Review of general topology

January 7, 2023

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Part I

Essential theory in general topology

Chapter 1

Topology and continuous map

1.1 Topology and basis

Definition 1.1.1 (Topology on a set)**.** Let X be a nonempty set. A collection $\mathcal{T}$ is called a topology on X if

(a) both X and the empty set belongs to $\mathcal{T}$,

(b) $\mathcal{T}$ is closed under arbitrary unions,

(c) $\mathcal{T}$ is closed under arbitrary finite intersections.

Definition 1.1.2 (Subbasis and basis of a topology)**.** A collection $\mathcal{B}$ is called a subbasis of the topology on X if $\mathcal{B}$ covers X. A collection $\mathcal{B}$ is called a basis of the topology on X if

(a) $\mathcal{B}$ covers X, i.e., $\mathcal{B}$ is a subbasis on X,

(b) given $B_1, B_2 \in \mathcal{B}$, there is another member $B_3 \in \mathcal{B}$ contained in $B_1 \cap B_2$.

The topology $\langle \mathcal{B} \rangle$ on X generated by the basis $\mathcal{B}$ is the following collection of subsets of X:

$$\langle \mathcal{B} \rangle = \left\{ U \subset X : \begin{array}{c} \text{Given } x \in U \text{, there is a basis member} \\ B \in \mathcal{B} \text{ such that } x \in B \subset U \end{array} \right\}.$$

(Remark how an open subset of a metric space is defined in the course of mathematical analysis.) In accordance with the definition of the first-countability, we will say that $\langle \mathcal{B} \rangle$ consists of all subsets of X based on $\mathcal{B}$.

Observation 1.1.3. Let X be a set and suppose $\mathcal{B}$ is a basis of X.

(a) The topology generated by $\mathcal{B}$ is the collection $\mathcal{C}$ of all unions of members in $\mathcal{B}$.

(b) The topology generated by $\mathcal{B}$ is the intersection $\mathcal{I}$ of all topologies on X containing $\mathcal{B}$. (Hence, the topology on X generated by $\mathcal{B}$ is the smallest topology on X containing $\mathcal{B}$.)

Proof. (a) By definition, it is clear that $\mathcal{C}$ is contained in $\langle \mathcal{B} \rangle$. To show the converse inclusion, suppose $U \in \langle \mathcal{B} \rangle$. By definition, for each $x \in U$, there is a basis member $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$, hence U is the union of B_x for $x \in U$.
(b) Because $\langle \mathcal{B} \rangle$ is a topology on X containing $\mathcal{B}$, $\mathcal{I}$ is contained in $\langle \mathcal{B} \rangle$. Conversely, by (a), every topology on X containing $\mathcal{B}$ also includes $\langle \mathcal{B} \rangle$. Thus, $\mathcal{I}$ contains $\langle \mathcal{B} \rangle$. $\square$

Lemma 1.1.4 (Containment criterion)**.** Let $\mathcal{B}, \mathcal{B}'$ be a basis of the topology $\mathcal{T}, \mathcal{T}'$ of X, respectively. Then the following are equivalent:

(a) $\mathcal{T}$ is finer than $\mathcal{T}'$.

(b) For each point $x \in X$ and a basis member $B' \in \mathcal{B}'$ containing x, there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Proof. Assume (a) and let x be a point of X and B' be a basis member containing x. Then $B' \in \mathcal{T}$, and there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Assume (b) and let U' be a member of $\mathcal{T}'$. For each point $p \in U'$, there is a basis member $B_p \in \mathcal{B}$ such that $p \in B_p \subset U'$. (There was a leap in the argument: Letting B'_p be a basis member of $\mathcal{B}'$ such that $p \in B'_p \subset U'$, by the assumption we have a basis member $B_p \in \mathcal{B}$ such that $x \in B_p \subset B'_p$.) Therefore, $U' \in \mathcal{T}$. □

Remark. (a) (subbasis) $\xrightarrow{\text{finite intersections}}$ (basis) $\xrightarrow{\text{arbitrary unions}}$ (topology)

(b) Suppose that $\mathcal{B}$ is a basis of a topology $\mathcal{T}$ on a set X. Then the topology generated by $\mathcal{B}$ as a 'subbasis' is $\mathcal{T}$. If $\mathcal{B}'$ is the collection of all finite intersections of the members of $\mathcal{B}$, then $\mathcal{B}' \supset \mathcal{B}$, so the topology $\mathcal{T}'$ generated by $\mathcal{B}'$ as a basis (i.e., by $\mathcal{B}$ as a subbasis) is finer than $\mathcal{T}$. To show that $\mathcal{T}$ is finer than $\mathcal{T}'$, it suffices to prove for each $x \in X$ and a basis member $B' \in \mathcal{B}'$ containing x there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$. Because $\mathcal{B}$ is a basis, if $B' = B_1 \cap \cdots \cap B_n$ for some $B_1, \cdots, B_n \in \mathcal{B}$ then there is a basis member $B_0 \in \mathcal{B}$ such that $x \in B_0 \subset B_1 \cap \cdots \cap B_n = B'$.

(c) Regarding containment, we have another proposition, given as follows:

> Let X be a topological space and $\mathcal{B}$ be a collection of sets open in X. Assume that for each open subset U of X and a point $x \in U$, there is a member $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Then $\mathcal{B}$ is a basis of the topology on X.

In fact, by assumption, $\mathcal{B}$ is a basis of a topology on X which is finer than the topology $\mathcal{T}$ on X. Because $\mathcal{B}$ is a subcollection of $\mathcal{T}$, the topology on X generated by $\mathcal{B}$ is coarser than $\mathcal{T}$, so $\mathcal{B}$ is a basis of $\mathcal{T}$.

1.2 Continuous maps

In this section, we assume X, Y, Z are topological spaces.

Definition 1.2.1 (Continuity)**.** The map $f : X \to Y$ is said to be continuous if $f^{-1}(U)$ is open in X whenever U is open in Y. If the map f is bijective and its inverse is also continuous, then f is called a homeomorphism. In addition, if f is injective and continuous, and the induced map $\widetilde{f} : X \to f(X)$ defined by $\widetilde{f}(a) = f(a)$ for all $a \in X$ is a homeomorphism, then f is called an embedding of X into Y. (It will be explained that such restriction of continuous maps are always continuous.)

Remark. (a) The procedure for checking continuity can be reduced to the members of a basis or a subbasis generating the topology on the codomain Y. (Why?)

(b) A homeomorphism naturally induces a bijection between the topologies on the domain and the codomain of the homeomorphism. Also, a bijective continuous map is a homeomorphism if and only if the map is an open map.

Theorem 1.2.2. Let X, Y be topological spaces and $f : X \to Y$ be a map. Then the following are equivalent:

(a) f is a continuous map.

(b) For any closed subset B of Y, $f^{-1}(B)$ is closed in X.

(c) For each $x \in X$ and a neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X such that $f(U) \subset V$.

(d) Whenever $U \subset X$, $f(\overline{U}) \subset \overline{f(U)}$.

Proof. (a)⇔(b): This follows directly by considering set complements.
(a)⇒(c): If $x \in X$ and V is a neighborhood of $f(x)$ in Y, then $f^{-1}(V)$ is a neighborhood of x in X whose image under f is V.

(c)⇒(a): Let V be an open subset of Y and let $U = f^{-1}(V) \subset X$. By assumption, for each $x \in X$, there is a neighborhood A_x of x in X such that $f(A_x) \subset V$. Then $A_x \subset U$, so U is open in X.

(a)⇒(d): Suppose $U \subset X$. We will show that if $x \in \overline{U}$ then $f(x) \in \overline{f(U)}$. If V is a neighborhood of $f(x)$ in Y, then $f^{-1}(V)$ is a neighborhood of x in X, so $f^{-1}(V)$ intersects U. Thus, $V \cap f(U) = f(f^{-1}(V) \cap U)$ is nonempty, so $f(x) \in \overline{f(U)}$, as desired.

(d)⇒(b): Let B be a closed subset of Y and let $A = f^{-1}(B)$. By assumption, we have $B = f(A) \subset f(\overline{A}) \subset \overline{f(A)} = \overline{B} = B$, so $B = f(\overline{A})$ and $\overline{A} \subset f^{-1}(B) = A$, proving that $A = \overline{A}$ is closed. □

If given maps $f : X \to Y$ and $g : Y \to Z$ are continuous, some naturally induced maps are also continuous, as indicated below. (Checking continuity is left as an exercise.)

Proposition 1.2.3. Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous.

(a) Every constant map is continuous.

(b) $g \circ f$ is continuous.

(c) Every restriction of a continuous map on a subspace is continuous. Also, if Y is a subspace of Z, then $\widetilde{f} : X \to Z$ defined by $\widetilde{f}(x) = f(x)$ for all $x \in X$ is continuous.

Problem 1.2.1. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution. It is not necessarily true; consider constant maps.

Theorem 1.2.4. Let A be a set; let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of spaces; and let $\{f_\alpha : A \to X_\alpha\}_{\alpha \in I}$ be an indexed family of functions.

(a) There is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the function f_α is continuous. In fact, the topology $\mathcal{T}$ is generated as a subbasis by the following collection:

$$\left\{ f_\alpha^{-1}(U_\alpha) : \alpha \in I \text{ and } U_\alpha \text{ is open in } X_\alpha \right\}.$$

(b) A map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each composition $f_\alpha \circ g$ is continuous.

Proof. (a) is almost clear; such topology necessarily contains the given collection. In proving (b), it suffices to prove if part. Suppose $g_\alpha := f_\alpha \circ g$ is continuous for each $\alpha \in I$. Given an open subset U of A, we have $g_\alpha^{-1}(A) = g^{-1}(f_\alpha^{-1}(A))$, completing the proof. □

Remark. The product topology on a product space satisfies the above properties; in fact, the product topology is the topology on A with

$$A = \prod_{\alpha \in I} X_\alpha \quad \text{and} \quad f_\alpha = \pi_\alpha \text{ for each } \alpha \in I.$$

Problem 1.2.2. Let (X, d) be a metric space.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' be a space having the same underlying set as X, and define the map $d' : X' \times X' \to \mathbb{R}$ by $d'(a, b) = d(a, b)$ for all $(a, b) \in X' \times X'$. Show that the topology on X' is finer than the topology on X, if d' is continuous. Deduce that the metric topology on X induced by the metric d is the coarsest topology on X relative to which d is continuous.

Solution. (a) Choose a point $(a, b) \in X \times X$ and write $r = d(a, b)$. We will show that given a positive real number ϵ there is a neighborhood of (a, b) whose image under d is contained in $(r - \epsilon, r + \epsilon)$. Let $V = B_d(a, \delta) \times B_d(b, \delta)$ with $\delta > 0$, which is a neighborhood of (a, b) in $X \times X$. Whenver $(p, q) \in V$, we have

$$d(p, q) \leq d(p, a) + d(a, b) + d(b, q) \quad \text{and} \quad d(a, b) \leq d(a, p) + d(p, q)_d(q, b,)$$

from which we obtain $r - 2\delta < d(p, q) < r + 2\delta$. Hence, by choosing $0 < \delta < \epsilon/3$ we have $d(V) \subset (r - \epsilon, r + \epsilon)$, as desired. Therefore, d is a continuous map.

(b) Choose $a \in X'$, and define the map $\epsilon_a : X' \to X' \times X'$ by $\epsilon_a(x) = (x, a)$ for $x \in X'$. Then ϵ_a is continuous, so $k := d' \circ \epsilon_a$ is continuous. Hence, $k^{-1}((-\infty, r)) = B_{d'}(a, r) = B_d(a, r)$ is an open subset of X' for all $r \in \mathbb{R}$ with $r > 0$. Therefore, the topology on X' contains $\{B_d(a, r) : a \in X, r > 0\}$, so the topology on X' is finer than the topology on X. The last assertion easily follows.

1.3 Topologies on function spaces

Let X_α be a topological space for each $\alpha \in I$, and write $X = \prod_{\alpha \in I} X_\alpha$.

Definition 1.3.1 (Box topology)**.** The topology on X generated by the following collection as a basis is called the box topology on X:

$$\left\{ \prod_{\alpha \in I} U_\alpha : \text{For each } \alpha \in I,\ U_\alpha \text{ is an open subset of } X_\alpha \right\}.$$

Indeed, the box topology on X is finer than the product topology on X. Also, it is easy to verify that the box topology on X can be generated as a basis by the following collection, given that $\mathcal{B}_\alpha$ is a basis of the topology on X for each $\alpha \in I$:

$$\left\{ \prod_{\alpha \in I} B_\alpha : \text{For each } \alpha \in I,\ B_\alpha \in \mathcal{B}_\alpha \right\}.$$

Remark. For X equipped with the product topology, we have observed that a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is convergent if and only if $(\pi_\alpha(x_n))_{n \in \mathbb{N}} \subset X_\alpha$ is convergent for each $\alpha \in I$. For X equipped with the box topology, however, if part is generally not true (only if part is true).

We now assume that (Y, d) is a metric space and I is an index set. Even Y is not metrizable, we could have considered the product and box topology on Y^I, but the following topology is considered only when a metric on Y is given.

Definition 1.3.2. Let I be a nonempty set and (Y, d) be a metric space.

(a) (Uniform metric) Define the map $\overline{\rho} : Y^I \times Y^I \to [0, \infty)$ by $(f, g) \mapsto \sup\left\{\overline{d}(f(x), g(x)) : x \in I\right\}$ for all $f, g \in Y^I$. The map $\overline{\rho}$ is a metric on Y^I, which is called the uniform metric corresponding to d.

(b) (Uniform topology) The metric topoogy on Y^I induced by the uniform metric corresponding to d is called the uniform topology induced by d.

Remark. (a) We use the standard bounded metric $\overline{d}$ induced by d to prevent the case where a distance between two distinct points diverges.

(b) As one might have expected, for a metric space (Y, d) and a nonempty set X, the convergence in Y^X in the uniform topology is called the uniform convergence. Similarly, the convergence in Y^X in the product topology is called the pointwise convergence.

Proposition 1.3.3. Let I be a nonempty set and (Y, d) be a metric space. Then, we have the following inclusions of the topologies on Y^I:

$$(\text{the product topology}) \subset (\text{the uniform topology}) \subset (\text{the box topology}).$$

In particular, if $I = \mathbb{N}$ and (Y, d) is the Euclidean metric space $(\mathbb{R}, d)$, then the above three topologies do not coincide.

Proof. **Step 1. Showing the inclusions.**
Given a basis member $\prod_{x \in I} U_x$ of the product topology on Y^I, we have $U_x = Y$ for all but finitely many values of x; namely, $x_1, \cdots, x_k$. Suppose $f \in \prod_{x \in I} U_x$. Because $f(x_i) \in U_{x_i}$ for each $i = 1, \cdots, k$, there is a positive real number $0 < r_i < 1$ such that $B_d(f(x_i), r_i) \subset U_{x_i}$. Letting $r = \min\{r_1, \cdots, r_k\}$, we have $f \in B_{\overline{\rho}}(f, r) \subset \prod_{x \in I} U_x$.

Given a basis member $B_{\overline{\rho}}(f,\epsilon)$ of the uniform topology on Y^I (with $0<\epsilon<1$) with a point g in the member, there is a positive real number $0<r<1$ such that $g \in B_{\overline{\rho}}(g,r) \subset B_{\overline{\rho}}(f,\epsilon)$. In fact, we also have $g \in \prod_{x\in I} B_d(g(x), r/2) \subset B_{\overline{\rho}}(g,r)$, proving the assertion.

Step 2. Investigating the given particular case.
Observe, when (Y,d) is the Euclidean metric space $(\mathbb{R},d)$ and $I=\mathbb{N}$, that $B_{\overline{\rho}}(0,1/2)$ cannot be an intersection of finitely many subsets of the form $\pi_n^{-1}(\mathbb{R})$ $(n\in\mathbb{N})$, and that $\prod_{n\in\mathbb{N}}(1/n,1/n)$ cannot be open in the uniform topology on $\mathbb{R}^{\mathbb{N}}$. □

To study further on the uniform topology, see Chapter 5.

1.4 Metrizable product spaces

Let X_n be a metrizable space for each $n\in\mathbb{N}$.

Notation. Given a metric d_n inducing the topology on X_n for each n, define the following metric on the product space $X := \prod_{n=1}^{\infty} X_n$ as follows:

$$\overline{\rho} : X\times X \to \mathbb{R},\ (x,y) \mapsto \sup_{n\in\mathbb{N}} \left\{\overline{d_n}(x_n,y_n)\right\},$$

$$D : X\times X \to \mathbb{R},\ (x,y) \mapsto \sup_{n\in\mathbb{N}} \left\{\frac{\overline{d_n}(x_n,y_n)}{n}\right\}.$$

The former metric is called the uniform metric on X and the latter metric is called the D-metric on X. Among them, the definition of the former metric generalizes to arbitrary product spaces: If $\{X_\alpha\}_{\alpha\in I}$ is an indexed family of metric spaces and $X=\prod_{\alpha\in I} X_\alpha$, then we may define the uniform metric $\rho : X\times X\to\mathbb{R}$ by $\rho(x,y) = \sup_{\alpha\in I}\{\overline{d}_\alpha(x_\alpha,y_\alpha)\}$ for $(x,y)\in X\times X$.

Theorem 1.4.1. Let (X_n,d_n) be a metric space for each $n\in\mathbb{N}$ and write $X=\prod_{n\in\mathbb{N}} X_n$. Then D-metric on X induces the product topology on X.

Proof. We first show that the topology $\mathcal{T}_D$ induced by the D-metric is finer than the product topology. Given a point $x\in X$ and a basis member $B=\prod_{n\in\mathbb{N}} B_n$ of the product topology with

$$\{n\in\mathbb{N} : B_n \neq X_n\} = \{n_1,\cdots,n_k\},$$

let us find a basis member of $\mathcal{T}_D$ which contains x and contained in B. For each $i=1,\cdots,k$, let r_i be a real number such that $0<r_i<1$ and $B_{d_{n_i}}(x_{n_i},r_i)\subset B_{n_i}$. Set

$$\epsilon = \min_{1\le i\le k}\left\{\frac{r_i}{n_i}\right\}$$

and suppose $y\in B_D(x,\epsilon)$. Then $\overline{d_n}(x_n,y_n)/n<\epsilon$ for all $n\in\mathbb{N}$. In particular, $\overline{d_{n_i}}(x_{n_i},y_{n_i}) < n_i\epsilon \le r_i$, so $y_{n_i}\in B_{d_{n_i}}(x_{n_i},r_i)$ for each all i. Hence, $x\in B_D(x,\epsilon)\subset B$, as desired.

Conversely, suppose a point x of X and a basis element $B_D(p,r)\in\mathcal{T}_D$ containing x are given. By choosing a small real number $0<\epsilon<1$, we can achieve $x\in B_D(x,\epsilon)\subset B_D(p,r)$. (Such leap is helpful in proving many other propositions regarding metric spaces.) Let N be a positive integer such that $1/N<\epsilon$. Let $B_n = B_{d_n}(x_n,\epsilon)$ for each positive integer $n<N$ and let $B_n = X_n$ otherwise. Then $x\in\prod_{n\in\mathbb{N}} B_n \subset B_D(x,\epsilon)\subset B_D(p,r)$, as desired. □

Remark. In particular, if each d_n havs values in $[0,1]$, then the map $\kappa : X\times X\to[0,1]$ defined by $\kappa(x,y)=\sum_{n\in\mathbb{N}} 2^{-n} d_n(x_n,y_n)$ is a metric on X which induces the product topology on X. Proving this statement is left as an exercise.

Observation 1.4.2. Every isometry of a metric space into a metric space is an embedding.

Proof. If $f: X\to Y$ is an isometry, then f is clearly injective and continuous. Because an open ball of radius $r>0$ in X is mapped onto an open ball of radius r in Y under f, f is an open map, so f is an embedding. □

We now introduce the structure of an open ball in $\mathbb{R}^{\mathbb{N}}$ which equips the uniform metric $\overline{\rho}$.

Proposition 1.4.3. Let $\overline{\rho}$ be the uniform metric on $\mathbb{R}^{\mathbb{N}}$. Given $x = (x_1, x_2, \cdots) \in \mathbb{R}^{\mathbb{N}}$ and a real number $0 < \epsilon < 1$, define

$$U(x, \epsilon) := \prod_{n=1}^{\infty} (x_n - \epsilon, x_n + \epsilon).$$

(a) $U(x, \epsilon) \neq B_{\overline{\rho}}(x, \epsilon)$ and $U(x, \epsilon)$ is not open in the uniform topology.

(b) $B_{\overline{\rho}}(x, \epsilon) = \bigcup_{0<r<\epsilon} U(x, r)$.

Proof. (a) Note that the point $(x_n + 2^{-n}\epsilon)_{n\in\mathbb{N}}$ is in $U(x, \epsilon)$ but not in $B_{\overline{\rho}}(x, \epsilon)$. Furthermore, no neighborhood of this point entirely lies in $U(x, \epsilon)$.

(b) For each real number $0 < r < \epsilon$, we have $U(x, r) \subset B_{\overline{\rho}}(x, \epsilon)$. Conversely, if $y \in B_{\overline{\rho}(x,\epsilon)}$, then $\overline{\rho}(x, y) < \epsilon$, so $\sup\{\overline{d_n}(x_n, y_n) : n \in \mathbb{N}\} = \delta$ for some $0 \leq \delta < \epsilon$. Hence, $B_{\overline{\rho}}(x, \epsilon) \subset U(x, r)$ for some real number r such that $\delta < r < \epsilon$. ☐

1.5 Quotient topology

1.5.1 Quotient maps

Definition 1.5.1 (Quotient map)**.** Let X and Y be topological spaces. A surjective map $p : X \to Y$ is called a quotient map if a subset U of Y is open if and only if $p^{-1}(U)$ is open in X.

Remark. Be noted that a quotient map need not be an open map.

Example 1.5.2. The following results will be helpful in practice, when one wishes to justify that a given surjective continuous map is a quotient map.

(a) A homeomorphism is an injective quotient map, and vice versa.

(b) An open (or closed) continuous surjection is a quotient map.

(c) For topological spaces X, Y Z, if $p : X \to Y$ and $q : Y \to Z$ are quotient maps, then $q \circ p : X \to Z$ is also a quotient map.

(d) Let $p : X \to Y$ be a continuous function. If there is a function continuous $f : Y \to X$ such that $p \circ f = id_Y$, then p is a quotient map. In short, a continuous map with a continuous right inverse is a quotient map.

(e) A retraction is a quotient map.[1]

(f) Cartesian product of quotient maps need not be a quotient map. See Problem 1.5.2.

For topological spaces X and Y, we say a subset A of X is saturated with respect to a surjective map $p : X \to Y$ when A contains every $p^{-1}(\{y\})$ that it intersects, i.e., A is the inverse image of a subset of Y. (In fact, because $A = p^{-1}(B)$ for some subset B of Y and $B = p(p^{-1}(B))$, we find that $A = p^{-1}(p(A))$.) To argue p is a quotient map is equivalent to argue that p is a continuous surjection mapping a saturated open (closed, respcetively) subsets of X onto an open (closed) subsets of Y.

A restriction of a quotient map to a subspace need not be a quotient map. One has, however, the following proposition:

Proposition 1.5.3. Let $p : X \to Y$ be a quotient map, and let A be a subspace of X that is saturated with respect to p. Let $q : A \to p(A)$ be the restriction of p to A.

(a) If A is open or closed in X, then q is a quotient map.

[1]Given a topological space X and its subspace A, a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ is called a retraction of X onto A.

(b) If p is an open map or a closed map, then q is a quotient map.

Proof. Our goal in either case is to show that a subset V of $p(A)$ is open in $p(A)$ whenever $q^{-1}(V)$ is open in A.

(a) Assume A is an open subset of X. Because A is saturated, $A = p^{-1}(p(A))$, so $p^{-1}(V) \subset p^{-1}(p(A)) = A$, which implies that $q^{-1}(V) = A \cap p^{-1}(V) = p^{-1}(V)$. Therefore, V is open in Y and in $p(A)$.

(b) Suppose p is an open map. Because $q^{-1}(V)$ is open in A, $q^{-1}(V) = A \cap U$ for some open subset U of X, and $V = p(q^{-1}(V)) = p(A \cap U)$. Because $p(A \cap U) = p(A) \cap p(U)$ (see the following remark), we find that V is open in $p(A)$.

The same results hold for closed cases, which can be proved by replacing the words "open" by "closed." □

Remark. Let $p : X \to Y$ be a quotient map and A be a subset of X saturated with respect to p. If B is a subset of X, then it is clear that $p(A \cap B) \subset p(A) \cap p(B)$. If $y \in p(A) \cap p(B)$, then $y = p(b)$ for some $b \in B$. Because A is saturated and $y \in p(A)$, $b \in A$, so $y = p(b) \in p(A \cap B)$, as desired.

In practice, it is difficult to check if a given surjective continuous map is a quotient map by merely checking the defining condition. The examples given in Example 1.5.2 will be helpful when justifying that a given surjective continuous map is a quotient map. Here, we provide proofs of (d) and (e).

Proof. (d) We need to justify that a subset U of Y is open in Y whenever $p^{-1}(U)$ is open in X. If U be a subset of Y such that $p^{-1}(U)$ is open, then $U = id_Y^{-1}(U) = f^{-1}(p^{-1}(U))$, so U is open in Y.

(e) Let A be a subspace of X and $p : X \to A$ be a retraction of X onto A. It suffices to check that a subset U of A is open in A whenever $p^{-1}(A)$ is open in X. Because $U = p(p^{-1}(U)) = p^{-1}(U) \cap A$, U is open in A. □

1.5.2 Quotient topology

Definition 1.5.4 (Quotient topology)**.** Let X be a topological space and A be a nonempty set. Given a surjection $p : X \to A$, there is a unique topology $\mathcal{T}$ on A relative to which p is a quotient map, which is given as

$$\mathcal{T} = \{U \subset A : p^{-1}(U) \text{ is open in } X\}. \tag{1.1}$$

This topology is called the quotient topology induced by p.

Problem 1.5.1. Check if the collection $\mathcal{T}$ in eq. (1.1) is a topology on A.

Definition 1.5.5 (Quotient space)**.** Let X be a topological space and let X^* be a partition of X. Let $p : X \to X^*$ be the natural projection, i.e., the surjection that maps each point of X to the member of X^* containing the point. In the quotient topology induced by p, the space X^* is called a quotient space.

Remark. The typical open set of X^* is a collection of equivalence classes whose union is an open set of X.

Theorem 1.5.6. Let $p : X \to Y$ be a quotient map. Let Z be a topological space and $f : X \to Z$ be a map which is constant on each fiber of p.[2] Then f induces a unique map $\overline{f} : Y \to Z$ such that $\overline{f} \circ p = f$. Indeed, f is surjective if and only if $\overline{f}$ is surjective. Furthermore,

(a) $\overline{f}$ is continuous if and only if f is continuous.

(b) $\overline{f}$ is a quotient map if and only if f is a quotient map.

$$\begin{tikzcd}
X \arrow[d, "p"'] \arrow[dr, "f"] & \\
Y \arrow[r, "\overline{f}"'] & Z
\end{tikzcd}$$

[2]By the fiber of the element $b \in B$ under a set map $f : A \to B$ we mean the inverse image of $\{b\}$ under f.

Proof. Well-definedness of $\overline{f}$ and its uniqueness are almost clear. Also, it is clear that f is surjective if and only if $\overline{f}$ is surjective. Note that $f^{-1} = p^{-1} \circ \overline{f}^{-1}$.

(a) It is clear that f is continuous if $\overline{f}$ is continuous. To show the converse, assume f is continuous and U is an open subset of Z. Since $f^{-1}(U)$ is open in X and p is a quotient map, $\overline{f}^{-1}(U)$ is also open in Y. Thus, $\overline{f}$ is a continuous map.

(b) It is clear that f is a quotient map if $\overline{f}$ is a quotient map. To show the converse, assume f is a quotient map. By (a), $\overline{f}$ is a continuous map, and it is clear that $\overline{f}$ is a surjection. Thus, it remains to show that a subset U of Z is open in Z if (and only if) $\overline{f}^{-1}(U)$ is open in Y. When $\overline{f}^{-1}(U)$ is open in Y, $f^{-1}(U) = (p^{-1} \circ \overline{f}^{-1})(U)$ is open in X, so U is open in Z, as desired. □

Corollary 1.5.7 (Homeomorphism of a quotient space)**.** Let $f : X \to Z$ be a surjective continuous map. Let X^* be the collection of the fibers of f, i.e., $X^* = \{f^{-1}(\{z\}) : z \in Z\}$. Give X^* the quotient topology induced by the natural projection map $p : X \to X^*$. Then, the map f induces a bijective continuous map $f^* : X^* \to Z$, which is a homeomorphism if and only if f is a quotient map.

Proof. Note that the natural projection map $p : X \to X^*$ is a quotient map. By the preceeding theorem, there is a well-defined continuous bijection $f^* : X^* \to Z$. Furthermore, f^* is a quotient map (hence, a homeomorphism) if and only if f is a quotient map. □

Remark. Under the setting given in the above corollary, we may use, for example, the following facts to distinguish two topological spaces.

(a) If Z is a Hausdorff space, then so is X^*.

(b) Suppose $h : A \to B$ is a continuous bijection. If h is compact and B is a Hausdorff space, then h is a homeomorphism.

1.5.3 Examples

Example 1.5.8. Define an equivalence relation $\sim$ on the plane $X = \mathbb{R}^2$ as follows:

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_1 + y_1^2 = x_2 + y_2^2.$$

(It is left as an exercise to check that $\sim$ is indeed an equivalence relation on X.) Let X^* be the corresponding quotient space. In fact, X^* is the set which consists of the collections of the form $f^{-1}(\{c\})$ for $c \in \mathbb{R}$, where $f(x, y) = x + y^2$. Because f is a continuous surjection onto $\mathbb{R}$, there is a unique continuous bijection f^* such that $f^* \circ p = f$, where $p : X \to X^*$ is the natural projection. Furthermore, the map $\iota : \mathbb{R} \to \mathbb{R}^2$ defined by $\iota(x) = (x, 0)$ for all $x \in \mathbb{R}$ is a continuous right inverse of f, so f is a quotient map. Therefore, X^* and $\mathbb{R}$ are homeomorphic.

