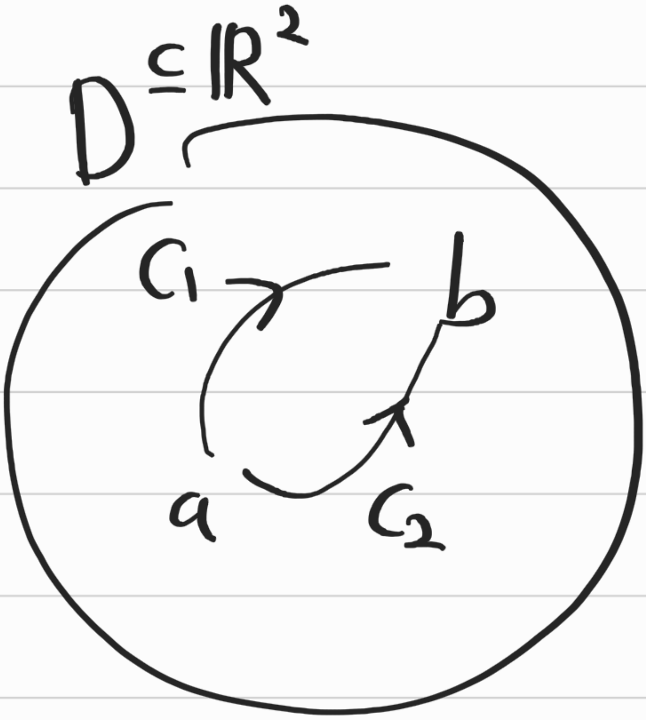
Def) Let $F: A \to B$ is a function. If $A = B = \mathbb{R}^n$, F is called a vector field.

<Notes> F is continuous if each component function is conti and it is differentiable if each component is differentiable.

Q: How to easily calculate $\int_C F\,dr$?

Like $\int_a^b f' = f(b) - f(a)$!



In general, $\int_{C_1} F\,dr \neq \int_{C_2} F\,dr$

DEFINITIONS Let **F** be a vector field defined on an open region *D* in space, and suppose that for any two points *A* and *B* in *D* the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path *C* from *A* to *B* in *D* is the same over all paths from *A* to *B*. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in *D*** and the field **F** is **conservative on *D***.

Now assume $F = \nabla f$ for some f (we call such a f a potential function of F)

We want to hold $\int_C F\,dr$

$$= \int_C \nabla f = \int_a^b \nabla f = f(b) - f(a).$$

THEOREM 1—Fundamental Theorem of Line Integrals Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Q: Does such f always exist?

A: No! (Continued)

Thm

For $F: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$,
$\exists f$ such that $F = \nabla f$
on D if D is open simply connected.

and F satisfies further good conditions:

Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

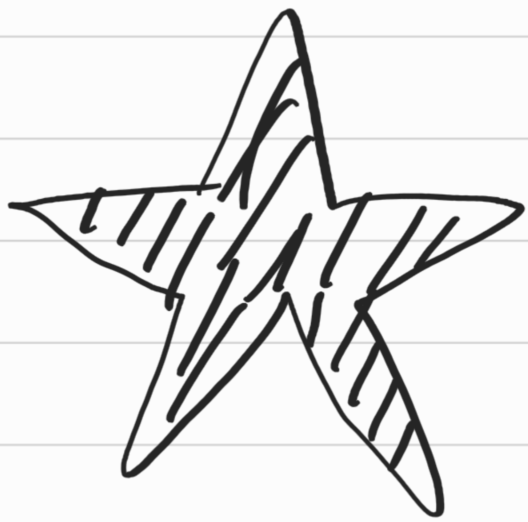
$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

Def) $R \subseteq \mathbb{R}^n$ is simply connected

if (i) R is path-connected

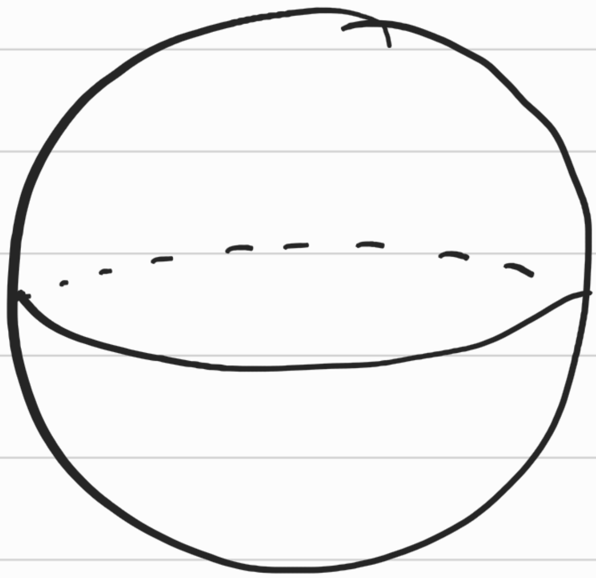
(ii) Every loop is contractible to a point.

