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3

91579



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NEW ZEALAND QUALIFICATIONS AUTHORITY
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SUPERVISOR'S USE ONLY

Level 3 Calculus, 2015

91579 Apply integration methods in solving problems

2.00 p.m. Wednesday 25 November 2015

Credits: Six

Achievement	Achievement with Merit	Achievement with Excellence
Apply integration methods in solving problems.	Apply integration methods, using relational thinking, in solving problems.	Apply integration methods, using extended abstract thinking, in solving problems.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should attempt ALL the questions in this booklet.

Show ALL working.

Make sure that you have the Formulae and Tables Booklet L3–CALCF.

If you need more space for any answer, use the page(s) provided at the back of this booklet and clearly number the question.

Check that this booklet has pages 2–16 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Low Merit

TOTAL

14

ASSESSOR'S USE ONLY

QUESTION ONE

- (a) Find $\int(\sqrt{x} + 6\cos 2x)dx$.

$$f'(x) = x^{\frac{1}{2}} + 6\cos 2x$$

$$f(x) = \frac{x^{\frac{3}{2}}}{1.5} + 3\sin 2x + C$$

11

- (b) Solve the differential equation $\frac{dy}{dx} = \frac{2}{x}$, given that when $x = 1, y = 3$.

$$\int 2/x$$

$$y = 2\ln x + C$$

$$3 = 2\ln 1 + C$$

$$3 = 0 + C$$

$$3 = C$$

$$y = 2\ln x + 3$$

11

- (c) If $\frac{dy}{dx} = \frac{e^{2x}}{4y}$ and $y = 4$ when $x = 0$, find the value of y when $x = 2$.

~~$y' = \frac{e^{2x}}{4y}$~~

~~$2y' = e^{2x}$~~

when $x = 0$

~~$2y' = \frac{e^{2x}}{2}$~~

~~$2y' = e^{2x}$~~

~~$2y' = 0.8$~~

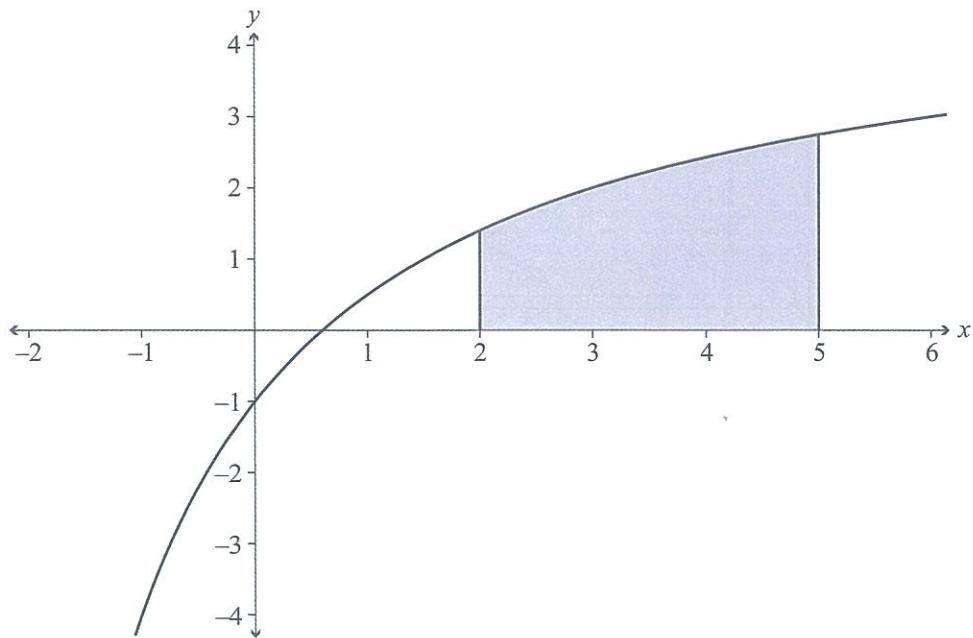
~~$y' = 0.4$~~

~~$y' = 0.25$~~

11

- (d) Use integration to find the area enclosed between the curve $y = \frac{5x-3}{x+3}$ and the lines $y = 0$, $x = 2$ and $x = 5$.

The area is shown shaded in the diagram below.



Show your working.

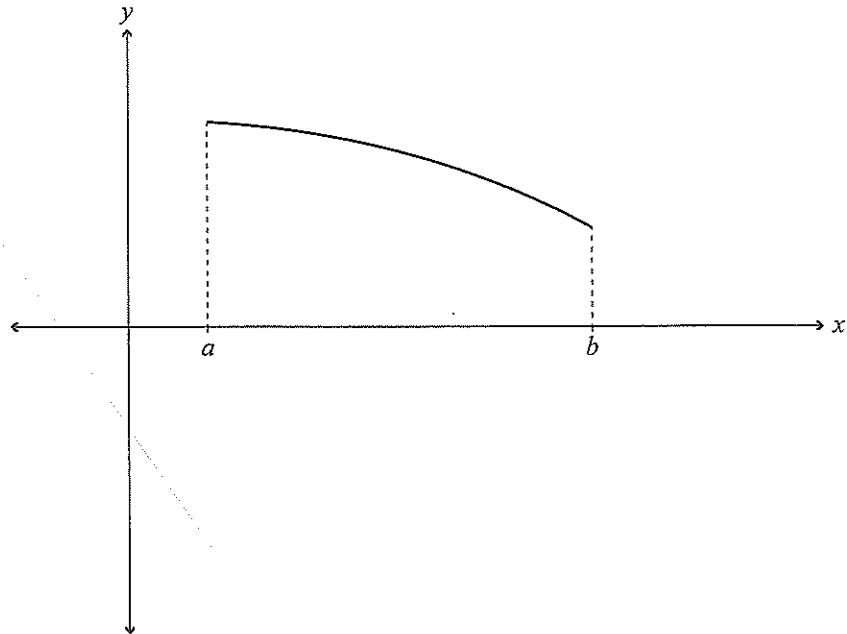
You must use calculus and give the results of any integration needed to solve this problem.

