Def) $\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

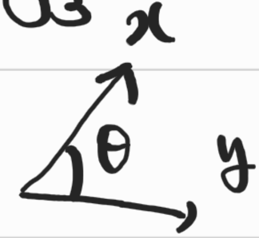
$$(x, y) \longmapsto x \cdot y,$$

where $x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$$x \cdot y \overset{\text{def}}{=} x_1 y_1 + x_2 y_2 + x_3 y_3$$

<Note> $x \cdot y = |x||y| \cos\theta$



Def) $\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

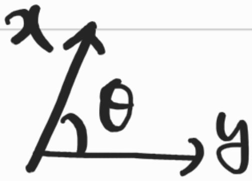
$$(x, y) \longrightarrow x \times y,$$

where $x = (x_1, x_2, x_3)$

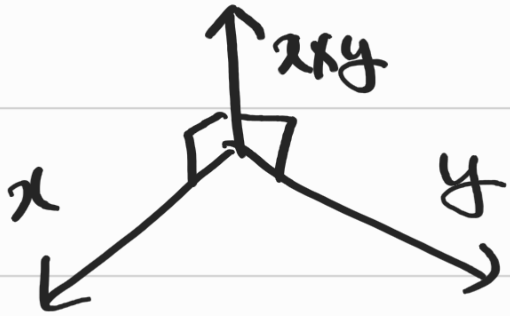
$y = (y_1, y_2, y_3)$

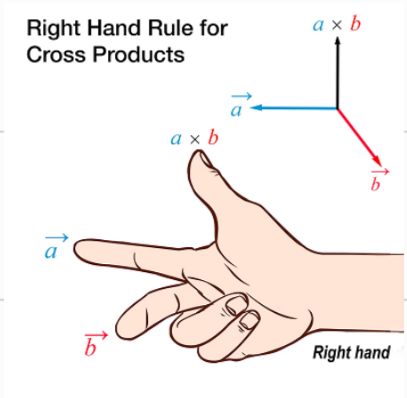
$$x \times y \overset{\text{def}}{=} \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2,\ x_1 y_3 - x_3 y_1,\ x_1 y_2 - x_2 y_1)$$

<Note> $|x \times y| = |x|\,|y| \sin\theta$

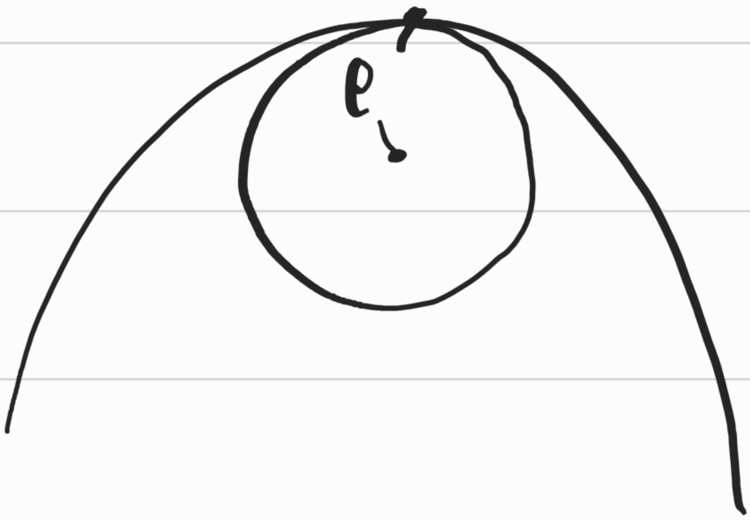


<Note2>





Def (Circle of curvature)

