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Technical communique

A delay-kernel-dependent approach to saturated control of linear systems with mixed delays ☆

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ABSTRACT

This article proposes a delay-kernel-dependent approach to deal with the saturated control problem of linear systems subject to mixed delays: discrete delay and distributed delay considering kernel. By combining the state vector and the distributed delay with kernel, a new polytopic representation strategy is used to cope with the nonlinear input saturation function. By choosing a Lyapunov–Krasovskii functional and applying an integral inequality both related to the distributed delay kernel, novel and less conservative results are provided to ensure the system stability. Finally, an example is simulated to display the advantages of the developed approach.

1. Introduction

Time-delay, usually categorized as discrete delay and distributed delay, is a general phenomenon frequently existing in many practical systems, such as networked control systems, biological systems, economic systems (Barforooshan, Derpich, Stavrou, & Ostergaard, 2020; Yan, Shen, Nguang, & Zhang, 2020). Regarding the stability analysis and controller design for time-delay systems, the derived results independent on the delay are generally more conservative than the results dependent on the delay. Over the past decades, some representative techniques like Wirtinger integral (Seuret & Gouaisbaut, 2013), Bessel–Legendre inequality (Seuret & Gouaisbaut, 2017; Seuret, Gouaisbaut, & Ariba, 2015) and their combination with reciprocally convex lemma (Park, Ko, & Jeong, 2011) have been addressed to reduce the design conservatism of linear systems with discrete delays or distributed delays.

On the other line, for practical control systems, input saturation exists in various physical devices due to the hardware limitation. In order to process the saturation nonlinearity, several effective tools such as sector-bound method (Yin, Seiler, & Arcak, 2021), polytopic representation method (Zhou, 2013) and anti-windup method (Li & Lin, 2014) are proposed. For linear systems with time-varying discrete delay and saturated control, under the polytopic representation framework, an auxiliary time-delay feedback is combined with the state feedback to represent the nonlinear input saturation in Chen, Fei, and Li (2015). In addition, a distributed-delay-dependent approach is further investigated in Chen, Fei, and Li (2016) for the saturated control issue of uncertain systems with discrete and distributed delays. Following the similar way, the impulsive control problem of nonlinear time-delay system subject to input saturation is developed in Li, Li, Ouyang, and Nguang (2020). Moreover, by extending the distributed-delay-dependent method to discrete-time systems with distributed state delay and fast-varying input delay, better results for saturated local stabilization have been obtained in Chen and Wang (2020). With the utilization of the discrete delay or distributed delay in the polytopic representation method, less conservative results can be obtained and larger estimated domain of attraction (DOA) can be achieved than the delay-independent strategy. Compared with the distributed delay system in Chen et al. (2016), the system with distributed delay kernel is more general and practical. For example, the kernel can be used to represent the probability density of stochastic network transmission delay. However, the above delay-dependent polytopic methods in Chen et al. (2015, 2016) and Li et al. (2020) are difficult to treat the saturated control issue of linear systems with discrete delay and distributed delay considering kernel.

Inspired by the precedent discussions, this article investigates the saturated control of linear systems with discrete delay and distributed delay considering kernel. The main contributions are given as:

(i) *A new polytopic delay-kernel-dependent approach utilizing the distributed delay with kernel is presented to deal with the saturation*

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nonlinearity. The existing method in Chen et al. (2016) can be viewed as an equivalent form of our approach under the limitation case (infinite divided delay intervals), which will lead to the curse of dimensionality. However, it can be avoided by our proposed approach.

(ii) *A delay-kernel-dependent Lyapunov–Krasovskii functional (LKF) and an integral inequality related to the kernel are applied to obtain the analysis and synthesis conditions. Compared with the delay-independent saturation handling strategy and the existing approach adopting Legendre polynomials to approximate the kernel (Seuret et al., 2015), the proposed delay-kernel-dependent approach is potential to derive less conservative results.*

The organization of this technical communique is given as follows. The problem formulation is presented in Section 2. In Section 3, the stability and controller design conditions are provided. Then, the advantages of the proposed approach are verified by some simulation results in Section 4. Section 5 shows the conclusions and future investigations.

