67800 | Probabilistic Methods in Artificial Intelligence | Ex 1

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Question 1 - Xor Distributions

Consider the following distribution over 3 binary variables X, Y, Z:

$$p(x,y,z) \quad = \begin{cases} 1/12 & x \oplus y \oplus z = 0 \\ 1/6 & x \oplus y \oplus z = 1 \end{cases}$$

where $\oplus$ denotes a XOR function.

Show that there is no BN graph structure $\mathcal{G}$ such that $\mathcal{I}(\mathcal{G}) = \mathcal{I}(p)$

Hint: Start by testing marginal independecies (ones of the form $X \perp\!\!\!\perp Y$)

Solution :

Let us assume by way of contradiction that for the given distribution function p, there indeed exists a Bayesian Network (BN) graph $\mathcal{G}$ such that $\mathcal{I}(\mathcal{G}) = \mathcal{I}(p)$.

Firstly, lets take the hints advice, and test the marginal distributions. Notice that our three random variables X, Y, Z are binary, and so, WLOG if we calculate $p(X = 0)$ its enough for calculating the marginal distribution of X since $p(X = 0) + p(X = 1) = 1$. With that in mind, lets proceed:

$$p(X=0) \overset{\text{Summing out } X}{=} \sum_{y \in Val(Y), z \in Val(Z)} p(0,y,z) \overset{Val(Y)=Val(Z)=\{0,1\}}{=} \sum_{y\in\{0,1\}} \sum_{z\in\{0,1\}} p(0,y,z) \overset{\text{Opening}}{=}$$

$$p(0,0,0) + p(0,0,1) + p(0,1,0) + p(0,1,1) \overset{\text{Rearange in xor grouping}}{=}$$

$$\overbrace{\sum_{y,z\in\{(0,0),(1,1)\}} p(0,y,z)}^{\text{Xor is 0}} + \overbrace{\sum_{y,z\in\{(0,1),(0,1)\}} p(0,y,z)}^{\text{Xor is 1}} =$$

$$\sum_{y,z\in\{(0,0),(1,1)\}} p(0,y,z|0 \oplus y \oplus z = 0) + \sum_{y,z\in\{(0,1),(0,1)\}} p(0,y,z|0 \oplus y \oplus z = 1) =$$

$$= \sum_{y,z\in\{(0,0),(1,1)\}} \frac{1}{12} + \sum_{y,z\in\{(0,1),(0,1)\}} \frac{1}{6} \overset{\text{2 components in each sum}}{=} \frac{2}{12} + \frac{2}{6} = \frac{1}{2}$$

$$\Rightarrow p(X=1) = \frac{1}{2}$$

As we can see the case of looking at the distribution of X is not special, and from the symmetry of the xor operation we can deduce that the marginal distributions of X, Y, Z are identical, and so we have done the calculation for all of them WLOG: $P(X) = P(Y) = P(Z)$.

Lets us now continue with the hint, and look at the dependency (or lack there of) of say, X, Y:

$$p(x|y) \overset{Bayes}{=} \frac{p(x,y)}{p(y)} \overset{\text{Summing out } Z}{=} \frac{\sum_{z\in Val(Z)} p(x,y,z)}{p(y)} \overset{\text{There are only two options: Either we get xor=0, or xor=1}}{=}$$

$$\frac{p(x,y,z|x\oplus y\oplus z=0)+p(x,y,z|x\oplus y\oplus z=1)}{p(y)}=\frac{\frac{1}{12}+\frac{1}{6}}{p(y)}=\frac{\frac{1}{4}}{p(y)}=\frac{\frac{1}{2}\cdot\frac{1}{2}}{p(y)}=\frac{p(x)\cdot\cancel{p(y)}}{\cancel{p(y)}}=p(x)$$

Meaning that $p(x|y)=p(x)$, and this implies that y gives us no new information. This also indicates that $X\perp Y$. And again from the symmetry of the xor operation we get that all three random variables X,Y,Z are independent.

Now, armed we our new knowledge ($P(X)=P(Y)=P(Z)=\frac{1}{2}$ and that $X\perp Y\perp Z$), we can get back to our initial assumption about the existence of $\mathcal{G}$.

$$\text{It is given that } p \text{ holds that : } p(x,y,z|x\oplus y\oplus z=0)=\frac{1}{12}.$$

Let us calculate the same proposition in the distribution $\mathcal{G}$ spaces: p_b which is defined by $p_b(x_1,...,x_n)=\prod_i p(x_i|x_{pa(i)})$

$$p_b(x,y,z|x\oplus y\oplus z=0)\overset{\text{According to xor truth table}}{=}$$

$$p_b(0,0,0)+p_b(0,1,1)+p_b(1,0,1)+p_b(1,1,0)\overset{p_b(x_1,...,x_n)=\prod_i p(x_i|x_{pa(i)})}{=}$$

$$\prod_{a\in\{X,Y,Z\}}p(a=0|x_{pa(i)})+\prod_{\substack{i\in\{0,1,1\}\\a\in\{X,Y,Z\}}}p(a=i|x_{pa(i)})+\prod_{\substack{i\in\{1,0,1\}\\a\in\{X,Y,Z\}}}p(a=i|x_{pa(i)})+\prod_{\substack{i\in\{1,1,0\}\\a\in\{X,Y,Z\}}}p(a=i|x_{pa(i)})\overset{X\perp Y\perp Z}{=}$$

$$\prod_{a\in\{X,Y,Z\}}p(a=0)+\prod_{a\in\{X,Y,Z\},i\in\{0,1,1\}}p(a=i)+\prod_{a\in\{X,Y,Z\},i\in\{1,0,1\}}p(a=i)+\prod_{a\in\{X,Y,Z\},i\in\{1,1,0\}}p(a=i)\overset{\text{Symmetry}}{=}$$

$$(\frac{1}{2})^3+(\frac{1}{2})^3+(\frac{1}{2})^3+(\frac{1}{2})^3=\frac{1}{2}$$

And so we get that $p(x,y,z|x\oplus y\oplus z=0)=\frac{1}{12}\neq\frac{1}{2}=p_b(x,y,z|x\oplus y\oplus z=0)\Rightarrow p_b\neq p$.
Which implies that our assumption that $\exists\mathcal{G}$ such that $\mathcal{I}(\mathcal{G})=\mathcal{I}(p)$ is wrong.