Review of general topology

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Contents

Part I

Essential theory in general topology

Chapter 1

Topology and continuous map

1.1 Topology and basis

Definition 1.1.1 (Topology on a set)**.** Let X be a nonempty set. A collection $\mathcal{T}$ is called a topology on X if

(a) both X and the empty set belongs to $\mathcal{T}$,

(b) $\mathcal{T}$ is closed under arbitrary unions,

(c) $\mathcal{T}$ is closed under arbitrary finite intersections.

Definition 1.1.2 (Subbasis and basis of a topology)**.** A collection $\mathcal{B}$ is called a subbasis of the topology on X if $\mathcal{B}$ covers X. A collection $\mathcal{B}$ is called a basis of the topology on X if

(a) $\mathcal{B}$ covers X, i.e., $\mathcal{B}$ is a subbasis on X,

(b) given $B_1, B_2 \in \mathcal{B}$, there is another member $B_3 \in \mathcal{B}$ contained in $B_1 \cap B_2$.

The topology $\langle \mathcal{B} \rangle$ on X generated by the basis $\mathcal{B}$ is the following collection of subsets of X:

$$\langle \mathcal{B} \rangle = \left\{ U \subset X : \begin{array}{c} \text{Given } x \in U \text{, there is a basis member} \\ B \in \mathcal{B} \text{ such that } x \in B \subset U \end{array} \right\}.$$

(Remark how an open subset of a metric space is defined in the course of mathematical analysis.) In accordance with the definition of the first-countability, we will say that $\langle \mathcal{B} \rangle$ consists of all subsets of X based on $\mathcal{B}$.

Observation 1.1.3. Let X be a set and suppose $\mathcal{B}$ is a basis of X.

(a) The topology generated by $\mathcal{B}$ is the collection $\mathcal{C}$ of all unions of members in $\mathcal{B}$.

(b) The topology generated by $\mathcal{B}$ is the intersection $\mathcal{I}$ of all topologies on X containing $\mathcal{B}$. (Hence, the topology on X generated by $\mathcal{B}$ is the smallest topology on X containing $\mathcal{B}$.)

Proof. We first prove (a). By definition, it is clear that $\mathcal{C}$ is contained in $\langle \mathcal{B} \rangle$. To show the converse inclusion, suppose $U \in \langle \mathcal{B} \rangle$. By definition, for each $x \in U$, there is a basis member $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$, hence U is the union of B_x for $x \in U$.

We now prove (b). Because $\langle \mathcal{B} \rangle$ is a topology on X containing $\mathcal{B}$, $\mathcal{I}$ is contained in $\langle \mathcal{B} \rangle$. Conversely, by (a), every topology on X containing $\mathcal{B}$ also includes $\langle \mathcal{B} \rangle$. Thus, $\mathcal{I}$ contains $\langle \mathcal{B} \rangle$. $\square$

Lemma 1.1.4 (Containment criterion)**.** Let $\mathcal{B}, \mathcal{B}'$ be a basis of the topology $\mathcal{T}, \mathcal{T}'$ of X, respectively. Then the followings are equivalent:

(a) $\mathcal{T}$ is finer than $\mathcal{T}'$.

(b) For each point $x \in X$ and a basis member $B' \in \mathcal{B}'$ containing x, there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Proof. Assume (a) and let x be a point of X and B' be a basis member containing x. Then $B' \in \mathcal{T}$, and there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Assume (b) and let U' be a member of $\mathcal{T}'$. For each point $p \in U'$, there is a basis member $B_p \in \mathcal{B}$ such that $p \in B_p \subset U'$. (There was a leap in the argument: Letting B'_p be a basis member of $\mathcal{B}'$ such that $p \in B'_p \subset U'$, by the assumption we have a basis member $B_p \in \mathcal{B}$ such that $x \in B_p \subset B'_p$.) Therefore, $U' \in \mathcal{T}$. $\square$

Remark. (a) (subbasis) $\xrightarrow{\text{finite intersections}}$ (basis) $\xrightarrow{\text{arbitrary unions}}$ (topology)

(b) Suppose that $\mathcal{B}$ is a basis of a topology $\mathcal{T}$ on a set X. Then the topology generated by $\mathcal{B}$ as a 'subbasis' is $\mathcal{T}$. If $\mathcal{B}'$ is the collection of all finite intersections of the members of $\mathcal{B}$, then $\mathcal{B}' \supset \mathcal{B}$, so the topology $\mathcal{T}'$ generated by $\mathcal{B}'$ as a basis (i.e., by $\mathcal{B}$ as a subbasis) is finer than $\mathcal{T}$. To show that $\mathcal{T}$ is finer than $\mathcal{T}'$, it suffices to prove for each $x \in X$ and a basis member $B' \in \mathcal{B}'$ containing x there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$. Becasue $\mathcal{B}$ is a basis, if $B' = B_1 \cap \cdots \cap B_n$ for some $B_1, \cdots, B_n \in \mathcal{B}$ then there is a basis member $B_0 \in \mathcal{B}$ such that $x \in B_0 \subset B_1 \cap \cdots \cap B_n = B'$.

1.2 Continuous maps

In this section, we assume X, Y, Z are topological spaces.

Definition 1.2.1 (Continuity)**.** The map $f : X \to Y$ is said to be continuous if $f^{-1}(U)$ is open in X whenever U is open in Y. If the map f is bijective and its inverse is also continuous, then f is called a homeomorphism. In addition, if f is injective and continuous, and the induced map $\tilde{f} : X \to f(X)$ defined by $\tilde{f}(a) = f(a)$ for all $a \in X$ is a homeomorphism, then f is called an embedding of X into Y. (It will be explained that such restriction of continuous maps are always continuous.)

Remark. (a) The procedure for checking continuity can be reduced to the members of a basis or a subbasis generating the topology on the codomain Y. (Why?)

(b) A homeomorphism naturally induces a bijection between the topologies on the domain and the codomain of the homeomorphism. Also, a bijective continuous map is a homeomorphism if and only if the map is an open map.

Theorem 1.2.2. Let X, Y be topological spaces and $f : X \to Y$ be a map. Then the followings are equivalent:

(a) f is a continuous map.

(b) For any closed subset B of Y, $f^{-1}(B)$ is closed in X.

(c) For each $x \in X$ and a neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X such that $f(U) \subset V$.

(d) Whenever $U \subset X$, $f(\overline{U}) \subset \overline{f(U)}$.

Proof. (a)$\Leftrightarrow$(b): This follows directly by considering set complements.

(a)$\Rightarrow$(c): If $x \in X$ and V is a neighborhood of $f(x)$ in Y, then $f^{-1}(V)$ is a neighborhood of x in X whose image under f is V.

(c)$\Rightarrow$(a): Let V be an open subset of Y and let $U = f^{-1}(V) \subset X$. By assumption, for each $x \in X$, there is a neighborhood A_x of x in X such that $f(A_x) \subset V$. Then $A_x \subset U$, so U is open in X.

(a)$\Rightarrow$(d): Suppose $U \subset X$. We will show that if $x \in \overline{U}$ then $f(x) \in \overline{f(U)}$. If V is a neighborhood of $f(x)$ in Y, then $f^{-1}(V)$ is a neighborhood of x in X, so $f^{-1}(V)$ intersects U. Thus, $V \cap f(U) = f(f^{-1}(V) \cap U)$ is nonempty, so $f(x) \in \overline{f(U)}$, as desired.

(d)$\Rightarrow$(b): Let B be a closed subset of Y and let $A = f^{-1}(B)$. By assumption, we have $B = f(A) \subset f(\overline{A}) \subset \overline{f(A)} = \overline{B} = B$, so $B = f(\overline{A})$ and $\overline{A} \subset f^{-1}(B) = A$, proving that $A = \overline{A}$ is closed. $\square$

If given maps $f : X \to Y$ and $g : Y \to Z$ are continuous, some naturally induced maps are also continuous, as indicated below. (Checking continuity is left as an exercise.)

Proposition 1.2.3. Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous.

(a) Every constant map is continuous.

(b) $g \circ f$ is continuous.

(c) Every restriction of a continuous map on a subspace is continuous. Also, if Y is a subspace of Z, then $\tilde{f} : X \to Z$ defined by $\tilde{f}(x) = f(x)$ for all $x \in X$ is continuous.

Problem 1.2.1. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution. It is not necessarily true; consider constant maps.

Theorem 1.2.4. Let A be a set; let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of spaces; and let $\{f_\alpha : A \to X_\alpha\}_{\alpha \in I}$ be an indexed family of functions.

(a) There is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the function f_α is continuous. In fact, the topology $\mathcal{T}$ is generated as a subbasis by the following collection:

$$\left\{ f_\alpha^{-1}(U_\alpha) : \alpha \in I \text{ and } U_\alpha \text{ is open in } X_\alpha \right\}.$$

(b) A map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each composition $f_\alpha \circ g$ is continuous.

Proof. (a) is almost clear; such topology necessarily contains the given collection. In proving (b), it suffices to prove if part. Suppose $g_\alpha := f_\alpha \circ g$ is continuous for each $\alpha \in I$. Given an open subset U of A, we have $g_\alpha^{-1}(A) = g^{-1}(f_\alpha^{-1}(A))$, completing the proof. $\square$

Remark. The product topology on a product space satisfies the above properties; in fact, the product topology is the topology on A with

$$A = \prod_{\alpha \in I} X_\alpha \quad \text{and} \quad f_\alpha = \pi_\alpha \text{ for each } \alpha \in I.$$

Problem 1.2.2. Let (X, d) be a metric space.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' be a space having the same underlying set as X, and define the map $d' : X' \times X' \to \mathbb{R}$ by $d'(a, b) = d(a, b)$ for all $(a, b) \in X' \times X'$. Show that the topology on X' is finer than the topology on X, if d' is continuous. Deduce that the metric topology on X induced by the metric d is the coarsest topology on X relative to which d is continuous.

Solution. (a) Choose a point $(a, b) \in X \times X$ and write $r = d(a, b)$. We will show that given a positive real number ϵ there is a neighborhood of (a, b) whose image under d is contained in $(r - \epsilon, r + \epsilon)$. Let $V = B_d(a, \delta) \times B_d(b, \delta)$ with $\delta > 0$, which is a neighborhood of (a, b) in $X \times X$. Whenever $(p, q) \in V$, we have

$$d(p, q) \leq d(p, a) + d(a, b) + d(b, q) \quad \text{and} \quad d(a, b) \leq d(a, p) + d(p, q)_d(q, b,)$$

from which we obtain $r - 2\delta < d(p, q) < r + 2\delta$. Hence, by choosing $0 < \delta < \epsilon/3$ we have $d(V) \subset (r - \epsilon, r + \epsilon)$, as desired. Therefore, d is a continuous map.

(b) Choose $a \in X'$, and define the map $\epsilon_a : X' \to X' \times X'$ by $\epsilon_a(x) = (x, a)$ for $x \in X'$. Then ϵ_a is continuous, so $k := d' \circ \epsilon_a$ is continuous. Hence, $k^{-1}((-\infty, r)) = B_{d'}(a, r) = B_d(a, r)$ is an open subset of X' for all $r \in \mathbb{R}$ with $r > 0$. Therefore, the topology on X' contains $\{B_d(a, r) : a \in X,\, r > 0\}$, so the topology on X' is finer than the topology on X. The last assertion easily follows.

1.3 Metrizable product spaces

Let X_n be a metrizable space for each $n \in \mathbb{N}$.

Notation. Given a metric d_n inducing the topology on X_n for each n, define the following metric on the product space $X := \prod_{n=1}^{\infty} X_n$ as follows:

$$\overline{\rho} : X \times X \to \mathbb{R},\ (x, y) \mapsto \sup_{n\in\mathbb{N}} \left\{ \overline{d_n}(x_n, y_n) \right\},$$
$$D : X \times X \to \mathbb{R},\ (x, y) \mapsto \sup_{n\in\mathbb{N}} \left\{ \frac{\overline{d_n}(x_n, y_n)}{n} \right\}.$$

The former metric is called the uniform metric on X and the latter metric is called the D-metric on X. Among them, the definition of the former metric generalizes to arbitrary product spaces: If $\{X_\alpha\}_{\alpha\in I}$ is an indexed family of metric spaces and $X = \prod_{\alpha\in I} X_\alpha$, then we may define the uniform metric $\rho : X \times X \to \mathbb{R}$ by $\rho(x, y) = \sup_{\alpha\in I}\{\overline{d}_\alpha(x_\alpha, y_\alpha)\}$ for $(x, y) \in X \times X$.

Theorem 1.3.1. The D-metric on X induces the product topology on X.

Proof. We first show that the topology $\mathcal{T}_D$ induced by the D-metric is finer than the product topology. Given a point $x \in X$ and a basis member $B = \prod_{n\in\mathbb{N}} B_n$ of the product topology with

$$\{n \in \mathbb{N} : B_n \neq X_n\} = \{n_1, \cdots, n_k\},$$

let us find a basis member of $\mathcal{T}_D$ which contains x and contained in B. For each n, let $C_n = B_{d_n}(x_n, r_n)$ be a neighborhood of x_n contained in B_n, where $C_n = X_n$ if and only if $B_n = X_n$ and $0 < r_n < 1$ for the other n's, and let $C = \prod_{n\in\mathbb{N}} C_n$. And let $\epsilon = \min_{1\le i\le k} \left\{ \frac{r_{n_i}}{n_i} \right\}$. Consider $G = B_D(x, \epsilon)$. If a point y of X is in G, then

$$\sup_{n\in\mathbb{N}} \left\{ \frac{\overline{d_n}(x_n, y_n)}{n} \right\} < \epsilon, \quad \overline{d_n}(x_n, y_n) < r_n,$$

so $y \in C \subset B$, i.e., $x \in G \subset B$.

Conversely, suppose a point x of X and a basis element $B_D(p, r) \in \mathcal{T}_D$ containing x are given. By choosing a small real number $\epsilon > 0$, we can achieve $x \in B_D(x, \epsilon) \subset B_D(p, r)$. Let N be a positive integer such that $1/N < \epsilon$. For each positive integer $n < N$, let $B_n = B_{d_n}(x_n, \epsilon)$; for each integer $n \geq N$, let $B_n = X_n$. Then $x \in \prod_{n\in\mathbb{N}} B_n \subset B_D(x, \epsilon) \subset B_D(p, r)$, as desired. □

Problem 1.3.1. Let (X, α) and (Y, β) be metric spaces. Show that every isometry from X into Y is an embedding.

Solution. Let f be an isometry from X into Y. It is clear that f is an injective continuous map. Because f maps an open ball of radius $r > 0$ in X onto an open ball of radius r in Y, f is an open map. Therefore, every isometry from X into Y is an embedding.

We now introduce the structure of an open ball in $\mathbb{R}^{\mathbb{N}}$ which equips the uniform metric $\overline{\rho}$.

Proposition 1.3.2. Let $\overline{\rho}$ be the uniform metric on $\mathbb{R}^{\mathbb{N}}$. Given $x = (x_1, x_2, \cdots) \in \mathbb{R}^{\mathbb{N}}$ and a real number $0 < \epsilon < 1$, define

$$U(x, \epsilon) := \prod_{n=1}^{\infty} (x_n - \epsilon, x_n + \epsilon).$$

(a) $U(x, \epsilon) \neq B_{\overline{\rho}}(x, \epsilon)$ and $U(x, \epsilon)$ is not open in the uniform topology.

(b) $B_{\overline{\rho}}(x, \epsilon) = \bigcup_{0<r<\epsilon} U(x, r)$.

Proof. (a) The point $(x_n + 2^{-n}\epsilon)_{n\in\mathbb{N}}$ is in $U(x, \epsilon)$ but not in $B_{\overline{\rho}}(x, \epsilon)$. Furthermore, no neighborhood of this point entirely lies in $U(x, \epsilon)$. Proving the details is left as an exercise.

(b) For each real number $0 < r < \epsilon$, we have $U(x,r) \subset B_{\overline{\rho}}(x,\epsilon)$. Conversely, if $y \in B_{\overline{\rho}(x,\epsilon)}$, then $\overline{\rho}(x,y) < \epsilon$, so $\sup\{\overline{d_n}(x_n,y_n) : n \in \mathbb{N}\} = \delta$ for some $0 \leq \delta < \epsilon$. Hence, $B_{\overline{\rho}}(x,\epsilon) \subset U(x,r)$ for some real number r such that $\delta < r < \epsilon$.

$\square$

Chapter 2

Connected spaces and compact spaces

2.1 Connected spaces

Definition 2.1.1. Let X be a topological space. A pair (A, B) is called a separation of X if $\{A, B\}$ is a partition of X by nonempty subsets of X. If X has no separation, then X is called a connected space.

Remark. The topological space X is connected if and only if only subset of X which is both open and closed in X are X and the empty set.

When discussing connectedness of the subspace A of X, according to the above definition, subspaces of A shall be considered. The following theorem implies that we can argue connectedness of the subspace A of X with regard to subspaces X.

Theorem 2.1.2 (Connected subspace)**.** Suppose X is a topological space and A is a subspace of X. Then (U, V) is a separation of X if and only if

(a) $\{U, V\}$ is a partition of Y by nonempty subsets of Y

(b) and neither of which contains a limit point of the other, i.e., $\overline{A} \cap B = A \cap \overline{B} = \varnothing$, where the closures are taken in X.

Proof. Suppose (U, V) is a separation of Y. Because U is both open and closed in Y, $U = \overline{U} \cap Y$. Since $U \cap V = \varnothing$, we have $\overline{U} \cap V = \varnothing$.

Conversely, assuming (a) and (b), we have $\overline{U} \cap Y = \overline{U} \cap (U \sqcup V) = U$, i.e., U is open in Y. □

Before introducing some examples of connected spaces, we first introduce a lemma stating regarding a connected subspace, which will be frequently used.

Lemma 2.1.3. If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within C or D.

Proof. Define $A := Y \cap C$ and $B := Y \cap D$. Since (C, D) is a separation of X, C and D are open (and closed) in X, so A and B are open in Y. If A and B are nonempty, then (A, B) is a separation of Y, a contradiction. □

Proposition 2.1.4. Suppose $\{U_\alpha\}_{\alpha \in I}$ is a collection of connected subspaces, and assume all the members have a common point. Then the union of the members of the collection is connected.

Proof. Suppose the union A of A_α's is not connected. Then there is a separation (U, V) of A. Each A_α resides entirely in U or V. Without loss of generality, assume A_{α_0} is in U for some $\alpha_0 \in I$. Since a common point is in U and not in V, all A_α's are in U, so $A \subset U$, a contradiction. □

Proposition 2.1.5. If A is a connected subspace of X, then adding some of its limit points keeps the space connected. To be precise, if $A \subset B \subset \overline{A}$, then B is connected.

Proof. Suppose B is not connected for some such B, and let (U, V) be a separation of B. Since A is a connected subspace of B, without loss of generality, $A \subset U$, and $\overline{A} \subset \overline{U}$. Because $\overline{U}$ and V are disjoint, $B \cap V = \varnothing$, a contradiction. □

Proposition 2.1.6. A continuous image of a connected space is connected.

Proof. Let X be a connected space and $f : X \to Y$ be a continuous map. Suppose $f(X)$ is not connected and let (U, V) be a separation of $f(X)$. Since U and V are open in $f(X)$, their preimages are open in X, forming a separation of X, a contradiction. □

The idea of the proof of the following proposition is worth remarking.

Proposition 2.1.7. A finite product of connected spaces is connected.

Solution. It suffices to prove for the product of two connected spaces, since the desired result can be obtained by mathematical induction. Given a point $(a, b) \in X_1 \times X_2$, where X_1 and X_2 are connected spaces, define the cross $C(a, b) := (\{a\} \times X_2) \cup (X_1 \times \{b\})$. Since each line in $C(a, b)$ is homeomorphic to X_1 or X_2, it is connected; by Proposition 2.1.4, the cross $C(a, b)$ is connected. Because

$$X_1 \times X_2 = \bigcup_{a \in X_1} C(a, b)$$

for any given $b \in X_2$ and the intersection of such $C(a, b)$'s is nonempty, by Proposition 2.1.4, $X_1 \times X_2$ is connected.

In fact, the above proposition extends to an arbitrary product.

Proposition 2.1.8. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of connected spaces. Then the product $X = \prod_{\alpha \in I} X_\alpha$ is connected.

Proof. We first fix a point $a = (a_\alpha)_{\alpha \in I} \in X$, and let

$$X_K = \{x \in X : x_\alpha = a_\alpha \text{ whenever } \alpha \in I \setminus K\}$$

for each finite subset K of I. In other words, X_K is a subset of X which consists of all the points permitting every possible value for indices in K only.

Step 1: The union of X_K's is connected

By definition, $X_K \approx \prod_{\alpha \in K} X_\alpha$. This also implies that the union Y of X_K's is also connected, since each X_K contains the point a.

