Measure theory and Lebesgue integral

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Chapter 1

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1.1 σ-algebras

We start this note with the definition of algebras and σ-algebras on a nonempty set.

Definition 1.1.1 (Algebra on a set)**.** Let X be a nonempty set. A nonempty collection $\mathcal{A}$ of subsets of X is called an algebra on X if

(a) $\mathcal{A}$ is closed under set complements

(b) $\mathcal{A}$ is closed under arbitrary finite unions.

Furthermore, if $\mathcal{A}$ is closed under arbitrary 'countable' unions, then $\mathcal{A}$ is called a σ-algebra on X. If $\mathcal{M}$ is a σ-algebra on the set X, then the tuple $(X, \mathcal{M})$ is also called a measurable space. Also, any set in $\mathcal{M}$ is called a measurable set.

Remark (A practical tip). Suppose $\mathcal{A}$ is a nonempty collection of subsets of X which is closed under set complements in X. To show $\mathcal{A}$ is a σ-algebra on X, it suffices to show that $\mathcal{A}$ is closed under arbitrary countable *disjoint* unions; if $\{A_n\}_n$ is a countable collection of members of $\mathcal{A}$, then the union of A_n's is the *disjoint* union of F_n's where

$$F_1 := A_1 \quad \text{and} \quad F_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k \text{ for } n \geq 2.$$

Remark (Algebras of subsets). Let $\mathcal{A}$ be an algebra on a set X. Then, $\mathcal{A}$ is a σ-algebra on X if and only if $\mathcal{A}$ is closed under arbitrary countable unions of sets in an ascending chain in $\mathcal{A}$.

To justify the assertion, it suffices to show that $\mathcal{A}$ is closed under arbitrary countable unions of members in $\mathcal{A}$, under the assumption that $\mathcal{A}$ is closed under arbitrary countable unions of sets in an ascending chain in $\mathcal{A}$. If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{A}$ and $F_k = \bigcup_{n=1}^{k} E_n$, then $F_1 \subset F_2 \subset \cdots$ and $\bigcup_n E_n = \bigcup F_n \in \mathcal{A}$, by assumption.

Example 1.1.2 (Restriction of a σ-algebra)**.** Let $\mathcal{M}$ be a σ-algebra on a nonempty set X. For a nonempty subset A of X, let

$$\mathcal{M}|_A = \{A \cap E : E \in \mathcal{M}\}.$$

Then $\mathcal{M}|_A$ is a σ-algebra on A, which is called the σ-algebra on A inherited (restricted) from X to A. (The same argument holds for algebras on X, too.)

As there were bases for topology or algebraic structures, we can consider a basis of a σ-algebra.

Definition 1.1.3 (Generated σ-algebra)**.** Suppose X is a nonempty set and E is a collection of subsets of X. The collection $\mathcal{M}(E)$ of the smallest σ-algebra containing E is called the σ-algebra on X generated by E.

Remark. Let X be a nonempty set and E be a collection of subsets of X.

(a) $\mathcal{M}(E)$ is the intersection of all σ-algebras on X containing E.

(b) If $E \subset \mathcal{M}(F)$, then $\mathcal{M}(E) \subset \mathcal{M}(F)$.

Because the proof is straightforward and easy, it is left as an exercise.

In particular, when X is a topological space, the σ-algebra on X generated by the topology on X is called the Borel σ-algebra on X.

Example 1.1.4 (The Borel σ-algebra on the extended real system $\overline{\mathbb{R}}$)**.** We define some basic operations over $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{\infty, -\infty\}$ as follows: For all $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{0\}$, we let

$$\infty + a = \infty, \quad c \cdot \infty = \infty, \quad 0 \cdot \infty = 0$$

and we do not define $\infty - \infty$. We can impose a metric on $\mathbb{R}$ by the following metric:

$$d : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, \infty), \quad (a, b) \mapsto |\arctan(a) - \arctan(b)|.$$

In fact, the metric topology on $\overline{\mathbb{R}}$ induced by d is generated as a basis by the following intervals:

$$(a, b),\ [-\infty, b),\ \text{and}\ (a, \infty]\ \text{with}\ -\infty \leq a < b \leq \infty.$$

The Borel σ-algebra on $\overline{\mathbb{R}}$ is generated by the intervals of the last two types. Also, it is equivalent to define

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subset \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}.$$

Problems

Problem 1.1.1 (Exercise 1.3)**.** Let $\mathcal{M}$ be an infinite σ-algebra on X. Show that $\mathcal{M}$ contains infinitely many nonempty and pairwise disjoint sets, and card($\mathcal{M}$) $\geq$ card($\mathbb{R}$)

Solution. We first show the existence of a nonempty set $E \in \mathcal{M}$ such that the restriction of $\mathcal{M}$ to $X \setminus E$ is still infinite.[1] This implies, by induction, the existence of infinitely many nonempty and pairwise disjoint sets in $\mathcal{M}$. Suppose $\mathcal{M}$ has no such E. In other words, suppose that the restriction of $\mathcal{M}$ to $X \setminus E$ is finite whenever $E \in \mathcal{M}$ is nonempty and $E \neq X$. Because the restriction of $\mathcal{M}$ to $X \setminus E$ is also finite, $\mathcal{M}$ is finite, which contradicts the assumption that $\mathcal{M}$ is infinite.

Let $\{E_n\}_{n \in \mathbb{N}} \in \mathcal{M}$ be an infinite collection of nonempty and pairwise disjoint members in $\mathcal{M}$. Define a function $f : \mathcal{M} \to \mathbb{R}$ as follows:

$$f(A) = \sum \{2^{-n} : A \cap E_n \neq \varnothing\}.$$

The function f is onto $[0, 1]$, so card($\mathcal{M}$) $\geq$ card($[0, 1]$) $\geq$ card($\mathbb{R}$).

1.2 Measures

Definition 1.2.1 (Measure)**.** Let $(X, \mathcal{M})$ be a measurable space. A function $\mu : \mathcal{M} \to [0, \infty]$ with $\mu(\varnothing) = 0$ is called a measure on $\mathcal{M}$ if μ is countably additive, i.e., if $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ is pairwise disjoint, then $\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$. If μ is a measure on $\mathcal{M}$, the tuple $(X, \mathcal{M}, \mu)$ is called a measure space.

Remark. The monotonicity and the countable subadditivity of measures, as illustrated in Proposition 1.2.4, are due to the countable additivity of measures. Whichever type of measure we consider (outer measures, premeasures, etc.) is assumed to satisfy the empty set condition, monotonicity, and an appropriate (sub)additivity; here, additivity implies monotonicity, and monotonicity implies an appropriate subadditivity.

Definition 1.2.2 (Some particular measures)**.** Let $(X, \mathcal{M})$ be a measurable space, and let $\mu : \mathcal{M} \to [0, \infty]$ be a function such that $\mu(\varnothing) = 0$. μ is called a finitely additive measure, if μ is finitely additive but not necessarily countably additive.

Assume $(X, \mathcal{M}, \mu)$ is a measure space.

[1] It is left as an exercise to show that the restriction $\mathcal{M}_E := \{E \cap M : M \in \mathcal{M}\}$ of $\mathcal{M}$ to $E \in \mathcal{M}$ is a σ-algebra on E.

(a) μ is called a finite measure, if $\mu(X) < \infty$ (so that $\mu(E) < \infty$ for all $E \in \mathcal{M}$).

(b) μ is called a σ-finite measure, if there is a countable collection $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ covering X such that $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$. Furthermore, if $E = \bigcup_n E_n$ with $E_n \in \mathcal{M}$ for each $n \in \mathbb{N}$ and $\mu(E_n) < \infty$ for each n, then E is said to be σ-finite for μ.

(c) μ is called a semifinite measure, if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there is $F \in \mathcal{M}$ with $F \subset E$ such that $0 < \mu(F) < \infty$.

Remark. (a) A finite measure is σ-finite, and a σ-finite measure is semifinite. (Why?)

(b) A semifinite measure $\mu : \mathcal{M} \to [0, \infty]$ requires every member $E \in \mathcal{M}$ of infinite measure for μ to contain a set $F \in \mathcal{M}$ such that $0 < \mu(F) < \infty$. In fact, when finding such F, we may restrict the lower bound for the measure of F. To be precise, if μ is semifinite, E is a measurable set of infinite measure, and C is any positive real number, then there is a measurable set F such that

$$F \subset E \quad \text{and} \quad C < \mu(F) < \infty.$$

See Problem 1.2.4.

Example 1.2.3. (a) Let $(X, \mathcal{M})$ be a measurable space and $f : X \to [0, \infty]$ be a function. The function $\mu_f : \mathcal{M} \to [0, \infty]$ defined by $\mu_f(A) = \sum_{x\in A} f(x)$ is a measure on $\mathcal{M}$.

(b) Let X be an infinite set and $\mathcal{M} = P(X)$. The function $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite is a finitely additive measure on $\mathcal{M}$, but it is not a measure on $\mathcal{M}$.

The following are basic properties of measures:

Proposition 1.2.4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \le \mu(F)$.

(b) (Subadditivity) If $E_n \in \mathcal{M}$ for each $n \in \mathbb{N}$, then $\mu\left(\bigcup_n E_n\right) \le \sum_n \mu(E_n)$.

(c) (Continuity from below) If $(E_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ is an ascending chain, then $\mu\left(\bigcup_n E_n\right) = \lim_{n\to\infty} \mu(E_n)$.

(d) (Continuity from above) If $(E_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ is a descending chain, then $\mu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \mu(E_n)$, provided that $\mu(E_1) < \infty$.

Proof. It is easy to check (a) and (b), so we will prove (c) and (d).
(c) Define $F_n := E_n \setminus E_{n-1}$ for each $n \in \mathbb{N}$, where $E_0 := \varnothing$. Then $\bigcup_n E_n = \bigsqcup_n F_n$, thus

$$\mu\left(\bigcup_n E_n\right) = \mu\left(\bigsqcup_n F_n\right) = \sum_n \mu(F_n) = \sum_n (\mu(E_n) - \mu(E_{n-1})) = \lim_{n\to\infty} \mu(E_n).$$

(d) Define $A_n = E_1 \setminus E_n$ for each $n \in \mathbb{N}$. Then $(A_n)_{n\in\mathbb{N}}$ is an ascending chain in $\mathcal{M}$, so

$$\mu\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mu(A_n) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n).$$

(Note that $(\mu(E_n))_{n\in\mathbb{N}}$ is convergent, since the sequence is decreasing and is bounded below.) Because $E_1 \setminus \bigcup_n A_n = \bigcap_n (E_1 \setminus A_n) = \bigcap_n E_n$, we have the desired identity. $\square$

Definition 1.2.5. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) (μ-Null set) A set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a μ-null set. If a statement about points $x \in X$ is true except for x in some μ-null set, then the statement is said to be true μ-almost everywhere on X.

(b) (Complete measure) A measure whose domain contains all subsets of μ-null sets is said to be complete.

Notation. Given a measure space $(X, \mathcal{M}, \mu)$, let $\mathcal{N}_\mu$ be the collection of all μ-null sets in $\mathcal{M}$, i.e.,

$$\mathcal{N}_\mu := \{E \in \mathcal{M} : \mu(E) = 0\}.$$

One can easily find that $\mathcal{N}_\mu$ is closed under countable unions.

Completeness of a measure can obviate annoying technical problems, and it can be achieved by enlarging the domain of μ as follows.

Theorem 1.2.6 (Completion of a measure)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space. Define

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \text{ is a subset of a } \mu\text{-null set}\}.$$

(a) $\overline{\mathcal{M}}$ is a σ-algebra on X containing $\mathcal{M}$.

(b) There is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Remark. The following proof seems quite similar to adjoining the fractional parts to the integer parts.

Proof. We first show that $\overline{\mathcal{M}}$ is a σ-algebra on X. It is clear that $\overline{\mathcal{M}}$ is closed under countable unions, since M and $\mathcal{N}_\mu$ are closed under countable unions. Thus, it remains to show that $\overline{\mathcal{M}}$ is closed under set complements. Suppose $E \in \mathcal{M}$ and F is a subset of $N \in \mathcal{N}_\mu$, and write $N = F \sqcup G$ (clearly, $G = N \setminus F$), $A = E \cup N \in \mathcal{M}$. Then $E \cup F = A \setminus G$ and one can show that

$$X \setminus (E \cup F) = (X \setminus A) \cup G \in \overline{\mathcal{M}},$$

which proves that $\overline{\mathcal{M}}$ is closed under set complements.

We now construct a measure $\overline{\mu}$ on $\overline{\mathcal{M}}$ which extends μ. Given $E \cup F \in \overline{\mathcal{M}}$ with E and F being assumed as earlier, one may suggest that $\overline{\mu}(E \cup F)$ be $\mu(E)$. In fact, this definition is unambiguous; if $E_1 \cup F_1 = E_2 \cup F_2$ with E_i and F_i being assumed correspondingly, we have $E_1 \subset E_2 \cup F_2 \subset E_2 \cup N_2$ so $\mu(E_1) \leq \mu(E_2)$, and $\mu(E_2) \leq \mu(E_1)$ by symmetry. It remains to check

(i) if the constructed map $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$, and

(ii) if such extension is unique.

It is clear that $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$ and is complete. It remains to prove the uniqueness part. Suppose $\widetilde{\mu} : \overline{\mathcal{M}} \to [0, \infty]$ is an extension of μ to a complete measure on $\overline{\mathcal{M}}$. Then $\overline{\mu}$ and $\widetilde{\mu}$ coincide on M. By monotonicity, we can find that $\widetilde{\mu}(E \cup F) = \mu(E)$, which proves the uniqueness. □

Remark. (a) In the preceeding theorem, $\overline{\mu}$ is called the completion of μ, and $\overline{\mathcal{M}}$ is called the completion of $\mathcal{M}$ with respect to μ (or simply called the μ-completion of $\mathcal{M}$). Indeed, a completion of a σ-algebra is determined by a measure on the σ-algebra.

(b) One should not be confused that the extension of μ to a complete measure is unique. The preceeding theorem states that the extension of μ to a complete measure "over $\overline{\mathcal{M}}$" is unique.

We move to the list of problems after introducing one remark, which will be helpful in solving some problems in this section.

Observation 1.2.7 (Maximal measurable subset of finite measure). Given a measure space $(X, \mathcal{M}, \mu)$, let E be a measurable set such that $s = \sup(\mu(R_E))$ is finite, where

$$R_E = \{A \subset E : A \text{ is measurable and } \mu(A) < \infty\}.$$

Our goal is to prove that there is a member A of R_E whose measure is s.

Let $(A_n)_{n\in\mathbb{N}} \subset R_E$ be a sequence such that $\mu(A_n) \to s$ as $n \to \infty$. (Why does such sequence exist?) Let $B_n = \bigcup_{i=1}^n A_i$ for each n. Because $A_n \subset B_n \in R_E$ for each n and $(B_n)_n$ is ascending, we have $\mu\left(\bigcup_n A_n\right) = \mu\left(\bigcup_n B_n\right) = \lim \mu(B_n) = s < \infty$. Therefore, the union of A_n for all n belongs to R, and its measure is s.

Problems

Problem 1.2.1 (Exercise 1.8)**.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $(E_n)_{n\in\mathbb{N}} \subset M$. Show that $\mu(\liminf_{n\to\infty} E_n) \le \liminf_{n\to\infty} \mu(E_n)$. Also, show that $\mu(\limsup_{n\to\infty} E_n) \ge \limsup_{n\to\infty} \mu(E_n)$ if $\mu(\bigcup_n E_n) < \infty$.

Solution. Remark that

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k\ge n} A_k \quad \text{and} \quad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\ge n} A_k$$

whenever A_n is a subset of a set X for each $n \in \mathbb{N}$. Since $\left(\bigcup_{k\ge n} A_k\right)_{n\in\mathbb{N}}$ is ascending, we have

$$\mu\left(\liminf_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu\left(\bigcap_{k\ge n} E_k\right) \le \lim_{n\to\infty} \inf_{k\ge n} \mu(E_k) = \liminf_{n\to\infty} \mu(E_n).$$

Similarly, $\left(\bigcup_{k\ge n} A_k\right)_{n\in\mathbb{N}}$ is descending and $\bigcup_{k\ge 1} E_k$ is finite for μ, we have

$$\mu\left(\limsup_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k\ge n} E_k\right) \ge \lim_{n\to\infty} \sup_{k\ge n} \mu(E_k) = \limsup_{n\to\infty} \mu(E_n).$$

Problem 1.2.2 (Exercise 1.11)**.** Let μ be a finitely additive measure on a measurable space $(X, \mathcal{M})$. Show that μ is a measure if and only if μ is continuous from below. Show also that μ is a measure if and only if μ is continuous from above, provided that $\mu(X) < \infty$.

Solution. It suffices to show in each case that continuity implies countable additivity. Let $\{E_n\}_{n\in\mathbb{N}}$ be a set of pairwise disjoint subsets of X belonging to $\mathcal{M}$.

(a) Define $F_n := \bigcup_{i=1}^n E_i$ for each $n \in \mathbb{N}$. Clearly, $(F_n)_n$ is an ascending chain in $\mathcal{M}$, hence

$$\mu\left(\bigsqcup_n E_n\right) = \mu\left(\bigcup_n F_n\right) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \sum_n \mu(E_n).$$

(b) In part (a), define further $G_n := X \setminus F_n$ for each $n \in \mathbb{N}$. Because $\mu(G_1) < \infty$ and $(G_n)_n$ is a descending chain in $\mathcal{M}$, we have

$$\mu\left(\bigcap_n G_n\right) = \lim_{n\to\infty} \mu(G_n) = \mu(X) - \sum_n \mu(E_n).$$

Because

$$\mu\left(\bigcap_n G_n\right) = \mu(X) - \mu\left(\bigcup_n F_n\right) = \mu(X) - \mu\left(\bigsqcup_n E_n\right),$$

we find that μ is a measure.

Problem 1.2.3 (Exercise 1.12)**.** Let (X, M, μ) be a finite measure. Assume E, F, $G \in \mathcal{M}$.

(a) Show that $\mu(E) = \mu(F) = \mu(E \cap F)$, if $\mu(E \triangle F) = 0$.

(b) Define a relation $\sim$ on $\mathcal{M}$ by $E \sim F$ if and only if $\mu(E \triangle F) = 0$. Show that $\sim$ denotes an equivalence relation on $\mathcal{M}$.

(c) Define a map $\rho : \mathcal{M} \times \mathcal{M} \to [0, \infty]$ by $\rho(E, F) = \mu(E \triangle F)$. Show that $\rho(E, G) \le \rho(E, F) + \rho(F, G)$, and deduce that ρ induces a metric on the space $\mathcal{M}/\sim$ of the equivalence classes. To be precise, the map $\overline{\rho} : (\mathcal{M}/\sim) \times (\mathcal{M}/\sim) \to [0, \infty]$ defined by $\overline{\rho}(\overline{E}, \overline{F}) = \rho(E, F)$ for all $\overline{E}$, $\overline{F} \in \mathcal{M}/\sim$ is a well defined metric on $\mathcal{M}/\sim$.

Solution. (a) If $\mu(E \triangle F) = 0$, then is $\mu(E \setminus F) = \mu(F \setminus E) = 0$, so $\mu(E) = \mu(F) = \mu(E \cap F)$.

(b) It is clear that $\sim$ is symmetric and reflexive. To show transitivity, assume $E \sim F$ and $F \sim G$. Note that $E \triangle G \subset (E \triangle F) \cup (F \triangle G)$ so $\mu(E \triangle G) \leq \mu(E \triangle F) + \mu(F \triangle G) = 0$.

(c) The triangular inequality with regard to ρ is explained in part (b). We now justify that $\overline{\rho}$ is a metric on $\mathcal{M}/\sim$. Suppose $E \sim A$ and $F \sim B$. Because $\rho(E, F) \leq \rho(E, A) + \rho(A, B) + \rho(B, F) = \rho(A, B)$ and $\rho(A, B) \leq \rho(E, F)$ by symmetry, the map $\overline{\rho}$ is well-defined. Therefore, $\overline{\rho}$ is a metric on $\mathcal{M}/\sim$, which follows directly from the definition of $\overline{\rho}$.

Problem 1.2.4 (Exercise 1.14)**.** Let $(X, \mathcal{M}, \mu)$ be a semifinite measure space, and assume $E \in \mathcal{M}$ is of infinite measure for μ. Show that for any $C > 0$ there is a subset $F \in \mathcal{M}$ of E such that $C < \mu(F) < \infty$.

