Review of general topology

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Contents

Part I

Essential theory in general topology

Chapter 1

Topology and continuous map

1.1 Topology and basis

Definition 1.1.1 (Topology on a set). Let X be a nonempty set. A collection $\mathcal{T}$ is called a topology on X if

(a) both X and the empty set belongs to $\mathcal{T}$,

(b) $\mathcal{T}$ is closed under arbitrary unions,

(c) $\mathcal{T}$ is closed under arbitrary finite intersections.

Definition 1.1.2 (Subbasis and basis of a topology). A collection $\mathcal{B}$ is called a subbasis of the topology on X if $\mathcal{B}$ covers X. A collection $\mathcal{B}$ is called a basis of the topology on X if

(a) $\mathcal{B}$ covers X, i.e., $\mathcal{B}$ is a subbasis on X,

(b) given $B_1, B_2 \in \mathcal{B}$, there is another member $B_3 \in \mathcal{B}$ contained in $B_1 \cap B_2$.

The topology $\langle \mathcal{B} \rangle$ on X generated by the basis $\mathcal{B}$ is the following collection of subsets of X:

$$\langle \mathcal{B} \rangle = \left\{ U \subset X : \begin{array}{c} \text{Given } x \in U, \text{ there is a basis member} \\ B \in \mathcal{B} \text{ such that } x \in B \subset U \end{array} \right\}.$$

(Remark how an open subset of a metric space is defined in the course of mathematical analysis.) In accordance with the definition of the first-countability, we will say that $\langle \mathcal{B} \rangle$ consists of all subsets of X based on $\mathcal{B}$.

Observation 1.1.3. Let X be a set and suppose $\mathcal{B}$ is a basis of X.

(a) The topology generated by $\mathcal{B}$ is the collection $\mathcal{C}$ of all unions of members in $\mathcal{B}$.

(b) The topology generated by $\mathcal{B}$ is the intersection $\mathcal{I}$ of all topologies on X containing $\mathcal{B}$. (Hence, the topology on X generated by $\mathcal{B}$ is the smallest topology on X containing $\mathcal{B}$.)

Proof. (a) By definition, it is clear that $\mathcal{C}$ is contained in $\langle \mathcal{B} \rangle$. To show the converse inclusion, suppose $U \in \langle \mathcal{B} \rangle$. By definition, for each $x \in U$, there is a basis member $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$, hence U is the union of B_x for $x \in U$.

(b) Because $\langle \mathcal{B} \rangle$ is a topology on X containing $\mathcal{B}$, $\mathcal{I}$ is contained in $\langle \mathcal{B} \rangle$. Conversely, by (a), every topology on X containing $\mathcal{B}$ also includes $\langle \mathcal{B} \rangle$. Thus, $\mathcal{I}$ contains $\langle \mathcal{B} \rangle$. $\square$

Lemma 1.1.4 (Containment criterion). Let $\mathcal{B}, \mathcal{B}'$ be a basis of the topology $\mathcal{T}, \mathcal{T}'$ of X, respectively. Then the following are equivalent:

(a) $\mathcal{T}$ is finer than $\mathcal{T}'$.

(b) For each point $x \in X$ and a basis member $B' \in \mathcal{B}'$ containing x, there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Proof. Assume (a) and let x be a point of X and B' be a basis member containing x. Then $B' \in \mathcal{T}$, and there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Assume (b) and let U' be a member of $\mathcal{T}'$. For each point $p \in U'$, there is a basis member $B_p \in \mathcal{B}$ such that $p \in B_p \subset U'$. (There was a leap in the argument: Letting B'_p be a basis member of $\mathcal{B}'$ such that $p \in B'_p \subset U'$, by the assumption we have a basis member $B_p \in \mathcal{B}$ such that $x \in B_p \subset B'_p$.) Therefore, $U' \in \mathcal{T}$. $\qquad\square$

Remark. (a) (subbasis) $\xrightarrow{\text{finite intersections}}$ (basis) $\xrightarrow{\text{arbitrary unions}}$ (topology)

(b) Suppose that $\mathcal{B}$ is a basis of a topology $\mathcal{T}$ on a set X. Then the topology generated by $\mathcal{B}$ as a 'subbasis' is $\mathcal{T}$. If $\mathcal{B}'$ is the collection of all finite intersections of the members of $\mathcal{B}$, then $\mathcal{B}' \supset \mathcal{B}$, so the topology $\mathcal{T}'$ generated by $\mathcal{B}'$ as a basis (i.e., by $\mathcal{B}$ as a subbasis) is finer than $\mathcal{T}$. To show that $\mathcal{T}$ is finer than $\mathcal{T}'$, it suffices to prove for each $x \in X$ and a basis member $B' \in \mathcal{B}'$ containing x there is a basis member $B \in \mathcal{B}$ such that $x \in B \subset B'$. Because $\mathcal{B}$ is a basis, if $B' = B_1 \cap \cdots \cap B_n$ for some $B_1, \cdots, B_n \in \mathcal{B}$ then there is a basis member $B_0 \in \mathcal{B}$ such that $x \in B_0 \subset B_1 \cap \cdots \cap B_n = B'$.

(c) Regarding containment, we have another proposition, given as follows:

Let X be a topological space and $\mathcal{B}$ be a collection of sets open in X. Assume that for each open subset U of X and a point $x \in U$, there is a member $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Then $\mathcal{B}$ is a basis of the topology on X.

In fact, by assumption, $\mathcal{B}$ is a basis of a topology on X which is finer than the topology $\mathcal{T}$ on X. Because $\mathcal{B}$ is a subcollection of $\mathcal{T}$, the topology on X generated by $\mathcal{B}$ is coarser than $\mathcal{T}$, so $\mathcal{B}$ is a basis of $\mathcal{T}$.

1.2 Continuous maps

In this section, we assume X, Y, Z are topological spaces.

Definition 1.2.1 (Continuity). The map $f : X \to Y$ is said to be continuous if $f^{-1}(U)$ is open in X whenever U is open in Y. If the map f is bijective and its inverse is also continuous, then f is called a homeomorphism. In addition, if f is injective and continuous, and the induced map $\widetilde{f} : X \to f(X)$ defined by $\widetilde{f}(a) = f(a)$ for all $a \in X$ is a homeomorphism, then f is called an embedding of X into Y. (It will be explained that such restriction of continuous maps are always continuous.)

Remark. (a) The procedure for checking continuity can be reduced to the members of a basis or a subbasis generating the topology on the codomain Y. (Why?)

(b) A homeomorphism naturally induces a bijection between the topologies on the domain and the codomain of the homeomorphism. Also, a bijective continuous map is a homeomorphism if and only if the map is an open map.

Theorem 1.2.2. Let X, Y be topological spaces and $f : X \to Y$ be a map. Then the following are equivalent:

(a) f is a continuous map.

(b) For any closed subset B of Y, $f^{-1}(B)$ is closed in X.

(c) For each $x \in X$ and a neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X such that $f(U) \subset V$.

(d) Whenever $U \subset X$, $f(\overline{U}) \subset \overline{f(U)}$.

Proof. (a)$\Leftrightarrow$(b): This follows directly by considering set complements.
(a)$\Rightarrow$(c): If $x \in X$ and V is a neighborhood of $f(x)$ in Y, then $f^{-1}(V)$ is a neighborhood of x in X whose image under f is V.

(c)⇒(a): Let V be an open subset of Y and let $U = f^{-1}(V) \subset X$. By assumption, for each $x \in X$, there is a neighborhood A_x of x in X such that $f(A_x) \subset V$. Then $A_x \subset U$, so U is open in X.

(a)⇒(d): Suppose $U \subset X$. We will show that if $x \in \overline{U}$ then $f(x) \in \overline{f(U)}$. If V is a neighborhood of $f(x)$ in Y, then $f^{-1}(V)$ is a neighborhood of x in X, so $f^{-1}(V)$ intersects U. Thus, $V \cap f(U) = f(f^{-1}(V) \cap U)$ is nonempty, so $f(x) \in \overline{f(U)}$, as desired.

(d)⇒(b): Let B be a closed subset of Y and let $A = f^{-1}(B)$. By assumption, we have $B = f(A) \subset f(\overline{A}) \subset \overline{f(A)} = \overline{B} = B$, so $B = f(\overline{A})$ and $\overline{A} \subset f^{-1}(B) = A$, proving that $A = \overline{A}$ is closed. $\qquad\square$

If given maps $f : X \to Y$ and $g : Y \to Z$ are continuous, some naturally induced maps are also continuous, as indicated below. (Checking continuity is left as an exercise.)

Proposition 1.2.3. Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous.

(a) Every constant map is continuous.

(b) $g \circ f$ is continuous.

(c) Every restriction of a continuous map on a subspace is continuous. Also, if Y is a subspace of Z, then $\widetilde{f} : X \to Z$ defined by $\widetilde{f}(x) = f(x)$ for all $x \in X$ is continuous.

Problem 1.2.1. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution. It is not necessarily true; consider constant maps.

Theorem 1.2.4. Let A be a set; let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of spaces; and let $\{f_\alpha : A \to X_\alpha\}_{\alpha \in I}$ be an indexed family of functions.