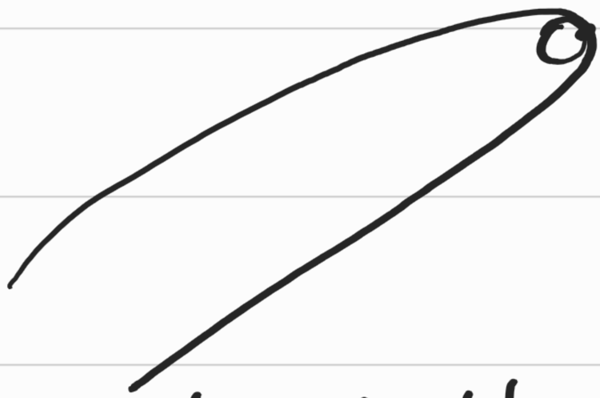


ρ = the radius of the above circle

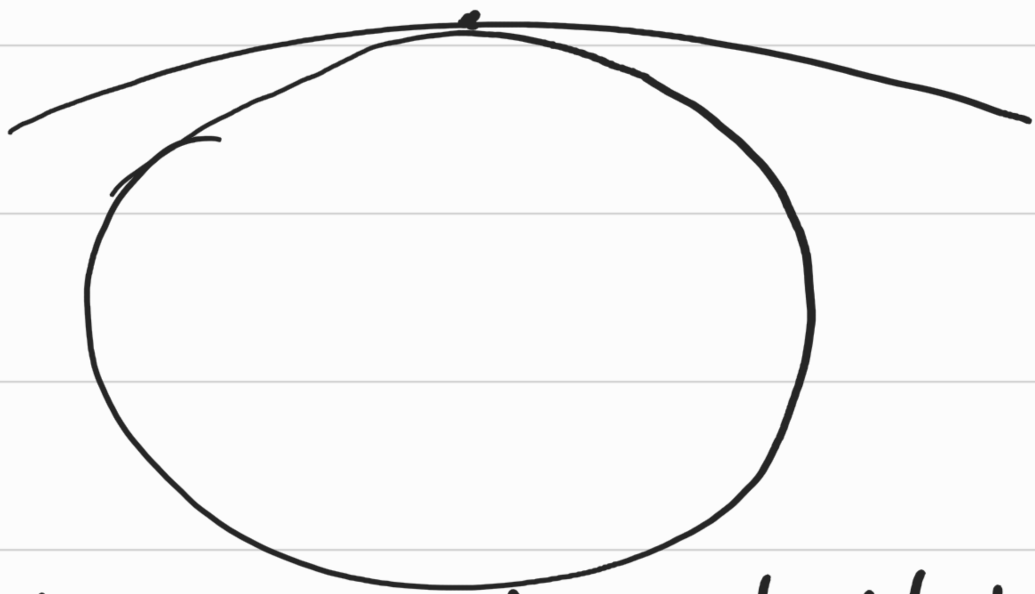
$\overset{\text{def}}{=} 1/\kappa$, where κ is the curvature at the given point.

<Note> $\kappa = \dfrac{|v \times a|}{|v|^3}$

<observation> If k is very large, we have



hence the circle should be small (radius $\frac{1}{k}$). On the other hand, if k is very small, we get