$$\begin{aligned}
 & \text{If } y = \frac{5x-3}{1} \times \frac{1}{x+3} \\
 & y = 5x-3 \times \left(\frac{1}{x+3}\right)^{-1} \\
 & \int_{2}^{5} (5x-3) \left(\frac{1}{x+3}\right)^{-1} dx \\
 & = \left[2.5x^2 - 3x \times \ln(x+3) \right]_2^5 \\
 & = (2.5(5)^2 - 3(5) \times \ln(5+3)) - (2.5(2)^2 - 3(2) \times \ln(2+3)) \\
 & = (62.5 - 15 \times 2.08) - (10 - 6 \times 1.61) \quad \text{2dp rounded} \\
 & = (62.5 - 31.2) - (10 - 9.66) \\
 & = 31.3 - 0.34 \\
 & = 30.96 \text{ units}^2
 \end{aligned}$$

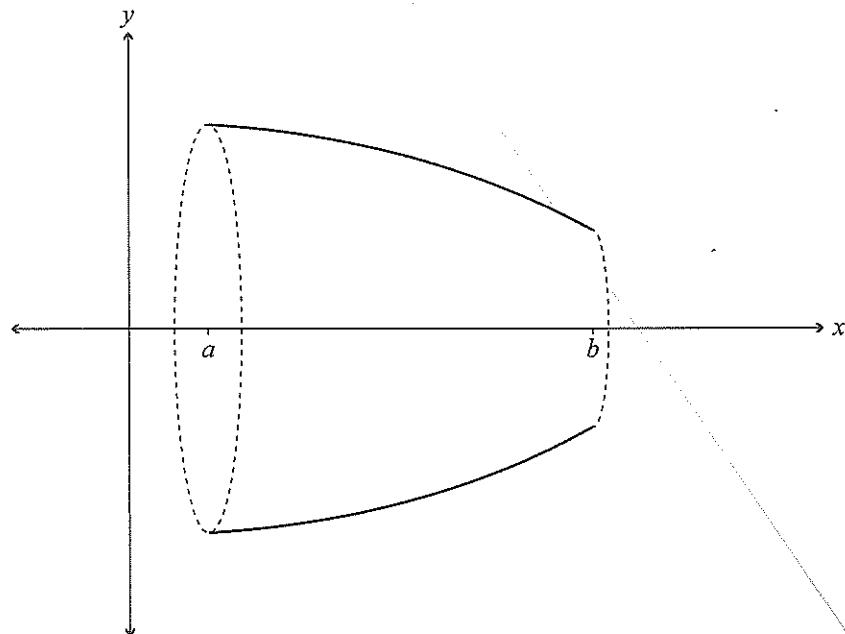
(e)

ASSESSOR'S
USE ONLY

Consider the curve defined by the function $y = f(x)$, bounded by $x = a$ and $x = b$.



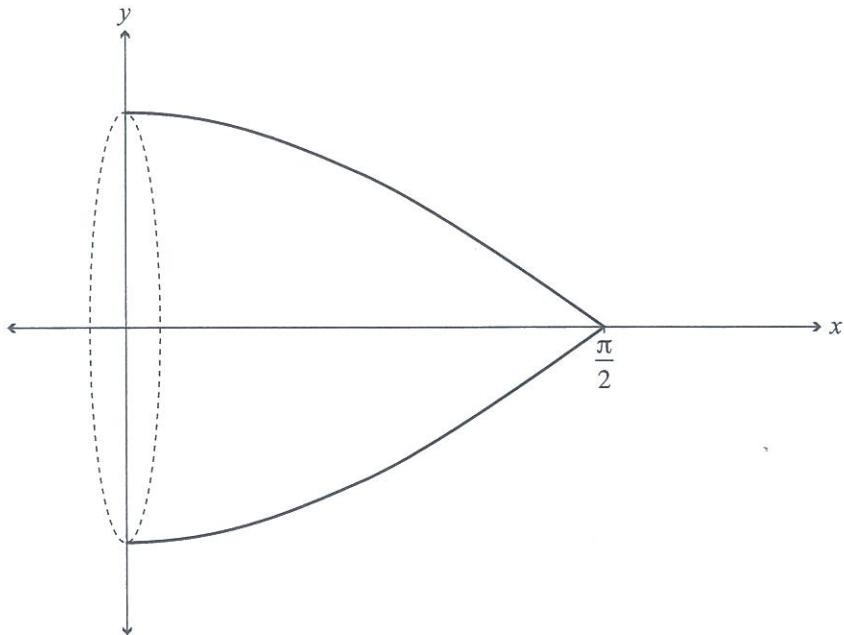
This portion of the curve is rotated around the x-axis, as shown below.



The volume created by this rotation is given by the formula

$$\text{Volume} = \pi \int_a^b (f(x))^2 dx$$

The graph below shows the function $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{2}$, rotated around the x -axis.



Find the volume created by this rotation.

You must use calculus and give the results of any integration needed to solve this problem.

$$\begin{aligned}
 &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \cos x^2 dx \\
 &= \pi \left[\frac{1}{3} \sin x^3 \right]_0^{\frac{\pi}{2}} \\
 &= \pi \left[\frac{1}{3} \sin \left(\frac{\pi}{2}\right)^3 - \frac{1}{3} \sin 0^3 \right] \\
 &= \pi \left[\frac{1}{3} \sin \left(\frac{\pi}{2}\right) \right] \\
 &= \pi \times \frac{1}{3} \sin \left(\frac{\pi}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \pi \int_0^{\frac{\pi}{2}} \cos x^2 dx \\
 &= \pi \left[\frac{1}{3} \sin x^3 \right]_0^{\frac{\pi}{2}} \\
 &= \pi \times (1 - 0) \\
 &= \pi
 \end{aligned}$$

N2

QUESTION TWO

- (a) Find $\int \left(3 - \frac{5}{x^2} \right) dx$.

$$\text{B } \int 3 - 5x^{-2} dx$$

$$3x - \frac{5x^{-1}}{-1}$$

- (b) Use the values given in the table below to find an approximation to $\int_1^{2.5} f(x) dx$, using the Trapezium Rule.

x	1	1.25	1.5	1.75	2	2.25	2.5
$f(x)$	0.3	0.7	1.65	1.9	2.35	1.7	1.1

$$= \frac{1}{2} 0.25 (0.3 + 1.1 + 2(0.7 + 1.65 + 1.9 + 2.35 + 1.7))$$

$$= 0.125 (18)$$

$$= 2.25$$

- (c) An object originally moving at a constant velocity suddenly starts to accelerate. From the start of the object's acceleration the motion of the object can be modelled by the differential equation

$$\frac{dv}{dt} = \frac{50t^2 - 80\sqrt{t}}{5\sqrt{t}} \text{ for } 0 \leq t \leq 20$$

where v is the velocity of the object in m s^{-1}

and t is the time in seconds after the object starts to accelerate.