(a) There is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the function f_α is continuous. In fact, the topology $\mathcal{T}$ is generated as a subbasis by the following collection:

$$\left\{ f_\alpha^{-1}(U_\alpha) : \alpha \in I \text{ and } U_\alpha \text{ is open in } X_\alpha \right\}.$$

(b) A map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each composition $f_\alpha \circ g$ is continuous.

Proof. (a) is almost clear; such topology necessarily contains the given collection. In proving (b), it suffices to prove if part. Suppose $g_\alpha := f_\alpha \circ g$ is continuous for each $\alpha \in I$. Given an open subset U of A, we have $g_\alpha^{-1}(A) = g^{-1}(f_\alpha^{-1}(A))$, completing the proof. $\qquad\square$

Remark. The product topology on a product space satisfies the above properties; in fact, the product topology is the topology on A with

$$A = \prod_{\alpha \in I} X_\alpha \quad \text{and} \quad f_\alpha = \pi_\alpha \text{ for each } \alpha \in I.$$

Problem 1.2.2. Let (X, d) be a metric space.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' be a space having the same underlying set as X, and define the map $d' : X' \times X' \to \mathbb{R}$ by $d'(a, b) = d(a, b)$ for all $(a, b) \in X' \times X'$. Show that the topology on X' is finer than the topology on X, if d' is continuous. Deduce that the metric topology on X induced by the metric d is the coarsest topology on X relative to which d is continuous.

Solution. (a) Choose a point $(a, b) \in X \times X$ and write $r = d(a, b)$. We will show that given a positive real number ϵ there is a neighborhood of (a, b) whose image under d is contained in $(r - \epsilon, r + \epsilon)$. Let $V = B_d(a, \delta) \times B_d(b, \delta)$ with $\delta > 0$, which is a neighborhood of (a, b) in $X \times X$. Whenver $(p, q) \in V$, we have

$$d(p, q) \leq d(p, a) + d(a, b) + d(b, q) \quad \text{and} \quad d(a, b) \leq d(a, p) + d(p, q) d(q, b,)$$

from which we obtain $r - 2\delta < d(p, q) < r + 2\delta$. Hence, by choosing $0 < \delta < \epsilon/3$ we have $d(V) \subset (r - \epsilon, r + \epsilon)$, as desired. Therefore, d is a continuous map.

(b) Choose $a \in X'$, and define the map $\epsilon_a : X' \to X' \times X'$ by $\epsilon_a(x) = (x, a)$ for $x \in X'$. Then ϵ_a is continuous, so $k := d' \circ \epsilon_a$ is continuous. Hence, $k^{-1}((-\infty, r)) = B_{d'}(a, r) = B_d(a, r)$ is an open subset of X' for all $r \in \mathbb{R}$ with $r > 0$. Therefore, the topology on X' contains $\{B_d(a, r) : a \in X, r > 0\}$, so the topology on X' is finer than the topology on X. The last assertion easily follows.

1.3 Topologies on function spaces

Let X_α be a topological space for each $\alpha \in I$, and write $X = \prod_{\alpha \in I} X_\alpha$.

Definition 1.3.1 (Box topology). The topology on X generated by the following collection as a basis is called the box topology on X:

$$\left\{ \prod_{\alpha \in I} U_\alpha : \text{For each } \alpha \in I, U_\alpha \text{ is an open subset of } X_\alpha \right\}.$$

Indeed, the box topology on X is finer than the product topology on X. Also, it is easy to verify that the box topology on X can be generated as a basis by the following collection, given that $\mathcal{B}_\alpha$ is a basis of the topology on X for each $\alpha \in I$:

$$\left\{ \prod_{\alpha \in I} B_\alpha : \text{For each } \alpha \in I, B_\alpha \in \mathcal{B}_\alpha \right\}.$$

Remark. For X equipped with the product topology, we have observed that a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is convergent if and only if $(\pi_\alpha(x_n))_{n \in \mathbb{N}} \subset X_\alpha$ is convergent for each $\alpha \in I$. For X equipped with the box topology, however, if part is generally not true (only if part is true).

We now assume that (Y, d) is a metric space and I is an index set. Even Y is not metrizable, we could have considered the product and box topology on Y^I, but the following topology is valid when a metric on Y is given. In other words, the following topology on Y^I (generally) depends on the choice of the metric on Y.

Definition 1.3.2.   (a) (Uniform metric) Define the map $\overline{\rho} : Y^I \times Y^I \to [0, \infty)$ by

$$(f, g) \mapsto \sup\{\overline{d}(f(x), g(x)) : x \in I\} \quad \text{for all } f, g \in Y^I.$$

The map $\overline{\rho}$ is a metric on Y^I, which is called the uniform metric.

(b) (Uniform topology) Let I be a nonempty set and (Y, d) be a metric space. The topology on Y^I generaged by the following collection as a basis is called the uniform topology on Y^I:

$$\left\{ B_{\overline{\rho}}(f, \epsilon) : f \in Y^I \text{ and } \epsilon > 0 \right\}$$

Proposition 1.3.3. Let I be a nonempty set and (Y, d) be a metric space. Then, we have the following inclusions of the topologies on Y^I:

$$(\text{the product topology}) \subset (\text{the uniform topology}) \subset (\text{the box topology}).$$

In particular, if $I = \mathbb{N}$ and (Y, d) is the Euclidean metric space $(\mathbb{R}, d)$, then the above three topologies do not coincide.

Proof. **Step 1. Showing the inclusions.**
Given a basis member $\prod_{x \in I} U_x$ of the product topology on Y^I, we have $U_x = Y$ for all but finitely many values of x; namely, $x_1, \cdots, x_k$. Suppose $f \in \prod_{x \in I} U_x$. Because $f(x_i) \in U_{x_i}$ for each $i = 1, \cdots, k$, there is a positive real number $0 < r_i < 1$ such that $B_d(f(x_i), r_i) \subset U_{x_i}$. Letting $r = \min\{r_1, \cdots, r_k\}$, we have $f \in B_{\overline{\rho}}(f, r) \subset \prod_{x \in I} U_x$.

Given a basis member $B_{\overline{\rho}}(f, \epsilon)$ of the uniform topology on Y^I (with $0 < \epsilon < 1$) with a point g in the member, there is a positive real number $0 < r < 1$ such that $g \in B_{\overline{\rho}}(g, r) \subset B_{\overline{\rho}}(f, \epsilon)$. In fact, we also have $g \in \prod_{x \in I} B_d(g(x), r/2) \subset B_{\overline{\rho}}(g, r)$, proving the assertion.

Step 2. Investigating the given particular case.
Observe, when (Y, d) is the Euclidean metric space $(\mathbb{R}, d)$ and $I = \mathbb{N}$, that $B_{\overline{\rho}}(0, 1/2)$ cannot be an intersection of finitely many subsets of the form $\pi_n^{-1}(\mathbb{R})$ $(n \in \mathbb{N})$, and that $\prod_{n \in \mathbb{N}} (1/n, 1/n)$ cannot be open in the uniform topology on $\mathbb{R}^{\mathbb{N}}$. $\qquad \square$

1.4 Metrizable product spaces

Let X_n be a metrizable space for each $n \in \mathbb{N}$.

Notation. Given a metric d_n inducing the topology on X_n for each n, define the following metric on the product space $X := \prod_{n=1}^{\infty} X_n$ as follows:

$$\bar{\rho} : X \times X \to \mathbb{R}, \ (x, y) \mapsto \sup_{n \in \mathbb{N}} \left\{ \overline{d_n}(x_n, y_n) \right\},$$

$$D : X \times X \to \mathbb{R}, \ (x, y) \mapsto \sup_{n \in \mathbb{N}} \left\{ \frac{\overline{d_n}(x_n, y_n)}{n} \right\}.$$

The former metric is called the uniform metric on X and the latter metric is called the D-metric on X. Among them, the definition of the former metric generalizes to arbitrary product spaces: If $\{X_\alpha\}_{\alpha \in I}$ is an indexed family of metric spaces and $X = \prod_{\alpha \in I} X_\alpha$, then we may define the uniform metric $\rho : X \times X \to \mathbb{R}$ by $\rho(x, y) = \sup_{\alpha \in I} \{\overline{d_\alpha}(x_\alpha, y_\alpha)\}$ for $(x, y) \in X \times X$.

Theorem 1.4.1. Let (X_n, d_n) be a metric space for each $n \in \mathbb{N}$ and write $X = \prod_{n \in \mathbb{N}} X_n$. Then D-metric on X induces the product topology on X.

Proof. We first show that the topology $\mathcal{T}_D$ induced by the D-metric is finer than the product topology. Given a point $x \in X$ and a basis member $B = \prod_{n \in \mathbb{N}} B_n$ of the product topology with

$$\{n \in \mathbb{N} : B_n \neq X_n\} = \{n_1, \cdots, n_k\},$$

let us find a basis member of $\mathcal{T}_D$ which contains x and contained in B. For each $i = 1, \cdots, k$, let r_i be a real number such that $0 < r_i < 1$ and $B_{d_{n_i}}(x_{n_i}, r_i) \subset B_{n_i}$. Set

$$\epsilon = \min_{1 \leq i \leq k} \left\{ \frac{r_i}{n_i} \right\}$$

and suppose $y \in B_D(x, \epsilon)$. Then $\overline{d_n}(x_n, y_n)/n < \epsilon$ for all $n \in \mathbb{N}$. In particular, $\overline{d_{n_i}}(x_{n_i}, y_{n_i}) < n_i \epsilon \leq r_i$, so $y_{n_i} \in B_{d_{n_i}}(x_{n_i}, r_i)$ for each all i. Hence, $x \in B_D(x, \epsilon) \subset B$, as desired.