Let $\sim$ denote the equivalence relation on X defined as follows:

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

(It is left as an exercise to check that $\sim$ is indeed an equivalence relation on X.) Now, X^* is the collection of the collections of the form $g^{-1}(\{c\})$ for $c \geq 0$, where $g(x, y) = x^2 + y^2$. Because g is surjective and continuous, there is a unique bijective continuous map $f^* : X^* \to [0, \infty)$. Because the map $r : [0, \infty) \to \mathbb{R}^2$ defined by $r(x) = (\sqrt{x}, 0)$ for all $x \in [0, \infty)$ is a continuous right inverse of g, g is a quotient map. Therefore, X^* and $[0, \infty)$ (equipped with the order topology) are homeomorphic.

Example 1.5.9. We will justify that D^2 and S^2 are homeomorphic. Define a map $f : D^2 \to S^2$ by

$$f \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} = \begin{pmatrix} \sin(\pi r)\cos(\theta) \\ \sin(\pi r)\sin(\theta) \\ \cos(\pi r) \end{pmatrix},$$

And let X^* be the partition of X defined by the collection of $f^{-1}(\{p\})$ for $p \in S^2$. Since f is surjective and continuous, the naturally induced map $f^* : X^* \to S^2$ is also surjective and continuous. To show that f^* is a homeomorphism, it suffices to show that f^* is a quotient map, which can be justified by showing that f is a quotient map. Indeed, f is a quotient map, since f is an open map.

Example 1.5.10. Let X^* be the quotient space $(S^1 \times S^1)/\sim$, where $(z,w) \sim (u,v)$ if and only if $zw = uv$. Our goal is to justify that $X^* \approx S^1$. Consider the function $f : S^1 \times S^1 \to S^1$ defined by $f(u,v) = uv$. Then, the collection of all preimages of singletons in S^1 under f is precisely the collection of all equivalence classes with respect to $\sim$. Because f is continuous and surjective, there is a unique bijective continuous map $f^* : X^* \to S^1$. To show that f^* is a homeomorphism, it suffies to show that f is a quotient map. Considering the map $\iota : S^1 \to S^1 \times S^1$ defined by $\iota(z) = (z,1)$, we can easily observe that ι is a continuous right inverse of f. Hence, f is a quotient map, and it follows that f^* is a homeomorphism.

Example 1.5.11. One can easily find that there are quotient maps for $S^2 \to D^2$, $D^2 \to D^1$, and $D^1 \to S^1$.

Problem 1.5.2. Let $\mathbb{R}_K$ be the set $\mathbb{R}$ equipped with the K-topology, and let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point. And let $p : \mathbb{R}_K \to Y$ be the quotient map.

(a) Show that Y satisfies the T_1 axiom but Y is not a Hausdorff space.

(b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map.

Solution. (a) Remark that the K-topology on $\mathbb{R}$ is generated as a basis by $\{(a,b) : a, b \in \mathbb{R} \text{ and } a < b\} \cup \{(a,b) - K : a, b \in \mathbb{R} \text{ and } a < b\}$.

Every singleton in Y is of the form $\{a\}$ for $a \in \mathbb{R} \setminus K$ or $\{K\}$. With respect to the natural projection $p : \mathbb{R}_K \to Y$, the preimages of both types of singletons are closed in $\mathbb{R}_K$, so Y satisfies the T_1 axiom.

To argue that Y is not a Hausdorff space, observe 0^* and k^*, where $k \in K$ and note that a typical open set in Y is the collection of equivalence classes belonging to Y whose union is an open set in $\mathbb{R}_K$. Given a neighborhood of 0^*, there is a basis member $(a,b) - K$ of $\mathbb{R}_K$ containing 0. If k_1 is a point of K such that $0 < k_1 < b$, then $k^* = k_1^*$, so a neighborhood of k^* in Y necessarily contains the projection of a neighborhood of k_1 in $\mathbb{R}_K$. This justifies that there is no pair of disjoint neighborhoods of 0^* and k^* in Y.

(b) Because Y is not a Hausdorff space, the diagonal of Y is not closed in $Y \times Y$. The preimage of the diagonal of Y under $p \times p$ is

$$\{(x,x) : x \in \mathbb{R} \setminus K\} \cup (K \times K) = \{(x,x) : x \in \mathbb{R}\} \cup (K \times K).$$

Since the diagonal of $\mathbb{R}$ is closed in $\mathbb{R} \times \mathbb{R}$, it is also closed in a finer space $\mathbb{R}_K \times \mathbb{R}_K$; $K \times K$ is obviously closed in $\mathbb{R}_K \times \mathbb{R}_K$. Hence, the preimage of the diagonal of Y under $p \times p$ is closed under $\mathbb{R}_K \times \mathbb{R}_K$. This is invalid, if $p \times p$ is a quotient map.

Chapter 2

Connected spaces and compact spaces

2.1 Connected spaces

Definition 2.1.1 (Separation of a topological space)**.** Let X be a topological space. A pair (A, B) is called a separation of X if $\{A, B\}$ is a partition of X by nonempty open subsets of X. If X has no separation, then X is called a connected space.

Remark. (a) Connectedness is a topological property, which is preserved under homeomorphisms.

(b) The topological space X is connected if and only if the only subsets of X which are both open and closed in X are $\varnothing$ and X.

When discussing connectedness of a subspace A of X according to the above definition, subsets of A shall be considered. The following theorem lets us argue the connectedness of a subspace A of X *in terms of subsets of* X.

Theorem 2.1.2 (Connected subspace)**.** Suppose X is a topological space and A is a subspace of X. Then (U, V) is a separation of A if and only if

(i) $\{U, V\}$ is a partition of A by nonempty subsets of A (here, U and V need not be open or closed in A or X)

(ii) and neither U nor V contains a limit point (in X) of the other, i.e., $\overline{U} \cap V = U \cap \overline{V} = \varnothing$. (The overline notations stand for the closure in X.)

Proof. Suppose (U, V) is a separation of Y. Because U is both open and closed in Y, $U = \overline{U} \cap Y$. Since $U \cap V = \varnothing$, we have $\overline{U} \cap V = \varnothing$. For the same reason, $U \cap \overline{V} = \varnothing$.

Conversely, assuming (i) and (ii), we have $\overline{U} \cap A = \overline{U} \cap (U \sqcup V) = U$, so U is closed in A. For the same reason, V is also closed in A, so (U, V) is a separation of A. ☐

We introduce a useful lemma regarding connected spaces, implying that a connected subspace lies entirely in a 'separation component.'

Lemma 2.1.3. If (C, D) is a separation of X and if Y is a connected subspace of X, then Y lies entirely within C or D.

Proof. Write $A = Y \cap C$ and $B = Y \cap D$. Since (C, D) is a separation of X, C and D are open (and closed) in X and in Y. Because Y is connected, either C or D is empty, as desired. ☐

Proposition 2.1.4. Suppose $\{U_\alpha\}_{\alpha \in I}$ is a collection of connected subspaces, and assume all the members have a common point. Then the union of the members of the collection is connected.

Proof. Suppose the union A of A_α for $\alpha \in I$ is not connected. Then there is a separation (U, V) of A, and each A_α resides entirely in either U or V. Without loss of generality, assume A_{α_0} is in U for some $\alpha_0 \in I$. Since a common point is in U and not in V, $A_\alpha \subset U$ for all $\alpha \in I$, so $A \subset U$, a contradiction. ☐

Proposition 2.1.5. If A is a connected subspace of X, then adding some of its limit points keeps the space connected. To be precise, if $A \subset B \subset \overline{A}$, then B is connected.

Proof. Suppose B is not connected and let (U, V) be a separation of B. Since A is a connected subspace of B, without loss of generality, $A \subset U$, so $\overline{A} \subset \overline{U}$. Because $\overline{U}$ and V are disjoint, $\overline{A} \cap V = \varnothing$, a contradiction. $\square$

Theorem 2.1.6. A continuous image of a connected space is connected.

Proof. Let X be a connected space and $f : X \to Y$ be a continuous map. Suppose $f(X)$ is not connected and let (U, V) be a separation of $f(X)$. Since U and V are open in $f(X)$, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Thus, $(f^{-1}(U), f^{-1}(V))$ is a separation of X, which contradicts the connectedness of X. $\square$

The idea of the proof of the following proposition is remarkable.

Theorem 2.1.7. A finite product of connected spaces is connected in the product topology.

Proof. It suffices to prove for the product of two connected spaces, for the desired result can be obtained by induction. Let X_1 and X_2 be connected spaces. Given a point $(a, b) \in X_1 \times X_2$, define the 'cross' $C(a, b)$ at (a, b) by $(\{a\} \times X_2) \cup (X_1 \times \{b\})$. Because $X_1 \times \{b\} \approx X_1$ and $\{a\} \times X_2 \approx X_2$ are connected, by Proposition 2.1.4, $C(a, b)$ is connected. Because

$$X_1 \times X_2 = \bigcup_{a \in X_1} C(a, b)$$

for any point b of X_2 and the intersection of $C(a, b)$ for $a \in X_1$ is nonempty, by Proposition 2.1.4, $X_1 \times X_2$ is connected. $\square$

In fact, the above proposition extends to an arbitrary product.

Theorem 2.1.8. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of connected spaces. Then the product $X = \prod_{\alpha \in I} X_\alpha$ is connected in the product topology.

Proof. We first fix a point $a = (a_\alpha)_{\alpha \in I} \in X$, and let

$$X_K = \{x \in X : x_\alpha = a_\alpha \text{ whenever } \alpha \in I \setminus K\}$$

for each finite subset K of I. In other words, X_K is the subset of X which consists of all points permitting all values only at each index in K.

Step 1: Showing that the union of all X_K is connected.
$X_K \approx \prod_{\alpha \in K} X_\alpha$ is connected by the preceeding theorem. Since each X_K contains a, the union Y of all X_K is connected.

Step 2: Showing that the closure of Y in X is X.
Suppose $p \in X$ and let W be a neighborhood of p. There is a basis member $B = \prod_{\alpha \in I} B_\alpha$ such that $p \in B \subset W$ (clearly, $J := \{\alpha \in I : B_\alpha \neq X_\alpha\} = \{\alpha_1, \cdots, \alpha_n\}$). Because B intersects X_J, B intersects Y, so $p \in \overline{Y}$ and $X = \overline{Y}$.

Therefore, the product space X is connected. $\square$

We give a problem in the textbook, with a solution using the cross we constructed in an earlier proof.

Problem 2.1.1. Let X, Y be connected spaces and A, B are nonempty proper subsets of X, Y, respectively. Show that $(X \setminus A) \times (Y \setminus B)$ is connected.

Solution. Let (p, q) be a point of $(X \setminus A) \times (Y \setminus B)$, and define

$$M := \bigcup_{x \in X \setminus A} C(x, q), \quad N := \bigcup_{y \in Y \setminus B} C(p, y).$$

It is easy to check that M, N are nonempty and contain (p, q), and $M \cup N = (X \setminus A) \times (Y \setminus B)$. Therefore, $(X \setminus A) \times (Y \setminus B)$ is connected.

Example 2.1.9. Since $\mathbb{R}$ is connected in the standard topology, $\mathbb{R}^{\mathbb{N}}$ is connected in the product topology. However, $\mathbb{R}^{\mathbb{N}}$ is not connected *in the uniform topology* (and in the box topology (why?)). To justify this assertion, let A be the set of all bounded sequences in $\mathbb{R}^{\mathbb{N}}$, and let B be the set of all unbounded sequences in $\mathbb{R}^{\mathbb{N}}$. It is clear by definition that $\mathbb{R}^{\mathbb{N}} = A \sqcup B$ and A, B are nonempty, and one can easily check that A and B are open in $\mathbb{R}^{\mathbb{N}}$. Thus, $\mathbb{R}^{\mathbb{N}}$ is not connected.

Theorem 2.1.10 (Intermediate value property)**.** Let X be a connected space, Y be an ordered set in the order topology, and let $f : X \to Y$ be a continuous map. If $a, b \in X$ and r is a point in Y between $f(a)$ and $f(b)$, there is a point $p \in X$ such that $f(p) = r$.

Proof. Suppose there is an intermediate value r such that $r \notin f(X)$. Then $f(X) = (f(X) \cap (-\infty, r)) \sqcup (f(X) \cap (r, \infty))$, forming a separation of $f(X)$. Here arises a contradiction, because the connectedness of X implies the connectedness of $f(X)$. □

Problem 2.1.2. Show that a connected metrizable space with more than one point is uncountable.[1]

Solution. Let X be a connected metrizable space with more than one point. Let d be a metric on X inducing the topology on X, and define a function $f : X \to \mathbb{R}$ by $f(x) = d(x, b)$ for $x \in X$, where $b \in X$ is given. Note that $f(X)$ is connected because X is connected and f is continuous. Thus, $f(X)$ and X are uncountable, for $f(a) \neq f(b)$.

2.2 Path-connected spaces

Definition 2.2.1 (Path-connected space)**.** A space X is said to be path-connected if given any two point a and b in X, there is a continuous map $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$. We call such f a path from a to b.

Proposition 2.2.2. A path-connected space is connected.

Proof. Suppose that a space X is path-connected but not connected. Then X has a separation (U, V). Choose a point $a \in U$ and $b \in V$, and let $f : [0,1] \to X$ be a path from a to b. Since $[0,1]$ is connected, the image of f is also connected. Then the image of f lies in U, for $f(0) = a \in U$; but $f(1) = b \notin U$, a contradiction. □

Remark. The converse of the preceeding proposition is not true in general. In other words, a connected space need not be path-connected. As a counterexample, consider the topologist's sine curve $\overline{S}$, where S is the subspace of $\mathbb{R}^2$ defined as

$$S := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : 0 < x \leq 1 \right\}$$

Clearly, S is connected, so is its closure $\overline{S}$ in $\mathbb{R}^2$. Nevertheless, $\overline{S}$ is not path-connected. (Why?)

Proposition 2.2.3. A continuous image of a path-connected space is path-connected.

Proof. Let $f : X \to Y$ be a continuous map, where X is a path-connected space. Suppose $p, q \in f(X)$ and let a and b be points of X such that $f(a) = p$ and $f(b) = q$. If γ is a path from a to b, then $f \circ \gamma$ is a path from p to q. □

Proposition 2.2.4. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of path-connected spaces with a point in common. Then the union of X_α for all $\alpha \in I$ is also path-connected.

Proof. Let p be a common point. Given two points x and y from the union, let f_1 and f_2 be paths lying in the union from x to p and from p to y, respectively. Concatenating f_2 after f_1 makes a path lying in the union from x to y. □

[1]The result of this problem will be generalized later. See Proposition 2.1.4.

Theorem 2.2.5. A product of path-connected spaces is path-connected in the product topology.

Proof. Let $\{X_\alpha\}_{\alpha\in I}$ be a collection of path-connected spaces, and let $X := \prod_{\alpha\in I} X_\alpha$. Given two points $a, b \in X$, let $f_\alpha : [0,1] \to X_\alpha$ be a path from a_α to b_α for each $\alpha \in I$. Define a map $F : [0,1] \to X$ by $F = (f_\alpha)_{\alpha\in I}$. Then the map F is a path from a to b relative to the product topology on X. □

Remark. Regarding connectedness, adding some of limit points keeps the space connected. This is not valid for path-connectedness. (Consider the topologist's sine curve.)

Problem 2.2.1. Show that if A is an open connected subspace of $\mathbb{R}^2$, then A is path-connected.

Solution. Given $p \in A$, let $C(p)$ denote the set of points in A which can be joined to p by a path in A. We will show that $C(p)$ is both open and closed in A; since A and $\varnothing$ are the only open (and closed) subspaces of A and $C(p) \neq \varnothing$, we have $C(p) = A$, so every point of A can be joined to p by a path in A.

We first show that $C(p)$ is open in A. Given a point $q \in C(p)$, let r be a positive real number such that $B(q,r) \in A$ (openness of A is used). Since any two points of a ball can be joined by a line segment (this idea will be proved when proving $C(p)$ is closed in A), $B(q,r) \subset C(p)$, i.e., $C(p)$ is open.

To show that $C(p)$ is closed in A is equivalent to show that $S(p)$ is open in A, where $S(p) = A \setminus C(p)$. Let q be a point of $S(p)$, and assume that $B(q,r) \not\subset S(p)$ for all real $r > 0$. If ϵ is a positive real number such that $B(q,\epsilon) \subset A$ (openness of A is used), there is a point $z \in B(q,\epsilon) \cap C(p)$, so p and q can be joined by a path in A via z. This contradicts the hypothesis that $q \notin C(p)$, and this proves that $S(p)$ is open in A, as desired.

Problem 2.2.2. Show that every co-countable subspace of $\mathbb{R}^2$ is path-connected.

Solution. Given two points a, b of a co-countable subspace A of $\mathbb{R}^2$, there are countably many lines passing $a \in A$ which intersects $\mathbb{R}^2 \setminus A$. Because there are uncountably many lines passing $a \in A$, we can choose a line l_1 passing a not intersecting $\mathbb{R}^2 \setminus A$. We can also find a line l_2 passing b which does not intersect $\mathbb{R}^2 \setminus A$ and not parallel to l_1. Then $l_1 \cup l_2$ contains a path from a to b lying in A.

2.3 Connected components and path-connected components

Definition 2.3.1 ((Path-)connected component)**.** Let X be a topological space and let $\sim$ and $\sim_{\mathsf{p}}$ denote the relation on X defined as follows:

$$x \sim y \text{ if and only if there is a connected subset } A \text{ of } X \text{ containing } x \text{ and } y,$$
$$x \sim_{\mathsf{p}} y \text{ if and only if there is a path from } x \text{ to } y \text{ lying in } X.$$

These relations are equivalence relations on X. Equivalence classes in $X/\sim$ and $X/\sim_{\mathsf{p}}$ are called a connected component of X and a path-connected component of X, respectively.

Proposition 2.3.2. Let X be a topological space. Then, the (path-)connected components of X form a partition of X and a (path-)connected subset of X is contained in precisely one of them. Furthermore, each (path-)connected component of X is (path-)connected.

Proof. Because $\sim$ and $\sim_{\mathsf{p}}$ are equivalence relations on X, the (path-)connected components of X form a partition of X.

We first prove the remaining part for connected case. First, if A is a connected subset of X, then it is obvious that A intersects only one connected component of X; if a and b are points of A, then C contains a and b, so $a \sim b$. To show that each connected component of X is connected, let C be a connected component of X and x_0 be a point of C. If x is a point of C, then there is a connected subset A_x of X containing x_0 and x. By the preceeding observation, $A_x \subset C$, and $C = \bigcup_{x\in C} A_x$, which is connected, since $x_0 \in A_x$ for all $x \in C$.

We now prove the remaining part for path-connected case. If A is a path-connected subset of X, then it is obvious that A intersects only one path-connected component of X; if a and b are points of A, then there is a path from a to b lying in X, so $a \sim_{\mathsf{p}} b$. It is obvious that a path-connected component of X is path-connected, because any two points of a path-connected component C of X can be joined by a curve γ lying in X and every point on γ is contained in C. □

Remark. Since a path-connected component of X is path-connected and connected, each path-connected is connected is contained in a connected component. Hence, the collection of the path-connected components of X is a refinement of the collection of the connected components of X.

Definition 2.3.3 (Local (path-)connectedness)**.** A space X is said to be locally (path-)connected at a point x of X if for every neighborhood U of x in X, there is a (path-)connected neighborhood V of x in X contained in U. If X is locally (path-)connected at every point of X, then X is said to be locally (path-)connected.

Remark. Global (path-)connectedness does not imply local (path-)connectedness. For example, the topologist's sine curve is connected but not locally connected.

Local (path-)connectedness can be defined in another way, given as follows:

Proposition 2.3.4. Let X be a topological space. Then X is locally (path-)connected if and only if for every open subset U of X each (path-)connected component of U is open in X.

Proof. Assume first that X is locally (path-)connected, and let U be a nonempty open subset of X, A be a (path-)connected component of U. To show A is open in X, let x be a point of A. Because $x \in U$, there is a (path-)connected neighborhood V of x such that $x \in V \subset U$. Since V is (path-)connected and V contains x, V is contained in A, proving that A is open in X. Assume conversely that each (path-)connected component of an open subset of X is open in X. For any point x of X and a neighborhood U of X, the (path-)connected component A of U containing x is open in X, proving that X is locally (path-)connected at every point of X. □

Theorem 2.3.5. Let X be a topological space. Then each path-connected component of X is contained in a connected component of X. When X is locally path-connected, the connected components of X and the path-connected components of X are the same.

Proof. The first half is already proved. Assume X is locally path-connected, and let x be a point of X, and let C and P be the connected and path-connected component of X containing x, respectively. Suppose $P \subsetneq C$, and define Q as the union of all path-connected components which intersect C, except for P. (Here, Q is nonempty, for Q is the union of all path-connected components containing the elements of $Q \setminus P \neq \varnothing$.) Since such path-connected components are connected and intersect C, they are contained in C. Hence, $C = P \sqcup Q$. Because X is locally path-connected, every path-connected component of X is open in X, so P and Q are open in X and they form a separation of C, a contradiction. □

Example 2.3.6. The connected components of $\mathbb{R}_l$ are the singletons, hence the path-connected components of $\mathbb{R}_l$ are the singletons; if C is a connected component of $\mathbb{R}_l$ and $a, b \in C$ and $a < b$, we have $C = ((-\infty, b) \cap C) \sqcup ([b, \infty) \cap C)$, which form a separation of C. Since a continuous map maps a connected space onto a connected space, the only continuous maps from $\mathbb{R}$ into $\mathbb{R}_l$ are constant functions.

Observation 2.3.7. As one might expect, two topological spaces with distinct cardinalities of (path-)connected components are not homeomorphic. Let X and Y be homeomorphic spaces with a homeomorphism $f : X \to Y$, and let $\mathcal{A}$ and $\mathcal{B}$ denote the collection of the (path-)connected components of X and Y, respectively. Suppose card$(\mathcal{A}) <$ card$(\mathcal{B})$, i.e., there is an injection but no bijection from $\mathcal{A}$ to $\mathcal{B}$. Because f is continuous, f maps a (path-)connected subset of X into a (path-)connected subset of Y. In particular, if $A \in \mathcal{A}$, then $f(A)$ is contained in precisely one member of $\mathcal{B}$, and this correspondence gives an injection from $\mathcal{A}$ to $\mathcal{B}$. Because such injection cannot be bijective, f fails to be surjective. Thus, we have card$(\mathcal{A}) \geq$ card(B), or equivalently, card$(\mathcal{B}) \leq$ card$(\mathcal{A})$. By symmetry, we find that card$(\mathcal{A}) =$ card$(\mathcal{B})$.

Example 2.3.8. Let $T = ([-1, 1] \times \{0\}) \cup (\{0\} \times [-1, 0]) \subset \mathbb{R}^2$. We will justify that T and the subspace $[0, 1]$ of $\mathbb{R}$ are not homeomorphic, by noticing that if $f : X \to Y$ is a homeomorphism and A is a subspace of X then $f|_A : A \to f(A)$ is a homeomorphism. Suppose T and $[0, 1]$ are homeomorphic and let $f : T \to [0, 1]$ be a homeomorphism. Since $S = T \setminus \{(0, 0)\}$ has three connected components, its image $f(T)$ also has three connected components. However, $f(T) = [0, 1] \setminus f((0, 0))$, which has at most two connected components. Therefore, T and $[0, 1]$ are not homeomorphic.

2.4 Compact spaces

Definition 2.4.1 (Compact space)**.** A topological space X is said to be compact if every covering of X by sets open in X has a finite subcover. If every such covering has a countable subcover, then X is called a Lindelöf space.

Remark. In fact, we can impose an alternative definition of compactness as follows:

> The topological space X is said to be compact, if for every collection $\mathcal{C}$ of closed sets in X with the finite intersection property, the intersection of the members of $\mathcal{C}$ is nonempty.

The equivalence of compactness and the above property can be checked by considering an appropriate contraposition.

As connectedness, compactness of subspaces can also be argued in larger spaces.

Theorem 2.4.2 (Compactness of subspaces)**.** Let A be a subspace of X. Then A is compact if and only if every covering of A by sets open in X has a finite subcover. (This statement is valid even if the words 'compact' are replaced by 'Lindelöf.')

Proof. Suppose A is compact, and $\mathcal{A} = \{A_\alpha\}_\alpha$ is a covering of A by sets open in X. Then the naturally induced collection $\{A_\alpha \cap A\}$ is a covering of A by sets open in A, and we can find a finite subcover. The corresponding finite subcover of $\mathcal{A}$ covers A.

Suppose conversely that every covering of A by sets open in X has a finite subcover, and let $\mathcal{A} = \{A_\alpha\}_\alpha$ be a covering of A by sets in A. Since each A_α can be written as $A \cap O_\alpha$ for some subset O_α open in X, the collection $\{O_\alpha\}_\alpha$ has a finite subcover. The corresponding finite subcover of $\mathcal{A}$ covers A. $\square$

Theorem 2.4.3. Regarding closedness and compactness of subspaces, the following statements holds:

(a) Every closed subspace of a compact space is compact.

(b) Every Hausdorff space is compactly normal. In other words, any two disjoint compact subspaces can be separated by (disjoint) neighborhoods.

(c) Every compact subspace of a Hausdorff space is closed. Hence, in a compact Hausdorff space, closedness and compactness of subspaces coincide. (In particular, every compact Hausdorff space is normal.)

Proof. (a) Let X be a compact space and Y be a closed subspace of X. Let $\mathcal{A} = \{A_\alpha\}_\alpha$ be a covering of Y by sets open in X. Because the collection $\mathcal{A} \cup \{X \setminus Y\}$ is also an open covering of X, the collection has a finite subcover. Among them, excluding $X \setminus Y$ gives a finite subcover of $\mathcal{A}$ which covers Y.

(b) We first prove that a Hausdorff space X is compactly regular. Let p be a point of X and A be a compact subspace of X not containing p. For each $a \in A$, let U_a and V_a be disjoint neighborhoods of a and p, respectively. Since A can be covered by finitely many U_a, the union of such U_a and the intersection of the corresponding V_a are disjoint neighborhoods of A and p, respectively. Thus, X is compactly regular.
To show that X is compactly normal, let A and B be disjoint compact subspaces of X. For each $a \in A$, let U_a and V_a be disjoint neighborhoods of a and B, respectively. Since A can be covered by finitely many U_a, the union of such U_a and the intersection of the corresponding V_a are disjoint neighborhoods of A and B, respectively. Therefore, X is compactly normal.

(c) If A is a compact subspace of a Hausdorff space X, then (b) implies that $X \setminus A$ is open. Thus, a compact subspace of a Hausdorff space is closed. The desired coincidence follows from (a), and the desired normality follows readily. $\square$

Proposition 2.4.4. A continuous image of a compact space is compact.

Proof. Straightforward, due to the compactness of the domain. $\square$

Proposition 2.4.5. Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. If A is a closed subspace of X, then A is compact, so $f(A)$ is compact (and closed in Y). □

Theorem 2.4.6. The product of finitely many compact spaces is compact.[2]

When proving the above theorem, the first version of the tube lemma will be used.

Lemma 2.4.7. (a) (The tube lemma) Consider the product space $X \times Y$, where Y is a compact space. If N is an open subspace of $X \times Y$ containing a slice $x_0 \times Y$ for some $\{x_0\} \in X$, then N contains a tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.

(b) (A generalized tube lemma) Let A and B be compact subspaces of X and Y, respectively, and let N be an open subspace of $X \times Y$ containing $A \times B$. Then, there are neighborhoods U of A in X and V of B in Y, respectively, such that $A \times B \subset U \times V \subset N$. ((a) readily follows from (b) when $A = \{x_0\}$ and $B = Y$.)

Proof of Lemma 2.4.7. (a) Since N is open, for each point $(x_0, y) \in \{x_0\} \times Y$, there is a neighborhood $A_y \times B_y$ of the point (x_0, y) contained in N, where A_y and B_y are open in X and Y, respectively. Because $\{x_0\} \times Y$ is homeomorphic to Y, $\{x_0\} \times Y$ is compact, so the slice can be covered by finitely many above basis members. For convinience, write such members as $A_i \times B_i$ for $i = 1, \cdots, n$. Letting $W = \bigcap_{i=1}^n A_i$, we have $\{x_0\} \times Y \subset W \times Y \subset N$.

(b) Since N is open, for each point $(a, t) \in \{a\} \times B$, there is a neighborhood $A_t^a \times B_t^a$ of the point (a, t) contained in N, where A_t^a and B_t^a are open in X and Y, respectively. Because the slice $\{a\} \times B$ is compact for each $a \in A$, the slice can be covered by finitely many neighborhoods; denote each of such neighborhoods by $A_i^a \times B_i^a$ for $i = 1, \cdots, n(a)$. Finally, define

$$U^a := \bigcap_{i=1}^{n(a)} A_i^a, \quad V^a := \bigcup_{i=1}^{n(a)} B_i^a.$$

Then, $U^a \times V^a$ is an open subset of $X \times Y$ such that $\{a\} \times Y \subset U^a \times V^a \subset N$. Since A is compact, there is a finite subcollection of $\{U^a\}_{a \in A}$ covering A; write it as $\{U^1, \cdots, U^k\}$. Finally, define

$$U := \bigcup_{i=1}^{k} U^k, \quad V := \bigcap_{i=1}^{k} V^k.$$

Then, U and V are open in X and Y, respectively, and $A \times B \subset U \times V \subset N$. □

Proof of Theorem 2.4.6. Let X, Y be compact spaces, and $\mathcal{A}$ be an open cover of $X \times Y$. Given $x_0 \in X$, because the slice $\{x_0\} \times Y$ is compact, there are finitely many members of $\mathcal{A}$ covering the slice $\{x_0\} \times Y$, and such members cover a tube $U(x_0) \times Y$ about $\{x_0\} \times Y$. (Here, $U(x_0)$ is assumed to be a neighborhood of x_0 in X.) Due to the compactness of X, there are finitely many points $x_1, \cdots, x_n$ such that $\bigcup_{i=1}^n U(x_i) = X$, and this justifies the compactness of $X \times Y$. The general result is deduced inductively. □

Before introducing some problems, we observe properties of ordered sets in the order topologies, regarding compactness.

