13.1.31 Show that the vector function

$r(t) = f(t)i + g(t)j + h(t)k$ is continuous at $t = t_0$ if and only if f, g, and h is continuous at $t = t_0$

<pf> ($\Rightarrow$) Suppose that r is continuous at $t = t_0$. Note that $|f(t) - f(t_0)|^2 \leq$

$|f(t) - f(t_0)|^2 + |g(t) - g(t_0)|^2 + |h(t) - h(t_0)|^2$

$= |r(t) - r(t_0)|^2 \longrightarrow 0$ as $t \to t_0$,

So $\lim_{t \to t_0} f(t) = f(t_0)$. Similarly,

we can show that $\lim_{t \to t_0} g(t) = g(t_0)$

and $\lim_{t \to t_0} h(t) = h(t_0)$.

($\Leftarrow$) Suppose f, g, and h are continuous at $t=t_0$. Note that $|r(t)-r(t_0)|^2 = [f(t)-f(t_0)]^2 + [g(t)-g(t_0)]^2 + [h(t)-h(t_0)]^2 \to 0$ as $t \to t_0$.

Therefore, $\lim_{t\to t_0} r(t) = r(t_0)$.

13.1.33. Show that if $r(t) = f(t)i + g(t)j + h(t)k$ is differentiable at $t=t_0$, then it is continuous at $t=t_0$ as well.

<pf> $$r(t) - r(t_0) = \frac{r(t)-r(t_0)}{t-t_0}\cdot(t-t_0)$$

$$\Longrightarrow \lim_{t\to t_0} r(t) - r(t_0) = \lim_{t\to t_0}\frac{r(t)-r(t_0)}{t-t_0}\lim_{t\to t_0}(t-t_0)$$

lim

$$= r'(t_0)\cdot 0 = 0$$

Hence, r is continuous at $t=t_0$.

13.2.41 (a) Show that if the vector functions $R_1(t)$ and $R_2(t)$ have identical derivatives on an interval I, then the functions differ by a constant vector value throughout I.

<pf> Let $r(t) = R_1(t) - R_2(t)$.

Then r is differentiable and have zero derivative. By the Mean Value Theorem for vector-valued functions, $r(t) = C$ for some constant vector C.

That is, $R_1(t) = R_2(t) + C$, as desired.

(b) Show that if $R(t)$ is any antiderivative of $r(t)$ on I, then any other derivative of r on I equals $R(t)+C$ for some constant vector C.

<pf> Suppose $R'(t)$ is any other antiderivative. Then R and R' have the same derivative $r(t)$ and it follows from part (a) that $R(t) = R'(t) + C$, as wished.

13.2.42 (Fundamental Theorem of Calculus, FTC) if R is any antiderivative of r on $[a,b]$, then $\int_a^b r(t)\,dt = R(b) - R(a)$.

<pf>

Lemma: if a vector function $r(t)$ is continuous for $a \le t \le b$, then $\frac{d}{dt}\int_a^t r(\tau)\,d\tau = r(t)$ at every point t of (a,b).

<pf> This is just a computation:

$$\frac{d}{dt}\int_a^t r(\tau)\,d\tau = \left[\frac{d}{dt}\int_a^t f(\tau)\,d\tau\right]i +$$

$r(\tau) = f(\tau)i + g(\tau)j + h(\tau)k$

$$\left[\frac{d}{dt}\int_a^t g(\tau)\,d\tau\right]j + \left[\frac{d}{dt}\int_a^t h(\tau)\,d\tau\right]k$$

$$\underset{\star}{=} f(t)i + g(t)j + h(t)k = r(t)$$

<Note> The lemma tells us that $\int_a^t r(\tau)$ is an antiderivative of $r(t)$.

Suppose $R(t)$ is an antiderivative of r. By the previous result, we know that

$$R(t) + C = \int_a^t r(\tau)\,d\tau \text{ for some } C.$$

Letting $t=a$ in the equation, it gives that $C = -R(a)$ and by pulgging $t=b$ in that, we eventually have

$$R(b) - R(a) = \int_a^b r(\tau)\,d\tau.$$