Notation: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ represent 2-norm and ∞-norm of vector, respectively. $He(X) = X + X^\top$, where $(\cdot)^\top$ means the transpose of X. $\Im(X, Y) = Y^\top XY$. $\mathbb{I}[1, a]$, $\otimes$ and $\mathrm{eig}_m(X)$ mean the set $\{1, \ldots, a\}$, Kronecker product and the maximum eigenvalue of matrix X, respectively.

2. Problem formulation

The considered system with discrete and distributed delays is given as:

$$\dot{x}(t) = Ax(t) + D_1x(t-d) + D_2\int_{-d}^{0} q(r)x(t+r)dr + B\mathbb{S}(u(t)), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ means the state, $u(t) \in \mathbb{R}^f$ means the control input, the nonlinear saturation function with unity level $\mathbb{S}(u(t))$ is represented as $\mathbb{S}(u(t)) = [\mathbb{S}(u_1(t))\cdots\mathbb{S}(u_g(t))\cdots\mathbb{S}(u_f(t))]^\top$, $\mathbb{S}(u_g(t)) = sgn(u_g(t))$
$\min\{1, |\bar{u}_g|\}$, A, B, D_1 and D_2 are system matrices.

Remark 1. For practical networked control systems, the network-induced communication delays are usually stochastic with some special distributions. To make full use of the stochastic feature of the communication delays, the distributed delay systems (1) can be applied to model the delayed networked control systems, where the probability distribution of stochastic communication delays can be described as the delay kernel $q(r)$. This application of the distributed delay systems has been reported in some existing literature (Yan, Gu, Park, & Xie, 2022; Yan, Gu, Park, Xie, & Dou, 2022).

The state feedback controller is constructed as:

$$u(t) = Kx(t), \quad (2)$$

where K is the controller gain to be designed.

Lemma 1 (Zhou, 2013). *For given integer $f \geq 1$ and function $\varphi(t) \in \mathbb{R}^{\overleftrightarrow{f}}$ satisfying $\|\varphi(t)\|_\infty \leq 1$, $\overleftrightarrow{f} = f2^{f-1}$, the function k_f defined as $k_f(0) = 0$ and*

$$k_f(s) = \begin{cases} k_f(s-1)+1, & C_s + C_j \neq I_f,\ \forall j \in \mathbb{I}[1, s] \\ k_f(j), & C_s + C_j = I_f,\ \exists j \in \mathbb{I}[1, s] \end{cases},$$

there exists $\mathbb{S}(u(t)) \in co\{C_su(t) + \mathcal{C}_s^-\varphi(t) : s \in \mathbb{I}[1, 2^f]\}$ holds for any $u(t) \in \mathbb{R}^f$, where co means the convex hull, $C_s^- = I - C_s$, $\mathcal{C}_s^- \triangleq e_{2^{f-1},k_f(s)} \otimes C_s^- \in \mathbb{R}^{f\times\overleftrightarrow{f}}$, $e_{2^{f-1},k_f(s)}$ means a row vector and its elements are 0 except for the $k_f(s)$-th element is 1.

We assume that $\exists\, F_1, F_2 \in \mathbb{R}^{\overleftrightarrow{f}\times n}$ such that

$$\|\varphi(t)\|_\infty = \left\| F_1x(t) + F_2\int_{-d}^{0} q(r)x(t+r)dr \right\|_\infty \leq 1. \quad (3)$$

In terms of Lemma 1, the nonlinear saturation function $\mathbb{S}(u(t))$ can be expressed as

$$\mathbb{S}(u(t)) = \sum_{s=1}^{2^f} \beta_s^t\Big[C_su(t) + \mathcal{C}_s^-\varphi(t)\Big], \quad (4)$$

where $\beta_s^t \geq 0$ and $\sum_{s=1}^{2^f}\beta_s^t = 1$.

Remark 2. In Eq. (4), the distributed delay with kernel $\int_{-d}^{0} q(r)x(t+r)dr$ is considered for the first time to describe the nonlinear saturation function. It is more general than the existing approach only related to state in Lv, Cao, Li, and Luo (2022) and Zhou (2013), and is potential to reduce the design conservatism.