Proposition 2.4.8. (a) Suppose X is an ordered set in the order topology, and assume that X is compact. Then X has both the greatest and the least element.

(b) (Extreme value theorem) Let $f : K \to Y$ be a continuous map, where K is a compact space and Y is an ordered set in the order topology. Then f attains the maximum and the minimum.

Proof. (a) If X has no least element, the open covering $\{(a, \infty) : a \in X\}$ of X contains no finite subcover of X, a contradiction. The same argument holds for the case where X has no greatest element, provided that (a, ∞) is replaced by $(-\infty, a)$ for each $a \in X$.

(b) The subspace $f(K)$ of Y is compact, and (a) implies that $f(K)$ has both the greatest and the least element. □

[2] In fact, this theorem extends to arbitrary products of compact spaces, which is called the Tychonoff theorem. The proof of the Tychonoff theorem will not be introduced in this note.

Proposition 2.4.9. Let X be an ordered set in the order topology. If X satisfies the least upper bound property if and only if every closed interval in X is compact.

Proof. The proof of only if part is not given in this note. Assume that every closed intervel in X is compact, i.e., $[a,b]$ is compact whenever $a,b \in X$ and $a < b$. Let S be a nonempty subset of X which is bounded above. We will show that S has the supremum (the least upper bound) in X. Let u be any upper bound of S in X, and consider the set

$$U := \bigcap_{a \in S} [a, u].$$

Then U is closed in X, so U is compact. Because U is a compact ordered set (in the order topology), U has the least element M.

We now show that M is the supremum of S. It is obvious that M is an upper bound of S in X; otherwise, there is an element $b \in S$ with $M < b$, which implies $M \notin U$, a contradiction. Assume that M is not the supremum of S. Then there is an upper bound v of S such that $v < M$. Because $v \geq a$ for all $a \in S$, we have $v \in U$, so $M \leq v$, a contradiction. $\square$

The first problem deals with the distance between a point and a subspace and neighborhoods of subspaces in a metric space. In particular, (c) implies that the original definition of the ϵ-neighborhood of a subspace and our intuition coincide.

Problem 2.4.1. Let (X,d) be a metric space and A be a nonempty subset of A.

(a) Show that $d\,(x, A) = 0$ if and only if $x \in \overline{A}$.

(b) Show that if A is compact, then $d\,(x, A) = d(x,a)$ for some $a \in A$, whenever $x \in X$.

(c) Define the ϵ-neighborhood of A in X to be the set

$$U(A,\epsilon) := \{x \in X : d\,(x, A) < \epsilon\}.$$

Show that $U(A,\epsilon)$ equals the union of the open balls $B_d(a,\epsilon)$ for $a \in A$.

(d) Assume that A is compact, and let U be an open set in X containing A. Show that U contains an ϵ-neighborhood of A for some $\epsilon > 0$.

(e) Show that the result in (d) need not hold if A is closed but not compact.

Solution. (a) Almost clear. If $x \in \overline{A}$, then whenever $k > 0$, there is a point $a \in A$ such that $d(x,a) < k$. Hence, $d\,(x, A) = 0$. Assuming conversely, whenever $k > 0$, there is a point $a \in A$ such that $d(x,a) < k$, so $B_d(x,k)$ intersects A. Hence, $x \in \overline{A}$.

(b) Given a point $x \in X$, define a function $f_x : A \to [0,\infty)$ by $f_x(a) = d(x,a)$ for $a \in A$. Since $|f_x(a) - f_x(b)| \leq d(a,b)$ whenever $a,b \in A$, f_x is (uniformly) continuous. Because the domain A of f_x is a compact space, f_x attains the minimum. Furthermore, $d\,(x, A) = \inf\{d(x,a) : a \in A\} = \min\{f_x(a) : a \in A\}$, so $d\,(x, A) = d(x,a)$ for some $a \in A$.

(c) It is clear that every ϵ-ball with the center in A is contained in the ϵ-neighborhood of A, so one inclusion is obvious. Suppose $u \in U(A,\epsilon)$ so that $d\,(u, A) < \epsilon$. It implies that the ϵ-ball centered at u intersects A at a point $a \in A$, so the ϵ-ball centered at a contains u. Therefore, $U(A,\epsilon)$ is contained in the union of the ϵ-balls with centers in A.

(d) Let $f : A \to \mathbb{R}$ be a function defined by $f(a) = d\,(a, X \setminus U)$.

(i) f is continuous; for $a,b \in A$, we have $f(a) \leq d(a,x) \leq d(a,b) + d(b,x)$ for all $x \in X \setminus U$, so $f(a) \leq d(a,b) + f(b)$, from which, by symmetry, we obtain $|f(a) - f(b)| \leq d(a,b)$.

(ii) $f(a) > 0$ for all $a \in A$; otherwise, if $f(p) = 0$ for some $p \in A$, then $p \in \overline{X \setminus U} = X \setminus U$, a contradiction.

Because A is compact, f attains the minimum $\delta > 0$. Hence, for example, if $\epsilon = \delta/2$, then $U(A, \epsilon)$ is contained in U.

(e) The x-axis in the xy-plane is closed but not compact, and $R = \{(x, y) \in \mathbb{R}^2 : y < e^{-|x|}\}$ is open and contains the x-axis. However, no neighborhood of the x-axis is contained in R.

Problem 2.4.2. Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then $Y := \bigcap_{A \in \mathcal{A}} A$ is connected.

Solution. Suppose Y is not connected, and let (C, D) be a separation of the intersection $Y = \bigcap_{A \in \mathcal{A}} A$. Because C and D are closed in Y and Y is closed in X, C and D are closed in X. Since X is normal, there are disjoint neighborhoods U and V of C and D, respectively.

Assume that $A \setminus (U \sqcup V)$ is empty for some $A \in \mathcal{A}$. Then $A \subset U \sqcup V$ and both $A \cap U, A \cap V$ are nonempty, for $Y \subset A$. Thus, A has a separation $(A \cap U, A \cap V)$, a contradiction; $A \setminus (U \sqcup V)$ is nonempty (i.e., $A \not\subset U \sqcup V$) for all $A \in \mathcal{A}$.

The collection $\mathcal{C} := \{A \setminus (U \sqcup V) : A \in \mathcal{A}\}$ is a simply ordered collection of sets closed in X, and $\mathcal{C}$ satisfies the finite intersection property. Because X is compact, the intersection of members in $\mathcal{C}$ is nonempty, implying that $Y \not\subset U \sqcup V$, a contradiction. Therefore, Y is connected.

2.5 Compact metrizable spaces

We first introduce some other types of compactness:

Definition 2.5.1. Let X be a space.

(a) X is said to be limit point compact, if every infinite subset of X has a limit point in X.

(b) X is said to be sequentially compact, if every (infinite) sequence of points of X has a convergent subsequence.

Remark. (a) Compactness implies limit point compactness.

(b) Let X be a metric space. If X is sequentially compact, then X is complete and totally bounded.[3]

Theorem 2.5.2. Suppose X is a metrizable space. Then the following are equivalent:

(a) X is compact.

(b) X is limit point compact.

(c) X is sequentially compact.

Proof. We already proved that (a) implies (b) and it is easy to show that (b) implies (c). Thus, it remains to prove (a) under (c), so we assume that X is sequentially compact. In this proof, we use the strategy by imposing X a metric d to understand X as a metric space, not just a metrizable space.

Step 1: Showing the existence of a Lebesgue number.
We wish to prove that for any open cover $\mathcal{A}$ of X there is a real number $\delta > 0$ such that every open set in X with diameter less than δ is contained in some member of $\mathcal{A}$. If there is no such δ, for each $n \in \mathbb{N}$, there is an open set C_n of diameter less than $1/n$ which is not contained in any member of an open cover $\mathcal{A}$ of X. Choosing a point $x_n \in C_n$ for each $n \in \mathbb{N}$, by the sequential compactness of X, there is a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$ (with the limit denoted by x).[4] Let A be a member of $\mathcal{A}$ containing x, and let $r > 0$ be a real number such that $B_d(x, r) \subset A$. If a positive integer i is large enough so that

$$d(x, x_{n_i}) < \frac{r}{2} \quad \text{and} \quad \frac{1}{n_i} < \frac{r}{2},$$

[3] Remark that both completeness and total boundedness are properties of metric spaces, for they can be discussed only when a metric on a metrizable space is given.

[4] It is natural to consider countably many objects when one should consider sequences.

then $C_{n_i} \subset A$, a contradiction.

Step 2: Deriving that X is compact.
Let $\mathcal{A}$ be an open cover of X and let $\delta > 0$ be a Lebesgue number for $\mathcal{A}$. Because a sequentially compact metric space is totally bounded, finitely many balls in X of radius $\delta/3$ cover X. So there is a finite subcollection of $\mathcal{A}$ which covers X. □

Remark. The result of Step 1 in the above proof is also known as the Lebesgue number lemma.

For metric spaces, the above equivalence reduces to the Heine-Borel theorem.

Theorem 2.5.3 (Heine-Borel theorem)**.** Let (X, d) be a metric space. Then the following are equivalent:

(a) X is compact.

(b) X is limit point compact.

(c) X is complete and totally bounded.

Proof. By the preceeding theorem, it suffices to show that (c) is equivalent to compactness. Since it is already observed that sequential compactness implies (c), it suffices to show that (c) implies any of compactnesses. For this, we will show that (c) implies sequential compactness.

Let $(x_n)_n \subset X$ be a sequence. Since X is sequentially compact, there is a ball B_1 in X of radius 2^{-1} which contains x_n for infinitely many $n \in N_1 \subset \mathbb{N}$. Because $X \cap B_1$ is also totally bounded, a ball B_2 with the center in $X \cap B_1$ of radius 2^{-2} contains x_n for infinitely many $n \in N_2 \subset N_1$. Continuing inductively, we can find $n_i \in N_i$ for each $i \in \mathbb{N}$ with $n_1 < n_2 < \cdots$. Then $d(x_{n_i}, x_{n_k}) \leq 2^{1-i}$ if $i < k$. Because X is complete, $(x_n)_n$ has a convergent subsequence. Therefore, (c) implies sequential compactness. □

Problem 2.5.1. Let U be an open subset of $\mathbb{C}$ which contains $\mathbb{D}$. Show that there is a positive real number $r > 1$ such that $B(0, r) \subset U$.

Solution. Suppose that there is no such $r > 1$. Then, we can choose $a_n \in B(0, 1 + 1/n) \setminus U$ for each $n \in \mathbb{N}$. Since $(a_n)_{n\in\mathbb{N}} \subset \overline{B(0, 2)}$, $(a_n)_n$ contains a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Letting α be the limit of $(a_{n_k})_k$, we find that α is a limit point of $\mathbb{C} \setminus U$. Because $\mathbb{C} \setminus U$ is closed, $\alpha \notin U$. On the other hand, $\alpha \in \partial\mathbb{D}$ (why?), we have $\alpha \in \partial\mathbb{D} \subset U$, a contradiction. Therefore, $B(0, r) \subset U$ for some real number $r > 1$.

Solution (Alternative solution). Note that the collection of 'polar rectangles' is a basis for the topology on $\mathbb{C}$, where a polar rectangle is of the form

$$a \leq \rho \leq b, \quad s \leq \theta \leq t,$$

where a, b, s, t are real numbers such that $a \leq b$. Because $\partial\mathbb{D}$ is compact, for each point $e^{ix} \in \partial\mathbb{D}\,(0 \leq x < 2\pi)$ there is a polar rectangle $P_x : a(x) \leq \rho \leq b(x), s(x) \leq \theta \leq t(x)$ such that $e^{ix} \in P_x \subset U$. Choosing finitely many members among P_x and letting r the smallest $b(x)$, we find that $B(0, r) \subset U$.

Problem 2.5.2. Let (X, d) be a metric space. Show that every isometry of X is surjective, when X is compact. (In fact, remarking the definition of an isometry, we may argue that every isometry of a compact metric space is an homeomorphism.)

Solution. Let $f : X \to X$ be an isometry of X, and suppose f is not surjective. Then there is a point $x \in X \setminus f(X)$. Because X is compact, $f(X)$ is compact, so $X \setminus f(X)$ is open in X. It implies the existence of a real number $r > 0$ such that $B_d(x, r) \cap f(X) = \varnothing$. Letting $x_0 = x$, define $x_n = f(x_{n-1})$ for each positive integer n, and let j, k be any positive integers with $j > k$. Then $d(x_j, x_k) = d(x_{j-1}, x_{k-1}) = \cdots = d(x_{j-k}, x) > r$, so the sequence $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence, which contradicts the compactness of X. Therefore, every isometry of a compact metric space is a homeomorphism.

2.6 Locally compact spaces and one-point compactification

Some of the properties which are most desired for a topological space to have are the space being metrizable or being a compact Hausdorff space. In this section, we impose a situation in which a topological space embeds into a compact Hausdorff space.

Definition 2.6.1 (Local compactness)**.** A space X is said to be locally compact at a point $a \in X$ if there is a compact subspace C of X containing a neighborhood of a.[5] If X is locally compact at every point of X, then X is said to be locally compact.

A simple observation follows:

Observation 2.6.2. Let $\{X_\alpha\}_\alpha$ be an indexed family of nonempty spaces.

(a) If $\prod_\alpha X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α. If the product space is compact, then each X_α is compact.

(b) The converse of the first statement in (a) is also true.

Proof. Write $X = \prod_\alpha X_\alpha$.
(a) Given a point $x \in X$, there is a compact subspace C of X and a basis member $\prod_\alpha B_\alpha$ such that $x \in \prod_\alpha B_\alpha \subset C$. Because $\pi_\alpha(C)$ is a compact subspace of X_α containing the neighborhood B_α of $x_\alpha \in X_\alpha$ for each α, each X_α is locally compact. Moreover, because $B_\alpha = X_\alpha$ for all but finitely many values of α, X_α is compact for all but finitely many values of α. In particular, if X is compact, then $X_\alpha = \pi_\alpha(X)$ is compact for all α.
(b) Given a point $x \in X$, for each α, use (local) compactness to find a compact subspace C_α of X_α containing a neighborhood B_α of x_α in X_α. When finding such subspaces, choose $B_\alpha = C_\alpha = X_\alpha$ whenever X_α is compact. Then, the product of B_α for all α is a basis member of X containing x; the product of C_α for all α is a compact subspace of X. □

Theorem 2.6.3 (Existence and uniqueness of a one-point compactification)**.** Let X be a space. Then X is locally compact and Hausdorff if and only if there is a space Y with the following properties:

(a) Y is a compact and Hausdorff space containing X as a subspace, i.e., Y is a compactification of X.

(b) $Y \setminus X$ is a one-point set.

Moreover, if Y_1 and Y_2 are such spaces, then Y_1 and Y_2 coincide on X and are homeomorphic, i.e., Y is uniquely determined up to equivalence.[6]

Proof. Since it is easier to check the uniqueness, we first explain the uniqueness (up to equivalence).

Step 1: Proving the uniqueness part.
Suppose Y_1 and Y_2 are compact Hausdorff spaces satisfying (a), (b), and (c). Let p and q denote the unique point of $Y_1 \setminus X$ and $Y_2 \setminus X$, respectively. Define a map $h : Y_1 \to Y_2$ by

$$h(x) = \begin{cases} x & (\text{if } x \in X) \\ q & (\text{otherwise, i.e., } x = p) \end{cases} .$$

We show h is a homeomorphism extending the identity map on X; and for this, it suffices to verify that h is a continuous map, because the openness of h will follow by symmetry. If U is an open subset of Y_2 contained in X, its preimage is U; because U is open in X and X is open in Y_1, U is open in Y_1. Assume U is an open subset of Y_2 containing q. Then the subset $C = Y_2 \setminus U$ is closed in Y_2, so C is compact and is contained in X. It follows that $h^{-1}(C) = C$ is a compact subspace of Y_1 and that $h^{-1}(C)$ is closed in Y_1. Therefore, h is a continuous map.

Step 2: Proving the existence part.
Suppose first that X is a locally compact Hausdorff space. Let p be any element not in X, and let $Y = X \sqcup \{p\}$. And impose a topology on Y by declaring the following subsets to be open in Y:

[5] To prevent a confusion, suppose the neighborhood of a is larger than the compact subspace C. In this case, we may choose $C = \{a\}$ for any topological space, making the notion of local compactness meaningless.

[6] Equivalence of compactness is introduced in Chapter 4.

(T1) Subsets which are open in X.

(T2) Subsets of the form $Y \setminus C$, where C is a compact subspace of X. (See Problem 2.6.1.)

It is left as an exercise to check that the above collection is a topology on Y. (See Problem 2.6.1.)

We first show that Y contains X as a subspace. (Clearly, the topology on X is coarser than the subspace topology on X inherited from Y.) By merely intersecting any subset of Y of either type with X, one can easily observe that those two topologies are equal.

To show that Y is compact, let $\mathcal{A}$ be any open cover of Y. Then there is a member $Y \setminus C \in \mathcal{A}$ of the second type which contains p. Since C is a compact subspace of X, finitely many members of $\mathcal{A}$ cover C. These members, together with $Y \setminus C$, cover Y, as desired.

Finally, we show that Y is a Hausdorff space, in which it suffices to show that p and any point a in X can be separated by disjoint open subsets of Y. Since X is locally compact, there is a compact subspace C of X containing a neighborhood U of a in X, and $(U, Y \setminus C)$ is a desired pair.

Step 3: Proving the converse.

Suppose such space Y exists for a space X. Being a subspace of the Hausdorff space Y, X is a Hausdorff space. Given a point $a \in X$, because Y is a Hausdorff space, there are neighborhoods U and V of a and p in Y which are disjoint. Since $Y \setminus V$ is a closed subspace of Y contained in X, $Y \setminus V$ is a compact subspace of X containing U. Therefore, X is locally compact. □

Problem 2.6.1. (a) Show that the collection imposed in Step 2 of the above proof is a topology on Y.

(b) Explain why we assume in the second type that C is compact, rather than closed in Y.

Solution. (a) What we need to do is to check the axioms of topology.

Clearly, $\varnothing$ is of the first type and Y is of the second type, so they belong to the collection.

A union of sets of the first type is an open subset of X, hence the union is of the first type. If $\{V_\beta = Y \setminus C_\beta\}_\beta$ is a collection of sets of the second type (each C_β is a compact subspace of X), their union is of the second type; because the union is $Y \setminus K$ (where $K = \bigcap_\beta C_\beta$) and K is a closed subspace of every C_β, K is compact. The union of a set U of the first type and a set $V = Y \setminus C$ of the second type (C is a compact subset of X) is of the second type, since

$$U \cup (Y \setminus C) = Y \setminus (C \setminus U)$$

and $C \setminus U$ is compact (because $C \setminus U$ is a closed subset of the compact subset C).

A finite intersection of sets of the first type is an open subset of X, hende the intersection is of the first type. If $(V_\beta = Y \setminus C_k)_{k=1}^n$ is a collection of sets of the second type (each C_k is a compact subspace of X), their intersection is of the second type, because the union is $Y \setminus K$, where $K = \bigcup_{k=1}^n C_k$ is compact. The intersection of a set U of the first type and a set $V = Y \setminus C$ of the second type (C is a compact subspace of X) is of the first type, since

$$U \cap (Y \setminus C) = U \setminus C$$

is open in X.

(b) If C in the second type is assumed to be closed subsets of X rather than being compact, then Y may not be compact, for a closed subset C of X may not be compact. (Remark that a compact subset of a Hausdorff space is closed.)

Let X be a locally compact Hausdorff space, and let Y be a space (which is unique up to equivalence) constructed as above.

- If X is compact, then X is a closed subspace of Y, so the closure of X in Y is X, which is a proper subset of Y.

- If X is not compact, then X is not closed in Y (why?), so the closure of X in Y is Y.

If Y is such a space and the closure of X in Y is Y, Y is called the one-point compactification (or the Alexandroff compactification) of X.

Remark. The above observation states that a locally compact Hausdorff space X has a one-point compactification if and only if X is not compact.

Remark. In Chapter 4, we will consider the general concept of compactification (before studying the Stone-Čech compactification) over completely regular spaces. Since one-point compactifications are also compactifications, it will be better if it is explained that a locally compact Hausdorff space X is completely regular. If the space X is compact, then X is normal, so X is completely regular; if X is not compact, then X is a subspace of its one-point compactification, so X is completely regular.

For Hausdorff spaces, local compactness can be intuitively considered as follows:

Proposition 2.6.4. Suppose X is a Hausdorff space. Then X is locally compact if and only if given $x \in X$ and its neighborhood U in X, there is a relatively compact neighborhood V of a in X whose closure in X is contained in U.[7]

Proof. Suppose X is a locally compact Hausdorff space, and suppose further that a point $x \in X$ together with its neighborhood U in X is given. Since X has a compactification Y such that $Y \setminus X$ is a singleton, the subset $C := Y \setminus U$ is a closed (of course, compact) subspace of Y. By regularity, there are disjoint neighborhoods V of x and W of C in Y. Then, V is a neighborhood of x in X contained in $Y \setminus W \subset U$. Furthermore, if the overline notation denotes the closure of in Y, we have $\overline{V} \subset \overline{Y \setminus W} = Y \setminus W \subset U$. So, the closure of V in X satisfies $X \cap \overline{V} = \overline{V} \subset Y \setminus W \subset U$. Because $\overline{V}$ is closed in Y, it follows that V is relatively compact in X.

The converse is clear, for we may let C be the closure of V in X. □

As a homeomorphism implies the topologically equivalence of two topological spaces, one might expect the one-point compactifications of two homeomorphic spaces to be homeomorphic.

Theorem 2.6.5. If $f : A \to B$ is a homeomorphism of locally compact Hausdorff spaces A and B, then f extends to a homeomorphism of the one-point compactifications of A and B.

Proof. Assume that A and B are not compact (otherwise, there is nothing to prove), and let X and Y be the one-point compactifications of A and B, respectively. If f were to be extended to X, such an extension (as a set map) should be given as the bijection $\widetilde{f} : X \to Y$, which is defined by

$$\widetilde{f}(x) = \begin{cases} f(x) & (\text{if } x \in A) \\ q & (\text{if } x = p) \end{cases},$$

where p and q are the unique elements of $X \setminus A$ and $Y \setminus B$, respectively. To verify that $\widetilde{f}$ is a continuous map, let V be an open subset of Y. If V is contained in B, $\widetilde{f}^{-1}(V) = f^{-1}(V)$ is open in A, hence in X. If V is not contained in B, $Y \setminus V$ is closed in Y (hence, compact), implying that $\widetilde{f}^{-1}(Y \setminus V) = f^{-1}(Y \setminus V)$ is compact (hence, closed in X), i.e., $f^{-1}(V)$ is open in X. By symmetry, it follows that $\widetilde{f}$ has an open map. This completes the proof. □

Example 2.6.6. We introduce some applications of Theorem 2.6.5.

(a) Since $\mathbb{R} \approx S^1 \setminus \{1\}$, the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1. Similarly, because $\mathbb{C} \approx \mathbb{R}^2 \approx S^2 \setminus \{n\}$ (where n is any point in S^2), the one-point compactification of $\mathbb{C}$ or $\mathbb{R}^2$ is homeomorphic to S^2. In addition, the one-point compactification of $\mathbb{C}^\times$ can be identified by gluing two distinct points of S^2.

(b) In (a), a widely-chosen homeomorphism between $\mathbb{R}^2$ and $S^2 \setminus \{n\}$ is the stereographic projection. Deleting one point in the domain and deleting the corresponding point in the codomain gives a restricted homeomorphism, which implies that a 2-dimensional ball with two punctures is homeomorphic to the plane with one puncture. In general, a 2-dimensional ball with n punctures for $n \in \mathbb{N}$ is homeomorphic to the plane with $(n-1)$ punctures.

(c) Since $\mathbb{N} \approx K$, where $K = \{1/n : n \in \mathbb{N}\}$, the one-point compactification of $\mathbb{N}$ is homeomorphic to $K \sqcup \{0\}$.

[7] By relatively compact we mean that the closure of the space is compact.

Chapter 3

Countability and separation axioms

3.1 Countability and separation axioms - Part 1

3.1.1 Definitions regarding countability and separation axioms

Definition 3.1.1 (Countability axioms)**.** Let X be a topological space.

(a) X is said to have a countable base at a point $x \in X$ if there is a countable collection $\mathcal{B}$ of neighborhoods of x in X such that every neighborhood of x contains at least one member of $\mathcal{B}$.

(b) X is said to be a first-countable space if every point of X has a countable base.

(c) X is said to be a second-countable space if X has a countable basis.

Remark that the second countability implies the first countability.

For a second-countable space, countability does not remain only on the existence of countable bases. In fact, countability presents in every basis of the topology on a second-countable space.

Proposition 3.1.2. If a space X is second-countable and $\mathcal{C}$ is any basis of X, then $\mathcal{C}$ has a countable subcollection which is a basis of X.

Proof. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis of X. For each $m, n \in \mathbb{N}$, whenever it is possible, choose a member $C_m^n \in \mathcal{C}$ such that $B_m \subset C_m^n \subset B_n$. To be precise, for each $n \in \mathbb{N}$ and a point $p \in B_n$, find all possible members $C \in \mathcal{C}$ such that $p \in C \subset B_n$. For each C, find a member $B_m \in \mathcal{B}$ such that $p \in B_m \subset C \subset B_n$. If a member of $\mathcal{C}$ satisfying the last inclusion is already found, discard the newly found member of $\mathcal{C}$; otherwise, let such C be denoted by C_m^n.

We first show that the collection $\mathcal{C}^* := \{C_m^n\}_{m,n}$ is a countable basis of X. The countability of $\mathcal{C}^*$ is obvious, and the construction of $\mathcal{C}^*$ asserts that $\mathcal{C}^*$ covers X. Assume that a point $p \in X$ belongs to two members $C_{m_1}^{n_1}$ and $C_{m_2}^{n_2}$ of $\mathcal{C}^*$. Let C_0 be a basis member of $\mathcal{C}$ such that $p \in C_0 \subset C_{m_1}^{n_1} \cap C_{m_2}^{n_2}$, and let $l \in \mathbb{N}$ be an index such that $p \in B_l \subset C_0$. By the above construction, for some integer $i \in \mathbb{N}$, we have $p \in B_i \subset C_i^l \subset B_l$, so $\mathcal{C}^*$ is a countable basis for a topology on X.

Because $\mathcal{C}^*$ is contained in $\mathcal{C}$, it suffices to prove that the topology generated by $\mathcal{C}^*$ is finer than the topology on X, which is clear by the above construction: Given a point x of X with a basis member $B_n \in \mathcal{B}$ containing x, there is a basis member $C_l^n \in \mathcal{C}$ containing x for some $l \in \mathbb{N}$. □

Definition 3.1.3 (Separation axioms)**.** A topological space X is called a Fréchet space (or a T_1 space) if every finite subset of X is closed in X. In the rest definitions, assume X is a Fréchet space.

(a) X is called a Hausdorff space (or a T_2 space) if given two distinct points a and b in X, there are disjoint neighborhoods of a and b.

(b) X is called a regular space (or a T_3 space) if given a point $a \in X$ and a nonempty closed subset $B \subset X$ not containing a, there are disjoint neighborhoods of a and B.

(c) X is called a completely regular space (or a $T_{3\frac{1}{2}}$ space) if, given a point $a \in X$ and a nonempty closed subset $B \subset X$ not containing a, there is a continuous function $f : X \to [0,1]$ such that $f(a) = 1$ and $f(B) = \{0\}$.

(d) X is called a normal space (or a T_4 space) if, given two nonempty disjoint subsets in X there are disjoint neighborhoods of those closed subsets.

Remark (Alternative definitions for some separabilities). Assume that X is a T_1 space.

(a) X is a regular space if and only if given a point $a \in X$ with its neighborhood U in X, there is a neighborhood of a in X whose closure in X is contained in U.

(b) X is a completely regular space if and only if given a point p of X and its neighborhood U in X, there is a continuous function $f : X \to [0,1]$ such that $f(p) = 1$ and $f(X \setminus U) = \{0\}$.

(c) X is a normal space if and only if given a nonempty closed subset $B \subset X$ with its neighborhood U in X, there is a neighborhood of B in X whose closure in X is contained in U.

In the above definitions, the condition that the closure of a neighborhood is contained in the larger open set corresponds to the exsitence of an appropriate neighborhood of a closed subset. For their proof, see Problem 3.2.1.

3.1.2 Basic properties of countabilities and separabilities

As a basis of a topology is powerful in describing the topology, second-countability is a strong condition.

Proposition 3.1.4. Suppose X is a second-countable space.

(a) X is a Lindelöf space, i.e., every open cover of X contains a countable subcover.

(b) X is a separable space, i.e., X has a countable dense subset.

Proof. (a) immediately deduced from the assumption. Collecting the points each of which is chosen from a basis member, the collection is dense in X, so X is separable. □

Remark. A space X is said to be hereditarily Lindelöf space if every subspace of X is a Lindelöf space.

The converse of Proposition 3.1.4 are valid when the space is metrizable.

Proposition 3.1.5. For metrizable spaces, second countability, separability, and being a Lindelöf space coincide. In other words,

(a) every metrizable Lindelöf space is second countable.

(b) every metrizable separable space is second countable.