If the original velocity of the object was 6 m s^{-1} , find the velocity of the object when $t = 4$.

You must use calculus and give the results of any integration needed to solve this problem.

$$C = 6$$

$$\begin{aligned}
 & \int 50t^2 - 80t^{1/2} \times 5t^{-1/2} dt \\
 &= \frac{50}{3}t^3 - \frac{80}{1.5}t^{1.5} \times \frac{5}{0.5}t^{-0.5} + C \\
 v &= \frac{50}{3}(4)^3 - \frac{80}{1.5}6(4)^{1.5} \times \frac{5}{0.5}(4)^{-0.5} + 6 \\
 &= 1066.67 - 426.67 \times 20 + 6 \\
 v &= 219.17
 \end{aligned}$$

- (d) In the town of Clarkeville, the rate at which the population, P , of the town changes at any instant is proportional to the population of the town at that instant.

- (i) Write a differential equation which models this situation.

$$\frac{dp}{dt} = kp$$

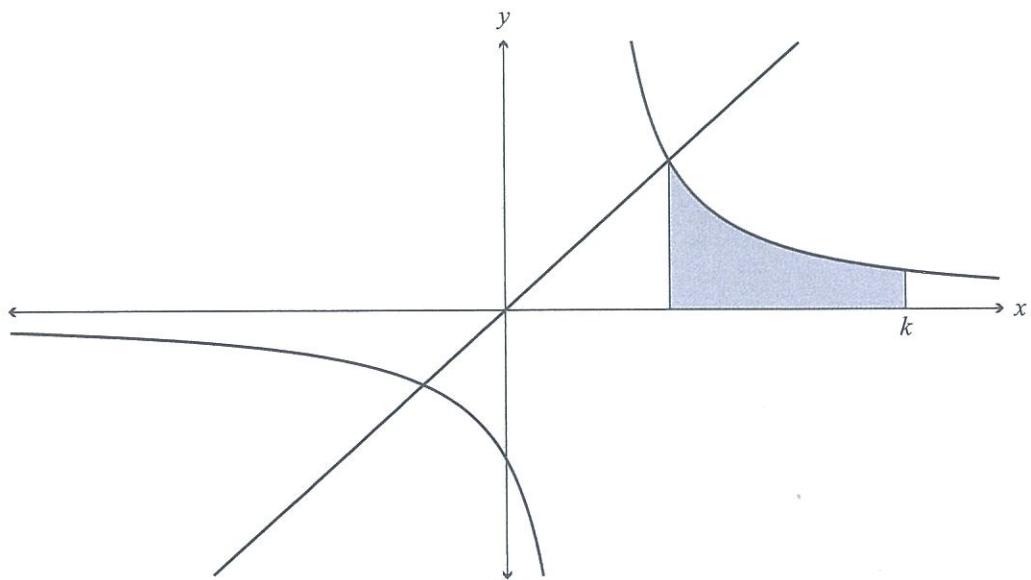
- (ii) At the start of 2000, the population of the town was 12 000.

At the start of 2010, the population of the town was 16 000.

Solve the differential equation in (i) to find the population the town will have at the start of 2025.

You must use calculus and give the results of any integration needed to solve this problem.

- (e) The graphs of $y = \frac{2}{x-1}$ and $y = x$ are shown on the axes below.



The shaded region has an area of 4 units squared.

Find the value of k .

You must use calculus and give the results of any integration needed to solve this problem.

$$\frac{2}{x-1} = x \quad \text{--- intersection graphs ---}$$

$$2 = x^2 - x$$

$$2 = x(x-1)$$

$$0 = x^2 - x - 2$$

$$x = 2 \quad \text{or} \quad x = -1$$

must be the one

$$\int_2^k \frac{2}{x-1} dx$$

$$= [2 \ln(x-1)]_2^k$$

$$4 = (2 \ln(k-1)) - (2 \ln(2-1))$$

$$4 = 2 \ln(k-1) - 0$$

$$4 = 2 \ln(k-1)$$

$$2 = \ln(k-1)$$

$$k-1 = e^2$$

$$k = e^2 + 1$$

$$\text{or } k = 8.39$$

E

E8

QUESTION THREE

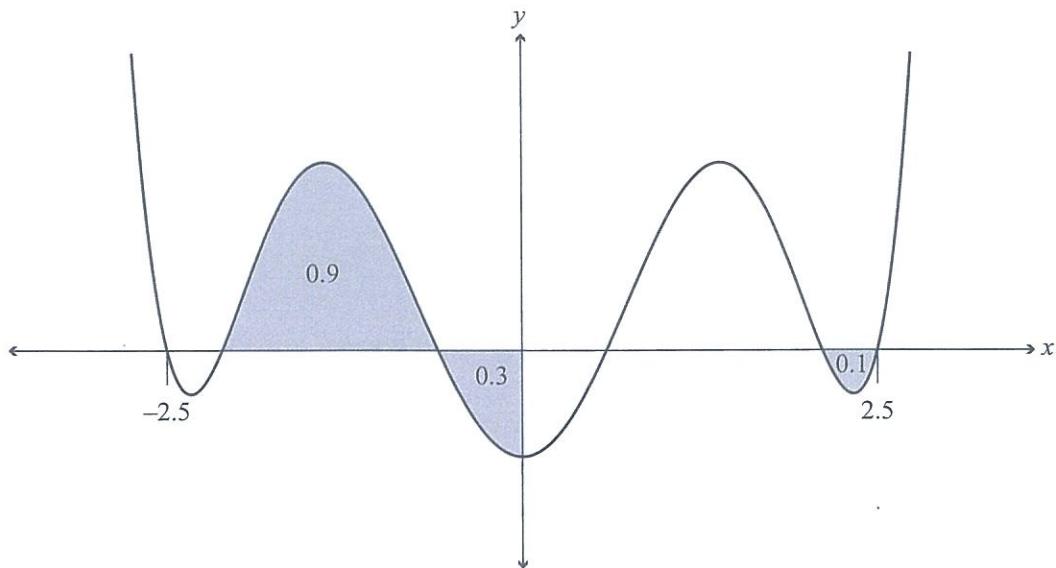
- (a) Find $\int ((x+4)^2 + 8e^{4x}) dx$.

$$\frac{(x+4)^3}{3} + 2e^{4x}$$

u

- (b) The graph of the function $y = f(x)$ below is symmetrical about the y -axis.