Remark 3. For $Nh = d$, a polytopic distributed-delay saturation representation approach in Chen et al. (2016) is given as

$$\|\varphi(t)\|_\infty = \left\| F_1x(t) + \sum_{j=1}^{N} F_{2j}\int_{-jh}^{-(j-1)h} x(t+r)dr \right\|_\infty \leq 1. \quad (5)$$

If $F_{2j} = F_2q_j$ and $q(-jh) \triangleq q_j$ are chosen, then we have

$$\|\varphi(t)\|_\infty = \left\| F_1x(t) + \sum_{j=1}^{N} F_2\int_{-jh}^{-(j-1)h} q_jx(t+r)dr \right\|_\infty \leq 1, \quad (6)$$

the limitation of which for $N \to \infty$ is equivalent to the form (3) in this work. Under such situation, it yields infinite dimensions of analysis conditions by using the approach in Chen et al. (2016), which fails to design the saturated controller for the distributed-kernel-based delay system (1).

In order to deal with the distributed delay item $\int_{-d}^{0} q(r)x(t+r)dr$, we define $q(r) = q_0(r)$ and construct

$$\mathbf{q}(r) = \begin{bmatrix} q_0(r) & \cdots & q_i(r) & \cdots & q_\kappa(r)\end{bmatrix}^\top,\ Q(r) = \mathbf{q}(r)\otimes I,$$
$$\mathbb{Q}(t) = \int_{-d}^{0} Q(r)x(t+r)dr,\ \mathcal{I} = \begin{bmatrix} I_n & 0_{n\times\kappa n}\end{bmatrix}. \quad (7)$$

Based on Feng and Nguang (2016), the basic principle to choose $q_i(r)$ is that the property $\dot{\mathbf{q}}(r) = \mathcal{Q}\mathbf{q}(r)$ should be satisfied. This implies that the elements $q_i(r)$ are the solutions of linear homogeneous differential equations with constant coefficients in the matrix $\mathcal{Q} \in \mathbb{R}^{n(\kappa+1)\times n(\kappa+1)}$. By combining (1), (2), (4) and (7), the closed-loop system is given as:

$$\begin{aligned}\dot{x}(t) = \sum_{s=1}^{2^f}\beta_s^t\Big[&(A + BC_sK + B\mathcal{C}_s^-F_1)x(t) \\ &+ D_1x(t-d) + (D_2 + B\mathcal{C}_s^-F_2)\mathcal{I}\mathbb{Q}(t)\Big].\end{aligned} \quad (8)$$

For further proceeding, a technical lemma is provided as follows.

Lemma 2 (Feng & Nguang, 2016). *For a matrix $M > 0 \in \mathbb{R}^{n\times n}$, $M = M^\top$ and the vector $\mathbf{q}(r)$ defined in (7), it yields*

$$\int_{-d}^{0}\Im\left(M, x(r)\right)dr \geq \Im\left(\mathbf{Q}\otimes M, \int_{-d}^{0} Q(r)x(r)dr\right) \quad (9)$$

with $\mathbf{Q}^{-1} = \int_{-d}^{0}\mathbf{q}(r)\mathbf{q}^\top(r)dr > 0$.

3. Main results

First, the stability conditions for system (8) with mixed delays and input saturation are formed in Theorem 4.

Theorem 4. For given constants α, d and any initial condition satisfying $\mathbb{V}(0) \le 1$, under the controller gain K, the system (8) is asymptotically stable, if there exist symmetric matrices $P > 0$, $M > 0$, $S > 0$, and matrices X, F_1 and F_2 such that

$$\Psi_s < 0, \tag{10}$$

$$\begin{bmatrix} 1 & \Theta_l^\top \\ \Theta_l & \mathbf{P} \end{bmatrix} \ge 0, \tag{11}$$

where $\mathbf{X} = \mathbf{e}_1^\top X + \alpha \mathbf{e}_2^\top X$, $\mathbb{S} = \mathbf{Q} \otimes S$, $\Theta_l = \begin{bmatrix} F_{1l} & F_{2l}\mathcal{I} \end{bmatrix}$,

$$\Psi_s = \Xi + He(\mathbf{X}\mathbf{Y}_s),\ \ \mathbf{P} = P + diag\{0, \mathbb{M}\},\ \ \mathbb{M} = \mathbf{Q} \otimes M,$$

$$\Xi = He(\Omega_1^\top P \Omega_2) + \Im(M + dS, \mathbf{e}_2) - \Im(M, \mathbf{e}_3) - \Im\left(\mathbb{S}, \mathbf{e}_q\right),$$