Proof. Let X be a metrizable space, and let d be a metric on X which induces the topology on X.
(a) For each $n \in \mathbb{N}$, the open cover $\{B_d\left(x, n^{-1}\right) : x \in X\}$ has a countable subcover; let each member of a countable subcover be denoted by $B^n_k = B_d(x^n_k, n^{-1})$ with $k \in \mathbb{N}$. We want to show that the collection $\mathcal{B} := \{B^n_k\}_{n,k\in\mathbb{N}}$ is a countable basis of the topology on X. Clearly, $\mathcal{B}$ is an open cover of X. If a point $p \in X$ is contained in two members $B^{n_1}_{k_1}$ and $B^{n_2}_{k_2}$, let r be the positive number which is the minimum among the following four positive numbers:

$$d(p, x^{n_1}_{k_1}), \quad \frac{1}{n_1} - d(p, x^{n_1}_{k_1}), \quad d(p, x^{n_2}_{k_2}), \quad \frac{1}{n_2} - d(p, x^{n_2}_{k_2}).$$

Suppose n is large so that $2/n < r$, and let j be an index such that B^n_j contains p. Then B^n_j is a member of $\mathcal{B}$ such that $p \in B^n_j \subset B^{n_1}_{k_1} \cap B^{n_2}_{k_2}$. Thus, $\mathcal{B}$ is a basis of a topology on X, and it is easy to show that $\mathcal{B}$ generates the metric topology on X induced by d.
(b) If $D = \{a_n\}_{n\in\mathbb{N}}$ is a countable dense subset of X, then $\{B_d(a_n, 1/k) : k \in \mathbb{N}\}$ is a basis of the topology on X. □

Example 3.1.6. Imposing $C^0([0,1],\mathbb{R})$ the sup metric so that $C^0([0,1],\mathbb{R})$ is considered a metric space, $C^0([0,1],\mathbb{R})$ is second countable, for it has a countable subset

$$\mathcal{S} := \left\{ f : [0,1] \to \mathbb{R} : \begin{array}{c} f \text{ is a linear spline with the rational knots } t_1, \cdots, t_k \text{ for} \\ \text{some } k \in \mathbb{N} \text{ and } f(t_i) \text{ is rational for } i = 1, \cdots, k \end{array} \right\}.$$

Given $f \in C^0([0,1],\mathbb{R})$ and a real number $\epsilon > 0$, there is a real number $\delta > 0$ such that $x, y \in [0,1]$ with $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. If P is any partition of $[0,1]$ with the norm less than δ, let s be a spline in $\mathcal{S}$ whose value at each knot t of s is a rational number satisfying $|s(t) - f(t)| < \epsilon$. Then, whenever $t \in [0,1]$ and t_i is a knot of s such that $|t - t_i| < \delta$, we have

$$|f(t) - s(t)| \le |f(t) - f(t_i)| + |f(t_i) - s(t_i)| + |s(t_i) - s(t)| < \epsilon + 2\epsilon + 2\epsilon = 5\epsilon.$$

This proves that $f \in B_{\overline{\rho}}(s, 5\epsilon)$, so $C^0([0,1],\mathbb{R})$ is separable. Therefore, the metric space $C^0([0,1],\mathbb{R})$ is second countable.

In fact, the collection $\mathcal{S}$ is a countable dense subset of $\mathbb{R}^{[0,1]}$, so $\mathbb{R}^{[0,1]}$ is separable. Later in this chapter, one can justify that $\mathbb{R}^{[0,1]}$ is not second-countable. Because $\mathbb{R}$ is regular, so is $\mathbb{R}^{[0,1]}$ is a regular space. Hence, if $\mathbb{R}^{[0,1]}$ is second-countable, then the Urysohn metrization theorem implies that $\mathbb{R}^{[0,1]}$ is metrizable, which is a contradiction.

We now introduce some inheritance properties.

Proposition 3.1.7 (Inheritance of countabilities)**.** Some countabilities are preserved as follow:

(a) First and second countabilities are inherited to subspaces and countable product spaces.

(b) Every closed subspace of a Lindelöf space is a Lindelöf space.

(c) Every open subspace of a separable space is a separable space.

Proof. See Problem 3.2.2. □

Proposition 3.1.8 (Inheritance of separabilities)**.** Some separabilities are preserved as follow:

(a) A subspace of a Hausdorff (regular, completely regular) space is a Hausdorff (regular, completely regular) spaces. (For a normal space X, a *closed* subspace of X is normal.)

(b) A product of Hausdorff (regular, completely regular) spaces is a Hausdorff (regular, completely regular) spaces.

(c) If the product space $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is a Hausdorff (regular, normal) space, then so is each X_α for $\alpha \in \mathcal{A}$.

Proof. See Problems 3.2.3 to 3.2.6. □

3.1.3 Examples of regular spaces

Example 3.1.9 (Ordered spaces are regular)**.** Let X be an ordered space, and let $U = (a, b)$ be a basis member in X containing $p \in X$.

(i) Suppose U consists of only one point p. Then, U is both open and closed in X, so we may choose $V = U$.

(ii) Suppose that (a, p) or (p, b) is empty and the other is nonempty. Without loss of generality, we assume that (p, b) is empty and (a, p) is nonempty. Choose a point $a' \in (a, p)$ and let $V = (a', b) = (a', p]$. Any point $x \in X$ with $x > p$ does not belong to the closure of V in X; (p, ∞) does not intersect U. Any point $x \in X$ with $x < a'$ does not belong to the closure of V in X; $(-\infty, a')$ does not intersect V. Hence, the closure of V in X is contained in $[a', p] \subset U$.

(iii) Suppose (a, p) and (p, b) are nonempty. Let a' and b' be points of X in (a, p) and (p, b), respectively, and let $V = (a', b')$. Clearly, V is a neighborhood of p in X, and it is easy to check that the closure of V in X is contained in U.

Therefore, every ordered space is a regular space. In fact, it is known that every ordered space is a normal space, whose proof will not be introduced in this note.

Example 3.1.10 (Locally compact Hausdorff spaces are (completely) regular)**.** Suppose X is a locally compact Hausdorff space, and let Y be a compact Hausdorff space containing X such that $Y \setminus X$ is a singleton. Then Y is normal, (completely) regular (illustrated in the following subsection), so X is also (completely) regular.

3.1.4 Examples of normal spaces

Theorem 3.1.11. Lindelöf regular spaces are normal.

Proof. Let A and B be disjoint closed subspaces of X. For each $a \in A$, let S_a be a neighborhood of a in X contained in $X \setminus B$. Using regularity, let U_a be a neighborhood of a in X whose closure in X is contained in S_a. And construct V_b for each $b \in B$ as we constructed U_a for each $a \in A$.

Even if $\{U_a\}_{a\in A}$ and $\{V_b\}_{b\in B}$ are open covers of A and B, respectively, the unions of the members in each collection need not be disjoint. (After drawing pictures on a sketchbook) one may wish to set

$$F_n := U_n \setminus \bigcup_{k=1}^{n} \overline{V_k}, \quad G_n := V_n \setminus \bigcup_{k=1}^{n} \overline{U_k}$$

for each $n \in \mathbb{N}$, and

$$F := \bigcup_{n=1}^{\infty} F_n, \quad G := \bigcup_{n=1}^{\infty} G_n.$$

We wish F and G to be disjoint neighborhoods of A and B, respectively. Clearly, F and G are neighborhoods of A and B, since

$$F = \bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} \overline{V_n} \supset A, \quad G = \bigcup_{n=1}^{\infty} V_n \setminus \bigcup_{n=1}^{\infty} \overline{U_n} \supset B.$$

To show $F \cap G = \varnothing$, suppose $F_n \cap G_k \neq \varnothing$ for some $n, k \in \mathbb{N}$; without loss of generality, we may assume $n \leq k$. If $p \in F_n \cap G_k$, then $p \in F_n \subset U_n$; on the other hand, G_k does not intersect $\overline{U_n}$, hence $p \notin G_k$. This proves that F and G separate A and B, so X is a normal space. □

Theorem 3.1.12. Metrizable spaces are normal.

Proof. Let X be a metrizable space and d be a metric on X inducing the topology on X. Let A and B be nonempty and disjoint closed subspaces of X. For each $a \in A$ and $b \in B$, let s_a and t_b be positive real numbers such that $B_d(a, s_a) \subset X \setminus B$ and $B_d(b, t_b) \subset X \setminus A$. Define

$$U := \bigcup_{a\in A} B_d(a, s_a/3), \quad V := \bigcup_{b\in B} B_d(b, t_b/3).$$

Then U and V are neighborhoods of A and B in X. If $U \cap V$ is nonempty, then for some $a \in A$ and $b \in B$, $B_d(a, s_a/3) \cap B_d(b, t_b/3)$ contains a point $z \in X$. We then have the following inequality:

$$d(a, b) \leq d(a, z) + d(z, b) < \frac{s_a}{3} + \frac{t_b}{3} < \max\{s_a, t_b\},$$

so $b \in B_d(a, s_a)$ or $a \in B_d(b, t_b)$, a contradiction. Therefore, $U \cap V$ is empty, proving that X is normal. □

Theorem 3.1.13. Well-ordered sets are normal. (In fact, every ordered space is normal.)

Proof. Let X be a well-orderd set.

Step 1: Proving that every subspace of X of the form $(x, y]$ is open in X.
Let $A = (x, y]$ be a subset of X. Then $X \setminus A = (-\infty, x] \sqcup (y, \infty)$; because (y, ∞) has the least element $y' \in X$, $X \setminus A = (-\infty, x] \sqcup [y', \infty)$ is closed in X.

Step 2: Proving that X is normal.
Let A and B be nonempty disjoint closed subspace of X.

(i) Assume that neither of them contains the least element m of X. For each $a \in A$, there is a neighborhood $(x_a, a]$ of a in X not intersecting B; for each $b \in B$, there is a neighborhood $(y_b, b]$ of b in X not intersecting A. Define

$$U := \bigcup_{a\in A} (x_a, a], \quad V := \bigcup_{b\in B} (y_b, b].$$

They are open in X and U covers A, and V covers B. If U and V are not disjoint, $(x_a, a] \cap (y_b, b]$ is nonempty for some $a \in A$ and $b \in B$, Without loss of generality, we may assume $a < b$; then we have $a \in (y_b, b]$, a contradicton.

(ii) Suppose, without loss of generality, A contains m. Because the singletone $\{m\} = [m, m] = (-\infty, m]$ is not only closed but also open in X, $A \setminus \{m\}$ is also closed in X. By the preceeding part, there are disjoint neighborhood U and V of $A \setminus \{m\}$ and B, respectively. Then, $U \cup \{m\}$ and $V \setminus \{m\}$ are disjoint neighborhoods of A and B, respectively.

Therefore, every well-ordered space is normal. □

Example 3.1.14 ($\mathbb{R}_l$ is normal)**.** Suppose A and B are nonempty disjoint closed subsets of $\mathbb{R}_l$. For each $a \in A$, let $[a, x_a)$ be a neighborhood of a in $\mathbb{R}_l$ contained in $\mathbb{R}_l \setminus B$, and for each $b \in B$, let $[b, y_b)$ be a neighborhood of b in $\mathbb{R}_l$ contained in $\mathbb{R}_l \setminus A$. Define

$$U := \bigcup_{a\in A} [a, x_a) \quad \text{and} \quad V := \bigcup_{b\in B} [b, y_b),$$

and assume $U \cap V \neq \varnothing$. Then $[a, x_a) \cap [b, y_b)$ is nonempty for some $a \in A$ and $b \in B$. Since $a \neq b$, we may assume $a < b$, which implies that $a < b < x_a < y_b$ and that b is contained in $[a, x_a)$, a contradiction. Therefore, $U \cap V = \varnothing$, proving that $\mathbb{R}_l$ is a normal space.

Example 3.1.15 (Compact Hausdorff spaces are normal)**.** Because closedness and compactness coincide in compact Hausdorff spaces and Hausdorff spaces are compactly normal, compact Hausdorff spaces are normal.

3.2 Countability and separation axioms - Part 2

In this section, we investigate some more properties regarding the countabilities and the separabilities of topological spaces. Such properties will be introduced in the form of problems.

Problem 3.2.1. Show that the suggested alternative definition for regular, completely regular, and normal spaces are equivalent to the original definitions.

Solution. We use the overline notation to denote the closure in X.

(a) Assume $B = \{x\}$ for a given point $x \in X$ in the proof of (c).

(b) (In this case, the proof is quite simple, for we do not have to consider larger subsets.) Let X be a completely regular space and suppose that a point x of X and its neighborhood U in X is given. Then $B = X \setminus U$ is closed in X, so there is a continuous map $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(B) = \{0\}$.

Assume conversely, and let x be a point of X and B be a closed subset of X not containing x. Then $U = X \setminus B$ is a neighborhood of x in X, so there is a continuous map $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) = \{0\}$.

(c) Let X be a normal space and suppose that a closed subset B of X and its neighborhood U in X are given. Then $C = X \setminus U$ is closed in X, so there are disjoint neighborhoods V of B and W of C in X. Then $B \subset V \subset X \setminus W$ and $\overline{V} \subset \overline{X \setminus W} = X \setminus W \subset X \setminus C = U$.

Assume conversely, and let B and C be disjoint closed subsets of X. If V is a neighborhood of B in X such that $\overline{V} \subset X \setminus C$, then V and $X \setminus \overline{V}$ are disjoint neighborhood of B and C in X.

Problem 3.2.2. Prove Proposition 3.1.7.

Solution. (a) Let Y be a subspace of X. If $\{B_n\}_{n\in\mathbb{N}}$ is a countable base at $p \in Y$ in X, then $\{B_n \cap Y\}_{n\in\mathbb{N}}$ is a countable base at p in Y; if $\{B_n\}_{n\in\mathbb{N}}$ is a countable basis of the topology on X, then $\{B_n \cap Y\}_{n\in\mathbb{N}}$ is a countable basis of the topology on Y.

For spaces $X_1, X_2, \cdots$, let $\mathcal{B}_n = \{B_{n,k}\}_{k\in\mathbb{N}}$ be a countable base at $x_n \in X_n$ in X_n. Then

$$\left\{ \prod_{j\in\mathbb{N}} U_j : \begin{array}{c} U_j \in \mathcal{B}_j \text{ for each } j \in \mathbb{N} \text{ and} \\ U_j = X_j \text{ for all but finitely many values of } j \end{array} \right\}$$

is a countable base at $x \in \prod_{n\in\mathbb{N}} X_n$, where $\pi_n(x) = x_n$ for all $n \in \mathbb{N}$. When each $\mathcal{B}_n$ is a countable basis of the topolgy on X_n, then the above collection is a countable basis of the topology on $\prod_{n\in\mathbb{N}} X_n$.

(b) Let Y be a closed subspace of a Lindelöf space X, and let $\{A_\alpha\}_\alpha$ be a covering of Y by sets open in Y (then, for each α, $A_\alpha = Y \cap O_\alpha$ for an open subset O_α of X). Since $X \setminus Y$ is open in X, countably many members A_α, together with $X \setminus Y$, cover X, as desired.

(c) Let X be a separable space and Y be an open subspace of X. And let D be a countable dense subset of X, and use the overline notation to denote the closure in X. We wish to show that $D \cap Y$ is a (countable) dense subset of Y, i.e., $\overline{D \cap Y} \cap Y = Y$. For this, it suffices to show $Y \subset \overline{D \cap Y}$. In fact, if $p \in Y$ and whenever V is a neighborhood of p in X contained in Y, then V contains a point of $D \cap Y$. Hence, every point of Y belongs to $\overline{D \cap Y}$, as desired.

Problem 3.2.3. Prove Proposition 3.1.8 for Hausdorff spaces.

Solution. All (a), (b), and (c) are obvious.

Problem 3.2.4. Prove Proposition 3.1.8 for regular spaces.

Solution. (a) Let X be a regular space and Y be a subspace of X. Assume that $p \in Y$ and C is a closed subset of Y not containing p. Because $C = Y \cap A$ for some subset A closed in X and A does not contain p, by regularity of X, there are disjoint neighborhoods U of p and V of A in X. Then $Y \cap U$ and $Y \cap V$ are disjoint neighborhoods of p and C in Y.

(b) Let X_α be a regular space for each $\alpha \in \mathcal{A}$ (here, $\mathcal{A}$ is an index set), and let $X = \prod_{\alpha\in\mathcal{A}} X_\alpha$. To justify the regularity of X, we make use of the alternative definition. Let x be a point of X and U be a neighborhood of x in X. Here, we may assume that $U = \prod_{\alpha\in\mathcal{A}} B_\alpha$, where B_α is open in X_α for each $\alpha \in \mathcal{A}$ and $B_\alpha = X_\alpha$ for all but finitely many values of $\alpha \in \mathcal{A}$. For each $\alpha \in \mathcal{A}$, use the regularity of X_α to find a neighborhood V_α of x_α in X_α whose closure in X_α is contained in B_α (when $B_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$). Then $V = \prod_{\alpha\in\mathcal{A}} V_\alpha$ is a neighborhood of x in X whose closure is contained in U.

(c) Given $\alpha \in \mathcal{A}$, let p_α be a point of X_α and C_α be a closed subset of X_α not containing p_α. Let x be any point of X whose α-component is p_α and let $K = \pi_\alpha^{-1}(C_\alpha)$. Because K is closed in X and does not contain x, by the regularity of X, there are disjoint neighborhoods U of x and V of K in X. Applying the result of the following remark (together with the homeomorphism), we can conclude that X_α is regular.

Remark (An embedding of a space into a product space). Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of topological spaces, and assume that X_α is a singleton whenever $\alpha \neq \beta$. Then X_β is homeomorphic to $X = \prod_{\alpha\in\mathcal{A}} X_\alpha$. Indeed, one can impose an explicit homeomorphism. Let s_α be the unique element of S_α for each $\alpha \neq \beta$, and let $\imath : X_\beta \to X$ be the map defined by

$$(\pi_\alpha \circ \imath)(x) = \begin{cases} x & (\text{if } \alpha = \beta) \\ s_\alpha & (\text{otherwise}) \end{cases} \quad \text{for all } x \in X_\beta.$$

Then $\imath$ is a desired homeomorphism. As an example, $\mathbb{R}^2$ and the plane $\{(x, y, z) \in \mathbb{R}^3 : y = 17\} \approx \mathbb{R}^2 \times \{17\}$ are homeomorphic.

Problem 3.2.5. Prove Proposition 3.1.8 for completely regular spaces.

Solution. (a) Let Y be a subspace of a completely regular space X, and let p be a point of Y and A be a closed subset of Y. Then $A = Y \cap B$ for some closed subset B of X, and B does not contain p. By the complete regularity of X, there is a continuous map $f : X \to [0,1]$ separating p and B. The restriction of f to Y separates p and A.

(b) Let X_α be a completely regular space for each $\alpha \in \mathcal{A}$ (here, $\mathcal{A}$ is an index set), and let $X = \prod_{\alpha\in\mathcal{A}} X_\alpha$. Let p be a point of X and U be a neighborhood of p in X. We may assume that $U = \prod_{\alpha\in\mathcal{A}} B_\alpha$, with B_α being an open subset of X_α for each $\alpha \in \mathcal{A}$ and $B_\alpha = X_\alpha$ for all but $\alpha_1, \cdots, \alpha_n \in \mathcal{A}$. For each $i = 1, \cdots, n$, let $f_i : X_{\alpha_i} \to [0,1]$ be a continuous map such that $f_i(p_{\alpha_i}) = 1$ and $f_i(X_{\alpha_i} \setminus B_{\alpha_i}) = \{0\}$. Then, the map $f : X \to [0,1]$ defined by $f(x) = f_1(x_{\alpha_1}) \times \cdots \times f_n(x_{\alpha_n})$ for $x \in X$ is a continuous map with the properties that $f(p) = 1$ and $f(X \setminus U) = \{0\}$.

Problem 3.2.6. Prove Proposition 3.1.8 for normal spaces.

Solution. (a) Suppose Y is a closed subspace of a normal space X. If A and B are disjoint closed subspaces of Y, then they are closed in X, too. By normality, there are disjoint open subspaces U and V in X containing A and B, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y containing A and B.

(c) Given $\alpha \in \mathcal{A}$, let B_α and C_α be disjoint closed subsets of X_α, and let $B = \pi_\alpha^{-1}(B_\alpha)$ and $C = \pi_\alpha^{-1}(C_\alpha)$. Because $B \cap C = \varnothing$, by the regularity of X, there are disjoint neighborhoods U of B and V of B in X. Again, applying the natural homeomorphism given in the previous remark, one can conclude that X_α is normal.

Problem 3.2.7. Let X be a completely regular space and A, B be disjoint closed subsets of X. Show that there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, if A is compact.

Solution. For each point $a \in A$, let $f_a : X \to [0,1]$ be a continuous function such that $f_a(a) = 0$ and $f_a(B) = \{1\}$. Note that the collection $\{f_a^{-1}([0,r)) : a \in A\}$ is an open cover of A by sets open in X, where r is any real number satisfying $0 < r < 1/2$. Thus, there are finitely many points $a_1, \cdots, a_n$ of A such that $\bigcup_{j=1}^n f_{a_i}^{-1}([0,r))$ covers A. Define the function $f : X \to [0,1]$ by $f(x) = f_1(x) \times \cdots \times f_n(x)$ for $x \in X$. Then $0 \leq f(a) < r$ for all $a \in A$ and $f(B) = \{1\}$. If $g : [0,1] \to [0,1]$ is the function defined by

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < r \\ \dfrac{x-r}{1-r} & \text{otherwise} \end{cases},$$

then the composite $g \circ f$ is a desired continuous map from X into $[0,1]$.

Problem 3.2.8. Let $f, g : X \to Y$ be continuous maps, where Y is a Hausdorff space. Show that $\{x \in X : f(x) = g(x)\}$ is a closed subspace of X.

Solution. We prove the closedness of the given subset by proving that $A := \{x \in X : f(x) \neq g(x)\}$ is open in X. Note that $A = \bigcup_{a,b\in Y,\, a\neq b}(f^{-1}(\{a\}) \cap g^{-1}(\{b\}))$. If, for each pair (a,b) of distinct points of Y, U_a and V_b are disjoint neighborhoods of a and b in Y, we have

$$A = \bigcup_{a,b\in Y,\, a\neq b} (f^{-1}(U_a) \cap g^{-1}(V_b)),$$

because $f^{-1}(U_a) \cap g^{-1}(V_b) = \bigcup_{s\in U_a} \left(f^{-1}(\{s\}) \cap g^{-1}(V_b)\right) = \bigcup_{s\in U_a,\, t\in V_b}(f^{-1}(\{s\}) \cap g^{-1}(\{t\}))$ for each pair (a,b). It easily follows that A is open in X.

Remark. Let X be a Hausdorff space and a continuous map $r : X \to A$ be a retraction. Because $A = \{x \in X : r(x) = id_X(x)\}$, so A is closed in X.

Problem 3.2.9. For a topological space X and its subset A, we say that A is *discrete* if every point of A is an isolated point of A in X. Show that an uncountable subset of a second countable space is indiscrete, i.e., every uncountable subset of a second countable space has a limit point.

Solution. Let $\mathcal{B}$ be a countable basis of the second countable space X, and suppose A is an uncountable discrete subset of X. For each point x in A, let $B_x \in \mathcal{B}$ be chosen so that $B_x \cap A = \{x\}$. Since the collection of B_x for $x \in A$ is countable, there are at most countably many distinct singletons $\{x\}$ for $x \in A$, a contradiction.

3.3 The Urysohn lemma

Theorem 3.3.1 (Urysohn lemma)**.** If X is a normal space, then any two nonempty disjoint closed subsets of X can be separated by a continuous function on X. To be precise, if A and B are nonempty disjoint closed subsets of X, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Remark. By the Urysohn lemma, every pair of closed subsets of X can be separated by disjoint open subsets of X (in short, X is normal) if and only if each such pair can be separated by a continuous function. (Here, if part is obvious.)

Proof. Let A and B be nonempty and disjoint closed subsets of X.

Step 1. Let P be the set of rational numbers in $[0, 1]$ and write $P = \{x_1, x_2, x_3, \cdots\}$ with $x_1 = 1$ and $x_2 = 0$. Define an open subset U_p for each $p \in P$ as follows:

(1) First, let $U_1 := X \setminus B$, which contains A. Second, by normality, we can find a neighborhood U_0 of A in X whose closure in X is contained in U_1.

(2) Suppose, for some positive integer $n \geq 2$, that we have constructed U_p for all $p \in P_n = \{x_1, \cdots, x_n\}$ satisfying the following rule, which is satisfied when $n = 2$:

$$\text{Whenever } p,\, q \in P_n \text{ and } p < q,\ \overline{U_p} \subset U_q. \tag{3.1}$$

Write $r = p_{n+1}$. We want U_r to satisfy eq. (3.1) with all U_p with indices in P_{n+1}. In this case, it suffices to care the immediate successor s and predecessor p of r in P_{n+1}; if U_r satisfies eq. (3.1) with U_p and U_s, then U_r satisfies eq. (3.1) with all the other U_p with $p \in P_{n+1}$. Since U_p and U_s satisfies $\overline{U_p} \subset U_s$, by normality, there is a neighborhoof U_r of $\overline{U_p}$ in X such that $\overline{U_r} \subset U_s$. For such U_r, eq. (3.1) is satisfied for P_{n+1}.

By induction, U_p can be defined for all $p \in P$ with eq. (3.1) being satisfied.

Step 2. We now define U_p for all rational numbers p as follows: Just let $U_p = \varnothing$ whenever $p < 0$, and $U_p = X$ whenever $p > 1$. Even in this case, eq. (3.1) is satisfied.

Step 3. For each point $x \in X$, define the set

$$C(x) := \{p \in \mathbb{Q} : x \in U_p\}.$$

For each x, the set $C(x)$ is bounded below by a nonnegative real number and it contains 1. Thus, the function $f : X \to [0, 1]$ defined by $f(x) = \inf C(x)$ for all $x \in X$ is well-defined.

Step 4. We now show that the function f is a continuous map mapping A onto $\{0\}$ and B onto $\{1\}$. First, since every point $a \in A$ belongs to U_0, $f(a) = 0$; since every point $b \in B$ does not belong to U_1, $f(b) = 1$. Before proving continuity, we note the following lemma:

$$x \in \overline{U_r} \text{ implies } f(x) \leq r, \text{ and } x \notin U_r \text{ implies } f(x) \geq r. \tag{3.2}$$

To check the continuity, fix a point $x \in X$ and let (a, b) be a neighborhood of $f(x)$ in $\mathbb{R}$. We want to find a neighborhood W of x in X whose image under f is contained in (a, b).[1] Using the density of $\mathbb{Q}$, we can find rational numbers c, d such that $a < c < f(x) < d < b$. Now, it suffices to find a neighborhood of x in X whose image under f is a subset of $[c, d]$. By the above lemma, if $c < f(x) < d$, then $x \in U_d \setminus \overline{U_c}$. Hence, $U_d \setminus \overline{U_c}$ is a neighborhood of x in X, and $f(U_d \setminus \overline{U_c}) \subset [c, d]$. Therefore, f is a continuous function from X into $[0, 1]$ mapping A into $\{0\}$ and B onto $\{1\}$. □

Proof of eq. (3.2)*.* If $x \in \overline{U_r}$, then $x \in U_s$ (i.e., $s \in C(x)$) for all $s \in \mathbb{Q}$ satisfying $r < s$, so $f(x) \leq r$. If $x \notin U_r$, then $x \notin U_p$ (i.e., $p \notin C(x)$) for all $p \in \mathbb{Q}$ satisfying $p \leq r$, so $f(x) \geq r$. □

In Problem 2.1.2, we have proved that every connected metrizable space having more than one point is uncountable. As we have studied in the preceeding section that every metrizable space is normal, one might want to generalize the result to a broader category of spaces. In the following problem, we will prove that every connected regular space with more than one point is uncountable.

[1] When trying to prove continuity with the original definition of continuity, you might encounter some technical problem.

Problem 3.3.1 (Nontrivial connected regular spaces are uncountable)**.** Prove that a connected normal space having more than one point is uncountable. And deduce that a connected regular space having more than one point is uncountable.

Solution. Assume X is a connected normal space having more than one point. By the Urysohn lemma, there is a continuous map $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, where A and B are nonempty disjoint closed subsets of X. Since X is connected, $f(X)$ is connected so $f(X) = [0,1]$, implying that X is uncountable.

Suppose there is a connected regular space X which is countable. It is obvious that X is a Lindelöf space, so X is a Lindelöf regular space; a normal space. It contradicts to the former result.

Another interesting property, regarding a partition of unity, will be introduced.

Definition 3.3.2 (Partition of unity)**.** Let $\{U_1, \cdots, U_n\}$ be a finite open covering of a topological space X. The collection of continuous functions

$$\phi_i : X \to [0,1] \text{ for } i = 1, \cdots, n$$

is called a partition of unity dominated by $\{U_1, \cdots, U_n\}$ if both of the following conditions are satisfied:

(i) The support of each ϕ_i is contained in U_i for $i = 1, \cdots, n$.

(ii) $\sum_{i=1}^{n} \phi_i(x) = 1$ for all $x \in X$.[2]

Remark. Some assumptions in the above definition is due to technical problems.

(a) The condition that $\{U_1, \cdots, U_n\}$ should cover X is obviously essential for defining a partition of unity. Otherwise, for any point $x \in X \setminus \bigcup_{i=1}^{n} U_i$, we have $\phi_i(x) = 0$ for $i = 1, \cdots, n$.

(b) The openness of each U_i is essential; otherwise, for example, when $U_1 = X = [0,1]$, it is allowed by assumption that the support of ϕ_1 to be X, such as $\phi_1 = id_X$, and this setting is problematic at the point 0 in X. Moreover, when the space X is assumed to be normal; in this case, there is a neighborhood of the support of ϕ_i in X which is contained in U_i, or one may apply the Urysohn lemma for the support of ϕ_i and $X \setminus U_i$.

Theorem 3.3.3 (Existence of finite partitions of unity)**.** Let $\{U_1, \cdots, U_n\}$ be a finite open covering of the normal space X. Then there is a partition of unity dominated by $\{U_1, \cdots, U_n\}$.

Proof. The main idea of the proof is to find an open cover $\{V_1, \cdots, V_n\}$ such that $\overline{V_i} \subset U_i$ for $i = 1, \cdots, n$, where, as usual, the overline is used to denote the closure in X. If this is proved, we may make use of another open over $\{W_1, \cdots, W_n\}$ such that $\overline{W_i} \subset V_i$ for $i = 1, \cdots, n$. Then, for each $i = 1, \cdots, n$, by the Urysohn lemma, there is a continuous map $\phi_i : X \to [0,1]$ such that $\phi_i(\overline{W_i}) = \{1\}$ and $\phi_i(X \setminus V_i) = \{0\}$; in this case, the support of ϕ_i is contained in the closure of V_i, which is contained in U_i, by construction. Because $\{W_1, \cdots, W_n\}$ covers X, the function $\Phi : X \to [0, \infty)$ defined by $\Phi = \sum_{i=1}^{n} \phi_i$ is positive, which proves the existence of finite partitions of unity.