The areas of the shaded regions are given.



Find $\int_{-2.5}^{2.5} f(x) dx$.

$$\int_{-2.5}^{2.5} f(x) dx$$

226 2.6

$$= [F'(x)]_{-2.5}^{2.5}$$

$$= (F'(2.5)) - (F'(-2.5))$$

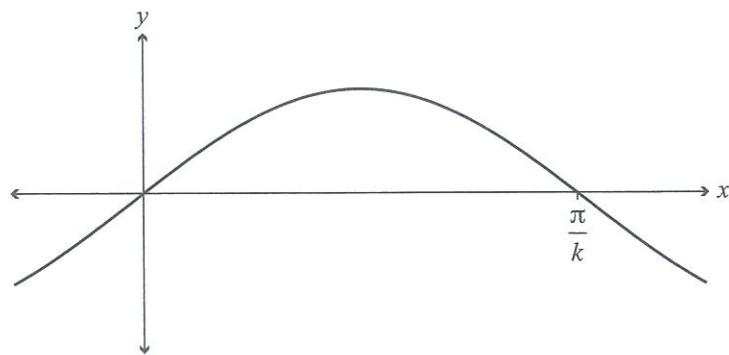
$$= 2 F'(2.5)$$

$$= 2 (1.3)$$

$$= 2.6$$

n

- (c) Find an expression in terms of k for the area bounded by the function $y = \sin kx$ and the x -axis, between $x = 0$ and $x = \frac{\pi}{k}$.

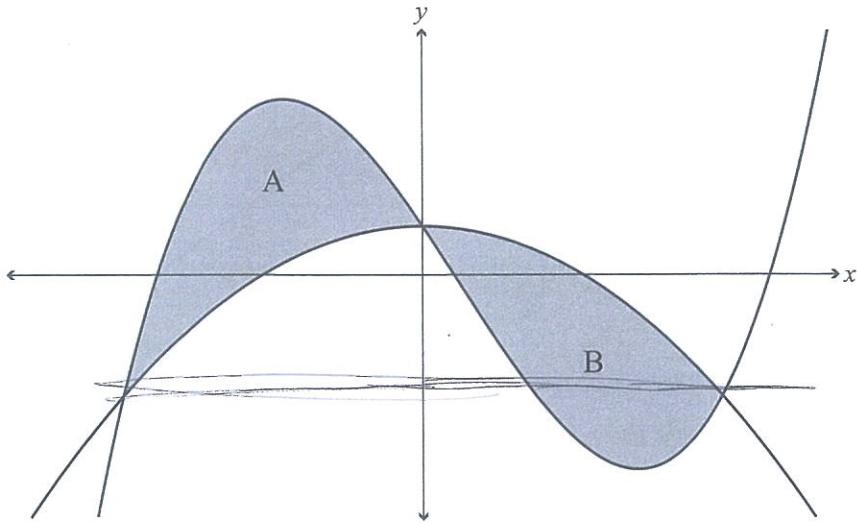


You must use calculus and give the results of any integration needed to solve this problem.

$$\begin{aligned}
 & \int_0^{\pi/k} \sin kx \, dx \\
 &= \left[-\frac{\cos kx}{k} \right]_0^{\pi/k} \\
 &= \left(-\frac{\cos k(\pi/k)}{k} \right) \\
 &= -\frac{\cos \pi}{k}
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{-\cos \pi}{k} \\
 &= \frac{1}{k}
 \end{aligned}$$

- (d) The graphs of $f(x) = -x^2 + 2$ and $g(x) = x^3 - x^2 - kx + 2$ are shown below.
 The graphs intersect and create two closed regions, A and B.



Show that these two regions have the same area.

You must use calculus and give the results of any integration needed to solve this problem.

$$-x^2 + 2 = x^3 - x^2 - kx + 2$$

$$0 = x^3 - kx$$

$$kx = x^3$$

$$\sqrt[3]{kx} = x$$

$$\int_{\sqrt[3]{kx}}^0 ((x^3 - x^2 - kx + 2) - (-x^2 + 2)) \, dx = A$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{1}{2}x^2 + 2x + \frac{x^3}{3} - 2x \right]_{\sqrt[3]{kx}}^0$$

$$A = \left[\frac{x^4}{4} - \frac{k}{2}x^2 \right]_{\sqrt[3]{kx}}^0$$

$$A = \frac{1}{4} \left(\frac{kx^{4/3}}{4} - \frac{k}{2} (kx)^{2/3} \right)$$

~~Doesn't have negative area~~

$$B = \int_0^{\sqrt[3]{kx}} (-x^2 + 2) - (x^3 - x^2 - kx + 2) \, dx$$

$$= \left[-\frac{x^3}{3} + 2x - \frac{x^4}{4} + \frac{x^3}{3} + \frac{k}{2}x^2 - 2x \right]_0^{\sqrt[3]{kx}}$$

$$= \left[-\frac{x^4}{4} + \frac{k}{2}x^2 \right]_0^{\sqrt[3]{kx}}$$

$$= \left(-\frac{kx^{4/3}}{4} + \frac{k}{2}(kx)^{2/3} \right)$$

$$A = -\left(\frac{kx^{4/3}}{4} - \frac{k}{2}(kx)^{2/3} \right) \quad \text{or} \quad A = -\frac{kx^{4/3}}{4} + \frac{k}{2}(kx)^{2/3}$$

~~$$B = -\frac{kx^{4/3}}{4} + \frac{k}{2}(kx)^{2/3}$$~~

- (e) An object starts from rest.

The object's acceleration is given by the formula $a = B(e^{kt})^2$

where a is the acceleration of the object in m s^{-2}

and t is the time, in seconds, from when the object started moving.

Show that the time that it takes the object to reach velocity v_0 is

$$t = \frac{1}{2k} \ln\left(\frac{2v_0 k + B}{B}\right)$$

You must use calculus and give the results of any integration needed to solve this problem.

n

A4

**Extra paper if required.
Write the question number(s) if applicable.**

QUESTION
NUMBER

ASSESSOR'S
USE ONLY

QUESTION
NUMBER

Extra paper if required.
Write the question number(s) if applicable.