Solution. Suppose that there is a member $E \in \mathcal{M}$ such that $\mu(E) = \infty$ and any measurable subset A of E satisfies $\mu(A) = \infty$ or $\mu(A) \leq \alpha$ for some real $\alpha > 0$. Then, the supremum s of R_E is finite, where R_E is defined as in 5. Hence, there is a member A of R_E such that $\mu(A) = s$. Because $\mu(E \setminus A) = \infty$, by semifiniteness, there is a member $B \subset E \setminus A$ belonging to $\mathcal{M}$ such that $0 < \mu(B) < \infty$. Then, $A \sqcup B \in R_E$ but $\mu(A \sqcup B) > s$, a contradiction.

Problem 1.2.5 (Exercise 1.15)**.** Given a measure μ on a measurable space $(X, \mathcal{M})$, define a map $\mu_0 : \mathcal{M} \to [0, \infty]$ by

$$\mu_0(E) := \sup\{\mu(F) : F \in \mathcal{M} \text{ is a subset of } E \text{ of finite measure for } \mu\}.$$

(a) Show that μ_0 is a semifinite measure on $\mathcal{M}$. μ_0 is called the semifinite part of μ.

(b) Show that if μ is semifinite, then $\mu = \mu_0$.

(c) Show that there is a measure ν on $\mathcal{M}$ which assumes only the values 0 and ∞ such that $\mu = \mu_0 + \nu$. (In general, such ν need not be unique.)

Solution. Remark that $\mu_0 \leq \mu$, and $\mu_0(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\mu(E) < \infty$.

(a) We first show that the map μ_0 is a measure on $\mathcal{M}$. Since $\mu_0(\varnothing) = 0$, it suffice to show that μ_0 is countably additive. Suppose a collection $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ is pairwise disjoint, and let B be the union of A_n for all n. Given a measurable subset E of B, we have $\mu(E) = \sum_n \mu(E \cap A_n) \leq \sum_n \mu_0(A_n)$, so $\mu_0(B) \leq \sum_n \mu_0(A_n)$. Conversely, given a measurable subset F_n of A_n of finite measure for each n, because every finite union of F_n's is a measurable subset of B of finite measure,

$$\sum_{n=1}^{N} \mu(F_n) = \mu\left(\bigsqcup_{n=1}^{N} F_n\right) \leq \mu_0(B) \quad \text{and} \quad \sum_{n=1}^{N} \mu_0(A_n) \leq \mu_0(B),$$

so $\sum_n \mu_0(A_n) \leq \mu_0(B)$. Hence, μ_0 is a measure on $\mathcal{M}$.

Semifiniteness of μ_0 is straightforward; if $\mu_0(E) = \infty$ for some $E \in M$, there is a measurable subset F of E with nonzero finite measure, and $\mu_0(F) = \mu(F) < \infty$.

(b) It follows from Problem 1.2.4 that $\mu(E) = \mu_0(E)$ whenever $E \in \mathcal{M}$ and $\mu(E) = \infty$. (How?)

(c) Assume such measure ν exists and $E \in \mathcal{M}$. If $\mu_0(E) = \mu(E) < \infty$, then $\nu(E) = 0$; if $\mu_0(E) < \infty$ bt $\mu(E) = \infty$, then $\nu(E) = \infty$. Also, if E is σ-finite, then $\nu(E) = 0$ by σ-additivity. Hence, to solve this problem, we need a set map from $\mathcal{M}$ onto $\{0, \infty\}$ satisfying the above properties.

Define a set map $\nu : \mathcal{M} \to \{0, \infty\}$ by

$$\nu(E) = \begin{cases} 0 & (\text{if } E \text{ is } \sigma\text{-finite for } \mu) \\ \infty & (\text{otherwise}) \end{cases}.$$

To check that ν satisfies the above properties, one must verify that $\mu_0(E) = \infty$ when $\mu(E) = \infty$ and E is σ-finite, where $E \in \mathcal{M}$. (This is left as an exercise.) ν clearly satisfies the empty set axiom.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of pairwise disjoint measurable sets. If A_n is σ-finite for all n, then $\bigsqcup_n A_n$ is also σ-finite, hence $\nu(\bigsqcup_n A_n) = 0 = \sum_n \nu(A_n)$. If A_k is not σ-finite for some $k \in \mathbb{N}$, then $\bigsqcup_n A_n$ is not σ-finite, so $\nu(\bigsqcup_n A_n) = \infty = \sum_n \nu(A_n)$.

So far, we have proved that ν is a measure. To check if $\mu = \mu_0 + \nu$, we only need to check $\mu(E) = \mu_0(E) + \nu(E)$ for all measurable sets E with $\mu_0(E) < \infty$; if $\mu(E) < \infty$, E is σ-finite for μ; otherwise, E is not σ-finite for μ.

1.3 Outer measures

Definition 1.3.1 (Outer measure)**.** Let X be a nonempty set. A function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ such that $\mu^*(\varnothing) = 0$ is called an outer measure on X if μ^* satisfies the following properties:

(a) (Monotonicity) If $A \subset B \subset X$, then $\mu^*(A) \le \mu^*(B)$.

(b) (Countable subadditivity) If $A_n \subset X$ for each $n \in \mathbb{N}$, then $\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n=1}^\infty \mu^*(A_n)$.

Indeed, measures are outer measures.

The most common way to obtain an outer measure is to start with a family $\mathcal{E}$ of "basic sets" on which a notion of measure is defined (such as rectangles in the plane) and then to approximate arbitrary sets "from the outside" by countable unions of members of $\mathcal{E}$. The presice construction is as follows:

Proposition 1.3.2 (Induced outer measure)**.** Let $\mathcal{E}$ be a subset of $\mathcal{P}(X)$ containing both $\varnothing$ and X, and let $\rho : \mathcal{E} \to [0,\infty]$ be a function. Define the function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by

$$\mu^*(A) = \inf\left\{\sum_{n=1}^\infty \rho(E_n) : \{E_n\}_n \text{ is a countable covering of } A \text{ by members of } \mathcal{E}\right\}. \tag{1.1}$$

Then, the map μ^* is an outer measure on X.

Proof. Since $\mathcal{E}$ contains X, the map μ^* is well defined. To show that μ^* is an outer measure on X, we need to check monotonicity and countable subadditivity.

The monotonicity is clear; if $A \subset B \subset X$ and $\{B_n\}_n$ is a countable covering of B by members of $\mathcal{E}$, because it covers A, we have $\mu^*(A) \le \sum_n \rho(B_n)$ and $\mu^*(A) \le \mu^*(B)$.

Given $A_n \subset X$ for each $n \in \mathbb{N}$, let $\{E_n(k)\}_{k\in\mathbb{N}}$ be a covering of A_n by members of $\mathcal{E}$ such that

$$\mu^*(A_n) \le \sum_k \rho(E_n(k)) < \mu^*(A_n) + \epsilon \cdot 2^{-n}$$

Since the union of $E_n(k)$ for all n, k covers the union of A_n for all $n \in \mathbb{N}$, we have

$$\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n,k} \rho(E_n(k)) = \sum_n \sum_k \rho(E_n(k)) < \sum_n (\mu^*(A_n) + \epsilon \cdot 2^{-n}) = \sum_n \mu^*(A_n) + \epsilon.$$

This proves that the induced map μ^* is an outer measure on X. $\square$

Remark. The outer measure on X constructed above from the function ρ is called the outer measure on X induced by ρ, and, if necessary, will be denoted by ρ^*.

Next step is to construct a measure on X when an outer measure μ^* on X is given. In fact, such construction is done by restricting the domain.

Definition 1.3.3 (μ^*-measurable set)**.** If μ^* is an outer measure on a nonempty set X, a subset A of X is called a μ^*-measurable set (or said to be measurable with respect to the outer measure μ^*) if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

for all $E \subset X$.

Remark. When μ^* is an outer measure on X, whenever $E,\, A \subset X$, we have $\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \setminus A)$. Thus, $A \subset X$ is μ^*-measurable if and only if $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \setminus A)$ for all $E \subset X$. (Here, we may assume $\mu^*(E) < \infty$.)

Here is an intuition inside the concept of μ^*-measurability. Suppose μ^* is an outer measure on X and A is a subset of X. To say A is 'measurable' in some sense, one may suggest that the 'interior' measure and the 'exterior' measure of A be equal, so that we can treat the identical value the measure of A. To write in the form of an identity, whenever E is a subset of X containing A, it should be satisfied that

$$\mu^*(A) = \mu^*(E) - \mu^*(E \setminus A) \tag{1.2}$$

where the left-hand side is the 'exterior' measure of A and the right-hand side is the 'interior' measure of A. μ^*-measurability is an extension of the idea in eq. (1.2); whenever E is a subset of X, the 'exterior' measure $\mu^*(A \cap E)$ of the portion of A contained in E must be equal to the 'interior' measure $\mu * (E) - \mu^*(E \setminus A)$ of the portion.

The following theorem justifies the earlier extension; it justifies that all μ^*-measurable sets are "well-behaved" relative to μ^*. Though it is labeled as a theorem, it is the nature of the collection of all subsets of X which are measurable with respect to μ^*.

Theorem 1.3.4 (Carathéodory's extension theorem)**.** If μ^* is an outer measure on X, the collection $\mathcal{M}^*$ of all μ^*-measurable sets is a σ-algebra on X. And the restriction of μ^* to $\mathcal{M}^*$ is a complete measure.[2]

Proof. Let $\mathcal{M}^*$ denote the collection of subsets of X which are μ^*-measurable.

Step 1: $\mathcal{M}^*$ is a σ-algebra.
It is clear that $\mathcal{M}^*$ is closed under set complements in X. Thus, it remains to show that $\mathcal{M}^*$ is closed under countable disjoint unions. (See the footnote in the definition of σ-algebras.) Whenever $A, B \in \mathcal{M}^*$ and $E \subset X$, because

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B) + \mu^*((E \setminus A) \cap B)) + \mu^*((E \setminus A) \setminus B) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)), \end{aligned}$$

$\mathcal{M}^*$ is closed under arbitrary finite set unions.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a collection of pairwise disjoint sets in $\mathcal{M}^*$. Define $B_n := \bigsqcup_{k=1}^n A_k$ for each $n \in \mathbb{N}$ and let B be the union of all A_n's. By induction, $\mu^*(E \cap B_n) = \sum_{k=1}^n \mu^*(E \cap A_k)$ for all $E \subset X$. Therefore,

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \setminus B_n) \ge \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \setminus B)$$

for all $n \in \mathbb{N}$, so

$$\mu^*(E) \ge \sum_n \mu^*(E \cap A_n) + \mu^*(E \setminus B) \ge \mu^*(E \cap B) + \mu^*(E \setminus B),$$

implying that $B = \bigsqcup_n A_n$ is also μ^*-measureble. Therefore, $\mathcal{M}^*$ is a σ-algebra on X.

Step 2: $\mu^*|_{\mathcal{M}^*}$ is a complete measure.
Let $\mu = \mu^*|_{\mathcal{M}^*}$, and we first justify that μ is a measure. Clearly, $\mu(\varnothing) = 0$. Let $\{A_n\}_n$ be a countable collection of members of $\mathcal{M}^*$ which are pairwise disjoint. Then $\mu(\bigsqcup_{n=1}^k) = \sum_{n=1}^k \mu(A_n)$ by induction, so $\mu(\bigsqcup_n A_n) \ge \sum_{n=1}^k \mu(A_n)$ for all $k \in \mathbb{N}$. Hence, $\mu(\bigsqcup_n A_n) \ge \sum_n \mu(A_n)$, and the σ-subadditivity of μ implies that μ is countably additive.

To show completeness, suppose A is a subset of a μ-null set. By monotonicity, $\mu^*(A) = 0$, hence $\mu^*(E) \ge \mu^*(E \setminus A) = \mu^*(E \cap A) + \mu^*(E \setminus A)$ for all $E \subset X$. Therefore, A is μ^*-measureble, i.e., $A \in \mathcal{M}^*$. □

Remark. (Function) $\xrightarrow{\text{consider infimums}}$ (Outer measure) $\xrightarrow{\text{restricting to } \mathcal{M}^*}$ (Complete measure), where $\mathcal{M}^*$ is the set of μ^*-measurable sets in X.

[2]One should remark that such complete measure need not be the completion of μ. In the following section, however, it will be proved that the completion of a measure and the Carathéodory extension of the measure coincide *if the measure is σ-finite.*

Problems

Problem 1.3.1 (Exercise 1.17)**.** Let μ^* be an outer measure on X and $\{A_n\}_n$ be a countable collection of pairwise disjoint μ^*-measureble sets. Show that $\mu^*(E \cap U) = \sum_n \mu^*(E \cap A_n)$, where $U = \bigsqcup_n A_n$.

Solution. It is clear from the countable subadditivity of outer measures that $\mu^*(E \cap U) \leq \sum_n \mu^*(E \cap A_n)$. To show the converse inequality, let $B_k = \bigsqcup_{n \geq k} A_n$, and note that

$$\begin{aligned}
\mu^*(E \cap U) &= \mu^*((E \cap U) \cap A_1) + \mu^*((E \cap U) \setminus A_1) = \mu^*(E \cap A_1) + \mu^*(E \cap B_1) \\
&= \mu^*(E \cap A_1) + \mu^*(E \cap A_2) + \mu^*(E \cap B_2) \\
&\vdots \\
&= \sum_{n=1}^{j} \mu^*(E \cap A_n) + \mu^*(E \cap B_j) \geq \sum_{n=1}^{j} \mu^*(E \cap A_n)
\end{aligned}$$

for all j so that the desired inequality holds. (Or one can prove the second inequality using $\mu^*(E \cap U) \geq \mu^*(E \cap \bigsqcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu^*(E \cap A_j)$.)

1.4 Premeasures

We introduce premeasures in this section which has some nice extension properties when generating a complete measure, where the generation is done as in Carathéodory extension, i.e., an outer measure from an appropriate basic function, and then a complete measure from the outer measure. We also introduce further properties when extending a premeasure μ_0 to a complete measure μ, when X is σ-finite for μ_0.

Definition 1.4.1 (Premeasures)**.** Let X be a nonempty set and $\mathcal{A}$ be an algebra on X. The function $\mu_0 : \mathcal{A} \to [0, \infty]$ with $\mu_0(\varnothing) = 0$ is called a premeasure on X if μ_0 is countably additive, i.e., whenever $\{A_n\}_n$ is a countable collection of pairwise disjoint members of $\mathcal{A}$ and $\bigsqcup_n A_n \in \mathcal{A}$, we have

$$\mu_0\left(\bigsqcup_n A_n\right) = \sum_n \mu_0(A_n).$$

Remark. As it was true for measures, the countable additivity of premeasures implies that premeasures are monotonic and appropriately countably subadditive.

As the first step, we induce an outer measure μ^* on X from the premeasure μ_0.

Proposition 1.4.2 (Extension of a premeasure)**.** Let $\mathcal{A}$ be an algebra on X, μ_0 be a premeasure on $\mathcal{A}$, and μ^* be the outer measure on X induced by μ_0 as in eq. (1.1) on page 8.

(a) μ^* extends μ_0, i.e., $\mu^*\lfloor_{\mathcal{A}} = \mu_0$.

(b) Every set in $\mathcal{A}$ is μ^*-measurable. Hence, every member of the σ-algebra generated by $\mathcal{A}$ is μ^*-measurable.

In short, the restriction of μ^* to μ^*-measurable sets is a complete measure which extends μ_0.

Proof. (a) To show that μ^* extends μ_0, it suffices to show that $\mu^*(E) \geq \mu_0(E)$ for all $E \in \mathcal{A}$; for this, one need to show that $\sum_n \mu_0(A_n) \geq \mu_0(E)$ whenever $\{A_n\}_n$ is a countable covering of E by members of $\mathcal{A}$. Let $\{F_n\}_n$ be the usual partition for $\bigcup_n A_n$. Since each F_n belongs to $\mathcal{A}$, $B_n := E \cap F_n$ is a member of $\mathcal{A}$ and the union of B_n for all n is E. Thus, by the σ-additivity of μ_0,

$$\mu_0(E) = \mu_0\left(\bigsqcup_n B_n\right) = \sum_n \mu_0(B_n) \leq \sum_n \mu_0(A_n),$$

from which it is followed that $\mu_0(E) \leq \mu^*(E)$.

(b) We will show that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)$ for all $E \subset X$ and $A \in \mathcal{A}$. By definition, given $\epsilon > 0$, we can find a countable covering $\{C_n\}_n$ of E by members of $\mathcal{A}$ such that

$$\mu^*(E) \leq \sum_n \mu_0(C_n) < \mu^*(E) + \epsilon.$$

Because $\mu_0(C_n) = \mu_0(C_n \cap A) + \mu_0(C_n \setminus A)$ for all n,

$$\begin{aligned} \mu^*(E) + \epsilon &> \sum_n (\mu_0(C_n \cap A) + \mu_0(C_n \setminus A)) \\ &= \sum_n \mu_0(C_n \cap A) + \sum_n \mu_0(C_n \setminus A) \geq \mu^*(E \cap A) + \mu^*(E \setminus A). \end{aligned}$$

Since ϵ is arbitrary, A is μ^*-measurable. The latter result follows, because the collection of μ^*-measurable sets is a σ-algebra on X.

It follows easily that the restriction of μ^* to μ^*-measurable sets extends μ_0. □

We now restrict the outer measure μ^* to $\langle \mathcal{A} \rangle$, the σ-algebra on X generated by the algebra $\mathcal{A}$ on X. As mentioned in (c) of the following theorem, when X is σ-finite for the premeasure μ_0, then the extension of μ_0 to a complete measure is unique. Moreover, in proving (c), we can observe how σ-finiteness can be enjoyed in some circumstances.

Theorem 1.4.3. Let $\mathcal{A}$ be an algebra on X, μ_0 be a premeasure on $\mathcal{A}$.

(a) There is a measure μ on $\langle \mathcal{A} \rangle$ whose restriction to $\mathcal{A}$ is the premeasure μ_0; namely, $\mu := \mu^*|_{\langle \mathcal{A} \rangle}$, where μ^* is the outer measure on X induced by μ_0.[3]

(b) If ν is another measure on $\langle \mathcal{A} \rangle$ extending μ_0, then $\nu \leq \mu$, with equality for all $E \in \langle \mathcal{A} \rangle$ such that $\mu(E) < \infty$.

(c) In particular, if μ_0 is a σ-finite premeasure, then μ is the unique extension of μ_0 to a measure on $\langle \mathcal{A} \rangle$.

Proof. (a) This follows directly from the Carathéodory extension theorem and Proposition 1.4.2.

(b) Suppose $E \in \mathcal{A}$ and $\{A_n\}_n$ be a countable covering of E by members of $\mathcal{A}$. Then $\nu(E) \leq \nu\left(\bigcup_n A_n\right) \leq \sum_n \nu(A_n) = \sum_n \mu_0(A_n)$, so $\nu(E) \leq \mu(E)$.

Let $A = \bigcup_n A_n$, where each A_n belongs to $\mathcal{A}$. Since every finite union of A_n's is a member of $\mathcal{A}$,

$$\nu(A) = \lim_{n\to\infty} \nu\left(\bigcup_{k=1}^n A_k\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^n A_k\right) = \mu(A)$$

Therefore, μ and ν coincide at every countable union of members of $\mathcal{A}$.

If E is a member of $\langle \mathcal{A} \rangle$ such that $\mu(E) < \infty$, there is a countable covering $\{B_n\}_n$ of E by members of $\mathcal{A}$ such that

$$\mu(E) \leq \sum_n \mu_0(B_n) < \mu(E) + \epsilon.$$

With $B = \bigcup_n B_n$, we have $\mu(B \setminus E) < \epsilon$ (why?), and

$$\mu(E) \leq \mu(B) = \nu(B) = \nu(E) + \nu(B \setminus E) \leq \nu(E) + \mu(B \setminus E) < \nu(E) + \epsilon.$$

Thus, $\nu \leq \mu$ with equality at every member of $\langle \mathcal{A} \rangle$ of finite measure for μ.

[3]Indeed, μ_0 extends to the restriction of μ^* to the collection of all μ^*-measurable sets.

(c) Finally, assume that μ_0 is a σ-finite premeasure on $\mathcal{A}$. Then there is a countable collection $\{A_n\}_n \subset \langle A \rangle$ such that $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ for each $n \in \mathbb{N}$. By partitioning, we may assume that $\{A_n\}_n$ is pairwise disjoint. If $E \in \langle \mathcal{A} \rangle$, we have

$$\mu(E) = \sum_n \mu(E \cap A_n) = \sum_n \nu(E \cap A_n) = \nu(E),$$

proving that μ is the unique extension of the premeasure μ_0 to a measure on $\langle \mathcal{A} \rangle$.