Step 2: The closure of Y in X is X

Suppose $p \in X$ and let W be a neighborhood of p. There is a basis member $B = \prod_{\alpha \in I} B_\alpha$ such that $p \in B \subset W$ (clearly, $J := \{\alpha \in I : B_\alpha \neq X_\alpha\} = \{\alpha_1, \cdots, \alpha_n\}$). Because B intersects X_J, we can conclude that $p \in \overline{Y}$.

By Step 1 and Step 2, the product space X is the closure of a connected space, so it is connected. □

We give a problem in the textbook, with a solution using the cross we constructed in this section.

Problem 2.1.1. Let X, Y be connected spaces and A, B are nonempty proper subsets of X, Y, respectively. Show that $(X \setminus A) \times (Y \setminus B)$ is connected.

Solution. Let (p, q) be a point of $(X \setminus A) \times (Y \setminus B)$, and define

$$M := \bigcup_{x \in X \setminus A} C(x, q), \quad N := \bigcup_{y \in Y \setminus B} C(p, y).$$

It is easy to check that M, N are nonempty and contain (p, q), and $M \cup N = (X \setminus A) \times (Y \setminus B)$. Therefore, $(X \setminus A) \times (Y \setminus B)$ is connected.

Example 2.1.9. Since $\mathbb{R}$ is connected in the standard topology, $\mathbb{R}^{\mathbb{N}}$ is connected in the product topology.

However, $\mathbb{R}^{\mathbb{N}}$ is not connected in the uniform topology. Let A be the set of all bounded sequences in $\mathbb{R}^{\mathbb{N}}$, and let B be the set of all unbounded sequences in $\mathbb{R}^{\mathbb{N}}$. Clearly, $\mathbb{R}^{\mathbb{N}} = A \sqcup B$ and A, B are nonempty. It is easy to check that A and B are open subsets. Thus, $\mathbb{R}^{\mathbb{N}}$ is not connected.

Because the box topology is finer than the uniform topology, $\mathbb{R}^{\mathbb{N}}$ is not connected in the box topology.

Theorem 2.1.10 (Intermediate value property)**.** Let X be a connected space; let Y be an ordered set in the order topology; and let $f : X \to Y$ be a continuous map. If $a, b \in X$ and r is a point in Y between $f(a)$ and $f(b)$, there is a point $p \in X$ such that $f(p) = r$.

Proof. Suppose there is an intermediate value r such that $r \notin f(X)$. Then $f(X) = (f(X) \cap (-\infty, r)) \sqcup (f(X) \cap (r, \infty))$, forming a separation of $f(X)$. However, $f(X)$ is connected, because X is connected. □

Problem 2.1.2. Show that a connected metric space with more than one point is uncountable.[1]

Solution. Given a point $a \in X$, define a function $f : X \to [0, \infty)$ by $f(x) = d(x, a)$ for $x \in X$, where d is a metric on a connected metric space X. Since f is continuous, $f(X)$ is connected. Since X has more than one point, $f(X)$ is uncountable, so X is uncountable.

2.2 Path-connected spaces

Definition 2.2.1 (Path-connected space)**.** A space X is said to be path-connected if given any two point a and b in X, there is a continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. We call such f a path from a to b.

Proposition 2.2.2. Path-connectedness implies connectedness, but not conversely.

Proof. Suppose X is a path-connected space, but X is not connected. Then X has a separation (U, V). Choose a point $a \in U$ and $b \in V$, and let $f : [0, 1] \to X$ be a path from a to b. Since $[0, 1]$ is connected, the image of f is also connected. Since $f(0) = a \in U$, the image of f lies in U, but $f(1) = b \notin U$, a contradiction.

To show that the connectedness does not always imply the path-connectedness, consider the topologist's sine curve $\overline{S}$, where S is the subspace of $\mathbb{R}^2$ defined as

$$S := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : 0 < x \leq 1 \right\}$$

Clearly, S is connected, so is its closure $\overline{S}$. Nevertheless, $\overline{S}$ is not path-connected. (Why?) □

Proposition 2.2.3. A continuous image of a path-connected space is path-connected.

Proof. Let $f : X \to Y$ be a continuous map, where X is a path-connected space. If $p, q \in f(X)$, there are points $a, b \in X$ such that $f(a) = p$ and $f(b) = q$. If γ is a path from a to b, then $f \circ \gamma$ is a path from p to q. □

Proposition 2.2.4. A product of path-connected spaces is path-connected.

Proof. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of path-connected spaces, and let $X := \prod_{\alpha \in I} X_\alpha$. Given two points $a, b \in X$, let $f_\alpha : [0, 1] \to X_\alpha$ be a path from a_α to b_α for each $\alpha \in I$. Define a map $F : [0, 1] \to X$ by $F = (f_\alpha)_{\alpha \in I}$. The map F is continuous relative to the product topology on X, so F is a path from a to b. □

Proposition 2.2.5. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of path-connected spaces with a point in common. Then the union of X_α's is also path-connected.

[1] The result of this problem will be generalized later. See Proposition 2.1.4 on page 9.

Proof. Let p be a common point. Given two points x and y from the union, let f_1 and f_2 be paths lying in the union from x to p and from p to y, respectively. Concatenating f_2 after f_1 makes a path lying in the union from x to y. □

Remark. Regarding connectedness, adding some of limit points keeps the space connected. This is not valid for path-connectedness. (Consider the topologist's sine curve)

Problem 2.2.1. Show that if A is an open connected subspace of $\mathbb{R}^2$, then A is path-connected.

Solution. Given $p \in A$, let $C(p)$ denote the set of points in A which can be joined to p by a path in A. We will show that $C(p)$ is both open and closed; since A and $\varnothing$ are the only open and closed subspaces of A and $C(p)$ is nonempty, it is forced that $C(p) = A$, i.e., every point of A can be joined to p by a path in A.

We first show that $C(p)$ is open. Given a point $x \in C(p)$, let r be a positive real number such that $B(x, r) \in A$ (openness of A is used). Since any two points of a ball can be joined by a line segment, $B(x, r) \subset C(p)$, i.e., $C(p)$ is open.

To show that $C(p)$ is closed, we show that $A \setminus C(p)$ is open. Assume $A \setminus C(p)$ is not open, and let y be a point of $A \setminus C(p)$ such that $B(y, \epsilon) \not\subset A \setminus C(p)$ for all $\epsilon > 0$. Because $y \in A$, there is a positive real number s such that $B(y, s) \subset A$. Let z be a point of $C(p) \cap B(y, s)$. Then, we can find a path from p to z and a path from z to y, a contradiction.

Hence, $C(p)$ is a subset of A which is open and closed in A. Therefore, $A = C(p)$, i.e., A is path-connected.

Problem 2.2.2. Show that every co-countable subspace of $\mathbb{R}^2$ is path-connected.

Solution. Given two points a, b of a co-countable subspace A of $\mathbb{R}^2$, there are countably many lines passing $a \in A$ which intersects $\mathbb{R}^2 \setminus A$. Because there are uncountably many lines passing $a \in A$, we can choose a line l_1 passing a not intersecting $\mathbb{R}^2 \setminus A$. We can also find a line l_2 passing b not intersecting $\mathbb{R}^2 \setminus A$ which is not parallel to l_1. We can thus find a path from a to b lying in A which lies in $l_1 \cup l_2$.

2.3 The topology of linear continuums

Definition 2.3.1. An ordered set L in the order topology is called a linear continuum if

(a) L satisfies the least upper bound property,

(b) and whenever $x, y \in L$ and $x < y$, there is a point $z \in (x, y)$.

Theorem 2.3.2. Suppose L is an ordered set in the order topology. Then the followings are equivalent:

(a) L is a linear continuum.

(b) L and its intervals and rays are connected.

Proof. Read your paper note. □

2.4 Compact spaces

Definition 2.4.1 (Compact space)**.** A topological space X is said to be compact if every covering of X by sets open in X has a finite subcover. If every such covering has a countable subcover, then X is said to be a Lindelöf space.

Remark. In fact, we can impose an alternative definition of compactness as follows:

The topological space X is said to be compact if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection of the members of $\mathcal{C}$ is nonempty.

Proof. Establish some contrapositions to check the above two statements are equivalent. The proof is left as an exercise. □

As connectedness, compactness of subspaces can also be argued in larger spaces.

Theorem 2.4.2 (Compactness of subspaces)**.** Let A be a subspace of X. Then A is compact if and only if every covering of A by sets open in X has a finite subcover. (This statement is valid if the words 'compact' are replaced by 'Lindelöf.')

Proof. Suppose A is compact, and $\mathcal{A} = \{A_\alpha\}_\alpha$ is a covering of A by sets open in X. Then the naturally induced collection $\{A_\alpha \cap A\}$ is a covering of A by sets open in A, and we can find a finite subcover. The corresponding finite subcover of $\mathcal{A}$ covers A.

Suppose conversely that every covering of A by sets open in X has a finite subcover, and let $\mathcal{A} = \{A_\alpha\}_\alpha$ be a covering of A by sets in A. Since each A_α can be written as $A \cap O_\alpha$ for some subset O_α open in X, the collection $\{O_\alpha\}_\alpha$ has a finite subcover. The corresponding finite subcover of $\mathcal{A}$ covers A. □

Theorem 2.4.3. Regarding closedness and compactness of subspaces, the following statements holds:

(a) Every closed subspace of a compact space is compact.

(b) Every Hausdorff space is compactly normal, i.e., any two disjoint compact subspaces can be separated by (disjoint) neighborhoods.

(c) Every compact subspace of a Hausdorff space is closed. (Hence, every compact Hausdorff space is normal.)

Solution. (a) Let X be a compact space and Y be a closed subspace of X. Let $\mathcal{A} = \{A_\alpha\}_\alpha$ be a covering of Y by sets open in X. Because the collection $\mathcal{A} \cup \{X \setminus Y\}$ is also an open covering of X, the collection has a finite subcover. Among them, excluding $X \setminus Y$ gives a finite subcover of $\mathcal{A}$ which covers Y.

(b) We first prove that a Hausdorff space X is compactly regular. Let p be a point of X and A be a compact subspace of X not containing p. For each $a \in A$, let U_a and V_a be disjoint neighborhoods of p and a, respectively. Since A can be covered by finitely many V_a's, the intersection of the corresponding U_a's is a neighborhood of p. Thus, X is compactly regular.

To show that X is compactly normal, let A, B be disjoint compact subspaces of X. For each $a \in A$, let U_a and V_a be disjoint neighborhoods of a and B, respectively. Since A can be covered by finitely many U_a's, the union of the corresponding U_a's and the intersection of the corresponding V_a's are disjoint neighborhoods of A and B, respectively. Therefore, X is compactly normal.

(c) If A is a compact subspace of a Hausdorff space X, then it follows from (b) that $X \setminus A$ is open, hence A is closed. In (a), we have shown that a closed subspace of a compact space is compact. Thus, compactness and closedness of a subspace of a compact Hausdorff space coincide. Hence, every compact Hausdorff space is normal.

Proposition 2.4.4. A continuous image of a compact space is compact.

Proof. Because this proposition is easy to prove, the proof will be left as an exercise. □

Proposition 2.4.5. Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. If A is a closed subspace of X, then A is compact, and $f(A)$ is a compact subspace of Y, hence $f(A)$ is a closed subspace of Y. Therefore, f is a homeomorphism. □

Theorem 2.4.6. The product of finitely many compact spaces is compact. (In fact, this theorem extends to arbitrary products of compact spaces, which is called the Tychonoff's theorem. The proof of the Tychonoff's theorem will not be introduced in this note; read your textbook.)

When proving the above theorem, the first version of the tube lemma will be used.

Lemma 2.4.7. (a) (The tube lemma) Consider the product space $X \times Y$, where Y is a compact space. If N is an open subspace of $X \times Y$ containing a slice $x_0 \times Y$ for some $\{x_0\} \in X$, then N contains a tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.

(b) (A generalization of the tube lemma) Let A and B be compact subspaces of X and Y, respectively, and let N be an open subspace of $X \times Y$ containing $A \times B$. Then, there are neighborhoods U and V of A in X and B in Y, respectively, such that $A \times B \subset U \times V \subset N$. (Our first version comes when $A = \{x_0\}$ and $B = Y$.)

Proof of Lemma 2.4.7. (a) Since N is open, for each point $(x_0, y) \in \{x_0\} \times Y$, there is a neighborhood $A_y \times B_y$ of the point (x_0, y) contained in N, where A_y and B_y are open in X and Y, respectively. Because $\{x_0\} \times Y$ is homeomorphic to Y, hence $\{x_0\} \times Y$ is also compact. Hence, the slice can be covered by finitely many above basis members. For convinience, write such members as $A_i \times B_i$ for $i = 1, \cdots, n$. Write $W = \bigcap_{i=1}^n A_i$, then $\{x_0\} \times Y \subset W \times Y \subset N$.

(b) Since N is open, for each point $(a, t) \in \{a\} \times B$, there is a neighborhood $A_t^a \times B_t^a$ of the point (a, t) contained in N, where A_t^a and B_t^a are open in X and Y, respectively. Because the slice $\{a\} \times B$ is compact for each $a \in A$, the slice can be covered by finitely many neighborhoods; write it as $A_i^a \times B_i^a$ for $i = 1, \cdots, n^a$. Finally, define

$$U^a := \bigcap_{i=1}^{n^a} A_i^a, \quad V^a := \bigcup_{i=1}^{n^a} B_i^a.$$

Then, $U^a \times V^a$ is an open set contained in N containing $\{a\} \times Y$. Since A is compact, finitely many U^a's cover A; write them as $U^1, \cdots, U^k$. Finally, define

$$U := \bigcup_{i=1}^{k} U^k, \quad V := \bigcap_{i=1}^{k} V^k.$$

Then, $U \times V$ is an open set contained in N containing $A \times B$.

This completes the proof of tube lemmas. □

Proof of Theorem 2.4.6. Let X, Y be compact spaces, and $\mathcal{A}$ be an open cover of $X \times Y$. Given $x_0 \in X$, because the slice $\{x_0\} \times Y$ is compact, there are finitely many members of $\mathcal{A}$ covering the slice $\{x_0\} \times Y$, and such members cover a tube $U(x_0) \times Y$ about $\{x_0\} \times Y$. Because finitely many $U(x_0)$'s cover X, the compactness of $X \times Y$ is justified. The general result can be obtained with help of mathematical induction. □

Before introducing some problems, we introduce some properties of compact ordered sets and a relevant theorem.

Remark (Compact ordered sets). Suppose X is an ordered set in the order topology, and assume X is compact.

(a) X has the greatest and the least element; otherwise, one can construct an open cover of X with no finite subcover.

(b) (Extreme value theorem) Let $f : K \to Y$ be a continuous map, where K is a compact space and Y is an ordered set in the order topology. Then f attains the maximum and the minimum. This is because the subspace $f(K)$ of Y is compact.

(c) If X satisfies the least upper bound property, then every closed interval in X is compact.

The first problem deals with the distance between a point and a subspace and neighborhoods of subspaces in a metric space. In particular, (c) implies that the original definition of the ϵ-neighborhood of a subspace and our intuition coincide.

Problem 2.4.1. Let (X, d) be a metric space and A be a nonempty subset of A.

(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

(b) Show that if A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$, whenever $x \in X$.

(c) Define the ϵ-neighborhood of A in X to be the set

$$U(A, \epsilon) := \{x \in X : d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

(d) Assume that A is compact, and let U be an open set in X containing A. Show that U contains an ϵ-neighborhood of A for some $\epsilon > 0$.

(e) Show that the result in (d) need not hold if A is closed but not compact.

Solution. (a) Almost clear. If $x \in \overline{A}$, then whenever $k > 0$, there is a point $a \in A$ such that $d(x, a) < k$. Hence, $d(x, A) = 0$. Assuming conversely, whenever $k > 0$, there is a point $a \in A$ such that $d(x, a) < k$, so $B_d(x, k)$ intersects A. Hence, $x \in \overline{A}$.

(b) Given a point $x \in X$, define a function $f_x : A \to [0, \infty)$ by $f_x(a) = d(x, a)$ for $a \in A$. Since $|f_x(a) - f_x(b)| \leq d(a, b)$ whenever $a, b \in A$, f_x is (uniformly) continuous. Because the domain A of f_x is a compact space, f_x attains the minimum. Furthermore, $d(x, A) = \inf\{d(x, a) : a \in A\} = \min\{f_x(a) : a \in A\}$, so $d(x, A) = d(x, a)$ for some $a \in A$.

(c) It is clear that every ϵ-ball with the center in A is contained in the ϵ-neighborhood of A, so one inclusion is obvious. Suppose $u \in U(A, \epsilon)$ so that $d(u, A) < \epsilon$. It implies that the ϵ-ball centered at u intersects A at a point $a \in A$, so the ϵ-ball centered at a contains u. Therefore, $U(A, \epsilon)$ is contained in the union of the ϵ-balls with centers in A.

(d) Let $f : A \to \mathbb{R}$ be a function defined by $f(a) = d(a, X \setminus U)$.

- f is continuous; for $a, b \in A$, we have $f(a) \leq d(a, x) \leq d(a, b) + d(b, x)$ for all $x \in X \setminus U$, so $f(a) \leq d(a, b) + f(b)$, from which, by symmetry, we obtain $|f(a) - f(b)| \leq d(a, b)$.
- $f(a) > 0$ for all $a \in A$; otherwise, if $f(p) = 0$ for some $p \in A$, then $p \in \overline{X \setminus U} = X \setminus U$, a contradiction.

Because A is compact, f attains the minimum $\delta > 0$. Hence, for example, if $\epsilon = \delta/2$, then $U(A, \epsilon)$ is contained in U.

(e) The x-axis in the xy-plane is closed but not compact, and the following open region contains the x-axis:

$$R := \{(x, y) \in \mathbb{R}^2 : y < e^{-|x|}\}.$$

No neighborhood of the x-axis is contained in R.

Problem 2.4.2. Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then $Y := \bigcap_{A \in \mathcal{A}} A$ is connected.

Solution. Suppose Y is not connected, and let (C, D) be a separation of the intersection $Y = \bigcap_{A \in \mathcal{A}} A$. Because C and D are closed in Y and Y is closed in X, C and D are closed in X. Since X is normal, there are disjoint neighborhoods U and V of C and D, respectively. Suppose $A \setminus (U \sqcup V)$ is empty for some $A \in \mathcal{A}$. Since A contains Y, $A \subset U \sqcup V$, A has a separation $(A \cap U, A \cap V)$, a contradiction. Therefore, $A \setminus (U \sqcup V)$ is nonempty for all $A \in \mathcal{A}$, and the collection $\mathcal{C}$ of $A \setminus (U \sqcup V)$ is a simply ordered collection of sets closed in X. Therefore, $\mathcal{C}$ satisfies the finite intersection property, and the intersection of members in $\mathcal{C}$ is nonempty, because X is compact. Thus, Y is not contained in $U \sqcup V$, a contradiction. Therefore, Y is connected.

2.5 Compact metrizable spaces

We first introduce some other types of compactness:

Definition 2.5.1. Let X be a space.

(a) X is said to be limit point compact, if every infinite subset of X has a limit point in X.

(b) X is said to be sequentially compact, if every infinite sequence with values in X has a convergent subsequence.

Remark. If X is sequentially compact, then X is complete, i.e., every Cauchy sequence in X is convergent.

Proposition 2.5.2. Compactness implies limit point compactness.

Proof. Left as an exercise. □

Theorem 2.5.3. Suppose X is a metrizable space. Then the followings are equivalent:

(a) X is compact.

(b) X is limit point compact.

(c) X is sequentially compact.

Proof. We already proved that (a) implies (b) and it is easy to show that (b) implies (c). The most difficult part of the proof is to show that (c) implies (a). Assume X is sequentially compact.

Step 1: Showing the existence of the Lebesgue number.

We wish to prove that for any open cover $\mathcal{A}$ of X there is a real number $\delta > 0$ such that every open set with diameter less than δ is contained in some member of $\mathcal{A}$. Suppose there is no such δ. Then, for each $n \in \mathbb{N}$, there is an open set C_n of diameter less than $1/n$ which is not contained in any member of an open cover $\mathcal{A}$ of X. Choose a point $x_n \in C_n$. By sequential compactness, the sequence $(x_n)_n$ has a convergent subsequence $(x_{n_k})_k$. Let x be the limit of the subsequence, and let A be a member of $\mathcal{A}$ containing x. And let $r > 0$ be a real number such that $B_d(x, r) \subset A$. If i is large enough so that

$$d(x, x_{n_i}) < \frac{r}{2} \quad \text{and} \quad \frac{1}{n_i} < \frac{r}{2},$$

then C_{n_i} lies in A, a contradiction.

Step 2: Showing that X is totally bounded.