Conversely, suppose a point x of X and a basis element $B_D(p, r) \in \mathcal{T}_D$ containing x are given. By choosing a small real number $0 < \epsilon < 1$, we can achieve $x \in B_D(x, \epsilon) \subset B_D(p, r)$. (Such leap is helpful in proving many other propositions regarding metric spaces.) Let N be a positive integer such that $1/N < \epsilon$. Let $B_n = B_{d_n}(x_n, \epsilon)$ for each positive integer $n < N$ and let $B_n = X_n$ otherwise. Then $x \in \prod_{n \in \mathbb{N}} B_n \subset B_D(x, \epsilon) \subset B_D(p, r)$, as desired. $\qquad \square$

Problem 1.4.1. Let (X, α) and (Y, β) be metric spaces. Show that every isometry from X into Y is an embedding.

Solution. If $f : X \to Y$ is an isometry, then f is clearly injective and continuous. Because an open ball of radius $r > 0$ in X is mapped onto an open ball of radius r in Y under f, f is an open map, so f is an embedding.

We now introduce the structure of an open ball in $\mathbb{R}^\mathbb{N}$ which equips the uniform metric $\bar{\rho}$.

Proposition 1.4.2. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\mathbb{N}$. Given $x = (x_1, x_2, \cdots) \in \mathbb{R}^\mathbb{N}$ and a real number $0 < \epsilon < 1$, define

$$U(x, \epsilon) := \prod_{n=1}^{\infty} (x_n - \epsilon, x_n + \epsilon).$$

(a) $U(x, \epsilon) \neq B_{\bar{\rho}}(x, \epsilon)$ and $U(x, \epsilon)$ is not open in the uniform topology.

(b) $B_{\bar{\rho}}(x, \epsilon) = \bigcup_{0 < r < \epsilon} U(x, r)$.

Proof. (a) Note that the point $(x_n + 2^{-n}\epsilon)_{n \in \mathbb{N}}$ is in $U(x, \epsilon)$ but not in $B_{\bar{\rho}}(x, \epsilon)$. Furthermore, no neighborhood of this point entirely lies in $U(x, \epsilon)$.

(b) For each real number $0 < r < \epsilon$, we have $U(x, r) \subset B_{\bar{\rho}}(x, \epsilon)$. Conversely, if $y \in B_{\bar{\rho}(x, \epsilon)}$, then $\bar{\rho}(x, y) < \epsilon$, so $\sup\{\overline{d_n}(x_n, y_n) : n \in \mathbb{N}\} = \delta$ for some $0 \leq \delta < \epsilon$. Hence, $B_{\bar{\rho}}(x, \epsilon) \subset U(x, r)$ for some real number r such that $\delta < r < \epsilon$. $\qquad \square$

Chapter 2

Connected spaces and compact spaces

2.1 Connected spaces

Definition 2.1.1 (Separation of a topological space)**.** Let X be a topological space. A pair (A, B) is called a separation of X if $\{A, B\}$ is a partition of X by nonempty open subsets of X. If X has no separation, then X is called a connected space.

Remark. (a) Connectedness is a topological property. Hence, connectedness is preserved under homeomorphisms.

 (b) The topological space X is connected if and only if the only subsets of X which are both open and closed in X are $\varnothing$ and X.

When discussing connectedness of a subspace A of X according to the above definition, subsets of A shall be considered. The following theorem lets us argue the connectedness of a subspace A of X *in terms of subsets of X.*

Theorem 2.1.2 (Connected subspace)**.** Suppose X is a topological space and A is a subspace of X. Then (U, V) is a separation of A if and only if

 (i) $\{U, V\}$ is a partition of A by nonempty subsets of A (here, U and V need not be open or closed in A or X)

 (ii) and neither U nor V contains a limit point (in X) of the other, i.e., $\overline{U} \cap V = U \cap \overline{V} = \varnothing$. (The overline notations stand for the closure in X.)

Proof. Suppose (U, V) is a separation of Y. Because U is both open and closed in Y, $U = \overline{U} \cap Y$. Since $U \cap V = \varnothing$, we have $\overline{U} \cap V = \varnothing$. For the same reason, $U \cap \overline{V} = \varnothing$.

Conversely, assuming (i) and (ii), we have $\overline{U} \cap A = \overline{U} \cap (U \sqcup V) = U$, so U is closed in A. For the same reason, V is also closed in A, so (U, V) is a separation of A. $\qquad\square$

We introduce a useful lemma regarding connected spaces, implying that a connected subspace lies entirely in a 'separation component.'

Lemma 2.1.3. If (C, D) is a separation of X and if Y is a connected subspace of X, then Y lies entirely within C or D.

Proof. Write $A = Y \cap C$ and $B = Y \cap D$. Since (C, D) is a separation of X, C and D are open (and closed) in X and in Y. Because Y is connected, either C or D is empty, as desired. $\qquad\square$

Proposition 2.1.4. Suppose $\{U_\alpha\}_{\alpha \in I}$ is a collection of connected subspaces, and assume all the members have a common point. Then the union of the members of the collection is connected.

Proof. Suppose the union A of A_α for $\alpha \in I$ is not connected. Then there is a separation (U, V) of A, and each A_α resides entirely in either U or V. Without loss of generality, assume A_{α_0} is in U for some $\alpha_0 \in I$. Since a common point is in U and not in V, $A_\alpha \subset U$ for all $\alpha \in I$, so $A \subset U$, a contradiction. $\qquad\square$

Proposition 2.1.5. If A is a connected subspace of X, then adding some of its limit points keeps the space connected. To be precise, if $A \subset B \subset \overline{A}$, then B is connected.

Proof. Suppose B is not connected and let (U, V) be a separation of B. Since A is a connected subspace of B, without loss of generality, $A \subset U$, so $\overline{A} \subset \overline{U}$. Because $\overline{U}$ and V are disjoint, $\overline{A} \cap V = \varnothing$, a contradiction. $\qquad\square$

Theorem 2.1.6. A continuous image of a connected space is connected.

Proof. Let X be a connected space and $f : X \to Y$ be a continuous map. Suppose $f(X)$ is not connected and let (U, V) be a separation of $f(X)$. Since U and V are open in $f(X)$, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Thus, $(f^{-1}(U), f^{-1}(V))$ is a separation of X, which contradicts the connectedness of X. $\qquad\square$

The idea of the proof of the following proposition is remarkable.

Theorem 2.1.7. A finite product of connected spaces is connected in the product topology.

Proof. It suffices to prove for the product of two connected spaces, for the desired result can be obtained by induction. Let X_1 and X_2 be connected spaces. Given a point $(a, b) \in X_1 \times X_2$, define the 'cross' $C(a, b)$ at (a, b) by $(\{a\} \times X_2) \cup (X_1 \times \{b\})$. Because $X_1 \times \{b\} \approx X_1$ and $\{a\} \times X_2 \approx X_2$ are connected, by Proposition 2.1.4, $C(a, b)$ is connected. Because

$$X_1 \times X_2 = \bigcup_{a \in X_1} C(a, b)$$

for any point b of X_2 and the intersection of $C(a, b)$ for $a \in X_1$ is nonempty, by Proposition 2.1.4, $X_1 \times X_2$ is connected. $\qquad\square$

In fact, the above proposition extends to an arbitrary product.

Theorem 2.1.8. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of connected spaces. Then the product $X = \prod_{\alpha \in I} X_\alpha$ is connected in the product topology.

Proof. We first fix a point $a = (a_\alpha)_{\alpha \in I} \in X$, and let

$$X_K = \{x \in X : x_\alpha = a_\alpha \text{ whenever } \alpha \in I \setminus K\}$$

for each finite subset K of I. In other words, X_K is the subset of X which consists of all points permitting all values only at each index in K.

Step 1: Showing that the union of all X_K is connected.
$X_K \approx \prod_{\alpha \in K} X_\alpha$ is connected by the preceeding theorem. Since each X_K contains a, the union Y of all X_K is connected.

Step 2: Showing that the closure of Y in X is X.
Suppose $p \in X$ and let W be a neighborhood of p. There is a basis member $B = \prod_{\alpha \in I} B_\alpha$ such that $p \in B \subset W$ (clearly, $J := \{\alpha \in I : B_\alpha \neq X_\alpha\} = \{\alpha_1, \cdots, \alpha_n\}$). Because B intersects X_J, B intersects Y, so $p \in \overline{Y}$ and $X = \overline{Y}$.

Therefore, the product space X is connected. $\qquad\square$

We give a problem in the textbook, with a solution using the cross we constructed in an earlier proof.

Problem 2.1.1. Let X, Y be connected spaces and A, B are nonempty proper subsets of X, Y, respectively. Show that $(X \setminus A) \times (Y \setminus B)$ is connected.