This completes the proof. □

Our last goal of this section is to prove the following:

> Given a *σ-finite* measure space $(X, \mathcal{M}, \mu)$, the completion of μ and the Carathéodory extension of μ coincide. In other words, the domain of the Carathéodory extension of μ is the completion of $\mathcal{M}$ with regard to μ.

Lemma 1.4.4. Let $\mathcal{A}$ be an algebra on X, μ_0 be a premeasure on $\mathcal{A}$, and μ^* be the outer measure induced by μ_0. Let $\mathcal{A}_\sigma$ denote the collection of all countable unions of the members of $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ denote the collection of all finite intersections of the members of $\mathcal{A}_\sigma$.

(a) For any $E \subset X$ and a real $\epsilon > 0$, there is a member $A \in \mathcal{A}_\sigma$ such that $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

(b) Suppose $E \subset X$ and $\mu^*(E) < \infty$. Then E is μ^*-measurable if and only if there is a member $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B \setminus E) = 0$.

(c) If μ_0 is *σ-finite*, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Proof. (a) Let $\{A_n\}_n$ be a countable covering of E by members of $\mathcal{A}$ such that $\sum_n \mu_0(A_n) \leq \mu^*(E) + \epsilon$. Then $E \subset \bigcup_n A_n \in \mathcal{A}_\sigma$ and $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n) = \sum_n \mu_0(A_n) \leq \mu^*(E) + \epsilon$.

(b) Note that every member of $\langle \mathcal{A} \rangle$ is μ^*-measurable.

Assume first that $\mu^*(E) < \infty$ and E is μ^*-measurable. For each $n \in \mathbb{N}$, let A_n be a member of $\mathcal{A}_\sigma$ such that $E \subset A_n$ and $\mu^*(A_n) < \mu^*(E) + n^{-1}$; and let $B = \bigcap_n A_n$. Clearly, $B \in \mathcal{A}_{\sigma\delta}$, $E \subset B$, and $\mu^*(B) = \mu^*(E)$. Because E is μ^*-measurable, $\mu^*(B) = \mu^*(E) + \mu^*(B \setminus E)$, so $\mu^*(B \setminus E) = 0$.

Assume conversely that there is a member $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B \setminus E) = 0$. Whenever $A \subset X$, we have $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B) \geq \mu^*(A \cap E) + \mu^*(A \setminus B)$. Because $\mu^*(A \setminus E) \leq \mu^*(A \setminus B) + \mu^*(A \cap (B \setminus E)) = \mu^*(A \setminus B)$, we have $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$. Thus, E is μ^*-measurable.

(c) Let $\{X_n\}_n$ be a countable covering of X by members of $\mathcal{A}$ such that $\mu_0(X_n) < \infty$ for all n, and define $E_n = E \cap X_n$; for each n, $\mu_0(E_n) < \infty$. Now assume E is μ^*-measurable. Then so is each E_n (because X_n is also μ^*-measurable), so there is a member $B_n \in \mathcal{A}_{\sigma\delta}$ such that $E_n \subset B_n$ and $\mu^*(B_n \setminus E_n) = 0$. The union B of B_n for all n satisfies $E \subset B$ and $\mu^*(B \setminus E) = 0$. Conversely, if such B for E exists, let $B_n = B \cap X_n$; $E_n = E \cap X_n \subset B_n$ and $\mu^*(B_n \setminus E_n) = 0$. Hence, each E_n is μ^*-measurable, so $E = \bigcup_n E_n$ is also μ^*-measurable.

This completes the proof of lemma. □

Theorem 1.4.5. If $(X, \mathcal{M}, \mu)$ is a *σ-finite* measure space, then the Carathéodory extension of μ is the completion of μ.

Proof. Note that the domain of the Carathéodory extension $\overline{\mu}$ of μ is the collection $\mathcal{M}^*$ of μ^*-measurable sets in X, where μ^* is the outer measure induced by μ.

We first show that $\mathcal{M}^*$ and the completion $\overline{\mathcal{M}}$ of $\mathcal{M}$ with regard to μ coincide. Assume first that $E \subset X$ is μ^*-measurable. Then there is a member $B \in \mathcal{M}_{\sigma\delta} = \mathcal{M}$ containing E such that $\mu^*(B \setminus E) = 0$. For each $j \in \mathbb{N}$, let $\{K_j(n)\}_n$ be a countable covering of $B \setminus E$ by members of $\mathcal{M}$ such that $\sum_n \mu(K_j(n)) < 1/j$. Letting $K_j = \bigcup_n K_j(n) \in \mathcal{M}$, we have $\mu(K_j) \leq \sum_n \mu(K_j(n)) < 1/j$; $\mu(K) = 0$ if $K = \bigcap_{j=1}^{\infty} K_j \in \mathcal{M}$. Moreover, $B \setminus E$ is a subset of the μ-null set K. Therefore, $B \supset E = B \setminus (B \setminus E) \supset B \setminus K$, i.e., $E \in \overline{\mathcal{M}}$ and $\mathcal{M}^* \leq \overline{\mathcal{M}}$. The converse inclusion is easy to verify, so we may conclude that $\overline{\mathcal{M}} = \mathcal{M}^*$.

Note that both $\overline{\mu}$ and the completion of μ are extensions of μ to a complete measure on $\overline{\mathcal{M}}$. By the uniqueness of such an extension, the completion of μ and $\overline{\mu}$ coincide. □

1.5 Borel measures on the real line

Notation. In this section, an interval of the form $(a, b]$ with $-\infty \le a \le b \le \infty$ is called an o-c interval, and the collection of o-c intervals is denoted by $\mathcal{A}$.

Remark. $\mathcal{A}$ is an algebra on $\mathbb{R}$, and the σ-algebra on $\mathbb{R}$ generated by $\mathcal{A}$ is the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ on $\mathbb{R}$.

Observation 1.5.1. Let μ be a finite Borel measure on $\mathbb{R}$ and let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by $F(x) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$. Then F is increasing and right continuous. Moreover, whenever a, b are real numbers with $a < b$, then $\mu((a, b]) = F(b) - F(a)$.

Sketch. Turning around the above observation; starting from an increasing and right continuous function F, we will construct a Borel measure μ_F.

Lemma 1.5.2. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function. If $(a_j, b_j]$ for $j = 1, \cdots, n$ are pairwise disjoint o-c intervals, let

$$\mu_0\left(\bigsqcup_{j=1}^{n}(a_j, b_j]\right) = \sum_{j=1}^{n}(F(b_j) - F(a_j)),$$

and $\mu_0(\varnothing) = 0$. Then μ_0 is a premeasure on $\mathcal{A}$.

Proof. **Step 1: Checking if μ_0 is well-defined.**
Let $\{I_i\}_{i=1}^{m}$ and $\{J_j\}_{j=1}^{n}$ be collections of pairwise disjoint o-c intervals with the same union I. If $I = (a, b]$ for some $a, b \in \mathbb{R}$, we can easily check that μ_0 coincide for those collections, i.e., μ_0 is well-defined for a single o-c interval. In general, because each I_i or J_j is an o-c interval and so is $I_i \cap J_j$, we have

$$\sum_{i} \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_{j} \mu_0(J_j).$$

Therefore, the function μ_0 is well-defined. Moreover, by construction, μ_0 is finitely additive.

Step 2: Showing that μ_0 is a premeasure on $\mathcal{A}$.
Since μ_0 satisfies the empty set condition, it remains to show that μ_0 is appropriately σ-additive. Let $\{I_n\}_{n=1}^{\infty}$ be a countable collection of pairwise disjoint o-c intervals with the union I in $\mathcal{A}$. Because $I \in \mathcal{A}$ so that I can be partitioned into finitely many pairwise disjoint o-c interval and μ_0 is finitely additive, without loss of generality, we may assume $I = (a, b]$ for some $a, b \in \mathbb{R}$. Then

$$\mu_0(I) = \mu_0\left(\bigsqcup_{n=1}^{N} I_n\right) + \mu_0\left(I \setminus \bigsqcup_{n=1}^{n} I_n\right) \ge \mu_0\left(\bigsqcup_{n=1}^{n} I_n\right) = \sum_{n=1}^{n} \mu_0(I_n),$$

so $\mu_0(\bigsqcup_n I_n) \ge \sum_n \mu_0(I_n)$. It now remains to show the converse inequality.

(i) First, assume that I is bounded. By the right continuity of F, given $\epsilon > 0$, there is $\delta > 0$ and $\delta_n > 0$ for each $n \in \mathbb{N}$ such that $F(a) \le F(a + \delta) < F(a) + \epsilon$ and $F(b_n) \le F(b_n + \delta_n) < F(b_n) + \epsilon \cdot 2^{-n}$. Then, the open intervals $(a_n, b_n + \delta_n)$ $(n \in \mathbb{N})$ cover the compact interval $[a + \delta, b]$, so finitely many intervals $(a_n, b_n + \delta_n)$ cover $[a + \delta, b]$; (after renumbering, if necessary) let such open intervals be written by $(a_i, b_i + \delta_i)$ for $i = 1, \cdots, N$, where

$$b_j + \delta_j \in (a_{j+1}, b_{j+1} + \delta_{j+1}) \text{ for } j = 1, \cdots, N - 1.$$

(Also, we may discard an interval which is contained in another interval.) Then the following string of inequalities holds:

$$\begin{aligned}
\mu_0(I) &= F(b) - F(a) \le F(b) - F(a+\delta) + \epsilon \\
&\le F(b_N + \delta_N) - F(a_1) + \epsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} (F(a_{j+1}) - F(a_j)) + \epsilon \\
&\le F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} (F(b_j + \delta_j) - F(a_j)) + \epsilon \\
&< \sum_{j=1}^{N} (F(b_j) - F(a_j) + \epsilon \cdot 2^{-j}) + \epsilon \\
&< \sum_n \mu_0(I_n) + 2\epsilon.
\end{aligned}$$

(ii) We now assume I is unbounded, i.e., $I = (-\infty, b]$ or $I = (a, \infty)$. For the case $I = (-\infty, b]$, given $M < \infty$, some finitely many intervals $(a_j, b_j + \delta_j)$ cover $[-M, b]$. The same reasoning proves $F(b) - F(-M) \le \sum_n \mu_0(I_n) + 2\epsilon$ for all $M > 0$. When $I = (a, \infty)$, for all $M > a$, we have $F(M) - F(a) \le \sum_n \mu_0(I_n) + 2\epsilon$.

Therefore, μ_0 is appropriately σ-additive, so μ_0 is a premeasure on $\mathcal{A}$. □

Theorem 1.5.3 (Uniqueness of a Borel measure on $\mathbb{R}$)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function.

(a) There is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$, i.e., there is a unique extension of μ_0 in the preceeding lemma to a Borel measure on $\mathbb{R}$.

(b) If G is another such function, we have $\mu_F = \mu_G$ if and only if $F - G$ is constant.

(c) Conversely, if μ is a Borel measure on $\mathbb{R}$ which is finite on all bounded Borel sets and we define

$$H(x) = \begin{cases} \mu((0, x]) & (x > 0) \\ 0 & (x = 0) \\ -\mu((x, 0]) & (x < 0) \end{cases},$$

then H is an increasing and right continuous function on $\mathbb{R}$, and $\mu = \mu_H$.

Proof. (a) A desired Borel measure on $\mathbb{R}$ should extend the premeasure μ_0 on $\mathcal{A}$ in the preceeding lemma; because $\mathbb{R}$ is σ-finite for μ_0, a Borel measure extending μ_0 is unique.

(b) Clear.

(c) Given a Borel measure μ on $\mathbb{R}$ which is finite on all bounded Borel sets, the suggested function H is well-defined, increasing, and right continuous. Because μ and μ_F coincide on $\mathcal{A}$ and μ is a unique extension of μ_0, we have $\mu = \mu_F$. □

Remark. Suppose $F : \mathbb{R} \to \mathbb{R}$ is an increasing and right continuous function. Because $\mathbb{R}$ is σ-finite for μ_F, the Carathéodory extension of μ_F is the completion of μ_F and the domain contains $\mathcal{B}_{\mathbb{R}}$. We shall usually denote this complete measure also by μ_F and call this the Lebesgue-Stieltjes measure associated to F.

In the rest of this section, we fix a (complete) Lebesgue-Stieltjes measure μ on $\mathbb{R}$ associated to an increasing and right continuous function F, and we denote $\mathcal{M}_\mu$ the domain of μ, i.e.,

$$\begin{aligned}
\mathcal{M}_\mu &= \text{(Carathéodory style)}\, \{E \subset \mathbb{R} : E \text{ is } (\mu_F)^*\text{-measurable}\} \\
&= (\mu_F\text{-completion style})\, \{E \cup F : E \in \mathcal{B}_{\mathbb{R}} \text{ and } F \text{ is a subset of a } \mu_F\text{-null set}\}.
\end{aligned}$$

Then, for any $E \in \mathcal{M}_\mu$, we have

$$\begin{aligned}
\mu(E) &= \inf\left\{\sum_n \mu_F(A_n) : E \subset \bigcup_n A_n, \text{ where } A_n \in \mathcal{B}_\mathbb{R}\right\} \\
&= \inf\left\{\sum_n (F(b_n) - F(a_n)) : E \subset \bigcup_n (a_n, b_n]\right\} \\
&= \inf\left\{\sum_n \mu((a_n, b_n]) : E \subset \bigcup_n (a_n, b_n]\right\},
\end{aligned}$$

which is easy to check. (Note that the Lebesgue-Stieltjes measure μ is extended from the measure μ_F, not directly from μ_0. It is left as an exercise to explain how the above equality could hold.)

We first check the o-c intervals in the above equation can be replaced by open intervals.

Proposition 1.5.4. For any $E \in \mathcal{M}_\mu$,

$$\mu(E) = \inf\left\{\sum_{n=1}^{\infty} \mu_F((a_n, b_n)) : E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\right\}.[4]$$

Proof. Let the value on the right hand side be denoted by $\nu(E)$. For each n, the open interval (a_n, b_n) is a union of pairwise disjoint o-c intervals $(c_n(j), c_n(j+1)]$ for $j \in \mathbb{N}$, where $c_n(1) = a_n$, $(c_n(j))_j$ is increasing, and $c_n(j) \nearrow b_n$ as $j \to \infty$. We then have

$$\sum_n \mu_F((a_n, b_n)) = \sum_{n,j} \mu_F((c_n(j), c_n(j+1)]) \geq \mu(E)$$

so that $\mu(E) \leq \nu(E)$. To show the converse inequality, suppose $\epsilon > 0$ is given and let $((a_n, b_n])_{n=1}^{\infty}$ be a countable covering of E by o-c intervals such that

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu_F((a_n, b_n]) < \mu(E) + \epsilon.$$

For each n, let $\beta_n > 0$ be a real number greater than b_n such that $F(\beta_n) < F(b_n) + \epsilon \cdot 2^{-n}$. Because $((a_n, \beta_n))_{n=1}^{\infty}$ covers E, we have

$$\sum_n \mu_F((a_n, \beta_n)) \leq \sum_n \mu_F((a_n, b_n]) + \epsilon \leq \mu(E) + 2\epsilon, \quad \text{so} \quad \nu(E) \leq \mu(E) + 2\epsilon.$$

Therefore, $\nu(E) \leq \mu(E)$, proving the identity. □

The above proposition introduces a method of computing $\mu(E)$, where $E \in \mathcal{M}_\mu$. Remarking that an open subset of $\mathbb{R}$ is a countable union of open intervals in $\mathbb{R}$, such computation reduces to the method given in the following proposition.

Proposition 1.5.5. If $E \in \mathcal{M}_\mu$, then

$$\begin{aligned}
\mu(E) &= \inf\{\mu(U) : U \supset E \text{ and } U \text{ is open in } \mathbb{R}\} \\
&= \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}.
\end{aligned}$$

Proof. The first identity is straightforward, because every open subset of $\mathbb{R}$ is a countable union of open intervals in $\mathbb{R}$. To prove the second identity, suppose first E is bounded and not closed; if E is closed, then the equality is clear, since E is compact. Considering $\overline{E} \setminus E$, we can find an open subset $U \subset \mathbb{R}$ containing

[4] In this proposition we try to distinguish μ and μ_F, which is, in fact, not necessary; all μ_F's in this proposition (including the proof) can be replaced by μ.

$\overline{E} \setminus E$ such that $\mu(U) < \mu(\overline{E} \setminus E) + \epsilon$. Setting $K = \overline{E} \setminus U$, we find that K is compact and is contained in E. Because $E \setminus K = E \cap U$, we have

$$\begin{aligned} \mu(K) &= \mu(E) - \mu(E \cap U) = \mu(E) - (\mu(U) - \mu(U \setminus E)) \\ &\geq \mu(E) - \mu(U) + \mu(\overline{E} \setminus E) \geq \mu(E) - \epsilon, \end{aligned}$$

proving the second equality for bounded E in $\mathcal{M}_\mu$. If E is unbounded, let $E_j = E \cap (j, j+1]$, which is bounded for each $j \in \mathbb{Z}$. For each $j \in \mathbb{Z}$, let K_j be a compact subspace of E_j such that $\mu(E_j) - \epsilon \cdot 3^{-|j|} < \mu(K_j) \leq \mu(E_j)$, and define H_n be the union of K_j's for $|j| \leq n$. Note that each H_n is a compact subspace of E and $\mu(H_n) \geq \mu\left(\bigcup_{j=-n}^{n} E_j\right) - 2\epsilon$. Because $\mu(E) = \lim \mu\left(\bigcup_{j=-n}^{n} E_j\right)$, we have $\mu(H_n) \geq \mu(E) - 3\epsilon$ for large n's. $\square$

We already checked that the Lebesgue-Stieltjes measure μ associated to F is the Carathéodory extension of the measure μ_F on $\mathcal{B}_\mathbb{R}$ and the completion of μ_F, which also explains the domain of the Lebesgue-Stieltjes measure μ. The following proposition states another way of expressing the domain of μ; all Borel sets or all sets in $\mathcal{M}_\mu$ are of a reasonably simple form modulo sets of measure zero.

Proposition 1.5.6. Suppose E is a subset of $\mathbb{R}$. Then, the following statements are equivalent:

(a) $E \in \mathcal{M}_\mu$, i.e, E belongs to the domian of the (complete) Lebesgue-Stieltjes measure μ.

(b) $E = V \setminus N_1$, where V is a G_δ set and $\mu(N_1) = 0$.

(c) $E = H \cup N_2$, where H is an F_σ set and $\mu(N_2) = 0$.

Proof. Since μ is complete on $\mathcal{M}_\mu$, (b) and (c) each imply (a). We will show that (a) implies (b) and (c). First, assume $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$. Using the preceeding proposition, for each $n \in \mathbb{N}$, let U_n and V_n be open and compact subsets of $\mathbb{R}$ such that

$$\begin{aligned} E \subset U_n, &\quad \mu(U_n) < \mu(E) + \frac{1}{n}, \\ E \supset K_n, &\quad \mu(K_n) > \mu(E) - \frac{1}{n}. \end{aligned}$$

Letting $V = \bigcap_{n=1}^\infty U_n$ and $H = \bigcup_{n=1}^\infty K_n$, we find that V is a G_δ set and H is an F_σ set and that $\mu(V) = \mu(E)$ and $\mu(H) = \mu(E)$. Because $E \subset V$ and $E \supset H$, $\mu(V \setminus E) = \mu(E \setminus H) = 0$. When $\mu(E) = \infty$, we may use the σ-finiteness of $\mathbb{R}$ relative to μ. For each $j \in \mathbb{Z}$, let $E_j = E \cap [j, j+1)$ and let $\{U_{j,k}\}_{k \in \mathbb{N}}$ be a countable collection of open sets in $\mathbb{R}$ such that

$$E_j \subset U_{j,k}, \quad \mu(E_j) \leq \mu(U_{j,k}) < \mu(E_j) + \frac{1}{2^{|j|}} \frac{1}{2^k}.$$

Define $V_k = \bigcup_{j \in \mathbb{Z}} U_{j,k}$; then V_k is an open subset of $\mathbb{R}$ containing E and $\mu(E) \leq \mu(V_k) < \mu(E) + 3 \cdot 2^{-k}$, so $V = \bigcap_{k \in \mathbb{N}} V_k$ is an open subset of $\mathbb{R}$ containing E such that $\mu(E) = \mu(V)$. Hence, $E = V \setminus (V \setminus E)$ is a desired form of identity. To show that (a) implies (c), note that $X \setminus E \in \mathcal{M}_\mu$ if $E \in \mathcal{M}_\mu$. By (b), $X \setminus E = V \setminus N_1$ for some G_δ set V and a μ-null set N_1, so $E = (X \setminus V) \cup N_1$ and (c) is deduced. $\square$

There is an approximation theorem for sets in $\mathcal{M}_\mu$ of finite measures, given as the following proposition. The proposition is not even an equivalence theorem, but it will play an essential role in proving the density of C^0 space in L^1 space (in the L^1 metric topology).