$$\Omega_1 = \begin{bmatrix} \mathbf{e}_2 \\ \mathbf{e}_q \end{bmatrix}, \Omega_2 = \begin{bmatrix} \mathbf{e}_1 \\ Q(0)\mathbf{e}_2 - Q(-d)\mathbf{e}_3 - \mathcal{Q}\mathbf{e}_q \end{bmatrix}, \mathbf{e}_q = \begin{bmatrix} \mathbf{e}_4 \\ \vdots \\ \mathbf{e}_{4+\kappa} \end{bmatrix},$$

$$\mathbf{Y}_s = -\mathbf{e}_1 + (A + BC_sK + BC_s^- F_1)\mathbf{e}_2 + D_1\mathbf{e}_3 + (D_2 + BC_s^- F_2)\mathcal{I}\mathbf{e}_q,$$

$$\mathbf{e}_h \triangleq \begin{bmatrix} 0_{n,n(h-1)} & I_n & 0_{n,n(4+\kappa-h)} \end{bmatrix},\quad h = 1, \ldots, 4+\kappa.$$

Proof. Defining $\chi^\top(t) \triangleq \left[\dot{x}^\top(t), x^\top(t), x^\top(t-d), \mathbb{Q}^\top(t)\right]$ and left- and right-multiplying the condition (10) from both sides by $\chi^\top(t)$ and $\chi(t)$, we have

$$\Im\big(\Psi_s, \chi(t)\big) < 0, \tag{12}$$

which further ensures

$$\sum_{s=1}^{2^f} \beta_s^t \left(\Im\big(\Psi_s, \chi(t)\big)\right) < 0. \tag{13}$$

By constructing $\mathbf{X} = X\mathbf{e}_1^\top + \alpha X \mathbf{e}_2^\top$, it is derived from system (8) that

$$\sum_{s=1}^{2^f} \beta_s^t \Im\big(\mathbf{X}\mathbf{Y}_s, \chi(t)\big) = 0. \tag{14}$$

Subtracting (13) by (14), one can get

$$\Im\big(\Xi, \chi(t)\big) = \Im\Big(He(\Omega_1^\top P \Omega_2), \chi(t)\Big) + \Im(M + dS, \mathbf{e}_2\chi(t)) - \Im(M, \mathbf{e}_3\chi(t)) - \Im\left(\mathbb{S}, \mathbf{e}_q\chi(t)\right) < 0. \tag{15}$$

Define $\eta(t) = [x^\top(t)\ \ \mathbb{Q}^\top(t)]^\top$ and construct an LKF as

$$\mathbb{V}(t) = \mathbb{V}_1(t) + \mathbb{V}_2(t), \tag{16}$$

where

$$\mathbb{V}_1(t) = \Im\big(P, \eta(t)\big), \mathbb{V}_2(t) = \int_{t-d}^{t} \Im(M + (r - t + d)S, x(r))dr.$$

By differentiating $\mathbb{V}(t)$, it gives

$$\dot{\mathbb{V}}(t) = He\left(\eta^\top(t) P \dot{\eta}(t)\right) + \Im(M + dS, x(t)) - \Im(M, x(t-d)) - \int_{-d}^{0} \Im(S, x(t+r))dr. \tag{17}$$

Applying Lemma 2 to handle the integral item, one gets

$$-\int_{-d}^{0} \Im(S, x(t+r))dr \le -\Im\left(\mathbb{S}, \mathbb{Q}(t)\right). \tag{18}$$

According to $\dot{\mathbf{q}}(r) = \mathcal{Q}\mathbf{q}(r)$, it leads to

$$\dot{\mathbb{Q}}(t) = Q(0)x(t) - Q(-d)x(t-d) - \mathcal{Q}\mathbb{Q}(t). \tag{19}$$

Based on (19), it yields

$$\eta(t) = \Omega_1\chi(t),\quad \dot{\eta}(t) = \Omega_2\chi(t). \tag{20}$$

From (15), (17)–(20), the system stability is ensured by

$$\dot{\mathbb{V}}(t) \le \Im\big(\Xi, \chi(t)\big) < 0. \tag{21}$$

According to (21), we get $\dot{\mathbb{V}}(t) < 0$ and $\mathbb{V}(0) > \mathbb{V}(t)$.
With the aid of Lemma 2, it leads to

$$\Im(\mathbf{P}, \eta(t)) \le \Im(P, \eta(t)) + \Im\left(\mathbb{M}, \mathbb{Q}(t)\right) + \int_{-d}^{0} \Im((r+d)S, x(t+r))dr \le \mathbb{V}(t). \tag{22}$$

