67731 | Convex Optimization and Applications | Ex 1

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Convex Sets

Which of the following is convex sets? prove or give a counter example:

1. $\{x \in \mathbb{R}^d | \alpha_i \leq x_i \leq \beta_i . i \in [d]\}$

The set is convex. Proof:

We can describe the aforementioned as two convex sets we have covered:

Denote: $A = \{x \in \mathbb{R}^d | \alpha_i \leq x_i\}, B = \{x \in \mathbb{R}^d | x_i \leq \beta_i\}$

Notice that both A, B are halfspaces, and so are convex.

We have learned in class that an intersection of convex sets, is convex it self, and so: B

$$\{x \in \mathbb{R}^d | \alpha_i \leq x_i \leq \beta_i . i \in [d]\} = \overbrace{A}^{\text{Convex}} \cap \overbrace{B}^{\text{Convex}} \rightarrow \overbrace{A \cap B}^{\text{Convex}} = \overbrace{\{x \in \mathbb{R}^d | \alpha_i \leq x_i \leq \beta_i . i \in [d]\}}^{\text{Convex}}$$

2. $\{x \in \mathbb{R}^d | \ \|x\|_0 \leq k\}$

The set is **not** convex. Counter example:

Let $k = 1$ and let $\mathbb{R}^d \ni v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbb{R}^d \ni v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$.

Its clear that both v_1, v_2 hold that $\forall i \in [2] : \|v_i\|_0 \leq k$ and so both are in the aforementioned set

Let us look at a convex combination of v_1, v_2: Let us choose $\theta = 0.5$

$$\theta \cdot v_1 + (1-\theta) \cdot v_2 = \begin{bmatrix} 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \| \begin{bmatrix} 0.5 \\ 0.5 \\ \vdots \\ 0 \end{bmatrix} \|_0 = 2 \neq 1 = k \rightarrow \theta \cdot v_1 + (1-\theta) \cdot v_2 \notin$$

The set$\rightarrow$ The set is not convex.

3. $\{x \in \mathbb{R}^d | \ \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in Y, x_0 \in \mathbb{R}^d\}$

The set is convex. Proof:

Notice that for each point $y \in Y$ what we are essentially creating here is a halfspace.

(To me this problem sounds alot like SVM, even though I might be exaggerating. And the point in Y which are closect to x_0 ar

(you can tell me how off I am, but for now lets get back to reality)

I will prove that for a point $y \in Y$ we get a convex set, and the convexity of the set will follow, since it will be an intersection o

Let $y \in Y$ and lets look at $\{x \in \mathbb{R}^d | \ \|x - x_0\|_2 \leq \|x - y\|_2\}$, and 2 points v, w in the set.

Let $0 \leq \theta \leq 1$.

Denote our candidate $c = \theta \cdot v + (1 - \theta) \cdot w$

Lets check: $\|c - x_0\|_2 \overset{?}{\leq} \|c - y\|_2$

4. $\{X \in \mathbb{R}^{d \times d} | rank(X) < d\}$

The set is **not** convex. Counter example:

Let us take $\delta_{i,j} = \begin{cases} 0 & i \neq j \\ f(i,j) & i = j \end{cases}$ where f is some function such that $f : \mathbb{N}^2 \to \{0, 1\}$

Also f holds that its image has both 0 and 1, meaning $\{0, 1\} \in Im(f)$

Let us construct $A \in \mathbb{R}^{d \times d}$ such that $A_{i,j} = \delta_{i,j}$.

Let us also construct $\eta_{i,j} = \begin{cases} 0 & i \neq j \\ g(i,j) & i = j \end{cases}$ where g is some function such that $f : \mathbb{N}^2 \to \{0, 1\}$ and says the oposite of f suc

$$g(i,j) = \begin{cases} 0 & f(i,j) = 1 \\ 1 & f(i,j) = 0 \end{cases}$$

And g also holds that $\{0, 1\} \in Im(g)$

Let us then construct $B \in \mathbb{R}^{d \times d}$ such that $B_{i,j} = \eta_{i,j}$.

Notice that by the way of construction, both A, B have a rank between 1 and $d - 1$.

This is true since both f, g hold that they are diagonal with a number of 1's between 1,$d - 1$.

Not only that, but they have "completing" ranks, s.t $rank(A) + rank(B) = d$.

This is again true because of the construction of f, g.

Notice that since $rank(A), rank(B) < d$ they are both is the set.

But for any convex combination of them (exluding $\theta \in \{0,1\}$) the result will be a full rank matrix

Since it will be a diagonal matrix, and so will not be in the set.