Solution. Let (p, q) be a point of $(X \setminus A) \times (Y \setminus B)$, and define

$$M := \bigcup_{x \in X \setminus A} C(x, q), \quad N := \bigcup_{y \in Y \setminus B} C(p, y).$$

It is easy to check that M, N are nonempty and contain (p, q), and $M \cup N = (X \setminus A) \times (Y \setminus B)$. Therefore, $(X \setminus A) \times (Y \setminus B)$ is connected.

Example 2.1.9. Since $\mathbb{R}$ is connected in the standard topology, $\mathbb{R}^{\mathbb{N}}$ is connected in the product topology. However, $\mathbb{R}^{\mathbb{N}}$ is not connected *in the uniform topology* (and in the box topology (why?)). To justify this assertion, let A be the set of all bounded sequences in $\mathbb{R}^{\mathbb{N}}$, and let B be the set of all unbounded sequences in $\mathbb{R}^{\mathbb{N}}$. It is clear by definition that $\mathbb{R}^{\mathbb{N}} = A \sqcup B$ and A, B are nonempty, and one can easily check that A and B are open in $\mathbb{R}^{\mathbb{N}}$. Thus, $\mathbb{R}^{\mathbb{N}}$ is not connected.

Theorem 2.1.10 (Intermediate value property)**.** Let X be a connected space, Y be an ordered set in the order topology, and let $f : X \to Y$ be a continuous map. If $a, b \in X$ and r is a point in Y between $f(a)$ and $f(b)$, there is a point $p \in X$ such that $f(p) = r$.

Proof. Suppose there is an intermediate value r such that $r \notin f(X)$. Then $f(X) = (f(X) \cap (-\infty, r)) \sqcup (f(X) \cap (r, \infty))$, forming a separation of $f(X)$. Here arises a contradiction, because the connectedness of X implies the connectedness of $f(X)$. $\qquad\square$

Problem 2.1.2. Show that a connected metrizable space with more than one point is uncountable.[1]

Solution. Let X be a connected metrizable space with more than one point. Let d be a metric on X inducing the topology on X, and define a function $f : X \to \mathbb{R}$ by $f(x) = d(x, b)$ for $x \in X$, where $b \in X$ is given. Note that $f(X)$ is connected because X is connected and f is continuous. Thus, $f(X)$ and X are uncountable, for $f(a) \neq f(b)$.

2.2 Path-connected spaces

Definition 2.2.1 (Path-connected space)**.** A space X is said to be path-connected if given any two point a and b in X, there is a continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. We call such f a path from a to b.

Proposition 2.2.2. A path-connected space is connected.

Proof. Suppose that a space X is path-connected but not connected. Then X has a separation (U, V). Choose a point $a \in U$ and $b \in V$, and let $f : [0, 1] \to X$ be a path from a to b. Since $[0, 1]$ is connected, the image of f is also connected. Then the image of f lies in U, for $f(0) = a \in U$; but $f(1) = b \notin U$, a contradiction. $\qquad\square$

Remark. The converse of the preceeding proposition is not true in general. In other words, a connected space need not be path-connected. As a counterexample, consider the topologist's sine curve $\overline{S}$, where S is the subspace of $\mathbb{R}^2$ defined as

$$S := \left\{ \left(x, \sin\left(\frac{1}{x} \right) \right) : 0 < x \leq 1 \right\}$$

Clearly, S is connected, so is its closure $\overline{S}$ in $\mathbb{R}^2$. Nevertheless, $\overline{S}$ is not path-connected. (Why?)

Proposition 2.2.3. A continuous image of a path-connected space is path-connected.

Proof. Let $f : X \to Y$ be a continuous map, where X is a path-connected space. Suppose $p, q \in f(X)$ and let a and b be points of X such that $f(a) = p$ and $f(b) = q$. If γ is a path from a to b, then $f \circ \gamma$ is a path from p to q. $\qquad\square$

Proposition 2.2.4. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of path-connected spaces with a point in common. Then the union of X_α for all $\alpha \in I$ is also path-connected.

Proof. Let p be a common point. Given two points x and y from the union, let f_1 and f_2 be paths lying in the union from x to p and from p to y, respectively. Concatenating f_2 after f_1 makes a path lying in the union from x to y. $\qquad\square$

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[1]The result of this problem will be generalized later. See Proposition 2.1.4.

Theorem 2.2.5. A product of path-connected spaces is path-connected in the product topology.

Proof. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of path-connected spaces, and let $X := \prod_{\alpha \in I} X_\alpha$. Given two points $a, b \in X$, let $f_\alpha : [0, 1] \to X_\alpha$ be a path from a_α to b_α for each $\alpha \in I$. Define a map $F : [0, 1] \to X$ by $F = (f_\alpha)_{\alpha \in I}$. Then the map F is a path from a to b relative to the product topology on X. $\qquad \square$

Remark. Regarding connectedness, adding some of limit points keeps the space connected. This is not valid for path-connectedness. (Consider the topologist's sine curve.)

Problem 2.2.1. Show that if A is an open connected subspace of $\mathbb{R}^2$, then A is path-connected.

Solution. Given $p \in A$, let $C(p)$ denote the set of points in A which can be joined to p by a path in A. We will show that $C(p)$ is both open and closed in A; since A and $\varnothing$ are the only open (and closed) subspaces of A and $C(p) \neq \varnothing$, we have $C(p) = A$, so every point of A can be joined to p by a path in A.

We first show that $C(p)$ is open in A. Given a point $q \in C(p)$, let r be a positive real number such that $B(q, r) \in A$ (openness of A is used). Since any two points of a ball can be joined by a line segment (this idea will be proved when proving $C(p)$ is closed in A), $B(q, r) \subset C(p)$, i.e., $C(p)$ is open.

To show that $C(p)$ is closed in A is equivalent to show that $S(p)$ is open in A, where $S(p) = A \setminus C(p)$. Let q be a point of $S(p)$, and assume that $B(q, r) \not\subset S(p)$ for all real $r > 0$. If ϵ is a positive real number such that $B(q, \epsilon) \subset A$ (openness of A is used), there is a point $z \in B(q, \epsilon) \cap C(p)$, so p and q can be joined by a path in A via z. This contradicts the hypothesis that $q \notin C(p)$, and this proves that $S(p)$ is open in A, as desired.

Problem 2.2.2. Show that every co-countable subspace of $\mathbb{R}^2$ is path-connected.

Solution. Given two points a, b of a co-countable subspace A of $\mathbb{R}^2$, there are countably many lines passing $a \in A$ which intersects $\mathbb{R}^2 \setminus A$. Because there are uncountably many lines passing $a \in A$, we can choose a line l_1 passing a not intersecting $\mathbb{R}^2 \setminus A$. We can also find a line l_2 passing b which does not intersect $\mathbb{R}^2 \setminus A$ and not parallel to l_1. Then $l_1 \cup l_2$ contains a path from a to b lying in A.

2.3 Connected components and path-connected components

Definition 2.3.1 ((Path-)connected component)**.** Let X be a topological space and let $\sim$ and $\sim_{\mathbf{p}}$ denote the relation on X defined as follows:

$$x \sim y \text{ if and only if there is a connected subspace } A \text{ of } X \text{ containing } x \text{ and } y,$$
$$x \sim_{\mathbf{p}} y \text{ if and only if there is a path from } x \text{ to } y \text{ lying in } X.$$

These relations are equivalence relations on X. Equivalence classes in $X/ \sim$ and $X/ \sim_{\mathbf{p}}$ are called a connected component of X and a path-connected component of X, respectively.

Proposition 2.3.2. Let X be a topological space. Then, the (path-)connected components of X form a partition of X and a (path-)connected subset of X is contained in precisely one of them. Furthermore, each (path-)connected component of X is (path-)connected.

Proof. Because $\sim$ and $\sim_{\mathbf{p}}$ are equivalence relations on X, the (path-)connected components of X form a partition of X.

We first prove the remaining part for connected case. First, if A is a connected subset of X, then it is obvious that A intersects only one connected component of X; if a and b are points of A, then C contains a and b, so $a \sim b$. To show that each connected component of X is connected, let C be a connected component of X and x_0 be a point of C. If x is a point of C, then there is a connected subset A_x of X containing x_0 and x. By the preceeding observation, $A_x \subset C$, and $C = \bigcup_{x \in C} A_x$, which is connected, since $x_0 \in A_x$ for all $x \in C$.

We now prove the remaining part for path-connected case. If A is a path-connected subset of X, then it is obvious that A intersects only one path-connected component of X; if a and b are points of A, then there is a path from a to b lying in X, so $a \sim_{\mathbf{p}} b$. It is obvious that a path-connected component of X is path-connected, because any two points of a path-connected component C of X can be joined by a curve γ lying in X and every point on γ is contained in C. $\qquad \square$

Remark. Since a path-connected component of X is path-connected and connected, each path-connected is connected is contained in a connected component. Hence, the collection of the path-connected components of X is a refinement of the collection of the connected components of X.

Definition 2.3.3 (Local (path-)connectedness). A space X is said to be locally (path-)connected at a point x of X if for every neighborhood U of x in X, there is a (path-)connected neighborhood V of x in X contained in U. If X is locally (path-)connected at every point of X, then X is said to be locally (path-)connected.