By using Schur complement to (11), it yields

$$\Theta_l^\top \Theta_l \le \mathbf{P},\quad l \in \mathbb{I}[1, \overleftrightarrow{f}]. \tag{23}$$

According to (23), it gives

$$\left|F_{1l}x(t) + F_{2l}\mathcal{I}\mathbb{Q}(t)\right|^2 = \Im\big(\Theta_l^\top \Theta_l, \eta(t)\big) \le \Im\big(\mathbf{P}, \eta(t)\big). \tag{24}$$

For any initial condition meeting $\mathbb{V}(t) \le \mathbb{V}(0) \le 1$, it is observed from (22) and (24) that $\left|F_{1l}x(t) + F_{2l}\mathcal{I}\mathbb{Q}(t)\right|^2 \le 1$ holds. This indicates that the assumption (3) is ensured. Therefore, the stability of system (8) is ensured for any initial state satisfying $\mathbb{V}(0) \le 1$. ■

Remark 5. It is unavoidable that the approximation error will be introduced by using Legendre polynomials to approximate the kernel $q(r)$ in Seuret et al. (2015). Compared with Seuret et al. (2015), the kernel-dependent integral inequality given in Lemma 2 is able to treat the distributed delay by excluding the approaching error and decrease the conservatism.

Second, in terms of the above theorem, the saturated controller design criteria formulated by linear matrix inequalities (LMIs) are deduced in Theorem 6.

Theorem 6. For given scalars α, and d, if there exist symmetric matrices $\hat{P} > 0$, $\hat{M} > 0$, $\hat{S} > 0$, and matrices L, Y, N_1 and N_2 such that for $\forall r \in \mathbb{I}[1, 2^f]$, $\forall l \in \mathbb{I}[1, \overleftrightarrow{f}]$, the following conditions hold:

$$\hat{\Psi}_s < 0, \tag{25}$$

$$\begin{bmatrix} 1 & \hat{\Theta}_l^\top \\ \hat{\Theta}_l & \hat{\mathbf{P}} \end{bmatrix} \ge 0, \tag{26}$$

where $\hat{\mathbf{X}} = \mathbf{e}_1^\top + \alpha\mathbf{e}_2^\top$, $\hat{\mathbb{S}} = \mathbf{Q} \otimes \hat{S}$, $\hat{\Theta}_l = \begin{bmatrix} N_{1l} & N_{2l}\mathcal{I} \end{bmatrix}$,

$$\hat{\Psi}_s = \hat{\Xi} + He(\hat{\mathbf{X}}\hat{\mathbf{Y}}_s),\ \ \hat{\mathbf{P}} = \hat{P} + diag\{0, \hat{\mathbb{M}}\},\ \ \hat{\mathbb{M}} = \mathbf{Q} \otimes \hat{M},$$

$$\hat{\Xi} = He(\Omega_1^\top \hat{P} \Omega_2) + \Im(\hat{M} + d\hat{S}, \mathbf{e}_2) - \Im(\hat{M}, \mathbf{e}_3) - \Im\left(\hat{\mathbb{S}}, \mathbf{e}_q\right),$$

$$\hat{\mathbf{Y}}_s = -Y\mathbf{e}_1 + (AY + BC_sL + BC_s^- N_1)\mathbf{e}_2 + D_1Y\mathbf{e}_3 + (D_2Y + BC_s^- N_2)\mathcal{I}\mathbf{e}_q,$$

then for any initial condition satisfying $\mathbb{V}(0) \le 1$, the asymptotic stability of system (8) can be ensured by the controller $K = LY^{-1}$.

Proof. Define $Y = X^{-1}$, $\hat{M} = \Im(M, Y)$, $\hat{S} = \Im(S, Y)$, $\hat{\mathbb{S}} = \Im(\mathbb{S}, I_{(\kappa+1)} \otimes Y)$, $\hat{\mathbb{M}} = \Im(\mathbb{M}, I_{(\kappa+1)} \otimes Y)$, $KY = L$, $\Theta_l(I_{(\kappa+2)} \otimes Y) = \begin{bmatrix} N_{1l} & N_{2l}\mathcal{I} \end{bmatrix}$.