And so the set is not convex.

5. $\{x \in \mathbb{R}^d | d(x,S) \leq d(x,T), s.t\ d(x,S) = \underset{y \in S}{min} \|x-y\|\}$

The set is **not** convex. Counter example:

Let $d = 1$

Let $S = \{1, -1\}, T = \{0\}$

Let $a = 1, b = -1$ be in the set since $\|1-(1)\| = 0 \leq 0 = \|-1-(-1)\|$.

But taking $\theta = 0.5$ gives us 0 which is closer to $T \rightarrow \underset{y \in S}{min}\|0-y\| = 1 \nleq 0 = \|0-0\| = \underset{y \in T}{min}\|x-y\|$

And so the set is not convex.

6. $\{X \in \mathbb{S}^d_+ | Tr(X) \leq r\}$

As the hint suggests, if with will show that $Tr(X)$ is a linear function we will be done since the set will be a halfspace and so v

Notice that $Tr(A+B) = \sum_i a_{i,i} + b_{i,i} = \sum_i a_{i,i} + \sum_i b_{i,i} = Tr(A) + Tr(B)$

And that: $Tr(cA) = \sum_i ca_{i,i} = c\sum_i a_{i,i} = c \cdot Tr(A)$.

And so, trace is a linear function, and the set is a halfspace, which is convex.

7. Prove that C is convex (according to the definition from the lecture) iff it contains every convex combination of its points.

$Solution:$

We shall prove on the convex set C by induction over the length of the convex combination n.

Base: Let $n = 1$,then for any point $x_1 \in C$ a convex combination must require $\theta = 1$ and we immidiatly get the derivation.

Step: Assume the statement is true for any $n-1$ and we shall prove for n

Let $x_1 - x_n \in C$.

We want to proove that $\sum_i \theta_i x_i \in C$ where $\sum_i \theta_i = 1$

From I.H we know that for $n-1$ the statement holds, and so we know that we can take

$$\sum_{i}^{n-1}\theta_i = 1 \rightarrow \text{ we can "renormalize" for the new expression } \theta_n \text{ thusly:}$$

$$\theta_i^{'} = \frac{\theta_i}{1-\theta_n}$$

And now:

$$\sum_{i}^{n}\theta_i x_i = \overbrace{(1-\theta_n)\sum_{i=1}^{n-1}\frac{\theta_i \cdot x_i}{1-\theta_n}}^{\in C \text{ via the I.H}} + \theta_n \overbrace{\cdot x_n}^{\text{Chosen from } C} \overbrace{\in}^{\sum_{i=1}^{n}\theta_i = 1} C$$

8. Prove that a convex hull is equivalent to $convS$: the smallest convex set that contains S

$Solution$

Let S be a set, and H be its convex hull, and K be the smallest convex set that contains S

$H \subset K$:

From Q7 we know that a convex set must contain all of the convex combinations in it.

Given that H is simply some convex combination of k points in S (which are also in H by definition)

K must contain this set otherwise its not convex.

$K \subset H$:

H is a set of convex combinations, and so is also convex (Q7 is iff) and so if K portrays to be the smallest convex set that conta

Then H is some convex combinations, and must be by definition larger or equal to the smallest convex set contaning S,

And so the statement is derived.

9. <u>**a**</u>

10. <u>**a**</u>

11. Let $f(\begin{bmatrix} x \\ t \end{bmatrix}) = \frac{Ax+b}{ct+d}$,such that $dom(f) = \{\begin{bmatrix} x \\ t \end{bmatrix} \mid ct+d > 0\}$

$Solution$:

Let $S \subset \mathbb{R}^d$ be a convex set, show that $\{f(x) \mid x \in S \cap dom(f)\}$ is convex.

Lets take the hints suggestion and look at $S_1 = \{\begin{bmatrix} A & 0 \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} \mid \begin{bmatrix} x \\ t \end{bmatrix} \in S\}$.

Notice that S is convex, and the operation $\begin{bmatrix} A & 0 \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix}$ is affine, meaning S_1 is a set of affine operation on a convex set - this is a convex set as well, and so S_1 is convex. This means that the enumerator if our output of f is convex. The denominator is clearly affine as well, and since $ct+d>0$ is required in $dom(f)$ we get that f is a composition of an 2 affine transformation, and packing them into a perspective function. Fortunately, S is convex, and a composition of affine functions, and perspective conserves convexity, and so $\{f(x) \mid x \in S \cap dom(f)\}$ is convex.