To complete the proof, it remains to justify the shrinking process can be done. The idea is to observe the portion of X which can be covered by only one member of $\{U_1, \cdots, U_n\}$. Let $A = X \setminus (U_2 \cup \cdots \cup U_n)$. Then A_1 is a closed subset of X which is contained in U_1. Hence, by normality, there is a neighborhood V_1 of A_1 whose closure in X is contained in U_1, and we have $X = V_1 \cup U_2 \cup \cdots \cup U_n$. In general, for an open cover $\{V_1, \cdots, V_{k-1}, U_k, U_{k+1}, \cdots, U_n\}$, define $A_k = X \setminus (V_1 \cup \cdots \cup V_{k-1}) \setminus (U_{k+1} \cup \cdots \cup U_n)$. Then A_k is a closed subset of X contained in U_k. Hence, by normality, there is a neighborhood V_k of A_k whose closure in X is contained in U_k, and we have $X = V_1 \cup \cdots V_k \cup U_{k+1} \cup \cdots U_n$. By induction, the shrinking process is justified. ☐

[2]This condition can be alleviated when $\Phi := \sum_{i=1}^{n} \phi_i > 0$, since we may consider ϕ_i/Φ in place of ϕ_i.

3.4 The embedding theorem

In this section, we introduce the embedding theorem, stating that every completely regular space embeds into a (possibly uncountable dimensional) Euclidean space $\mathbb{R}^I$ for some index set I. As an application of the embedding theorem, we will prove the Urysohn metrization theorem, which states that every second-countable regular space is metrizable. When proving the Urysohn metrization theorem, the fact that a countable dimensional Euclidean space $\mathbb{R}^{\mathbb{N}}$ is metrizable will be applied.

Theorem 3.4.1 (The embedding theorem)**.** Let X be a completely regular space and let $\{f_\alpha\}_{\alpha\in I}$ be a collection of continuous functions from X into $\mathbb{R}$ separating points and closed subsets of X. Define the function $F : X \to \mathbb{R}^I$ by $F = (f_\alpha)_{\alpha\in I}$. Then F denotes an embedding of X into $\mathbb{R}^I$. Furthermore, if each f_α maps X into $[0, 1]$, then F denotes an embedding of X into $[0, 1]^I$.

Proof. Because $\mathbb{R}^I$ (or $[0, 1]^I$) equips the product topology, F is continuous. If a and b are distinct points of X, there is an index $i \in I$ such that $f_i(a) \neq 0$ and $f_i(b) = 0$, so F is injective. Thus, to show that F is an embedding of X, it suffices to show that F maps an open subset of X onto an open subset of $F(X)$.

Assume U is an open subset of X, and let z be a point of $F(U)$, and let x be the point of U such that $F(x) = z$. The goal is to find a neighborhood of z in $F(X)$ which is contained in $F(U)$. Let $i \in I$ be an index such that $f_i(x) \neq 0$ and $f_i(X \setminus U) = \{0\}$, and set $W = \pi_i^{-1}(\mathbb{R} \setminus \{0\}) \cap F(X)$.

(i) Clearly, W is open in $F(X)$ and W contains z.

(ii) For a point $y \in W$, let a be a point in X such that $F(a) = y$. Since $y \in \pi_i^{-1}(\mathbb{R} \setminus \{0\})$, we have $f_i(a) = \pi_i(y) \neq 0$, so $a \notin X \setminus U$, implying $y \in F(U)$. In other words, $W \subset F(U)$.

So $F(U)$ is open in $F(X)$, which proves that F is an embedding of X into $\mathbb{R}^I$ (or into $[0, 1]^I$). □

We now introduce the Urysohn metrization theorem, whose result is stronger than the result of the Urysohn lemma.

Theorem 3.4.2 (Urysohn metrization theorem)**.** Every second countable regular space is metrizable.

To prove this theorem, we first prove the following countability lemma.

Lemma 3.4.3. Let X be a second countable regular space. Then there is a countable family of continuous functions from X into $[0, 1]$ separating points of X from closed subsets of X.

Proof of Lemma 3.4.3. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis of the topology on X. For each pair (m, n) of positive integers for which $\overline{B_m} \subset B_n$, by the Urysohn lemma, there is a continuous function $f_{m,n}$ from X into $[0, 1]$ such that $f(\overline{B_m}) = \{1\}$ and $f(X \setminus B_n) = \{0\}$. The collection of such continuous function is countable, and it separates points of X from closed subsets of X. (Remark how we obtained alternative definitions of some separabilities of topological spaces.) □

Proof of Theorem 3.4.2. By Lemma 3.4.3 and the embedding theorem, X embeds into $\mathbb{R}^{\mathbb{N}}$, which is metrizable. □

Example 3.4.4. Suppose X is a compact Hausdorff space. If X is metrizable, one can easily deduce that X is second countable, using the compactness of X. The converse implication that X is metrizable when X is second countable is an immediate result of the Urysohn metrization theorem, because a compact Hausdorff space is normal.

In the following chapter, the embedding theorem is applied when studying the Stone-Čech compactification of a completely regular space, to which every continuous function into a compact Hausdorff space extends continuously (and uniquely[3]).

[3] The uniqueness follows from the assumption that the codomain is a Hausdorff space.

Chapter 4

Compactification of completely regular spaces

4.1 Compactification

In this section, we deals with a general class of compactification, which covers a one-point compactification.[1]

Definition 4.1.1 (Compactification)**.** Given a space X, a space Y is called a compactification of X if the following conditions are satisfied:

(i) Y is a compact Hausdorff space containing X as a subspace.

(ii) The closure of X in Y is Y.

Two compactifications Y_1 and Y_2 of X is said to be equivalent if there is a homeomorphism from Y_1 into Y_2 extending the identity map on X.

Remark that a topological space which has a compactification is necessarily completely regular. Hence, it is natural to wonder if every completely regular space has a compactification. Theorem 4.1.2 suggests that every completely regular space has a compactification; it first suggests that every space which embeds into a compact Hausdorff space has a compactification, and second that if such compactification is required to satisfy an "extension property" then such compactification is unique up to equivalence.

Theorem 4.1.2 (The compactification induced by an embedding)**.** Let X be a space which admits an embedding h of X into a compact Hausdorff space Z. Then, there is a unique (up to equivalence) compactification Y of X with the following property: There is an embedding $H : Y \hookrightarrow Z$ extending h.[2]

$$\begin{tikzcd} X \arrow[r, hook, "\text{inclusion}"] \arrow[dr, hook, "h"'] & Y \arrow[d, hook, dashed, "H"] \\ & Z \end{tikzcd}$$

Such compactification Y is called the compactification of X induced by the embedding $h : X \hookrightarrow Z$.

Remark. A compactification of a completely regular space X may not be unique; what is unique up to equivalence is a compactification of X induced by an embedding of X into a particular compact Hausdorff space.

Proof. We first prove the existence of a compactification Y of X induced by $h : X \hookrightarrow Z$, and then we prove its uniqueness.

[1] Generally, the compactification covered in this note is referred to as a Hausdorff compactification, for a compactification is required to be a Hausdorff space.

[2] Here, Y is unique up to equivalence, but H need not be unique.

Step 1. Finding a homeomorphic copy of a compactification of X.
Let $X_0 := h(X) \approx X$ and Y_0 be the closure $\overline{X_0}$ of X_0 in Z. To argue that Y_0 is a compactification of X_0, we check the axioms of compactificaitions. Because Y_0 is a closed subset of Z, Y_0 is a compact Hausdorff space. Also, Y_0 contains X_0 as a subspace; every open subset of X_0 is of the form $X_0 \cap O = X_0 \cap (Y_0 \cap O)$, where O is an open subset of Z. Finally, the closure of X_0 in Y_0 is clearly $Y_0 \cap \overline{X_0} = Y_0$. Therefore, Y_0 is a compactification of X_0.

Step 2. Finding a compactification of X.
To find a compactification Y of X, we seek to find a space Y such that (X, Y) and (X_0, Y_0) are homeomorphic. Let A be any set disjoint from X which is in bijection with $Y_0 \setminus X_0$ (say $k : A \to Y_0 \setminus X_0$ is such a bijection), and define $Y = X \sqcup A$. Define a bijective map $H : Y \to Y_0$ by

$$H(x) = \begin{cases} h(x) & (\text{if } x \in X) \\ k(x) & (\text{if } x \in A) \end{cases}$$

Topologize Y by declaring that $U \subset Y$ is open in Y if and only if $H(U)$ is open in Y_0. (Indeed, the collection induced by such declaration is a topology on Y.) This topologization makes H a homeomorphism. It is easy to justify that Y is a compactification of X; it is because H extends h and $h : X \to X_0$ and $H : Y \to Y_0$ are homeomorphisms. To be brief, it is because the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "h", "\approx"'] \arrow[d, hook, "\text{inclusion}"'] & X_0 \arrow[d, hook, "\text{inclusion}"] \\ Y \arrow[r, "\approx", "H"'] & Y_0 \end{tikzcd}$$

Step 3. Justifying the uniqueness of a compactification of X
Suppose Y_1 and Y_2 are compactifications of X which extends h to embeddings from Y_1 and Y_2 into Z, respectively. Denote such embeddings by $H_1 : Y_1 \hookrightarrow Z$ and $H_2 : Y_2 \hookrightarrow Z$, respectively. By restricting the codomains of H_1 and H_2, we find that $H_2 \circ H_1^{-1}$ denotes a homeomorphism between Y_1 and Y_2 extending the identity map on X. Therefore, Y_1 and Y_2 are equivalent. ☐

Corollary 4.1.3. A topological space X has a compactification if and only if X is completely regular.

Proof. Since it is already proved that a topological space with a compactification is completely regular, it remains to justify that every completely regular space has a compactification. If X is a completely regular space, by the embedding theorem, X embeds into $[0,1]^I$ for some I, which is a compact Hausdorff space. By Theorem 4.1.2, the embedding of X into $[0,1]^I$ induces a compactification of X. ☐

4.2 The Stone-Čech compactification

Throughout this section, we assume that X is a completely regular space, unless stated otherwise.

Finding a compactification Y of a completely regular space X to which every continuous map from X into $\mathbb{R}$ is continuously extended is a basic problem. Regarding this, if a given real-valued function on X were to be extended, then the function must have been bounded.

The idea to find such a compactification of X is to apply Theorem 4.1.2. To be precise, we first find an appropriate embedding of X in terms of all the bounded continuous functions on X. Theorem 4.1.2 then asserts the existence of an extended embedding of a compactification of X, which surely is in terms of the "extended" bounded continuous functions on Y. This idea is applicable, since we know from the embedding theorem that a completely regular space can be embedded into $[0,1]^I$ for some I, which is a compact space.

Write $I = C_b^0(X, \mathbb{R})$ for convinience. For each $\alpha \in I$, let $B_\alpha = [\inf(\alpha), \sup(\alpha)] \subset \mathbb{R}$, and define

$$B = \prod_{\alpha \in I} B_\alpha.$$

Since I separates points of X from closed subsets of X, the map $F : X \to B$ defined by $F = (f)_{f\in I}$ is an embedding of X into the compact Hausdorff space B. By Theorem 4.1.2, there is a unique compactification Y of X such that F extends to an embedding F_* of Y into B; namely, the compactification of X induced by the embedding F. The f-component f_* of F_* is a desired extension of f, since $f_* = \pi_f \circ F_*$ is continuous. This proves the existence of a continuous extension of f to a compactification of X. Remark that the uniqueness of a continuous extension of f to a compactification of X is guaranteed by the assumption that the codomain is a Hausdorff space.

$$\begin{tikzcd}
 & X \arrow[dl, hook', "\text{inclusion}"'] \arrow[d, "F"] \arrow[dr, "f"] & \\
Y \arrow[r, hook, "F_*"'] & B \arrow[r, "\pi_f"'] & \mathbb{R}
\end{tikzcd}$$

To sum up, by considering an embedding F with regard to $C_b^0(X, \mathbb{R})$ and considering the compactification Y of X induced by F, we could show that every function in $C_b^0(X, \mathbb{R})$ extends uniquely to a continuous function on Y. The above observation is summarized as the following proposition:

Proposition 4.2.1. Let X be a completely regular space. Then there is a compactification Y of X, to which every function $f \in C_0^b(X, \mathbb{R})$ extends continuously to Y uniquely.

We wish to consider a compactification of a completely regular space which satisfies a more general property. Such compactification would be defined in terms of a universal property.

Definition 4.2.2 (The Stone-Čech compactification of a topological space)**.** Let X be a topological space (not necessarily a completely regular space). A pair $(\beta(X), \iota)$ of a compact Hausdorff space $\beta(X)$ and a continuous map $\iota : X \to \beta(X)$ is called a Stone-Čech compactification of X, if the pair satisfies the following universal property: For any compact Hausdorff space K and a continuous map $f : X \to K$, there is a unique continuous map $f_* : \beta(X) \to K$ such that $f_* \circ \iota = f$.

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "f"'] & \beta(X) \arrow[d, dashed, "f_*"] \\
 & K
\end{tikzcd}$$

Note that a Stone-Čech compactification of a topological space X is unique up to homeomorphism (but may not be unique up to equivalence), provided that it exists. While the Stone-Čech compactification could be considered for topological spaces which are not completely regular, in this note, we only consider the Stone-Čech compactification of completely regular spaces.

Theorem 4.2.3. Let X be a completely regular space. Then a Stone-Čech compactification of X exists. Furthermore, a Stone-Čech compactification of X is unique up to equivalence.

The following proposition states that a compactification Y of X in Proposition 4.2.1, together with the inclusion of X into Y, is a Stone-Čech compactification of X.

Proposition 4.2.4 (Existence part)**.** Let X be a completely regular space and Y be a compactification of X satisfying the property in Proposition 4.2.1. Then, for any compact Hausdorff space K and a continuous map $f : X \to K$, there is a unique continuous map $f_* : Y \to K$ extending f.

Proof. Remark that a compact Hausdorff space is completely regular, so K embeds into $[0, 1]^I$ for some I. Let $e : K \hookrightarrow [0, 1]^I$ be such an embedding. To apply the result of Proposition 4.2.1, we may consider the composition $g := e \circ f : X \to e(K) \subset [0, 1]^I$ rather than f, because g is into a Euclidean space so every projection of g is into $\mathbb{R}$.

For each $\alpha \in I$, let $(g_\alpha)_* : Y \to [0, 1]$ be the unique continuous extension of $g_\alpha : X \to [0, 1]$. Then, the map $g_* : Y \to [0, 1]^I$ defined by $g_* = ((g_\alpha)_*)_{\alpha\in I}$ is the unique continuous extension of $g : X \to [0, 1]^I$.

To assert that $f_* := e^{-1} \circ g_*$ is a desired extension of f, it suffices to show that f_* is well-defined, by justifying that $g_*(Y)$ is contained in $e(K)$. Because the closure of X in Y is Y and $e(K)$ is compact,

$$g_*(Y) \subset \overline{g_*(X)} = \overline{g(X)} \subset \overline{e(K)} = e(K).$$

Therefore, f_* is well-defined and f_* is a desired extension of $f : X \to K$ to Y. The uniqueness follows from the assumption that K is a Hausdorff space. □

Proposition 4.2.5 (Uniqueness part)**.** Let X be a completely regular space.

(a) A Stone-Čech compactification of X is unique up to homeomorphism.

(b) A compactification of X which is a Stone-Čech compactification of X is unique up to equivalence.

Proof. (a) is clear, since the Stone-Čech compactification of a topological space is defined by a universal property. (b) can be similarly proved, if one applies the assumption that the given maps are inclusions. □

So far, we have proved the existence and the uniqueness of the Stone-Čech compactification of a completely regular space. In fact, it is known that the Stone-Čech compactification exists not only for a completely regular space but also for a topological space.

Here are some basic properties regarding the Stone-Čech compactification of a completely regular space. When we constructed the Stone-Čech compactification of a completely regular space, we adopted a compactification of the space so that the map of the space into the compactification is the inclusion (an embedding, to be general). The following proposition states that the map from the space into its Stone-Čech compactification is always an embedding.

Proposition 4.2.6. Let X be a completely regular space and $(Y, \imath)$ be the Stone-Čech compactification of X. Then $\imath$ is an embedding of X into Y.

Proof. By Proposition 4.2.4, there is a compactification Y of X which is a Stone-Čech compactification of X. Let $(\beta(X), \imath)$ be a Stone-Čech compactification of X and $\jmath : X \to Y$ be the inclusion. Let $\jmath_* : Y \to \beta(X)$ be the unique continuous map such that $\jmath_* \circ \imath = \jmath$, which is induced by the universal property of the Stone-Čech compactification. Since $\beta(X)$ and Y are homeomorphic, it follows that $\jmath_*$ is a homeomorphism, proving that $\jmath$ is an embedding. □

Next, we investigate the maximality of the Stone-Čech compactification of a completely regular space.

Proposition 4.2.7. Let X be a completely regular space and Y be any compactification of X. Then there is a continuous closed surjective map $g : \beta X \to Y$, where $(\beta X, \imath)$ is the Stone-Čech compactification of X. (In fact, if $\jmath : X \hookrightarrow Y$ is the inclusion map, the continuous map $f : \beta X \to Y$ induced by $\jmath$ is a continuous closed surjective map.)

Remark. Since a continuous map which is closed and surjective is a quotient map, it follows that every compactification of X is homeomorphic to a quotient of βX.

Proof. Let $\imath$ be the embedding of X into βX and $\jmath$ be the inclusion of X into Y. By the universal property of the Stone-Čech compactification, there is a unique continuous map $f : \beta X \to Y$ such that $f \circ \imath = \jmath$. Since any continuous map from a compact space into a Hausdorff space is a closed map, f is a closed map. Because the image of f is a closed subset of Y containing $(f \circ \imath)(X) = \imath(X) = X$ and X is dense in Y, we conclude that f is surjective. □

An interesting example is introduced.

Example 4.2.8. Remark that $\mathbb{N}$ equips the discrete topology, so every set map from $\mathbb{N}$ into any topological space is continuous; in particular, $\mathbb{N}$ is completely regular. In this example, we prove that card$(\beta\mathbb{N}) \geq$ card(I^I), where $I = [0, 1] \subset \mathbb{R}$ and $(\beta X, \iota)$ is the Stone-Čech compactification of $\mathbb{N}$.

Let $D = \{d_n\}_{n \in \mathbb{N}}$ be a countable dense subset of I^I (Why is I^I separable?) and let $f : \mathbb{N} \to I^I$ be the map defined by $n \mapsto d_n$ for all $n \in \mathbb{N}$. Since I^I is a compact Hausdorff space, by the universal property of the Stone-Čech compactification, there is a unique continuous map $f_* : \beta\mathbb{N} \to I^I$ such that $f_* \circ \iota = f$. Since the image of f_* is a closed subset of I^I containing $f(\mathbb{N}) = D$, we find that card$(\beta\mathbb{N}) \geq$ card(I^I).

We end this section with defining a functor from **Top** to **CHaus**, where **CHaus** is the category of the compact Hausdorff spaces, with morphisms being the continuous maps of compact Hausdorff spaces into compact Hausdorff spaces. For a topological space X, let $\beta(X)$ be the Stone-Čech compactification of X. For a continuous map $f : X \to Y$, define $\beta(f) : \beta(X) \to \beta(Y)$ as follows. Let $(\beta(X), \imath)$ and $(\beta(Y), \jmath)$ be the Stone-Čech compactifications of X and Y, respectively. Then there is a unique continuous map $(\jmath \circ f)_* : \beta(X) \to \beta(Y)$ satisfying $(\jmath \circ f)_* \circ \imath = \jmath \circ f$; define $\beta(f) = (\jmath \circ f)_*$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\imath"'] \arrow[dr, "\jmath \circ f"] & Y \arrow[d, "\jmath"] \\ \beta(X) \arrow[r, "\beta(f) := (\jmath \circ f)_*"'] & \beta(Y) \end{tikzcd}$$

Note that the above correspondence is well-defined (when choosing up to homeomorphism), and the universal property of the Stone-Čech compactification naturally explains the functoriality of β. Hence, one may argue as follows.

Proposition 4.2.9. The map β : **Top** $\to$ **CHaus** defined above is a covariant functor.

4.3 Problems

Problem 4.3.1. Let X be a completely regular space. Show that X is connected if and only if its Stone-Čech compactification is connected.

Solution. βX is connected when X is connected, since the closure of X in βX is βX and the connectedness is preserved under closing subspaces. Thus, it remains to show that X is connected when βX is connected. Suppose X is disconnected and write $X = A \sqcup B$, where A and B are nonempty disjoint open subsets of X. Let $f : X \to \{0, 1\}$ be the function defined $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$, and let $\beta f : \beta X \to \{0, 1\}$ be the continuous function induced by f. (Note that $\{0, 1\}$ is a finite set, so it is a compact Hausdorff space.) Since βX is connected, the image of βf is $[0, 1]$ in $\mathbb{R}$, a contradiction. Therefore, X is connected if βX is connected.

Problem 4.3.2. Let X be a discrete space and let $(\beta X, \iota)$ be the Stone-Čech compactification of X.

(a) Show that if A is a subset of X then the closures of A and $X \setminus A$ in βX are disjoint.

(b) Show that if U is an open subset of βX then so is its closure in βX.

(c) Show that βX is totally disconnected.

Solution.

Problem 4.3.3. Let X be a completely regular space.

(a) Suppose X is normal and identify X as a subset of βX (or choose βX as a compactification of X satisfying the universal property of the Stone-Čech compactification). Show that y is not a limit point of X in βX.

(b) Show that βX is not metrizable if X is completely regular but not compact.

Solution.

Chapter 5

Complete metric spaces and the space of continuous maps

5.1 Complete metric spaces

5.1.1 Complete metric spaces and uniform metrics

Definition 5.1.1 (Complete metric space)**.** Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in X is called a Cauchy sequence if the following condition is satisfied: For any positive real number ϵ, there is a positive integer N such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. The metric space (X, d) is called a complete metric space if every Cauchy sequence in X is convergent.

Remark. The completeness of a metric space is a metric property, rather than a topological property. Hence, the completeness may not be preserved under homeomorphisms.

Some basic properties regarding the completeness of metric spaces are given as follows. Proving them is left as an exercise. See Problem 5.1.1.

Proposition 5.1.2. Let (X, d) and (X_n, d_n) be complete metric spaces for $n \in \mathbb{N}$.

(a) (Reduction of criterion) X is complete if and only if every Cauchy sequence in X has a convergent subsequence.

(b) (Completeness and closedness) Let Y be any nonempty subset of X and let $e = d|_{Y\times Y}$. Then (Y, e) is complete if and only if Y is closed in X.

(c) (Completeness and countable products) The product space $Z = \prod_{n\in\mathbb{N}} X_n$ is complete under the metric $D : Z \times Z \to [0, \infty)$, which is defined by

$$D(x, y) = \sup_{n\in\mathbb{N}} \left\{ \frac{1}{n} \overline{d_n}(x_n, y_n) \right\} \quad (x, y \in Z),$$

where $\overline{d_n}$ is the standard bounded metric induced by d_n for each $n \in \mathbb{N}$. In fact, defining a metric $\kappa : Z \times Z \to [0, \infty)$ by $\kappa(x, y) = \sum_{n\in\mathbb{N}} 2^{-n}\overline{d_n}(x_n, y_n)$ for all $x, y \in Z$, Z is also complete under κ.

Let (X, d) be a metric space (not necessarily being complete) and I be a nonempty index set. In Definition 1.3.2, we have defined the uniform metric $\overline{\rho}$ on X^I correspoinding to d, which is defined by $\overline{\rho}(f, g) = \sup \left\{ \overline{d}(f(a), g(a)) : a \in I \right\}$. Remark that the metric topology induced by the uniform metric is finer than the product topology and coarser than the box topology.

In (c) of Proposition 5.1.2, the completeness of the product of complete metric spaces is ensured only for countable products. The following theorem ensures the completeness for uncountable products, when the the uniform topology is considered.

Theorem 5.1.3. Let X be a nonempty set. If (Y, d) is a complete metric space, then $(Y^X, \overline{\rho})$ is complete.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y^X. Because $\overline{\rho}(f_n, f_m) \to 0$ as $n, m \to \infty$, for each $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Hence, $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent. To prove that $(f_n)_{n\in\mathbb{N}}$ is convergent in Y^X, first observe that $d(f_n(t), f(t)) \le d(f_n(t), f_p(t)) + d(f_p(t), f(t))$ for any point t of X and any positive integer p. Let N be an integer such that $n, m \ge N$ implies $\overline{\rho}(f_n, f_m) < \epsilon/2$; given a point x of X, let M be an integer such that $d(f_p(x), f(x)) < \epsilon/2$ whenever $p \ge M$. Then, whenever $p \ge M, N$, we have $d(f_n(x), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore, $(f_n)_{n\in\mathbb{N}}$ converges to f in Y^X. □

Theorem 5.1.4 (Complete spaces of functions)**.** Let X be a topological space, (Y, d) be a complete metric space, and assume that Y^X equips the uniform metric $\overline{\rho}$ corresponding to d.

(a) $C(X, Y)$ and $B(X, Y)$ are closed subspaces of Y^X.

(b) Hence, if Y is complete, then $C(X, Y)$ and $B(X, Y)$ are complete.

Proof. Because (b) is immediate from Theorem 5.1.3 and (a), it suffices to prove (a). Assume first that $f \in Y^X$ is a limit point of $C(X, Y)$ in Y^X. Then, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C(X, Y)$ satisfying $\overline{\rho}(f_n, f) \to 0$ as $n \to \infty$. Being a uniform limit of a sequence of continuous maps, f is also continuous. The proof for $B(X, Y)$ is also easy, which is left as an exercise. □

For $B(X, Y)$, one may impose the metric $\rho : B(X, Y) \times B(X, Y) \to [0, \infty)$ defined by

$$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

This metric is called the supremum (or 'sup') metric corresponding to d. In particular, when X is compact, since $C(X, Y) \subset B(X, Y)$, one may make use of the supremum metric, rather than the uniform metric. In fact, when X is compact, the supremum metric on $C(X, Y)$ is the standard bounded metric corresponding to the uniform metric on $C(X, Y)$. In other words, $\overline{\rho}(f, g) = \min\{1, \rho(f, g)\}$ for all $f, g \in C(X, Y)$. Its justification is left as an exercise.

5.1.2 Isometric embedding of a metric space

Recall that a map h from a metric space (X, d_X) into (Y, d_Y) satisfying $d_Y(h(a), h(b)) = d_X(a, b)$ for all $a, b \in X$ is called an isometric embedding of X into Y. If $h : X \to Y$ is an isometric embedding and Y is complete, the closure of $h(X)$ in Y is also complete.

When trying to define objects, one should be interested in their existence, uniqueness, and exactness.

Theorem 5.1.5 (Existence of a completion of a metric space)**.** Let (X, d) is a metric space, and consider the map $\phi : X \to B(X, \mathbb{R})$ defined by

$$\phi(x) = l_x \text{ for } x \in X,$$

where $l_x : X \to \mathbb{R}$ maps $t \in X$ to $d(t, x) - d(t, a)$. (Here, a is a given point of X, so $|l_x(t)| \le d(x, a)$ for all $t \in X$) Then ϕ is an isometric embedding of X into $B(X, \mathbb{R})$, where $B(X, \mathbb{R})$ equips the sup metric. Because $\mathbb{R}$ is complete, the closure of $\phi(X)$ in $B(X, \mathbb{R})$ is also complete. Therefore, every metric space has a completion.

Theorem 5.1.6 (Uniqueness of a completion of a metric space)**.** Let $h_1 : X \hookrightarrow Y_1$ and $h_2 : X \hookrightarrow Y_2$ be isometric embeddings of X into complete metric spaces (Y_1, d_1) and (Y_2, d_2). Denote the closure of $h_i(X)$ in Y_i by $\overline{h_i(X)}$ $(i = 1, 2)$. Then, there is an isometric homeomorphism $H : \overline{h_1(X)} \to \overline{h_2(X)}$ which equals $h_2 \circ h_1^{-1}$ on $h_1(X)$. Hence, the completion of X is unique up to an isometric homeomorphism.

$$
\begin{tikzcd}
h_1(X) \arrow[d, hook] \arrow[rr, bend left, "h_2\circ h_1^{-1}", "\text{isometric}"'] & X \arrow[l, "h_1"', "\text{isometric}"] \arrow[r, "h_2", "\text{isometric}"'] & h_2(X) \arrow[d, hook] \\
\overline{h_1(X)} \arrow[rr, "H", "\text{isometric}"'] \arrow[d, hook] & & \overline{h_2(X)} \arrow[d, hook] \\
Y_1 & & Y_2
\end{tikzcd}
$$

Proof of Theorem 5.1.5. Since it is checked that $l_x \in B(X,\mathbb{R})$, the map ϕ is well defined. Thus, it remains to check that ϕ is an isometric embedding of X into $B(X,\mathbb{R})$. For this, it suffices to show that $\overline{\rho}(l_x,l_y) = d(x,y)$ for all $x,y \in X$, where ρ is the uniform metric on $B(X,\mathbb{R})$. Observing that $l_x(t) - l_y(t) = d(t,x) - d(t,y)$, we can find that $\rho(l_x,l_y) \le d(x,y)$ and $l_x(x) - l_y(x) = -d(x,y)$, proving that $\rho(l_x,l_y) = d(x,y)$. $\square$

Proof of Theorem 5.1.6. Remark that $\overline{h_i(X)}$ is a complete metric space for $i = 1,2$. Define a map $H : \overline{h_1(X)} \to \overline{h_2(X)}$ as follows:

(1) When $x \in h_1(X)$, let $H(x) := h_2(h_1^{-1}(x))$. Since h_1 is a bijection between X and $h_1(X)$, the above setting is not ambiguous.

(2) When $x \in \overline{h_1(X)} \setminus h_1(X)$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $h_1(X)$ which converges to x, and let $H(x) := \lim H(x_n)$.

   This setting seems to be ambiguous, because the resulting $H(x)$ might differ by the choice of $(x_n)_{n\in\mathbb{N}}$ (also, even the existence of the limit could be inquired).