ASSESSOR'S
USE ONLY

QUESTION
NUMBER

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91579 Apply integration methods in solving problems

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High Merit

TOTAL

20

ASSESSOR'S USE ONLY

QUESTION ONE

- (a) Find $\int (\sqrt{x} + 6 \cos 2x) dx$.

$$\int x^{1/2} + 6 \cos 2x \, dx$$

$$\frac{2}{3} x^{3/2} + 3 \sin 2x + C$$

~~/~~

- (b) Solve the differential equation $\frac{dy}{dx} = \frac{2}{x}$, given that when $x = 1, y = 3$.

$$y = 2 \ln |x| + C$$

$$3 = 2 \ln 1 + C$$

$$\ln 1 = 0$$

$$\therefore C = 3$$

$$y = 2 \ln |x| + 3$$

~~/~~

- (c) If $\frac{dy}{dx} = \frac{e^{2x}}{4y}$ and $y = 4$ when $x = 0$, find the value of y when $x = 2$.

$$\int 4y \, dy = \int e^{2x} \, dx$$

$$2y^2 = \frac{1}{2} e^{2x} + C$$

$$32 = \frac{1}{2} + C$$

$$C = 31.5$$

$$2y^2 = \frac{1}{2} e^{2x} + 31.5$$

$$2y^2 = \frac{1}{2} e^4 + 31.5$$

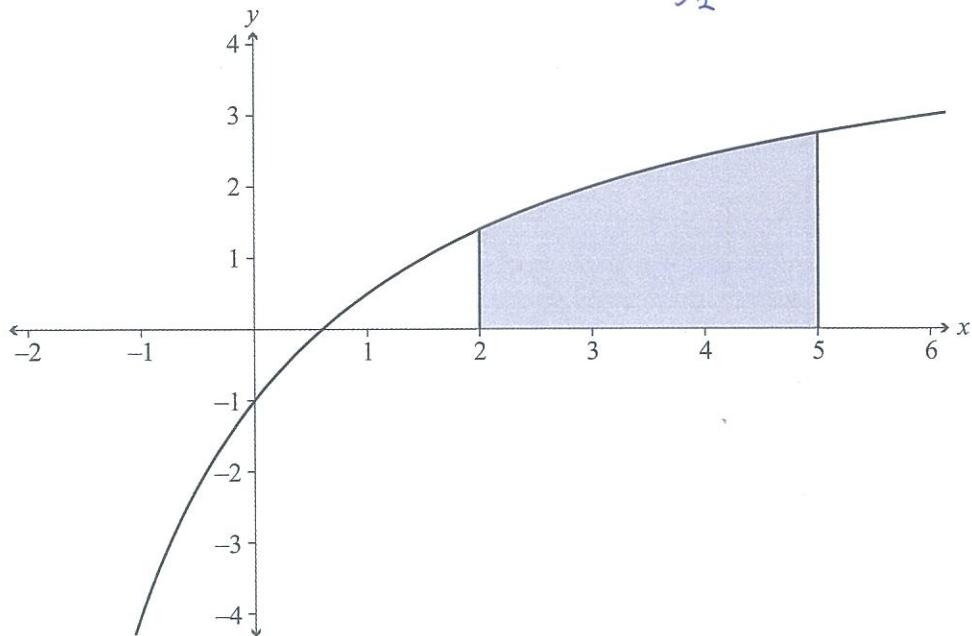
$$y = \pm 5.422$$

~~/~~

- (d) Use integration to find the area enclosed between the curve $y = \frac{5x-3}{x+3}$ and the lines $y = 0$, $x = 2$ and $x = 5$.

The area is shown shaded in the diagram below.

$$\int_2^5 \frac{5x-3}{x+3} dx$$



Show your working.

You must use calculus and give the results of any integration needed to solve this problem.