Proposition 1.5.7. Suppose $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$. Given $\epsilon > 0$, there is a set F which is a finite union of open intervals such that $\mu(E \triangle F) < \epsilon$.

Proof. Let U be an open set in $\mathbb{R}$ containing E such that $\mu(E) \leq \mu(U) < \mu(E) + \epsilon$, and let $((a_n, b_n))_n$ be the countable collection of pairwise disjoint open intervals in $\mathbb{R}$ whose union is U. Because $\mu(U)$ is finite, there is an integer N such that $\sum_{n > N} \mu((a_n, b_n)) < \epsilon$. Letting $F = \bigsqcup_{n=1}^N (a_n, b_n)$, we have

$$\mu(E \setminus F) \leq \mu(U \setminus F) < \epsilon, \quad \mu(F \setminus E) \leq \mu(U \setminus E) < \epsilon.$$

Hence, $\mu(E \triangle F) < 2\epsilon$, as desired. $\square$

Notation. When μ is the Lebesgue-Stieltjes measure on $\mathbb{R}$ associated to the identity map on $\mathbb{R}$, μ is called the Lebesgue measure, and denoted by m. Furthermore, the domain $\mathcal{M}_m$ of the Lebesgue measure m is denoted by $\mathcal{L}$. Sometimes, we also call the restriction of m to $\mathcal{B}_{\mathbb{R}}$ the Lebesgue measure, too.

Among the most significant properties of Lebesgue measure are its invariance under translations and simple behavior under dilations.

Proposition 1.5.8. If $E \in \mathcal{L}$ and $s, r \in \mathbb{R}$, then $E + s$, $rE \in \mathcal{L}$, and $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Proof. Almost clear. □

Example 1.5.9 (Topological magnitude and measure need not be consistent)**.** Let $\{r_i\}$ be an enumeration of the rational numbers in $[0, 1]$, and given $\epsilon > 0$, let I_i be the open interval centered at r_i of length $2^{-i}\epsilon$; then set $U = (0, 1) \cap \bigcup_{i=1}^{\infty} I_i$. The set U is open in $\mathbb{R}$ and dense in $[0, 1]$, but $m(U) \leq \epsilon$. The set $K = [0, 1] \setminus I$ is closed in $\mathbb{R}$ and nowhere dense in $\mathbb{R}$, i.e., the closure of K in $\mathbb{R}$ has no interior point. Thus, K is *topologically small*, but $\mu(K) \geq 1 - \epsilon$.

Example 1.5.10 (Cantor set)**.** Let C denote the Cantor set. The following statements are basic properties regarding the Cantor set.

(a) C is compact, nowhere dense in $\mathbb{R}$, and totally disconnected.[5] Moreover, C has no isolated points.

(b) $m(C) = 0$.

(c) card(C) = card$(\mathbb{R})$. Let $f : C \to [0, 1]$ be the monotonically increasing function defined as follows:

> If $0.a_1a_2a_3\cdots$ is the base-3 expansion of $x \in C$ with $a_i = 0$ or $a_i = 2$ for all $i \in \mathbb{N}$, let $0.b_1b_2b_3\cdots$ with $b_i = a_i/2$ be the base-2 expansion of $f(x)$. (In fact, this definition contains a gap.)

It is clear that whenever $x, y \in C$ and $x < y$ we have $f(x) < f(y)$, unless both x and y are the endpoints of one interval which is deleted from $[0, 1]$ to form C; in this case, we have $f(x) = f(y)$. By declaring f to be constant on each interval deleted from $[0, 1]$ to form C, we can extend f to $[0, 1]$. Because f is monotonically increaing on $[0, 1]$ and is onto $[0, 1]$, f is surprisingly a continuou function. Such f is called the Cantor function.

Problems

Problem 1.5.1 (Exercise 1.30)**.** Suppose $E \in \mathcal{L}$ and $m(E) > 0$. Show that for any real $0 < \alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Solution. Assume first that the statement is valid whenever $E \in \mathcal{L}$ and $m(E) < \infty$. If $E \in \mathcal{L}$ and $m(E) = \infty$, there is an integer $j > 0$ such that $F = E \cap (-j, j)$ is of nonzero (finite) measure. Finding an open interval I for F, we have $m(E \cap I) \geq m(F \cap I) > \alpha m(I)$.

It remains to prove the statement for $m(E) < \infty$. Let U be an open subset of $\mathbb{R}$ containing E such that $m(E) \leq m(U) < (1 + \rho)m(E)$ for some $\rho > 0$. Since U is open in $\mathbb{R}$, there is a (unique) countable collection $\{(a_n, b_n)\}_{n=1}^{\infty}$ of pairwise disjoint open intervals in $\mathbb{R}$ with the union U. Letting $E_j = E \cap (a_n, b_n)$ for each $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} m((a_n, b_n)) < (1 + \rho) \sum_{n=1}^{\infty} m(E_n)$$

Hence, for some integer $j \in \mathbb{N}$, we have $m((a_j, b_j)) < (1 + \rho)m(E \cap (a_j, b_j))$. This proves the statement when letting $\rho = \alpha^{-1} - 1$.

[5] A nonempty subset E of a topological space X is said to be totally disconnected if the only connected subspaces of E are the singletons.

Problem 1.5.2 (Exercise 1.31)**.** Suppose $E \in \mathcal{L}$ and $m(E) > 0$. Show that $E - E$ contains an open interval centered at 0.

Solution. Let α be a real number such that $0 < \alpha < 1$. Then there is an open interval I in $\mathbb{R}$ such that $m(E \cap I) > \alpha m(I)$. Let $E_0 = E \cap I$ and assume that $E_0 - E_0$ contains no open interval centered at 0. Then, whenever $\epsilon > 0$, there is a positive real number $a < \epsilon$ for which E_0 and $a + E_0$ are disjoint. (Why?) For such $0 < a < \epsilon$, from the inclusion $E_0 \sqcup (a + E_0) \subset I \cup (a + I)$, we have $2m(E_0) \leq a + m(I)$, i.e., $2\alpha m(I) < m(I) + a$. Here arises a contradiction, when $1/2 < \alpha < 1$. Therefore, $E_0 - E_0$ contains an open interval centered at 0, proving the statement.

Problem 1.5.3. Let A be the subset of $[0, 1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

Solution. Oberve that $A \subset [0, 0.4) \sqcup [0.5, 1]$. Proceeding as constructing the Cantor set, we have $m(A) \leq 0.9^n$ for all positive integer n, implying that $m(A) = 0$.

1.6 Product σ-algebras

When we studied general topology, we learned how to impose a topology on the set defined as the Cartesian product of sets. There is a corresponding counterpart in the theory of measure, called the product σ-algebra.

Definition 1.6.1 (Product σ-algebra)**.** Given measurable spaces $(X_\alpha, \mathcal{M}_\alpha)$ with $\alpha \in I$, the product σ-algebra on the product $X = \prod_{\alpha \in I} X_\alpha$ is defined as the σ-algebra on X generated by the following collection:

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha \text{ for } \alpha \in I\}.$$

And the product σ-algebra on X is denoted by $\bigotimes_{\alpha \in I} \mathcal{M}_\alpha$.

Remark. Given a collection of maps $\{f_\alpha : A \to X_\alpha\}_{\alpha \in I}$ where each X_α is a topological space, we have imposed A the smallest topology relative to which each f_α is continuous; the topology generated by $\{f_\alpha^{-1}(U_\alpha) : \alpha \in I \text{ and } U_\alpha \text{ is open in } X_\alpha\}$, and the topology on $A = \prod_{\alpha \in I} X_\alpha$ with $f_\alpha = \pi_\alpha$ for each $\alpha \in I$ is the product topology on $\prod_{\alpha \in I} X_\alpha$. Similar argument is appliable in the theory of σ-algebras; given a collection of maps $\{f_\alpha : A \to X_\alpha\}_{\alpha \in I}$ where each $(X_\alpha, \mathcal{M}_\alpha)$ is a measurable space, we impose A the smallest σ-algebra relative to which each f_α is measurable; the σ-algebra generated by

$$\{f_\alpha^{-1}(E_\alpha) : \alpha \in I \text{ and } E_\alpha \in \mathcal{M}_\alpha\},$$

and the topology on $A = \prod_{\alpha \in I} X_\alpha$ with $f_\alpha = \pi_\alpha$ for each $\alpha \in I$ is the product σ-algebra on $\prod_{\alpha \in I} X_\alpha$.

We now introduce some propositions regarding product σ-algebras. The first proposition states that the generators of a product σ-algebra reduce to the products of members in σ-algebras, when there are countably many σ-algebras. (Remark the counterpart in the topology.)

Proposition 1.6.2. If I is countable, then $\bigotimes_{\alpha \in I} \mathcal{M}_\alpha$ is the σ-algebra on X generated by $\{\prod_{\alpha \in I} E_\alpha : E_\alpha \in \mathcal{M}_\alpha \text{ for } \alpha \in I\}$.

Proof. The countability of σ-algebras asserts that both $\{\prod_{\alpha \in I} E_\alpha : E_\alpha \in \mathcal{M}_\alpha \text{ for } \alpha \in I\}$ and $\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha \text{ for } \alpha \in I\}$ generates the same σ-algebra on $\prod_{\alpha \in I} X_\alpha$. □

Remark that a product topology is generated as a subbais by all preimages of open sets under canonical projections and as a subbasis by all preimages of basis members under canonical projections. Likewisely, a product σ-algebra is generated by all preimages under canonical projections of all sets generating σ-algebras. (Again, remark the counterpart in the topology.)

Proposition 1.6.3 (Reduction to generators)**.** Suppose $\mathcal{M}_\alpha$ is generated by $\mathcal{F}_\alpha$ for each $\alpha \in I$. Then $\bigotimes_\alpha \mathcal{M}_\alpha$ is generated by $\mathcal{F} := \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{F}_\alpha \text{ for all values of } \alpha\}$. Furthermore, if I is countable, then $\bigotimes_\alpha \mathcal{M}_\alpha$ is generated by $\{\prod_\alpha E_\alpha : E_\alpha \in \mathcal{F}_\alpha \text{ for all values of } \alpha\}$.

Proof. Since it is clear that $\langle \mathcal{F} \rangle \le \bigotimes_\alpha \mathcal{M}_\alpha$, it is required to show the converse inclusion. What we want to achieve is the following:

A generator $\pi_\alpha^{-1}(E_\alpha)$ of $\bigotimes_\alpha \mathcal{M}_\alpha$, where $\alpha \in I$ and $E_\alpha \in \mathcal{M}_\alpha$, belongs to $\langle \mathcal{F} \rangle$, i.e., $\pi_\alpha^{-1}(E_\alpha) \in \langle \mathcal{F} \rangle$.

For this, we seek to prove that $E_\alpha \in \mathcal{M}_\alpha$ belongs to the following collection for any $\alpha \in I$:

$$\{U \subset X_\alpha : \pi_\alpha^{-1}(U) \in \langle \mathcal{F} \rangle\}. \tag{1.3}$$

The above collection contains $\mathcal{F}_\alpha$ and is a σ-algebra on X_α, thus it contains $\mathcal{M}_\alpha$. Hence, whenever $\alpha \in I$ and $E_\alpha \in \mathcal{M}_\alpha$, we have $\pi_\alpha^{-1}(E_\alpha) \in \langle \mathcal{F} \rangle$, proving the first statement.

Letting the second collection in the statement be denoted by $\mathcal{H}$, all we need to prove is $\langle \mathcal{H} \rangle$ contains the product σ-algebra, which is as clear as the preceeding proposition. □

Remark. Given a subbasis of a topology, every member of the topology is constructive; collecting every finite intersection of the members of the subbasis gives a basis, and collecting all arbitrary unions of the members of the basis gives the topology. However, there is no corresponding proposition regarding forming σ-algebra from generators.

For such non-constructive objects, we may detour in proving some properties and set a collection as in eq. (1.3). To review, what we wanted to show was that every generator for the product σ-algebra to be in $\langle \mathcal{F} \rangle$, i.e., whenever $\alpha \in I$ and $E \in \mathcal{M}_\alpha$, we wanted $\pi_\alpha^{-1}(E_\alpha) \in \langle \mathcal{F} \rangle$. Hence, we set an appropriate collection which can be compared with $\mathcal{F}_\alpha$ as

$$\{U \subset X_\alpha : \pi_\alpha^{-1}(U) \in \langle \mathcal{F} \rangle\},$$

and then proved that the above collection contains $\mathcal{M}_\alpha$.

The lesson from this remark is that sometimes it is helpful to observe the statement to be proved and set an appropriate test set. Such strategy would be adopted in some propositions throughout this document.

Proposition 1.6.4. Let X_i be metric spaces for $i = 1, 2, \cdots, n$, and let $X = \prod_i X_i$, equipped with the product metric. Then $\bigotimes_i \mathcal{B}_{X_i}$ is contained in $\mathcal{B}_X$. When each X_i is separable, then $\bigotimes_i \mathcal{B}_{X_i} = \mathcal{B}_X$.

Proof. Note that $\bigotimes_i \mathcal{B}_{X_i} = \left\langle \pi_i^{-1}(E_i) : E_i \text{ is open in } X_i \text{ for } 1 \le i \le n \right\rangle$ and each $\pi_i^{-1}(E_i)$ is open in X, so $\bigotimes_i \mathcal{B}_{X_i} \subset \mathcal{B}_X$. Now, assume each X_i contains a countable dense subset D_i, and consider the following collection:

$$\left\{ \prod_{i=1}^n B_{X_i}(x_i, n^{-1}) : x_i \in D_i \text{ and } n \in \mathbb{N} \right\}.$$

The above collection is countable and generates the metric topology on X. Hence, every set open in X is a countable union of members in the above collection. Therefore, $\mathcal{B}_X$ is contained in $\bigotimes_{i=1}^n \mathcal{B}_{X_i}$ and $\mathcal{B}_X = \bigotimes_i \mathcal{B}_{X_i}$. □

1.7 Monotone class lemma

Definition 1.7.1 (Monotone class)**.** A monotone class on a set X is a collection M of subsets of X which is closed under countable monotone unions and countable monotone intersections.

Theorem 1.7.2 (Monotone class lemma for sets)**.** Let $\mathcal{A}$ be an algebra of sets on X. Then the monotone class generated by $\mathcal{A}$ is precisely the σ-algebra on X generated by $\mathcal{A}$.

Proof. Let $\mathcal{C}$ denote the monotone class generated by $\mathcal{A}$ and $\mathcal{M}$ denote the σ-algebra generated by $\mathcal{A}$. It is clear by definition that $\mathcal{C} \subset \mathcal{M}$, so it remains to prove $\mathcal{M} \subset \mathcal{C}$, and it suffices to prove that $\mathcal{C}$ is an algebra on X; it then implies that $\mathcal{C}$ is a σ-algebra on X (containing $\mathcal{A}$). To this end, we will justify the following statement:

For each member $E \in \mathcal{M}$, define

$$\mathcal{M}(E) := \{F \in \mathcal{M} : E \setminus F,\, F \setminus S,\, E \cap F \in \mathcal{M}\}.$$

Then $\mathcal{M}(E) = \mathcal{M}$ for all $E \in \mathcal{M}$.

(If the above statement is true, then $\mathcal{M}$ is closed under arbitrary finite intersections, and because $X \in \mathcal{A} \subset \mathcal{M}$, $\mathcal{M}$ is closed under set complements.)

To prove the above statement, one should notice the followings:

(i) For $E, F \in \mathcal{M}$, we have $E \in \mathcal{M}(F)$ if and only if $F \in \mathcal{M}(E)$.

(ii) $\mathcal{M}(E)$ is a monotone class on X whenever $E \in \mathcal{M}$. (Check.)

If A is a member of $\mathcal{A}$, then $B \in \mathcal{M}(A)$ for all $B \in \mathcal{A}$, so it follows that $\mathcal{M} \subset \mathcal{M}(A)$, i.e., $\mathcal{M}(A) = \mathcal{M}$ for all $A \in \mathcal{A}$. Thus, for a given member $E \in \mathcal{M}$, because $E \in \mathcal{M}(A)$ for all $A \in \mathcal{A}$, we have $A \in \mathcal{M}(E)$ for all $A \in \mathcal{A}$, implying that $\mathcal{M} = \mathcal{M}(E)$, as desired. □

1.8 Further topics in measure theory

1.8.1 A subset of $\mathbb{R}$ which is not Lebesgue measurable

Here, we introduce a subset of $\mathbb{R}$ which is not Lebesgue measurable. If a set is Lebesgue measurable, then its Lebesgue measure is invariant under translation, rotation, and symmetry.

To begin with, we define a relation $\sim$ on $[0,1)$ by declaring that $x \sim y$ if and only if $x - y \in \mathbb{Q}$. (Check that this relation is an equivalence relation on $[0,1)$.) Let N be any subset of $[0,1)$ which contains precisely one member of each equivalence class; let $R = [0,1) \cap \mathbb{Q}$. For each $r \in R$, define

$$N_r = \{x + r : x \in N \cap [0, 1-r)\} \cup \{x + r - 1 : x \in N \cap [1-r, 1)\}.$$

(Idea: Shifting and moving that sticks out beyond $[0,1)$.) Then $N_r \subset [0,1)$, and every $x \in [0,1)$ belongs to precisely one N_r. (Why?) Assuming N is Lebesgue measurable, by the idea of forming N_r, we can easily find that $m(N_r) = m(N)$. Because $[0,1)$ is the disjoint union of N_r's for $r \in R$, we find that $1 = m([0,1)) = \sum_{r \in R} m(N_r) = \sum_{r \in R} m(N)$, which is impossible. Therefore, the subset N is not Lebesgue measurable.

1.8.2 Locally measurable sets and saturated measures

Definition 1.8.1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) (Locally measurable set) A subset E of X is said to be locally measurable if $E \cap A$ is measurable whenever A is a measurable set such that $\mu(A) < \infty$. (Remark that every measurable set is locally measurable; using the notation in (b), we have $\mathcal{M} \subset \widetilde{\mathcal{M}}$.)

(b) (Saturated measure) Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. μ is said to be saturated if $\mathcal{M} = \widetilde{\mathcal{M}}$.

Proposition 1.8.2. (a) $\widetilde{\mathcal{M}}$ is a σ-algebra.

(b) If μ is σ-finite, then μ is saturated.

Proof. (a)

(b)

□

Define the set map $\widetilde{\mu} : \widetilde{\mathcal{M}} \to [0, \infty]$ by $\widetilde{\mu}(E) = \mu(E)$ for all $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, and we call $\widetilde{\mu}$ the saturation of μ.

Proposition 1.8.3. The saturation of a complete measure is also complete.

Proof. □

Chapter 2

Integration

2.1 Measurable functions

Definition 2.1.1 (Measurable function)**.** Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces. A map $f : X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$ whenever $E \in \mathcal{N}$. (When the σ-algebras on X and Y are understood, f is just called a measurable function.)

It is easy to check that the composition of two measurable functions is also measurable, as it held for continuous functions. Remarking that we could reduce to basis members for checking the continuity of a given map, one might wish to establish the counterpart in the theory of measurable functions.

Lemma 2.1.2 (Reduction to generators)**.** Suppose as in the above definition, and assume is generated by $\mathcal{F}$. Then a map $f : X \to Y$ is measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{F}$.

Proof. We only need to show the if part; we want $f^{-1}(N) \in \mathcal{M}$ for all $N \in \mathcal{N}$. Consider the collection

$$\{E \in \mathcal{N} : f^{-1}(E) \in \mathcal{M}\}.$$

Because $\mathcal{M}$ is a σ-algebra (on Y) and the above collection contains $\mathcal{F}$, it contains $\mathcal{N}$. $\square$

Corollary 2.1.3. Suppose X and Y are topological spaces and $f : X \to Y$ is a continuous map. Then f is a $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable function.

Proof. The topology on Y generates the Borel σ-algebra on Y. $\square$

Definition 2.1.4. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be a measurable space.

(a) A real or complex-valued map f defined on X is called a $\mathcal{M}$-measurable map if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ or $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable. In particular, when $X = \mathbb{R}$, f is said to be Lebesgue (or Borel) measurable if f is $\mathcal{L}$ (or $\mathcal{B}_{\mathbb{R}}$)-measurable.[1]

(b) Suppose $E \in \mathcal{M}$. We say a map $f : X \to Y$ is measurable on E if $f|_E$ is measurable, where E equips the restriction of $\mathcal{M}$ to E as the σ-algebra.