Remark. Global (path-)connectedness does not imply local (path-)connectedness. For example, the topologist's sine curve is connected but not locally connected.

Local (path-)connectedness can be defined in another way, given as follows:

Proposition 2.3.4. Let X be a topological space. Then X is locally (path-)connected if and only if for every open subset U of X each (path-)connected component of U is open in X.

Proof. Assume first that X is locally (path-)connected, and let U be a nonempty open subset of X, A be a (path-)connected component of U. To show A is open in X, let x be a point of A. Because $x \in U$, there is a (path-)connected neighborhood V of x such that $x \in V \subset U$. Since V is (path-)connected and V contains x, V is contained in A, proving that A is open in X. Assume conversely that each (path-)connected component of an open subset of X is open in X. For any point x of X and a neighborhoof U of X, the (path-)connected component A of U containing x is open in X, proving that X is locally (path-)connected at every point of X. $\qquad\square$

Theorem 2.3.5. Let X be a topological space. Then each path-connected component of X is contained in a connected component of X. When X is locally path-connected, the connected components of X and the path-connected components of X are the same.

Proof. The first half is already proved. Assume X is locally path-connected. Let x be a point of X, and let C and P be the connected and path-connected component of X containing x, respectively. Suppose $P \subsetneq C$, and define Q as the union of all path-connected components which intersect C, except for P. (Here, Q is nonempty, for Q is the union of all path-connected components containing the elements of $Q \setminus P \neq \varnothing$.) Since such path-connected components are connected and intersect C, they are contained in C. Hence, $C = P \cup Q$. Because X is locally path-connected, every path-connected component of X is open in X, so P and Q are open in X. Because P and Q are disjoint, they form a separation of C, a contradiction. $\qquad\square$

Example 2.3.6. The connected components of $\mathbb{R}_l$ are the singletons, hence the path-connected components of $\mathbb{R}_l$ are the singletons; if C is a connected component of $\mathbb{R}_l$ and $a, b \in C$ and $a < b$, we have $C = ((-\infty, b) \cap C) \sqcup ([b, \infty) \cap C)$, which form a separation of C. Since a continuous map maps a connected space onto a connected space, the only continuous maps from $\mathbb{R}$ into $\mathbb{R}_l$ are constant functions.

Example 2.3.7. Indeed, two topological spaces with different number of (path-)connected components are not homeomorphic. Suppose X and Y are homeomorphic topological spaces with m and n (path-)connected components, where $1 \leq m < n < \infty$, with a homeomorphism $f : X \to Y$. Since a continuous map maps a (path-)connected space onto a (path-)connected space, each of m (path-)connected components of X is mapped into a (path-)connected component of Y, so f fails to be surjective, a contradiction.

Example 2.3.8. Let $T = ([-1, 1] \times \{0\}) \cup (\{0\} \times [-1, 0]) \subset \mathbb{R}^2$. We will justify that T and the subspace $[0, 1]$ of $\mathbb{R}$ are not homeomorphic, by noticing that if $f : X \to Y$ is a homeomorphism and A is a subspace of X then $f|_A : A \to f(A)$ is a homeomorphism. Suppose T and $[0, 1]$ are homeomorphic and let $f : T \to [0, 1]$ be a homeomorphism. Since $S = T \setminus \{(0, 0)\}$ has three connected components, its image $f(T)$ also has three connected components. However, $f(T) = [0, 1] \setminus f((0, 0))$, which has at most two connected components. Therefore, T and $[0, 1]$ are not homeomorphic.

2.4 The topology of linear continuums

Definition 2.4.1. An ordered set L in the order topology is called a linear continuum if

(a) L satisfies the least upper bound property,

(b) and whenever $x, y \in L$ and $x < y$, there is a point $z \in (x, y)$.

Theorem 2.4.2. Suppose L is an ordered set in the order topology. Then the following are equivalent:

(a) L is a linear continuum.

(b) L and its intervals and rays are connected.

Proof. Read your paper note. $\qquad\qquad\square$

2.5 Compact spaces

Definition 2.5.1 (Compact space)**.** A topological space X is said to be compact if every covering of X by sets open in X has a finite subcover. If every such covering has a countable subcover, then X is called a Lindelöf space.

Remark. In fact, we can impose an alternative definition of compactness as follows:

> The topological space X is said to be compact, if for every collection $\mathcal{C}$ of closed sets in X with the finite intersection property, the intersection of the members of $\mathcal{C}$ is nonempty.

The equivalence of compactness and the above property can be checked by considering an appropriate contraposition.

As connectedness, compactness of subspaces can also be argued in larger spaces.

Theorem 2.5.2 (Compactness of subspaces)**.** Let A be a subspace of X. Then A is compact if and only if every covering of A by sets open in X has a finite subcover. (This statement is valid even if the words 'compact' are replaced by 'Lindelöf.')

Proof. Suppose A is compact, and $\mathcal{A} = \{A_\alpha\}_\alpha$ is a covering of A by sets open in X. Then the naturally induced collection $\{A_\alpha \cap A\}$ is a covering of A by sets open in A, and we can find a finite subcover. The corresponding finite subcover of $\mathcal{A}$ covers A.

Suppose conversely that every covering of A by sets open in X has a finite subcover, and let $\mathcal{A} = \{A_\alpha\}_\alpha$ be a covering of A by sets in A. Since each A_α can be written as $A \cap O_\alpha$ for some subset O_α open in X, the collection $\{O_\alpha\}_\alpha$ has a finite subcover. The corresponding finite subcover of $\mathcal{A}$ covers A. $\qquad\square$

Theorem 2.5.3. Regarding closedness and compactness of subspaces, the following statements holds:

(a) Every closed subspace of a compact space is compact.

(b) Every Hausdorff space is compactly normal. In other words, any two disjoint compact subspaces can be separated by (disjoint) neighborhoods.

(c) Every compact subspace of a Hausdorff space is closed. Hence, in a compact Hausdorff space, closedness and compactness of subspaces coincide. (In particular, every compact Hausdorff space is normal.)

Proof. (a) Let X be a compact space and Y be a closed subspace of X. Let $\mathcal{A} = \{A_\alpha\}_\alpha$ be a covering of Y by sets open in X. Because the collection $\mathcal{A} \cup \{X \setminus Y\}$ is also an open covering of X, the collection has a finite subcover. Among them, excluding $X \setminus Y$ gives a finite subcover of $\mathcal{A}$ which covers Y.

(b) We first prove that a Hausdorff space X is compactly regular. Let p be a point of X and A be a compact subspace of X not containing p. For each $a \in A$, let U_a and V_a be disjoint neighborhoods of a and p, respectively. Since A can be covered by finitely many U_a, the union of such U_a and the

intersection of the corresponding V_a are disjoint neighborhoods of A and p, respectively. Thus, X is compactly regular.

To show that X is compactly normal, let A and B be disjoint compact subspaces of X. For each $a \in A$, let U_a and V_a be disjoint neighborhoods of a and B, respectively. Since A can be covered by finitely many U_a, the union of such U_a and the intersection of the corresponding V_a are disjoint neighborhoods of A and B, respectively. Therefore, X is compactly normal.

(c) If A is a compact subspace of a Hausdorff space X, then (b) implies that $X \setminus A$ is open. Thus, a compact subspace of a Hausdorff space is closed. The desired coincidence follows from (a), and the desired normality follows readily. □

Proposition 2.5.4. A continuous image of a compact space is compact.

Proof. Left as an exercise. □

Proposition 2.5.5. Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. If A is a closed subspace of X, then A is compact, so $f(A)$ is compact (and closed in Y). □

Theorem 2.5.6. The product of finitely many compact spaces is compact.[2]

When proving the above theorem, the first version of the tube lemma will be used.

Lemma 2.5.7. (a) (The tube lemma) Consider the product space $X \times Y$, where Y is a compact space. If N is an open subspace of $X \times Y$ containing a slice $x_0 \times Y$ for some $\{x_0\} \in X$, then N contains a tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.

(b) (A generalized tube lemma) Let A and B be compact subspaces of X and Y, respectively, and let N be an open subspace of $X \times Y$ containing $A \times B$. Then, there are neighborhoods U of A in X and V of B in Y, respectively, such that $A \times B \subset U \times V \subset N$. ((a) readily follows from (b) when $A = \{x_0\}$ and $B = Y$.)

Proof of Lemma 2.5.7. (a) Since N is open, for each point $(x_0, y) \in \{x_0\} \times Y$, there is a neighborhood $A_y \times B_y$ of the point (x_0, y) contained in N, where A_y and B_y are open in X and Y, respectively. Because $\{x_0\} \times Y$ is homeomorphic to Y, $\{x_0\} \times Y$ is compact, so the slice can be covered by finitely many above basis members. For convinience, write such members as $A_i \times B_i$ for $i = 1, \cdots, n$. Letting $W = \bigcap_{i=1}^{n} A_i$, we have $\{x_0\} \times Y \subset W \times Y \subset N$.