$$u = x + 3$$

$$\therefore \int_2^5 \frac{5(u-3)-3}{u} du$$

$$\int_2^5 \frac{5u-15-3}{u} du$$

$$\int_2^5 \frac{5u-18}{u} du \rightarrow \int_2^5 5 - \frac{18}{u} du$$

$$[5u - 18 \ln|u|]_2^5$$

$$[5(x+3) - 18 \ln|x+3|]_2^5$$

$$(25 + 15 - 18 \ln 8) - (10 + 15 - 18 \ln 5)$$

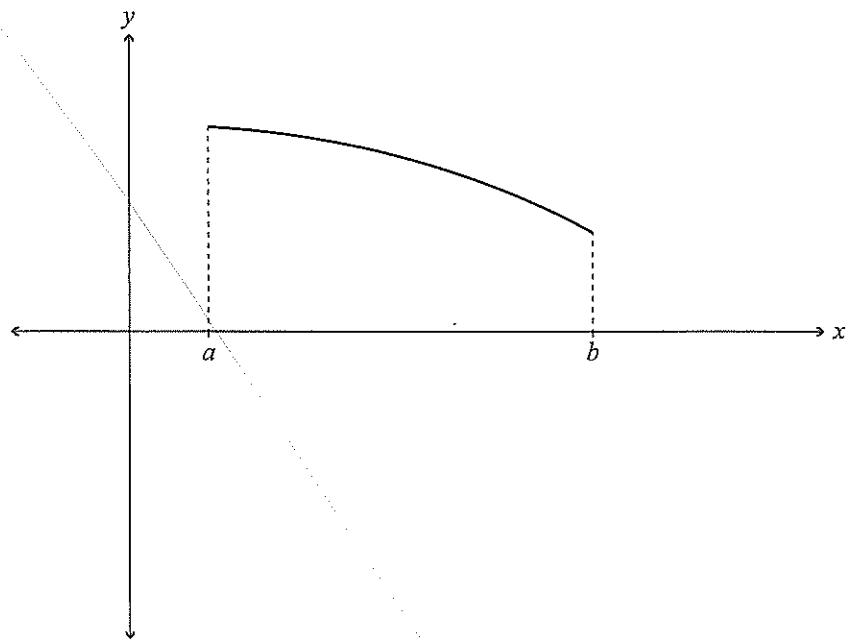
$$2.57 + 3.96988$$

$$6.54 \text{ units}^2 \quad (3sf)$$

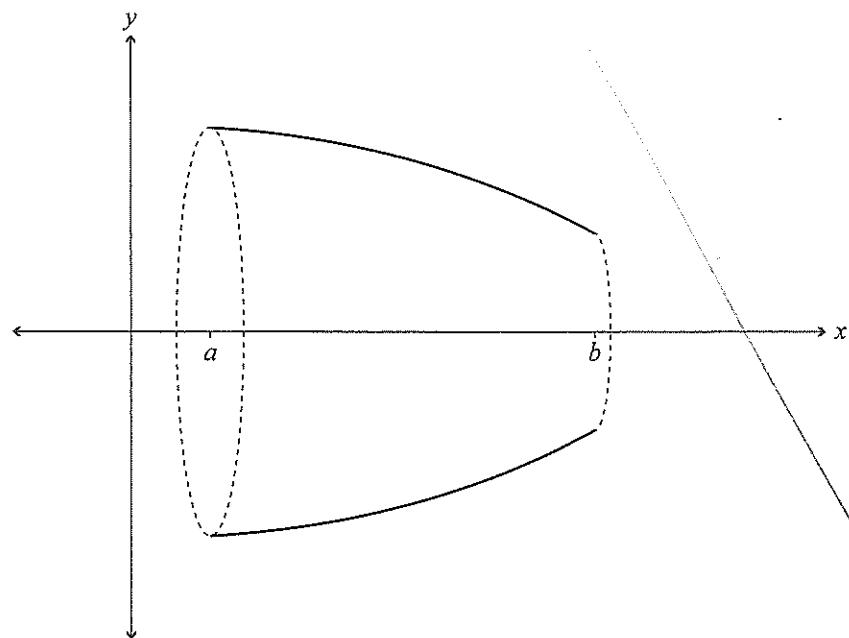
(e)

ASSESSOR'S
USE ONLY

Consider the curve defined by the function $y = f(x)$, bounded by $x = a$ and $x = b$.



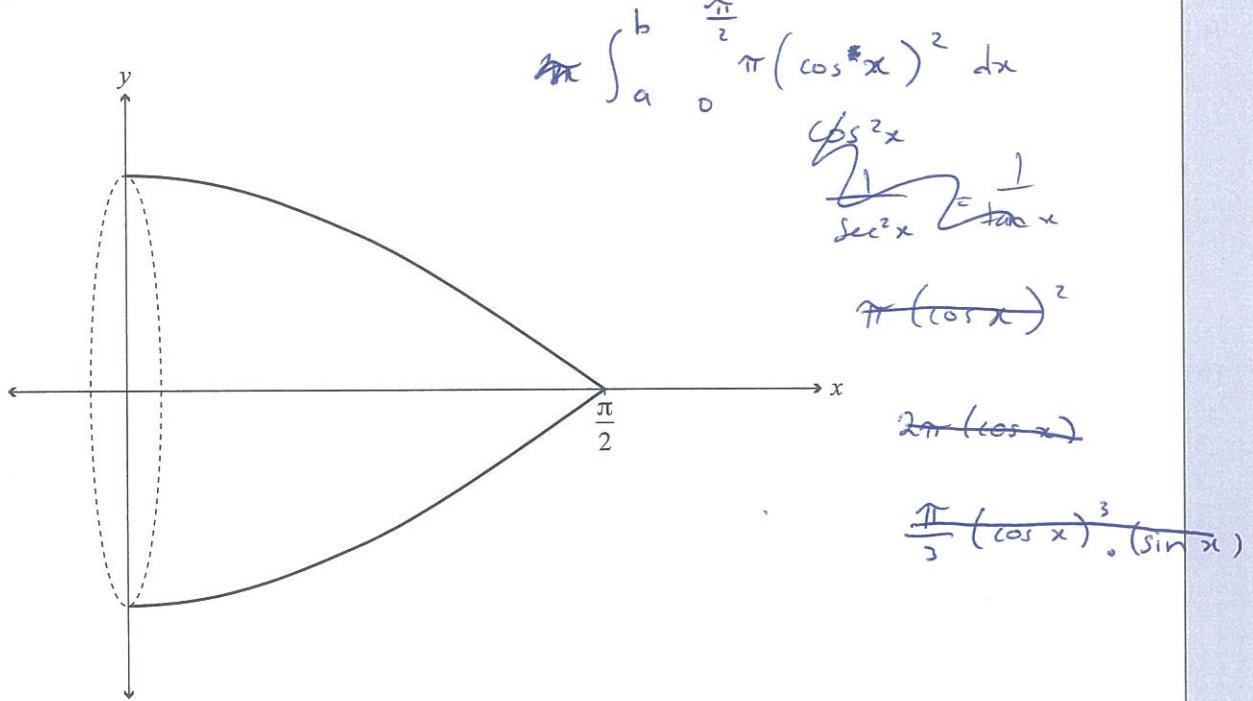
This portion of the curve is rotated around the x-axis, as shown below.



The volume created by this rotation is given by the formula

$$\text{Volume} = \pi \int_a^b (f(x))^2 dx$$

The graph below shows the function $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{2}$, rotated around the x-axis.



Find the volume created by this rotation.

You must use calculus and give the results of any integration needed to solve this problem.

$$\text{Volume} = \pi \int_0^{\pi/2} (\cos x)^2 dx$$

$$\text{Volume} = \pi \int_0^{\pi/2} -\cos^2 x + (\cos x)(\cos x) dx$$

$$f = \cos x \quad g = \cos x$$

$$f' = -\sin x \quad g' = -\sin x$$

$$\text{Volume} = \pi \int_0^{\pi/2} \frac{1}{\sec^2 x} dx$$

$$\text{Volume} = \pi \int_0^{\pi/2} \frac{1}{\tan x} \pi \left[\frac{1}{\tan x} \right]_0^{\pi/2}$$

$$\tan 0 = 0$$

$$\therefore \pi \left[\frac{1}{\tan x} \right]_0^{\pi/2} \quad \therefore \pi \left(\frac{\cos x}{\sin x} \right)$$

$$\left[\frac{\pi}{3} (\cos x)^3 \cdot \sin(x) \right]_0^{\pi/2}$$

$$\left(\frac{\pi}{3} (\cos \frac{\pi}{2})^3 \cdot \sin(\frac{\pi}{2}) \right) - \left(\frac{\pi}{3} (\cos 0)^3 \cdot \sin(0) \right)$$

M

M6

QUESTION TWO

- (a) Find $\int \left(3 - \frac{5}{x^2}\right) dx$.