   - Because $d_2(H(x_n),H(x_m)) = d_1(x_n,x_m)$ for all $n,m \in \mathbb{N}$, $(H(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\overline{h_2(X)}$. Since $\overline{h_2(X)}$ is complete, $(H(x_n))_{n\in\mathbb{N}}$ is convergent, with the limit in $\overline{h_2(X)}$.
   - Suppose $(a_j)_{j\in\mathbb{N}}$ is another sequence in $h_1(X)$ converging to x. Letting $X = \lim H(x_n)$ and $A = \lim H(a_j)$, we find that

$$
\begin{aligned}
d_2(X,A) &\le d_2(X,H(x_n)) + d_2(H(x_n),H(a_j)) + d_2(H(a_j),A) \\
&\le d_2(X,H(x_n)) + d_1(x_n,a_j) + d_2(H(a_j),A) \xrightarrow{n,j\to\infty} 0,
\end{aligned}
$$

   so $X = A$ and the above setting of $H(x)$ is not ambiguous.

We just established a map H from $\overline{h_1(X)}$ into $\overline{h_2(X)}$. It remains to show that H is an isometry. For this, it suffices to show that H is an isometric homeomorphism, and H coincides $h_2 \circ h_1^{-1}$ on $h_1(X)$.[1]

(a) Given $a,b \in \overline{h_1(X)}$, let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be seqences in $h_1(X)$ converging to a and b, respectively. Then $d_2(H(a),H(b)) = \lim d_2(H(a_n),H(b_n)) = \lim d_1(a_n,b_n) = d_1(a,b)$, so H is isometric and injective. To check the surjectivity, let y be a point of $\overline{h_2(X)}$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $h_2(X)$ converging to y. Letting $x_n = h_1(h_2^{-1}(y_n))$ for each $n \in \mathbb{N}$, the limit of $(x_n)_{n\in\mathbb{N}}$ exists (say $x \in \overline{h_1(X)}$ is the limit), and we have $H(x) = \lim H(x_n) = \lim y_n = y$.

(b) Clearly, H extends $h_2 \circ h_1^{-1}$.

Therefore, there is an isometric homeomorphism between two completions of X. In other words, there is a unique completion of a metric space up to isometric homeomorphism. $\square$

To emphasize the result of Theorems 5.1.5 and 5.1.6, we leave the definition of the completion of a metric space.

[1] Since an isometric embedding is an open map, it suffices to show that H is an isometric bijection.

Definition 5.1.7 (The completion of a metric space)**.** Let (X, d) be a metric space, and let $h : (X, d) \to (Y, e)$ be an isometric embedding, where (Y, e) is a complete metric space. The closure of the image of h in Y is called the completion of X. The completion of a metric space always exists, and it is unique up to isometric homeomorphism.

5.1.3 Problems

Problem 5.1.1. Prove Proposition 5.1.2.

Solution. (a) Easy.

(b) If Y is a nonempty closed supset of X and $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, then it converges, due to the completeness of X; and the limit belongs to Y, due to the closedness of Y in X, which implies the completeness of Y. Conversely, if (Y, e) is complete and y is a limit point of Y in X, then for each $n \in \mathbb{N}$, there is a point $y_n \in B_d(y, 1/n)$; since the sequence $(y_n)_{n\in\mathbb{N}}$ converges to y and Y is complete, $y \in Y$, so Y is closed in X.

(c) Remark that D and ρ induces the product topology on Z. If $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Z, it easily follows from the definition of D or ρ that $(\pi_k(z_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in X_k for each $k \in \mathbb{N}$. Since each X_k is a complete metric space, each sequence $(\pi_k(z_n))_{n\in\mathbb{N}}$ in X_k is convergent, hence the sequence $(z_n)_{n\in\mathbb{N}}$ is convergent in Z.

5.2 Pointwise equicontinuous collection of continuous maps

Definition 5.2.1 (Equicontinuity)**.** Let X be a topological space, (Y, d) be a metric space, and let $\mathcal{F}$ be a subset of $C(X, Y)$, i.e., a collection of continuous functions from X into Y.

(a) (Pointwise equicontinuity) The collection $\mathcal{F}$ is said to be equicontinuous at a point $x_0 \in X$ if, for any $\epsilon > 0$, there is a neighborhood U of x_0 in X such that $x \in U$ and $f \in \mathcal{F}$ implies $d(f(x), f(x_0)) < \epsilon$. If $\mathcal{F}$ is equicontinuous at every point of X, then $\mathcal{F}$ is said to be pointwise equicontinuous.

(b) (Uniform equicontinuity) Suppose that X is a metric space and d_X be the metric on X inducing the topology on X. The collection $\mathcal{F}$ is said to be uniformly equicontinuous if, for every $\epsilon > 0$, there is $\delta > 0$ such that $a, b \in X$ with $d_X(a, b) < \delta$ and $f \in \mathcal{F}$ implies $d(f(a), f(b)) < \epsilon$.

Remark. In the course of introduction to mathematical analysis, we studied the following version of Ascoli's theorem:

> Suppose K is a compact metric space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued continuous functions (continuous functions with values in a complete metric space, respectivley) on K. If $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded (totally bounded under the sup metric) and uniformly equicontinuous, then $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded and $(f_n)_{n\in\mathbb{N}}$ contains a uniformly convergent subsequence.

Lemma 5.2.2. Suppose that X is a topological space and (Y, d) is a metric space, and assume $C(X, Y)$ equips the uniform topology. If $\mathcal{F} \subset C(X, Y)$ is totally bounded under the uniform metric, then $\mathcal{F}$ is pointwise equicontinuous under d.

Proof. Fix a positive real number $0 < \epsilon < 1$ and write $\mathcal{F} = \bigcup_{i=1}^{N} B_{\overline{\rho}}(f_i, \epsilon)$, where $f_i \in \mathcal{F}$ for each i. To show pointwise equicontinuity, we fix a point $p \in X$ and choose a neighborhood U_i of p in X such that $x \in U_i$ implies $d(f_i(x), f_i(p)) < \epsilon$; when $U = \bigcap_{i=1}^{N} U_i$, then U is still a neighborhood of p in X and whenever $x \in U$ we have $d(f_i(x), f_i(p)) < \epsilon$ for all i. Given $f \in \mathcal{F}$, choose an index i such that $\overline{\rho}(f, f_i) < \epsilon$. Because

$$d(f(x), f(p)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(p)) + d(f_i(p), f(p)) < 3\epsilon$$

whenever $x \in U$. Therefore, every totally bounded subset of $C(X, Y)$ is pointwise equicontinuous. □

The preceeding lemma states that a totally bounded (in the uniform metric) subset of continuous functions is pointwise equicontinuous. Its reverse is generally not true; for example, the countable collection $(f_n : \mathbb{R} \to \mathbb{R},\, x \mapsto n)_{n\in\mathbb{N}}$. The following lemma states that the converse to the preceeding lemma is valid if both domain and codomain are compact.

Lemma 5.2.3. Let X be a topological space and (Y, d) be a metric space. Assume that both X and Y are compact. If the subset $\mathcal{F}$ of $C(X, Y)$ is pointwise equicontinuous, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.

Remark. Suppose X is a topological space and (Y, d) is a metric space. When we consider the space $C(X, Y)$, we first consider the uniform metric $\overline{\rho}$. If X is assumed to be compact, we can also consider the sup metric, which will be denoted by ρ.

(a) As the following properties are particularly related to points within arbitrarily small distance and both $\overline{\rho}$ and ρ coincide at every pair of continuous functions within distance less than 1, they are common properties of $C(X, Y)$ under each metric: completeness, total boundedness. (For example, if $C(X, Y)$ under the uniform metric is complete, then so is $C(X, Y)$ under the sup metric, and vice versa.)

(b) The boundedness of $C(X, Y)$ is not common; while the space may be unbounded under the sup metric, the space is clearly bounded under the uniform metric; $C(X, Y) = B_{\overline{\rho}}(0, 2)$, where 0 in the first argument is the zero map.

Proof. By the preceeding remark, it suffices to justify the total boundedness under the sup metric ρ.

Using pointwise equicontinuity, for each point $a \in X$, find a neighborhood D_a of a in X such that $x \in D_a$ and $f \in \mathcal{F}$ implies $d(f(x), f(a)) < \epsilon$. Since X is compact, we can cover X with finitely many D_a, suppose $\{D_{a_i}\}_{i=1}^{k}$ covers X and write $U_i = D_{a_i}$. On the other hand, cover Y with finitely many open balls $V_1, \cdots, V_m$ of diameter less than ϵ.

Let J be the collection of all functions from $\{1, \cdots, k\}$ into $\{1, \cdots, m\}$. Given $\alpha \in J$, if there is a function $f \in \mathcal{F}$ such that $f(a_i) \in V_{\alpha(i)}$ for all $i = 1, \cdots, k$, choose one such function $f \in \mathcal{F}$ and label it f_α. The collection $\{f_\alpha\}$ is indexed by a subset I of J and is thus finite.

Choose a map $f \in \mathcal{F}$. Because $\{V_1, \cdots, V_m\}$ covers Y, for each i, $f(a_i)$ belongs to some V_j; write $f(a_i) \in V_{\beta(i)}$, and consider the function f_β. Given a point $x \in X$, there is an index i such that $x \in U_i$, and

$$d(f(x), f_\beta(x)) \le d(f(x), f(a_i)) + d(f(a_i), f_\beta(a_i)) + d(f_\beta(a_i), f_\beta(x)).$$

The first and the last term in the right-hand side are, respectivley, less than ϵ (pointwise equicontinuity) and the second term is also less than ϵ (the diameter of V_i is less than ϵ), so $d(f(x), f_\beta(x)) < 3\epsilon$. This implies that $f \in B_\rho(f_\beta, 4\epsilon)$. Therefore, $\mathcal{F}$ is covered by the open balls $B_\rho(f_\alpha, 4\epsilon)$ for $\alpha \in I$, so $\mathcal{F}$ is totally bounded under the sup metric. □

The above theory, together with the theory developed in the following section, will be used in the last section when we prove Ascoli's theorem.

5.3 Topologies on the space of continuous functions

In this section, we first impose topologies on $C(X, Y)$ to which we can correspond pointwise convergence and uniform convergence. Throughout this section, unless stated otherwise, X and Y are assumed to be topological spaces.

5.3.1 Topology of pointwise convergence and uniform topology

Definition 5.3.1 (Topology of pointwise convergence)**.** The topology on Y^X generated by the collection of subsets

$$S(x, U) := \{f \in Y^X : f(x) \in U\} \quad (x \in X,\ U \text{ is open in } X)$$

as a subbasis is called the topology of pointwise convergence, or the point-open topology.

Note that the topology on Y^X of pointwise convergence is just the product topology on Y^X, since $S(x,U) = \pi_x^{-1}(U)$ for each $x \in X$ and open subspace U of X. Thus, from now on, we say a sequence $(f_n)_{n\in\mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ pointwise when the sequence converges to f in the topology of pointwise convergence, as the convergence in the uniform topology is called the uniform convergence.

Remark. (a) The pointwise (uniform, respectively) convergence coincides our old concept of pointwise (uniform) convergence.

(b) (Review of Lemma 5.2.2) If a subset of $C(X,Y)$ is totally bounded under the uniform metric, then the subset is pointwise equicontinuous under d.

5.3.2 Topology of compact convergence

Still, we assume that Y is a metric space.

Definition 5.3.2 (Topology of compact convergence)**.** The topology on Y^X generated by the collection of the subsets

$$B_C(f,\epsilon) := \left\{ g \in Y^X : \sup_{x\in C} d(f(x),g(x)) < \epsilon \right\} \quad \left(\begin{array}{c} C \text{ is a compact subspace of } X, \\ f \in Y^X,\ \epsilon > 0 \end{array} \right) \tag{5.1}$$

as a basis is called the topology of compact convergence or the topology of uniform convergence on compact sets.[2]

Remark. The collection of sets in eq. (5.1) is indeed a basis of the topology on Y^X of compact convergence, and its justification is given here.

(1) It is clear that the collection covers Y^X.

(2) Given two basis members $B_C(f,a)$ and $B_K(g,b)$, where C and K are compact subspaces of X and $a,b > 0$ and a point $u \in B_C(f,a) \cap B_K(g,b)$, we wish to find a basis member B such that $u \in B \subset B_C(f,a) \cap B_K(g,b)$.

For a point $u \in B_C(f,a)$, if $\delta = a - \sup\{d(f(x),u(x)) : x \in C\} > 0$, then $B_C(u,\delta) \subset B_C(f,a)$.

Using the above observation, we can find a small positive real numbera α such that $B_C(u,\alpha) \subset B_C(f,a)$ and $B_K(u,\alpha) \subset B_K(g,b)$, and $B_{C\cup K}(u,\alpha)$ is a desired basis member.

From now on, we say a sequence $(f_n)_{n\in\mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ compactly when the sequence converges to f in the topology of compact convergence, i.e., $f_n \to f$ uniformly on every compact subspace of X.

Theorem 5.3.3 (Inclusions regarding topologies on Y^X)**.** Assume (Y,d) is a metric space. Then,

(uniform topology) $\supset$ (topology of compact convergence) $\supset$ (topology of pointwise convergence).

Furthermore, if X is compact, then the first two topologies coincide; if X is discrete, then the last two coincide.

Proof. We first show the inclusion.

(1) (The uniform topology is finer than the topology of compact convergence) Given a basis member $B_C(f,\epsilon)$ of the topology of compact convergence and its point g, we can find a positive real number $r < 1$ such that $B_C(g,r) \subset B_C(f,\epsilon)$. It is clear that $g \in B_{\overline{\rho}}(g,r) \subset B_C(g,r) \subset B_C(f,\epsilon)$.

(2) (The topology of compact convergence is finer than the topology of pointwise convergence) Given a basis member $S(x, B_d(p,\epsilon))$ of the topology of pointwise convergence and its point g, because $d(g(x),p) < \epsilon$, there is a positive real number r such that $B_d(g(x),r) \subset B_d(p,\epsilon)$. Because $\{x\}$ is compact, it is clear that $g \in B_{\{x\}}(g,r/2) \subset S(x, B_d(p,\epsilon))$.

[2] The supremum of $d(f(x),g(x))$ for $x \in C$ is finite since C is compact.

We now show the coincidence under each case. When X is compact, then $B_\rho(f,\epsilon)$ is a basis member of the topology of compact convergence as well as a basis member of the uniform topology, proving the first coincidence. When X is discrete and $B_C(f,\epsilon)$ is a basis member of the topology of compact convergence, the compact subspace C of X is necessarily a finite subset of X; hence, writing $C = \{x_1, \cdots, x_n\}$, we have $B_C(f,\epsilon) = \bigcap_{i=1}^n S(x_i, B_d(f(x_i),\epsilon))$. □

Before studying further properties of the topology of compact convergence, we introduce a topological space in which a subset is open if and only if its restriction to any compact subspace is open in the compact subspace.

Definition 5.3.4 (Compactly generated space)**.** A topological space X is said to be compactly generated if a subspace A of X is open when the following property is satisfied:

$$A \cap C \text{ is open in } C \text{ whenever } C \text{ is a compact subspace of } X.$$

Remark. We may replace the word "open" by "closed," by considering set complements.

Example 5.3.5. Some examples of compactly generated spaces are introduced here.

(a) (Locally compact spaces) Suppose X is a locally compact space and let A be a subset of X such that $A \cap C$ is open in C whenever C is a compact subspace of X. We wish to justify that A is an open subspace of X.

Given $x \in A$, choose a neighborhood U of x in X that lies in a compact subspace C of X. Since $A \cap C$ is open in C, $A \cap U = (A \cap C) \cap U$ is open in U, and hence in X. Then $A \cap U$ is a neighborhood of x in X contained in A, so A is open in X.

(b) (First-countable spaces) Suppose X is a first-countable space and let B be a subset of X such that $B \cap C$ is closed in C whenever C is a compact subspace of X. We wish to justify that B is a closed subset of X.

Let x be a point of $\overline{B}$, where the overline notation means the closure in X. Since X has a countable base at x, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points in B converging to x. The subspace $C = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, so $B \cap C$ is closed in C. Because $B \cap C$ contains all x_n, it also contains x, so $x \in B$. Therefore, B is closed.

For pointwise convergence, the limit map of continuous maps need not be continuous; this is valid for uniform convergence. For compact convergence, the assertion is valid when the domain is compactly generated, which is justified by the below theorem after a lemma.

Lemma 5.3.6. If X is compactly generated, then a map $f : X \to Y$ is continuous if and only if for each compact subspace C of X, the restriction $f|_C$ is continuous.

Proof. Only if part is obvious, and if part is also clear if one remarks that $(f|_C)^{-1}(U) = C \cap f^{-1}(U)$ whenever $U \subset Y$. □

Theorem 5.3.7. Let X be a topological space, (Y,d) be a metric space. Then $C(X,Y)$ is closed in Y^X in the topology of compact convergence. (Hence, if a sequence $(f_n)_{n\in\mathbb{N}} \subset C(X,Y)$ converges to f compactly, then f is continuous.)

Proof. We need to show that a limit point f of $C(X,Y)$ in Y^X belongs to $C(X,Y)$. By the preceeding lemma, it suffices to show that $f|_C$ is continuous whenever C is a compact subspace of X. Given a compact subspace C of X, by considering the neighborhoods $B_C(f,1/n)$ of f in Y^X, we can find a sequence $(g_n^C)_{n\in\mathbb{N}}$ of points in $C(X,Y)$ such that $g_n^C \in B_C(f,1/n)$. In this case, $g_n^C|_C \to f|_C$ uniformly, so $f|_C$ is continuous. □

Remark. The space $C(X,Y)$ (X is a topological space and (Y,d) is a metric space) is closed in Y^X in the uniform topology, or in the topology of compact convergence when it is further assumed that X is compactly generated.

5.3.3 Compact-open topology

When we introduced the uniform topology and the topology of compact convergence, we had to assume that Y is a metric space (and the topologies seem to depend on the choice of the metric on Y). It is natural to ask whether either of these topologies can be extended to the case where Y is an arbitrary topological space. It is known that there is no satisfactory answer to this question for the space Y^X; fortunately, one can prove something for its subspace $C(X, Y)$.

To study such case, in this subsection we assume X and Y are topological spaces.

Definition 5.3.8 (Compact-open topology)**.** The topology on $C(X, Y)$ generated by the collection of the subsets of the form

$$S(C, U) := \{f \in C(X, Y) : f(C) \subset U\} \quad (C \text{ is a compact subspace of } X \text{ and } U \text{ is open in } X)$$

as a subbasis is called the compact-open topology.

Remark. (a) The above definition of compact-open topology naturally extends to Y^X, which is not of our interest.

(b) Clearly, the compact-open topology is finer than the topology of pointwise convergence (the point-open topology).

Observation 5.3.9. When (Y, d) is a metric space, the compact-open topology on $C(X, Y)$ and the topology of compact convergence on $C(X, Y)$ coincide.

Proof. The following result helps in this proof:

> Suppose A is a compact subspace of X and V is an open subspace of X containing A. Then, there is $\epsilon > 0$ such that V contains the ϵ-neighborhood of A.

We first prove that the topology of compact convergence is finer than the compact-open topology. Given a subbasis member $S(C, U)$ of the compact-open topology (C is compact and U is open in X) and its point f, note that f is continuous so that $f(C)$ is compact. Because $f(C) \subset U$, there is $\epsilon > 0$ such that the ϵ-neighborhood of $f(C)$ is contained in U. Therefore, $f \in B_C(f, \epsilon) \subset S(C, U)$.

Now we prove that the compact-open topology is finer than the topology of compact convergence. Given a basis member $B_C(f, \epsilon)$ of the topology of compact convergence and its point g, without loss of generality, it suffices to find a member A of the topology of compact convergence such that $f \in A \subset B_C(f, \epsilon)$.[3] For each point $x \in C$, there is a neighborhood U_x of x in X such that

$$t \in \overline{U_x} \text{ implies } d(f(t), f(x)) < \epsilon/3.^{4}$$

Using the compactness of C, choose finitely many $V_i = U_{x_i}$ $(i = 1, \cdots, n)$ covering C. If we write $C_i = \overline{V_i} \cap C$ for each i, the finite intersection $\bigcap_{i=1}^n S(C_i, B_d(f(x_i), \epsilon/3))$ contains f and is contained in $B_C(f, \epsilon)$. $\square$

In the beginning of this subsection, it is mentioned as a statement that the compact convergence topology (as well as the uniform topology) seems to depend on the metric on the codomain. By Observation 5.3.9, we can obtain the following coincidences:

Corollary 5.3.10. If Y is a metric space (so that the compact convergence topology and the compact-open topology coincide), the compact convergence topology on $C(X, Y)$ is independent of the metric on Y. Hence, if it is further assumed that X is compact (so that the uniform topology and the compact convergence topology coincide), the uniform topology on $C(X, Y)$ is independent of the metric on Y.

[3] There is no loss of generality, since we can find a positive real number r such that $B_C(g, r) \subset B_C(f, \epsilon)$.

[4] By letting U_x be a neighborhood of x in X such that $t \in U_x$ implies $d(f(t), f(x)) < \epsilon/4$, the result follows from the inclusion $f(\overline{U_x}) \subset \overline{f(U_x)}$.

Continuity of evaluation maps under the compact-open topology on $C(X,Y)$

We end this section with some theory which will be helpful when studying Ascoli's theorem in the following section.

Definition 5.3.11 (Evaluation maps)**.** Let A be a nonempty set and P be a collection of maps from A into a nonempty set B. The map $\mathrm{ev}_P : A \times P \to B$ defined by $\mathrm{ev}_P(a,f) = f(a)$ for $a \in A, f \in P$ is called the evaluation map with maps in P. If the context is clear, we may omit the subscript.

Example 5.3.12. Let X and Y be topological spaces and consider the evaluation map $\mathrm{ev} : X \times C(X,Y) \to Y$. Suppose further that there is a metric d on Y inducing the topology on Y. Show that, under the compact-open topology on $C(X,Y)$, that the evaluation map is continuous.

Theorem 5.3.13. Let X be a locally compact Hausdorff space and suppose $C(X,Y)$ equips the compact-open topology. Then the evaluation map $\mathrm{ev} : X \times C(X,Y) \to Y$ is continuous.

Proof. To show the continuity of the evaluation map, it suffices to show that there is a neighborhood of $(a,f) \in X \times C(X,Y)$ which is mapped into an arbitrarily given neighborhood W of $f(a)$ in Y. Using the continuity of f, choose a neighborhood U of a in X such that $f(U) \subset W$. By hypothesis, there is a neighborhood V of a whose closure in X is a compact subspace of U. The box $U \times S(\overline{U}, W)$ is a desired neighborhood of (a,f). □

A consequence of this theorem is the theorem that follows.

Definition 5.3.14. Given a map $f : X \times Z \to Y$, there is a corresponding map $F : Z \to Y^X$, defined by $(F(z))(x) = f(x,z)$ for all $x \in X$ and $z \in Z$. Conversely, given $F : Z \to Y^X$, the above equation defines a corresponding map $f : X \times Z \to Y$. We say that F is the map of Z into $C(X,Y)$ which is induced by f.

Theorem 5.3.15. Let X and Y be topological spaces and give $C(X,Y)$ the compact-open topology. If $f : X \times Z \to Y$ is continuous, then so is its induced map $F : Z \to C(X,Y)$. The converse holds if X is a locally compact Hausdorff space.

Proof. We first prove that f is continuous when X is a locally compact Hausdorff space and F is continuous. Note that if $\alpha_i : A_i \to B_i$ denotes a continuous map for each $i = 1, \cdots, n$ then $\alpha_1 \times \cdot \alpha_n$ is also continuous, and remark the following commutative diagram.

$$\begin{tikzcd} X \times Z \arrow[r, "\mathrm{id}_Z \times F"'] \arrow[rr, bend left, "f"] & C(X,Y) \arrow[r, "\mathrm{ev}"'] & Y \end{tikzcd}$$

We now prove that F is continuous if f is continuous. For this, we fix a point $z_0 \in Z$ and seek to find a neighborhood W of z_0 in Z such that $F(W) \subset S(C,U)$, where $S(C,U)$ is any subbasis member of the compact-open topology on $C(X,Y)$ which contains $F(z_0)$, i.e., $f(C \times \{z_0\}) \subset U$. By definition, for each $c \in C$, $f(c, z_0) \in U$; by continuity, there is an open box $A_c \times B_c$ in $X \times Z$ containing (c, z_0) which is mapped into U under f. Since $C \times \{z_0\}$ is compact, this slice can be covered by finitely many boxes of the form $A_c \times B_c$; write $C \times \{z_0\} \subset \bigcup_{i=1}^n (A_{c_i} \times B_{c_i})$ and $B = \bigcap_{i=1}^n B_{c_i}$. Clearly, B is a neighborhood of z_0 in Z and $f(C \times B) \subset U$, i.e., $F(B) \in S(C,U)$, as desired. □

5.4 Ascoli's theorem

Throughout this section, we assume that X is a topological space and (Y,d) is a metric space, unless stated otherwise. In this section, we study a generalized Ascoli's theorem, which characterizes the compact subspaces of $C(X,Y)$ in the topology of compact convergence. Also, some particular versions of Ascoli's theorem will be introduced.

5.4.1 Generalized Ascoli's theorem

Theorem 5.4.1 (Ascoli's theorem)**.** Let X be a topological space and (Y, d) be a metric space. Give $C(X, Y)$ the topology of compact convergence, and let $\mathcal{F}$ be a subset of $C(X, Y)$.

(a) Suppose that $\mathcal{F}$ is pointwise equicontinuous under d and pointwise relatively compact. Then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$; to say equivalently, $\mathcal{F}$ is relatively compact.

(b) The converse of (a) is valid if X is a locally compact Hausdorff space.

Remark. For a Hausdorff space A, a subspace B of A is contained in a compact subspace of A if and only if A is relatively compact. Since if part is clear, we assume that B is contained in a compact subpsace K of A. Note that the closure $\overline{B}$ of B in X is contained in the closure $\overline{K}$ of K in X. Since A is a Hausdorff space, every compact subspace of A is closed, so $\overline{K} = K$. Hence, $\overline{A} \subset K$, implying that $\overline{A}$ is a closed subspace of K. Therefore, A is relatively compact.

Proof of (a). Impose Y^X the product topology, which equals to the topology of pointwise convergence. Note that Y^X is a Hausdorff space and that $C(X, Y)$ (which equips to topology of compact convergence) is not a subspace of Y^X. And let $\mathcal{G}$ denote the closure of $\mathcal{F}$ in Y^X. (It will be clear in Step 1 why we consider the closure of $\mathcal{C}$ in Y^X, not in $C(X, Y)$.)

Our proof then consists of the following four steps:

- Step 1: Showing that $\mathcal{G}$ is a compact subspace of Y^X.
- Step 2: Showing that each member of $\mathcal{G}$ is continuous, and $\mathcal{G}$ is pointwise equicontinuous under d.
- Step 3: The topology on Y^X and the topology on $C(X, Y)$ coincide on $\mathcal{G}$.
- Step 4: Concluding the proof of (a).

Step 1. For each point $a \in X$, let C_a denote the closure of $\mathcal{F}(a)$ in Y. By assumption, each C_a is compact, hence the product $C = \prod_{a \in X} C_x$ is also compact. Because $\mathcal{F} \subset C \subset Y^X$, the closure of $\mathcal{F}$ in Y^X is also contained in C; because Y^X is a Hausdorff space, $\mathcal{G}$ is a compact subspace of Y^X.

Step 2. It suffices to check that $\mathcal{G}$ is pointwise equicontinuous under d. Given $p \in X$, let V_p be a neighborhood of p in X such that $f \in \mathcal{F}$ and $x \in V_p$ implies $d(f(x), f(p)) < \epsilon$. Given $g \in \mathcal{G}$ and $x \in V_p$, we can find $h \in \mathcal{F}$ such that $h \in \pi_p^{-1}(B_d(g(p), \epsilon)) \cap \pi_x^{-1}(B_d(g(x), \epsilon))$, i.e., $d(h(p), g(p)) < \epsilon$ and $d(h(x), h(x)) < \epsilon$. In this case,

$$d(g(x), g(p)) \leq d(g(x), h(x)) + d(h(x), h(p)) + d(h(p), g(p)) < 3\epsilon,$$

which proves that $\mathcal{G}$ is pointwise equicontinuous.

Step 3. It is clear that the topology on Y^X restricted to $\mathcal{G}$ is coarser than the topology on $C(X, Y)$ restricted to $\mathcal{G}$. Thus, it remains to show that the converse inclusion; for this, given a basis member $B_K(f, \epsilon) \cap \mathcal{G}$ of the latter topology (assume $f \in \mathcal{G}$), we need to find a basis member B of the pointwise convergence topology on Y^X such that $f \in B \cap \mathcal{G} \subset B_K(f, \epsilon) \cap \mathcal{G}$. (How could we skip some procedures?)

Using pointwise equicontinuity of $\mathcal{G}$, for each point $x \in X$, let U_x be a neighborhood of x in X such that $t \in U_x$ and $g \in \mathcal{G}$ implies $d(g(t), g(x)) < \epsilon/4$; using compactness of K, cover K with finitely many neighborhoods $U_{p_1}, \cdots, U_{p_n}$. Then, choose

$$B = \bigcap_{i=1}^{n} \pi_{p_i}^{-1}(B_d(f(p_i), \epsilon/4))$$

so that whenever $g \in B \cap \mathcal{G}$ we have $d(g(p_i), f(p_i)) < \epsilon/4$ for all i. If $g \in B \cap \mathcal{G}$ and $x \in K$, there is an index i such that $x \in U_{p_i}$, so

$$d(g(x), f(x)) \leq d(g(x), g(p_i)) + d(g(p_i), f(p_i)) + d(f(p_i), f(x)) < \frac{3}{4}\epsilon.$$

This implies that $g \in B_K(f, \epsilon) \cap \mathcal{G}$, as desired.