Suppose X is a sequentially compact space which is not totally bounded. We can choose countably many points $x_1, x_2, \cdots \in X$ such that $\bigcup_{n=1}^{\infty} B_d(x_n, \epsilon) \subsetneq X$ for some $\epsilon > 0$. Since X is sequentially compact, we can find a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$, a contradiction.

Step 3: Deriving that X is compact.

Let $\mathcal{A}$ be an open cover of X. By the result of Step 1, there is a Lebesgue number $\delta > 0$ for the collection $\mathcal{A}$. By the result of Step 2, finitely many balls in X of radius $\dfrac{2\delta}{5}$ cover X. Therefore, a finite subcover of $\mathcal{A}$ covers X. □

In metric spaces, the above equivalence reduces to the Heine-Borel theorem.

Theorem 2.5.4 (Heine-Borel theorem)**.** Let (X, d) be a metric space. Then the followings are equivalent:

(a) X is compact.

(b) X is limit point compact.

(c) X is complete and totally bounded.

Proof. By the preceeding theorem, it suffices to show that (c) is equivalent to compactness. Since it is already observed that sequential compactness implies (c), it suffices to show that (c) implies any of compactnesses. For this, we will show that (c) implies sequential compactness.

Let $(x_n)_n \subset X$ be a sequence. Since X is sequentially compact, there is a ball B_1 in X of radius 2^{-1} which contains x_n's for infinitely many $n \in N_1 \subset \mathbb{N}$. Because $X \cap B_1$ is also totally bounded, a ball B_2 with the center in $X \cap B_1$ of radius 2^{-2} contains x_n's for infinitely many $n \in N_2 \subset N_1$. Continuing inductively, we can find $n_i \in N_i$ for each $i \in \mathbb{N}$ with $n_1 < n_2 < \cdots$. Then $d(x_{n_i}, x_{n_k}) \leq 2^{1-i}$ if $i < k$. Because X is complete, $(x_n)_n$ has a convergent subsequence. Therefore, (c) implies sequential compactness. $\square$

Problem 2.5.1. Let U be an open subset of $\mathbb{C}$ which contains $\mathbb{D}$. Show that there is a positive real number $r > 1$ such that $B(0, r) \subset U$.

Proof 1. Suppose that there is no such $r > 1$. Then, we can choose $a_n \in B(0, 1 + 1/n) \setminus U$ for each $n \in \mathbb{N}$. Since $(a_n)_{n\in\mathbb{N}} \subset \overline{B(0, 2)}$, $(a_n)_n$ contains a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Letting α be the limit of $(a_{n_k})_k$, we find that α is a limit point of $\mathbb{C} \setminus U$. Because $\mathbb{C} \setminus U$ is closed, $\alpha \notin U$. On the other hand, $\alpha \in \partial\mathbb{D}$ (why?), we have $\alpha \in \partial\mathbb{D} \subset U$, a contradiction. Therefore, $B(0, r) \subset U$ for some real number $r > 1$. $\square$

Proof 2. Note that the collection of 'polar rectangles' is a basis for the topology on $\mathbb{C}$, where a polar rectangle is of the form

$$a \leq \rho \leq b, \quad s \leq \theta \leq t,$$

where a, b, s, t are real numbers such that $a \leq b$. Because $\partial\mathbb{D}$ is compact, for each point $e^{ix} \in \partial\mathbb{D}$ $(0 \leq x < 2\pi)$ there is a polar rectangle $P_x : a(x) \leq \rho \leq b(x), s(x) \leq \theta \leq t(x)$ such that $e^{ix} \in P_x \subset U$. Choosing finitely many members among P_x's and letting r the smallest $b(x)$, we find that $B(0, r) \subset U$. $\square$

2.6 Locally compact spaces and one-point compactification

Some of the properties which are most desired for a topological space to have are the space being metrizable or being a compact Hausdorff space. In this section, we impose a situation in which a topological space can be embedded into a sompact Hausdorff space.

Definition 2.6.1 (Local compactness)**.** A space X is said to be locally compact at the point $a \in X$ if there is a compact subspace C of X containing a neighborhood of a. If X is locally compact at every point of X, then X is said to be locally compact.

One property regarding local compactness:

Proposition 2.6.2. Let $\{X_\alpha\}_\alpha$ be an indexed family of nonempty spaces.

(a) If $\prod_\alpha X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α. If the product space is compact, then each X_α is compact.

(b) The converse of the first statement in (a) is also true.

Proof. Remark that projection maps are continuous. Let $X = \prod_\alpha X_\alpha$, and we first prove (a). Given a point $x \in X$, there is a compact subspace C of X containing a basis member $\prod_\alpha B_\alpha$ that contains x. For each α, $\pi_\alpha(C)$ is a compact subspace of X_α, which contains the neighborhood B_α of $x_\alpha \in X_\alpha$. Becuase for all but finitely many values of α it is satisfied that $B_\alpha = X_\alpha$, it is proved that each X_α is locally compact and all but finitely many of X_α's are compact. In particular, if X is compact, then each $X_\alpha = \pi_\alpha(X)$ is compact.

Given a point $x \in X$, for each α, use local compactness and compactness to find a compact subspace $C_\alpha \subset X_\alpha$ such that C_α contains a neighborhood B_α of x_α in X_α. When finding such subspaces, we choose $B_\alpha = C_\alpha = X_\alpha$ whenever X_α is compact. Then, the product of B_α's is a basis member of X containing x; the product of C_α's is a compact subspace of X. $\square$

Theorem 2.6.3 (Existence and uniqueness of a one-point compactification)**.** Let X be a space. Then X is locally compact and Hausdorff if and only if there is a space Y with the following properties:

(a) Y is a compact and Hausdorff space containing X as a subspace, i.e., Y is a compactification of X.

(b) $Y \setminus X$ is a one-point set.

Moreover, if Y_1 and Y_2 are such spaces, then they coincides on X and they are homeomorphic, i.e., Y is uniquely determined up to equivalence.[2]

Proof. Since it is easier to check the uniqueness, we first explain the uniqueness (up to equivalence).

Step 1: Proving the uniqueness

Suppose Y_1 and Y_2 are compact Hausdorff spaces satisfying (a), (b), and (c). Let p and q denote the unique point of $Y_1 \setminus X$ and $Y_2 \setminus X$, respectively. Define a map $h : Y_1 \to Y_2$ by

$$h(x) = x \text{ if } x \in X, \text{ and } h(p) = q.$$

We show h is a homeomorphism extending the identity map on X; and for this, it remains to verify that h is a continuous map, because the openness of h will follow from symmetry. Suppose first that $U \subset Y_2$ is an open subset contained in X. Clearly, its preimage is U, and because U is open in X and X is open in Y_1, U is open in Y_1. Now, suppose that $U \subset Y_2$ is an open subset containing q. Let $C = Y_2 \setminus U$ be the complement of U in Y_2, which is contained in X. The preimage of C is C; because C is closed and compact in Y_2, C is a compact subspace of X, a compact subspace of Y_1, so C is closed in Y_1, too. Thus, h is a continuous map.

Step 2: Proving the existence

Suppose first that X is a locally compact Hausdorff space. Let p be any element not in X, and let $Y = X \sqcup \{p\}$. And impose a topology on Y by declaring the following sets to be open in Y:

(T1) Subsets which are open in X.

(T2) Subsets of the form $Y \setminus C$, where C is a compact subspace of X.

It is left as an exercise to check that the above collection is a topology on Y. (See Problem 2.6.1)

We first show that Y contains X as a subspace. Clearly, the topology on X is coarser than the subspace topology on X inherited from Y. Conversely, the subspace topology on X inherited from Y is also coarser than the topology on X, which can be easily checked by intersecting X with the sets of either type.

To show that Y is a compact space, let $\mathcal{A}$ be any open cover of Y. There is a member $A \in \mathcal{A}$ containing p, and the susspace $Y \setminus A$ is closed in Y. Since $Y \setminus A$ is contained in the Hausdorff space X, $Y \setminus A$ is compact. This proves the compactness of Y.

Finally, we show that Y is a Hausdorff space, in which it remains to show that p and any point a of X can be separated by disjoint sets open in Y. Since X is locally compact, there is a compact subspace C of X containing a neighborhood U of a in X. $(U, Y \setminus C)$ is a desired pair.

Step 3: Proving the converse

Suppose such space Y exists for a space X. Being a subspace of the Hausdorff space Y, X is also a Hausdorff space. Given a point $a \in X$, because Y is a Hausdorff space, there are neighborhoods U and V of a and the unique point $p \in Y \setminus X$ in Y which are disjoint. Since $Y \setminus V$ is a closed subspace of Y contained in X, $Y \setminus V$ is a compact subspace of X containing the neighborhood U of a. Therefore, X is locally compact. □

Problem 2.6.1. Show that the collection imposed in the proof of the existence part is a topology on Y.

Solution. **Step 1. Cheking the first axiom**

Clearly, $\varnothing$ is of the first type and Y is of the second type, so they belong to the collection.

Step 2. Checking the axiom of union

A union of sets of the first type is an open subset of X, hence the union is of the first type. If $(V_\beta = Y \setminus C_\beta)_\beta$ is a collection of sets of the second type (each C_β is a compact subspace of X), their

[2] Equivalence of compactness is introduced in Chapter 4.

union is of the second type, because the union is $Y \setminus K$, where $K = \bigcap_{\beta} C_\beta$, and K is a closed subspace of C_β's, so K is compact. The union of a set U of the first type and a set $V = Y \setminus C$ of the second type (C is a compact subspace of X) is of the second type, since

$$U \cup (Y \setminus C) = Y \setminus (C \setminus U)$$

and $C \setminus U$ is compact (why?).

Step 3. Checking the axiom of intersection

A finite intersection of sets of the first type is an open subset of X, hende the intersection is of the first type. If $(V_\beta = Y \setminus C_k)_{k=1}^n$ is a collection of sets of the second type (each C_k is a compact subspace of X), their intersection is of the second type, because the union is $Y \setminus K$, where $K = \bigcup_{k=1^n} C_k$ is compact. The intersection of a set U of the first type and a set $V = Y \setminus C$ of the second type (C is a compact subspace of X) is of the first type, since

$$U \cap (Y \setminus C) = U \setminus C$$

is open in X.

Let X be a locally compact Hausdorff space, and let Y be a space (which is unique up to equivalence) constructed as above. If X is compact, then X is a closed subspace of Y, so the closure of X in Y is X, a proper subset of Y. On the other hand, if X is not compact, as X is a subspace of a compact Hausdorff space Y, X is not closed ini Y, hence the closure of X in Y is Y. If Y is such a space and the closure of X in Y is Y, Y is called the one-point compactification (or the Alexandroff compactification) of X.

Remark. The above observation states that a locally compact Hausdorff space X has a one-point compactification if and only if X is not compact.

Remark. In Chapter 4, we will consider the general concept of compactification (before studying the Stone-Čech compactification) over completely regular spaces. Since one-point compactifications are also compactifications, here we briefly explain that the locally compact Hausdorff space X is completely regular. If the space X is compact, then X is normal, so X is completely regular; if X is not compact, then X is a subspace of its one-point compactification, so X is completely regular.

When the given space is a Hausdorff, the following property can be used as an alternative definition of local compactness.

Proposition 2.6.4. Suppose X is a Hausdorff space. Then X is locally compact if and only if given $x \in X$ and its neighborhood U in X, there is a precompact neighborhood V of a in X whose closure in X is contained in U.

Proof. Suppose X is a locally compact Hausdorff space, and suppose further that a point $x \in X$ together with its neighborhood U in X is given. Since X has a compactification Y such that $Y \setminus X$ has the unique point p, the subspace $C := Y \setminus U$ is a closed (hence, compact) subspace of Y containing p. Using regularity, one can find a neighborhood V of x and a neighborhood W of C which are disjoint. Then V is a neighborhood of x in X contained in $Y \setminus W \subset U$. Furthermore, denoting the closure of V in Y by $\overline{V}$, the closure of V in X satisfies $X \cap \overline{V} \subset \overline{V} \subset Y \setminus W \subset U$. It is clear that the closure of V in X is compact, since X is a Hausdorff space.

The converse of the statement is clear, because the set $C = \overline{V}$ is a desired compact subspace of X for X to be locally compact. $\square$

As homeomorphism implies that the given two spaces are "topologically equivalent," one might expect one-point compactifications of two homeomorphic spaces to be homeomorphic.

Theorem 2.6.5. If $f : A \to B$ denotes a homeomorphism of locally compact Hausdorff spaces, then f extends to a homeomorphism of their one-point compactifications.

Proof. Let X and Y be the one-point compactifications of A and B, respectively. Define $\tilde{f} : X \to Y$ by

$$\tilde{f}(a) = f(a) \text{ if } a \in A, \text{ and } \tilde{f}(p) = q,$$

where p and q are the unique elements of $X \setminus A$ and $Y \setminus B$, respectively. Clearly, $\tilde{f}$ is a bijection. Also, $\tilde{f}$ is a continuous map; if $V \subset B$ is an open subset of B, then $\tilde{f}^{-1}(V) = f^{-1}(V)$ is open in A, hence $\tilde{f}^{-1}(V)$ is open in X; if $C \subset B$ is a compact subset of B, then $\tilde{f}^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is open in X (because $f^{-1}(C)$ is a compact subspace of A). It is easy to show that $\tilde{f}$ has a continuous inverse, which completes the proof. □

Example 2.6.6. We introduce some applications of Theorem 2.6.5.

(a) Since $\mathbb{R} \approx S^1 \setminus \{1\}$, the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1.

(b) Since $\mathbb{N} \approx K$, where $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, the one-point compactification of $\mathbb{N}$ is homeomorphic to $K \sqcup \{0\}$.

Chapter 3

Countability and separation axioms

3.1 Countability and separation axioms

3.1.1 Definitions regarding countability and separation axioms

Definition 3.1.1 (Countability axioms)**.** Let X be a topological space.

(a) X is said to have a countable base at the point $x \in X$, if there is a countable collection $\{B_n\}_{n\in\mathbb{N}}$ of open subsets of X with the following property:

For each open set $A \subset X$ there is a positive integer n such that B_n is contained in A.

(b) X is said to be a first-countable space (or to be first countable) if every point of X has a countable base.

(c) X is said to be a second-countable space (or to be second countable) if X has a countable basis.

Remark that the second countability implies the first countability.

Remark (A stronger property of second-countability). The definition of the second-countability states that a space has a countable basis. In fact, if the space X is second-countable and $\mathcal{C}$ is a basis of X, then $\mathcal{C}$ has a countable subcollection which is a basis of X.

Proof. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a coundtable basis of X. For each $m, n \in \mathbb{N}$, whenever it is possible, choose a member $C_{m,n} \in \mathcal{C}$ such that $B_m \subset C_{m,n} \subset B_n$. Indeed, for each $n \in \mathbb{N}$, there is such index $m \in \mathbb{N}$; for each point $p \in X$ and an index $n \in \mathbb{N}$ such that $p \in B_n$, there is a basis member $\gamma_n \in \mathcal{C}$ such that $p \in \gamma_n \subset B_n$, and there is an index $m \in \mathbb{N}$ such that $p \in B_m \subset \gamma_n$.

We first show that the collection $\mathcal{C}^* := \{C_{m,n}\}_{m,n}$ is a countable basis of X. The countability of $\mathcal{C}^*$ is obvious, and the observation in the preceeding paragraph implies that $\mathcal{C}^*$ covers X. Assume that a point $p \in X$ belongs to two members $C_{m,n}, C_{j,k} \in \mathcal{C}^*$. Let C_0 be a basis member of $\mathcal{C}$ such that $p \in C_0 \in C_{m,n} \cap C_{j,k}$, and let $l \in \mathbb{N}$ be an index such that $p \in B_l \subset C_0$. Using the idea introduced in the preceeding paragraph again, we can find an integer $i \in \mathbb{N}$ such that $p \in B_i$ and $C_{i,l}$ exists. The member $C_{i,l}$ is one of a basis member in $\mathcal{C}^*$ we have been seek to find.

Cleraly, since $\mathcal{C}^*$ is contained in $\mathcal{C}$, the topology generated by $\mathcal{C}^*$ is coarser than the topology on X. Conversrly, given $x \in X$ with a basis member $B_n \in \mathcal{B}$ containing x, there is a basis member $C_{l,n} \in \mathcal{C}$ containing x for some $l \in \mathbb{N}$. Therefore, $\mathcal{C}^*$ is a countable basis of X which is contained in $\mathcal{C}$. □

Definition 3.1.2 (Separation axioms)**.** Let X be a topological space. X is said to be a Kolomogrov space (or a T_1 space) if every finite subset of X is closed in X. In the rest definitions, assume X is a Kolomogrov space.

(a) X is said to be a Hausdorff space (or a T_2 space) if given two distinct points a and b in X, there are disjoint neighborhoods of a and b.

(b) X is said to be a regular space (or a T_3 space) if given a point $a \in X$ and a nonempty closed subset $B \subset X$ not containing a, there are disjoint neighborhoods of a and B.

(c) X is said to be a completely regular space (or a $T_{3\frac{1}{2}}$ space) if given a point $a \in X$ and a nonempty closed subset $B \subset X$ not containing a, there is a continuous function $f : X \to [0, 1]$ such that $f(a) = 1$ and $f(B) = \{0\}$.

(d) X is said to be a normal space (or a T_4 space) if given two nonempty disjoint subsets in X there are disjoint neighborhoods of those closed subsets.

Remark (Alternative definitions for some separabilities in terms of locality). Suppose X is a T_1 space. Some separabilities on X may be defined alternatively as following statements written in terms of locality:

(a) X is a regular space if and only if given a point $a \in X$ with its neighborhood U in X, there is a neighborhood of a in X whose closure in X is contained in U.

(b) X is a completely regular space if and only if given a point p of X and its neighborhood U in X, there is a continuous function $f : X \to [0, 1]$ such that $f(p) = 1$ and $f(X \setminus U) = \{0\}$.

(c) X is a normal space if and only if given a nonempty closed subset $B \subset X$ with its neighborhood U in X, there is a neighborhood of B in X whose closure in X is contained in U.

Proving the above equivalences is left as an exercise.

3.1.2 Basic properties of countabilities and separabilities

As one can expect from the fact that a basis of the topology plays a major role in theory, second-countability is a strong condition.

Proposition 3.1.3. Suppose X is a second-countable space.

(a) X is a Lindelöf space, i.e., every open covering of X has a countable subcover.

(b) X is a separable space, i.e., X has a coundtable dense subset.

Proof. Left as an exercise. □

The converse implications in Proposition 3.1.3 are sometimes valid. One of such a particular case is when the space is metrizable, which is also one of a desired property.

Proposition 3.1.4. For metrizable spaces, second countability, separability, and being a Lindelöf space coincide. In other words,

(a) every metrizable Lindelöf space is second countable.

(b) every metrizable separable space is second countable.

Proof. Let X be a metrizable space, and let d be a metric on X which induces the topology on X.

(a) For each $n \in \mathbb{N}$, the open cover $\{B_d\left(x, n^{-1}\right) : x \in X\}$ has a countable subcover; let each member of a countable subcover be denoted by $B^n_k = B_d(x^n_k, n^{-1})$ with $k \in \mathbb{N}$. We want to show that the collection $\mathcal{B} := \{B^n_k\}_{n,k\in\mathbb{N}}$ is a countable basis of the topology on X. Clearly, $\mathcal{B}$ is an open cover of X. If a point $p \in X$ is contained in two members $B^{n_1}_{k_1}$ and $B^{n_2}_{k_2}$, let r be the positive number which is the minimum among the following four positive numbers:

$$d(p, x^{n_1}_{k_1}), \quad \frac{1}{n_1} - d(p, x^{n_1}_{k_1}), \quad d(p, x^{n_2}_{k_2}), \quad \frac{1}{n_2} - d(p, x^{n_2}_{k_2}).$$

Suppose n is large so that $\dfrac{2}{n} < r$, and let j be an index such that B^n_j contains p. Then B^n_j is a member of $\mathcal{B}$ such that $p \in B^n_j \subset B^{n_1}_{k_1} \cap B^{n_2}_{k_2}$. Thus, $\mathcal{B}$ is a basis of a topology on X. It is clear that the topology on X generated by $\mathcal{B}$ is coarser than the topology on X. The converse inclusion is also clear, when considering how $\mathcal{B}$ consists of. (Proving details are left as an exercise.)

(b) Let $D = \{a_n\}_{n\in\mathbb{N}}$ be a countable dense subset of X. The following countable collection

$$\{B_d(a_n, 1/k) : k \in \mathbb{N}\}$$

is a basis of the topology on X. (Think why.)

Hence, when the space is metrizable, Lindelöf condition or separability implies second-countability. □

We now introduce some inheritance properties.