$$3 - 5x^{-2}$$

$$3x + 5x^{-1} + C$$

- (b) Use the values given in the table below to find an approximation to $\int_1^{2.5} f(x) dx$, using the Trapezium Rule.

x	1	1.25	1.5	1.75	2	2.25	2.5
$f(x)$	0.3	0.7	1.65	1.9	2.35	1.7	1.1

$$\frac{1}{2}(0.25) [0.3 + 1.1 + 2(0.7 + 1.65 + 1.9 + 2.35 + 1.7)]$$

$$\frac{1}{2}(0.25)(18)$$

2.25 units²

- (c) An object originally moving at a constant velocity suddenly starts to accelerate. From the start of the object's acceleration the motion of the object can be modelled by the differential equation

$$\frac{dv}{dt} = \frac{50t^2 - 80\sqrt{t}}{5\sqrt{t}} \text{ for } 0 \leq t \leq 20$$

where v is the velocity of the object in m s^{-1}

and t is the time in seconds after the object starts to accelerate.

If the original velocity of the object was 6 m s^{-1} , find the velocity of the object when $t = 4$.

You must use calculus and give the results of any integration needed to solve this problem.

$$\begin{aligned}\frac{dv}{dt} &= 10t^{3/2} - 16 \\ v &= 4t^{5/2} - 16t + C \quad C = 6 \\ v &= 4(4)^{5/2} - 16(4) + 6 \\ &= 128 - 64 + 6 \\ &= 70 \text{ ms}^{-1}\end{aligned}$$

- (d) In the town of Clarkeville, the rate at which the population, P , of the town changes at any instant is proportional to the population of the town at that instant.

- (i) Write a differential equation which models this situation.

$$\frac{dP}{dt} = kP$$

- (ii) At the start of 2000, the population of the town was 12 000.

At the start of 2010, the population of the town was 16 000.

Solve the differential equation in (i) to find the population the town will have at the start of 2025.

You must use calculus and give the results of any integration needed to solve this problem.

$$P = Ae^{kt}$$

$$\int \frac{1}{P} dP = \int k dt$$

$$\ln P = kt + c$$

$$P = e^{kt+c} = e^{kt} \cdot e^c \quad e^c = A$$

$$P = Ae^{kt}$$

$$A = 12000$$

$$P = 12000 e^{kt}$$

$$\text{When } t=10 \quad P = 12000 e^{10k} = 16000$$

$$\therefore k = 0.0287682072$$

$$\therefore P = 12000 e^{0.0287682072t}$$

$$\text{when } t = 25$$

$$P = 12000 e^{0.0287682072(25)}$$

$$P = 24633.6$$

$$P = 24633 \text{ people}$$

- (e) The graphs of $y = \frac{2}{x-1}$ and $y = x$ are shown on the axes below.

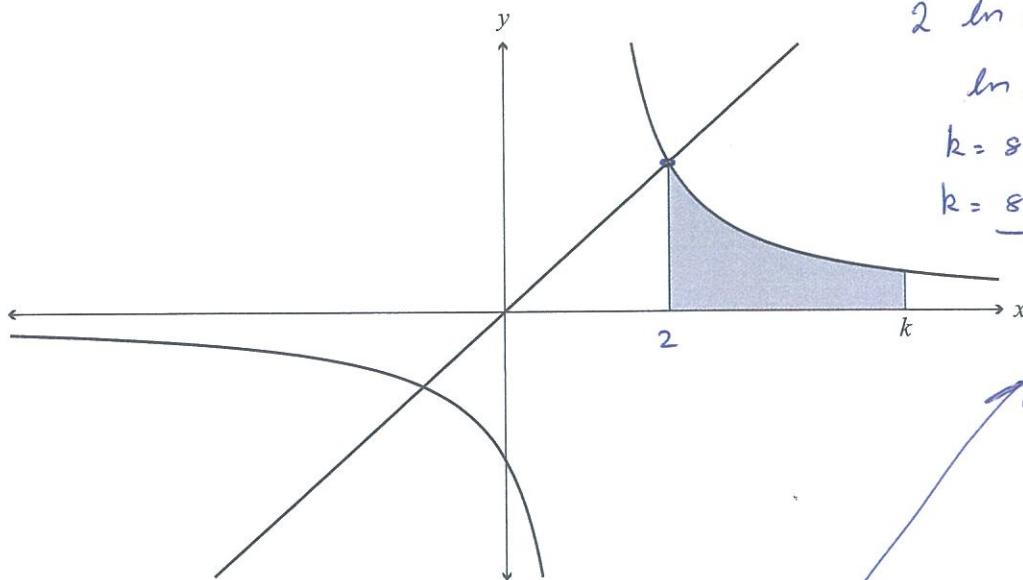
$$2 \int_2^k [2 \ln |x-1|] dx$$

$$2 \ln |k-1| = 4$$

$$\ln |k-1| = 2$$

$$k = e^2 = 7.389056099$$

$$k = 8.389$$



The shaded region has an area of 4 units squared.

Find the value of k .

You must use calculus and give the results of any integration needed to solve this problem.

$$x = \frac{2}{x-1} \quad x^2 - x = 2 \quad x^2 - x - 2 = 0$$

$$x(x-1)=2$$

$$x = \frac{1 \pm \sqrt{1+8}}{2} \quad x = \frac{\frac{1}{2} \pm \frac{1}{2}\sqrt{9}}{2}$$

$$x = 0.5 \pm 1.5$$

$$x = -1 \text{ or } 2$$

$$\int_2^k \frac{2}{x-1} dx = 4$$

$$\left[2 \ln |x-1| \right]_2^k = 4$$

$$(2 \ln k - 2k) - (2 \ln 2 - 4) = 4$$

$$2 \ln k - 2k = 1.386294361$$

$$k - e^k = 2 \quad \ln k - k = -0.6931471805$$

$$\ln k = 0.6931471805 + k$$

$$k = e^{0.6931471805 + k}$$

ANSWER:

