Asymptotic Statistics Ch 11 (10/8)

Def) Let T be a random variable with $ET^2 < \infty$. Given a family $\mathcal{S}$ of random variables, $\hat{S}$ is the projection of T onto $\mathcal{S}$ iff $\hat{S} \in \mathcal{S}$ and $\hat{S}$ minimizes $S \longmapsto E(T-S)^2$ $S \in \mathcal{S}$.

proposition: Suppose $\mathcal{S}$ is a linear space and all elements have finite second moment

(i) $\hat{S}$ is the projection of T iff

$$E(T-\hat{S})S = 0 \quad \forall S \in \mathcal{S}$$

(ii) If $\tilde{S}$ is another projection, then $\hat{S} = \check{S}$ a.s.

(iii) If $c \in \mathcal{S}$, $ET = E\hat{S}$ and $Cov(T-\hat{S}, S) = 0$ $\forall S \in \mathcal{S}$.

<pf> $\Rightarrow$) $E(T-\hat{S}-\alpha S)^2 - E(T-\hat{S})^2 = -2\alpha E(T-\hat{S})S + \alpha^2 \cdot ES^2 \geq 0$ $\forall \alpha$.

$\Rightarrow E(T-\hat{S})S = 0$

$\Leftarrow$) $E(T-S)^2 = E(T-\hat{S})^2 + 2E(T-\hat{S})\underbrace{(\hat{S}-S)}_{\in \mathcal{S}} + E(\hat{S}-S)^2 \geq E(T-\hat{S})^2$ (note: equality holds iff $\hat{S} \overset{a.s.}{=} S$)

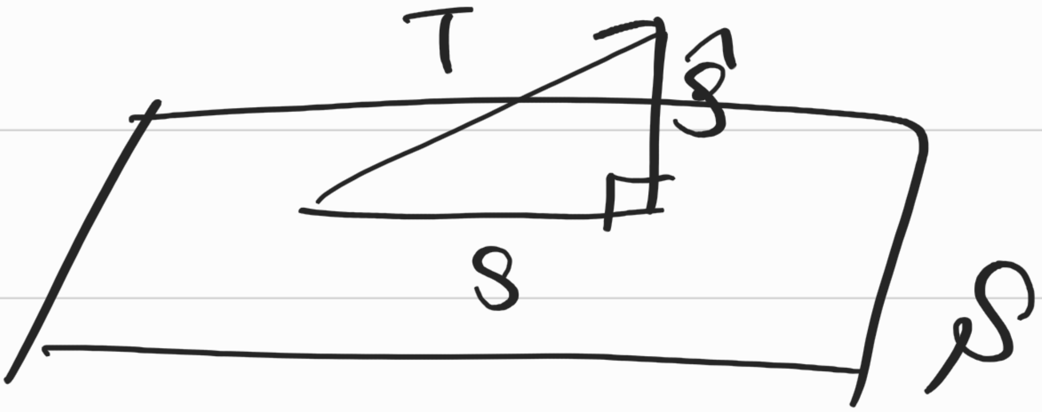
If $c \in \mathcal{S}$, $E(T-\hat{S}) \cdot c = 0 \Rightarrow$

$ET = E\hat{S}$ and $Cov(T-\hat{S}, S)$

$= \left(E\,\cancel{(T-\hat{S})S}^{\,0} - \cancel{E(T-\hat{S})}^{\,0}ES\right)/\cdots$

$= 0$ ///

<Remark> $E(T-\hat{S})S = 0 \quad \forall s \in \mathcal{S}$



<Remark> It may not exist!

Theorem: Let T_n be random variables with projections $\hat{S}_n$ onto $\mathcal{S}_n$. If $\mathrm{Var}\,T_n / \mathrm{Var}\,\hat{S}_n \to 1$, then $\dfrac{T_n - ET_n}{sd\,T_n} - \dfrac{\hat{S}_n - E\hat{S}_n}{sd\,\hat{S}_n} \xrightarrow{P} 0$

<pf> (i) $E[\star] = 0$

(ii) $Var[\star] = 2 - 2\dfrac{Cov(T_n, \hat{S}_n)}{sd(T_n)\, sd(\hat{S}_n)}$

$= 2 - 2\dfrac{1}{sd(T_n)\, sd(\hat{S}_n)}\left\{E T_n \hat{S}_n - E T_n E\hat{S}_n\right\} = 2 - 2\dfrac{sd(\hat{S}_n)}{sd(T_n)} \to 0$

$E(T_n - \hat{S}_n)\hat{S}_n = 0$ assumption

$\Rightarrow E T_n \hat{S}_n = E\hat{S}_n^2$

$\Rightarrow 0 \in \mathcal{S}_n \Rightarrow ET_n = E\hat{S}_n$ ///

$\mathcal{S} = \{$ all measurable function of $Y\}$.

EX minimizes $E(X-a)^2$

$\Rightarrow E X|Y = g_0(Y)$ minimizes $E(X - g(Y))^2$

$\Rightarrow E X|Y$ is the projection of X onto $\mathcal{S}$.

$\Leftrightarrow \; E(X - E X|Y) g(Y) = 0 \;\; \forall g$

prop

Example 1: $g \equiv 1 \Rightarrow EX = E E X|Y$.

Example 2: If $X = f(Y)$, then $EX|Y = X$, because $E[X - g(Y)]^2$ is minimized at $g = f$.

Example 3: If $X \perp Y$, then $EX|Y = EX$, because $E(X - EX) g(Y) = 0$.

Example 4: $E(f(Y) X | Y) = f(Y) E X|Y$, because $E\left(f(Y)X - f(Y) EX|Y\right) g(Y)$

$= E(X - EX|Y) f(Y) g(Y)$

$= 0$

Theorem (Hájek projection): Let $X_1, \cdots, X_n$ be independent and $\mathcal{S} = \{\sum_1^n g_i(X_i) \mid g_i\}$. The projection of T onto $\mathcal{S}$ is $\hat{S} = \sum_{i=1}^n E(T|X_i) - (n-1)ET$.

$$\langle \cdot \rangle\ E(T-\hat{S}) \sum g_i(X_i) \overset{\text{WTS}}{=} 0$$

$$\text{ETS}: E(T-\hat{S})\, g_j(X_j) = 0$$

$$\Rightarrow E\{T - (\sum ET|X_i - (n-1)ET)\} g_j(X_j)$$

$$= E\{T - \sum_{i \neq j} ET|X_i + (n-1)ET\} g_j(X_j)$$

$$-E\{(ET|X_j) g_j(X_j)\}$$

$$E\ E\{T - \sum_{i \neq j} ET|X_i + (n-1)ET\} g_j(X_j)|X_j$$

$$= E[E(T - \sum_{i \neq j} ET|X_i + (n-1)ET)|X_j$$

$$E g_j(X_j)|X_j] = g_j(X_j)$$

$$= E\left[E\,T \mid X_j - \sum_{i \neq j} \underbrace{E\,E(T \mid X_i) \mid X_j}_{\overset{\uparrow}{=}\ E\,E\,T \mid X_i = ET} + (n-1)\,ET\right]$$

$X_j \perp X_i$

$$= E(E\,T \mid X_j)$$

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