Proposition 3.1.5 (Inheritance of countabilities)**.** Some countabilities are preserved as follows:

(a) First and second countabilities are inherited to subspaces and countable product spaces.

(b) Every closed subspace of a Lindelöf space is also a Lindelöf space.

(c) Every open subspace of a separable space is also a separable space.

Proof. In this note, we only prove part (c). Let D be a countable dense subset of the separable space X, and let A be an open subset of X. We want to check if the closure of $A \cap D$ in A is A; in other words, if $\overline{A \cap D} \cap A = A$. Suppose $a \in A$ and U is a neighborhood of a contained in A. Then U intersects D, hence U intersects $A \cap D$. □

Proposition 3.1.6 (Inheritance of separabilities)**.** (a) A subspace of a Hausdorff (regular, completely regular) space is a Hausdorff (regular, completely regular) spaces.

(b) A product of Hausdorff (regular, completely regular) spaces is a Hausdorff (regular, completely regular) spaces.

(c) For the normal space X, a closed subspace of X is normal.

(d) If the product space $\prod_{\alpha\in A} X_\alpha$ is a Hausdorff (regular, normal) space, then so is each X_α for $\alpha \in A$.

Proof. (a) Since the result is clear when the space X is a Hausdorff space, assume X is a regular space and Y is a subspace of X. Let x be a point of Y and B be a closed subspace of Y not containing x. Because $Y \cap \overline{B} = B$, $x \notin \overline{B}$. Using regularity, x and $\overline{B}$ can be separated by two disjoint sets U and V open in X. Thus, $Y \cap U$ and $Y \cap V$ form a desired pair.

(b) We also prove for regular case only. Suppose $\{X_\alpha\}_{\alpha\in A}$ is a collection of regular spaces, and let $X = \prod_{\alpha\in A} X_\alpha$. Suppose $p \in X$ and $U \subset X$ is a neighborhood of p in X. Let $\prod_{\alpha\in A} V_\alpha \subset X$ be a basis member containing p and contained in U. (Here, $\{\alpha \in A : V_\alpha \neq X_\alpha\} = \{\alpha_1, \cdots, \alpha_n\}$.) For each $\alpha \in A$, use regularity to find a neighborhood W_α of p_α in X_α whose closure in X_α is contained in V_α, with $W_{\alpha_i} = X_{\alpha_i}$ for $i = 1, \cdots, n$. Then, $\prod_{\alpha\in A} W_\alpha$ is a neighborhood of p in X, and its closure satisfies

$$\overline{\prod_{\alpha\in A} W_\alpha} = \prod_{\alpha\in A} \overline{W_\alpha} \subset \prod_{\alpha\in A} V_\alpha \subset U.$$

Therefore, the product space X is also a regular space.

(c) Suppose Y is a closed subspace of the normal space X. If A and B are disjoint closed subspaces of Y, then they are closed in X, too. By normality, there are disjoint open subspaces U and V in X containing A and B, respectively. So $Y \cap U$ and $Y \cap V$, which are open subspaces of Y, separate A and B. Therefore, a closed subspace of a normal space is normal.

The proof for complete regular spaces and the proof of (d) are not given in this section. □

3.1.3 Examples of regular spaces

Example 3.1.7 (Ordered spaces are regular)**.** Let X be an ordered space, and let $U = (a, b)$ be a basis member in X containing $p \in X$.

(i) Suppose U consists of only one point p. Then, U is both open and closed in X, so we may choose $V = U$.

(ii) Suppose that (a, p) or (p, b) is empty. Without loss of generality, we assume that (p, b) is empty and (a, p) is nonempty. Choose a point $a' \in (a, p)$ and let $V = (a', b) = (a', p]$. Any point $x \in X$ with $x > p$ does not belong to the closure of V in X; (p, ∞) does not intersect U. Any point $x \in X$ with $x < a'$ does not belong to the closure of V in X; $(-\infty, a')$ does not intersect V. Hence, the closure of V in X is contained in $[a', p] \subset U$.

(iii) Suppose (a, p) and (p, b) are nonempty. Let a' and b' be points of X in (a, p) and (p, b), respectively, and let $V = (a', b')$. Clearly, V is a neighborhood of p in X, and it is now easy to check that the closure of V in X is contained in U.

By from (i) to (iii), every ordered space is a regular space. In fact, it is known that every ordered space is a normal space, whose proof will not be introduced in this note.

Example 3.1.8 (Locally compact Hausdorff spaces are regular)**.** Suppose X is a locally compact Hausdorff space. Without loss of generality, we may assume X is not compact. (You will see why.) Let Y be the one-point compactification of X. Since Y is a compact Hausdorff space, Y is normal and regular, as illustrated in the following subsection. Hence, X, a subspace of Y, is also regular.

3.1.4 Examples of normal spaces

Theorem 3.1.9. Lindelöf regular spaces are normal.

Proof. Let A and B be disjoint closed subspaces of X. For each $a \in A$, let S_a be a neighborhood of a in X contained in $X \setminus B$. Using regularity, let U_a be a neighborhood of a in X whose closure in X is contained in S_a. And construct V_b for each $b \in B$ as we constructed U_a for each $a \in A$.

Even if $\{U_a\}_{a \in A}$ and $\{V_b\}_{b \in B}$ are open covers of A and B, respectively, the unions of the members in each collection need not be disjoint. (By drawing a picture on a sketchbook) we may set

$$F_n := U_n \setminus \bigcup_{k=1}^{n} \overline{V_k}, \quad G_n := V_n \setminus \bigcup_{k=1}^{n} \overline{U_k}$$

for each $n \in \mathbb{N}$, and

$$F := \bigcup_{n=1}^{\infty} F_n, \quad G := \bigcup_{n=1}^{\infty} G_n.$$

We wish F and G to be disjoint neighborhoods of A and B, respectively. Clearly, F and G are neighborhoods of A and B, since

$$F = \bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} \overline{V_n} \supset A, \quad G = \bigcup_{n=1}^{\infty} V_n \setminus \bigcup_{n=1}^{\infty} \overline{U_n} \supset B.$$

Also, $F \cap G = \varnothing$; otherwise, $F_n \cap G_k \neq \varnothing$ for some $n, l \in \mathbb{N}$. Without loss of generality, we may assume $n \leq k$. If $p \in F_n \cap G_k$, then $p \in F_n \subset U_n$; on the other hand, G_k does not intersect $\overline{U_n}$, hence $p \notin G_k$.

Therefore, F and G separate A and B and X is a normal space. □

Theorem 3.1.10. Metrizable spaces are normal.

Proof. Let X be a metrizable space and d be a metric on X inducing the topology on X. Let A and B be nonempty and disjoint closed subspaces of X. For each $a \in A$ and $b \in B$, let s_a and t_b be positive real numbers such that $B_d(a, s_a) \subset X \setminus B$ and $B_d(b, t_b) \subset X \setminus A$. Define

$$U := \bigcup_{a \in A} B_d(a, s_a/3), \quad V := \bigcup_{b \in B} B_d(b, t_b/3).$$

Then U and V are neighborhoods of A and B in X. If $U \cap V$ is nonempty, then $B_d(a, s_a/3) \cap B_d(b, t_b/3)$ contains a point $z \in X$. We then have the following inequality:

$$d(a, b) \leq d(a, z) + d(z, b) < \frac{s_a}{3} + \frac{t_b}{3} < \max\{s_a, t_b\},$$

so $b \in B_d(a, s_a)$ or $a \in B_d(b, t_b)$, a contradiction. Therefore, $U \cap V$ is empty, proving that X is normal. □

Theorem 3.1.11. Well-ordered sets are normal. (In fact, every ordered space is normal.)

Proof. Let X be a well-orderd set.
(Step 1: Proving that every subspace of X of the form $(x, y]$ is open in X)

Let $A = (x, y]$ be a subset of X. Then $X \setminus A = (-\infty, x] \sqcup (y, \infty)$; because (y, ∞) has the least element $y' \in X$, $X \setminus A = (-\infty, x] \sqcup [y', \infty)$ is closed in X.
(Step 2: Proving that X is normal)

Let A and B be nonempty disjoint closed subspace of X.

(i) Assume that neither of them contains the least element m of X. For each $a \in A$, there is a neighborhood $(x_a, a]$ of a in X not intersecting B; for each $b \in B$, there is a neighborhood $(y_b, b]$ of b in X not intersecting A. Define

$$U := \bigcup_{a \in A} (x_a, a], \quad V := \bigcup_{b \in B} (y_b, b].$$

They are open in X and U covers A, and V covers B. If U and V are not disjoint, $(x_a, a] \cap (y_b, b]$ is nonempty for some $a \in A$ and $b \in B$, Without loss of generality, we may assume $a < b$; then we have $a \in (y_b, b]$, a contradicton.

(ii) Suppose, without loss of generality, A contains m. Because the singletone $\{m\} = [m, m] = (-\infty, m]$ is not only closed but also open in X, $A \setminus \{m\}$ is also closed in X. By the preceeding part, there are disjoint neighborhood U and V of $A \setminus \{m\}$ and B, respectively. Then, $U \cup \{m\}$ and $V \setminus \{m\}$ are disjoint neighborhoods of A and B, respectively.

Therefore, every well-ordered space is normal. □

Example 3.1.12 ($\mathbb{R}_l$ is normal)**.** Suppose A and B are disjoint closed subspaces of $\mathbb{R}_l$. For each $a \in A$, let $[a, x_a)$ be a neighborhood of a in $\mathbb{R}_l$ contained in $\mathbb{R}_l \setminus B$; for each $b \in B$, let $[b, y_b)$ be a neighborhood of b in $\mathbb{R}_l$ contained in $\mathbb{R}_l \setminus A$. Now set

$$U := \bigcup_{a \in A} [a, x_a), \quad V := \bigcup_{b \in B} [b, y_b).$$

To show normality, it remains to show that U and V are disjoint. If $U \cap V \neq \varnothing$, then $[a, x_a) \cap [b, y_b)$ is nonempty for some $a \in A$ and $b \in B$. Since $a \neq b$, we may assume $a < b$; then $a < b < x_a < y_b$, and this implies that the point b of B is contained in $[a, x_a)$, a contradiction. Therefore, $\mathbb{R}_l$ is a normal space.

Example 3.1.13 (Compact Hausdorff spaces are normal)**.** Because closedness and compactness coincide in compact Hausdorff spaces and compact Hausdorff spaces are compactly normal, compact Hausdorff spaces are normal.

3.2 Countability and separation axioms (Problems)

Problem 3.2.1. Show that the suggested alternative definition for regular and normal spaces are equivalent to the original definitions.

Solution. Suppose X is a regular space. We first show part (a). Given a point $a \in X$ with its neighborhood U in X, consider the closed subspace $B := X \setminus U$ of X. Using the regularity of X, one can find a neighborhood V of a in X and a neighborhood W of B in X which are disjoint. Then $\overline{V} \subset X \setminus W \subset X \setminus B = U$, proving the only if part. To show the if part, suppose whenever a point a of X with its neighborhood U in X is given, there is a neighborhood of a in X whose closure in X is contained in U exists. Given a point $x \in X$ and a closed subspace B of X not containing x, consider the neighborhood $U := X \setminus B$ of x in X. Let V be a neighborhood of x in X whose closure is contained in U. Then V and $X \setminus \overline{V}$ are disjoint open subsets of X separating x and B, proving the if part.

Part (b) is almost clear, and part (c) follows if we replace points of X with closed subspaces of X.

Problem 3.2.2 (Tube lemma for Lindelöf spaces)**.**

Solution.

Problem 3.2.3. Prove Proposition 3.1.6 for complete regular spaces.

(a) Let X be a completely regular space and Y be a subspace of X. If $a \in Y$ and B is a closed subspace of Y not containing a, then $B = Y \cap \overline{B}$, so $a \notin \overline{B}$. Therefore, there is a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(\overline{B}) = \{1\}$. The restriction of f on Y separates a and B.

(b) Suppose $\{X_\alpha\}_{\alpha \in A}$ is a collection of completely regular spaces and write $X = \prod_{\alpha \in A} X_\alpha$. Let a be a point of X and B be a closed subspace of X not containing a. Then a is contained in the open subspace $X \setminus B$, so there is a basis member $\prod_{\alpha \in A} O_\alpha$ about a in X contained in $X \setminus B$, with $O_\alpha \neq X_\alpha$ if and only if $\alpha = \alpha_1, \cdots, \alpha_n$. For each α_i $(i = 1, \cdots, n)$, let $f_i : X_{\alpha_i} \to [0, 1]$ be a continuous function such that $f_i(a_{\alpha_i}) = 1$ and $f_i(X_{\alpha_i} \setminus O_{\alpha_i}) = \{0\}$. Then, define the function $f : X \to [0, 1]$ by $f(x) = f_1(x_{\alpha_1}) \times \cdots \times f_n(x_{\alpha_n})$ for $x \in X$. The function f is continuous and $f(a) = 1$, and f maps B onto $\{0\}$.

Problem 3.2.4. Prove part (d) of Proposition 3.1.6.

Solution. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of T_1-spaces such that the product space $X := \prod_{\alpha \in A} X_\alpha$ is a Hausdorff (regular, normal) space. We show that each X_α is also a Hausedorff (regular, normal) space.

(1) When X is a Hausdorff space and p, q are disjoint points of X, let U_β and V_β be disjoint open sets in X_β separating p_β and q_β, where $\beta \in A$. Then $\prod_{\alpha \in A} U_\alpha$ and $\prod_{\alpha \in A} V_\alpha$ are desired sets, where $U_\alpha = X_\alpha$ and $V_\alpha = X_\alpha$ when $\alpha \neq \beta$.

(2) Suppose X is a regular space, and let p be a point of X. Given $\beta \in A$, let U_β be a neighborhood of p_β in X_β and $U_\alpha = X_\alpha$ for all $\alpha \neq \beta$. Since $\prod_{\alpha \in A} U_\alpha$ is a neighborhood of p in X, by normality, there is a basis member $\prod_{\alpha \in A} V_\alpha$ containing p with the closure in X contained in $\prod_{\alpha \in A} U_\alpha$. Projecting onto X_β, we find that V_β is a neighborhood of p_β whose closure is contained in U_β. Therefore, if the product space is regular, then so is each X_α.

(3) Given $\beta \in A$, let B_β be a closed subspace of X_β and let $B_\alpha = X_\alpha$ for all $\alpha \neq \beta$. Also, let U_β be a neighborhood of B_β in X_β and let $U_\alpha = X_\alpha$ for all $\alpha \neq \beta$. Then, the subspace $B := \prod_{\alpha \in A} B_\alpha$ is a closed subspace of X contained in the open subspace $U := \prod_{\alpha \in A} U_\alpha$ of X. By normality, there is a neighborhood V of B in X whose closure in X is contained in U.

Define $F_\beta = \prod_{\alpha \in A} Z_\alpha$, where $Z_\beta = X_\beta$ and $Z_\alpha = \{x_\alpha\}$ for some $x_\alpha \in X_\alpha$ when $\alpha \neq \beta$. Then, the map $\imath : X_\beta \to F_\beta$ defined by

$$(\pi_\alpha \circ \imath)(p) = \begin{cases} p & \text{if } \alpha = \beta \\ x_\alpha & \text{otherwise} \end{cases}$$

for all $p \in X_\beta$ is a homeomorphism. Note that $F_\beta \cap V$ is open in F_β and contains $F_\beta \cap B$ (which is closed in F_β) and that the closure of $F_\beta \cap V$ in F_β is contained in $F_\beta \cap U$ (which is open in F_β). Because $\imath^{-1}(F_\beta \cap B) = B_\beta$ and $\imath^{-1}(F_\beta \cap V) = V_\beta$, the open subspace $\imath^{-1}(F_\beta \cap V)$ of X_β is a neighborhood of B_β in X_β whose closure in X_β is contained in U_β. Therefore, X_β is a normal space.

Problem 3.2.5. Let $f, g : X \to Y$ be continuous maps, where Y is a Hausdorff space. Show that $\{x \in X : f(x) = g(x)\}$ is a closed subspace of X.

Solution.

3.3 The Urysohn lemma

Theorem 3.3.1 (Urysohn lemma)**.** If X is a normal space, then any two (nonempty) disjoint closed subspaces of X can be separated by a continuous function on X. To be precise, if A and B are (nonempty) disjoint closed subspaces of X, there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Remark. The Urysohn lemma implies that if every pair of closed sets in X can be separated by disjoint open sets, i.e., X is normal, then each such pair can be separated by a continuous function. The converse is trivial. (Why?)

Proof of Theorem 3.3.1. Let A and B be nonempty and disjoint closed subsets of X.

Step 1. Let P be the set of rational numbers in $[0,1]$ and write $P = \{x_1, x_2, x_3, \cdots\}$ with $x_1 = 1, x_2 = 0$. Now define an open subset U_p for $p \in P$ as follows:

(1) Fitst, let $U_1 := X \setminus B$, which contains A. Second, by normality, we can find a neighborhood U_0 of A in X whose closure is in U_1.

(2) Suppose we have constructed U_p's for $p \in P_n$, where $P_n = \{x_1, \cdots, x_n\}$ for $n \geq 3$ such that $p, q \in P_n$ with $p < q$ implies

$$\overline{U_p} \subset U_q. \tag{3.1}$$

For convinience, denote $r = p_{n+1}$. We want U_r to satisfy eq. (3.1) with all U_p's with indices in P_{n+1}. For this, it suffices to care the immediate successor s and predecessor p of r in P_{n+1}; if U_r satisfies eq. (3.1) with U_p and U_s, then U_r satisfies eq. (3.1) with all the other U_p's with $p \in P_{n+1}$.

Since U_p and U_s satisfies $\overline{U_p} \subset U_s$, by normality, we can find a neighborhoof U_r of the closed subspace $\overline{U_p}$ in X such that $\overline{U_r} \subset U_s$. For such U_r, eq. (3.1) is satisfied for every pair of U_p's with $p \in P_{n+1}$.

By induction, we have U_p defined for all $p \in P$.

Step 2. We now define U_p for all rational numbers p as follows: just let $U_p = \varnothing$ whenever $p < 0$ and $U_p = X$ whenever $p > 1$. For such U_p's with $p \in \mathbb{Q}$, eq. (3.1) is satisfied.

Step 3. For each point $x \in X$, define the set

$$C(x) := \{p \in \mathbb{Q} : x \in U_p\}.$$

For each x, the set $C(x)$ is bounded (below by a nonnegative real number, and above by 1). Thus, we may define the function $f : X \to [0,1]$ by $f(x) = \inf C(x)$ for $x \in X$.

Step 4. We now show that the function f is a continuous map mapping A onto $\{0\}$ and B onto $\{1\}$. First, since every point $a \in A$ belongs to U_0, $f(a) = 0$; since every point $b \in B$ does not belong to U_1, $f(b) = 1$. Before proving continuity, we note the following lemma:

$$x \in \overline{U_r} \text{ implies } f(x) \leq r, \text{ and } x \notin U_r \text{ implies } f(x) \geq r.$$

To check the continuity, fix a point $x \in X$ and let (a, b) be a neighborhood of $f(x)$ in $\mathbb{R}$. We want to find a neighborhood W of x in X whose image under f is contained in (a, b).[1] Using the density of $\mathbb{Q}$, we can find rational numbers c, d such that $a < c < f(x) < d < b$. If we can find a neighborhood of x in X whose image under f is a subset of $[c, d]$, the proof ends. By the above lemma, if $c < f(x) < d$, then $x \in U_d \setminus \overline{U_c}$. Hence, $W := U_d \setminus \overline{U_c}$ is a neighborhood of x in X. Furthermore, if $y \in W$, then $c \leq f(y) \leq d$, hence W is a neighborhood of x in X with the image in (a, b). Therefore, f is a continuous function from X into $[0, 1]$ mapping A into $\{0\}$ and B onto $\{1\}$. □

Problem 3.3.1. Prove the lemma introduced in the fourth step of the proof of Theorem 3.3.1.

Solution. Suppose first that $x \in \overline{U_r}$. Whenever $r < p$, since $x \in \overline{U_r} \subset U_p$, $C(x)$ contains $p \in \mathbb{Q}$ with $p > r$; $f(x) = \inf C(x) \leq r$. Conversely, if $x \notin U_r$, whenever $p < r$, since $\overline{U_p} \subset U_r$, $p \notin C(x)$ for all $p < r$ and it implies that $f(x) = \inf C(x) \geq r$.

The following problem deals with a property of completely regular spaces, and its solution does not require the Urysohn lemma.