Remark. (a) It must be noted that Borel measurability is preserved under compositions while Lebesgue meaurability is not.

(b) Let f be a function from X into $\overline{\mathbb{R}}$ and let $A = f^{-1}(\mathbb{R})$. Then f is measurable if and only if f is measurable on $\mathbb{R}$ and both $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ are measurable sets in X.

Proposition 2.1.5. Let $(X, \mathcal{M})$ and $(Y_\alpha, \mathcal{N}_\alpha)\,(\alpha \in A)$ be measurable spaces, and let

$$Y = \prod_{\alpha \in A} Y_\alpha, \quad \mathcal{N} = \bigotimes_{\alpha \in A} \mathcal{N}_\alpha.$$

Let $f_\alpha : X \to Y_\alpha$ be a map for each $\alpha \in A$ and $f : X \to Y$ be the map such that $\pi_\alpha \circ f = f_\alpha$ for all α. Then f is $(\mathcal{M}, \mathcal{N})$-measurable if and only if f_α is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for each $\alpha \in A$.

[1] Of course, one can extend this definition for a metric (or metrizable) space as the codomain. Nevertheless, in this note, a measurable function will denote a real or complex-valued function only.

Proof. Because π_α is $(\mathcal{N}, \mathcal{N}_\alpha)$-measurable, f_α is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for each $\alpha \in A$, provided that f is $(\mathcal{M}, \mathcal{N})$-measurable. To show the converse, note that $f^{-1}(\pi_\alpha^{-1}(E_\alpha)) = f_\alpha^{-1}(E_\alpha)$ is a member of $\mathcal{M}$ whenever $\alpha \in A$ and $E_\alpha \in \mathcal{N}_\alpha$. □

Corollary 2.1.6. A complex-valued function f on X is $\mathcal{M}$ measurable if and only if $\mathrm{Re}\,(f)$ and $\mathrm{Im}\,(f)$ are $\mathcal{M}$-measurable. (Here, $(X, \mathcal{M})$ is a measurable space.)

Proof. It follows from $\mathcal{B}_\mathbb{C} = \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R}$. □

We now prove that the complex-valued measurable functions (with a given domain) form a $\mathbb{C}$-algebra (when $\mathbb{R}$ is given as the codomain, then the collection is an $\mathbb{R}$-algebra).

Lemma 2.1.7. Suppose $(X, \mathcal{M})$ is a measurable space and $f, g : X \to F$ are $\mathcal{M}$-measurable functions, where $F = \mathbb{R}$ or $F = \mathbb{C}$. Then $f + g$, fg, $cf\,(c \in \mathbb{R})$ are $\mathcal{M}$-measurable.

Proof. The usual addition and multiplication in $\mathbb{R}$ and $\mathbb{C}$ are continuous. □

Proposition 2.1.8. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$, then the functions

$$g_1(x) = \sup_n f_n(x), \quad g_2(x) = \inf_n f_n(x)$$
$$g_3(x) = \limsup_n f_n(x), \quad g_4(x) = \liminf_n f_n(x)$$

are all measurable. If $\lim_n f_n(x)$ exists for every $x \in X$, then $f = \lim_n f_n$ is also measurable.

Proof. Remark that $g_1^{-1}(a, \infty] = \bigcup_n f_n^{-1}(a, \infty]$ and $g_2^{-1}[-\infty, a) = \bigcup_n f_n^{-1}[-\infty, a)$ whenever $a \in \overline{\mathbb{R}}$. □

Now we introduce the definition of a simple function and its significance in the integral theory.

Definition 2.1.9 (Simple function)**.** Let $(X, \mathcal{M})$ be a measurable function. A function $s : X \to \mathbb{C}$ is called a simple function if s is measurable and has a finite range. Indeed, s is a simple function if and only if the range of s is $\{a_1, \cdots, a_k\}$ for some complex numbers $a_1, \cdots, a_k$ and $s = \sum_{k=1}^n a_k \chi_{E_k}$, where $E_k = f^{-1}(E_k)$ is measurable.

We now introduce a remarkable observation that an $\overline{\mathbb{R}}$-valued (or complex-valued) measurable function f defined on a measurable space $(X, \mathcal{M})$ can be approximated by a simple function. This observation, especially together with the monotone convergence theorem which will be introduced in the following section, is essential in establishing the integral theory.

Theorem 2.1.10. Let $(X, \mathcal{M})$ be a measurable space.

(a) If $f : X \to [0, \infty]$ is a measurable function, then there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions on X such that

   (i) $0 \le \phi_1 \le \phi_2 \le \cdots \le f$,

   (ii) $\phi_n \to f$ pointwise on X,

   (iii) and $\phi_n \to f$ uniformly on any set on which f is bounded.

(b) If $f : X \to \mathbb{C}$ is a measurable function, then there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions such that

   (i) $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$,

   (ii) $\phi_n \to f$ pointwise on X,

   (iii) and $\phi_n \to f$ uniformly on any set on which f is bounded.

Proof. In proving (a), for each $n \in \mathbb{N}$ and $0 \leq k \leq 2^{2n} - 1$, define

$$E_n^k := f^{-1}([k \cdot 2^{-n}, (k+1) \cdot 2^{-n})), \quad F_n := f^{-1}([2^n, \infty]).$$

And let

$$s_n := \sum_{k=1}^{2^{2n}-1} \frac{k}{2^n} \chi_{E_n^k} + 2^n \chi_{F_n}.$$

Clearly, $0 \leq s_1 \leq s_2 \leq \cdots \leq f$ and $s_n \to f$ pointwise. Moreover, given a subset of X over which $|f| < 2^j$ for some positive integer j, $|f - s_m| \leq 2^{-m}$ whenever $m \geq j$; this proves the desired uniform convergence. In proving (b), decompose f into the real and the imaginary parts and decompose each part into the nonnegative and the negative parts, and then use the result of (a). □

If μ is a measure on a measurable space $(X, \mathcal{M})$, then one may wish to except μ-null sets from consideration. In this respect, it is simpler when μ is complete.

Proposition 2.1.11. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. The following statements each are valid if and only if μ is complete:

(a) If $f : X \to Y$ is a measurable function and $g : X \to Y$ is a function such that $f = g$ μ-a.e., then g is measurable.

(b) Let Y be a topological space and $\mathcal{N}$ be the Borel σ-algebra on Y. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions from X into Y and $f_n \to f$ μ-a.e., then f is measurable.

Proof. (a) Assume first that μ is complete and let g be a function on X such that $f = g$ μ-a.e. Letting $D = \{x \in X : f(x) \neq g(x)\}$, then D is a μ-null set. Hence, whenever $E \in \mathcal{N}$, $g^{-1}(E)$ and $f^{-1}(E)$ differ by at most D, i.e., $f^{-1}(E) \setminus D \subset g^{-1}(E) \subset f^{-1}(E) \cup D$. Therefore, $g^{-1}(E) \in \mathcal{M}$ and g is measurable.

Assume conversely and let N be a subset of a μ-null set. Letting $f = 0$ and $g = \chi_N$, by hypothesis, g is measurable, hence $N \in \mathcal{M}$ and μ is complete.

(b) Assume first that μ is complete and suppose $f_n \to f$ μ-a.e. Letting D be the set of a point x of X such that $f_n(x) \nrightarrow f(x)$. Then $(X \setminus D) \cap f^{-1}(V) = (X \setminus D) \cap \liminf_{n\to\infty} f_n^{-1}(V)$, thus

$$(\liminf_{n\to\infty} f_n^{-1}(V)) \setminus D \subset f^{-1}(V) \subset (\liminf_{n\to\infty} f_n^{-1}(V)) \cup D$$

and $f^{-1}(V) \in \mathcal{M}$. This proves that f is measurable.

Assume conversely and let N be a subset of a μ-null set. Then $0 \to \chi_N$ μ-a.e., so N is measurable by hypothesis.

This completes the proof of the equivalences. □

On the other hand, the following result shows that one is unlikely to commit any serious blunders by forgetting to worry about completeness of the measure. Later in this chapter, the following lemma will be applied to identify $L(\mu)$ and $L(\mu)$, where $(X, \overline{\mathcal{M}}, \overline{\mu})$ is the completion of $(X, \mathcal{M}, \mu)$.

Lemma 2.1.12. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f = g$ $\overline{\mu}$-a.e..

Remark. Roughly speaking, according to this lemma, we may discard $\overline{\mu}$-null sets of X to find an $\mathcal{M}$-measurable function which coincides f $\overline{\mu}$-almost everywhere.

Proof. We first check the statement for simple functions.

Step 1: The proof for simple functions.

Suppose f attains $a \in \mathbb{C}$ only. Because f is $\overline{\mathcal{M}}$-measurable, $f = a\chi_E$ for some $E \in \overline{\mathcal{M}}$. By definition, $E = M \cup N$ for some $M \in \mathcal{M}$ and a subset N of a μ-null set. Letting $g = a\chi_M$, we have $f = g$ $\overline{\mu}$-almost everywhere. The case for simple functions is now obvious.

Step 2: Completing the proof.
Assume f is $\overline{\mathcal{M}}$-measurable, and let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of simple functions given as in Theorem 2.1.10. For each $n \in \mathbb{N}$, using the result of Step 1, let φ_n be an $\mathcal{M}$-measurable simple function on X such that $\phi_n = \varphi_n$ $\overline{\mu}$-almost everywhere. Let D_n be the subset of a point x of X such that $\phi_n(x) \neq \varphi_n(x)$, and let D be the union of D_n for all n. Then $\overline{\mu}(D) = 0$ and $\phi_n = \varphi_n$ on $X \setminus D$ for all n, so $f = \lim_n \varphi_n$ on $X \setminus D$. If N is a subset of X which belongs to $\mathcal{M}$ such that $D \subset N$ and one defines $g = \chi_{X\setminus N} \lim_n \phi_n$, we find that g is a measurable function which coincides f $\overline{\mu}$-almost everywhere. □

Problems

Problem 2.1.1 (Exercise 2.3)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on a measurable space $(X, \mathcal{M})$. Show that the collection $E := \{x \in X : \lim_n f_n(x) \text{ exists}\}$ is a measurable set.

Solution. Suppose first that every f_n is real-valued. Then E is the collection of points x in X such that $\liminf_n f_n(x) = \limsup_n f_n(x)$. Letting $s := \limsup_n f_n$ and $i = \liminf_n f_n$, we have $E = (s-i)^{-1}(\{0\})$. Because s and i are measurable functions, E is a measurable function.

Now, assume that at least one f_n is complex-valued function. Then $x \in E$ if and only if both $\mathrm{Re}\,(f_n)$ and $\mathrm{Im}\,(f_n)$ are convergent at x. Using the preceeding result, we can easily derive the property.

Problem 2.1.2 (Exercise 2.5)**.** Assume that $(X, \mathcal{M})$ is a measurable space and $X = A \cup B$ where A, $B \in \mathcal{M}$. Show that a function f on X is measurable if and only if f is measurable on A and B.

Solution. Easy.

Remark (Forming a measurable limit function). Suppose $(f_n)_n$ is a sequence of measurable functions, and let E be the subset of the domain on which $(f_n)_n$ is convergent. Let f be a map defined on X such that $f(x) = \lim_n f_n(x)$ for all $x \in E$. By Problem 2.1.1, E is a measurable set, so $f|_E$ is measurable on E. To find a 'measurable' map f on the entire domain, we just let $f(x) = 0$ for all $x \in X \setminus E$. Because f is then measurable on E and $X \setminus E$, by Problem 2.1.2, such map f is measurable on the entire domain. Such construction of "almost convergent" limit funciton will be done in later sections.

Problem 2.1.3 (Exercise 2.6)**.** Show that the supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable (unless the σ-algebra on X is very special).

Solution. We use the set we found when we proved the existence of a subset of $\mathbb{R}$ which is not Lebesgue measurable. Impose an equivalence relation $\sim$ on $[0, 1]$ by declaring $x \sim y$ if and only if $x - y \in \mathbb{Q}$, and let N be the collection of all representatives. Then N is not Lebesgue measurable (so it is necessarily uncountable). For each $\alpha \in N$, let $f_\alpha = \chi_{\{\alpha\}}$. Then the supremum of $\{f_\alpha\}_{\alpha\in N}$ is χ_N, which is not measurable if the domain equips the Borel σ-algebra (if the domain equips the discrete σ-topology, then any function on the domain is measurable).

Problem 2.1.4 (Exercise 2.8)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonic function. Show that f is measurable.

Solution. Remark that the Borel σ-algebra on $\mathbb{R}$ is generated by closed rays in $\mathbb{R}$ and that $f^{-1}([a, \infty))$ is a closed ray in $\mathbb{R}$.

2.2 Integration of nonnegative functions

Throughout this section, we fix a measure space $(X, \mathcal{M}, \mu)$, and we define

Definition 2.2.1 (The class of nonnegative measurable functions)**.** We define

$$L^+ := (\text{the space of all measurable functions from } X \text{ to } [0, \infty]).$$

2.2.1 Integration of nonnegative measurable simple functions

If $\phi \in L^+$ is a simple function with the standard representation $\phi = \sum_{j=1}^n a_j \chi_{E_j}$, we define the integral of ϕ with respect to μ by

$$\int \phi \, d\mu := \sum_{j=1}^n a_j \mu(E_j),$$

where $d\mu$ would be omitted if the context is clear. In addition, if $A \in \mathcal{M}$, we define $\int_A \phi$ to be $\int \phi \chi_A$.

Some basic, and seemingly obvious, statements regarding integrations of simple functions now follow.

Proposition 2.2.2. Let ϕ and φ be simple functions in L^+.

(a) If $c \geq 0$, then $\int c\phi = c \int \phi$.

(b) $\int (\phi + \varphi) = \int \phi + \int \varphi$.

(c) If $\phi \leq \varphi$, then $\int \phi \leq \int \varphi$.

(d) The map $A \mapsto \int_A \phi$ defined for all $A \in \mathcal{M}$ is a measure on $\mathcal{M}$.

Proof. Straightforward. □

2.2.2 Integration of nonnegative measurable functions

Given a function $f \in L^+$, we define the integral of f by

$$\int f := \sup \left\{ \int s : s \text{ is simple and } 0 \leq s \leq f \right\}.$$

Theorem 2.2.3 (Monotone convergence theorem)**.** Suppose $(f_n)_{n \in \mathbb{N}} \subset L^+$ is monotonically increasing. If $f := \lim_n f_n (= \sup_n f_n)$, then $\int f = \lim_n \int f_n$.

Proof. Remark that $f = \sup_{n \in \mathbb{N}} f_n \in L^+$ and $\int f \geq \lim_n \int f_n$. To prove the equality, fix a constant $0 < \rho < 1$ and set $E_n := \{x \in X : f_n(x) \geq \rho f(x)\}$ for each $n \in \mathbb{N}$. Note that the sequence $\{E_n\}_{n \in \mathbb{N}}$ is increasing and $\bigcup_{n \in \mathbb{N}} E_n = X$. (Why?) We also have the following inequality:

$$\int f_n \geq \int_{E_n} f_n \geq \rho \int_{E_n} f, \quad \text{so} \quad \lim_n \int f_n \geq \rho \lim_n \int_{E_n} f \geq \rho \lim_n \int_{E_n} s,$$

where s is a simple function in L^+ such that $0 \leq s \leq f$. By (d) of Proposition 2.2.2, we have $\lim_n \int_{E_n} s = \int s$, thus $\lim_n \int f_n \geq \rho \int s$ and $\lim_n \int f_n \geq \rho \int f$. This proves the theorem. □

Remark. In the last section, we studied that for a (real or complex-valued) measurable function f on X, there is a monotonically increaisng sequence of (simple) functions in L^+ which converges to f pointwise on X (and uniformly on any set on which f is bounded). Thus, we may freely use the monotone convergence theorem for functions in L^+.

Using the monotone convergence theorem, we can prove that the properties in Proposition 2.2.2 holds accordingly for nonsimple functions in L^+. Proving them is left as an exercise.

We introduce some basic but helpful properties. Justitifications are left as exercises.

Example 2.2.4. (a) Suppose $f \in L^+$. Then $\int f = 0$ if and only if $f = 0$ μ-almost everywhere.

(b) Suppose $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. If $f_n(x)$ increases to $f(x)$ for μ-almost every $x \in X$, then $\int f = \lim_{n \to \infty} \int f_n$.

Proposition 2.2.5 (Fatou's lemma)**.** If $(f_n)_{n \in \mathbb{N}}$ is any sequence in L^+, then

$$\int \left(\liminf_{n \to \infty} f_n \right) \leq \liminf_{n \to \infty} \int f_n.$$

Example 2.2.6. (a) If $(f_n)_n$ is any sequence in L^+, $f \in L^+$, and $f_n \to f$ μ-almost everywhere, then $\int f \leq \liminf_n \int f_n$.

(b) If $f \in L^+$ and $\int f < \infty$, then $f^{-1}(\{\infty\})$ is μ-null and $f^{-1}([0, \infty))$ is a σ-finite for μ.

Problems

Problem 2.2.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in L^+ and $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Show that $\int_E f = \lim \int_E f_n$ whenever $E \in \mathcal{M}$. Explain why this equation may fail if $\int f = \lim \int f_n = \infty$.

Solution. Whenever $E \in \mathcal{M}$, by Fatou's lemma, we have

$$\int_E f \le \liminf \int_E f_n \quad \text{and} \quad \int_{X\setminus E} f \le \liminf \int_{X\setminus E} f_n$$

so $\int f \le \liminf \int_E f_n + \liminf \int_{X\setminus E} f_n = \liminf \int f_n$ and $\int_E f = \lim \int_E f_n$.

Let $f_n = \chi_{(0,\infty)} + n^2\chi_{(-1/n,0)}$ for each $n \in \mathbb{N}$. Then $\int_{(-1,0)} f = 0$ but $\lim \int_{(-1,0)} f_n = \infty$.

Problem 2.2.2. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $f \in L^+$, and define $\lambda : \mathcal{M} \to [0, \infty]$ by $\lambda(E) = \int_E f \, d\mu$ for all $E \in \mathcal{M}$.

(a) Show that λ is a measure on $\mathcal{M}$.

(b) Prove that $\int f \, d\lambda = \int fg \, d\mu$ for all $g \in L^+$.

2.3 Integration of complex functions - Part 1

Again, throughout this section, we fix a measure space $(X, \mathcal{M}, \mu)$.

2.3.1 Integration of real-valued measurable functions

Remark. By real-valued we mean the case where the codomain is $\mathbb{R}$ or $\overline{\mathbb{R}}$.

Given a measurable function $f : X \to \overline{\mathbb{R}}$, we define the integral of f with regard to μ by

$$\int f := \int f^+ - \int f^-.$$

(Remark that f^+ and f^- are measurable, because f is measurable.) We are clearly concerned with the case where $\int f^+$ and $\int f^-$ are both finite.

Definition 2.3.1 (Integrable real-valued function)**.** A real-valued measurable function f on X is said to be integrable if

$$\int f^+ \text{ and } \int f^- \text{ are both finite, i.e., } \int |f| \text{ is finite.}$$

Furthermore, given $E \in \mathcal{M}$, we say a real-valued function f on X is integrable on E if $f\chi_E$ is measurable and $\int_E |f| < \infty$.

Notation. Let $(X, \mathcal{M}, \mu)$ be a measure space. The collection of real-valued integrable functions (including such functions with values in $\overline{\mathbb{R}}$) is denoted by $L_r(X, \mathcal{M}, \mu)$.

As real or complex-valued measurable functions form an algebra over $\mathbb{R}$ or $\mathbb{C}$, one might wish to check if there is a corresponding result for real-valued or complex-valued integrable functions.

Proposition 2.3.2. (a) If f, g are real-valued measurable functions on X and $f \le g$, then $\int f \le \int g$.

(b) Under the usual operations, L_r is an $\mathbb{R}$-vector space, and the integral is an $\mathbb{R}$-linear functional on L_r.

Proof. (a) $f^+ \le g^+$ and $f^- \ge g^-$, thus $\int f = \int f^+ - \int f^- \le \int g^+ - \int g^- = \int g$.

(b) To verify that L_r is an $\mathbb{R}$-vector space, suppose $a, b \in \mathbb{R}$, $f, g \in L_r$, and observe that $|af + bg| \leq |a||f| + |b||g|$. We now show that the integral is an $\mathbb{R}$-linear functional on L_r. Let $h = f + g$ and observe that $h^+ - h^- = f^+ - f^- + g^+ - g^-$ so $h^+ + f^- + g^- = h^- + f^+ + g^+$. The latter identity explains the additivity. $\mathbb{R}$-scalar multiplicativity easily follows.