12. Let $S \subset \mathbb{R}^{d-1}$ be a convex set, show that: $\{x \mid f(x) \in S, x \in dom(f)\}$ is convex.

$Solution$:

This is basically requiring that the inverse of a conserves conserving function also conserves convexity (since I have shown in Q11 that f conserves convexity), and we have shown this is class.

13. Let $K \subset V$ be a convex cone, and $K^* = \{x \in V | \langle x, y \rangle \geq 0, \forall y \in K\}$ be the dual cone. Prove that the dual cone is a convex cone:

$$Solution:$$

Let $a, b \in K^*, y \in K, 0 \leq \theta \leq 1$. We need to prove that $\theta a + (1-\theta)b \in K^*$

Let us look at $\langle \theta a + (1-\theta)b, y \rangle \overset{\text{Inner product is linear in the first component}}{=} \theta \overbrace{\langle a, y \rangle}^{\geq 0} + (1-\theta)\overbrace{\langle b, y \rangle}^{\geq 0} \geq 0$
$\rightarrow \langle \theta a + (1-\theta)b, y \rangle \in K^* \rightarrow K^*$ is convex.

14. Let K a subspace of V. Show that the dual cone is its orthogonal complement.

$$Solution:$$

$$V \not\subset V^*:$$

Let $0 \neq v \in V$ then $\langle v, -v \rangle < 0$ and so v cannot be in $V^* \rightarrow v \notin V^*$

$$V \perp V^*:$$

Let $v \in V$ and let o be in $V's$ orthogonal complement, then $\langle v, o \rangle = 0$ by definitions, and so $o \in V^* \rightarrow V^*$ =orthogonal complement of V.

15. Show that $\mathbb{R}_+^{n*} = \mathbb{R}_+^n$.

$$Solution:$$

$$\mathbb{R}_+^n \subset \mathbb{R}_+^{n*}:$$

For $x_1, x_2 \in \mathbb{R}_+^n$ notice that $\langle x_1, x_2 \rangle \geq 0 \rightarrow x_1 \in \mathbb{R}_+^{n*}$

$$\mathbb{R}_+^{n*} \subset \mathbb{R}_+^n:$$

For $x_1 \in \mathbb{R}_+^{n*}$ and $x_2 \in \mathbb{R}_+^n$ notice that $\langle x_1, x_2 \rangle \geq 0 \rightarrow x_1 \in \mathbb{R}_+^n$

16. Show that $\mathbb{S}_+^{n*} = \mathbb{S}_+^n$.

$$Solution:$$

We have defined in EX0 that an inner product of two matrices in the trace of their multiplication, and so:

Let $A, B \in \mathbb{S}_+^n$ that means that $\langle A, B \rangle = Tr(A^\intercal B) = Tr(AB) \overbrace{\geq}^{A,B \in \mathbb{S}_+^n} 0$ and so $B \in \mathbb{S}_+^{n*}$.
Let $X \notin \mathbb{S}_+^n$ that means that each eigen value of X is negative, and since Tr preserves in similar matrices, we can consider that X has its eigen values on its diagonal. And so we can conclude that $X \notin \mathbb{S}_+^{n*}$.

17. **Let $K = \{(x,t) | \, \|x\|_2 \leq t\}$, show that $K^* = K$.**

$$Solution:$$

a

Which of the following is convex functions? prove or give a counter example:

18. Any norm $\|\cdot\|$

It is a convex function, proof: Directly derived from the triangle inequality:

$$\forall x, y, 0 \leq \theta \leq 1:$$

$$\|\theta x + (1-\theta)y\| \leq \theta\|x\| + (1-\theta)\|y\|$$

19. $f(x) = x_1 \cdot x_2$

Not a convex function. since:

$$|\nabla^2 f| = |\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}| = -1 < 0 \rightarrow f \text{ is not convex.}$$

20. $f(x) = \frac{x_1}{x_2}$

Not a convex function. since:

$$\nabla f = [\frac{1}{x_2}, \frac{-x_1}{x_2^2}]$$

$$\nabla^2 f = \begin{bmatrix} 0 & \frac{-1}{x_2^2} \\ \frac{-1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

$$|\nabla^2 f| = |\begin{bmatrix} 0 & \frac{-1}{x_2^2} \\ \frac{-1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}| = \frac{-1}{x_2^4} < 0 \rightarrow f \text{ is not convex.}$$

21. $f(x) = \frac{x_1^2}{x_2}, x_2 \geq 0.$

It is convex, proof:

$$\nabla f = [\frac{2x_1}{x_2}, \frac{-x_1^2}{x_2^2}]$$

$$\nabla f^2 = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} = \frac{2}{x_2} \begin{bmatrix} 1 & \frac{-x_1}{x_2} \\ \frac{-x_1}{x_2} & \frac{x_1^2}{x_2^2} \end{bmatrix}$$