(b) Since N is open, for each point $(a, t) \in \{a\} \times B$, there is a neighborhood $A_t^a \times B_t^a$ of the point (a, t) contained in N, where A_t^a and B_t^a are open in X and Y, respectively. Because the slice $\{a\} \times B$ is compact for each $a \in A$, the slice can be covered by finitely many neighborhoods; denote each of such neighborhoods by $A_i^a \times B_i^a$ for $i = 1, \cdots, n(a)$. Finally, define

$$U^a := \bigcap_{i=1}^{n(a)} A_i^a, \quad V^a := \bigcup_{i=1}^{n(a)} B_i^a.$$

Then, $U^a \times V^a$ is an open subset of $X \times Y$ such that $\{a\} \times Y \subset U^a \times V^a \subset N$. Since A is compact, there is a finite subcollection of $\{U^a\}_{a \in A}$ covering A; write it as $\{U^1, \cdots, U^k\}$. Finally, define

$$U := \bigcup_{i=1}^{k} U^k, \quad V := \bigcap_{i=1}^{k} V^k.$$

Then, U and V are open in X and Y, respectively, and $A \times B \subset U \times V \subset N$. □

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[2]In fact, this theorem extends to arbitrary products of compact spaces, which is called the Tychonoff theorem. The proof of the Tychonoff theorem will not be introduced in this note.

Proof of Theorem 2.5.6. Let X, Y be compact spaces, and $\mathcal{A}$ be an open cover of $X \times Y$. Given $x_0 \in X$, because the slice $\{x_0\} \times Y$ is compact, there are finitely many members of $\mathcal{A}$ covering the slice $\{x_0\} \times Y$, and such members cover a tube $U(x_0) \times Y$ about $\{x_0\} \times Y$. (Here, $U(x_0)$ is assumed to be a neighborhood of x_0 in X.) Due to the compactness of X, there are finitely many points $x_1, \cdots, x_n$ such that $\bigcup_{i=1}^{n} U(x_i) = X$, and this justifies the compactness of $X \times Y$. The general result is deduced inductively. $\qquad\square$

Before introducing some problems, we observe properties of ordered sets in the order topologies, regarding compactness.

Proposition 2.5.8. (a) Suppose X is an ordered set in the order topology, and assume that X is compact. Then X has both the greatest and the least element.

(b) (Extreme value theorem) Let $f : K \to Y$ be a continuous map, where K is a compact space and Y is an ordered set in the order topology. Then f attains the maximum and the minimum.

Proof. (a) If X has no least element, the open covering $\{(a, \infty) : a \in X\}$ of X contains no finite subcover of X, a contradiction. The same argument holds for the case where X has no greatest element, provided that (a, ∞) is replaced by $(-\infty, a)$ for each $a \in X$.

(b) The subspace $f(K)$ of Y is compact, and (a) implies that $f(K)$ has both the greatest and the least element. $\qquad\square$

Proposition 2.5.9. Let X be an ordered set in the order topology. If X satisfies the least upper bound property if and only if every closed interval in X is compact.

Proof. The proof of only if part is not given in this note. Assume that every closed interval in X is compact, i.e., $[a, b]$ is compact whenever $a, b \in X$ and $a < b$. Let S be a nonempty subset of X which is bounded above. We will show that S has the supremum (the least upper bound) in X. Let u be any upper bound of S in X, and consider the set

$$U := \bigcap_{a \in S} [a, u].$$

Then U is closed in X, so U is compact. Because U is a compact ordered set (in the order topology), U has the least element M.

We now show that M is the supremum of S. It is obvious that M is an upper bound of S in X; otherwise, there is an element $b \in S$ with $M < b$, which implies $M \notin U$, a contradiction. Assume that M is not the supremum of S. Then there is an upper bound v of S such that $v < M$. Because $v \geq a$ for all $a \in S$, we have $v \in U$, so $M \leq v$, a contradiction. $\qquad\square$

The first problem deals with the distance between a point and a subspace and neighborhoods of subspaces in a metric space. In particular, (c) implies that the original definition of the ϵ-neighborhood of a subspace and our intuition coincide.

Problem 2.5.1. Let (X, d) be a metric space and A be a nonempty subset of A.

(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

(b) Show that if A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$, whenever $x \in X$.

(c) Define the ϵ-neighborhood of A in X to be the set

$$U(A, \epsilon) := \{x \in X : d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

(d) Assume that A is compact, and let U be an open set in X containing A. Show that U contains an ϵ-neighborhood of A for some $\epsilon > 0$.

(e) Show that the result in (d) need not hold if A is closed but not compact.

Solution. (a) Almost clear. If $x \in \overline{A}$, then whenever $k > 0$, there is a point $a \in A$ such that $d(x, a) < k$. Hence, $d(x, A) = 0$. Assuming conversely, whenever $k > 0$, there is a point $a \in A$ such that $d(x, a) < k$, so $B_d(x, k)$ intersects A. Hence, $x \in \overline{A}$.

(b) Given a point $x \in X$, define a function $f_x : A \to [0, \infty)$ by $f_x(a) = d(x, a)$ for $a \in A$. Since $|f_x(a) - f_x(b)| \leq d(a, b)$ whenever $a, b \in A$, f_x is (uniformly) continuous. Because the domain A of f_x is a compact space, f_x attains the minimum. Furthermore, $d(x, A) = \inf\{d(x, a) : a \in A\} = \min\{f_x(a) : a \in A\}$, so $d(x, A) = d(x, a)$ for some $a \in A$.

(c) It is clear that every ϵ-ball with the center in A is contained in the ϵ-neighborhood of A, so one inclusion is obvious. Suppose $u \in U(A, \epsilon)$ so that $d(u, A) < \epsilon$. It implies that the ϵ-ball centered at u intersects A at a point $a \in A$, so the ϵ-ball centered at a contains u. Therefore, $U(A, \epsilon)$ is contained in the union of the ϵ-balls with centers in A.

(d) Let $f : A \to \mathbb{R}$ be a function defined by $f(a) = d(a, X \setminus U)$.

- f is continuous; for $a, b \in A$, we have $f(a) \leq d(a, x) \leq d(a, b) + d(b, x)$ for all $x \in X \setminus U$, so $f(a) \leq d(a, b) + f(b)$, from which, by symmetry, we obtain $|f(a) - f(b)| \leq d(a, b)$.
- $f(a) > 0$ for all $a \in A$; otherwise, if $f(p) = 0$ for some $p \in A$, then $p \in \overline{X \setminus U} = X \setminus U$, a contradiction.

Because A is compact, f attains the minimum $\delta > 0$. Hence, for example, if $\epsilon = \delta/2$, then $U(A, \epsilon)$ is contained in U.

(e) The x-axis in the xy-plane is closed but not compact, and $R = \{(x, y) \in \mathbb{R}^2 : y < e^{-|x|}\}$ is open and contains the x-axis. However, no neighborhood of the x-axis is contained in R.

Problem 2.5.2. Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then $Y := \bigcap_{A \in \mathcal{A}} A$ is connected.

Solution. Suppose Y is not connected, and let (C, D) be a separation of the intersection $Y = \bigcap_{A \in \mathcal{A}} A$. Because C and D are closed in Y and Y is closed in X, C and D are closed in X. Since X is normal, there are disjoint neighborhoods U and V of C and D, respectively.

Assume that $A \setminus (U \sqcup V)$ is empty for some $A \in \mathcal{A}$. Then $A \subset U \sqcup V$ and both $A \cap U, A \cap V$ are nonempty, for $Y \subset A$. Thus, A has a separation $(A \cap U, A \cap V)$, a contradiction; $A \setminus (U \sqcup V)$ is nonempty (i.e., $A \not\subset U \sqcup V$) for all $A \in \mathcal{A}$.

The collection $\mathcal{C} := \{A \setminus (U \sqcup V) : A \in \mathcal{A}\}$ is a simply ordered collection of sets closed in X, and $\mathcal{C}$ satisfies the finite intersection property. Because X is compact, the intersection of members in $\mathcal{C}$ is nonempty, implying that $Y \not\subset U \sqcup V$, a contradiction. Therefore, Y is connected.

2.6 Compact metrizable spaces

We first introduce some other types of compactness:

Definition 2.6.1. Let X be a space.

(a) X is said to be limit point compact, if every infinite subset of X has a limit point in X.

(b) X is said to be sequentially compact, if every (infinite) sequence of points of X has a convergent subsequence.

Remark. (a) Compactness implies limit point compactness.

(b) Let X be a metric space. If X is sequentially compact, then X is complete and totally bounded.[3]

Theorem 2.6.2. Suppose X is a metrizable space. Then the following are equivalent:

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[3]Remark that both completeness and total boundedness are properties of metric spaces, for they can be discussed only when a metric on a metrizable space is given.

(a) X is compact.

(b) X is limit point compact.

(c) X is sequentially compact.

Proof. We already proved that (a) implies (b) and it is easy to show that (b) implies (c). Thus, it remains to prove (a) under (c), so we assume that X is sequentially compact. In this proof, we use the strategy by imposing X a metric d to understand X as a metric space, not just a metrizable space.