Remark that L_r need not be an algebra on $\mathbb{R}$. For example, when the Lebesgue measure space on $\mathbb{R}$ is given and $f, g \in L_r$ where

$$f(x) = \begin{cases} x^{-2} & (\text{if } x > 0) \\ 0 & (\text{otherwise}) \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x & (\text{if } 0 \leq x \leq 1) \\ 0 & (\text{otherwise}) \end{cases},$$

then $f, g \in L_r$ but $fg \notin L_r$. □

2.3.2 Integration of complex-valued measurable functions

Given a measurable function $f : X \to \mathbb{C}$, we define the integral of f with regard to μ by

$$\int f := \int \mathrm{Re}\,(f) + i \int \mathrm{Im}\,(f)\,.$$

(Remark that $\mathrm{Re}\,(f)$ and $\mathrm{Im}\,(f)$ are measurable, because f is measurable.) We are concerned with the case where $\int \mathrm{Re}\,(f)$ and $\int \mathrm{Im}\,(f)$ are integrable.

Definition 2.3.3 (Integrable complex-valued functions)**.** A complex-valued function f on X is called an integrable function, if

(a) f is a measurable function, and

(b) $\int \mathrm{Re}\,(f)$ and $\int \mathrm{Im}\,(f)$ are both integrable.[2]

Also, given a complex-valued function f and $E \in \mathcal{M}$, we say f is integrable on E if $f\chi_E$ is measurable and $\int_E |f| < \infty$.

Notation. Let $(X, \mathcal{M}, \mu)$ be a measure space. The collection of complex-valued integrable functions is denoted by $L_c(X, \mathcal{M}, \mu)$.

As L_r is an $\mathbb{R}$-vector space, L_c is a $\mathbb{C}$-vector space.

Proposition 2.3.4. L_c is a $\mathbb{C}$-vector space, and the integral on L_c is a $\mathbb{C}$-linear functional.

Proof. To verify that L_c is a $\mathbb{C}$-vector space, suppose $a, b \in \mathbb{C}$, $f, g \in L_c$, and observe that $|af + bg| \leq |a||f| + |b||g|$. The integral is clearly $\mathbb{C}$-linear; the additivity follows from the definition of the integral, and the $\mathbb{C}$-scalar multiplicativity follows easily. □

Proposition 2.3.5. If $f \in L_c$, then $|\int f| \leq \int |f|$.

Proof. Let $e^{i\theta}$ be the sign of $\int f$ and α be its complex conjugate. Because $|\int f| = \alpha \int f = \int(\alpha f)$ is real,

$$\left|\int f\right| = \mathrm{Re}\left(\int(\alpha f)\right) = \int \mathrm{Re}\,(\alpha f) \leq \int |\mathrm{Re}\,(\alpha f)\,| \leq \int |\alpha f| = \int |f|,$$

proving the inequality. □

Observation 2.3.6. If $f \in L_c$, then the set $E = \{x \in X : f(x) \neq 0\}$ is the union of all A_n for $n \in \mathbb{N}$, where $A_n = \{x \in X : 1/n < |f(x)| < n\}$. Since $\mu(A_n)/n \leq \int_{A_n} f < \infty$, each A_n has a finite measure for μ. Hence, E is a σ-finite measurable set.

We have constructed the completion of a measure space $(X, \mathcal{M}, \mu)$ by adjoining all subsets of μ-null sets. Hence, one might expect that the measurability, integrability, and even the value of the respective integrals of a function $f : X \to \mathbb{C}$ are the same.

[2]In other words, $\int |\mathrm{Re}\,(f)\,|$ and $\int |\mathrm{Im}\,(f)\,|$ are both finite, i.e., $\int |f|$ is finite.

Lemma 2.3.7. Let $(X, \mathcal{M}, \mu)$ be a measure space with the completion $(X, \overline{\mathcal{M}}, \overline{\mu})$, and let $f : X \to \mathbb{C}$ be a function.

(a) If f is $\mathcal{M}$-measurable, then f is $\overline{\mathcal{M}}$-measurable. (Conversely, if f is $\overline{\mathcal{M}}$-measurable, then there is a μ-measurable function $f_0 : X \to \mathbb{C}$ which coincides f $\overline{\mu}$-almost everywhere. See Lemma 2.1.12.)

(b) Assume f is $\mathcal{M}$-measurable. Then $\int f\, d\mu = \int f\, d\overline{\mu}$. In particular, f is integrable with respect to μ if and only if f is integrable with respect to $\overline{\mu}$.

Proof. (a) is straightforward. To prove (b), it suffices to prove the identity for the case where $f \in L^+$. Indeed, it is clear that $\int f\, d\mu \le \int f\, d\overline{\mu}$. To prove the converse inequality, let $s = \sum_{j=1}^n a_j \chi_{E_j}$ be a simple function which is $\overline{\mathcal{M}}$-measurable and $0 \le s \le f$ (assume that $E_1, \cdots, E_n$ are pairwise disjoint). Because $E_1, \cdots, E_n \in \overline{\mathcal{M}}$, there are sets $A_1, \cdots, A_n \in \mathcal{M}$ such that $A_j \subset E_j$ and $\mu(A_j) = \overline{\mu}(E_j)$ for $j = 1, \cdots, n$. Then

$$\int s\, d\overline{\mu} = \int \sum_{j=1}^n a_j \chi_{A_j}\, d\mu \le \int f\, d\mu,$$

so $\int f\, d\mu = \int f\, d\overline{\mu}$. The last assertion is a direct conclusion of the preceeding identity. □

Changing values of a function on a μ-null set makes no difference in integral.

Lemma 2.3.8. Suppose $f, g \in L_c$. Then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ if and only if $\int |f - g| = 0$ if and only if $f = g$ μ-almost everywhere.

Proof. It is clear that $\int |f - g| = 0$ if and only if $f = g$ μ-almost everywhere and $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ under any of the other assumptions.

Suppose $\int_E f = \int_E g$ for all $E \in \mathcal{M}$. In fact, it suffices to check the equivalence for the case $f, g \in L_r$. Letting $h = f - g \in L_r$, then $\int_E h = 0$ whenever $E \in \mathcal{M}$. If $E = \{x \in X : h(x) \ge 0\} \in \mathcal{M}$, then $\int_E h = 0$ so $h^+ = 0$ μ-a.e.; similarly, $h^- = 0$ μ-a.e., hence $f = g$ μ-a.e. □

According to Lemma 2.3.8, if $f \in L_c$ is finite almost everywhere and one wishes to find $\int f$, then we may re-define $f(x) = 0$ when $f(x)$ was infinite.

Definition 2.3.9 (L^1 space)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space. Define a relation $\sim$ on L_c by

$$f \sim g \text{ if and only if } f = g \ \mu\text{-almost everywhere.}$$

Then $\sim$ denotes an equivalence relation on L_c, and the set of equivalence representatives $L_c / \sim$ is denoted by $L^1(X, \mathcal{M}, \mu)$.[3]

Remark. L^1 is still an $\mathbb{R}$ (and $\mathbb{C}$)-vector space under the usual addition and scalar multiplication. Considering the equivalence relation in constructing L^1, it will make no confusion when $f \in L^1$ means that f is an integrable function which is defined μ-almost everywhere.

We introduce two further advantages of L^1.

(a) (Identifying $L^1(\mu)$ and $L^1(\overline{\mu})$) Let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be the completion of $(X, \mathcal{M}, \mu)$. Suppose $f \in L^1(\overline{\mu})$ and let g be an $\mathcal{M}$-measurable function such that $f = g$ $\overline{\mu}$-almost everywhere. Then the equivalence classes of f and g are the same, inducing the same class in $L^1(\overline{\mu})$. So, there is a one-to-one correspondence between $L^1(\overline{\mu})$ and $L^1(\mu)$:

$$L^1(\mu) \to L^1(\overline{\mu}) : h \mapsto h, \qquad L^1(\overline{\mu}) \to L^1(\mu) : f \mapsto g$$

Hence, we may identify $L^1(\mu) = L^1(\overline{\mu})$ and let $f \in L^1(\mu)$ mean that f is a $\overline{\mu}$-almost everywhere defined function which is integrable with regard to $\overline{\mu}$. Furthermore, by Lemma 2.3.7, the above identification works appropriately when integrating a given function in L^1.

[3] Such construction is valid for L_r as well as L_c. If necessary, we shall use L^1_c and L^1_r; otherwise, we regard $L^1 = L^1_c$.

(b) (A metric on L^1) The map $\rho : L^1 \times L^1 \to [0, \infty)$ defined by

$$(f, g) \mapsto \int |f - g| \quad \text{(for all } f, g \in L^1)$$

is a metric on L^1. We shall refer to convergence with respect to this metric as convergence in L^1, i.e., $f_n \to f$ in L^1 if and only if $\int |f_n - f| \to 0$.

We end this section after studying a convergence theorem in L^1, which seems to be quite strong.

Theorem 2.3.10 (The Lebesgue dominated convergence theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in L^1, which converges to f μ-almost everywhere. Suppose that there is an integrable function $g : X \to [0, \infty]$ such that $|f_n| \le g$ μ-almost everywhere for all $n \in \mathbb{N}$. Then $f \in L^1$ and $f_n \to f$ in L^1, hence $\int f_n \to \int f$.

Proof. Clearly, f is a measurable function and $\int |f| \le \int |g| < \infty$, so $f \in L^1$. Because $|f_n - f| \le 2g$, we may use Fatou's lemma as follows:

$$\liminf_{n\to\infty} \int (2g - |f_n - f|) \ge \int \liminf_{n\to\infty} (2g - |f_n - f|) = \int 2g.$$

Hence, $\limsup_n \int |f_n - f| = -\liminf_{n\to\infty} \int (-|f_n - f|) = 0$ and $f_n \to f$ in L^1. $\square$

Problems

Problem 2.3.1 (2nd textbook, 1.5.8)**.** Suppose $f \in L^1$ and let $E_a = \{x \in X : |f(x)| > a\}$ for each $a \in (0, \infty)$. Prove that $a \cdot \mu(E_a) \to 0$ as $a \searrow 0$.

Solution. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0, and define $f_n = a_n \chi_{E_{a_n}}$ for each n. Becaue each f_n is a measurable function and $|f_n| \le f$ for all n, by the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} (a_n \cdot \mu(E_{a_n})) = \lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int 0 = 0.$$

Since this equality holds for any sequence $(a_n)_n \subset [0, \infty)$ converging to 0, we obtain the desired result.

Problem 2.3.2 (2nd textbook, 1.5.9)**.** Let $f : E \to [0, \infty]$ be a function where E is a measurable set of finite measure for μ. For each $n \in \mathbb{N}$, let $E_n = \{x \in X : |f(x)| \ge n\}$. Show that f is integrable if and only if $\sum_n \mu(E_n) < \infty$. Investigate when $\mu(E) = \infty$.

Solution. Letting $g = \sum_n n\chi_{E_n}$, it is clear from $0 \le g \le f$ that $\sum_n \mu(E_n) = \int g \le \int f < \infty$. To show the converse implication, assume $\sum_n \mu(E_n) < \infty$. Because $\mu(E) < \infty$, $f \le g + 1$, so $\int f \le \int (g + 1) = \sum_n \mu(E_n) + \mu(E) < \infty$, hence f is integrable, i.e., $f \in L^1$.

Under the assumption that $\mu(E) = \infty$, it is still valid that $g \le f$ so $\sum_n \mu(E_n) < \infty$ if f is integrable. The converse may not hold; consider the function $f : (0, \infty) \to [0, \infty]$ define by $f(x) = 1/\sqrt{x}$, whose integral diverges. For each positive integer n, we have $E_n = (0, 1/n^2)$, so $\sum_n \mu(E_n) = \pi^2/6$.

Problem 2.3.3 (2nd textbook, 1.5.10)**.** Show that

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f(x) \cos(\epsilon x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

whenever $f \in L^1$.

Solution. For any sequence $(\epsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ converging to 0, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos(\epsilon_n x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

Since the result holds for any $(\epsilon_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ converging to 0, we obtain the desired result.

Problem 2.3.4 (Exercise 2.26)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and assume $f \in L^1$. Prove that for any real number $\epsilon > 0$ there is a real number $\delta > 0$ satisfying the following property: Whenever $E \in \mathcal{M}$ and $\mu(E) < \delta$ then $\int_E |f| < \epsilon$.

Solution. It suffices to prove the statement for $f \in L^+$; then, when $f \in L^1_r$ we have $\int_E |f| \leq \int_E f_+ + \int_E f_- < 2\epsilon$, and if $f \in L^1_c$ we have $\int_E |f| \leq \int_E |\mathrm{Re}\,(f)\,| + \int_E |\mathrm{Im}\,(f)\,| < 4\epsilon$. For each $n \in \mathbb{N}$, define the function $s_n : X \to [0, \infty]$ by $s_n(x) = \min\{f(x), n\}$ for all $x \in X$. Since $f \in L^1$ and $|s_n| \leq f$, by the Lebesgue dominated convergence theorem, $s_n \to f$ in L^1. Hence, there is a positive integer N such that $\|f - s_N\|_{L^1} < \epsilon/2$. Furthermore, if $0 < \delta < \epsilon/N$, then $\int_E |s_N| \leq \epsilon/2$ whenever $E \in \mathcal{M}$ and $\mu(E) < \delta$. Therefore, $\int_E |f| \leq \int_E |f - s_N| + \int_E s_N \leq \epsilon$, as desired.

2.4 Integration of complex functions - Part 2

In this section, as a sequel of the preceeding section, we introduce some applications of the Lebesgue dominated convergence theorem. Unless stated otherwise, we assume that a measure space $(X, \mathcal{M}, \mu)$ is given.

2.4.1 Approximation of an L^1-function

As a Riemann integrable function $f : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$) can be approximated by a simple function or a continuous function with a compact support, an L^1-function can also be approximated by a simple or continuous function.

Theorem 2.4.1. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $f \in L^1$.

(a) For any $\epsilon > 0$, there is a simple integrable function $s = \sum_{j=1}^n a_j \chi_{E_j}$ such that $\int |f - s| < \epsilon$. That is, the collection of the equivalence classes among simple integrable functions is dense in L^1.

(b) If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$, each set E_j in (a) can be taken to be a finite union of open intervals. Moreover, there is a continuous function g that vanishes outside a bounded interval such that $\int |f - g| < \epsilon$. Hence, any $f \in L^1$ can be approximated by a C^∞-function in L^1-sense.

Proof. (a) Let $(s_n)_{n=1}^\infty$ be a sequence of simple functions given as in Theorem 2.1.10. Because $f \in L^1$, by the Lebesgue dominated convergence theorem, $s_n \to f$ in L^1.

(b) Without loss of generality, we may assume $a_j \neq 0$ for all j. Because $s \in L^1$, $\mu(E_j) < \infty$ for each j. Thus, for each j, given $\epsilon > 0$, there is a set A_j which is a finite union of open intervals such that $\mu(E_j \triangle A_j) < \epsilon$. Letting $t = \sum_{j=1}^n a_j \chi_{A_j}$, we have $\int |f - t| \leq \int |f - s| + \int |s - t| < \epsilon(1 + 2\sum_{j=1}^n |a_j|)$. This justifies that each E_j can be chosen to be a finite union of open intervals. The density of C^0 (and C^∞) in L^1 now easily follows. (How?) $\square$

2.4.2 Completeness of L^1

Our goal in this subsection is to prove that L^1 is a Banach space. We start this subsection with the following essential tool, which is also introduced in Chapter 5 of the textbook. The 'tool' tells us that a normed vector space (with the metric which is induced by the norm) is complete (i.e., a Banach space) if and only if every absolutely convergent series of vectors in the space is convergent.

Lemma 2.4.2 (Checking completeness of a vector space)**.** Let $(V, \|\cdot\|)$ be a normed vector space and impose V the metric induced by $\|\cdot\|$. Then, the following are equivalent:

(a) V is a Banach space.

(b) Every absolutely convergent series of vectors in V is convergent. To be precise, $(v_n)_{n \in \mathbb{N}}$ is a sequence of vectors in V such that $\sum_n \|v_n\|$ is convergent, then $\sum_n v_n$ is convergent.

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of vectors in V such that $\sum_n \|v_n\|$ is convergent. Because the sequence $(\sum_{k=1}^n v_k)_{n\in\mathbb{N}}$ is a Cauchy sequence in V and V is complete, the sequence is convergent in V.

Assume (b) and let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V. For each positive integer j, there is an integer $f(j)$ such that

(i) $f(j) < f(j+1)$ for all j, and

(ii) $n, m \geq f(j)$ implies $\|v_n - v_m\| < 2^{-j}$.

For each $n \in \mathbb{N}$, define $y_n = v_{f(n+1)} - v_{f(n)}$. Then $\|y_n\| \leq 2^{-n}$, so $\sum_n \|y_n\| \leq 1$. By hypothesis, the series $\sum_n y_n$ is convergent, so the subsequence $(v_{f(n)})_{n\in\mathbb{N}}$ of the Cauchy sequence $(v_n)_n$ is convergent. Therefore, V is complete. $\square$

Next, we study another convergence theorem derived from the Lebesgue dominated convergence theorem, which is called Levi's convergence theorem. In fact, together with Lemma 2.4.2, the Levi's convergence theorem justifies that L^1 is a Banach space.

Proposition 2.4.3 (Levi's convergence theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in L^1 such that $\sum_n \int |f_n| < \infty$. Then, the series $\sum_n f_n$ has the following types of convergence:

(a) $\sum_n f_n$ converges absolutely μ-almost everywhere (hence, pointwise μ-almost everywhere).

(b) $\sum_n f_n \in L^1$, and $\sum_{k=1}^n f_k \to \sum_n f_n$ in L^1.

Proof. Because each $|f_n|$ belongs to L^+, by the monotone convergence theorem, $\int \sum_n |f_n| = \sum_n \int |f_n| < \infty$, hence $\sum_n f_n$ converges absolutely μ-almost everywhere. (In fact, $\sum_n |f_n| \in L^+$.) Because $\sum_n f_n$ is measurable and $\int |\sum_n f_n| \leq \int \sum_n |f_n| < \infty$, $\sum_n f_n \in L^1$. Moreover, because $|\sum_{k=1}^n f_k| \leq \sum_n |f_n|$, by the Lebesgue dominated convergence theorem, we find that $\sum_{k=1}^n f_k \to \sum_n f_n$ in L^1. $\square$

Even though Levi's convergence theorem can be applied to some computations of integrals, there is one remarkable observation; the assumption states that the series of norms $\sum_n \|f_n\|_1$ is finite and the result states that the series of vectors $\sum_n f_n$ is convergent, which is part (b) of Lemma 2.4.2. Therefore, it can be deduced that L^1 is a Banach space.

Theorem 2.4.4. L^1 is a Banach space.

As L^1 is a Banach space, L^p for $1 \leq p \leq \infty$ is known to be a Banach space. It will be proved at the end of this chapter.

2.4.3 Riemann integrable functions

In this subsection, we fix the measure space to be the Lebesgue measure space $(\mathbb{R}, \mathcal{L}, m)$ or its restriction to a compact interval $[a, b]$. Our goal is to prove that a Riemann integrable function $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable and that f is Riemann integrable if and only if f is continuous m-almost everywhere.

Suppose $f : [a, b] \to \mathbb{R}$ is a (bounded and) Riemann integrable function. Given $\epsilon > 0$, there is a partition P_0 of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$ whenever P is a refinement of P_0. For a partition $P = \{a = x_0, x_1, \cdots, x_{n-1}, x_n = b\}$ of $[a, b]$, define

$$U_P = f(a)\chi_{\{a\}} + \sum_{i=1}^n M_i \chi_{(x_{i-1}, x_i]} \quad \text{and} \quad L_P = f(a)\chi_{\{a\}} + \sum_{i=1}^n m_i \chi_{(x_{i-1}, x_i]},$$

where $M_i = \sup\{f(x) : x_{i-1} < x \leq x_i\}$ and $m_i = \inf\{f(x) : x_{i-1} < x \leq x_i\}$. What we should notice is that U_P and L_P are simple functions on $[a, b]$ if P is a partition of $[a, b]$. Furthermore, if $(P_n)_n$ is an incresaing sequence of partitions of $[a, b]$ which are finer than P_0 with the property that $\|P_n\| \to 0$ as $n \to \infty$, then $(U_{P_n})_{n\in\mathbb{N}}$ and $(L_{P_n})_{n\in\mathbb{N}}$ are sequences of simple functions such that

$$L_{P_1} \leq L_{P_2} \leq \cdots \leq f \leq \cdots \leq U_{P_2} \leq U_{P_1}.$$

Remark. (a) Each U_{P_n} and L_{P_n} is an integrable simple function, which is bounded.