Step 4. Using the results from Step 1 to Step 3, we derive that $\mathcal{F}$ is contained in a compact subspace of $C(X,Y)$. By Step 1 and Step 2, the closure $\mathcal{G}$ of $\mathcal{F}$ in Y^X is a compact subspace of Y^X (in the product topology) which is contained in $C(X,Y)$; by Step 3, $\mathcal{G}$ is also a subspace of $C(X,Y)$ (in the topology of compact convergence). Therefore, $\mathcal{F}$ is contained in a compact subspace $\mathcal{G}$ of $C(X,Y)$. □

Proof of (b). Assume that X is a locally compact Hausdorff space, and let $\mathcal{H}$ be a compact subspace of $C(X,Y)$ which contains $\mathcal{F}$. We show that $\mathcal{H}$ is pointwise equicontinuous under d and pointwise (relatively) compact, from which it follows that $\mathcal{F}$ is pointwise equicontinuous under d and pointwise relatively compact.

We first show that $\mathcal{H}$ is pointwise compact. For this, we fix a point $a \in X$, and consider the following composition:

$$\begin{tikzcd}
C(X,Y) \arrow[r, hook, "\imath_a"] & X \times C(X,Y) \arrow[r, "\mathsf{ev}"] & Y \\
H \arrow[u, hook] \arrow[rr, "\jmath"'] & & H(a) \arrow[u, hook]
\end{tikzcd},$$

where $\imath_a$ maps $f \in C(X,Y)$ to $(a,f) \in X \times C(X,Y)$. Because $\imath_a$ is continuous for all $a \in X$ and ev is continuous (see Theorem 5.3.13, and note that the compact convergence topology on $C(X,Y)$ and the compact-open topology on the space coincide). Hence, $\jmath$ maps the compact subspace $\mathcal{H}$ of $C(X,Y)$ onto a compact $\mathcal{H}(a)$ of Y.

We now show that $\mathcal{H}$ is pointwise equicontinuous under d; for this, we fix a point $a \in X$ and let A be a compact subspace of X containing a neighborhood of a in X. Then it suffices to check that the following restriction

$$\mathcal{R} := \{f|_A : f \in \mathcal{H}\}$$

is equicontinuous at a. Give $C(A,Y)$ the compact convergence topology. Because A is compact, the compact convergence topology on $C(A,Y)$ and the uniform topology on $C(A,Y)$ coincide. Also, it can be easily verified that the restriction map $r : C(X,Y) \to C(A,Y)$ is continuous (see the following problem), which implies that $\mathcal{R} = r(\mathcal{H})$ is a compact subspace of $C(A,Y)$. Hence, $\mathcal{R}$ is totally bounded under the uniform metric on $C(A,Y)$ induced by d, so $\mathcal{R}$ is pointwise equicontinuous under d. □

Problem 5.4.1. Show that the restriction map $r : C(X,Y) \to C(A,Y)$ in the proof of (b) above is continuous.

Solution. Given $f \in C(X,Y)$ and a basis member $B_2 := B_K(f,\epsilon)$ of the topology on $C(A,Y)$, because K is a compact subspace of A, K is a compact subspace of X. Hence, one can consider the basis member $B_1 := B_K(f,\epsilon)$ of the topology on $C(X,Y)$, and this basis member is mapped onto B_2 by the restriction map.

5.4.2 Some particular versions of Ascoli's theorem

We first introduce the following classical version of Ascoli's theorem. The following remark plays as a technically essential tool in this subsection, and it worths considering the remark as a lemma:

Remark. When (Y,d) is a complete metric space, relative compactness and total boundedness for subspaces of Y coincide.

Theorem 5.4.2 (Classical version of Ascoli's theorem)**.** Let X be a compact space; let $(\mathbb{R}^n, d)$ denote the Euclidean space in either the square metric or the Euclidean metric; give $C(X,\mathbb{R}^n)$ the corresponding uniform topology (which coincides the compact convergence topology). For a subset $\mathcal{F}$ of $C(X,\mathbb{R}^n)$, the following are equivalent:

(a) $\mathcal{F}$ is pointwise equicontinuous under d and pointwise bounded.

(b) $\mathcal{F}$ is relatively compact in $C(X,\mathbb{R}^n)$.

Proof. Since boundedness and total boundedness coincide in $\mathbb{R}$, it is clear that (a) implies (b). To prove the converse implication, assume that $\mathcal{F}$ is relatively compact in $C(X, \mathbb{R}^n)$ and let $\mathcal{G}$ be the closure of $\mathcal{F}$ in $C(X, \mathbb{R}^n)$. What we want to show is the following:

$$\mathcal{G} \text{ is pointwise equicontinuous under } d \text{ and pointwise bounded.}$$

We first prove that $\mathcal{G}$ is pointwise bounded. Since $\mathcal{G}$ is a compact subspace of $C(X, Y)$, given $\epsilon > 0$, there are finitely many maps $f_1, \cdots, f_n \in \mathcal{F}$ such that $\mathcal{G} \subset \bigcup_{i=1}^n B_{\overline{\rho}}(f_i, \epsilon)$, thus $\mathcal{G}$ is pointwise bounded. Now we prove pointwise equicontinuity. Suppose $g \in \mathcal{G}$ and let $f \in \mathcal{F}$ be a point such that $\overline{\rho}(g, f) < \epsilon/3$; given a point $p \in X$, let V be a neighborhood of p in X such that $x \in V$ and $h \in \mathcal{F}$ implied $d(h(x), h(p)) < \epsilon/3$. We then have

$$d(g(x), g(p)) \leq d(g(x), f(x)) + d(f(x), f(p)) + d(f(p), g(p)) < \epsilon$$

for all $x \in V$, proving pointwise equicontinuity of $\mathcal{G}$. ☐

Corollary 5.4.3. Let X be a compact space; let d denote either the square metric or the Euclidean metric on $\mathbb{R}^n$; give $C(X, \mathbb{R}^n)$ the corresponding uniform topology (which coincides the compact convergence topology). For a subset $\mathcal{F}$ of $C(X, \mathbb{R}^n)$, the following are equivalent:

(a) $\mathcal{F}$ is compact.

(b) $\mathcal{F}$ is closed in $C(X, \mathbb{R}^n)$, pointwise equicontinuous under d, and bounded under the sup metric ρ.

Proof. Note the following observations:

- By Ascoli's theorem, $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is pointwise equicontinuous under d and pointwise totally bounded.
- By Heine-Borel theorem, $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is complete and totally bounded under the sup metric.
- Because $C(X, Y)$ is complete under the sup metric, closedness and completeness coincide for subspaces of $C(X, Y)$.

Hence, (a) implies (b). To show that (b) implies (a), it suffices to check that boundedness of $\mathcal{F}$ implies pointwise total boundedness of $\mathcal{F}$. Since $\mathcal{F} \subset B_{\overline{\rho}}(0, R)$ for some real number $R > 0$, $\mathcal{F}(a) \subset [-R, R]$ for all $a \in X$. Being a subspace of a totally bounded space, $\mathcal{F}(a)$ is also totally bounded for each $a \in X$, as desired. ☐

As the second type, we assume that X is a compact space and (Y, d) is a complete metric spcae.

Theorem 5.4.4. Let X be a compact space, (Y, d) be a complete metric space, and $\mathcal{F}$ be a subset of $C(X, Y)$. Give $C(X, Y)$ the compact convergence topology (which coincides the uniform topology).

(a) If $\mathcal{F}$ is pointwise equicontinuous and pointwise totally bounded, then $\mathcal{F}$ is relatively compact.

(b) The converse of (a) holds if X is a Hausdorff space.

Remark. Suppose X is a compact Hausdorff space. Then $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is pointwise equicontinuous and pointwise totally bounded.

Proof. The results follow from Ascoli's theorem and the first remark. ☐

As the third type, we assume that X is a locally compact Hausdorff space and (Y, d) is a complete metric space.

Theorem 5.4.5. Suppose X is a locally compact Hausdorff space and (Y, d) is a complete metric space. Give $C(X, Y)$ the compact convergence topology. Then a subset $\mathcal{F}$ of $C(X, Y)$ is relatively compact if and only if $\mathcal{F}$ is pointwise totally bounded and pointwise equicontinuous under d.

Proof. Note again that pointwise total boundedness and pointwise relative compactness coincide, since (Y,d) is complete. $\square$

Corollary 5.4.6 (Generalized Arzelà's theorem)**.** Let X be a σ-compact Hausdorff space and (Y,d) be a complete metric space. Give $C(X,Y)$ the compact convergence topology. If a sequence $(f_n)_{n\in\mathbb{N}} \subset C(X,Y)$ is pointwise equicontinuous and pointwise totally bounded under d, then the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence which converges compactly in $C(X,Y)$.

Proof. $\square$

Corollary 5.4.7 (Arzelà's theorem of undergraduate mathematical analysis)**.** Let X be a compact Hausdorff space and (Y,d) be a complete metric space. Give $C(X,Y)$ the compact convergence topology (which coincides the uniform topology). If a sequence $(f_n)_{n\in\mathbb{N}} \subset C(X,Y)$ is pointwise equicontinuous and pointwise totally bounded under d, then the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence which converges compactly (or uniformly, since X is assumed to be compact) in $C(X,Y)$.

Proof. By the preceeding theorem, $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $C(X,Y)$. Because $C(X,Y)$ is a Hausdorff space, the closure of $(f_n)_{n\in\mathbb{N}}$ is closed, which justifies the assertion. $\square$

Remark. When Y is given as $\mathbb{R}$ or $\mathbb{C}$, total boundedness reduces to boundedness, since they are equivalent on those metric spaces.

Chapter 6

Introduction to algebraic topology

6.1 Basic theory on homotopies

Throughout this section, unless stated otherwise, I denotes the unit interval $[0, 1]$ in $\mathbb{R}$, and both X and Y are, as usual, assumed to be topological spaces.

Definition 6.1.1 (Homotopy of continuous maps)**.** Let f and g be continuous maps from X into Y. A continuous map $F : X \times I \to Y$ is called a homotopy in Y between f and g if

$$F(\cdot, 0) = f(\cdot) \text{ and } F(\cdot, 1) = g(\cdot).$$

In this case, two continuous maps f and g are said to be homotopic in Y, and it is denoted by $f \simeq g$. In particular, if g is a constant map, then f is said to be nulhomotopic.

Example 6.1.2. Suppose there is a path in a topological space X from a to b. Then e_a and e_b are homotopic, where $e_x : I \to X$ with $x \in X$ is the constant map with the range $\{x\}$. In particular, in a path-connected space, every pair of two constant maps is homotopic.

Definition 6.1.3 (Path homotopy of paths)**.** Given a topological space X, let f and g be paths from $x_0 \in X$ to $x_1 \in X$. A continuous map $F : I \times I \to X$ is called a path homotopy in X between f and g if

$$F(\cdot, 0) = f \text{ and } F(\cdot, 1) = g, \text{ and } F(0, \cdot) = x_0 \text{ and } F(1, \cdot) = x_1.$$

In this case, two paths f and g are said to be path homotopic in X, and it is denoted by $f \simeq_{\mathsf{p}} g$.

Example 6.1.4. Two paths in $\mathbb{R}^2$ from $(-1, 0)$ to $(1, 0)$ along the upper and the lower hemicircle are path homotopic in $\mathbb{R}^2$, while they are not path homotopic in $\mathbb{R}^2 \setminus \{0\}$.

Some easy observations now follow.

Observation 6.1.5. $\simeq$ and $\simeq_{\mathsf{p}}$ are equivalence relations on $C^0(X, Y)$ and $C^0(I, X)$, respectively. (Justification is left as an exercise.)

Notation. For a path $f : I \to X$, $[f]$ denotes the equivalence class in $C^0(I, X)/\simeq_{\mathsf{p}}$ containing f, i.e., the collection of all paths which are path homotopic to f in X. Such equivalence class $[f]$ is called the path homotopy class of f.

When proving Observation 6.1.5, one may have adopted the idea to concatenate two curves. Such operation will be called the product of two paths.

Definition 6.1.6 (Product of paths)**.** Let $f, g : I \to X$ be paths such that the final point of f and the initial point of g are the same, i.e., $f(1) = g(0)$. The product $f * g$ of f and g is defined to be the map defined by

$$(f * g)(t) = \begin{cases} f(2t) & (\text{if } 0 \le t \le 1/2) \\ g(2t - 1) & (\text{if } 1/2 \le t \le 1) \end{cases},$$

which is a path in X.

Definition 6.1.7 (Product of path homotopy classes)**.** Let $[f]$ and $[g]$ be path homotopy classes such that $f(1) = g(0)$. We define the product $[f] * [g]$ of $[f]$ and $[g]$ by $[f] * [g] = [f * g]$.

Remark. Indeed, the above definition of the product of two path homotopy classes is well-defined, provided that two curves can be concatenated: If $f \simeq_{\mathsf{p}} f_\star$ and $g \simeq_{\mathsf{p}} g_\star$, then $f * g \simeq_{\mathsf{p}} f_\star * g_\star$. It can be easily verified as follows: If F and G are path homotopies between f and $f_\star$ and g and $g_\star$, respectively, then the map $H : I \times I \to X$ defined by

$$H(s,t) = \begin{cases} F(2s,t) & (\text{if } 0 \le s \le 1/2) \\ G(2s-1,t) & (\text{if } 1/2 \le s \le 1) \end{cases}$$

is a path homotopy between $f * g$ and $f_\star * g_\star$.

Other simple but essential observations, which require tedious proof, follow.

Observation 6.1.8. Let $k : X \to Y$ be a continuous map and F be a path homotopy in X between two paths f and g.

(a) $k \circ F$ is a path homotopy between $k \circ f$ and $k \circ g$.

(b) If $f(1) = g(0)$, then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Proof. Clear. □

Observation 6.1.9. The operation $*$ on path homotopy classes has the following properties:

(a) (Associativity) If $[f] * ([g] * [h])$ is defined, then so is $([f] * [g]) * [h]$, and they are equal.

(b) (Right and left identities) Given a point x of X, let e_x denote the constant function from I into X with the range $\{x\}$. If f is a path in X from x_0 to x_1, then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(c) (Inverse) Given a path f in X from x_0 to x_1, let $\overline{f}$ be the reverse of f. Then $[f] * [\overline{f}] = [e_{x_1}]$ and $[\overline{f}] * [f] = [e_{x_0}]$.

Proof. Every statement is obvious, but we provide a technical proof.

(a) The two products are defined if and only if $f(1) = g(0)$ and $g(1) = h(0)$. In this case, we may explicitly give a homotopy in X between $f * (g * h)$ and $(f * g) * h$ to obtain that $[f] * ([g] * [h]) = [f * (g * h)] = [(f * g) * h] = ([f] * [g]) * [h]$, which is quite tedious.

(b) To show $f \simeq_{\mathsf{p}} e_{x_0} * f$, it suffices to show that $\mathit{id}_I \simeq_{\mathsf{p}} e_0 * \mathit{id}_I$, where $e_0 : I \to I$ is the constant map with the image $\{0\}$. The latter path homotopy easily follows, since I is convex. For the same reasoning, we have $\mathit{id}_I \simeq_{\mathsf{p}} \mathit{id}_I * e_1$ so that $f \simeq_{\mathsf{p}} f * e_{x_1}$.

(c) Remark that $f * \overline{f} = (f \circ \mathit{id}_I) * (f \circ \overline{\mathit{id}_I}) = f \circ (\mathit{id}_I * \overline{\mathit{id}_I}) \simeq_{\mathsf{p}} f \circ e_0 = e_{x_0}$, and the same reasoning proves that $\overline{f} * f \simeq_{\mathsf{p}} e_{x_1}$. □

Remark. In the textbook, the following (obvious) statement is introduced as a theorem.

> Let f be a path in X and let $a_0, a_1, \cdots, a_n$ be numbers such that $0 = a_0 < a_1 < \cdots < a_n = 1$. Let $f_i : I \to X$ be the path in X defined by $f_i = f \circ r_i$, where $r_i : I \to [a_{i-1} \circ a_i]$ for each $i = 1, \cdots, n$. Then $[f] = [f_1] * \cdots * [f_n]$.

A corollary which will be labeled a theorem of the preceeding observation is that the first homotopy group (called the fundamental group, in general) of a topological space relative to a point is a group. The definition of the first homotopy group and further topics will be introduced in the folloiwng section, and we end this section with some problems.

Problem 6.1.1. Let X, Y, Z be topological spaces and $h, h' : X \to Y$ and $k, k' : Y \to Z$ be continuous maps which are homotopic, respectively. Show that $k \circ h$ and $k' \circ h'$ are homotopic.

Solution. Let $H : X \times I \to Y$ and $K : Y \times I \to Z$ denote homotopies in Y and Z between h and h', and between k and k', respectively. Consider the map $G : X \times I \to Z$ defined by $G(x,t) = K(H(x,t),t)$ for $(x,t) \in X \times I$. Then G is continuous, and $G(\cdot, 0) = K(H(\cdot, 0), 0) = k \circ h$, and $G(\cdot, 1) = K(H(\cdot, 1), 1) = k' \circ h'$. Therefore, G is a homotopy in Z between $k \circ h$ and $k' \circ h'$.

Remark. Because a continuous map is homotopic to itself, we have $k \circ h \simeq k \circ h'$ and $k \circ h \simeq k' \circ h$.

Problem 6.1.2. Let X and Y be topological spaces and $[X, Y]$ be the set of homotopy classes of continuous maps from X into Y.

(a) Show that $[X, I]$ has a single element.

(b) Show that if Y is path connected then $[I, Y]$ has a single element.

Solution. (a) Every continuous map from X into I is homotopic to the zero map in I.
(b) Given a continuous map $f : I \to Y$, we may consider f a path in Y. Let γ be a path in Y from a to $f(0)$, where a is a given point of Y. Define a map $H : I \times I \to Y$ by $H(x, t) = (\gamma * f)(((1-t)(x+1))/2)$, which is continuous. Then $H(x, 0) = f(x)$ and $H(x, 1) = \gamma(0) = g$, so H is a homotopy in Y between f and the constant map on X with the image $\{g\}$, which proves that $[I, Y]$ has a single element.

Problem 6.1.3. A topological space X is said to be contractible if the identity map of X is nulhomotopic.

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path-connected.

(c) Show that the set $[X, Y]$ has a single element if Y is contractible.

(d) Show that the set $[X, Y]$ has a single element if X is contractible and Y is path-connected.

Solution. (a) Consider the map $H : X \times I \to X$ defined by $(x, t) = (1-t)x$ for $(x, t) \in X \times I$ with $X = I$ or $X = \mathbb{R}$.
(b) Let $H : X \times I \to X$ be a homotopy in X between *id*$_X$ and a constant map c whose range is x_0 for some $x_0 \in X$. It suffices to show the existence of a path in X from x to x_0 for each $x \in X$. In fact, for each point x in X, the map $H(x, \cdot)$ is a path in X from x to x_0.
(c) Assume *id*$_Y$ and a constant map $c : Y \to Y$ on Y are homotopic in Y (write the range of c as $\{y_0\}$). Then *id*$_Y \circ f$ and $c \circ f$ are homotopic in Y, where the former composition is f and the latter composition is the constant map with the range $\{y_0\}$, so $[X, Y]$ has a single element.
(d) The reasoning used in (c) gives that for any continuous map $f : X \to Y$ we have $f \circ$ *id*$_X \simeq f \circ c$ for the constant function $c : X \to X$ with the range $\{x_0\}$. This gives the homotopy $f \simeq f(x_0)$, where the constant is abused in the right-hand side. Even though the right-hand side varies as f varies, because any two constant maps from Y into Y are homotopic (because Y is path-connected), it follows that $f \simeq a$ for a given point a in Y and that $[X, Y]$ has a single element.

6.2 The fundamental group

Notation. The category of pointed spaces (the pair of a topological space and a point of the space) is denoted by $\mathbf{Top}_*$, where the morphisms of $\mathbf{Top}_*$ are the continuous maps preserving base points, i.e., for any two objects (X, a) and (Y, b) of $\mathbf{Top}_*$, $f : (X, a) \to (Y, b)$ is a morphism from (X, a) to (Y, b) if f is a continuous map and $f(a) = b$.

Definition 6.2.1 (Fundamental group (First homotopy group))**.** Let X be a topological space and x_0 be a point of X. The path homotopy classes of loops bases at x_0, denoted by $\pi_1(X, x_0)$, with the product operation introduced in the preceeding section, is called the fundamental group (or the first homotopy group) of X relative to the base point x_0.

Theorem 6.2.2. For a topological space X and a point x_0 of X, $(\pi_1(X, x_0), *)$ is a group.

Proof. The product is well-defined on $\pi_1(X, x_0)$, for every path is a loop baset at x_0. The operation is associative, $[e_{x_0}]$ is an(the) identity, and $[\overline{f}]$ is the inverse of $[f]$ for any loop f based at x_0. $\square$

Remark. From now on, the operator $*$ will not be denoted if there is no confusion. The bracket notation for equivalence classes shall be used, though.

Example 6.2.3. The fundamental group of the unit ball B^n in $\mathbb{R}^n$ is trivial, because every loop in B^n based at $x_0 \in B^n$ is path-homotopic to the constant path x_0.

Proposition 6.2.4. Let (X_i, x_i) be pointed spaces for $i = 1, \cdots, n$. Then there is a group isomorphism $\phi : \pi_1\left(\prod_{i=1}^n X_i, (x_i)_{i=1}^n\right) \to \prod_{i=1}^n \pi_1(X_i, x_i)$ such that $\phi[f] = ([\pi_i \circ f])_{i=1}^n$ for all $[f] \in \pi_1\left(\prod_{i=1}^n X_i, (x_i)_{i=1}^n\right)$.

One might ask how the fundamental group of a topological space may depend on the base point. Given two points x_0 and x_1 in X for which there is a path in X from x_0 to x_1, define a map $\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\widehat{\alpha}[f] = [\overline{\alpha}][f][\alpha].$$

It is left as an exercise to show that $\widehat{\alpha}$ is a group isomorphism. Furthermore, if C is the path-connected component of X containing x_0, then $\pi_1(X, x_0) = \pi_1(C, x_0)$. Hence, the fundamental group depends on only the path-connected component containing the base point, and such fundamental group do not give us information whatever about the rest of the space. Also, we may omit to specify a base point when considering a fundamental group of a path-connected space, because any two fundamental groups are isomorphic.

We now verify that pi_1 is a (covariant) functor from $\mathbf{Top}_*$ into $\mathbf{Grp}$. Suppose $h : (X, x_0) \to (Y, y_0)$ is a continuous map (in other words, a morphism from (X, x_0) to (Y, y_0)) and f is a loop in X based at x_0. Then $h \circ f$ is a loop in Y based at y_0. Furthermore, because $h \circ f_1$ and $h \circ f_2$ are path homotopic in Y if f_1 and f_2 are path homotopic in X, we find that $h \circ \gamma_1 \simeq_{\mathsf{p}} h \circ \gamma_2$ in Y when γ_1 and γ_2 are path homotopic loops in X based at x_0. Therefore, the map $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by

$$h_*[\gamma] = [h \circ \gamma]$$

is a well-defined group homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. (Check that h_* is a group homomorphism.) And one can easily find that the map $h \mapsto h_*$ preserves composition and identities: Letting (X, a), (Y, b), and (Z, c) be objects of $\mathbf{Top}_*$,

(i) whenever f is a morphism from (X, a) to (Y, b) and g is a morphism from (Y, b) to (Z, c), we have $(g \circ f)_* = g_* \circ f_*$, and

(ii) $(id_{(X,a)})_* = id_{\pi_1(X,a)}$.

Our observation can be summarized as follows:

Theorem 6.2.5. Let π_1 be the map from $\mathbf{Top}_*$ to $\mathbf{Grp}$, which maps an object (X, x_0) of $\mathbf{Top}_*$ to $\pi_1(X, x_0)$ and a morphism $f : (X, x_0) \to (Y, y_0)$ in $\mathbf{Top}_*$ to f_*. Then π_1 is a (covariant) functor from $\mathbf{Top}_*$ to $\mathbf{Grp}$.

And an easy observation:

Observation 6.2.6. Assume $h : (X, p) \to (Y, q)$ is an isomorphism in $\mathbf{Top}_*$, i.e., a pointed space isomorphism. Then $\pi_1(h) = h_*$ is a group isomorphism.

We solve a (technical) notational problem here. Suppose $h : X \to Y$ is a continuous map and let $h_1 : (X, x_1) \to (Y, y_1)$ and $h_2 : (X, x_2) \to (Y, y_2)$ be morphisms of pointed spaces such that $h_1 = h = h_2$ as continuous maps. Even though we distinguish h_1 and h_2 if $x_1 \neq x_2$, we know that if x_1 and x_2 lies in the same path-connected component of X then $\pi_1(X, x_1) = \pi_1(X, x_2)$. Thus, we may identify $(h_1)_*$ and $(h_2)_*$. The following proposition introduces a circumstance where such identification is possible.

Proposition 6.2.7. Suppose X is a path-connected space, and let $h : X \to Y$ be a continuous map. For any two points x_1 and x_2 of X, write $y_1 = h(x_1)$ and $y_2 = h(x_2)$, and let $h_1 : (X, x_1) \to (Y, y_1)$ and $h_2 : (X, x_2) \to (Y, y_2)$ be the morphisms in $\mathbf{Top}_*$ such that $h_1 = h = h_2$ as continuous maps. Then

$$\widehat{\beta} \circ (h_1)_* = (h_2)_* \circ \widehat{\alpha},$$

where α is any path in X from x_1 to x_2 and β is the path in Y from y_1 to y_2 defined by $\beta = h \circ \alpha$.

$$\begin{tikzcd}
\pi_1(X, x_1) \arrow[r, "\pi_1(h_1)=(h_1)_*"] \arrow[d, "\widehat{\alpha}"'] & \pi_1(Y, y_1) \arrow[d, "\widehat{\beta}"] \\
\pi_1(X, x_2) \arrow[r, "\pi_1(h_2)=(h_2)_*"'] & \pi_1(Y, y_2)
\end{tikzcd}$$

Proof. Easy. □

We end this section with some interesting circumstances.

Definition 6.2.8 (Simply connected space)**.** A path-connected space is said to be simply connected if its fundamental group is trivial.

Example 6.2.9. (a) Because every loop in B^n or in $\mathbb{R}^n$ is path homotopic to a point, B^n and $\mathbb{R}^n$ is simply connected.

(b) A star convex subset of $\mathbb{R}^k$ is simply connected.

Proposition 6.2.10. Let X be a path-connected space. Then the fundamental group of X is abelian if and only if for any two points p and q of X and any two paths α and β in X from p to q we have $\widehat{\alpha} = \widehat{\beta}$.

Proof. Let p and q be any two points of X. Assume first that $\pi_1(X, p)$ is abelian, and let α and β be two paths in X from p to q. Then, whenever f is a loop in X based at p, we have

$$[\alpha\overline{\beta}][f][\beta\overline{\alpha}] = [\alpha\overline{\beta}][\beta\overline{\alpha}][f] = [e_p][f] = [f],$$

from which we obtain $\widehat{\alpha}[f] = \widehat{\beta}[f]$ for all loops f in X based at p. Conversely, assume $\widehat{\alpha} = \widehat{\beta}$ whenever α and β are paths in X from p to q, where p and q are arbitrary points in X. Given two loops f and g based at p and a path α in X from p to q, define $\beta = f * \alpha$. Because β is a path in X from p to q, we have $\widehat{\alpha} = \widehat{\beta}$, so $[\overline{\alpha}g\alpha] = [\overline{\beta}g\beta] = [\overline{\alpha}\overline{f}gf\alpha]$, from which we obtain $[f][g] = [g][f]$. □

6.3 Covering spaces

Definition 6.3.1 (Covering space)**.** Let $p : E \to B$ be a surjective continuous map. An open subset U of B is said to be evenly covered by p if

(i) there is a partition $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ of $p^{-1}(U)$ into open subsets of E, and

(ii) the restriction of p to V_α denotes a homeomorphism from V_α onto U for each $\alpha \in \mathcal{A}$.

A continuous surjection $p : E \to B$ is called a covering map and E is called a covering space of B if every point of B admits a neighborhood of B which is evenly covered by p.

In the following proposition, some basic properties of covering maps are introduced.

Proposition 6.3.2. Let $p : E \to B$ be a covering map.

(a) An open subset of an open subset of B which is evenly covered by p is also evenly covered by p.

(b) For each point b of B, $p^{-1}(\{b\})$ has the discrete topology.

(c) p is an open map.

Example 6.3.3. (a) (Projection) Let $\pi_X : X \times Y \to X$ be the canonical projection map. If Y has the discrete topology, then π_X is a covering map.

(b) Let $p : E \to B$ be a continuous surjection and U be a nonempty subset of B which is evenly covered by p. If U is connected, then the partition of $p^{-1}(U)$ into slices is unique, because each slice is a connected component of $p^{-1}(U)$.

(c) Let $p : E \to B$ be a covering map. If B_0 is a subspace of B and $E_0 = p^{-1}(B_0)$, then $p_0 := p|_{E_0} : E_0 \to B_0$ is a covering map.

$$\begin{tikzcd}
E \arrow[r, "p", "\text{covering}"'] \arrow[d, dash] & B \arrow[d, dash] \\
E_0 \arrow[r, "\text{covering}", "p_0"'] & B_0
\end{tikzcd}$$

(d) Let $p_k : E_k \to B_k$ be a covering map for $k = 1, \cdots, n$. Then $p_1 \times \cdots \times p_n : E_1 \times \cdots \times E_n \to B_1 \times \cdots \times B_n$ is a covering map.

Problem 6.3.1. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps, and let $p = r \circ q$. Show that p is a covering map, provided that every fiber of r is finite.

Solution.

Problem 6.3.2. Let $p : E \to B$ be a covering map.

(a) Show that E is a Hausdorff, regular, completely regular, or locally compact Hausdorff space, provided that B is so.

(b) Show that E is compact if B is compact and every fiber of p is finite.

Solution.

6.4 The fundamental group of the circle

Remark. In this chapter, when considering a lift of a continuous map $f : X \to B$, we assume that a covering map $p : E \to B$ is given and we consider a lift $\widetilde{f} : X \to E$ of f to E. In other words, unless stated otherwise, we consider a map $\widetilde{f} : X \to E$ such that $p \circ \widetilde{f} = f$.