Pre- and post-multiplying (10) with $\mathcal{Y}^\top = I_{n(\kappa+4)} \otimes Y^\top$ and $\mathcal{Y}$, we derive the condition (25).

Following the similar way in the above, the condition (26) is also obtained, which completes the proof. ■

Next, an optimization problem is proposed to derive a larger estimation of DOA ($\mathbb{A}_\sigma$) when the controller is designed. According to the chosen LKF (16), we have

$$\begin{aligned}\mathbb{V}_1(t) &\leq \eta^\top(t)P\eta(t) \leq \eta^\top(t)diag\{\Omega_0, \Omega_1\}\eta(t)\\ &\leq x^\top(t)\mathrm{eig}_m(\Omega_0)x(t) + \mathbb{Q}^\top(t)\mathrm{eig}_m(\Omega_1)\mathbb{Q}(t)\\ &\leq \left(\mathrm{eig}_m(\Omega_0) + d\ \mathrm{eig}_m(\Omega_1)\mathrm{eig}_m(\mathbf{Q}^{-1})\right)\|x(t)\|_2^2, \end{aligned}\quad (27)$$

$$\mathbb{V}_2(t) \leq \left(d\ \mathrm{eig}_m(M) + d^2\mathrm{eig}_m(S)\right)\|x(t)\|_2^2. \quad (28)$$

Then, the bound of $\mathbb{A}_\delta$ can be estimated by

$$\begin{aligned}\mathbb{V}(0) \leq &\left(\mathrm{eig}_m(\Omega_0) + d\varpi\,\mathrm{eig}_m(\Omega_1)\right.\\ &\left.+\ d\ \mathrm{eig}_m(M) + d^2\mathrm{eig}_m(S)\right)\sigma^2,\end{aligned} \quad (29)$$

where $\|x(0)\|_2 \leq \sigma$, $\varpi = \mathrm{eig}_m(\mathbf{Q}^{-1})$.

As in Chen et al. (2016), the constraint $Y^{-\top}Y^{-1} \leq \nu I$ with a variable scalar $\nu > 0$ is considered, which is ensured by

$$\begin{bmatrix} \nu I & I \\ I & He(Y) - I \end{bmatrix} \geq 0. \quad (30)$$

Define $\hat{\Omega}_0 = \Im(\Omega_0, Y^{-1})$, $\hat{\Omega}_1 = \Im(\Omega_1, I_{(\kappa+1)} \otimes Y)$ and $\hat{\Omega} \triangleq diag\{\hat{\Omega}_0, \hat{\Omega}_1\}$ and let

$$\hat{P} \leq \hat{\Omega},\quad \hat{\Omega}_i \leq \gamma_i I,\ i = 0, 1,\quad \hat{M} \leq a_1 I,\quad \hat{S} \leq a_2 I. \quad (31)$$

Thus, the maximization of $\mathbb{A}_\delta$ in Theorem 6 can be optimized by the following optimization problem:

Problem 1. $\min_{\hat{P},\hat{M},\hat{S},\Omega_i,L_1,L_2,N_1,N_2,Y,\nu,\gamma_0,\gamma_1,a_1,a_2}\ \rho$ subject to LMIs (30)–(31), (25)–(26) where

$$\rho = \epsilon\nu + (\gamma_0 + d\varpi\gamma_1 + da_1 + d^2a_2)$$

and ϵ is a weighting parameter. Consequently, the maximum σ is derived by $\sigma_{\max} = \sqrt{1/\mho}$, where $\mho = \mathrm{eig}_m(\Omega_0) + d\varpi\,\mathrm{eig}_m(\Omega_1) + d\ \mathrm{eig}_m(M) + d^2\mathrm{eig}_m(S)$.