(b) Letting $U = \lim_n U_{P_n}$ and $L = \lim_n L_{P_n}$, both U and L are measurable and $L \le f \le U$ on $[a,b]$. Furthermore, because U and L are bounded, they are Lebesgue integrable.

(c) The Lebesgue integral and the Riemann integral coincide for all U_{P_n} and L_{P_n}. Furthermore,

$$\lim_{n\to\infty} \int_{[a,b]} U_{P_n} = \lim_{n\to\infty} \int_a^b U_{P_n} = \overline{\int_a^b} f \quad \text{and} \quad \lim_{n\to\infty} \int_{[a,b]} L_{P_n} = \lim_{n\to\infty} \int_a^b L_{P_n} = \underline{\int_a^b} f.$$

Because all the functions are bounded, we may use the monotone convergence theorem, which gives $\int_{[a,b]} U = \int_{[a,b]} L = \int_a^b f$. Because $L \le f \le U$, this coincidence of the integrals implies that $U = f = L$ m-almost everywhere. Therefore, f is Lebesgue integrable and $\int_{[a,b]} f = \int_a^b f$.

We now investigate the condition under which a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable, with the notations given above. Assume f is Riemann integrable. Then $L = f = U$ m-almost everywhere. Let $P = \bigcup_{n=1}^\infty P_n$ and $E \subset [a,b]$ be the set on which $L \ne U$. If $x \in [a,b] \setminus (P \cup E)$, then $L(x) = U(x)$ since $x \notin E$; and one can deduce the continuity of f at x using $x \notin P$. Hence, f is continuous m-almost everywhere. Conversely, assume f is continuous at m-almost every point of $[a,b]$ and let E be the set of points in $[a,b]$ at which f is discontinuous. And let $(P_n)_{n=1}^\infty$ be any increasing sequence of partitions of $[a,b]$ such that $\|P_n\| \to 0$ as $n \to \infty$. If $x \in [a,b] \setminus E$, by continuity of f at x, we have $U(x) = L(x)$, so $L = f = U$ m-almost everywhere. Because m is complete, f is measurable, and

$$\overline{\int_a^b} f = \lim \int_a^b U_{P_n} = \lim \int_{[a,b]} U_{P_n} = \int_{[a,b]} U = \int_{[a,b]} L = \lim \int_{[a,b]} L_{P_n} = \lim \int_a^b L_{P_n} = \underline{\int_a^b} f.$$

Therefore, f is Riemann integrable on $[a,b]$ if f is continuous m-almost everywhere.

We summarize the above results as the following theorem.

Theorem 2.4.5. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

(a) If f is Riemann integrable, then f is Lebesgue integrable, and these two integrals of f are the same.

(b) f is Rieamann integrable if and only if f is continuous m-almost everywhere.

2.4.4 A differentiation theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Theorem 2.4.6. Let $f : X \times [a,b] \to \mathbb{C}\,(-\infty < a < b < \infty)$ be a function. Suppose that $f(\cdot, t) : X \to \mathbb{C}$ is integrable for each $t \in [a,b]$ and let $F(t) = \int f(x,t)\, d\mu(x) = \int f(\cdot, t)$.

(a) Suppose that there exists $g \in L^1$ such that $|f(x,t)| \le g(x)$ for all x, t. If $f(x, \cdot)$ is continuous at $t_0 \in [a,b]$ for each x, then F is continuous at t_0. In particular, if $f(x,\cdot)$ is continuous for each x, then F is continuous.

(b) Suppose that $f_t = \partial_t f$ exists and there is a function $h \in L^1$ such that $|f_t(x,t)| \le h(x)$ for all x, t. Then F is differentiable and $F'(t) = \int f_t(x,t)\, d\mu(x)$.

Proof. (a) Note that F is continuous at $t_0 \in [a,b]$ if and only if $F(p_n) \to F(t_0)$ whenever $(p_n)_n$ is any sequence in $[a,b]$ with the limit t_0.[4] Becasue $(f(x,p_n) - f(x,t_0))_n$ is a sequence of integrable functions bounded by $2g$, by the Lebesgue dominated convergence theorem , we have $\lim_n (F(p_n) - F(t_0)) = \int \lim_n (f(\cdot, p_n) - f(\cdot, t_0)) = \int 0 = 0$.

(b) Let $(p_n)_n$ be, again, a sequence of points of $[a,b]$ with the limit t_0. Then

$$f_t(x,t_0) = \lim_{n\to\infty} \frac{f(x,p_n) - f(x,t_0)}{p_n - t_0}.$$

[4] This approach is quite natural and essential, because the Lebesgue dominated convergence theorem requires the collection of integrable functions to be countable.

By the mean value theorem , for each n there is a point $\gamma_n \in [a,b]$ such that $\dfrac{f(x,p_n)-f(x,t_0)}{p_n-t_0} = f_t(x,\gamma_n)$, hence $\left|\dfrac{f(x,p_n)-f(x,t_0)}{p_n-t_0}\right| \le h(x)$. Also, because each $\dfrac{f(x,p_n)-f(x,t_0)}{p_n-t_0}$ is measurable, it is integrable. Therefore, by the Lebesgue dominated convergence theorem again, we have

$$\lim_{n\to\infty}\frac{F(p_n)-F(t_0)}{p_n-t_0} = \int \lim_{n\to\infty}\frac{f(x,p_n)-f(x,t_0)}{p_n-t_0}\,d\mu(x) = \int f_t(x,t_0)\,d\mu(x),$$

proving the differentiablility.

This completes the proof. □

Remark. Suppose the domain of f were given as $X\times I$, where I is an interval in $\mathbb{R}$, rather than a compact interval. If f or F satisfies (a) or (b) on any subset $X\times[a,b]$ with $[a,b]\subset I$, then f or F satisfies (a) or (b) on its domain.

2.5 Modes of convergence

Definition 2.5.1. Suppose $f_n : X\to\mathbb{C}$ is a measurable function for each $n\in\mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}KO}$ is said to be Cauchy in measure if, given $\alpha>0$,

$$\mu(\{x\in X : |f_n(x)-f_m(x)|\ge\alpha\})\to 0\,\text{as}\,n,m\to\infty.$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is said to converge to $f : X\to\mathbb{C}$ in measure if for each positive real number α,

$$\mu(\{x\in X : |f_n(x)-f(x)|\ge\alpha\})\to 0 \text{ as } n\to\infty.$$

Remark. We may consider pointwise convergences 'vertical' convergences, while we may consider convergences with respect to 'measures' 'horizontal' convergences.

Remark. (a) Suppose $(f_n : X\to\mathbb{C})_{n\in\mathbb{N}}$ is a sequnece of measurable functions which converges to f in measure. Then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, since $\{x\in X : |f_n(x)-f_m(x)|\ge\epsilon\}$ is contained in

$$\{x\in X : |f_n(x)-f(x)|\ge\epsilon/2\}\cup\{x\in X : |f(x)-f_m(x)|\ge\epsilon/2\}$$

and the μ-measure of the last two sets tends to 0 as $n,m\to\infty$.

(b) However, convergence almost everywhere does not imply convergence in measure, and vice versa. For example, when $f_n=\chi_{(0,\infty)} : \mathbb{R}\to\mathbb{C}$, then $f_n\to 0$ pointwise as $n\to\infty$ but not in measure; when

$$\begin{gathered} f_1=\chi_{[0,1/2]},\ f_2=\chi_{[1/2,1]},\\ f_3=\chi_{[0,1/4]},\ f_4=\chi_{[1/4,2/4]},\ f_5=\chi_{[2/4,3/4]},\ f_6=\chi_{[3/4,1]},\\ \cdots, \end{gathered}$$

$f_n\to 0$ in measure but $(f_n)_{n\in\mathbb{N}}$ is not pointwise convergent.

Observation 2.5.2. If $f_n\to f$ in L^1, then $f_n\to f$ in measure; for any $\epsilon\in\mathbb{R}^{>0}$, we have $\epsilon\cdot\mu(\{x\in X : |f_n(x)-f(x)|\ge\epsilon\})\le\int|f_n-f|\to 0$ as $n\to\infty$.

Observation 2.5.3 (The set of convergences). Let X be a topological space and (Y,d) be a metric space. Suppose a sequence $(f_n)_{n\in\mathbb{N}}$ of functions from X into Y is given. Then, for a point $x\in X$, $f_n(x)\to f(x)$ for a function $f : X\to Y$ if and only if for each $\epsilon>0$ there is an integer $N(\epsilon)$ such that $n\ge N(\epsilon)$ implies $d(f_n(x),f(x))<\epsilon$. The latter statement for a given positive real number ϵ is valid if and only if a point x of X belongs to $\bigcap_{n=k}^{\infty}\{x\in X : d(f_n(x),f(x))<\epsilon\}$ for some positive integer k, i.e.,

$$x\in\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\{x\in X : d(f_n(x),f(x))<\epsilon\}.$$

Hence, the collection of a point x of X such that $f_n(x) \to f(x)$ as $n \to \infty$ is given by

$$\bigcap_{\epsilon>0} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in X : d(f_n(x), f(x)) < \epsilon\}.$$

Of course, the intersection for $\epsilon > 0$ can be replaced by $j \in \mathbb{N}$, when ϵ is replaced by, such as, $1/j$ or 2^{-j}.

Theorem 2.5.4. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of complex-valued measurable functions and $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure.

(a) There is a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ which converges to a measurable function f almost everywhere.

(b) $f_n \to f$ in measure as $n \to \infty$. Moreover, if f_n converges to a function g in measure, then $f = g$ almost everywhere.

Proof. Because $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, for each positive integer t, there is a positive integer $N(t)$ such that $N(t) < N(t+1)$ and $\mu(\{x \in X : |f_n(x) - f_k(x)| \geq 2^{-t}\}) < 2^{-t}$ whenever $n, k \geq N(t)$. Write $g_t = f_{N(t)}$; then $\mu(\{x \in X : |g_t(x) - g_s(x)| \geq 2^{-t}\}) < 2^{-t}$ whenever $s \geq t$.

Being motivated from Observation 2.5.3, for each positive integer t, set

$$E_t = \{x \in X : |g_t(x) - g_{t+1}(x)| \geq 2^{-t}\}, \quad F_t = \bigcup_{s\geq t} E_s, \quad F = \bigcap_{t=1}^{\infty} F_t.$$

(i) Suppose $x \in X \setminus F$. Then $x \notin F_t$ for some positive integer t, and $x \notin F_s$ for all integers $s \geq t$. Hence, it follows that whenever a, b are positive integers greater than or equal to t, then $|g_a(x) - g_b(x)| < 2^{1-t}$. Therefore, the sequence $(g_t(x))_{t\in\mathbb{N}}$ is a Cauchy sequence with values in $\mathbb{C}$.

(ii) Because $\mu(F_t) \leq \sum_{s\geq t} \mu(E_s) \leq 2^{1-t}$, $\mu(F) = \lim_t \mu(F_t) = 0$.

Because each g_t is a measurable function defined on X, there is a measurable function f defined on X such that $g_t \to f$ on $X \setminus E$ as $t \to \infty$. (For example, one may set $f = \lim_t g_t$ on $X \setminus E$ and $f = 0$ on E.) This proves (a) of the theorem.

Remark from the above (i) that whenever $x \in X \setminus F_t$ for some integer t, we have $|f(x) - g_t(x)| = \lim_s |g_s(x) - g_t(x)| \leq 2^{1-t}$. This implies that $g_t \to f$ in measure as $t \to \infty$, because $\mu(F_t) \to 0$ as $t \to \infty$. Remark also that $\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$ is contained in the union

$$\{x \in X : |f_n(t) - g_t(x)| \geq \epsilon/2\} \cup \{x \in X : |g_t(x) - f(x)| \geq \epsilon/2\},$$

where the measure of both sets decay as $n, t \to \infty$. Thus, $f_n \to f$ in measure as $n \to \infty$. Furthermore, if $f_n \to g$ in measure as $n \to \infty$, because $\{x \in X : |f(x) - g(x)| \geq \epsilon\}$ is contained in the union

$$\{x \in X : |f(x) - f_n(x)| \geq \epsilon/2\} \cup \{x \in X : |f_n(x) - g(x)| \geq \epsilon/2\},$$

we have $\mu(\{x \in X : |f(x) - g(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$. It follows that $f = g$ μ-almost everywhere. $\square$

Corollary 2.5.5. If $f_n \to f$ in L^1, there is a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_j} \to f$ almost everywhere.

Proof. By assumption, $f_n \to f$ in measure, hence there is a subsequence of $(f_n)_{n\in\mathbb{N}}$ which converges to f almost everywhere. $\square$

We now turn our attention to a uniform convergence of a sequence which converges almost everywhere.

Lemma 2.5.6. Suppose $\mu(X) < \infty$, and let $(f_n)_{n\in\mathbb{N}}$ be a complex-valued measurable functions defined on X such that $f_n \to f$ μ-almost everywhere. Then, for any $\epsilon > 0$ and $\delta > 0$, there is an integer N and a measurable subset $A \in \mathcal{M}$ of X satisfying the following properties:

(i) $\mu(A) < \delta$.

(ii) Whenever $n \geq N$ and $x \in X \setminus A$, we have $|f_n(x) - f(x)| < \epsilon$.

Proof. Again, as motivated by Observation 2.5.3, consider the set of convergences

$$C = \bigcap_{\alpha>0} \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \{x \in X : |f_n(x) - f(x)| < \alpha\}$$

of the sequence $(f_n)_{n\in\mathbb{N}}$. Since $\mu(X \setminus C) = 0$, we have $\mu\left(X \setminus \bigcup_{k=1}^{\infty} A_k\right) = 0$, where

$$A_k = \bigcap_{n \geq k} \{x \in X : |f_n(x) - f(x)| < \epsilon\}.$$

Since $\mu(X) < \infty$ and $(X \setminus \bigcup_{k=1}^{j} A_k)_{j\in\mathbb{N}}$ is decreasing, we have $\lim_j \mu(X \setminus \bigcup_{k=1}^{j} A_k) = 0$, so there is a positive integer N such that $\mu(X \setminus \bigcup_{k=1}^{j} A_k) < \delta$ whenever $j \geq N$. Letting $A = X \setminus \bigcup_{k=1}^{N} A_k$, we find easily observe that whenever $x \in X \setminus A$ then $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. $\square$

Theorem 2.5.7 (Egoroff's theorem)**.** Suppose as in Lemma 2.5.6. Then, for every $\epsilon > 0$, there is a measurable subset E of X such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $X \setminus E$.

Proof. By Lemma 2.5.6, for each $k \in \mathbb{N}$ there is a measurable subset A_k of X and a positive integer N_k with the following properties:

(i) $\mu(A_k) < 2^{-k}\epsilon$.

(ii) If $x \in X \setminus A_k$ and $n \geq N_k$, then $|f_n(x) - f(x)| < 1/k$.

And set $A = \bigcup_{k=1}^{\infty} A_k$. Then $\mu(A) \leq \sum \mu(A_k) < \epsilon$. Moreover, if $x \in X \setminus A$, then $x \in X \setminus A_k$ for all $k \in \mathbb{N}$; hence, whenever $n \geq N_k$, we have $|f_n(x) - f(x)| < 1/k$. Since such inequality holds for all $x \in X \setminus A$ and for each $k \in \mathbb{N}$, we conclude that $f_n \to f$ uniformly on $X \setminus A$. $\square$

Definition 2.5.8. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a measure space into a metric space. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be uniformly convergent μ-almost everywhere in X, if for every $\epsilon > 0$ there is a measurable subset A of X such that $\mu(A) < \epsilon$ and $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on $X \setminus A$.

Theorem 2.5.9 (Lusin's theorem)**.** Let A be a nonempty Lebesuge measurable subset of $\mathbb{R}$ such that $m(A) < \infty$, and let f be a complex-valued function defined on A. For any $\epsilon > 0$, there is a compact subset K of $\mathbb{R}$ contained in A such that $m(A \setminus K) < \epsilon$ and $f|_K$ is continuous.

Remark. Since the case where f is complex-valued easily follows from the case where f is real-valued, we may assume that f is real-valued.

Proof 1. Let $(V_n)_{n\in\mathbb{N}}$ be an enumeration of the open intervals with rational endpoints, which is a basis of the topology on $\mathbb{R}$. For each $n \in \mathbb{N}$, let K_n and K_n' be compact subsets of A such that $K_n \subset f^{-1}(V_n)$, $K_n' \subset A \setminus f^{-1}(V_n)$, and $m(A \setminus (K_n \cup K_n')) < 2^{-n}\epsilon$. Letting $K = \bigcap_{n\in\mathbb{N}}(K_n \cup K_n')$, we have

$$m(A \setminus K) = m\left(\bigcup_{n\in\mathbb{N}} (A \setminus (K_n \cup K_n'))\right) \leq \sum_{n\in\mathbb{N}} m(A \setminus (K_n \cup K_n')) < \epsilon.$$

To show the continuity of f over K, let x be a point of K and j be any integer such that $f(x) \in V_j$. Then x is a point of $K \setminus K_j'$, which is open in K, and $f(K \setminus K_j') \subset f((K_j \cup K_j') \setminus K_j') \subset f(K_j) \subset V_j$. Therefore, $f|_K$ is continuous. $\square$

2.6 Product measures

For two measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$, let $\mathcal{A}$ be the collection of all finite disjoint unions of the members of $\{E \times F : E \in \mathcal{M},\ F \in \mathcal{N}\}$. What we seek is a measure on the measurable space $(X \times Y, \mathcal{M} \otimes \mathcal{N})$, and it will be constructed by defining a premeasure on $\mathcal{A}$; we set the function $\mu_0 : \mathcal{A} \to [0, \infty]$ to be the σ-additive function such that $\mu_0(E \times F) = \mu(E)\nu(F)$ whenever $E \in \mathcal{M}$ and $F \in \mathcal{N}$. (This function is a premeasure on $\mathcal{A}$.) Then there is a measure on $\mathcal{M} \otimes \mathcal{N}$ which extends μ_0 (obtained by first taking the Carathéodory extension and then restricting to $\mathcal{M} \otimes \mathcal{N}$), which will be denoted by $\mu \times \nu$.

The construction of a measure for finitely many measurable spaces can be done analogously.

Definition 2.6.1 (Product measures). Suppose $(X_i, \mathcal{M}_i, \mu_i)$ is a measure space for $i = 1, \cdots, n$. The product measure $\mu_1 \times \cdots \mu_n$ is a measure on the product space $X_1 \times \cdots \times X_n$ extending the premeasure $\mu_0 : \mathcal{A} \to [0, \infty]$, where $\mathcal{A}$ is the collection of all finite disjoint unions of cubes(rectangles) and $\mu_0(E_1 \times \cdots \times E_n) = \mu_1(E_1) \cdot \cdots \cdot \mu_n(E_n)$ whenever $E_i \in \mathcal{M}_i$ for $i = 1, \cdots, n$.

Furthermore, if X and Y are σ-finite so that $X \times Y$ is σ-finite, then $\mu \times \nu$ is the unique extension of ρ to a measure.

Notation. Given two measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ and a subset E of $X \times Y$, we define the x-section E_x and the y-section E^y of E by

$$E_x := \{y \in Y : (x, y) \in E\}, \quad E^y := \{x \in X : (x, y) \in E\}.$$

Also, if f is a function on $X \times Y$, we define the x-section f_x and the y-section f^y of f by the function on Y and X, resepectively, satisfying

$$f_x(y) = f^y(x) = f(x, y).$$

Thus, for example, $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.

Proposition 2.6.2. (a) If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$.

(b) If f is an $\mathcal{M} \otimes \mathcal{N}$-measurable function, then f_x is $\mathcal{N}$-measurable and f^y is $\mathcal{M}$-measurable.

Proof. Let $\mathcal{T}$ be the collection of subsets of $X \times Y$ for which (a) is valid. We wish to show that $\mathcal{M} \otimes \mathcal{N}$ is contained in $\mathcal{T}$. One can easily observe that $\mathcal{T}$ is a σ-algebra on $X \times Y$ and that $\mathcal{T}$ contains $\mathcal{A}$. (b) easily follows from (a). $\square$

Our main result in this section is regarding integration with respect to the product measure, assuming that the given measure spaces are σ-finite.