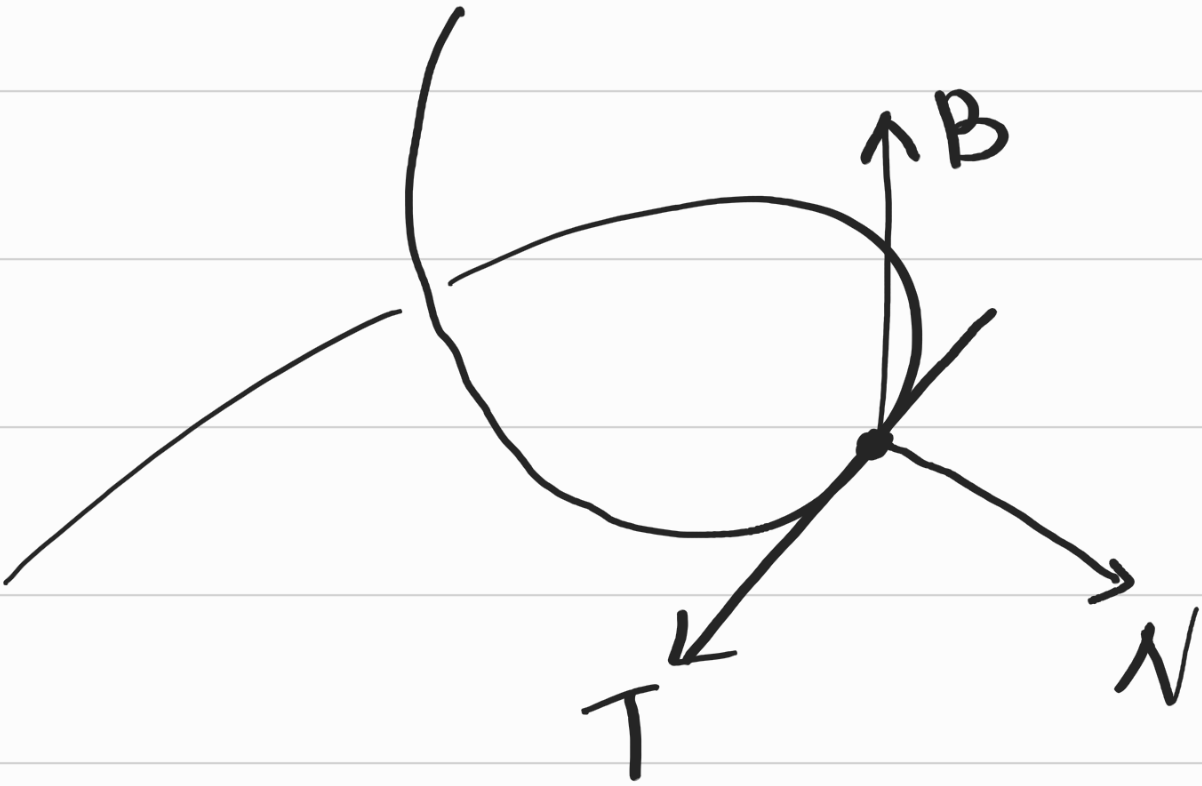


It implies the radius should be large.

Consequently, e measures how similar

the curve is to the circle at the given point (local sense)

Observation:



$\Downarrow$

$$\kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$$

$$\tau = -\frac{dB}{ds} \cdot N$$

(i) $\kappa \uparrow$

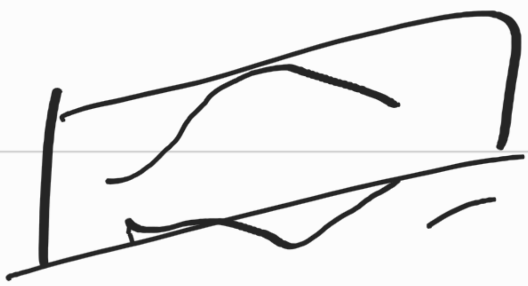
(ii) $\kappa \downarrow$

(iii) $\kappa = 0$

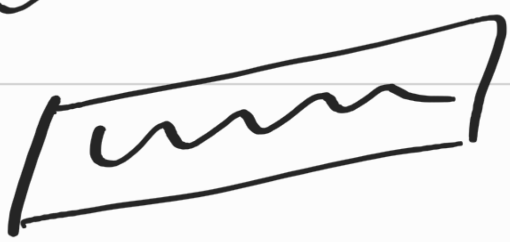
(straight line)

