Course 67800 Spring 2021

Probabilistic Methods in Artificial Intelligence

Problem Set 1: Bayesian Networks

Deadline: Thursday 15/4/21, 23:59

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Q1 - Xor Distributions

Consider the following distribution over 3 binary variables X, Y, Z:

$$p(x, y, z) = \begin{cases} 1/12 & x \oplus y \oplus z = 0 \\ 1/6 & x \oplus y \oplus z = 1 \end{cases}$$

where $\oplus$ denotes a XOR function.

Show that there is no BN graph structure $\mathcal{G}$ such that $\mathcal{I}(\mathcal{G}) = \mathcal{I}(p)$

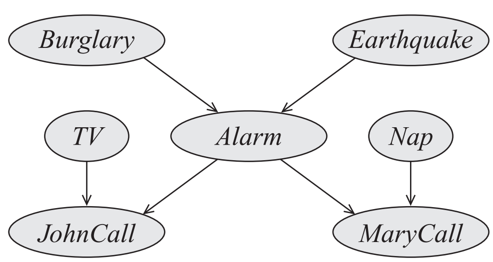
Hint: Start by testing marginal independecies (ones of the form $X \perp\!\!\!\perp Y$)

Q2 - Importance of Being Acyclic

Show that if we relax the requirement that the BN graph has no cycles, it is no longer guaranteed that $p_{\mathcal{B}}(X_1, \ldots, X_n) = \prod_{i=1}^{n} p_i \left(X_i \mid \mathrm{Pa}_{X_i}^{\mathcal{G}} \right)$ is a valid probability distribution.

Specifically, give an example of a BN $\mathcal{B} = \langle \mathcal{G}, p_{\mathcal{B}} \rangle$, where $\mathcal{G}$ has cycles, and show that $\sum_{x_1, \ldots, x_n} p_{\mathcal{B}}(x_1, \ldots, x_n) \neq 1$

Q3 - Removing a Variable From a Bayesian Network



1. Consider the Burglary Alarm network given above (It's a bit different than the example we saw in class - create a suitable story if you wish). Construct a Bayesian network over all nodes except the Alarm node that is a minimal I-Map for the marginal distribution over the remaining variables (B, E, N, T, J, M).
   Specifically, construct a new graph over the remaining variables which models all of the dependencies (active paths) from the original network in which A is unobserved (it's an I-map) and in which no edge can be removed without creating new independencies (it's minimal).

2. Generalize the procedure you used above to an arbitrary network. More precisely, assume we are given a BN $\mathcal{B}$, an ordering $X_1, \ldots, X_n$ that is consistent with the topological ordering of the variables in $\mathcal{B}$, and a node X_i to be removed. Specify a network $\mathcal{B}'$ which is consistent with this ordering and which is a minimal I-Map of $p_{\mathcal{B}}(X_{-i}) = p_{\mathcal{B}}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$. Your answer should be an explicit specification of the set of parents for each variable in $\mathcal{B}'$.

Q4 - Towards Inference in Bayesian Networks

Suppose you have a Bayes net over variables $X_1, \ldots, X_n$ and all variables except X_i are observed. Using the chain rule and the network's conditional independence assumptions, find an efficient way to compute $p\left(X_i \mid \boldsymbol{x}_{-i}\right) = p\left(X_i \mid x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)$.
In particular, your calculation should not require evaluating the full joint distribution.

Q5 - I-equivalence

In this question, we'll investigate when two BN graphs encode the same set of independencies

Definition. We say that two BN graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are *I-Equivalent* if $\mathcal{I}\left(\mathcal{G}_1\right) = \mathcal{I}\left(\mathcal{G}_2\right)$.

Definition. Given a directed graph $\mathcal{G}$, we define its *skeleton* as the undirected graph that results from removing all arrows in $\mathcal{G}$.

1. Show that having the same skeleton is insufficient for I-equivalence. i.e. give a counter example of two graphs with the same skeleton but a different set of independencies.

2. Prove that if two BN graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same skeleton **and** the same set of v-structures then they are I-equivalent. In other words, you can flip the directionality of any edge that does not participate in a v-structure and this will not affect the graph's encoded independencies.

3. Show that the converse to (2) doesn't hold. i.e. find two two BN graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ which are I-equivalent, but which do not have the same skeleton and set of v-structures.