Problem 3.3.2. Let X be a completely regular space and A, B be disjoint closed subspaces of X. Show that if A is compact, then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Solution. For each point $a \in A$, let $f_a : X \to [0, 1]$ be a continuous function such that $f_a(a) = 0$ and $f_a(B) = \{1\}$. Note that the collection $\{f_a^{-1}([0, r)) : a \in A\}$ is an open cover of A by sets open in X, where r is a real number such that $0 < r < \dfrac{1}{2}$. Thus, A can be covered by finitely many sets $f_{a_i}^{-1}([0, r))$ for $i = 1, \cdots, n$. Define the function $f : X \to [0, 1]$ by $f(x) = f_1(x) \times \cdots \times f_n(x)$ for $x \in X$. Then $0 \leq f(a) < r$ for all $a \in A$ and $f(B) = \{1\}$. If $g : [0, 1] \to [0, 1]$ is the function defined by

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < r \\ \dfrac{x - r}{1 - r} & \text{otherwise} \end{cases},$$

then the composite $g \circ f$ is a desired continuous map from X into $[0, 1]$.

In Problem 2.1.2, we have proved that every connected metrizable space having more than one point is uncountable. As we have studied in the preceeding section that every metrizable space is normal, one might want to generalize the result to a broader category of spaces. In the following problem, we will prove that every connected regular space with more than one point is uncountable.

Problem 3.3.3 (Nontrivial connected regular spaces are uncountable)**.** Prove that a connected normal space having more than one point is uncountable. And deduce that a connected regular space having more than one point is uncountable.

Solution. Suppose X is a connected normal space having more than one point. Since every singletone is closed, there is a continuous map $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$, where a, b are disjoint points of X. Since X is connected, $f(X)$ is connected, so $f(X) = [0, 1]$. Therefore, X is necessarily uncountable.

Assume there is a connected regular space X with more than one point which is countable. It is clear that X is Lindelöf, so X is a Lindelöf regular space; a normal space. It contradicts to the former result.

3.4 The embedding theorem

In this section, we introduce the embedding theorem, stating that every completely regular space can be embedded into a (possibly uncountable dimensional) Euclidean space $\mathbb{R}^I$ for some I. As an application of the embedding theorem, we will prove Urysohn metrization theorem, stating that every second-countable regular space is metrizable, using the fact that a countable dimensional Euclidean space $\mathbb{R}^{\mathbb{N}}$ is metrizable.

[1] When trying to prove continuuity with the original definition of continuity, you might encounter some technical problem.

Theorem 3.4.1 (The embedding theorem)**.** Let X be a completely regular space and let $\{f_\alpha\}_{\alpha\in I}$ be a collection of continuous functions from X into $\mathbb{R}$ separating points and closed subspaces of X. Define the function $F : X \to \mathbb{R}^I$ by $F = (f_\alpha)_{\alpha\in I}$. Then F denotes an embedding of X into $\mathbb{R}^I$. Furthermore, if each f_α maps X into $[0, 1]$, then F denotes an embedding of X into $[0, 1]^I$.

Proof. It is clear that F is continuous, since $\mathbb{R}^I$ (or $[0, 1]^I$) equips the product topology. If a and b are disjoint points of X, there are indices $i, j \in I$ such that $f_i(a) = 1$, $f_i(b) = 0$ and $f_j(a) = 0$, $f_j(b) = 1$, hence F is injective. Therefore, to show that F is an embedding of X, it remains to show that F is a homeomorphism from X into its image $F(X)$. For this, we need to show that F maps an open set in X onto an open set in $F(X)$.

Suppose U is an open subset of X, and let z be a point of $F(U)$, and let x be a point of U such that $F(x) = z$. We seek to find a neighborhood of z in $F(X)$ contained in $F(U)$. Let $i \in I$ be an index such that $f_i(x) > 0$ and $f_i(X \setminus U) = \{0\}$, and set $W = \pi_i^{-1}((0,\infty)) \cap F(X)$.

(a) Clearly, W is open in $F(X)$ and W contains z.

(b) If $y \in W$, there is a point $a \in X$ such that $F(a) = y$. Since $y \in \pi_i^{-1}((0,\infty))$, we have $f_i(a) = \pi_i(y) > 0$, so $a \notin X \setminus U$, implying $y \in F(U)$. Thus, W is a neighborhood of z in $F(X)$ which is contained in $F(U)$.

Therefore, F denotes an embedding of X into $\mathbb{R}^I$ (or into $[0, 1]^I$). □

We now introduce the Urysohn metrization theorem, whose statement is stronger than every second countable regular space is normal.

Theorem 3.4.2 (Urysohn metrization theorem)**.** Every second countable regular space is metrizable.

To prove this theorem, we first prove the following countability lemma.

Lemma 3.4.3. Let X be a second countable regular space. Then there is a countable family of continuous functions from X into $[0, 1]$ separating points of X from closed subsets of X.

Proof of Lemma 3.4.3. Let $\{B_n\}$ be a countable basis of the topology on X. Whenever possible, let $f_{m,n}$ be a continuous function from X into $[0, 1]$ such that $f(\overline{B_m}) = \{1\}$ and $f(X\setminus B_n) = \{0\}$, where $\overline{B_m} \subset B_n$ (this could be done by applying Urysohn lemma). The collection of such continuous function will separate points of X from closed subsets of X, and the collection is countable. □

Proof of Theorem 3.4.2. By the embedding theorem, X can be embedded into $\mathbb{R}^{\mathbb{N}}$. Since $\mathbb{R}^{\mathbb{N}}$ is a countable product of metrizable spaces, $\mathbb{R}^{\mathbb{N}}$ is metrizable, and its subspaces are also metrizable. □

In the following chapter, the embedding theorem is applied when studying the Stone-Čech compactification of a completely regular space, to which evert continuous function into a compact Hausdorff space extends continuously (and uniquely[2]).

[2] The uniqueness follows from the assumption that the codomain is a Hausdorff space.

Chapter 4

The Stone-Čech compactification

4.1 Compactification

In this section, we introduce a general definition of a compactification, unlike when we studied a one-point compactification.

Definition 4.1.1 (Compactification)**.** Given a space X, a space Y is called a compactification of X if

(a) Y is a compact Hausdorff space containing X as a subspace, and

(b) the closure of X in Y is Y.

Two compactifications Y_1 and Y_2 of X is said to be equivalent if there is a homeomorphism from Y_1 into Y_2 extending the identity map on X.

Remark. If the space X has a compactification Y, because X is a subspace of Y, X is necessarily completely regular.

It is natural to wonder if every completely regular space has a compactification. Hopefully, every completely regular space has a compactification, as justified by the lemma which will be introduced soon. The lemma states first that every space which can be embedded into a compact Hausdorff space has a compactification, and second that if such compactification is required to satisfy an "extension property" given as in the lemma, then such compactificaiton is unique up to equivalence.

Lemma 4.1.2 (Compactification extending an embedding)**.** Let X be a space such that there is an embedding h of X into a compact Hausdorff space Z. Then, there is a unique (up to equivalence) compactification Y of X with the following property: There is an embedding $H : Y \hookrightarrow Z$ extending h.[1]

$$\begin{tikzcd} X \arrow[r, hook, "\text{inclusion}"] \arrow[dr, hook, "h"'] & Y \arrow[d, hook, dashed, "H"] \\ & Z \end{tikzcd}$$

Such compactification Y is called the compactificaiton of X induced by h.

Proof. We first prove the existence of a compactification Y of X induced by $h : X \hookrightarrow Z$, and then we prove its uniqueness.

(a) Let $X_0 := h(X) \approx X$ and Y_0 be the closure $\overline{X_0}$ of X_0 in Z. To argue that Y_0 is a compactification of X_0, we check the axioms of compactificaitions. Because Y_0 is a closed subset of Z, Y_0 is a compact Hausdorff space. Also, Y_0 contains X_0 as a subspace; every open subset of X_0 is of the form $X_0 \cap O = X_0 \cap (Y_0 \cap O)$, where O is an open subset of Z. Finally, the closure of X_0 in Y_0 is clearly $Y_0 \cap \overline{X_0} = Y_0$. Therefore, Y_0 is a compactification of X_0.

[1] Here, Y is unique up to equivalence, but H need not be unique.

To find a compactification Y of X, we seek to find a space Y such that (X, Y) and (X_0, Y_0) are homeomorphic. Let A be any set disjoint from X which is in bijection with $Y_0 \setminus X_0$ (say $k : A \to Y_0 \setminus X_0$ is such a bijection), and define $Y = X \sqcup A$. Define a bijective map $H : Y \to Y_0$ by

$$\begin{aligned} H(x) &= h(x) \quad \text{if } x \in X, \\ H(a) &= k(a) \quad \text{if } a \in A. \end{aligned}$$

Topologize Y by declaring that $U \subset Y$ is open in Y if and only if $H(U)$ is open in Y_0. (Indeed, the collection induced by such declaration is a topology on Y.) This topologization makes H a homeomorphism. It is easy to justify that Y is a compactification of X; it is because H extends h and $h : X \to X_0$ and $H : Y \to Y_0$ are homeomorphisms. To be brief, it is because the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h", "\approx"'] \arrow[d, hook, "\imath"'] & X_0 \arrow[d, hook, "\imath_0"] \\ Y \arrow[r, "\approx", "H"'] & Y_0 \end{tikzcd},$$

where $\imath$ is the inclusion embedding of X into Y and $\imath_0$ is the inclusion embedding of X_0 into Y_0.

(b) Suppose Y_1 and Y_2 are compactifications of X which extends h to embeddings from Y_1 and Y_2 into Z, respectively. Denote such embeddings by $H_1 : Y_1 \hookrightarrow Z$ and $H_2 : Y_2 \hookrightarrow Z$, respectively. By restricting the codomains of H_1 and H_2, we find that $H_2 \circ H_1^{-1}$ denotes a homeomorphism between Y_1 and Y_2 extending the identity map on X. Therefore, Y_1 and Y_2 are equivalent.

Therefore, if X is a space with an embedding h into a compact Hausdorff space Z, then X has a compactification Y such that there is an embedding of Y into Z extending h, and such a compactification exist uniquely up to equivalence. $\square$

Remark. We now jestify that every completely regular space has a compactification. If X is a completely regular space, by the embedding theorem, X could be embedded into $[0,1]^I$ for some I, which is a compact Hausdorff space. Applying Lemma 4.1.2 to this situation gives a compactification of X. Therefore, a space X has a compactification if and only if X is completely regular.

4.2 The Stone-Čech compactification

It is highly recommended to review the embedding theorem before studying this section. Also, since this section covers a specific type of compactification, throughout this section we assume that X is a completely regular space, even if there is no mention about it.

It is known that finding a compactification Y of a (completely regular) space X to which every continuous map from X into $\mathbb{R}$ is continuously extended is a basic problem. Regarding this, if a given real-valued function on X were to be extended, then the function must have been bounded.

The idea we use to find such a compactification of X is to apply Lemma 4.1.2. To be precise, we first find an appropriate embedding of X in terms of all the bounded continuous functions on X. Lemma 4.1.2 then asserts the existence of an extended embedding of a compactification of X, which surely is in terms of the "extended" bounded continuous functions on Y. This idea is applicable, since we know from the embedding theorem that a completely regular space can be embedded into $[0,1]^I$ for some I, which is a compact space.

For convinience, write $I = C_b^0(X, \mathbb{R})$ and for each $\alpha \in I$, let $B_\alpha = [\inf(\alpha), \sup(\alpha)] \subset \mathbb{R}$, and define

$$B = \prod_{\alpha \in I} B_\alpha.$$

Since I separates points of X from closed subsets of X, the map $F : X \to B$ defined by $F = (f)_{f \in I}$ is an embedding of X into the compact Hausdorff space B. By Lemma 4.1.2, there is a (unique) compactification Y of X such that F extends to an embedding F_* of Y into B, and the f-component f_* of F_* is a desired extension of f, since $f_* = \pi_f \circ F_*$ is continuous. The uniqueness of an extension of f easily follows from the assumption that the codomain $\mathbb{R}$ is a Hausdorff space.

To sum up, by considering an embedding F with regard to $C_b^0(X, \mathbb{R})$ and considering the compactification Y of X induced by F, we could show that every function in $C_b^0(X, \mathbb{R})$ can be uniquely extended to a continuois function on Y. The above observation is summarized as the following proposition:

Proposition 4.2.1 (Real-version of Stone-Čech compactification)**.** Let X be a completely regular space. There is a compactification Y of X satisfying the following property: Any function $f \in C_0^b(X, \mathbb{R})$ extends continuously to Y uniquely.

What we have shown is the existence of a compactification Y of X to which every bounded real-valued continuous function on X extends continuously to Y uniquely. Our goal is to find a compactification satisfying a more general property, which turns out to be a universal property:

Theorem 4.2.2 (Universal property of the Stone-Čech compactification)**.** Let X be a completely regular space. Then, there is a unique (up to equivalence) compactification $\beta(X)$ of X satisfying the following universal property: For any compact Hausdorff space K and a continuous map $f : X \to K$, there is a unique continuous map $f_* : \beta(X) \to K$ extending f. In other words, the pair $(\beta(X), \imath)$ ($\imath$ is the inclusion embedding of X into $\beta(X)$) makes the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, hook, "\imath\,\text{(the inclusion)}"] \arrow[dr, "\text{continuous}", "f"'] & \beta(X) \arrow[d, dashed, "\text{continuous}"', "f_*"] \\
& K
\end{tikzcd}.$$

We call such an extension $\beta(X)$ (or the pair $(\beta(X), \imath)$) the Stone-Cěch compactification of X.

If there exists such Y for each X, then the Stone-Čech compactification can be defined by the above universal property. For this, one needs to show the existence part, rather than the uniqueness (up to equivalence) part.[2] The following proposition states that a compactification Y of X in Proposition 4.2.1 is, in fact, a compactification of X to which every continuous map from X to a compact Hausdorff space extends continuously and uniquely.

Proposition 4.2.3 (Existence part)**.** Let X be a completely regular space and Y be the compactification of X satisfying the property in Proposition 4.2.1. Then every continuous map from X to a compact Hausdorff space extends to Y continuously and uniquely. To be precise, for any compact Hausdorff space K and a continuous map $f : X \to K$, there is a unique continuous map $f_* : Y \to K$ extending f.

Proof. Remark that a compact Hausdorff space is completely regular, so we can embed K into $[0,1]^I$ for some I (let $e : K \hookrightarrow [0,1]^I$ be such an embedding). So we consider the composition $g := e \circ f : X \to e(K) \subset [0,1]^I$ rather than f.

For each $\alpha \in I$, find the unique extension $(g_\alpha)_* : Y \to [0,1]$ of $g_\alpha : X \to [0,1]$. Then, the map $g_* : Y \to [0,1]^I$ defined by $g_* = ((g_\alpha)_*)_{\alpha \in I}$ is the unique continuous extension of $g : X \to [0,1]^I$.

Since we have composited e at the left of f to obtain g, to assert that $f_* := e^{-1} \circ g_*$ is a desired extension of f, we need to show that f_* is well-defined. Then, it naturally follows that f_* is a unique continuous extension of f. Because the closure of X in Y is Y, we have

$$g_*(Y) \subset \overline{g_*(X)} = \overline{g(X)} \subset \overline{e(K)} = e(K).$$

Therefore, f_* is well-defined, so f_* is a desired extension of $f : X \to K$. The uniqueness follows from the assumption that K is a Hausdorff space. $\square$

Problem 4.2.1. Show that a Stone-Cěch compactification of a completely regular space X is unique up to equivalence.

[2] Any object defined by a universal property exists uniquely up to some sense (for example, isomorphism or equivalence).

Solution. Let Y_1 and Y_2 be two Stone-Cěch compactifications of X, and let $\imath_k$ denote the inclusion embedding of X into Y_k for $k = 1, 2$. Clearly, each embedding is continuous. Hence, the universal property of Y_1 gives the continuous extension $(\imath_2)_\star : Y_1 \to Y_2$ of $\imath_2$ and the universal property of Y_2 gives the continuous extension $(\imath_1)_* : Y_2 \to Y_1$, and these extensions surely extends the identity map on X. Since the universal property of Y_1 gives a unique extension, the extension $(\imath_1)_* : Y_1 \to Y_1$ of $\imath_1$ is forced to be the identity map on Y_1. Therefore, $(\imath_1)_\star \circ (\imath_2)_*$ is the identity map on Y_1. Similarly, $(\imath_2)_* \circ (\imath_1)_\star$ is the identity map on Y_2. Therefore, $(\imath_1)_\star$ and $(\imath_2)_*$ are homeomorphisms, proving that Y_1 and Y_2 are equivalent. See the following commutative diagrams:

$$
\begin{tikzcd}
 & Y_1 \arrow[d, "(\imath_2)_*"'] \arrow[dd, bend left, "(\imath_1)_* = \mathsf{id}_{Y_1}"] \\
X \arrow[ur, hook, "\imath_1"] \arrow[r, hook, "\imath_2"'] \arrow[dr, hook, "\imath_1"'] & Y_2 \arrow[d, "(\imath_1)_\star"'] \\
 & Y_2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & Y_2 \arrow[d, "(\imath_1)_\star"'] \arrow[dd, bend left, "(\imath_2)_\star = \mathsf{id}_{Y_2}"] \\
X \arrow[ur, hook, "\imath_2"] \arrow[r, hook, "\imath_1"'] \arrow[dr, hook, "\imath_2"'] & Y_1 \arrow[d, "(\imath_2)_*"'] \\
 & Y_1
\end{tikzcd}
.
$$

Chapter 5

Complete metric spaces and the space of continuous maps

5.1 Complete metric spaces

5.1.1 Complete metric spaces and uniform metrics

Definition 5.1.1 (Complete metric space)**.** Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in X is called a Cauchy sequence, if given $\epsilon > 0$, there is an integer $N > 0$ such that $n, m > N$ implies $d(x_n, x_m) < \epsilon$. The space X is called a complete metric space if every Cauchy sequence in X is convergent.

Remark. (a) (Reduction of criterion) Let (X, d) be a metric space. X is a complete metric space if and only if every Cauchy sequence in X has a convergent subsequence.

(b) Suppose that (X, d) is a complete metric space. Then a closed subspace of X is complete: every Cauchy sequence in a closed subspace C of X is a Cauchy sequence in X, hence it has the limit, which should belong to C. Conversely, a complete subspace of X is closed in X: If A is a complete subspace of X and p is a limit point of A, we can construct a Cauchy sequence of points in A which converges to p, and completeness implies that $p \in A$. Therefore, for a complete metric space, closedness and completeness of subspaces coincide.

(c) The product of countably many complete metric spaces is, under the induced D-metric, complete: Note that a sequence $(x_n)_n$ converges to x in the product space if and only if $(\pi_k(x_n))_n$ converges to x_n for all k. Because $(x_n)_n$ is a Cauchy sequenc in the product space, it can be easily shown that each $(\pi_k(x_n))_n$ is a Cauchy sequence in the k-th complete metric space.

Contrary to the part (c) in the preceeding remark, one cannot assert that $\mathbb{R}^I$ is complete for arbitrary I, when $\mathbb{R}^I$ equips the product topology.

Definition 5.1.2 (Uniform metric)**.** Given a metric space (X, d) and an index set I, we define the uniform metric $\overline{\rho}$ corresponding to d on X^I as follows:

$$\overline{\rho} : X \times X \to [0, \infty), \quad (x, y) \mapsto \sup\{\overline{d}(x_\alpha, y_\alpha) : \alpha \in I\}.$$

One should remark that the topology induced by the uniform metric if finer than the product topology and coarser than the box topology.

In particular, if $f, g : I \to X$, then $\overline{\rho}(f, g) = \sup\{d(f(a), g(a)) : a \in I\}$.

Theorem 5.1.3. If (Y, d) is a complete metric space, then $(Y^X, \overline{\rho})$ is complete, where X is an index set.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y^X. We first assert that $(f_n)_n$ is pointwise convergent; because $\overline{\rho}(f_n, f_m) \to 0$ as $n, m \to \infty$, for each $x \in X$, $(f_n(x))_n$ is a Cauchy sequence in Y, justifying the assertion. We now prove that $f_n \to f$ in $\overline{\rho}$, i.e., uniformly.[1] Let N be an integer such that $n, m \geq N$

[1] Indeed, the convergence in $\overline{\rho}$ and the uniform convergence coincide.

implies $\overline{\rho}(f_n, f_m) < \epsilon/2$; given $x \in X$, let p be an integer not less than N such that $d(f_p(x), f(x)) < \epsilon/2$. Then,

$$d(f_n(x), f(x)) \leq d(f_n(x), f_p(x)) + d(f_p(x), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon, \tag{5.1}$$

so $\overline{\rho}(f_n, f) \leq \epsilon$. Therefore, $f_n \to f$ uniformly. □

Theorem 5.1.4 (Complete spaces of functions). Let X be a topological space, (Y, d) be a complete metric space, and assume that Y^X equips the uniform metric $\overline{\rho}$ corresponding to d.

(a) $C(X, Y)$ and $B(X, Y)$ are closed subspaces of Y^X.

(b) Hence, if Y is complete, then $C(X, Y)$ and $B(X, Y)$ are complete.