(i) $|\tau| \uparrow$

(ii) $|\tau| \downarrow$

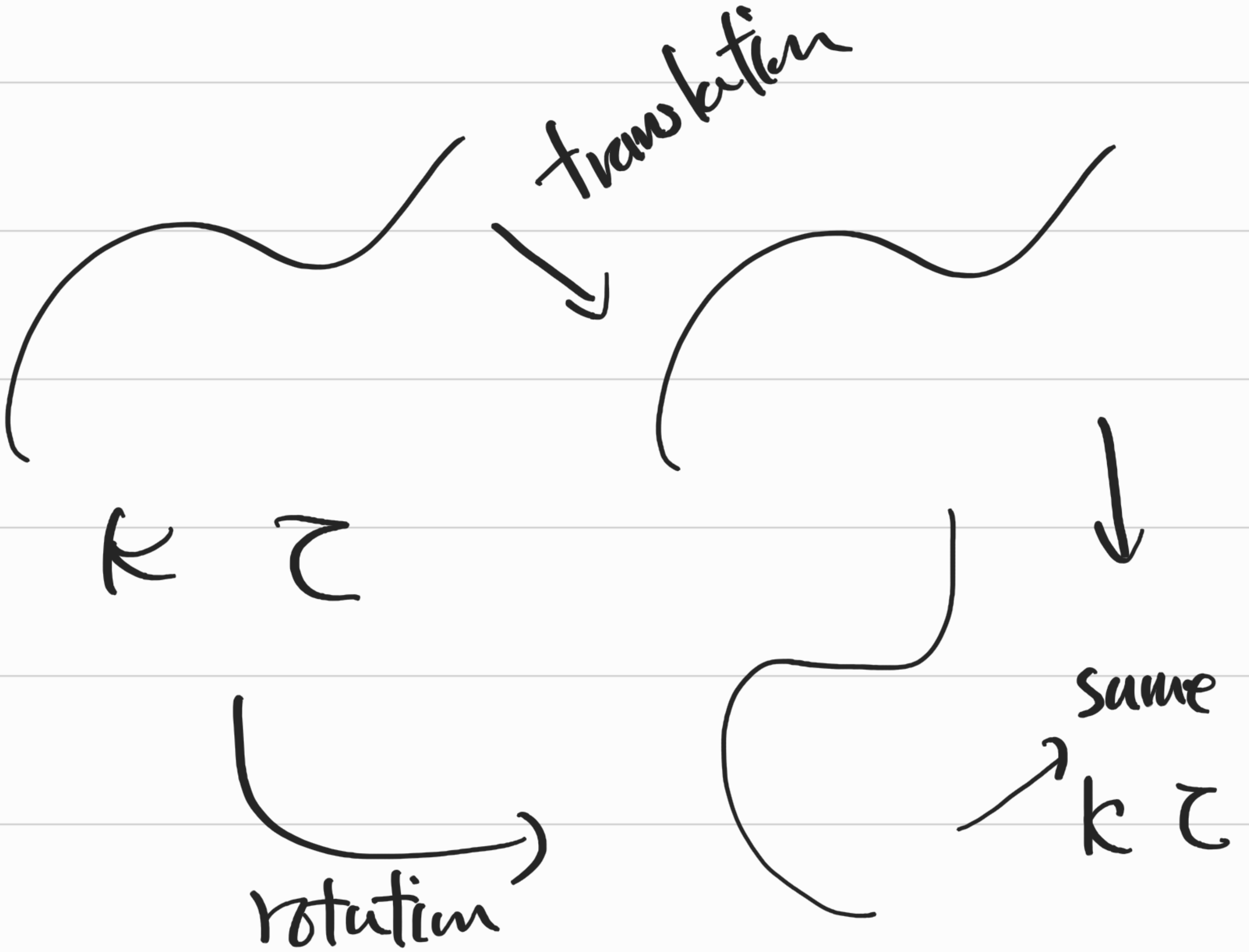


(iii) $\tau = 0$



in plane

The fundamental theorem of space curves: Given $\kappa \geq 0$ and $\tau \in \mathbb{R}$, we can find a curve whose curvature is κ and torsion is τ. The uniqueness is also true up to translations and rotations.



14.2.46 By considering different paths of approach, show that the function have no limit as $(x, y) \to (0, 0)$.

10 points

$$g(x, y) = \frac{x^2 - y}{x - y}$$

Strategy 1) Consider three paths :

$x = 0$, $y = 0$, and $y = x$

$\downarrow$

Strategy 2) Consider $y = kx$ (or $x = ky$)

$\downarrow$

Strategy 3) Consider $x = r\cos\theta$ and $y = r\sin\theta$

$\downarrow$

Strategy 4) Maybe, the limit exists!

Example 1) $f(x,y) = \dfrac{x^2}{x^2+y^2}$

Along $x=0 \Rightarrow f(x,y) = 0$

Along $y=0 \Rightarrow f(x,y) = x \to 0$

Along $x=y \Rightarrow f(x,y) = \dfrac{1}{2} \not\to 0$

Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

Example 2) $f(x,y) = \dfrac{x^3}{x^2+y^2}$

Along $y=kx \Rightarrow \dfrac{x^3}{(1+k^2)x^2} \to 0$

$x = r\cos\theta,\ y = r\sin\theta \Rightarrow \dfrac{r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= r\cos^3\theta \to 0,$

because $|r\cos^3\theta| \leq |r| \to 0$ (Sandwich theorem)

Therefore, $\lim f(x,y) = 0.$

Example 3) $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$

Along $y=kx \Rightarrow \dfrac{2kx^3}{x^2(x^2+k^2)} = \dfrac{2kx}{x^2+k^2} \to 0$

Along $y=x^2 \Rightarrow \dfrac{2x^4}{2x^4} = 1 \not\to 0$

Therefore $\lim f(x,y)$ does not exist

<Remark> Example 3 means that the following statement is NOT true

「If the limit may exist along every straight line (or ray), then the limit exists」

Example 4 (2020 exam) Determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not.

$$f(x,y) = \frac{\sin x^3(y^3+\pi)}{x^2+y^2}$$

<pf> You will arive at impossiblity of the three strategies. So, the limit may exist.