Example I:

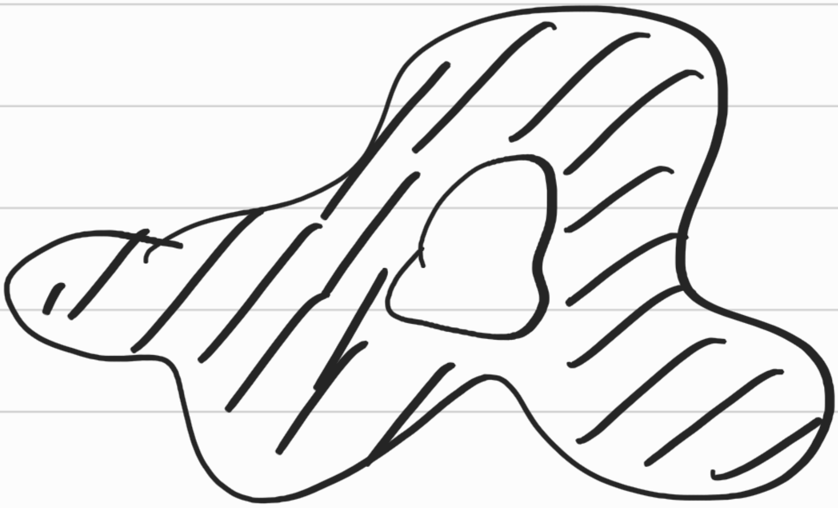


Example 2:

$x^2+y^2+z^2=1$

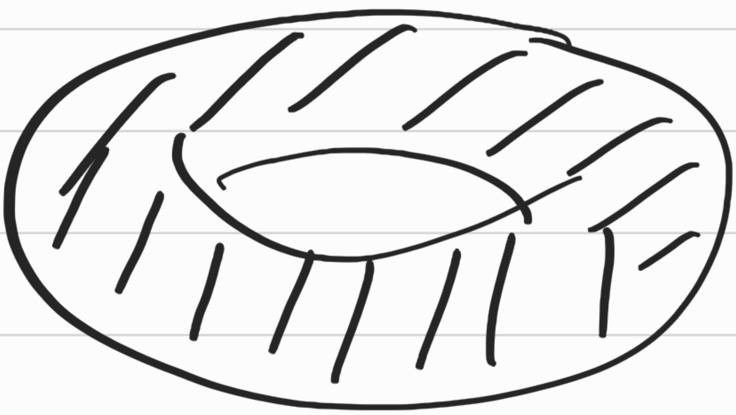


Example 3:

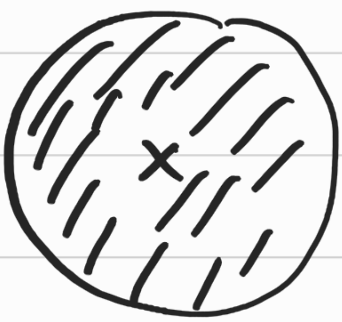


Example 4:

donut



Example 5:



$\{x^2+y^2 \leq 1\}$
$-\{(0,0)\}$

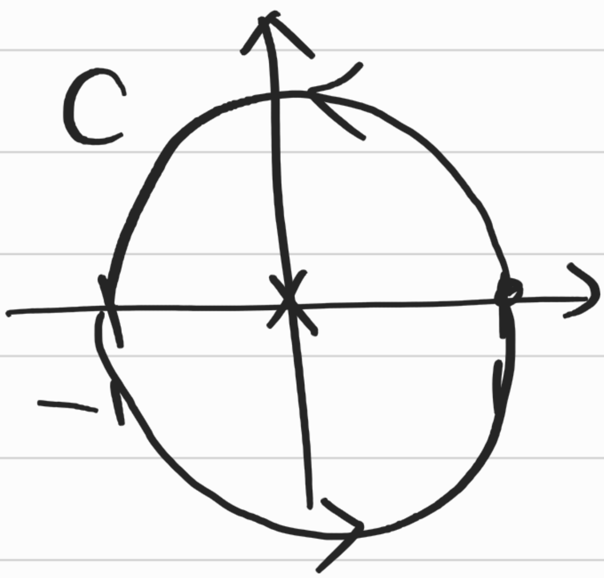
Q: Does such f always exist?

A: partially true! only on an open simply connected domain.

Example: Let $F : D' \longrightarrow \mathbb{R}^2$

$$(x,y) \longmapsto \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Assume F has a potential function.



Consider $\int_C F\,dr$,

so we have

$$\int_C F\,dr =$$

$$\int_C \nabla f\,dr = \int_{(1,0)}^{(1,0)} \nabla f\,dr$$

$$= f(1,0) - f(1,0) = 0$$

Fundamental Theorem of Line Integral.

But, $\int_c F\, dr = \int_0^{2\pi} F \cdot \frac{dr}{dt}\, dt$

$c = (\cos t, \sin t)$
$0 \leq t \leq 2\pi$

$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \neq 0!!$$

Therefore F is not conservative.

Why? $\Rightarrow$ D' is not simply connected!