<pf> By the Taylor's thm

$$\phi(\theta+h) - \phi(\theta) = \phi'_\theta(h) + \|h\|\, o(1)$$

$$\underset{h = T_n - \theta}{\Rightarrow} \phi(T_n) - \phi(\theta) = \phi'_\theta(T_n-\theta) + \|T_n-\theta\|\, o_p(1)$$

$$\underset{\times r_n}{\Rightarrow} r_n(\phi(T_n)-\phi(\theta)) = \phi'_\theta(r_n(T_n-\theta)) + r_n\|T_n-\theta\|\, o_p(1)$$

$\xrightarrow{D} \phi'_\theta(T)$ as $n \to \infty$, because

$r_n(T_n-\theta) \xrightarrow{D} T \underset{\text{prohorov's thm}}{\Rightarrow} r_n(T_n-\theta)$ is uniformly tight

$\Leftrightarrow r_n\|T_n-\theta\| = O_p(1)$ and $O_p(1)\,o_p(1) = o_p(1)$

Let $\alpha_i = EX^i$ for $i = 1, 2, 3, 4$.

(Observation 2) $\sqrt{n}\left(\begin{bmatrix}\bar{X}\\ \overline{X^2}\end{bmatrix} - \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}\right)$

$$\xrightarrow{D} N_2 \sim N_2\left(\begin{bmatrix}0\\ 0\end{bmatrix}, \begin{bmatrix} & \\ & \alpha_4 - \alpha_2^2\end{bmatrix}\right)$$

(Observation 1) $S^2 = \overline{X^2} - \bar{X}^2 = \phi(\bar{X}, \overline{X^2})$

$\phi(x, y) = y - x^2$

$$\Rightarrow \sqrt{n}\left(\phi(\bar{X}, \overline{X^2}) - \phi(\alpha_1, \alpha_2)\right)$$

Delta Method

$$\xrightarrow{D} \phi'_{(\alpha_1, \alpha_2)} N_2 = -2\alpha_1 T_1 + T_2$$

$N_2 = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$

Let $Y_i = X_i - \alpha_1$.

(Observation 2') $$\sqrt{n}\left(\begin{bmatrix}\bar{Y}\\ \overline{Y^2}\end{bmatrix} - \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}\right)$$

$$\xrightarrow{D} N_2' \sim N_2\left(\begin{bmatrix}0\\ 0\end{bmatrix}, \begin{bmatrix} & \\ & \mu_4-\mu_2^2\end{bmatrix}\right)$$

(Observation 1') $$S^2 = \overline{Y^2} - \bar{Y} = \phi(\bar{Y}, \overline{Y^2})$$

$\phi(x,y) = y - x^2$

$$\Rightarrow \sqrt{n}\left(\phi(\bar{Y}, \overline{Y^2}) - \phi(\mu_1, \mu_2)\right)$$

Delta Method

$$\xrightarrow{D} \phi'_{\mu_1,\mu_2} N_2' =$$

$N_2' = \begin{bmatrix} T_1' \\ T_2' \end{bmatrix}$

$$= -2\mu_1 T_1' + T_2' = T_2' \sim N(0, \mu_4 - \mu_2^2)$$

$\mu_1 = 0$

Fact 1: $$\frac{\chi^2_{n-1} - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{D} N(0,1)$$

$$\Rightarrow \frac{\chi^2_{n,\alpha} - (n-1)}{\sqrt{2(n-1)}} \longrightarrow z_\alpha$$

Fact 2: $$\sqrt{n}\left(\frac{S^2}{\mu_2} - 1\right) \xrightarrow{D} N\Big(0, \underbrace{\frac{\mu_4}{\mu_2^2} - 1}_{k+2}\Big)$$

Observation: For $\mu_2 = 1$, $P(nS^2 > \chi^2_{n\alpha})$

$$= P\left(\frac{\sqrt{n}\left(\frac{S^2}{\mu_2} - 1\right)}{\sqrt{k+2}} > \sqrt{\frac{2}{k+2}}\ \frac{\chi^2_{n,\alpha} - n}{\sqrt{2n}}\right)$$

$$\underset{n}{\approx} P\left(Z > z_\alpha \cdot \sqrt{\frac{2}{k+2}}\right)$$

(Observation 1) $l_n = \phi(\overline{X}, \overline{X^2}, \overline{X^3})$

$$\phi(a,b,c) = \frac{c - 3ab + 2a^3}{(b-a^2)^{3/2}}$$

Observation 2) $\sqrt{n}\left(\begin{bmatrix}\overline{X}\\ \overline{X^2}\\ \overline{X^3}\end{bmatrix} - \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}\right)$

$\xrightarrow{D} N_3 \sim N_3\left(\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}, \begin{bmatrix} \quad \end{bmatrix}\right)$

Let $Y_i = \dfrac{X_i - \alpha_1}{\sigma}$. Then

(Observation 1') $l_n = \phi(\overline{Y}, \overline{Y^2}, \overline{Y^3})$

(observation 2') $\sqrt{n}\left(\begin{bmatrix}\overline{Y}\\ \overline{Y^2}\\ \overline{Y^3}\end{bmatrix} - \begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix}\right)$

$\xrightarrow{D} N_3' \sim N_3\left(\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}, \begin{bmatrix} ? \end{bmatrix}\right)$

$$\Longrightarrow \sqrt{n}\left(\phi(\overline{Y}, \overline{Y^2}, \overline{Y^3}) - \phi(\mu_1, \mu_2, \mu_3)\right)$$

Delta Method

$$\xrightarrow{D} \phi'_{\mu_1,\mu_2,\mu_3} N_3'$$

$$= N(0, 6)$$

Under H_0

$(\lambda = K = 0)$

$\mu_1 = 0$

$\mu_2 = 1.$

Observation 1: $\sqrt{n}(r_n - e) \xrightarrow{D} N(0, (1-e^2)^2)$.

Observation 2: $\sqrt{n}(\phi(r_n) - \phi(e))$

$\xrightarrow{D} N(0, \phi'(e)(1-e^2)^2)$

Assume $\phi'(e)(1-e^2)^2 = 1$.

Then we have $\phi'(e) = \frac{1}{(1-e^2)^2}$

$\Longrightarrow \phi(e) = \int \frac{1}{(1-e^2)^2}$

$\int$

$= \tanh^{-1}(e)$

Consequently, if $\phi(\rho) = \tan^{-1}(\rho)$,

then $\sqrt{n}\left(\phi(r_n) - \phi(\rho)\right) \xrightarrow{D} N(0,1)$

$$\Rightarrow \text{CI for } \phi(\rho) = \phi(r_n) \pm \frac{z_{\alpha/2}}{\sqrt{n}}$$

$$\Rightarrow \text{CI for } \rho = \phi^{-1}\left(\phi(r_n) \pm \frac{z_{\alpha/2}}{\sqrt{n}}\right)$$

Such a ϕ is called variance-stabilizing transformation.

$$\cos \bar{X} - \cos 0 = -\frac{1}{2}(\bar{X} - 0)^2 + R$$

Note: $\sqrt{n}\,\bar{X} \xrightarrow{D} N(0,1)$

$$n\bar{X}^2 \longrightarrow \chi^2_1$$

$$\Rightarrow -2n(\cos \bar{X} - \cos 0) = n\bar{X}^2 + R$$

$$\xrightarrow{D} \chi^2_1$$