Step 1: Showing the existence of a Lebesgue number.
We wish to prove that for any open cover $\mathcal{A}$ of X there is a real number $\delta > 0$ such that every open set in X with diameter less than δ is contained in some member of $\mathcal{A}$. If there is no such δ, for each $n \in \mathbb{N}$, there is an open set C_n of diameter less than $1/n$ which is not contained in any member of an open cover $\mathcal{A}$ of X. Choosing a point $x_n \in C_n$ for each $n \in \mathbb{N}$, by the sequential compactness of X, there is a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$ (with the limit denoted by x).[4] Let A be a member of $\mathcal{A}$ containing x, and let $r > 0$ be a real number such that $B_d(x, r) \subset A$. If a positive integer i is large enough so that

$$d(x, x_{n_i}) < \frac{r}{2} \quad \text{and} \quad \frac{1}{n_i} < \frac{r}{2},$$

then $C_{n_i} \subset A$, a contradiction.

Step 2: Deriving that X is compact.
Let $\mathcal{A}$ be an open cover of X and let $\delta > 0$ be a Lebesgue number for $\mathcal{A}$. Because a sequentially compact metric space is totally bounded, finitely many balls in X of radius $\delta/3$ cover X. So there is a finite subcollection of $\mathcal{A}$ which covers X. $\qquad \square$

Remark. The result of Step 1 in the above proof is also known as the Lebesgue number lemma.

For metric spaces, the above equivalence reduces to the Heine-Borel theorem.

Theorem 2.6.3 (Heine-Borel theorem)**.** Let (X, d) be a metric space. Then the following are equivalent:

(a) X is compact.

(b) X is limit point compact.

(c) X is complete and totally bounded.

Proof. By the preceeding theorem, it suffices to show that (c) is equivalent to compactness. Since it is already observed that sequential compactness implies (c), it suffices to show that (c) implies any of compactnesses. For this, we will show that (c) implies sequential compactness.

Let $(x_n)_n \subset X$ be a sequence. Since X is sequentially compact, there is a ball B_1 in X of radius 2^{-1} which contains x_n for infinitely many $n \in N_1 \subset \mathbb{N}$. Because $X \cap B_1$ is also totally bounded, a ball B_2 with the center in $X \cap B_1$ of radius 2^{-2} contains x_n for infinitely many $n \in N_2 \subset N_1$. Continuing inductively, we can find $n_i \in N_i$ for each $i \in \mathbb{N}$ with $n_1 < n_2 < \cdots$. Then $d(x_{n_i}, x_{n_k}) \le 2^{1-i}$ if $i < k$. Because X is complete, $(x_n)_n$ has a convergent subsequence. Therefore, (c) implies sequential compactness. $\qquad \square$

Problem 2.6.1. Let U be an open subset of $\mathbb{C}$ which contains $\mathbb{D}$. Show that there is a positive real number $r > 1$ such that $B(0, r) \subset U$.

Solution. Suppose that there is no such $r > 1$. Then, we can choose $a_n \in B(0, 1 + 1/n) \setminus U$ for each $n \in \mathbb{N}$. Since $(a_n)_{n \in \mathbb{N}} \subset \overline{B(0, 2)}$, $(a_n)_n$ contains a convergent subsequence $(a_{n_k})_{k \in \mathbb{N}}$. Letting α be the limit of $(a_{n_k})_k$, we find that α is a limit point of $\mathbb{C} \setminus U$. Because $\mathbb{C} \setminus U$ is closed, $\alpha \notin U$. On the other hand, $\alpha \in \partial \mathbb{D}$ (why?), we have $\alpha \in \partial \mathbb{D} \subset U$, a contradiction. Therefore, $B(0, r) \subset U$ for some real number $r > 1$.

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[4]It is natural to consider countably many objects when one should consider sequences.

Solution (Alternative solution). Note that the collection of 'polar rectangles' is a basis for the topology on $\mathbb{C}$, where a polar rectangle is of the form

$$a \leq \rho \leq b, \quad s \leq \theta \leq t,$$

where a, b, s, t are real numbers such that $a \leq b$. Because $\partial\mathbb{D}$ is compact, for each point $e^{ix} \in \partial\mathbb{D}$ ($0 \leq x < 2\pi$) there is a polar rectangle $P_x : a(x) \leq \rho \leq b(x), s(x) \leq \theta \leq t(x)$ such that $e^{ix} \in P_x \subset U$. Choosing finitely many members among P_x and letting r the smallest $b(x)$, we find that $B(0, r) \subset U$.

Problem 2.6.2. Let (X, d) be a metric space. Show that every isometry of X is surjective, when X is compact. (In fact, remarking the definition of an isometry, we may argue that every isometry of a compact metric space is an homeomorphism.)

Solution. Let $f : X \to X$ be an isometry of X, and suppose f is not surjective. Then there is a point $x \in X \setminus f(X)$. Because X is compact, $f(X)$ is compact, so $X \setminus f(X)$ is open in X. It implies the existence of a real number $r > 0$ such that $B_d(x, r) \cap f(X) = \varnothing$. Letting $x_0 = x$, define $x_n = f(x_{n-1})$ for each positive integer n, and let j, k be any positive integers with $j > k$. Then $d(x_j, x_k) = d(x_{j-1}, x_{k-1}) = \cdots = d(x_{j-k}, x) > r$, so the sequence $(x_n)_{n \in \mathbb{N}}$ has no convergent subsequence, which contradicts the compactness of X. Therefore, every isometry of a compact metric space is a homeomorphism.

2.7 Locally compact spaces and one-point compactification

Some of the properties which are most desired for a topological space to have are the space being metrizable or being a compact Hausdorff space. In this section, we impose a situation in which a topological space embeds into a compact Hausdorff space.

Definition 2.7.1 (Local compactness). A space X is said to be locally compact at a point $a \in X$ if there is a compact subspace C of X containing a neighborhood of a. If X is locally compact at every point of X, then X is said to be locally compact.

A simple observation follows:

Observation 2.7.2. Let $\{X_\alpha\}_\alpha$ be an indexed family of nonempty spaces.

(a) If $\prod_\alpha X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α. If the product space is compact, then each X_α is compact.

(b) The converse of the first statement in (a) is also true.

Proof. Write $X = \prod_\alpha X_\alpha$.
(a) Given a point $x \in X$, there is a compact subspace C of X and a basis member $\prod_\alpha B_\alpha$ such that $x \in \prod_\alpha B_\alpha \subset C$. Because $\pi_\alpha(C)$ is a compact subspace of X_α containing the neighborhood B_α of $x_\alpha \in X_\alpha$ for each α, each X_α is locally compact. Moreover, because $B_\alpha = X_\alpha$ for all but finitely many values of α, X_α is compact for all but finitely many values of α. In particular, if X is compact, then $X_\alpha = \pi_\alpha(X)$ is compact for all α.
(b) Given a point $x \in X$, for each α, use (local) compactness to find a compact subspace C_α of X_α containing a neighborhood B_α of x_α in X_α. When finding such subspaces, choose $B_\alpha = C_\alpha = X_\alpha$ whenever X_α is compact. Then, the product of B_α for all α is a basis member of X containing x; the product of C_α for all α is a compact subspace of X. $\qquad\square$

Theorem 2.7.3 (Existence and uniqueness of a one-point compactification). Let X be a space. Then X is locally compact and Hausdorff if and only if there is a space Y with the following properties:

(a) Y is a compact and Hausdorff space containing X as a subspace, i.e., Y is a compactification of X.

(b) $Y \setminus X$ is a one-point set.

Moreover, if Y_1 and Y_2 are such spaces, then Y_1 and Y_2 coincide on X and are homeomorphic, i.e., Y is uniquely determined up to equivalence.[5]

Proof. Since it is easier to check the uniqueness, we first explain the uniqueness (up to equivalence).

Step 1: Proving the uniqueness part.

Suppose Y_1 and Y_2 are compact Hausdorff spaces satisfying (a), (b), and (c). Let p and q denote the unique point of $Y_1 \setminus X$ and $Y_2 \setminus X$, respectively. Define a map $h : Y_1 \to Y_2$ by

$$h(x) = \begin{cases} x & \text{(if } x \in X) \\ q & \text{(otherwise, i.e., } x = p) \end{cases}.$$

We show h is a homeomorphism extending the identity map on X; and for this, it suffices to verify that h is a continuous map, because the openness of h will follow by symmetry. If U is an open subset of Y_2 contained in X, its preimage is U; because U is open in X and X is open in Y_1, U is open in Y_1. Assume U is an open subset of Y_2 containing q. Then the subset $C = Y_2 \setminus U$ is closed in Y_2, so C is compact and is contained in X. It follows that $h^{-1}(C) = C$ is a compact subspace of Y_1 and that $h^{-1}(C)$ is closed in Y_1. Therefore, h is a continuous map.

Step 2: Proving the existence part.

Suppose first that X is a locally compact Hausdorff space. Let p be any element not in X, and let $Y = X \sqcup \{p\}$. And impose a topology on Y by declaring the following subsets to be open in Y:

(T1) Subsets which are open in X.

(T2) Subsets of the form $Y \setminus C$, where C is a compact subspace of X. (See Problem 2.7.1.)

It is left as an exercise to check that the above collection is a topology on Y. (See Problem 2.7.1.)

We first show that Y contains X as a subspace. (Clearly, the topology on X is coarser than the subspace topology on X inherited from Y.) By merely intersecting any subset of Y of either type with X, one can easily observe that those two topologies are equal.