E8

QUESTION THREE

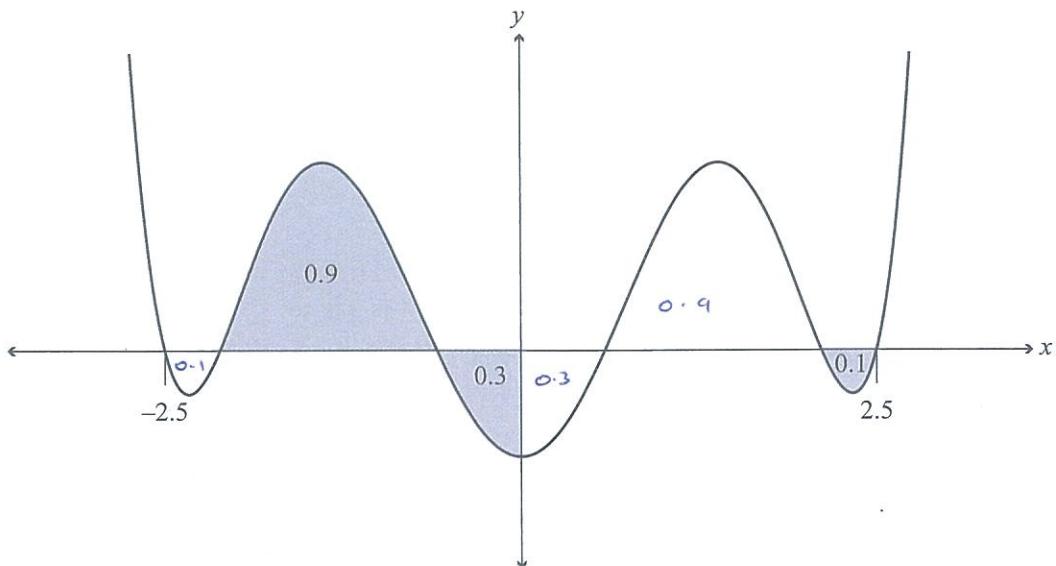
$$x^2 + 8x + 66 + 8e^{4x}$$

- (a) Find $\int ((x+4)^2 + 8e^{4x}) dx$.

$$y = \frac{1}{3}x^3 + 4x^2 + 16x + 2e^{4x} + C$$

- (b) The graph of the function $y = f(x)$ below is symmetrical about the y -axis.

The areas of the shaded regions are given.



Find $\int_{-2.5}^{2.5} f(x) dx$.

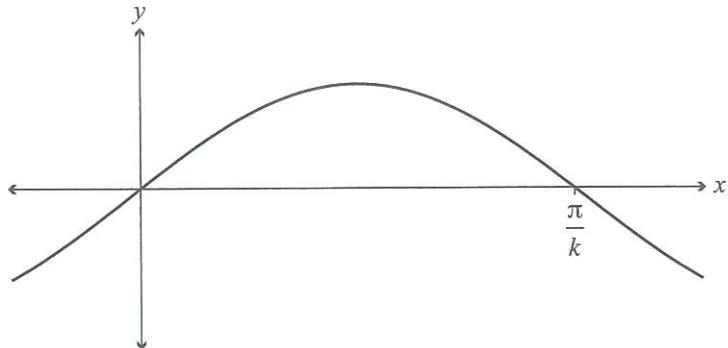
$$0.9 + 0.3 + 0.3 + 0.9 + 0.1 + 0.1$$

2.6 units²

- (c) Find an expression in terms of k for the area bounded by the function $y = \sin kx$

and the x -axis, between $x = 0$ and $x = \frac{\pi}{k}$.

$$-\frac{1}{k} \cos kx$$



You must use calculus and give the results of any integration needed to solve this problem.

WWT

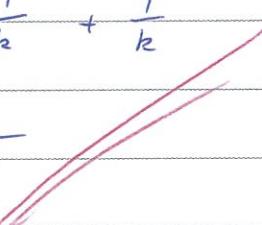
$$\int_0^{\pi/k} \sin kx \, dx$$

$$\left[-\frac{1}{k} \cos kx \right]_0^{\pi/k}$$

$$-\frac{1}{k} \cos \pi + \frac{1}{k}$$

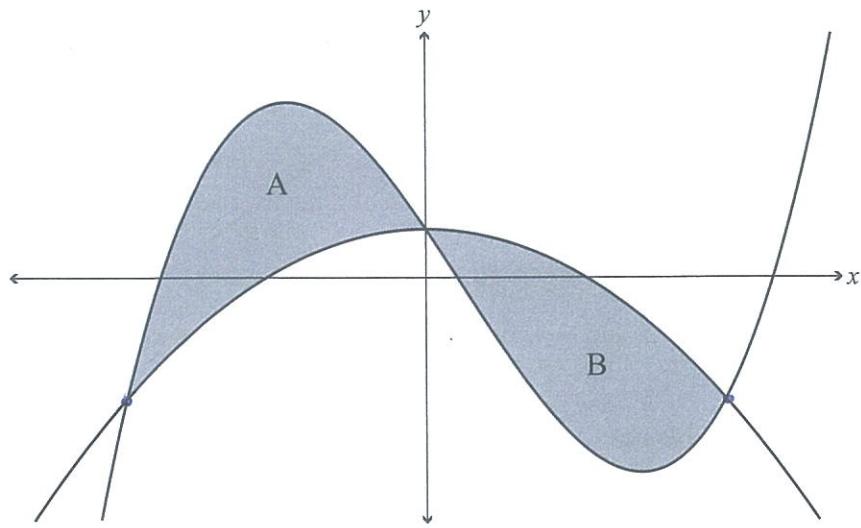
$$\frac{1}{k} + \frac{1}{k}$$

$$A = \frac{2}{k}$$



R

- (d) The graphs of $f(x) = -x^2 + 2$ and $g(x) = x^3 - x^2 - kx + 2$ are shown below.
The graphs intersect and create two closed regions, A and B.



Show that these two regions have the same area.

You must use calculus and give the results of any integration needed to solve this problem.

A
Y

n

- (e) An object starts from rest.

The object's acceleration is given by the formula $a = B(e^{kt})^2$

where a is the acceleration of the object in m s^{-2}

and t is the time, in seconds, from when the object started moving.

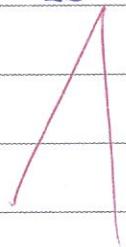
Show that the time that it takes the object to reach velocity v_0 is

$$t = \frac{1}{2k} \ln\left(\frac{2v_0 k + B}{B}\right)$$

You must use calculus and give the results of any integration needed to solve this problem.

$$V = \frac{\beta}{2k} e^{2kt} + C$$

$$C = 0$$



✓

M6

**Extra paper if required.
Write the question number(s) if applicable.**

QUESTION
NUMBER

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QUESTION
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