4. Example

The system parameters are chosen as

$$A_1 = \begin{bmatrix} 0.2 & 0 \\ 0.4 & -0.7 \end{bmatrix}, D_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0 & -0.5 \end{bmatrix}, D_2 = \begin{bmatrix} -0.3 & 0.1 \\ 0 & -0.6 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0.5 \\ 1 & -1 \end{bmatrix},\quad q(r) = -(10/d)^2 re^{\frac{10r}{d}},\quad r \in [-d,\ 0],$$

where the kernel $q(r)$ satisfies Gamma distribution. For $\kappa = 1$, to construct the vector $\mathbf{q}(r)$ meeting the property $\dot{\mathbf{q}}(r) = \mathcal{Q}\mathbf{q}(r)$, the other term can be chosen as $q_1(r) = -\frac{10}{d}e^{\frac{10r}{d}}$. Then, the vector $\mathbf{q}(r)$ and $\mathcal{Q}$ are obtained as $\mathbf{q}(r) = \begin{bmatrix} -(10/d)^2re^{\frac{10r}{d}} \\ -\frac{10}{d}e^{\frac{10r}{d}} \end{bmatrix}$, $\mathcal{Q} = \begin{bmatrix} \frac{10}{d} & \frac{10}{d} \\ 0 & \frac{10}{d} \end{bmatrix}$.

By choosing $\epsilon = 3 \times 10^4$, $\alpha = 1$, and the unity saturation level $\bar{u}_1 = \bar{u}_2 = 1$, the optimal estimation of DOA ($\mathbb{A}_\sigma$) for different d derived by our delay-kernel-dependent approach solved by Problem 1 and the existing delay-independent approach are shown in Table 1. From this table, one observes that larger DOA can be obtained by our approach than the traditional approach without considering the distributed delay term. This means that the delay-kernel-dependent term and the auxiliary matrix F_2 are helpful in reducing design conservatism.

The controller gain for $d = 0.5s$ is computed as $K = \begin{bmatrix} -0.8381 & -0.5168; -0.4335 & -0.0669 \end{bmatrix}$.

Table 1
Comparison of the estimation of DOA ($\mathbb{A}_\sigma$) for different d.

d/s	0.1	0.5	1
Delay-independent method (Lv et al., 2022; Zhou, 2013)	2.0556	1.5078	1.1990
Delay-kernel-dependent method (4)	2.2179	2.0243	1.8329

Table 2
Comparison of the estimation of DOA ($\mathbb{A}_\sigma$) for $D_2 = 0$.

d/s	0.1	0.5	1
Delay-independent method (Lv et al., 2022; Zhou, 2013)	1.1863	0.7538	0.5676
Delay-kernel-dependent method (4)	1.4331	1.3581	1.2629

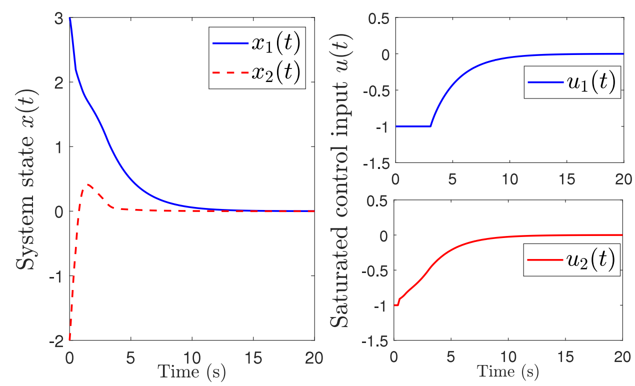

Fig. 1. The responses of state $x(t)$ (left) and saturated control input $u(t)$ (right) for $h = 0.5$ s.

In simulation, under the initial condition $x(0) = [3, -2]^\top$, the responses of the state $x(t)$ and saturated control input $u(t)$ for $d = 0.5s$ are illustrated in Fig. 1. From this figure, one observes that the designed saturated controller is effective to guarantee the stability when input saturation happens.

In addition, the considered distributed delay system (1) is reduced to conventional discrete delay system by setting $D_2 = 0$. In this case, the corresponding comparison results are obtained in the following table.

From Table 2, one observes that the estimations of DOA obtained by our proposed method are larger than the results obtained by existing delay-independent method. These comparison results further illustrate that our proposed method is effective for systems with discrete delay.

5. Conclusion

This paper has studied the saturated control issue of linear systems with discrete and distributed delays. A delay-kernel-dependent approach is proposed to handle the nonlinear saturation function. Then, sufficient LMI conditions to solve the saturated controller are obtained. Lastly, some simulation results demonstrate that our proposed delay-kernel-dependent approach is less conservative than the existing delay-independent method. Note that the continuous-time system is considered in this work. In the future, how to extend the proposed delay-kernel-dependent saturated control method to discrete-time systems deserves further studies.

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