Proof. Because part (b) follows directly from part (a), it suffices to prove part (a). Assume first that $f \in Y^X$ is a limit point of $C(X, Y)$. Then, there is a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous functions from X into Y, which converges to f uniformly. Because the convergence is uniform, f is necessarily continuous. The proof for $B(X, Y)$ is also easy, which is left as an exercise. □

For example, when X is compact, we may impose the supremum(sup) metric

$$\|\cdot\| : C(X, Y) \times C(X, Y) \to [0, \infty), \quad (f, g) \mapsto \sup\{d(f(x), g(x)) : x \in X\}$$

on $C(X, Y)$, and such imposition is general.

5.1.2 Isometric embedding of a metric space

Recall that we call a map h from a metric space (X, d_X) into (Y, d_Y) such that $d_Y(h(a), h(b)) = d_X(a, b)$ for all $a, b \in X$ an isometric embedding of X into Y. If $h : X \to Y$ is an isometric embedding and Y is complete, the closure of $h(X)$ in Y is also a complete metric space, and we call this space the completion of X.

When trying to define objects, one should be interested in their existence, uniqueness, and exactness.

Theorem 5.1.5 (Existence of a completion of a metric space). Let (X, d) is a metric space, and consider the map $\phi : X \to B(X, \mathbb{R})$ defined by

$$\phi(x) = l_x \text{ for } x \in X,$$

where $l_x : X \to \mathbb{R}$ maps $t \in X$ to $d(t, x) - d(t, a)$ (here, a is a given point of X, so $|l_x(t)| \leq d(x, a)$ for all $t \in X$). Then ϕ denotes an isometric embedding of X into $B(X, \mathbb{R})$. Because $\mathbb{R}$ is complete, the closure of $\phi(X)$ in $B(X, \mathbb{R})$ is complete, i.e., every metric space has a completion.

Theorem 5.1.6 (Uniqueness of a completion of a metric space). Let $h_1 : X \hookrightarrow Y_1$ and $h_2 : X \hookrightarrow Y_2$ be isometric embeddings of X into complete metric spaces (Y_1, d_1) and (Y_2, d_2). Denote the closure of $h_i(X)$ in Y_i by $\overline{h_i(X)}$ $(i = 1, 2)$. Then, there is an isometry $H : \overline{h_1(X)} \to \overline{h_2(X)}$ which equals $h_2 \circ h_1^{-1}$ on $h_1(X)$.[2] Hence, the completion of X is unique up to an isometry.

$$\begin{tikzcd}
h_1(X) \arrow[d, hook] \arrow[rr, bend left, "h_2 \circ h_1^{-1}", "\text{isometry}"'] & X \arrow[l, "h_1", "\text{isometry}"'] \arrow[r, "h_2", "\text{isometry}"'] & h_2(X) \arrow[d, hook] \\
\overline{h_1(X)} \arrow[rr, "H", "\text{isometry}"'] \arrow[d, hook] & & \overline{h_2(X)} \arrow[d, hook] \\
Y_1 & & Y_2
\end{tikzcd}.$$

[2]By an isometry we mean an isometric homeomorphism.

Proof of Theorem 5.1.5. Since it is checked that $l_x \in B(X, \mathbb{R})$, the map ϕ is well defined. Thus, it remains to check that ϕ is an isometric embedding of X into $B(X, \mathbb{R})$. For this, it suffices to show that $\overline{\rho}(l_x, l_y) = d(x, y)$ for all $x, y \in X$, where $\overline{\rho}$ is the uniform metric on $B(X, \mathbb{R})$. Observing that $l_x(t) - l_y(t) = d(t, x) - d(t, y)$, we can find that $\overline{\rho}(l_x, l_y) \le d(x, y)$ and $l_x(x) - l_y(x) = -d(x, y)$, proving that $\overline{\rho}(l_x, l_y) = d(x, y)$. ☐

Proof of Theorem 5.1.6. One should remark that $\overline{h_i(X)}$ is a complete metric space ($i = 1, 2$). Define a map $H : \overline{h_1(X)} \to \overline{h_2(X)}$ as follows:

(1) When $x \in h_1(X)$, let $H(x) := h_2(h_1^{-1}(x))$. Since h_1 is a bijection between X and $h_1(X)$, the above setting is not ambiguous.

(2) When $x \in \overline{h_1(X)} \setminus h_1(X)$, set a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $h_1(X)$, converging to x (note that x is in a closure) and let $H(x) := \lim H(x_n)$.

    This setting seems to be ambiguous, because the resulting $H(x)$ might differ by the choice of $(x_n)_n$ (also, even the existence of the limit could be inquired).

    - As for the existence of the limit, note that $d_2(Hx_n, Hx_m) = d_1(x_n, x_m)$ so that the sequence $(Hx_n)_n$ is a Cauchy sequence in $\overline{h_2(X)}$.
    - As for the unambiguity, suppose $(a_j)_{j\in\mathbb{N}}$ is another sequence in $h_1(X)$ converging to x. Letting $X = \lim H(x_n)$ and $A = \lim H(a_j)$, we find that

    $$\begin{aligned} d_2(X, A) &\le d_2(X, H(x_n)) + d_2(H(x_n), H(a_j)) + d_2(H(a_j), A) \\ &\le d_2(X, H(x_n)) + d_1(x_n, a_j) + d_2(H(a_j), A) \xrightarrow{n,j\to\infty} 0, \end{aligned}$$

    so $X = A$ and the above setting of $H(x)$ is not ambiguous.

We just established a map H from $\overline{h_1(X)}$ into $\overline{h_2(X)}$. It remains to show that H is an isometry. For this, it suffices to show that H is bijective and isometric, and H coincides $h_2 \circ h_1^{-1}$ on $h_1(X)$.

(a) Given $a, b \in \overline{h_1(X)}$, let $(a_n)_n$ and $(b_n)_n$ be seqences in $h_1(X)$ converging to a and b, respectively. Because $d_2(Ha, Hb) \le d_2(Ha, Ha_n) + d_2(Ha_n, Hb_n) + d_2(Hb_n, Hb)$ and $d_2(Ha_n, Hb_n) \le d_2(Ha_n, Ha) + d_2(Ha, Hb) + d_2(Hb, Hb_n)$, we have $\lim d_2(Ha_n, Hb_n) = d_2(Ha, Hb)$. Therefore, $d_2(Ha, Hb) = \lim d_2(Ha_n, Hb_n) = \lim d_1(a_n, b_n) = d_1(a, b)$. This implies that H is isometric and injective. The surjectivity is easily checked. (How?)

(b) Clearly, H extends $h_2 \circ h_1^{-1}$.

Therefore, there is an isometry between two completions of X. In other words, there is a unique completion of a metric space up to an isometry. ☐

5.2 Pointwise equicontinuous collection of continuous maps

Definition 5.2.1 (Equicontinuity)**.** Let X be a topological space, (Y, d) be a metric space, and let $\mathcal{F}$ be a subset of $C(X, Y)$, i.e., a collection of continuous functions from X into Y.

(a) (Pointwise equicontinuity) The collection $\mathcal{F}$ is said to be equicontinuous at a point $x_0 \in X$ if, for any $\epsilon > 0$, there is a neighborhood U of x_0 in X such that $x \in U$ and $f \in \mathcal{F}$ implies $d(f(x), f(x_0)) < \epsilon$. If $\mathcal{F}$ is equicontinuous at every point of X, then $\mathcal{F}$ is said to be pointwise equicontinuous.

(b) (Uniform equicontinuity) Suppose that X is a metric space and d_X be the metric on X inducing the topology on X. The collection $\mathcal{F}$ is said to be uniformly equicontinuous if, for every $\epsilon > 0$, there is $\delta > 0$ such that $a, b \in X$ with $d_X(a, b) < \delta$ and $f \in \mathcal{F}$ implies $d(f(a), f(b)) < \epsilon$.

Remark. In the course of introduction to mathematical analysis, we studied the following version of Ascoli's theorem:

Suppose K is a compact metric space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued continuous functions (continuous functions with values in a complete metric space, respectivley) on K. If $(f_n)_n$ is pointwise bounded (totally bounded under the sup metric) and uniformly equicontinuous, then $(f_n)_n$ is uniformly bounded and $(f_n)_n$ contains a uniformly convergent subsequence.

Lemma 5.2.2. Suppose that X is a topological space and (Y, d) is a metric space, and assume $C(X, Y)$ equips the uniform topology. If $\mathcal{F} \subset C(X, Y)$ is totally bounded under the uniform metric, then $\mathcal{F}$ is pointwise equicontinuous under d.

Proof. Fix a positive real number $0 < \epsilon < 1$ and write $\mathcal{F} = \bigcup_{i=1}^{N} B_{\overline{\rho}}(f_i, \epsilon)$, where $f_i \in \mathcal{F}$ for each i. To show pointwise equicontinuity, we fix a point $p \in X$ and choose a neighborhood U_i of p in X such that $x \in U_i$ implies $d(f_i(x), f_i(p)) < \epsilon$; when $U = \bigcap_{i=1}^{N} U_i$, then U is still a neighborhood of p in X and whenever $x \in U$ we have $d(f_i(x), f_i(p)) < \epsilon$ for all i. Given $f \in \mathcal{F}$, choose an index i such that $\overline{\rho}(f, f_i) < \epsilon$. Because

$$d(f(x), f(p)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(p)) + d(f_i(p), f(p)) < 3\epsilon$$

whenever $x \in U$. Therefore, every totally bounded subset of $C(X, Y)$ is pointwise equicontinuous. □

The preceeding lemma states that a totally bounded (in the uniform metric) subset of continuous functions is pointwise equicontinuous. Its reverse is generally not true; for example, the countable collection $(f_n : \mathbb{R} \to \mathbb{R},\ x \mapsto n)_{n\in\mathbb{N}}$. The following lemma states that the converse to the preceeding lemma is valid if both domain and codomain are compact.

Lemma 5.2.3. Let X be a topological space and (Y, d) be a metric space. Assume that both X and Y are compact. If the subset $\mathcal{F}$ of $C(X, Y)$ is pointwise equicontinuous, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.

Remark. Suppose X is a topological space and (Y, d) is a metric space. When we consider the space $C(X, Y)$, we first consider the uniform metric $\overline{\rho}$. If X is assumed to be compact, we can also consider the sup metric, which will be denoted by ρ.

(a) As the following properties are particularly related to points within arbitrarily small distance and both $\overline{\rho}$ and ρ coincide at every pair of continuous functions within distance less than 1, they are common properties of $C(X, Y)$ under each metric: completeness, total boundedness. (For example, if $C(X, Y)$ under the uniform metric is complete, then so is $C(X, Y)$ under the sup metric, and vice versa.)

(b) The boundedness of $C(X, Y)$ is not common; while the space may be unbounded under the sup metric, the space is clearly bounded under the uniform metric; $C(X, Y) = B_{\overline{\rho}}(0, 2)$, where 0 in the first argument is the zero map.

Proof. According to the preceeding remark, it suffices to justify the total boundedness under the sup metric ρ.

Using pointwise equicontinuity, for each point $a \in X$, find a neighborhood D_a of a in X such that $x \in D_a$ and $f \in \mathcal{F}$ implies $d(f(x), f(a)) < \epsilon$. Since X is compact, we can cover X with finitely many D_a's, suppose $\{D_{a_i}\}_{i=1}^{k}$ covers X and write $U_i = D_{a_i}$. On the other hand, cover Y with finitely many open balls $V_1, \cdots, V_m$ of diameter less than ϵ.

Let J be the collection of all functions from $\{1, \cdots, k\}$ into $\{1, \cdots, m\}$. Given $\alpha \in J$, if there is a function $f \in \mathcal{F}$ such that $f(a_i) \in V_{\alpha(i)}$ for all $i = 1, \cdots, k$, choose one such function $f \in \mathcal{F}$ and label it f_α. The collection $\{f_\alpha\}$ is indexed by a subset I of J and is thus finite.

Choose a map $f \in \mathcal{F}$. Because V_j's $(j = 1, \cdots, m)$ cover Y, for each i, $f(a_i)$ belings to some V_j; write $f(a_i) \in V_{\beta(i)}$, and consider the function f_β. Given a point $x \in X$, there is an index i such that $x \in U_i$, and

$$d(f(x), f_\beta(x)) \leq d(f(x), f(a_i)) + d(f(a_i), f_\beta(a_i)) + d(f_\beta(a_i), f_\beta(x)).$$

The first and the last term in the right-hand side are, respectivley, less than ϵ (pointwise equicontinuity) and the second term is also less than ϵ (the diameter of V_i is less than ϵ), so $d(f(x), f_\beta(x)) < 3\epsilon$. This implies that $f \in B_\rho(f_\beta, 4\epsilon)$. Therefore, $\mathcal{F}$ is covered by the open balls $B_\rho(f_\alpha, 4\epsilon)$ for $\alpha \in I$, so $\mathcal{F}$ is totally bounded under the sup metric. □

The above theory, together with the theory developed in the following section, will be used in the last section when we prove Ascoli's theorem.

5.3 Topologies on the space of continuous functions

In this section, we first impose topologies on $C(X,Y)$ to which we can correspond pointwise convergence and uniform convergence. Throughout this section, unless stated otherwise, X and Y are assumed to be topological spaces.

5.3.1 Topology of pointwise convergence and uniform topology

Definition 5.3.1 (Topology of pointwise convergence)**.** The topology on Y^X generated by the collection of subsets

$$S(x,U) := \{f \in Y^X : f(x) \in U\} \quad (x \in X, U \text{ is open in } X)$$

is called the topology of pointwise convergence, or the point-open topology.

From now on, we say a sequence $(f_n)_{n\in\mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ pointwise when the sequence converges to f in the topology of pointwise convergence.

Remark. (a) Indeed, the topology on Y^X of pointwise convergence and the product topology on Y^X coincide; note that $S(x,U) = \pi_x^{-1}(U)$ for all $x \in X$ and open subspace U of X.

(b) The pointwise convergence coincides our old concept of pointwise convergence. To be specific, when $(f_n)_{n\in\mathbb{N}} \subset Y^X$, $f_n \to f$ pointwise if and only if $f_n(x) \to f(x)$ for each $x \in X$.

When considering uniform convergence, the codomain must be given as a metric space.

Definition 5.3.2 (Uniform topology)**.** Assume Y is a metric space with the metric d on Y. The uniform topology on Y is the topology on Y^X induced by the uniform metric $\overline{\rho}$ induced by d.

From now on, we say a sequence $(f_n)_{n\in\mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ uniformly when the sequence converges to f in the uniform topology.

Remark. (a) The uniform convergence coincides our old concept of uniform convergence.

(b) (Review of Theorem 5.1.4) Suppose $(f_n)_{n\in\mathbb{N}} \subset C(X,Y)$, where Y is a metric space. If $f_n \to f$ uniformly, then $f \in C(X,Y)$; it is because $C(X,Y)$ is a closed subspace of Y^X, when Y^X is assumed to equip the uniform metric.

(c) (Review of Lemma 5.2.2) If a subset of $C(X,Y)$ is totally bounded under the uniform metric, then the subset is pointwise equicontinuous under d.

Another remark: When X is a topological space and Y is a metric space, the uniform topology on Y^X is finer than the topology on Y^X of pointwise convergence.

5.3.2 Topology of compact convergence

Still, we assume that Y is a metric space.

Definition 5.3.3 (Topology of compact convergence)**.** The topology on Y^X generated by the collection of the subsets

$$B_C(f,\epsilon) := \left\{ g \in Y^X : \sup_{x\in C} d(f(x), g(x)) < \epsilon \right\} \ (C \text{ is a compact subspace of } X, f \in Y^X, \epsilon > 0) \quad (5.2)$$

as a basis is called the topology of compact convergence or the topology of uniform convergence on compact sets.[3]

Remark. The collection of sets in eq. (5.2) is indeed a basis of the topology on Y^X of compact convergence, and its justification is given here.

(1) It is clear that the collection covers Y^X.

(2) Given two basis members $B_C(f,a)$ and $B_K(g,b)$, where C and K are compact subspaces of X and $a,b>0$ and a point $u \in B_C(f,a) \cap B_K(g,b)$, we wish to find a basis member B such that $u \in B \subset B_C(f,a) \cap B_K(g,b)$.

   For a point $u \in B_C(f,a)$, if $\delta = a - \sup\{d(f(x),u(x)) : x \in C\} > 0$, then $B_C(u,\delta) \subset B_C(f,a)$.

   Using the above observation, we can find a small positive real numbera α such that $B_C(u,\alpha) \subset B_C(f,a)$ and $B_K(u,\alpha) \subset B_K(g,b)$, and $B_{C\cup K}(u,\alpha)$ is a desired basis member.

From now on, we say a sequence $(f_n)_{n\in\mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ compactly when the sequence converges to f in the topology of compact convergence, i.e., $f_n \to f$ uniformly on every compact subspace of X.

Theorem 5.3.4 (Inclusions regarding topologies on Y^X)**.** Assume (Y,d) is a metric space. Then,

$$\text{(uniform topology)} \supset \text{(topology of compact convergence)} \supset \text{(topology of pointwise convergence)}.$$

Furthermore, if X is compact, then the first two topologies coincide; if X is discrete, then the last two coincide.

Proof. We first show the inclusion.

(1) (The uniform topology is finer than the topology of compact convergence) Given a basis member $B_C(f,\epsilon)$ of the topology of compact convergence and its point g, we can find a positive real number $r<1$ such that $B_C(g,r) \subset B_C(f,\epsilon)$. It is clear that $g \in B_{\overline{\rho}}(g,r) \subset B_C(g,r) \subset B_C(f,\epsilon)$.

(2) (The topology of compact convergence is finer than the topology of pointwise convergence) Given a basis member $S(x, B_d(p,\epsilon))$ of the topology of pointwise convergence and its point g, because $d(g(x),p) < \epsilon$, there is a positive real number r such that $B_d(g(x),r) \subset B_d(p,\epsilon)$. Because $\{x\}$ is compact, it is clear that $g \in B_{\{x\}}(g,r/2) \subset S(x, B_d(p,\epsilon))$.

We now show the coincidence under each case. When X is compact, then $B_\rho(f,\epsilon)$ is a basis member of the topology of compact convergence as well as a basis member of the uniform topology, proving the first coincidence. When X is discrete and $B_C(f,\epsilon)$ is a basis member of the topology of compact convergence, the compact subspace C of X is necessarily a finite subset of X; hence, writing $C = \{x_1,\cdots,x_n\}$, we have $B_C(f,\epsilon) = \bigcap_{i=1}^n S(x_i, B_d(f(x_i),\epsilon))$. □

Before studying further properties of the topology of compact convergence, we introduce a topological space in which a subset is open if and only if its restriction to any compact subspace is open in the compact subspace.

Definition 5.3.5 (Compactly generated space)**.** A topological space X is said to be compactly generated if a subspace A of X is open when the following property is satisfied:

$$A \cap C \text{ is open in } C \text{ whenever } C \text{ is a compact subspace of } X.$$

Remark. We may replace the word "open" by "closed," by considering set complements.

Example 5.3.6. Some examples of compactly generated spaces are introduced here.

[3] The supremum of $d(f(x),g(x))$ for $x \in C$ is finite since C is compact.

(a) (Locally compact spaces) Suppose X is a locally compact space and let A be a subset of X such that $A \cap C$ is open in C whenever C is a compact subspace of X. We wish to justify that A is an open subspace of X.

Given $x \in A$, choose a neighborhood U of x in X that lies in a compact subspace C of X. Since $A \cap C$ is open in C, $A \cap U = (A \cap C) \cap U$ is open in U, and hence in X. Then $A \cap U$ is a neighborhood of x in X contained in A, so A is open in X.

(b) (First-countable spaces) Suppose X is a first-countable space and let B be a subset of X such that $B \cap C$ is closed in C whenever C is a compact subspace of X. We wish to justify that B is a closed subset of X.

Let x be a point of $\overline{B}$, where the overline notation means the closure in X. Since X has a countable base at x, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points in B converging to x. The subspace $C = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, so $B \cap C$ is closed in C. Because $B \cap C$ contains all x_n's, it also contains x, so $x \in B$. Therefore, B is closed.

For pointwise convergence, the limit map of continuous maps need not be continuous; this is valid for uniform convergence. For compact convergence, the assertion is valid when the domain is compactly generated, which is justified by the below theorem after a lemma.

Lemma 5.3.7. If X is compactly generated, then a map $f : X \to Y$ is continuous if and only if for each compact subspace C of X, the restriction $f|_C$ is continuous.

Proof. Only if part is obvious, and if part is also clear if one remarks that $(f|_C)^{-1}(U) = C \cap f^{-1}(U)$ whenever $U \subset Y$. □

Theorem 5.3.8. Let X be a topological space, (Y, d) be a metric space. Then $C(X, Y)$ is closed in Y^X in the topology of compact convergence. (Hence, if a sequence $(f_n)_{n\in\mathbb{N}} \subset C(X, Y)$ converges to f compactly, then f is continuous.)

