrix} A & 0 \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix}$ is affine, meaning S_1 is a set of affine operation on a convex set - this is a convex set as well, and so S_1 is convex. This means that the enumerator if our output of f is convex. The denominator is clearly affine as well, and since $ct + d > 0$ is required in $dom(f)$ we get that f is a composition of an 2 affine transformation, and packing them into a perspective function. Fortunately, S is convex, and a composition of affine functions, and perspective conserves convexity, and so $\{f(x) \mid x \in S \cap dom(f)\}$ is convex.

12. Let $S \subset \mathbb{R}^{d-1}$ be a convex set, show that: $\{x \mid f(x) \in S, x \in dom(f)\}$ is convex.

$$Solution:$$

This is basically requiring that the inverse of a conserves conserving function also conserves convexity (since I have shown in Q11 that f conserves convexity), and we have shown this is class.

13. Let $K \subset V$ be a convex cone, and $K^* = \{x \in V | \langle x, y \rangle \geq 0, \forall y \in K\}$ be the dual cone. Prove that the dual cone is a convex cone:

$$Solution:$$

Let $a, b \in K^*, y \in K, 0 \leq \theta \leq 1$. We need to prove that $\theta a + (1-\theta)b \in K^*$

Let us look at $\langle \theta a + (1-\theta)b, y \rangle \overset{\text{Inner product is linear in the first component}}{=} \theta \overbrace{\langle a, y \rangle}^{\geq 0} + (1-\theta)\overbrace{\langle b, y \rangle}^{\geq 0} \geq 0$
$\to \langle \theta a + (1-\theta)b, y \rangle \in K^* \to K^*$ is convex.

14. Let K a subspace of V. Show that the dual cone is its orthogonal complement.

$$Solution:$$

$$V \not\subset V^*:$$

Let $0 \neq v \in V$ then $\langle v, -v \rangle < 0$ and so v cannot be in $V^* \to v \notin V^*$

$$V \perp V^*:$$

Let $v \in V$ and let o be in $V's$ orthogonal complement, then $\langle v, o \rangle = 0$ by definitions, and so $o \in V^* \to V^*$ =orthogonal complement of V.

15. Show that $\mathbb{R}_+^{n*} = \mathbb{R}_+^n$.

$$Solution:$$

$$\mathbb{R}_+^n \subset \mathbb{R}_+^{n*}:$$

For $x_1, x_2 \in \mathbb{R}_+^n$ notice that $\langle x_1, x_2 \rangle \geq 0 \to x_1 \in \mathbb{R}_+^{n*}$

$$\mathbb{R}_+^{n*} \subset \mathbb{R}_+^n:$$

For $x_1 \in \mathbb{R}_+^{n*}$ and $x_2 \in \mathbb{R}_+^n$ notice that $\langle x_1, x_2 \rangle \geq 0 \to x_1 \in \mathbb{R}_+^n$

16. Show that $\mathbb{S}_{+}^{n*} = \mathbb{S}_{+}^{n}$.

$$Solution:$$

We have defined in EX0 that an inner product of two matrices in the trace of their multiplication, and so:

Let $A, B \in \mathbb{S}_{+}^{n}$ that means that $\langle A, B \rangle = Tr(A^{\intercal}B) = Tr(AB) \overbrace{\geq}^{A,B\in\mathbb{S}_{+}^{n}} 0$ and so $B \in \mathbb{S}_{+}^{n*}$.
Let $X \notin \mathbb{S}_{+}^{n}$ that means that each eigen value of X is negative, and since Tr preserves in similar matrices, we can consider that X has its eigen values on its diagonal. And so we can conclude that $X \notin \mathbb{S}_{+}^{n*}$.