And since $x_2 \geq 0$ what is left to show is that $\begin{bmatrix} 1 & \frac{-x_1}{x_2} \\ \frac{-x_1}{x_2} & \frac{x_1^2}{x_2^2} \end{bmatrix}$ is PSD.

$$|\begin{bmatrix} 1 & \frac{-x_1}{x_2} \\ \frac{-x_1}{x_2} & \frac{x_1^2}{x_2^2} \end{bmatrix}| = \frac{x_1^2}{x_2^2} - \frac{x_1^2}{x_2^2} = 0 \geq 0$$

$$Tr(\begin{bmatrix} 1 & \frac{-x_1}{x_2} \\ \frac{-x_1}{x_2} & \frac{x_1^2}{x_2^2} \end{bmatrix}) = 1 + \frac{x_1^2}{x_2^2} \geq 0$$

And so both the trace and the determinant are non negative, meaning ∇f^2 is PSD

22. **Let $f(x) = -(\prod_{i=1}^n x_i)^{\frac{1}{n}}$. Proof that $v^\intercal H_f v = C_x(n\sum \frac{v_i^2}{x_i^2} - (\sum \frac{v_i}{x_i})^2)$**

$$v^\intercal H_f v = v^\intercal \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} v$$

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23. Use Cauchy Schwarz to show that $(n\sum\frac{v_i^2}{x_i^2}-(\sum\frac{v_i}{x_i})^2)\geq 0$

$$Solution:$$

Let us recall that Cauchy Schwarz says that $\|u\|^2\|w\|^2\geq\langle u,w\rangle^2 \Leftrightarrow \|u\|^2\|w\|^2-\langle u,w\rangle^2\geq 0$.
Let us use $u=(1,1...,1)^\intercal, w=(\frac{v_1}{x_1},...,\frac{v_n}{x_n})^\intercal$,and using Cauchy Schwarz we get:

$\overbrace{\|u\|^2\|w\|^2}^{n\sum\frac{v_i^2}{x_i^2}}-\overbrace{\langle u,w\rangle^2}^{(\sum\frac{v_i}{x_i})^2}\geq 0 \Longrightarrow (n\sum\frac{v_i^2}{x_i^2}-(\sum\frac{v_i}{x_i})^2)\geq 0.$

24. Conclude that f is convex.

$$Solution:$$

We have seen in Q22 that the Hessian of f is made up of $C_x(n\sum\frac{v_i^2}{x_i^2}-(\sum\frac{v_i}{x_i})^2)$ Where $C_x$is non negative, and we have shown in Q23 that $n\sum\frac{v_i^2}{x_i^2}-(\sum\frac{v_i}{x_i})^2$ is non negative as well. Meaning that f's Hessian is non negative - meaning its PSD - meaning f is convex.

25. Prove that if f is convex then ∇f is monotonic increasing.

$$Solution:$$

A criteria for convexity is that $\nabla^2 f$ is PSD which means that ∇f is monotonic increasing... But I guess we can prove it from definition:

Let f be convex, and so lets look at the expression we want to prove:

$$(\nabla f(x)^\intercal-\nabla f(y)^\intercal)(x-y)\overset{?}{\geq} 0\Leftrightarrow$$

$$\nabla f(x)^\intercal(x-y)-\nabla f(y)^\intercal(x-y)\overset{?}{\geq} 0\Leftrightarrow$$

And if we remember the definition of high dimensional convexity, with regard to the first order derivative we get:

$$f(x)\geq f(y)+\nabla f(x)^\intercal(y-x)$$

$$f(y)\geq f(x)+\nabla f(x)^\intercal(x-y)$$

Summing up these equations yields:

$$\nabla f(x)^\intercal(x-y)-\nabla f(y)^\intercal(x-y)+f(x)+f(y)\leq f(x)+f(y)\Leftrightarrow$$

$$\nabla f(x)^\intercal(x-y)-\nabla f(y)^\intercal(x-y)\leq 0$$

Which is awfully close to what we want to prove.

But if we remember that the statement in the question is regarding any $x,y\in dom(f)$ we realize that the order isn't important and we can flip it to get:

$$\nabla f(x)^\intercal(x-y)-\nabla f(y)^\intercal(x-y)\leq 0\Leftrightarrow$$

$$\nabla f(y)^\intercal(y-x)-\nabla f(x)^\intercal(y-x)\geq 0$$

Which is what we wanted.