Proof. We need to show that a limit point f of $C(X, Y)$ in Y^X belongs to $C(X, Y)$. By the preceeding lemma, it suffices to show that $f|_C$ is continuous whenever C is a compact subspace of X. Given a compact subspace C of X, by considering the neighborhoods $B_C(f, 1/n)$ of f in Y^X, we can find a sequence $(g_n^C)_{n\in\mathbb{N}}$ of points in $C(X, Y)$ such that $g_n^C \in B_C(f, 1/n)$. In this case, $g_n^C|_C \to f|_C$ uniformly, so $f|_C$ is continuous. □

Remark. The space $C(X, Y)$ (X is a topological space and (Y, d) is a metric space) is closed in Y^X in the uniform topology, or in the topology of compact convergence when it is further assumed that X is compactly generated.

5.3.3 Compact-open topology

When we introduced the uniform topology and the topology of compact convergence, we had to assume that Y is a metric space (and the topologies seem to depend on the choice of the metric on Y). It is natural to ask whether either of these topologies can be extended to the case where Y is an arbitrary topological space. It is known that there is no satisfactory answer to this question for the space Y^X; fortunately, one can prove something for its subspace $C(X, Y)$.

To study such case, in this subsection we assume X and Y are topological spaces.

Definition 5.3.9 (Compact-open topology)**.** The topology on $C(X, Y)$ generated by the collection of the subsets of the form

$$S(C, U) := \{f \in C(X, Y) : f(C) \subset U\} \quad (C \subset X \text{ is compact, } U \subset X \text{ is open})$$

as a subbasis is called the compact-open topology.

Remark. (a) The above definition of compact-open topology naturally extends to Y^X, which is not of our interest.

(b) Clearly, the compact-open topology is finer than the topology of pointwise convergence (the point-open topology).

Observation 5.3.10. When (Y, d) is a metric space, the compact-open topology on $C(X, Y)$ and the topology of compact convergence on $C(X, Y)$ coincide.

Proof. The following result helps in this proof:

Suppose A is a compact subspace of X and V is an open subspace of X containing A. Then, there is $\epsilon > 0$ such that V contains the ϵ-neighborhood of A.

We first prove that the topology of compact convergence is finer than the compact-open topology. Given a subbasis member $S(C, U)$ of the compact-open topology (C is compact and U is open in X) and its point f, note that f is continuous so that $f(C)$ is compact. Because $f(C) \subset U$, there is $\epsilon > 0$ such that the ϵ-neighborhood of $f(C)$ is contained in U. Therefore, $f \in B_C(f, \epsilon) \subset S(C, U)$.

Now we prove that the compact-open topology is finer than the topology of compact convergence. Given a basis member $B_C(f, \epsilon)$ of the topology of compact convergence and its point g, without loss of generality, it suffices to find a member A of the topology of compact convergence such that $f \in A \subset B_C(f, \epsilon)$.[4] For each point $x \in C$, there is a neighborhood U_x of x in X such that

$$t \in \overline{U_x} \text{ implies } d(f(t), f(x)) < \epsilon/3.^5$$

Using the compactness of C, choose finitely many $V_i = U_{x_i}$ $(i = 1, \cdots, n)$ covering C. If we write $C_i = \overline{V_i} \cap C$ for each i, the finite intersection $\bigcap_{i=1}^n S(C_i, B_d(f(x_i), \epsilon/3))$ contains f and is contained in $B_C(f, \epsilon)$. □

In the beginning of this subsection, it is mentioned as a statement that the compact convergence topology (as well as the uniform topology) seems to depend on the metric on the codomain. By Observation 5.3.10, we can obtain the following coincidences:

Corollary 5.3.11. If Y is a metric space (so that the compact convergence topology and the compact-open topology coincide), the compact convergence topology on $C(X, Y)$ is independent of the metric on Y. Hence, if it is further assumed that X is compact (so that the uniform topology and the compact convergence topology coincide), the uniform topology on $C(X, Y)$ is independent of the metric on Y.

Continuity of evaluation maps under the compact-open topology on $C(X, Y)$

We end this section with some theory which will be helpful when studying Ascoli's theorem in the following section.

Definition 5.3.12 (Evaluation maps)**.** Let A be a nonempty set and P be a collection of maps from A into a nonempty set B. The map $\mathrm{ev}_P : A \times P \to B$ defined by $\mathrm{ev}_P(a, f) = f(a)$ for $a \in A, f \in P$ is called the evaluation map with maps in P. If the context is clear, we may omit the subscript.

Example 5.3.13. Let X and Y be topological spaces and consider the evaluation map $\mathrm{ev} : X \times C(X, Y) \to Y$. Suppose further that there is a metric d on Y inducing the topology on Y. Show that, under the compact-open topology on $C(X, Y)$, that the evaluation map is continuous.

Theorem 5.3.14. Let X be a locally compact Hausdorff space and suppose $C(X, Y)$ equips the compact-open topology. Then the evaluation map $\mathrm{ev} : X \times C(X, Y) \to Y$ is continuous.

Proof. To show the continuity of the evaluation map, it suffices to show that there is a neighborhood of $(a, f) \in X \times C(X, Y)$ which is mapped into an arbitrarily given neighborhood W of $f(a)$ in Y. Using the continuity of f, choose a neighborhood U of a in X such that $f(U) \subset W$. By hypothesis, there is a neighborhood V of a whose closure in X is a compact subspace of U. The box $U \times S(\overline{U}, W)$ is a desired neighborhood of (a, f). □

[4] There is no loss of generality, since we can find a positive real number r such that $B_C(g, r) \subset B_C(f, \epsilon)$.

[5] By letting U_x be a neighborhood of x in X such that $t \in U_x$ implies $d(f(t), f(x)) < \epsilon/4$, the result follows from the inclusion $f(\overline{U_x}) \subset \overline{f(U_x)}$.

A consequence of this theorem is the theorem that follows.

Definition 5.3.15. Given a map $f : X \times Z \to Y$, there is a corresponding map $F : Z \to Y^X$, defined by $(F(z))(x) = f(x, z)$ for all $x \in X$ and $z \in Z$. Conversely, given $F : Z \to Y^X$, the above equation defines a corresponding map $f : X \times Z \to Y$. We say that F is the map of Z into $C(X, Y)$ which is induced by f.

Theorem 5.3.16. Let X and Y be topological spaces and give $C(X, Y)$ the compact-open topology. If $f : X \times Z \to Y$ is continuous, then so is its induced map $F : Z \to C(X, Y)$. The converse holds if X is a locally compact Hausdorff space.

Proof. We first prove that f is continuous when X is a locally compact Hausdorff space and F is continuous. Note that if $\alpha_i : A_i \to B_i$ denotes a continuous map for each $i = 1, \cdots, n$ then $\alpha_1 \times \cdot \alpha_n$ is also continuous, and remark the following commutative diagram:

$$\begin{tikzcd} X \times Z \arrow[r, "\mathrm{id}_Z \times F"'] \arrow[rr, bend left, "f"] & C(X,Y) \arrow[r, "\mathrm{ev}"'] & Y \end{tikzcd} .$$

We now prove that F is continuous if f is continuous. For this, we fix a point $z_0 \in Z$ and seek to find a neighborhood W of z_0 in Z such that $F(W) \subset S(C, U)$, where $S(C, U)$ is any subbasis member of the compact-open topology on $C(X, Y)$ which contains $F(z_0)$, i.e., $f(C \times \{z_0\}) \subset U$. By definition, for each $c \in C$, $f(c, z_0) \in U$; by continuity, there is an open box $A_c \times B_c$ in $X \times Z$ containing (c, z_0) which is mapped into U under f. Since $C \times \{z_0\}$ is compact, this slice can be covered by finitely many boxes of the form $A_c \times B_c$; write $C \times \{z_0\} \subset \bigcup_{i=1}^n (A_{c_i} \times B_{c_i})$ and $B = \bigcap_{i=1}^n B_{c_i}$. Clearly, B is a neighborhood of z_0 in Z and $f(C \times B) \subset U$, i.e., $F(B) \in S(C, U)$, as desired. $\square$

5.4 Ascoli's theorem

Throughout this section, we assume that X is a topological space and (Y, d) is a metric space, unless stated otherwise. In this section, we study a generalized Ascoli's theorem, which characterizes the compact subspaces of $C(X, Y)$ in the topology of compact convergence. Also, some particular versions of Ascoli's theorem will be introduced.

5.4.1 Generalized Ascoli's theorem

Theorem 5.4.1 (Ascoli's theorem)**.** Let X be a topological space and (Y, d) be a metric space. Give $C(X, Y)$ the topology of compact convergence, and let $\mathcal{F}$ be a subset of $C(X, Y)$.

(a) Suppose that $\mathcal{F}$ is pointwise equicontinuous under d and pointwise relatively compact. Then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$; to say equivalently, $\mathcal{F}$ is relatively compact.

(b) The converse of (a) is valid if X is a locally compact Hausdorff space.

Remark. For a Hausdorff space A, a subspace B of A is contained in a compact subspace of A if and only if A is relatively compact. Since if part is clear, we assume that B is contained in a compact subpsace K of A. Note that the closure $\overline{B}$ of B in X is contained in the closure $\overline{K}$ of K in X. Since A is a Hausdorff space, every compact subspace of A is closed, so $\overline{K} = K$. Hence, $\overline{A} \subset K$, implying that $\overline{A}$ is a closed subspace of K. Therefore, A is relatively compact.

Proof of (a). Impose Y^X the product topology, which equals to the topology of pointwise convergence. Note that Y^X is a Hausdorff space and that $C(X, Y)$ (which equips to topology of compact convergence) is not a subspace of Y^X. And let $\mathcal{G}$ denote the closure of $\mathcal{F}$ in Y^X. (It will be clear in Step 1 why we consider the closure of $\mathcal{C}$ in Y^X, not in $C(X, Y)$.)

Our proof then consists of the following four steps:

- Step 1: Showing that $\mathcal{G}$ is a compact subspace of Y^X.

- Step 2: Showing that each member of $\mathcal{G}$ is continuous, and $\mathcal{G}$ is pointwise equicontinuous under d.
- Step 3: The topology on Y^X and the topology on $C(X,Y)$ coincide on $\mathcal{G}$.
- Step 4: Concluding the proof of (a).

(Step 1: Showing that $\mathcal{G}$ is a compact subspace of Y^X.)

For each point $a \in X$, let C_a denote the closure of $\mathcal{F}(a)$ in Y. By assumption, each C_a is compact, hence the product $C = \prod_{a\in X} C_x$ is also compact. Because $\mathcal{F} \subset C \subset Y^X$, the closure of $\mathcal{F}$ in Y^X is also contained in C; because Y^X is a Hausdorff space, $\mathcal{G}$ is a compact subspace of Y^X.

(Step 2: Showing that each member of $\mathcal{G}$ is continuous, and $\mathcal{G}$ is pointwise equicontinuous under d.)

It suffices to check that $\mathcal{G}$ is pointwise equicontinuous under d. Given $p \in X$, let V_p be a neighborhood of p in X such that $f \in \mathcal{F}$ and $x \in V_p$ implies $d(f(x), f(p)) < \epsilon$. Given $g \in \mathcal{G}$ and $x \in V_p$, we can find $h \in \mathcal{F}$ such that $h \in \pi_p^{-1}(B_d(g(p),\epsilon)) \cap \pi_x^{-1}(B_d(g(x),\epsilon))$, i.e., $d(h(p), g(p)) < \epsilon$ and $d(h(x), h(x)) < \epsilon$. In this case,

$$d(g(x), g(p)) \le d(g(x), h(x)) + d(h(x), h(p)) + d(h(p), g(p)) < 3\epsilon,$$

which proves that $\mathcal{G}$ is pointwise equicontinuous.

(Step 3: The topology on Y^X and the topology on $C(X,Y)$ coincide on $\mathcal{G}$.)

It is clear that the topology on Y^X restricted to $\mathcal{G}$ is coarser than the topology on $C(X,Y)$ restricted to $\mathcal{G}$. Thus, it remains to show that the converse inclusion; for this, given a basis member $B_K(f,\epsilon) \cap \mathcal{G}$ of the latter topology (assume $f \in \mathcal{G}$), we need to find a basis member B of the pointwise convergence topology on Y^X such that $f \in B \cap \mathcal{G} \subset B_K(f,\epsilon) \cap \mathcal{G}$. (How could we skip some procedures?)

Using pointwise equicontinuity of $\mathcal{G}$, for each point $x \in X$, let U_x be a neighborhood of x in X such that $t \in U_x$ and $g \in \mathcal{G}$ implies $d(g(t), g(x)) < \epsilon/4$; using compactness of K, cover K with finitely many neighborhoods $U_{p_1}, \cdots, U_{p_n}$. Then, choose

$$B = \bigcap_{i=1}^{n} \pi_{p_i}^{-1}(B_d(f(p_i), \epsilon/4))$$

so that whenever $g \in B \cap \mathcal{G}$ we have $d(g(p_i), f(p_i)) < \epsilon/4$ for all i. If $g \in B \cap \mathcal{G}$ and $x \in K$, there is an index i such that $x \in U_{p_i}$, so

$$d(g(x), f(x)) \le d(g(x), g(p_i)) + d(g(p_i), f(p_i)) + d(f(p_i), f(x)) < \frac{3}{4}\epsilon.$$

This implies that $g \in B_K(f,\epsilon) \cap \mathcal{G}$, as desired.

(Step 4: Concluding the proof of (a).)

Using the results from Step 1 to Step 3, we derive that $\mathcal{F}$ is contained in a compact subspace of $C(X,Y)$. By Step 1 and Step 2, the closure $\mathcal{G}$ of $\mathcal{F}$ in Y^X is a compact subspace of Y^X (in the product topology) which is contained in $C(X,Y)$; by Step 3, $\mathcal{G}$ is also a subspace of $C(X,Y)$ (in the topology of compact convergence). Therefore, $\mathcal{F}$ is contained in a compact subspace $\mathcal{G}$ of $C(X,Y)$. $\square$

Proof of (b). Assume that X is a locally compact Hausdorff space, and let $\mathcal{H}$ be a compact subspace of $C(X,Y)$ which contains $\mathcal{F}$. We show that $\mathcal{H}$ is pointwise equicontinuous under d and pointwise (relatively) compact, from which it follows that $\mathcal{F}$ is pointwise equicontinuous under d and pointwise relatively compact.

We first show that $\mathcal{H}$ is pointwise compact. For this, we fix a point $a \in X$, and consider the following composition:

$$\begin{tikzcd} C(X,Y) \arrow[r, hook, "\imath_a"] & X \times C(X,Y) \arrow[r, "\mathrm{ev}"] & Y \\ H \arrow[u, hook] \arrow[rr, "\jmath"'] & & H(a) \arrow[u, hook] \end{tikzcd},$$

where $\imath_a$ maps $f \in C(X,Y)$ to $(a,f) \in X \times C(X,Y)$. Because $\imath_a$ is continuous for all $a \in X$ and ev is continuous (see Theorem 5.3.14, and note that the compact convergence topology on $C(X,Y)$ and the compact-open topology on the space coincide). Hence, $\jmath$ maps the compact subspace $\mathcal{H}$ of $C(X, Y)$ onto a compact $\mathcal{H}(a)$ of Y.

We now show that $\mathcal{H}$ is pointwise equicontinuous under d; for this, we fix a point $a \in X$ and let A be a compact subspace of X containing a neighborhood of a in X. Then it suffices to check that the following restriction

$$\mathcal{R} := \{f|_A : f \in \mathcal{H}\}$$

is equicontinuous at a. Give $C(A, Y)$ the compact convergence topology. Because A is compact, the compact convergence topology on $C(A, Y)$ and the uniform topology on $C(A, Y)$ coincide. Also, it can be easily verified that the restriction map $r : C(X, Y) \to C(A, Y)$ is continuous (see the following problem), which implies that $\mathcal{R} = r(\mathcal{H})$ is a compact subspace of $C(A, Y)$. Hence, $\mathcal{R}$ is totally bounded under the uniform metric on $C(A, Y)$ induced by d, so $\mathcal{R}$ is pointwise equicontinuous under d. $\square$

Problem 5.4.1. Show that the restriction map $r : C(X, Y) \to C(A, Y)$ in the proof of (b) above is continuous.

Solution. Given $f \in C(X, Y)$ and a basis member $B_2 := B_K(f, \epsilon)$ of the topology on $C(A, Y)$, because K is a compact subspace of A, K is a compact subspace of X. Hence, one can consider the basis member $B_1 := B_K(f, \epsilon)$ of the topology on $C(X, Y)$, and this basis member is mapped onto B_2 by the restriction map.

5.4.2 Some particular versions of Ascoli's theorem

We first introduce the following classical version of Ascoli's theorem. The following remark plays as a technically essential tool in this subsection, and it worths considering the remark as a lemma:

Remark. When (Y, d) is a complete metric space, relative compactness and total boundedness for subspaces of Y coincide.

Theorem 5.4.2 (Classical version of Ascoli's theorem)**.** Let X be a compact space; let $(\mathbb{R}^n, d)$ denote the Euclidean space in either the square metric or the Euclidean metric; give $C(X, \mathbb{R}^n)$ the corresponding uniform topology (which coincides the compact convergence topology). For a subset $\mathcal{F}$ of $C(X, \mathbb{R}^n)$, the followings are equivalent:

(a) $\mathcal{F}$ is pointwise equicontinuous under d and pointwise bounded.

(b) $\mathcal{F}$ is relatively compact in $C(X, \mathbb{R}^n)$.

Proof. Since boundedness and total boundedness coincide in $\mathbb{R}$, it is clear that (a) implies (b). To prove the converse implication, assume that $\mathcal{F}$ is relatively compact in $C(X, \mathbb{R}^n)$ and let $\mathcal{G}$ be the closure of $\mathcal{F}$ in $C(X, \mathbb{R}^n)$. What we want to show is the following:

$$\mathcal{G} \text{ is pointwise equicontinuous under } d \text{ and pointwise bounded.}$$

We first prove that $\mathcal{G}$ is pointwise bounded. Since $\mathcal{G}$ is a compact subspace of $C(X, Y)$, given $\epsilon > 0$, there are finitely many maps $f_1, \cdots, f_n \in \mathcal{F}$ such that $\mathcal{G} \subset \bigcup_{i=1}^n B_{\overline{\rho}}(f_i, \epsilon)$, thus $\mathcal{G}$ is pointwise bounded. Now we prove pointwise equicontinuity. Suppose $g \in \mathcal{G}$ and let $f \in \mathcal{F}$ be a point such that $\overline{\rho}(g, f) < \epsilon/3$; given a point $p \in X$, let V be a neighborhood of p in X such that $x \in V$ and $h \in \mathcal{F}$ implied $d(h(x), h(p)) < \epsilon/3$. We then have

$$d(g(x), g(p)) \leq d(g(x), f(x)) + d(f(x), f(p)) + d(f(p), g(p)) < \epsilon$$

for all $x \in V$, proving pointwise equicontinuity of $\mathcal{G}$. $\square$

Corollary 5.4.3. Let X be a compact space; let d denote either the square metric or the Euclidean metric on $\mathbb{R}^n$; give $C(X, \mathbb{R}^n)$ the corresponding uniform topology (which coincides the compact convergence topology). For a subset $\mathcal{F}$ of $C(X, \mathbb{R}^n)$, the followings are equivalent:

(a) $\mathcal{F}$ is compact.

(b) $\mathcal{F}$ is closed in $C(X, \mathbb{R}^n)$, pointwise equicontinuous under d, and bounded under the sup metric ρ.

Proof. Note the following observations:

- By Ascoli's theorem, $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is pointwise equicontinuous under d and pointwise totally bounded.
- By Heine-Borel theorem, $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is complete and totally bounded under the sup metric.
- Because $C(X, Y)$ is complete under the sup metric, closedness and completeness coincide for subspaces of $C(X, Y)$.

Hence, (a) implies (b). To show that (b) implies (a), it suffices to check that boundedness of $\mathcal{F}$ implies pointwise total boundedness of $\mathcal{F}$. Since $\mathcal{F} \subset B_{\overline{\rho}}(0, R)$ for some real number $R > 0$, $\mathcal{F}(a) \subset [-R, R]$ for all $a \in X$. Being a subspace of a totally bounded space, $\mathcal{F}(a)$ is also totally bounded for each $a \in X$, as desired. □

As the second type, we assume that X is a compact space and (Y, d) is a complete metric spcae.

Theorem 5.4.4. Let X be a compact space, (Y, d) be a complete metric space, and $\mathcal{F}$ be a subset of $C(X, Y)$. Give $C(X, Y)$ the compact convergence topology (which coincides the uniform topology).