idea) $\dfrac{\sin x^3(y^3+\pi)}{x^2+y^2} = \dfrac{\sin x^3(y^3+\pi)}{x^3(y^3+\pi)} \dfrac{x^3(y^3+\pi)}{x^2+y^2}$

(i) We claim $\dfrac{\sin x^3(y^3+\pi)}{x^3(y^3+\pi)} \to 1$

Let $\varepsilon > 0$. Since $\sin t/t \to 1$, $\exists \delta_1 > 0$ s.t $|t| < \delta_1 \Rightarrow \left|\sin t/t - 1\right| < \varepsilon$.

Because $x^3(y^3+\pi) \xrightarrow{(\text{why?})} 0$ as $(x,y)\to(0,0)$,

$\exists \delta_2$ s.t $\sqrt{x^2+y^2} < \delta_2 \Rightarrow |x^3(y^3+\pi)| < \delta_1$

Hence $\sqrt{x^2+y^2} < \delta_2 \Rightarrow \left| \frac{\sin x^3(y^3+\pi)}{x^3(y^3+\pi)} - 1 \right| < \varepsilon$,

because $\sqrt{x^2+y^2} < \delta_2 \Rightarrow |x^3(y^3+\pi)| < \delta_1$.

This proves (i).

(ii) We claim that $\frac{x^3(y^3+\pi)}{x^2+y^2} \to 0$

Note that

The Sandwich Theorem for functions of two variables states that if $g(x, y) \le f(x, y) \le h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L.$$

,

So $\left| \frac{x^3(y^3+\pi)}{x^2+y^2} \right| \leq \left| \frac{x^3}{x^2}(y^3+\pi) \right|$

$= |x||y^3+\pi| \to 0$

Therefore, $\frac{x^3(y^3+\pi)}{x^2+y^2} \to 0$ ///

14.3.72 Let $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{if } (x,y) = 0. \end{cases}$

20 points

a. [**10 points**] Show that $\frac{\partial f}{\partial y}(x,0) = x$ for all x, and $\frac{\partial f}{\partial x}(0,y) = -y$ for all y.

b. [**10 points**] Show that $\frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0)$

Solution1. **a.** For $y \neq 0$ **(+1 points)**[1],

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) =& y\frac{x^2-y^2}{x^2+y^2} + xy\frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)} \\ \Rightarrow \frac{\partial f}{\partial x}(0,y) =& -y \end{aligned} \qquad \textbf{(+2 points)}$$

Also, for $x \neq 0$ **(+1 points)**

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) =& x\frac{x^2-y^2}{x^2+y^2} + xy\frac{-2y(x^2+y^2) - 2y(x^2-y^2)}{x^2+y^2} \\ \Rightarrow \frac{\partial f}{\partial y}(x,0) =& x \end{aligned} \qquad \textbf{(+2 points)}$$

For $(x,y) = 0$,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} = 0 \end{aligned} \qquad \textbf{(+2 points)}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} = 0 \end{aligned} \qquad \textbf{(+2 points)}$$

Therefore,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,y) =& -y \quad \forall y \\ \frac{\partial f}{\partial y}(x,0) =& x \quad \forall x \end{aligned}$$

14.4.51 **Differentiating Integrals** Under mild continuity restrictions, it is true that if

10 points

$$F(x) = \int_a^b g(t,x)dt,$$

then $F'(x) = \int_a^b g_x(t,x)dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t,x)dt$$

by letting

$$G(u,x) = \int_a^u g(t,x)dt,$$

where $u = f(x)$. Find the derivatives of the function

$$F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}dt$$

sketch) $F'(x) = \underbrace{\int_0^{x^2}}_{d/dx} \underbrace{\sqrt{t^4+x^3}dt}_{d/dx}$

<pf>

Let $u = x^2$. Then $F'(x) = \frac{d}{dx}\int_0^u \sqrt{t^4+x^3}dt$

$= \frac{d}{du}\left(\int_0^u \sqrt{t^4+x^3}\,dt\right) \cdot \frac{du}{dx} + \int_0^u \frac{d}{dx}\sqrt{t^4+x^3}dt$

$= \sqrt{u^4+x^3}\,(2x) + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4+x^3}}dt$

$= 2x\sqrt{x^8+x^3} + \int_0^{x^2}$ 〃 ///