Theorem 6.4.1 (Uniqueness of a lifting)**.** Let $p : E \to B$ be a covering map and $f : X \to B$ be a continuous map. Assume that X is a connected space, and suppose there are liftings g_1 and g_2 of f to E which coincide at a point of X. Then $g_1 = g_2$.

$$\begin{tikzcd} & (E, e_0) \arrow[d, two heads, "p\,(\text{covering})"] \\ (X, x_0) \arrow[ur, "g_1 = g_2"] \arrow[r, "f"'] & (B, b_0) \end{tikzcd}$$

Proof. Let $A_= = \{x \in X : g_1(x) = g_2(x)\}$ and $A_{\neq} = \{x \in X : g_1(x) \neq g_2(x)\}$; we wish to show that both $A_=$ and $A_{\neq}$ are open in X.

Assume first that $a \in A_=$, and write $e = g_1(a) = g_2(a)$. Let U be a neighborhood of $f(a)$ in B which is evenly covered by p, and let V be the slice of $p^{-1}(U)$ which contains the point e. Finally, set $W = g_1^{-1}(V) \cap g_2^{-1}(V)$, which is a neighborhood of a in X. And it follows that $W \subset A_=$, because whenever $w \in W$, we have $g_1(w), g_2(w) \in V$. Therefore, $W \subset A_=$, proving that $A_=$ is open in X.

Now, assume that $b \in A_{\neq}$, and let U be a neighborhoof of $f(b)$ in B which is evenly covered by p. In this step, let V_1 and V_2 be the slice of $p^{-1}(U)$ which contains $g_1(b)$ and $g_2(b)$, respectively. By the definition of slices, it follows that V_1 and V_2 are disjoint. Letting $W = g_1^{-1}(V_1) \cap g_2^{-1}(V_2)$, W is a neighborhood of b in X, which is contained in $A_{\neq}$, so $A_{\neq}$ is open in X.

Because g_1 and g_2 collapse at a point of X, $A_=$ is nonempty, so we conclude that $A_{\neq} = \varnothing$ so $g_1 = g_2$, because X is connected. ☐

Example 6.4.2. The above theorem asserts that two liftings over a connected spaces which coincide at a point are necessarily identical. This does not hold when two liftings do not coincide. For instance, consider the case where $p : \mathbb{R} \to S^1$ is the natural covering map and $X = [0, 1]$. When $f : X \to S^1$ is defined by $f(s) = \exp 2\pi i s$ for all $s \in [0, 1]$, there are two distinct liftings $g_1, g_2 : [0, 1] \to \mathbb{R}$ of f to $\mathbb{R}$, e.g.,

$$g_1(x) = x \text{ and } g_2(x) = x + 1 \text{ for all } x \in [0, 1].$$

Observe that these two liftings do not coincide at any point of $[0, 1]$.

The above lemma deals with the uniqueness of a lifting over a connected space, provided that a lifting exists. The following lemma deals with both the existence and the uniqueness of a lifting of a path.

Theorem 6.4.3 (Path lifting theorem)**.** Let $p : (E, e_0) \to (B, b_0)$ be a covering map. Any path $f : (I, 0) \to (B, b_0)$ in B has a unique lifting to a path $\widetilde{f} : (I, 0) \to (E, e_0)$ in E.

$$\begin{tikzcd}
 & (E,e_0) \arrow[d, two heads, "p\text{ (covering)}"] \\
(I,0) \arrow[ur, "\widetilde{f}\text{ (unique)}"] \arrow[r, "f"'] & (B,b_0)
\end{tikzcd}$$

Remark. The path lifting theorem is quite obvious when B is evenly covered by p, for we may choose the restriction of p to the slice containing e_0 and composite the inverse of the restriction to the path. Even if B need not be evenly covered, because p is a covering map, we shall make use of open subsets of B which are evenly coveredby p.

Proof. We start the proof by covering B by the open subsets U which are evenly covered by p. Then the open subsets $f^{-1}(U)$ of I cover I, and the Lebesgue number lemma implies that there is a positive real number δ such that any open subset of I of diameter less than δ is contained in at least one $f^{-1}(U)$. So there is a division $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ of I such that $f([s_i, s_{i+1}])$ is contained in any U for $i = 0, 1, \cdots, n-1$.

We shall define a lifting $\widetilde{f} : I \to E$ of f and prove the uniqueness step by step. By assumption, we have $\widetilde{f}(0) = e_0$. Now we suppose $\widetilde{f}(s)$ is defined for all $s \in [0, s_i]$, where $0 \le i \le n-1$. Let U_i be any open subset of B which is evenly covered by p containing $f([s_i, s_{i+1}])$, and let V_i be the slice of $p^{-1}(U_i)$ containing $\widetilde{f}(s_i)$. Since $p|_{V_i} : V_i \to U_i$ is a homeomorphism, we may define $\widetilde{f}(s) = (p|_{V_i})^{-1}(f(s))$ for $s \in [s_i, s_{i+1}]$, which defines a continuous map on $[0, s_{i+1}]$. This proves the existence of a lifting of f to a path in B beginning at b_0.

We now prove the uniqueness part. Let g_2 be a lifting of f to a path in E beginning at e_0, and write $g_1 = \widetilde{f}$. Suppose $g_1 = g_2$ on $[0, s_i]$, where $0 \le i \le n-1$, and let U_i and V_i be defined as above. Note that $p(g_1([s_i, s_{i+1}])) = p(g_2([s_i, s_{i+1}])) = f([s_i, s_{i+1}]) \subset U_i$ and $g_1(s_i) = g_2(s_i) \in V_i$. Because $g_1([s_i, s_{i+1}])$ and $g_2([s_i, s_{i+1}])$ are connected, they are contained in V_i, so for each $s \in [s_i, s_{i+1}]$ we have $g_1(s) = (p|_{V_i})^{-1}(f(s)) = g_2(s)$. Hence, by induction, we have $g_1 = g_2$ on I. $\square$

Theorem 6.4.4 (Homotopy lifting theorem)**.** Let $p : (E, e_0) \to (B, b_0)$ be a covering map. Let the map $F : (I \times I, (0, 0)) \to (B, b_0)$ be a continuous map. Then there is a unique lifting of F to a continuous map $\widetilde{F} : (I \times I, (0, 0)) \to (E, e_0)$. In particular, if F is a path homotopy, then so is $\widetilde{F}$.

$$\begin{tikzcd}
 & (E,e_0) \arrow[d, two heads, "p\text{ (covering)}"] \\
(I \times I,(0,0)) \arrow[ur, "\widetilde{F}\text{ (unique)}"] \arrow[r, "F"'] & (B,b_0)
\end{tikzcd}$$

Proof. For each point of B, consider a neighborhood which is evenly covered by p. Considering the preimages under F, note that $I \times I$ is compact so that we may apply the Lebesgue number lemma to find a partition

$$0 = s_0 < s_1 < \cdots < s_{m-1} < s_m = 1 \text{ and } 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1,$$

where each rectangle is mapped into an open subset of B which is evenly covered by p. $\square$

Definition 6.4.5 (Lifting correspondence)**.** Let $p : (E, e_0) \to (B, b_0)$ be a covering map. Given an element $[f] \in \pi_1(B, b_0)$, let $\widetilde{f}$ be the lifting of f to the path in E beginning at e_0, and let $\phi[f]$ denote the endpoint $\widetilde{f}(1)$ of $\widetilde{f}$.[1] Then $\phi : \pi_1(B, b_0) \to p^{-1}(\{b_0\})$ is a well-defined set map, which is called the lifting correspondence derived from the covering map p and e_0.

[1] Note that the lifting of a loop need not be a loop.

Theorem 6.4.6. Let $p : (E, e_0) \to (B, b_0)$ be a covering map. If E is path-connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(\{b_0\})$ is surjective. In particular, if E is simply connected, then ϕ is bijective.

Proof. Assume E is path-connected, and let $\widetilde{\gamma} : I \to E$ be a curve in E from e_0 to e_1, where e_1 is any point of $p^{-1}(\{b_0\})$. Then $[p \circ \widetilde{\gamma}] \in \pi_1(B, b_0)$ and $\phi[p \circ \gamma] = \widetilde{\gamma}(1) = e_1$. Assuming that E is simply connected and $\phi[\gamma_1] = \phi[\gamma_2]$ for some loops γ_1 and γ_2 based at b_0, we find that their respective liftings $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ to E are paths in E from e_0 to a point e_1 in $\phi^{-1}(\{b_0\})$. Because E is simply connected, it follows that $\widetilde{\gamma_1} \simeq_{\mathsf{p}} \widetilde{\gamma_2}$ in E, hence $\gamma_1 = p \circ \widetilde{\gamma_1}$ and $\gamma_2 = p \circ \widetilde{\gamma_2}$ are path homotopic in B. This proves the injectivity of ϕ when E is simply connected. $\square$

Theorem 6.4.7. The fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

Proof. Use the map $\phi : \pi_1(S^1, e^0) \to p^{-1}(\{0\})$ defined as above; because $\mathbb{R}$ is simply connected, it remains to verify that ϕ is a group homomorphism. $\square$

Example 6.4.8. Define a relation $\sim$ on S^2 by declaring $x \sim y$ for $x, y \in S^2$ if and only if either $x = y$ or $x = -y$. Then $\sim$ denotes an equivalence relation on S^2. Write $X = S^2/\sim$ denote the quotient space, and let α be any shortest path in S^2 from the north pole to the south pole and let β be the constant loop in S^2 based at the north pole. Because their projections onto X are loops but have distinct valued under ϕ, we conclude that X is not simply connected.

Theorem 6.4.9. Let $p : (E, e_0) \to (B, b_0)$ be a covering map.

(a) The homomorphism $p_* = \pi_1(p)$ is a group monomorphism.

(b) Let $H = \mathit{im}\,(p_*)$. Then the lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0),$$

which is bijective if E is path-connected.

(c) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

Proof. (a) It suffices to prove the injectivity of p_*. Suppose $p_*[\gamma] = [e_{b_0}]$, i.e., $p \circ \gamma \simeq_{\mathsf{p}} e_{b_0}$. Then the lifting of $p \circ \gamma$ and e_{b_0} to E with e_0 at 0 are path homotopic. Because γ and e_{e_0} is the unique such lifting of $p \circ \gamma$ and e_{b_0}, respectively, we find that $[\gamma]$ is the identity element of $\pi_1(E, e_0)$. Therefore, p_* is a group monomorphism.
(b) ——@
(c) ——@ $\square$

Part II

Further theory in general topology

Chapter 7

Topological groups

7.1 Topological groups

Definition 7.1.1 (Topological groups)**.** A group G is called a topological group if G is also a topological space such that both multiplication and inversion are continuous. (The continuity axiom can be replaced by the following axiom: The map from $G \times G$ into G defined by $(a, b) \mapsto ab^{-1}$ is continuous.)

Remark. A continuous group homomorphism is called a topological group homomorphism.

Let G be a topological group. For any point a of G, the following maps

$$\begin{aligned} \rho_a : G \to G, &\quad g \mapsto ga \\ \lambda_a : G \to G, &\quad g \mapsto ag \\ \gamma_a : G \to G, &\quad g \mapsto aga^{-1} \end{aligned}$$

will be frequently used in this chapter to investigate properties of topological groups. Indeed, the above maps are group isomorphisms which are also homeomorphisms.

Observation 7.1.2. Some direct conclusions from the result that the above maps are group-isomorhphic homeomorphisms are listed here:

(a) G is a homogeneous space, i.e., for every pair x, y of points of G, there is a homeomorphism of G onto itself mapping x to y.

(b) Every neighborhood W of g in G can be written as $W = Ug = gV$, where $U = Wg^{-1}$ and $V = g^{-1}W$ are neighborhoods of the identity.[1]

(c) Let G, K be topological groups and $f : G \to K$ be a group homomorphism. Then f is continuous on G if f is continuous at a point of G.

While (a) is direct and (b) is clear, (c) seems to be explained. Assume that f is continuous at $p \in G$, and choose $x \in G$ and let W be a neighborhood of $y := f(x)$ in K. Then $f(p)y^{-1}W$ is a neighborhood of $f(p)$ in K, hence there is a neighborhood U of p in G such that $f(U) \subset f(p)y^{-1}W$. The neighborhood $xp^{-1}U$ of x in G, as desired, is mapped into W under f.

Example 7.1.3. (a) The (abelian) groups $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{R}^{>0}, \cdot)$ are topological groups. (As usual, impose each space the order topology.)

(b) $(S^1, \cdot)$ is a topological group. (As usual, let $A \subset S^1$ be a basis member if A is the intersection of an open ball in $\mathbb{C}$ and S^1.)

(c) The general linear group $GL_n(\mathbb{R})$ is a topological group, when it is topologized naturally as a subspace of $\mathbb{R}^{n^2}$.

Topological groups behave well in the context of products.

[1] A neighborhood of the identity in a topological group is called an identity neighborhood.

Proposition 7.1.4. Suppose that $(G_\alpha)_{\alpha\in I}$ is a collection of topological groups.

(a) The product $G := \prod_\alpha G_\alpha$, endowed with the product topology, is a topological group.

(b) For each index $\alpha \in I$, the projection map π_α is an open topological group homomorphism.

(c) (A universal property) For every topological group H and topological group homomorphisms $f_\alpha : H \to G$ for $\alpha \in I$, there is a unique topological group homomorphism $f : H \to G$ such that $\pi_\alpha \circ f = f_\alpha$ for each $\alpha \in I$. In short, there is a unique topological group homomorphism $f : H \to G$ satisfying the following commutative diagram.

$$\begin{tikzcd}
 & \prod_{\alpha\in I} G_\alpha \arrow[dr, "\pi_\gamma"] & \\
H \arrow[ur, "f"] \arrow[rr, "f_\gamma"'] & & G_\gamma
\end{tikzcd}$$

Proof. To show that the product of topological groups is a topological group, it suffices to check continuity of multiplication and inversion: being endowed with the product topology and each multiplication and inversion are continuous, the multiplication and inversion in G are also continuous. Because each projection is open and continuous, it is clear that each projection is an open topological group homomorphism. The existence and uniqueness of a "group homomorphism" f is due to a universal property of product groups; its continuity follows from the equation $\pi_\alpha \circ f = f_\alpha$ for $\alpha \in I$. $\square$

Topological groups also behave well in the context of subgroups and closures.

Proposition 7.1.5. A subgroup H of a topological groups G is a topological group. Also, the closure $\overline{H}$ of H in G is a subgroup of G, hence a topological group. In addition, if H is a normal subgroup of G, then $\overline{H}$ is also a normal subgroup of G.

Proof. It is easy to check that subgroups of a topological group is a topological group. To check that $\overline{H}$ is a subgroup of G, it suffices to check if $ab^{-1} \in \overline{H}$ for any $a, b \in \overline{H}$. Considering the following continuous map $\kappa : G \times G \to G$ defined by $\kappa(x, y) = xy^{-1}$, we have $\kappa(\overline{H} \times \overline{H}) = \kappa(\overline{H \times H}) \subset \overline{\kappa(H \times H)} = \overline{H \times H} = \overline{H} \times \overline{H}$, so H is a subgroup of G. Finally, assume H is a normal subgroup of G. To show that $\overline{H}$ is a normal subgroup of G, it suffices to check if $ghg^{-1} \in \overline{H}$ for all $g \in G$ and $h \in \overline{H}$; consider the continuous map γ_g for $g \in G$, and observe that $\gamma_g(\overline{H}) \subset \overline{\gamma_g(H)} = \overline{H}$. $\square$

Regarding subgroups, we introduce the following proposition.

Proposition 7.1.6. Let G be a topological group and H be a subgroup of G.

(a) The subgroup H is open in G if it contains a nonempty open set.

(b) If H is open in G, then H is closed in G.

(c) The subgroup H is closed in G if and only if there is an open subset U of G such that $H \cap U$ is a nonempty closed subspace of U.

Proof. Let H be a subgroup of G. To prove (a), suppose H contains an open subspace U of G. Then H is open in G, since $H = UH = \bigcup_{h\in H} Uh$. To prove (b), assume H is open in G. Because $G \setminus H = \bigcap_{a\in G\setminus H} aH$ is open, H is closed. When proving (c), note that one way is clear. Let U be an open subset of G such that $U \cap H$ is a nonempty closed subspace of U. Letting $V = U \cap \overline{H}$ where the overline is used to denote the closure in G, V is an open subset of $\overline{H}$ such that $V \cap H = U \cap H$ is a nonempty closed subspace of V, hence of $\overline{H}$. Because H is dense in $\overline{H}$, it can be easily checked that $V \cap H$ should be empty and $V \subset H$. Because V is an open subspace of $\overline{H}$, H turns out to be an open subspace of $\overline{H}$, hence a closed subspace of $\overline{H}$. Because the closure of H in $\overline{H}$ is $\overline{H}$, we have $\overline{H} = H$, i.e. H is closed in G, as desired. $\square$

Remark. As an undesired result obtained in the proof of (a) is the following:

For a topological group G and its open subset U, XU and UX are open for any subset X of G.

Using this result, we can derive that the maps $(x, y) \mapsto xy$ and $(x, y) \mapsto x^{-1}y$ defined on $G \times G$ are open (and continuous).

We end this section with the discovery that a T_1-topological space is necessarily a regular space.

Theorem 7.1.7. A topological space satisfying a T_1-axiom is a Hausdorff space and a regular space.

To prove this, we need the following lemma:

Lemma 7.1.8. If U is a neighborhood of $e \in G$ in G, there is a symmetric identity neighborhood V such that $VV \subset U$. (We say a subset A of G is symmetric if $A^{-1} = A$.)

Proof of Lemma 7.1.8. Because multiplication is continuous, there is an identity neighborhood V_1 such that $V_1V_1 \subset U$; because inversion is continuous, there is an identity neighborhood W such that $V := WW^{-1} \subset V_1$. Therefore, $VV \subset U$, and it is easy to check that V is a symmetric identity neighborhood. □

Proof of Theorem 7.1.7. Suppose that every singletone in G is closed and choose two distinct points $x, y \in G$.

Want: an identity neighborhood V in G such that $xy^{-1} \notin VV$.

If the above desire is satisfied, we then have $xy^{-1} \neq u^{-1}v$ for all $u, v \in V$, $ux \neq vy$, thus $Vx \cap Vy = \varnothing$. Using the preceeding lemma, one can find a symmetric identity neighborhood V such that $VV \subset G \setminus \{xy^{-1}\}$. This proves that G is a Hausdorff space.

It remains to prove that a T_1-topological group is regular. For this, it suffices to show the existence of a symmetric identity neighborhood $Vx \cap VA = \varnothing$, where A is a closed subspace of G and x is a point of G not contained in A. As in the first paragraph, our goal is to find an identity neighborhood V in G such that $ax^{-1} \notin VV$ for all $a \in A$, and for this we find such V satisfying $VV \subset G \setminus Ax^{-1}$ by using the preceeding lemma. This proves that G is a regular space. □

7.2 Quotients of topological groups

Suppose that H is a subgroup of a topological group G. We give the quotient G/H the quotient topology induced by the projection map $p : G \to G/H$. Then a subset of G/H is open if and only if its preimage under p is open in G.

Proposition 7.2.1. Let G be a topological group and H be a subgroup of G. Then, the projection map $p : G \to G/H$ is an open quotient map, hence G/H is a quotient space of G. Moreover, the quotient space G/H is Hausdorff if and only if H is closed in G, and G/H is discrete if and only if H is open in G.

Proof. Since G/H is given the quotient topology induced by p, p is clearly a quotient map. If U is an open subspace of G, then $p(U)$ is open in G/H, since $p^{-1}(p(U)) = UH$ is open in G and p is a quotient map. Because p is chosen so that every point of G is mapped to an element of G/H that contains the point, G/H is a quotient space of G.

It is clear that H is closed in G if G/H is a Hausdorff space, since $H = p^{-1}(\{H\})$ and $\{H\}$ is a closed singletone in G/H. Conversely, if H is closed in G, because $H = p^{-1}(\{H\})$, we find that $\{H\}$ is closed in G/H, hence every translation of $\{H\}$ is closed, i.e., G/H is a Hausdorff space.

Similarly, it is clear that H is open in G if G/H is discrete. Converse implication is now almost clear. □

Theorem 7.2.2. Let G be a topological group.

(a) If N is a normal subgroup of G, then the group G/N is a topological group with respect to the quotient topology on G/N.

(b) The quotient map $p : G \to G/N$ is an open topological group homomorphism.

(c) Moreover, the topological group G/N is a Hausdorff space if and only if N is closed. Hence, in particular, $G/\overline{N}$ is a Hausdorff topological group.

Proof. To check that G/N is a topological group, it remains to check if multiplication and inversion are continuous, which are easy to check. The remaining assertions follow from the preceeding proposition. $\square$

Example 7.2.3. Consider the abelian group $(\mathbb{R}, +)$ and its normal subgroup $\mathbb{Z}$. The quotient group $(\mathbb{R}/\mathbb{Z}, +)$ is homeomorphic to $(S^1, \cdot)$

We end this section with a morphism theorem.

Theorem 7.2.4 (Morphism theorem for topological groups)**.** Let $f : G \to H$ be a topological group homomorphism, and suppose that the normal subgroup N of G is contained in *ker*(f) Then f factors through the open topological group homomorphism $p : G \to G/N$ by a unique topological group homomorphism $\overline{f}$, satisfying the following commutative diagram.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & K \\ G/N \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

Furthermore, f is open if and only if $\overline{f}$ is open.

Proof. The existence and uniqueness of such group homomorphism $\overline{f}$ is already proved. Since $p^{-1} \circ \overline{f}^{-1} = f^{-1}$ and p is a quotient map, $\overline{f}$ is easily turned out to be continuous, so $\overline{f}$ is a topological group homomorphism. It can also be checked easily that f is open if and only if $\overline{f}$ is open. $\square$

7.3 Further properties of topological groups

Chapter 8

Topological vector spaces

8.1 Topological vector spaces

8.2 Further properties of topological vector spaces

Chapter 9

Tietze extension theorem

9.1 Tietze extension theorem

Theorem 9.1.1 (Tietze extension theorem)**.** Let X be a normal space and let A be a closed subset of X.

(a) Any continuous map of A into $[a, b] \subset \mathbb{R}$ extends to a continuous map of X into $[a, b]$.

(b) Any continuous map of A into $\mathbb{R}$ extends to a continuous map of X into $\mathbb{R}$.

Proof. The theorem shall be proved by constructing a continuous map of X extending f. To this end, we construct a sequence of continuous functions which converges to f uniformly on X and approximates f on A more and more closely as $n \to \infty$. Then the limit function is continuous on X and it extends f.

Step 1. For convinience, we may assume that f is onto $[-r, r]$ for some positive real number r. Our first goal is to show the existence of a continuous map $g_1 : X \to [-r/3, r/3]$ such that $|f - g_1| \leq 2r/3$ on A. Write $L = f^{-1}([-r, -r/3])$ and $U = f^{-1}([r/3, r])$, which are closed subsets of X. Because X is normal, by the Urysohn lemma, there is a continuous map $g_1 : X \to [-r/3, r/3]$ such that $g(L) = -r/3$ and $g(U) = r/3$. Then g_1 satisfies the desired property.

Step 2. We now prove part (a) of the theorem. After finding g_1, because $|f - g_1| \leq 2r/3$, we can find a continuous map $g_2 : X \to [-(2/3)^2 r, (2/3)^2 r]$ such that $|(f - g_1) - g_2| \leq (2/3)^2 r$ on A. By induction, for each $n \in \mathbb{N}$, there is a continuous map $g_n : X \to [-(2/3)^n r, (2/3)^n r]$ such that $|f - (g_1 + \cdots + g_n)| \leq (2/3)^n r$ on A. Moreover, the series $\sum g_n$ is absolutely (hence, uniformly) convergent on X. Therefore, defining the map $g : X \to [-r, r]$ by $g = \sum g_n$, we find that g is a continuous map (being the limit of a uniformly convergent sequence of continuous maps) and that $f = g$ on A.

Step 3. We finish the proof with a proof of part (b) of the theorem. Since the open interval $(-1, 1)$ in $\mathbb{R}$ and $\mathbb{R}$ are homeomorphic, we may assume that f is into $(-1, 1)$ (by compositing homeomorphisms). Part (a) of the theorem gives a continuous extension $g : X \to [-1, 1]$ of f. Since we wish to obtain a continuous extension of f with values in $(-1, 1)$, we may wish to preserve the values of f (or equivalently. g) on A and remove the values 1 and -1 from g. To this end, let $D = g^{-1}(\{\pm 1\})$, which is closed in X. Because A and D are disjoint closed subsets of X, by the Urysohn lemma, there is a continuous map $h : X \to [0, 1]$ such that $h(A) = \{1\}$ and $h(D) = \{0\}$. Defining a map $k : X \to [-1, 1]$ by $k(x) = g(x)h(x)$ for all $x \in X$, one can easily find that k is a continuous map into $(-1, 1)$ such that $k = f$ on A. Therefore, k is a desired continuous extension of f to X. $\square$

Remark. Assume all conditions in the Tietze extension theorem, except for that f is into $\mathbb{C}$, not into $\mathbb{R}$. By considering the real and imaginary part of f separately, one can deduce that there is a continuous extension of f to X. Also, by considering componentwise, the Tietze extension theorem can be generalized to the case where f is into $\mathbb{R}^I$ or $[0, 1]^I$ for some index set I.

Chapter 10

Introduction to manifolds

10.1 Introduction to manifolds

Definition 10.1.1. (a) (Locally Euclidean space) A topological space X is called locally Euclidean if there is a non-negative integer n such that every point in X has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

(b) (m-manifold) A second countable Hausdorff space in which every point has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^m$ is called an m-manifold.

(c) (Support of a real-valued function) Given a real valued function $\phi : X \to \mathbb{R}$, the support of ϕ is defined to be the closure of $\phi^{-1}(\mathbb{R} \setminus \{0\})$ in X.

Remark (Chart, atlas). In theory of differential manifold, a second countable Hausdorff space which is a locally Euclidean space is called a manifold. For this definition, we first observe the notions which follow, after assuming that X is a topological space.

(a) (Chart) A pair (U, φ) of an open subset U of X and a homeomorphism φ from U onto an open subset of a Euclidean space is called a chart.[1]

(b) (Atlas) An indexed family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is called an atlas for X if it covers X.

One should remark that a topological space X is a locally Euclidean space if and only if X admits an atlas of which the codomain of each chart is $\mathbb{R}^n$ for some non-negative integer n.

Remark (The separability of manifolds). We wish to call, for example, a 1-manifold a curve, a 2-manifold a surface, and so on. Consider the line with two origins, which is second countable, locally Euclidean, but not a Hausdorff space. Drawing the line with two origins, nobody would wish to call it a curve, so the condition of being a Hausdorff space is essential in defining manifolds.

Proposition 10.1.2. Every manifold is completely regular, hence, metrizable.

Proof. Let M be an m-manifold for some positive integer m. It suffices to show that M is locally compact; then M is a locally compact Hausdorff space, a completely regular space, and the metrizability is immediate from the Urysohn metrization theorem. Let x be a point of M and A be a neighborhood of x in M which embeds into an open subset W of $\mathbb{R}^m$; let $f : A \to W$ be such homeomorphism. Because $\mathbb{R}^m$ is a locally compact Hausdorff space, there is a relatively compact neighborhood V of $f(x)$ in $\mathbb{R}^m$ whose closure in $\mathbb{R}^m$ is contained in W. Then $f^{-1}(V)$ is open in A, hence, in M. Because $f^{-1}(\overline{V})$ is a compact subspace of A, it is a compact subspace of M which contains a neighborhood of x, namely, $f^{-1}(V)$. This proves that M is locally compact. $\square$

In dealing with the support of a function $f : X \to \mathbb{R}$, because the support K of f is defined to be closed in X, the argument that $X \setminus K$ will be frequently used. In particular, whenever x is a point in $X \setminus K$, there is a neighborhood U of x on which f vanishes.

[1] A chart is also called a coordinate chart, coordinate patch, coordinate map

In the beginning of this section, we proved that an m-manifold is completely regular, hence metrizable. In fact, because an m-manifold X is second countable and completely regular, X embeds into $\mathbb{R}^{\mathbb{N}}$ or $[0,1]^{\mathbb{N}}$, which is infinite dimensional. We wish to reduce the dimension to a finite number, which is possible when X is a compact manifold.

Theorem 10.1.3. If M is a compact m-manifold with a positive integer m, then X embeds into $\mathbb{R}^N$ for some positive integer N.

Proof. Because M is a compact m-manifold, there is a finite open cover $\{U_1, \cdots, U_n\}$ of X such that U_i is homeomorphic to an open subset of $\mathbb{R}^m$ for $i = 1, \cdots, n$; let $g_i : U_i \hookrightarrow \mathbb{R}^m$ denote such homeomorphism for $i = 1, \cdots, n$. Remark that M is a compact Hausdorff space, so M is normal. Hence, there is a partition $\{\phi_1, \cdots, \phi_n\}$ of unity dominated by $\{U_1, \cdots, U_n\}$. Letting A_i be the support of ϕ_i for $i = 1, \cdots, n$, define a map $h_i : X \to \mathbb{R}^m$ for each $i = 1, \cdots, n$ by

$$h_i(x) = \begin{cases} \phi_i(x) \cdot g_i(x) & \text{for } x \in U_i \\ 0 & \text{for } x \in X \setminus A_i \end{cases}.$$

Remark that h_i is well-defined and continuous, for $h_i|_{U_i}$ and $h_i|_{X \setminus A_i}$ are continuous.

Define a map $F : X \to \mathbb{R}^{(m+1)n}$ by

$$F(x) = (\phi_1(x), \cdots, \phi_n(x), h_1(x), \cdots, h_n(x)).$$

It is clear that F is continuous, so it remains to show that F is injective to show F is an embedding; because X is compact and the codomain is a Hausdorff space, F is a closed map. Suppose $F(x) = F(y)$ for some points x and y in X. Then $\phi_i(x) = \phi_i(y)$ for $i = 1, \cdots, n$, so $\phi_i(x) = \phi_i(y)$ for some i and $x, y \in U_i$. Then $\phi_i(x) \cdot g_i(x) = h_i(x) = h_i(y) = \phi_i(y) \cdot g_i(y)$, so $g_i(x) = g_i(y)$. Because g_i is injective, it follows that $x = y$, as desired. $\square$

Part III

Some other contents