(a) If $\mathcal{F}$ is pointwise equicontinuous and pointwise totally bounded, then $\mathcal{F}$ is relatively compact.

(b) The converse of (a) holds if X is a Hausdorff space.

Remark. Suppose X is a compact Hausdorff space. Then $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is pointwise equicontinuous and pointwise totally bounded.

Proof. The results follow from Ascoli's theorem and the first remark. □

As the third type, we assume that X is a locally compact Hausdorff space and (Y, d) is a complete metric space.

Theorem 5.4.5. Suppose X is a locally compact Hausdorff space and (Y, d) is a complete metric space. Give $C(X, Y)$ the compact convergence topology. Then a subset $\mathcal{F}$ of $C(X, Y)$ is relatively compact if and only if $\mathcal{F}$ is pointwise totally bounded and pointwise equicontinuous under d.

Proof. Note again that pointwise total boundedness and pointwise relative compactness coincide, since (Y, d) is complete. □

Corollary 5.4.6 (Generalized Arzelà's theorem)**.** Let X be a σ-compact Hausdorff space and (Y, d) be a complete metric space. Give $C(X, Y)$ the compact convergence topology. If a sequence $(f_n)_{n\in\mathbb{N}} \subset C(X, Y)$ is pointwise equicontinuous and pointwise totally bounded under d, then the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence which converges compactly in $C(X, Y)$.

Proof. It is not solved, yet. □

Corollary 5.4.7 (Arzelà's theorem of undergraduate mathematical analysis)**.** Let X be a compact Hausdorff space and (Y, d) be a complete metric space. Give $C(X, Y)$ the compact convergence topology (which coincides the uniform topology). If a sequence $(f_n)_{n\in\mathbb{N}} \subset C(X, Y)$ is pointwise equicontinuous and pointwise totally bounded under d, then the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence which converges compactly (or uniformly, since X is assumed to be compact) in $C(X, Y)$.

Proof. By the preceeding theorem, $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $C(X, Y)$. Because $C(X, Y)$ is a Hausdorff space, the closure of $(f_n)_{n\in\mathbb{N}}$ is closed, which justifies the assertion. □

Remark. When Y is given as $\mathbb{R}$ or $\mathbb{C}$, total boundedness reduces to boundedness, since they are equivalent on those metric spaces.

Part II

Further theory in general topology

Chapter 6

Quotient topology

6.1 Quotient topology

6.1.1 Quotient maps

Definition 6.1.1 (Quotient map)**.** Let X and Y be topological spaces. A surjective map $p : X \to Y$ is called a quotient map if a subset U of Y is open if and only if $p^{-1}(U)$ is open in X.

Example 6.1.2. Here are some examples of quotient maps.

(a) A homeomorphism is an injective quotient map, and vice versa.

(b) An open (or closed) continuous surjection is a quotient map.

(c) A continuous map which has a right inverse is a quotient map.

(d) A retraction is a quotient map.[1]

For topological spaces X and Y, we say a subset A of X is saturated with respect to the surjective map $p : X \to Y$ when A contains every $p^{-1}(\{y\})$ that it intersects, i.e., A is the inverse image of a subset of Y. To say p is a quotient map is equivalent to saying that p is a continuous surjection mapping a saturated open (closed, respcetively) subspaces of X onto an open (closed) subspaces of Y.

A restriction of a quotient map to a subspace need not be a quotient map. One has, however, the following proposition:

Proposition 6.1.3. Let $p : X \to Y$ be a quotient map, and let A be a subspace of X that is saturated with respect to p. Let $q : A \to p(A)$ be the restriction of p to A.

(a) If A is open or closed in X, then q is a quotient map.

(b) If p is an open map or a closed map, then q is a quotient map.

Proof. We first verify the following two equations:

$$q^{-1}(V) = p^{-1}(V) \quad (V \subset p(A))$$
$$p(U \cap A) = p(U) \cap p(A) \quad (U \subset X).$$

To check the first equation, we note that since $V \subset p(A)$ and A is saturated, $p^{-1}(V)$ is contained in A, hence $q^{-1}(V) = p^{-1}(V)$. To check the second equation, we note the inclusion $p(U \cap A) \subset p(U) \cap p(A)$. If $y \in p(U) \cap p(A)$, then $y = p(u) = p(a)$ for some $u \in U$ and $a \in A$. Because A is saturated, $p^{-1}(\{y\})$ is contained in A, so $u \in A$.

(a) Suppose A is an open subspace of X. We need to show that a subset V of $p(A)$ is open whenever $q^{-1}(V)$ is open in A. Because $q^{-1}(V)$ is open in A and A is open in X, $q^{-1}(V)$ is open in X. Because A is saturated and $V \subset p(A)$, $q^{-1}(V) = p^{-1}(V)$, so V is open in Y and $p(A)$.

[1] Given a topological space X and its subspace A, a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ is called a retraction of X onto A.

(b) Suppose p is an open map. Again, we need to show that a subset V of $p(A)$ is open whenever $q^{-1}(V)$ is open in A. Since $q^{-1}(V) = p^{-1}(V)$ and $q^{-1}(V)$ is open in A, $q^{-1}(V) = U \cap A$ for some open subspace U of X. Hence, $V = q(U \cap A) = p(U \cap A) = p(U) \cap p(A)$, which is open in $p(A)$.

The same results hold for closed cases, which can be proved by replacing the words "open" by "closed." $\square$

Remark. (a) It is clear that the composite of quotient maps is again a quotient map.

(b) Cartesian product of quotient maps need not be a quotient map.

6.1.2 Quotient topology

Definition 6.1.4 (Quotient topology)**.** Let X be a topological space and A be a nonempty set. Given a surjection $p : X \to A$, there is a unique topology $\mathcal{T}$ on A relative to which p is a quotient map, which is given as

$$\mathcal{T} = \{U \subset A : p^{-1}(A) \text{ is open in } X\}. \tag{6.1}$$

This topology is called the quotient topology induced by p.

Problem 6.1.1. Check if the collection $\mathcal{T}$ in eq. (6.1) is a topology on A.

Definition 6.1.5 (Quotient space)**.** Let X be a topological space and let X^* be a partition of X. Let $p : X \to X^*$ be a surjection that maps each point of X to the element of X^* containing the point. In the quotient topology induced by p, the space X^* is called a quotient space.

Remark. The typical open set of X^* is a collection of equivalence classes whose union is an open set of X.

Theorem 6.1.6. Let $p : X \to Y$ be a quotient map. Let Z be a topological space and $f : X \to Z$ be a map which is constant on each set $p^{-1}(\{y\})$ for $y \in Y$. Then f induces a map $\overline{f} : Y \to Z$ such that $\overline{f} \circ p = f$. Furthermore,

(a) $\overline{f}$ is continuous if and only if f is continuous.

(b) $\overline{f}$ is a quotient map if and only if f is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "f"] & \\ Y \arrow[r, "\overline{f}"'] & Z \end{tikzcd}$$

Proof. Well-definedness of $\overline{f}$ is almost clear. To prove (a) and (b), note that $f^{-1} = p^{-1} \circ (\overline{f})^{-1}$.

(a) It is clear that f is continuous if $\overline{f}$ is continuous. To show the converse, assume f is continuous and U is an open subspace of Z. Since $f^{-1}(U)$ is open by continuity and p is a quotient map, $\overline{f}^{-1}(U)$ is also open, hence $\overline{f}$ is a continuous map.

(b) It is clear that f is a quotient map if $\overline{f}$ is a quotient map. To show the converse, assume f is a quotient map. By (a), $\overline{f}$ is a continuous map, and it is easy to check that $\overline{f}$ is a surjection. It, thus, remains to show that $U \subset Z$ is open if $\overline{f}^{-1}(U)$ is open. When $\overline{f}^{-1}(U)$ is open, $p^{-1} \circ \overline{f}^{-1}(U)$ is open, and its image under f is open, and the image is given as $(f \circ p^{-1} \circ \overline{f})(U) = U$, as desired.

Therefore, if f is constant on each set $p^{-1}(\{y\})$ for $y \in Y$, then $\overline{f}$ is well-defined. Furthermore, $\overline{f}$ is continuous if and only if f is continuous; $\overline{f}$ is a quotient map if and only if f is a quotient map. $\square$

Corollary 6.1.7. Let $f : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:

$$X^* = \{f^{-1}(\{z\}) : z \in Z\}.$$

Give X^* the quotient topology.

(a) The map f induces a bijective continuous map $f^* : X^* \to Z$, which is a homeomorphism if and only if f is a quotient map.

(b) If Z is a Hausdorff space, then so is X^*.

Proof. Letting $p : X \to X^*$ be the natural projection, p is a quotient map. By the preceeding theorem, f^* is a well-defined surjective continuous map. By definition of X^*, f^* is injective. Furthermore, f^* is a quotien map (hence, a homeomorphism) if and only if f is a quotient map.

Given two distinct points of X^*, their images under f are distinct points of Z, hence there are disjoint neighborhoods separating those points; their preimage under f are disjoint neighborhoods in X^* separating the given two points. $\square$

6.2 Further properties regarding quotient maps

Chapter 7

Topological groups

7.1 Topological groups

Definition 7.1.1 (Topological groups)**.** A group G is called a topological group if G is also a topological space such that both multiplication and inversion are continuous. (The continuity axiom can be replaced by the following axiom: The map from $G \times G$ into G defined by $(a, b) \mapsto ab^{-1}$ is continuous.)

Remark. A continuous group homomorphism is called a topological group homomorphism.

Let G be a topological group. For any point a of G, the following maps

$$\begin{aligned} \rho_a : G \to G, &\quad g \mapsto ga \\ \lambda_a : G \to G, &\quad g \mapsto ag \\ \gamma_a : G \to G, &\quad g \mapsto aga^{-1} \end{aligned}$$

will be frequently used in this chapter to investigate properties of topological groups. Indeed, the above maps are group isomorphisms which are also homeomorphisms.

Observation 7.1.2. Some direct conclusions from the result that the above maps are group-isomorhphic homeomorphisms are listed here:

(a) G is a homogeneous space, i.e., for every pair x, y of points of G, there is a homeomorphism of G onto itself mapping x to y.

(b) Every neighborhood W of g in G can be written as $W = Ug = gV$, where $U = Wg^{-1}$ and $V = g^{-1}W$ are neighborhoods of the identity.[1]

(c) Let G, K be topological groups and $f : G \to K$ be a group homomorphism. Then f is continuous on G if f is continuous at a point of G.

While (a) is direct and (b) is clear, (c) seems to be explained. Assume that f is continuous at $p \in G$, and choose $x \in G$ and let W be a neighborhood of $y := f(x)$ in K. Then $f(p)y^{-1}W$ is a neighborhood of $f(p)$ in K, hence there is a neighborhood U of p in G such that $f(U) \subset f(p)y^{-1}W$. The neighborhood $xp^{-1}U$ of x in G, as desired, is mapped into W under f.

Example 7.1.3. (a) The (abelian) groups $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{R}^{>0}, \cdot)$ are topological groups. (As usual, impose each space the order topology.)

(b) $(S^1, \cdot)$ is a topological group. (As usual, let $A \subset S^1$ be a basis member if A is the intersection of an open ball in $\mathbb{C}$ and S^1.)

(c) The general linear group $GL_n(\mathbb{R})$ is a topological group, when it is topologized naturally as a subspace of $\mathbb{R}^{n^2}$.

Topological groups behave well in the context of products.

[1] A neighborhood of the identity in a topological group is called an identity neighborhood.

Proposition 7.1.4. Suppose that $(G_\alpha)_{\alpha\in I}$ is a collection of topological groups.

(a) The product $G := \prod_\alpha G_\alpha$, endowed with the product topology, is a topological group.

(b) For each index $\alpha \in I$, the projection map π_α is an open topological group homomorphism.

(c) (A universal property) For every topological group H and topological group homomorphisms $f_\alpha : H \to G$ for $\alpha \in I$, there is a unique topological group homomorphism $f : H \to G$ such that $\pi_\alpha \circ f = f_\alpha$ for each $\alpha \in I$. In short, there is a unique topological group homomorphism $f : H \to G$ satisfying the following commutative diagram:

$$\begin{tikzcd} & \prod_{\alpha\in I} G_\alpha \arrow[dr, "\pi_\gamma"] & \\ H \arrow[ur, "f"] \arrow[rr, "f_\gamma"'] & & G_\gamma \end{tikzcd}.$$

Proof. To show that the product of topological groups is a topological group, it suffices to check continuity of multiplication and inversion: being endowed with the product topology and each multiplication and inversion are continuous, the multiplication and inversion in G are also continuous. Because each projection is open and continuous, it is clear that each projection is an open topological group homomorphism. The existence and uniqueness of a "group homomorphism" f is due to a universal property of product groups; its continuity follows from the equation $\pi_\alpha \circ f = f_\alpha$ for $\alpha \in I$. $\square$

Topological groups also behave well in the context of subgroups and closures.

Proposition 7.1.5. A subgroup H of a topological groups G is a topological group. Also, the closure $\overline{H}$ of H in G is a subgroup of G, hence a topological group. In addition, if H is a normal subgroup of G, then $\overline{H}$ is also a normal subgroup of G.

Proof. It is easy to check that subgroups of a topological group is a topological group. To check that $\overline{H}$ is a subgroup of G, it suffices to check if $ab^{-1} \in \overline{H}$ for any $a, b \in \overline{H}$. Considering the following continuous map $\kappa : G \times G \to G$ defined by $\kappa(x, y) = xy^{-1}$, we have $\kappa(\overline{H} \times \overline{H}) = \kappa(\overline{H \times H}) \subset \overline{\kappa(H \times H)} = \overline{H \times H} = \overline{H} \times \overline{H}$, so H is a subgroup of G. Finally, assume H is a normal subgroup of G. To show that $\overline{H}$ is a normal subgroup of G, it suffices to check if $ghg^{-1} \in \overline{H}$ for all $g \in G$ and $h \in \overline{H}$; consider the continuous map γ_g for $g \in G$, and observe that $\gamma_g(\overline{H}) \subset \overline{\gamma_g(H)} = \overline{H}$. $\square$

Regarding subgroups, we introduce the following proposition.

Proposition 7.1.6. Let G be a topological group and H be a subgroup of G.

(a) The subgroup H is open in G if it contains a nonempty open set.

(b) If H is open in G, then H is closed in G.

(c) The subgroup H is closed in G if and only if there is an open subset U of G such that $H \cap U$ is a nonempty closed subspace of U.

Proof. Let H be a subgroup of G. To prove (a), suppose H contains an open subspace U of G. Then H is open in G, since $H = UH = \bigcup_{h\in H} Uh$. To prove (b), assume H is open in G. Because $G \setminus H = \bigcap_{a\in G\setminus H} aH$ is open, H is closed. When proving (c), note that one way is clear. Let U be an open subset of G such that $U \cap H$ is a nonempty closed subspace of U. Letting $V = U \cap \overline{H}$ where the overline is used to denote the closure in G, V is an open subset of $\overline{H}$ such that $V \cap H = U \cap H$ is a nonempty closed subspace of V, hence of $\overline{H}$. Because H is dense in $\overline{H}$, it can be easily checked that $V \cap H$ should be empty and $V \subset H$. Because V is an open subspace of $\overline{H}$, H turns out to be an open subspace of $\overline{H}$, hence a closed subspace of $\overline{H}$. Because the closure of H in $\overline{H}$ is $\overline{H}$, we have $\overline{H} = H$, i.e. H is closed in G, as desired. $\square$

Remark. As an undesired result obtained in the proof of (a) is the following:

For a topological group G and its open subset U, XU and UX are open for any subset X of G.

Using this result, we can derive that the maps $(x, y) \mapsto xy$ and $(x, y) \mapsto x^{-1}y$ defined on $G \times G$ are open (and continuous).

We end this section with the discovery that a T_1-topological space is necessarily a regular space.

Theorem 7.1.7. A topological space satisfying a T_1-axiom is a Hausdorff space and a regular space.

To prove this, we need the following lemma:

Lemma 7.1.8. If U is a neighborhood of $e \in G$ in G, there is a symmetric identity neighborhood V such that $VV \subset U$. (We say a subset A of G is symmetric if $A^{-1} = A$.)

Proof of Lemma 7.1.8. Because multiplication is continuous, there is an identity neighborhood V_1 such that $V_1V_1 \subset U$; because inversion is continuous, there is an identity neighborhood W such that $V := WW^{-1} \subset V_1$. Therefore, $VV \subset U$, and it is easy to check that V is a symmetric identity neighborhood. □

Proof of Theorem 7.1.7. Suppose that every singletone in G is closed and choose two distinct points $x, y \in G$.

$$\text{Want: an identity neighborhood } V \text{ in } G \text{ such that } xy^{-1} \notin VV.$$

If the above desire is satisfied, we then have $xy^{-1} \neq u^{-1}v$ for all $u, v \in V$, $ux \neq vy$, thus $Vx \cap Vy = \varnothing$. Using the preceeding lemma, one can find a symmetric identity neighborhood V such that $VV \subset G \setminus \{xy^{-1}\}$. This proves that G is a Hausdorff space.

It remains to prove that a T_1-topological group is regular. For this, it suffices to show the existence of a symmetric identity neighborhood $Vx \cap VA = \varnothing$, where A is a closed subspace of G and x is a point of G not contained in A. As in the first paragraph, our goal is to find an identity neighborhood V in G such that $ax^{-1} \notin VV$ for all $a \in A$, and for this we find such V satisfying $VV \subset G \setminus Ax^{-1}$ by using the preceeding lemma. This proves that G is a regular space. □

7.2 Quotients of topological groups

Suppose that H is a subgroup of a topological group G. We give the quotient G/H the quotient topology induced by the projection map $p : G \to G/H$. Then a subset of G/H is open if and only if its preimage under p is open in G.

Proposition 7.2.1. Let G be a topological group and H be a subgroup of G. Then, the projection map $p : G \to G/H$ is an open quotient map, hence G/H is a quotient space of G. Moreover, the quotient space G/H is Hausdorff if and only if H is closed in G, and G/H is discrete if and only if H is open in G.

Proof. Since G/H is given the quotient topology induced by p, p is clearly a quotient map. If U is an open subspace of G, then $p(U)$ is open in G/H, since $p^{-1}(p(U)) = UH$ is open in G and p is a quotient map. Because p is chosen so that every point of G is mapped to an element of G/H that contains the point, G/H is a quotient space of G.

It is clear that H is closed in G if G/H is a Hausdorssf space, since $H = p^{-1}(\{H\})$ and $\{H\}$ is a closed singletone in G/H. Conversely, if H is closed in G, because $H = p^{-1}(\{H\})$, we find that $\{H\}$ is closed in G/H, hence every translation of $\{H\}$ is closed, i.e., G/H is a Hausdorff space.

Similarly, it is clear that H is open in G if G/H is discrete. Converse implication is now almost clear. □

Theorem 7.2.2. Let G be a topological group.

(a) If N is a normal subgroup of G, then the group G/N is a topological group with respect to the quotient topology on G/N.

(b) The quotient map $p : G \to G/N$ is an open topological group homomorphism.

(c) Moreover, the topological group G/N is a Hausdorff space if and only if N is closed. Hence, in particular, $G/\overline{N}$ is a Hausdorff topological group.

Proof. To check that G/N is a topological group, it remains to check if multiplication and inversion are continuous, which are easy to check. The remaining assertions follow from the preceeding proposition. $\square$

Example 7.2.3. Consider the abelian group $(\mathbb{R}, +)$ and its normal subgroup $\mathbb{Z}$. The quotient group $(\mathbb{R}/\mathbb{Z}, +)$ is homeomorphic to $(S^1, \cdot)$

We end this section with a morphism theorem.

Theorem 7.2.4 (Morphism theorem for topological groups)**.** Let $f : G \to H$ be a topological group homomorphism, and suppose that the normal subgroup N of G is contained in *ker*(f) Then f factors through the open topological group homomorphism $p : G \to G/N$ by a unique topological group homomorphism $\overline{f}$, satisfying the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & K \\ G/N \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

Furthermore, f is open if and only if $\overline{f}$ is open.

Proof. The existence and uniqueness of such group homomorphism $\overline{f}$ is already proved. Since $p^{-1} \circ \overline{f}^{-1} = f^{-1}$ and p is a quotient map, $\overline{f}$ is easily turned out to be continuous, so $\overline{f}$ is a topological group homomorphism. It can also be checked easily that f is open if and only if $\overline{f}$ is open. $\square$

7.3 Further properties of topological groups

Chapter 8

Topological vector spaces

8.1 Topological vector spaces

8.2 Further properties of topological vector spaces

Chapter 9

Tietze extension theorem

Chapter 10

Manifolds and partition of unity

Chapter 11

Fundamental groups