Theorem 2.6.3. Suppose $(X, \mathcal{N}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

(a) For each $E \in \mathcal{M} \otimes \mathcal{N}$, the maps

$$h : X \to [0, \infty],\ x \mapsto \nu(E_x) \quad \text{and} \quad w : Y \to [0, \infty],\ y \mapsto \mu(E^y)$$

are measurable functions on X and Y, resepectively.

(b) Furthermore, $(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$.

Proof. **Step 1.** Suppose the theorem is proven for finite measure spaces $(X, (M), \mu)$ and $(Y, \mathcal{N}, \nu)$, and assume the measure spaces are σ-finite, i.e., $X = \bigcup_{n \in \mathbb{N}} X_n$ and $Y = \bigcup_{j \in \mathbb{N}} Y_j$ with $\mu(X_n), \nu(Y_j) < \infty$ for all $n, j \in \mathbb{N}$. (Without loss of generality, we may assume further that $(X_n)_{n \in \mathbb{N}}$ and $(Y_j)_{j \in \mathbb{N}}$ are increasing.) Then $h|_{X_n}, w|_{Y_j}$ are measurable for all $n, j \in \mathbb{N}$, implying that h and w are measurable. Also, after writing $E_n = E \cap (X_n \times Y_n)$ for each $n \in \mathbb{N}$, we have

$$(\mu \times \nu)(E_n) = \int \mu((E_n)^y)\, d\nu(y) = \int \nu((E_n)_x)\, d\mu(x)$$

and the monotone convergence theorem induces a desired result.

Step 2. Hence, we now assume that the given measure spaces are finite. We will prove the theorem by showing that the following collection containes $\mathcal{M} \otimes \mathcal{N}$:

$$\mathcal{T} := \left\{ E \subset X \times Y : \begin{array}{c} h, w \text{ are measurable and} \\ (\mu \times \nu)(E) = \int \mu(E^y)\, d\nu(y) = \int \nu(E_x)\, d\mu(x) \end{array} \right\}$$

One can easily check that $\mathcal{T}$ contains $\mathcal{A}$. To show that $\mathcal{T}$ contains $\mathcal{M} \otimes \mathcal{N}$, we shall make use of the monotone class lemma; we will show that $\mathcal{T}$ is a monotone class on $X \times Y$.

Let $(E_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{T}$. (a) is valid by the continuity of measures; (b) is valid by the monotone convergence theorem. For a decreasing sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$, the finiteness of X and Y are essential. In this case, (a) is valid by the continuity of measures, and (b) is valid by the Lebesgue dominated convergence theorem. □

In the above theorem, when s is a simple function on $X \times Y$, we have

$$\int s\, d(\mu \times \nu) = \iint s_x\, d\nu d\mu = \iint s^y\, d\mu d\nu.$$

Thus, the following theorem is naturally deduced.

Theorem 2.6.4 (The Fubini-Tonelli theorem)**.** Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

(a) (Tonelli's theorem) If $f \in L^+(X \times Y)$, then the functions

$$g : X \to [0, \infty],\ x \mapsto \int f_x\, d\nu \quad \text{and} \quad h : Y \to [0, \infty],\ y \mapsto \int f^y\, d\mu$$

are in $L^+(X)$ and $L^+(Y)$, resepectively, and

$$\int f\, d(\mu \times \nu) = \int \left(\int f\, d\nu \right)\, d\mu = \int \left(\int f\, d\mu \right)\, d\nu$$

(b) (Fubini's theorem) If $f \in L^1(X \times Y)$, then the functions

$$g : X \to \mathbb{C},\ x \mapsto \int f_x\, d\nu \quad \text{and} \quad h : Y \to \mathbb{C},\ y \mapsto \int f^y\, d\mu$$

are in $L^1(X)$ and $L^1(Y)$, resepectively, and

$$\int f\, d(\mu \times \nu) = \int \left(\int f\, d\nu \right)\, d\mu = \int \left(\int f\, d\mu \right)\, d\nu$$

Remark. Indeed, when one is interested in a properties of the L^p space $(1 \leq p \leq \infty)$, one reduces the case to L^1 space, to L^+ space, and to the space of simple functions.

Proof. Since (b) follows from (a) by considering componentwise, it suffices to prove (a). Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real-valued simple functions found conventionally for f. Then $g(x) = \int f_x\, d\nu = \int \lim(s_n)_x\, d\nu = \lim \int (s_n)_x\, d\nu$, and the preceeding theorem implies that g is a measurable function. The preceeding theorem also implies $\int g\, d\mu = \int \left(\int \lim(s_n)_x\, d\nu \right)\, d\mu = \lim \iint (s_n)_x\, d\nu d\mu = \lim \int s_n\, d(\mu \times \nu) = \int f\, d(\mu \times \nu)$. The same reasoning proves for h. □

Even if μ, ν are complete measures, $\mu \times \nu$ is almost never complete: If $A \in \mathcal{M}$ such that $\mu(A) = 0$ and $B \in \mathcal{P}(Y) \setminus \mathcal{N}$, then $A \times B \notin \mathcal{M} \otimes \mathcal{N}$ (otherwise, its possibe x-section B belongs to $\mathcal{N}$). However, $A \times B$ is a subset of the following $(\mu \times \nu)$-null set: $A \times Y$. If one wishes to work with complete measures, of course, one can consider the completion of $\mu \times \nu$. In this setting the relationship between the measuability of a function on $X \times Y$ and the measurability of its x-sections and y-sections is not so simple. However, the Fubini-Tonelli theorem is still valid when suitably reformulated:

Theorem 2.6.5 (The Fubini-Tonelli theorem for complete measures)**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete σ-finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. Let f be an $\mathcal{L}$-measurable satisfying either (i) $f \geq 0$ or (ii) $f \in L^1(\lambda)$.

(a) Then f_x is $\mathcal{N}$-measurable and f^y is $\mathcal{M}$-measurable for almost every x and y. In case (ii), f_x and f^y are integrable for almost every x and y.

(b) Moreover, the functions $g : X \to [0, \infty]$, $x \mapsto \int f_x \, d\nu$ and $h : Y \to [0, \infty]$, $y \mapsto \int f^y \, d\mu$ are measurable. In case (ii), g and h are also integrable. Finally,

$$\int f \, d\lambda = \iint f(x, y) \, d\mu(x) \, d\nu(y) = \iint f(x, y) \, d\nu(y) \, d\mu(x).$$

Proof. $\int f \, d\lambda = \int f \, d(\mu \times \nu)$. □

Problems

Problem 2.6.1 (Exercise 2.46)**.** Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, and μ, ν be the Lebesgue measure and the counting measure on X and Y, resepectively. Let D be the diagonal of $[0, 1]$. Find $\iint \chi_D \, d\mu d\nu$, $\iint \chi_D \, d\nu d\mu$ and $\int \chi_D \, d(\mu \times \nu)$.

Solution. It is easy to compute that $\iint \chi_D \, d\mu d\nu = \int 0 \, d\nu = 0$ and $\iint \chi_D \, d\nu d\mu = \int 1 \, d\nu = 1$. To compute $\int \chi_D \, d(\mu \times \nu)$, remark that $\int \chi_D \, d(\mu \times \nu)$ is the infimum of $\sum_{n=1}^{\infty} \mu(A_n)\nu(B_n)$, where $(A_n \times B_n)_{n \in \mathbb{N}}$ covers D. Since D is uncountable, there is a positive integer k such that $\mu(A_k) > 0$ and $\nu(B_k) = \infty$, from which it follows that $\int \chi_D \, d(\mu \times \nu) = \infty$.

2.7 Basic theory of L^p spaces for $1 \leq p \leq \infty$

Since we have shown that L^1 is a Banach space, we give a generalization of the observation, called the Riesz-Fischer theorem, which states that L^p is a Banach space whenever $1 \leq p \leq \infty$. We start the section with the definition of L^p space for $1 \leq p < \infty$.

Definition 2.7.1 (L^p space for $1 < p < \infty$)**.** Given a measure space $(X, \mathcal{M}, \mu)$ and a real number $p > 1$, we consider the following collection:

$$L^p_c := \{f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_p < \infty\},$$

where $\|f\|_p = (\int |f|^p)^{1/p}$ for all measurable function f. The space L^p is defined as the collection of equivalence classes on L^p_c, where the equivalence relation $\sim$ on L^p_c is defined as follows:

$$f \sim g \text{ if and only if } f = g \ \mu\text{-almost everywhere. (Here, } f, g \in L^p_c.)$$

As L^1 is, L^p under the above definition is clearly a $\mathbb{C}$-vector space. To argue as in L^1 that $\|\cdot\|_p$ is a norm on L^p, it suffices to check the following inequality:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \text{ for all } f, g \in L^p.$$

This inequality is called the Minkowski inequality, which will be proved in this section.

Proposition 2.7.2 (Jensen's inequality)**.** Suppose a function $f : [0, 1] \to (a, b) \subset \mathbb{R}$ is integrable and $\phi : (a, b) \to \mathbb{R}$ is convex, where $-\infty \leq a \leq b \leq \infty$. Then

$$\phi\left(\int_0^1 f\right) \leq \int_0^1 (\phi \circ f).$$

Proof. For convinience, write $\alpha = \int_0^1 f \in (a, b)$. Since ϕ is a convex function on (a, b), if we let $\beta = \sup\left\{\dfrac{\phi s - \phi\alpha}{s - \alpha} : a < s < \alpha\right\}$, then for each $s \in (a, \alpha)$ we have $\phi s - \phi\alpha \leq \beta(s - \alpha)$. This inequality holds even if $\alpha < s < b$ (how?), so $(\phi \circ f)(x) - \phi(\alpha) \leq \beta(f(x) - \alpha)$ for all $x \in [0, 1]$. Therefore,

$$\int_0^1 ((\phi \circ f) - \phi(\alpha)) \geq 0,$$

proving the inequality. □

Observation 2.7.3. Because the exponential function $f(x) = e^x$ $(x \in \mathbb{R})$ is convex, $\exp(\int_0^1 f) \leq \int_0^1 e^f$. In particular, given a partition $\{0 = x_0, x_1, \cdots, x_n = 1\}$ of $[0, 1]$, if we define the function $f : [0, 1] \to \mathbb{R}$ by $f = \sum_{i=1}^n s_i \chi_{(x_{i-1}, x_i]}$ $(x_i \in \mathbb{R})$ and let $a_i = x_i - x_{i-1}$ for each i, we have

$$e^{a_1 s_1 + \cdots + a_n s_n} \leq a_1 e^{s_1} + \cdots + a_n e^{s_n}.$$

Thus, if we let $e^{s_i} = t_i$, we obtain the following result:

If $a_i > 0$ and $t_i \geq 0$ for each $1 \leq i \leq n$ and $a_1 + \cdots + a_n = 1$, then

$$t_1^{a_1} \cdot \cdots \cdot t_n^{a_n} \leq a_1 t_1 + \cdots + a_n t_n.$$

Proposition 2.7.4. Suppose $p, q \in [1, \infty)$ such that $p^{-1} + q^{-1} = 1$ and $f, g : E \to [0, \infty]$ are measurable functions defined on a measurable set E.

(a) (Hölder's inequality) $\int fg \leq \left(\int f^p\right)^{1/p} \left(\int g^q\right)^{1/q}$. Therefore, if u, v are measurable functions, then $\|uv\|_1 \leq \|u\|_p \|v\|_q$.

(b) (Minkowski's inequality) $\left(\int (f + g)^p\right)^{1/p} \leq \left(\int f^p\right)^{1/p} + \left(\int g^p\right)^{1/p}$. Therefore, if u, v are measurable functions, then $\|u + v\|_p \leq \|u\|_p + \|v\|_p$; hence, L^p is a normed $\mathbb{C}$-vector space.

Proof. (a) The inequality is valid if $\int f^p = 0$ or $\int g^q = 0$. Thus, we may assume $A := \left(\int f^p\right)^{1/p}$ and $B := \left(\int g^q\right)^{1/q}$ are positive. Define $F := f/A$ and $G := g/B$. By the preceeding observation, we have $FG \leq p^{-1}F^p + q^{-1}G^q$. By integrating, we obtain $\int FG \leq 1$, giving the desired inequality.

(b) In this proof, we use the norm notation for convinience. By Hölder's inequality, we have $\int f \cdot (f+g)^{p-1} \leq \|f\|_p \left\|(f + g)^{p-1}\right\|_q$ and $\int g \cdot (f+g)^{p-1} \leq \|g\|_p \left\|(f + g)^{p-1}\right\|_q$. Because $(p-1)q = pq - q = (p+q) - q = p$, we have

$$\int (f + g)^p \leq \left(\int (f + g)^p\right)^{1/q} (\|f\|_p + \|g\|_p).$$

When $\int (f+g)^p = 0$, there is nothing to prove; when it is nonzero, we obtain the desired inequality. □

Problem 2.7.1. Given $f \in L^p$ and $g \in L^q$, where p, q are positive real numbers with $1/p + 1/q = 1$, show that $\|fg\|_1 = \|f\|_p \|g\|_q$ if and only if $A|f|^p = B|g|^q$ almost everywhere, where A and B are some real numbers such that $A, B \geq 0$ and $A + B > 0$.

Solution. Define F and G as in the proof. Because $FG \leq p^{-1}F^p + q^{-1}G^q$, the eqaultiy holds if and only if $FG = p^{-1}F^p + q^{-1}G^q$ almost everywhere.

Remark. For $a, b > 0$, the equaltiy $ab = p^{-1}a^p + q^{-1}b^q$ holds if and only if $a^p = ab = b^q$.

Therefore, the desired equality holds if and only if $A^p|f|^p = B^q|g|^q$ almost everywhere, where $A = c \cdot \|f\|_p$ and $B = c \cdot \|g\|_q$ with $c > 0$.

We now define another vector space L^∞.

Definition 2.7.5 (L^∞ space). (a) Given a measurable function $f : X \to [0, \infty]$, define

$$\operatorname{ess\,sup} f := \inf\{\alpha \in \mathbb{R} : \mu(f^{-1}((\alpha, \infty])) = 0\}.$$

If the set in the above definition is empty, we let $\operatorname{ess\,sup} f = \infty$.

(b) Given a measurable function f, define $\|f\|_\infty := \operatorname{ess\,sup}|f|$. And define L^∞ be the collection of the equivalence classes on

$$I^\infty := \{f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_\infty < \infty\},$$

where the equivalence relation $\sim$ on I^∞ is given by $f \sim g$ if and only if $f = g$ μ-almost everywhere.

Some necessary observations regarding the essential supremum:

Observation 2.7.6. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Suppose $f : X \to \mathbb{C}$ is a measurable function. Then $|f| \leq \alpha$ μ-almost everywhere if and only if $\|f\|_\infty \leq \alpha$.

(b) Suppose $(f_n : X \to \mathbb{C})_{n\in\mathbb{N}}$ is a sequence of measurable functions and $f : X \to \mathbb{C}$ is a measurable function. Show that $\|f_n - f\|_\infty \to 0$ if and only if there is a μ-null set $E \in \mathcal{M}$ for which $f_n \to f$ uniformly on $X \setminus E$.

(c) Assume that μ is a Borel measure on X assigning positive values to all open subsets of X. Show that $\|f\|_\infty = \|f\|_{C^0}$ whenever $f : X \to \mathbb{C}$ is continuous.

Proof. (a) Clear.
(b) $\|f_n - f\|_\infty \to 0$ if and only if for any positive integer k there is an integer $N(k) > 0$ such that $n \geq N(k)$ implies $\|f_n - f\|_\infty < 1/k$, which is equivalent to the situation where $E_n(k) := \{x \in X : |f_n(x) - f(x)| \geq 1/k\}$ is μ-null whenever $n \geq N(k)$. To rewrite the latter statement, $|f_n(x) - f(x)| < 1/k$ whenever $x \in X \setminus E_n(k)$ and $n \geq N(k)$, i.e., $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on $X \setminus E$, where $E = \bigcup_{k\in\mathbb{N}} \bigcup_{n\geq N(k)} E_n(k)$.
(c) Let B denote the set of nonnegative real numbers a such that $\mu(\{x \in X : |f(x)| > a\}) = 0$. Since f is continuous, $m \in B$ if and only if $\{x \in X : |f(x)| > m\} = \varnothing$ (why?), i.e., $|f| \leq m$ on X. This implies that B is the collection of upper bounds of $f(X)$, so $\|f\|_\infty$ is the least upper bound of $f(X)$, i.e., the supremum of $f(X)$. □

Theorem 2.7.7 (Riesz-Fischer theorem)**.** For each $p \in [1, \infty]$, L^p is a Banach space.

Proof for $1 \leq p < \infty$. To show that L^p is a Banach space is equivalent to show that every absolutely convergent sequence is convergent. Suppose $(f_n)_{n\in\mathbb{N}} \subset L^p$ is absolutely convergent, and let $A_N = \sum_{n=1}^N |f_n|$ for each $N \in \mathbb{N}$ and $A = \sum |f_n|$. The monotone convergence theorem and Minkowski's inequality imply

$$\int A^p = \lim \int A_N^p \leq \lim \left(\sum_{n=1}^N \|f_n\|_p \right)^p < \infty,$$

so $\sum f_n$ converges absolutely almost everywhere, i.e., A converges almost everywhere, and $A \in L^p$. Then $\left|\sum_{n=1}^N f_n\right| \leq A_N \leq A$, so the Lebesgue dominated convergence theorem implies $\left(\left(\sum_{n=1}^N f_n\right)^p\right)_{N\in\mathbb{N}}$ is convergent in L^1, i.e., $\left(\sum_{n=1}^N f_n\right)_{N\in\mathbb{N}}$ is convergent in L^p. This proves that L^p is a Banach space for all $p \in [1, \infty)$. □

Another proof for $1 \leq p < \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in L^p, and find an increasing sequence $(n(k))_k$ of positive integers such that $\|f_a - f_b\|_p < 2^{-k}$ whenever $a, b \geq n(k)$.[5] Set

$$g_i = \sum_{k=1}^i |f_{n(k+1)} - f_{n(k)}|, \quad g = \sum_{k=1}^\infty |f_{n(k+1)} - f_{n(k)}|.$$

[5] Such extraction is a widely used strategy when one deals with a Cauchy sequence.

By Minkowski's inequality, $\|g_i\|_p \leq 1$ for all i; by Fatou's lemma, $\|g\|_p = \int(\lim_i |g_i|^p)^{1/p} \leq \lim_i \|g_i\|_p \leq 1$, so g is finite almost everywhere and the series

$$f_{n(1)}(x) + \sum_{k=1}^{\infty}(f_{n(k+1)}(x) - f_{n(k)}(x))$$

converges almost everywhere. Let f denote the limit function of the above series (put $f = 0$ whenever the series diverges). Finally, let m be an integer such that $a, b \geq m$ implies $\|f_a - f_b\| < \epsilon$. For each integer $a \geq m$, by Fatou's lemma, we have

$$\int |f - f_a|^p \leq \liminf_{k \to \infty} \int |f_{n(k)} - f_a|^p \leq \epsilon^p.$$

Therefore, $f - f_m \in L^p$ and $\|f - f_n\|_p \to 0$ as $n \to \infty$. □

Proof for $p = \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in L^∞. To achieve a limit function, for each positive integer n and m, let $A_{n,m} = \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$. Then $A := \bigcup_{n,m\in\mathbb{N}} A_{n,m}$ is μ-null, and $|f_n - f_m| \leq \|f_n - f_m\|_\infty$ on $X \setminus A$. For the limit function to be in L^∞, let $B_n = \{x \in X : |f_n(x)| > \|f_n\|_\infty\}$ for each $n \in \mathbb{N}$, and let $B = \bigcup_{n\in\mathbb{N}} B_n$, which is μ-null. Defining $E = A \cup B$, which is μ-null, the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent on $X \setminus E$; let $f : X \to \mathbb{C}$ be the function defined by $f(x) = 0$ for $x \in E$ and $f(x) = \lim f_n(x)$ for $x \in X \setminus E$. Then $f \in L^\infty$, and if $N(\epsilon)$ is a positive integer for a given real number $\epsilon > 0$ such that $n, m \geq N(\epsilon)$ implies $\|f_n - f_m\|_\infty < \epsilon$, we have $|f_n - f| \leq \epsilon$ on $X \setminus E$ whenever $n \geq N(\epsilon)$. It follows that $\|f_n - f\|_\infty \to 0$. □

Chapter 3

Signed measures and differentiation

3.1 Signed measures

Definition 3.1.1 (Signed measure).

Theorem 3.1.2 (Hahn decomposition theorem).

Theorem 3.1.3 (Jordan decomposition theorem).

3.2 The Lebesgue-Radon-Nykodym lemma

Definition 3.2.1 (Absolute continuity of a measure).