To show that Y is compact, let $\mathcal{A}$ be any open cover of Y. Then there is a member $Y \setminus C \in \mathcal{A}$ of the second type which contains p. Since C is a compact subspace of X, finitely many members of $\mathcal{A}$ cover C. These members, together with $Y \setminus C$, cover Y, as desired.

Finally, we show that Y is a Hausdorff space, in which it suffices to show that p and any point a in X can be separated by disjoint open subsets of Y. Since X is locally compact, there is a compact subspace C of X containing a neighborhood U of a in X, and $(U, Y \setminus C)$ is a desired pair.

Step 3: Proving the converse.

Suppose such space Y exists for a space X. Being a subspace of the Hausdorff space Y, X is a Hausdorff space. Given a point $a \in X$, because Y is a Hausdorff space, there are neighborhoods U and V of a and p in Y which are disjoint. Since $Y \setminus V$ is a closed subspace of Y contained in X, $Y \setminus V$ is a compact subspace of X containing U. Therefore, X is locally compact. □

Problem 2.7.1. (a) Show that the collection imposed in Step 2 of the above proof is a topology on Y.

(b) Explain why we assume in the second type that C is compact, rather than closed in Y.

Solution. (a) What we need to do is to check the axioms of topology.

Clearly, $\varnothing$ is of the first type and Y is of the second type, so they belong to the collection.

A union of sets of the first type is an open subset of X, hence the union is of the first type. If $\{V_\beta = Y \setminus C_\beta\}_\beta$ is a collection of sets of the second type (each C_β is a compact subspace of X), their union is of the second type; because the union is $Y \setminus K$ (where $K = \bigcap_\beta C_\beta$) and K is a closed subspace of every C_β, K is compact. The union of a set U of the first type and a set $V = Y \setminus C$ of the second type (C is a compact subset of X) is of the second type, since

$$U \cup (Y \setminus C) = Y \setminus (C \setminus U)$$

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[5]Equivalence of compactness is introduced in Chapter 4.

and $C \setminus U$ is compact (because $C \setminus U$ is a closed subset of the compact subset C).

A finite intersection of sets of the first type is an open subset of X, hende the intersection is of the first type. If $(V_\beta = Y \setminus C_k)_{k=1}^n$ is a collection of sets of the second type (each C_k is a compact subspace of X), their intersection is of the second type, because the union is $Y \setminus K$, where $K = \bigcup_{k=1}^n C_k$ is compact. The intersection of a set U of the first type and a set $V = Y \setminus C$ of the second type (C is a compact subspace of X) is of the first type, since

$$U \cap (Y \setminus C) = U \setminus C$$

is open in X.

(b) If C in the second type is assumed to be closed subsets of X rather than being compact, then Y may not be compact, for a closed subset C of X may not be compact. (Remark that a compact subset of a Hausdorff space is closed.)

Let X be a locally compact Hausdorff space, and let Y be a space (which is unique up to equivalence) constructed as above.

- If X is compact, then X is a closed subspace of Y, so the closure of X in Y is X, which is a proper subset of Y.

- If X is not compact, then X is not closed in Y (why?), so the closure of X in Y is Y.

If Y is such a space and the closure of X in Y is Y, Y is called the one-point compactification (or the Alexandroff compactification) of X.

Remark. The above observation states that a locally compact Hausdorff space X has a one-point compactification if and only if X is not compact.

Remark. In Chapter 4, we will consider the general concept of compactification (before studying the Stone-Čech compactification) over completely regular spaces. Since one-point compactifications are also compactifications, it will be better if it is explained that a locally compact Hausdorff space X is completely regular. If the space X is compact, then X is normal, so X is completely regular; if X is not compact, then X is a subspace of its one-point compactification, so X is completely regular.

For Hausdorff spaces, local compactness can be intuitively considered as follows:

Proposition 2.7.4. Suppose X is a Hausdorff space. Then X is locally compact if and only if given $x \in X$ and its neighborhood U in X, there is a relatively compact neighborhood V of a in X whose closure in X is contained in U.[6]

Proof. Suppose X is a locally compact Hausdorff space, and suppose further that a point $x \in X$ together with its neighborhood U in X is given. Since X has a compactification Y such that $Y \setminus X$ is a singleton, the subset $C := Y \setminus U$ is a closed (of course, compact) subspace of Y. By regularity, there are disjoint neighborhoods V of x and W of C in Y. Then, V is a neighborhood of x in X contained in $Y \setminus W \subset U$. Furthermore, if the overline notation denotes the closure of in Y, we have $\overline{V} \subset \overline{Y \setminus W} = Y \setminus W \subset U$. So, the closure of V in X satisfies $X \cap \overline{V} = \overline{V} \subset Y \setminus W \subset U$. Because $\overline{V}$ is closed in Y, it follows that V is relatively compact in X.

The converse is clear, for we may let C be the closure of V in X. $\square$

As a homeomorphism implies the topologically equivalence of two topological spaces, one might expect the one-point compactifications of two homeomorphic spaces to be homeomorphic.

Theorem 2.7.5. If $f : A \to B$ is a homeomorphism of locally compact Hausdorff spaces A and B, then f extends to a homeomorphism of the one-point compactifications of A and B.

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[6]By relatively compact we mean that the closure of the space is compact.

Proof. Assume that A and B are not compact (otherwise, there is nothing to prove), and let X and Y be the one-point compactifications of A and B, respectively. If f were to be extended to X, such an extension (as a set map) should be given as the bijection $\widetilde{f} : X \to Y$, which is defined by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{(if } x \in A) \\ q & \text{(if } x = p) \end{cases},$$

where p and q are the unique elements of $X \setminus A$ and $Y \setminus B$, respectively. To verify that $\widetilde{f}$ is a continuous map, let V be an open subset of Y. If V is contained in B, $\widetilde{f}^{-1}(V) = f^{-1}(V)$ is open in A, hence in X. If V is not contained in B, $Y \setminus V$ is closed in Y (hence, compact), implying that $\widetilde{f}^{-1}(Y \setminus V) = f^{-1}(Y \setminus V)$ is compact (hence, closed in X), i.e., $f^{-1}(V)$ is open in X. By symmetry, it follows that $\widetilde{f}$ has an open map. This completes the proof. $\qquad\square$

Example 2.7.6. We introduce some applications of Theorem 2.7.5.

(a) Since $\mathbb{R} \approx S^1 \setminus \{1\}$, the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1. Similarly, because $\mathbb{C} \approx \mathbb{R}^2 \approx S^2 \setminus \{n\}$ (where n is any point in S^2), the one-point compactification of $\mathbb{C}$ or $\mathbb{R}^2$ is homeomorphic to S^2. In addition, the one-point compactification of $\mathbb{C}^\times$ can be identified by gluing two distinct points of S^2.

(b) Since $\mathbb{N} \approx K$, where $K = \{1/n : n \in \mathbb{N}\}$, the one-point compactification of $\mathbb{N}$ is homeomorphic to $K \sqcup \{0\}$.

Chapter 3

Countability and separation axioms

3.1 Countability and separation axioms

3.1.1 Definitions regarding countability and separation axioms

Definition 3.1.1 (Countability axioms)**.** Let X be a topological space.

(a) X is said to have a countable base at a point $x \in X$ if there is a countable collection $\mathcal{B}$ of neighborhoods of x in X such that every neighborhood of x contains at least one member of $\mathcal{B}$.

(b) X is said to be a first-countable space if every point of X has a countable base.

(c) X is said to be a second-countable space if X has a countable basis.

Remark that the second countability implies the first countability.

For a second-countable space, countability does not remain only on the existence of countable bases. In fact, countability presents in every basis of the topology on a second-countable space.

Proposition 3.1.2. If a space X is second-countable and $\mathcal{C}$ is any basis of X, then $\mathcal{C}$ has a countable subcollection which is a basis of X.

Proof. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable basis of X. For each $m, n \in \mathbb{N}$, whenever it is possible, choose a member $C_m^n \in \mathcal{C}$ such that $B_m \subset C_m^n \subset B_n$. To be precise, for each $n \in \mathbb{N}$ and a point $p \in B_n$, find all possible members $C \in \mathcal{C}$ such that $p \in C \subset B_n$. For each C, find a member $B_m \in \mathcal{B}$ such that $p \in B_m \subset C \subset B_n$. If a member of $\mathcal{C}$ satisfying the last inclusion is already found, discard the newly found member of $\mathcal{C}$; otherwise, let such C be denoted by C_m^n.

We first show that the collection $\mathcal{C}^* := \{C_m^n\}_{m,n}$ is a countable basis of X. The countability of $\mathcal{C}^*$ is obvious, and the construction of $\mathcal{C}^*$ asserts that $\mathcal{C}^*$ covers X. Assume that a point $p \in X$ belongs to two members $C_{m_1}^{n_1}$ and $C_{m_2}^{n_2}$ of $\mathcal{C}^*$. Let C_0 be a basis member of $\mathcal{C}$ such that $p \in C_0 \subset C_{m_1}^{n_1} \cap C_{m_2}^{n_2}$, and let $l \in \mathbb{N}$ be an index such that $p \in B_l \subset C_0$. By the above construction, for some integer $i \in \mathbb{N}$, we have $p \in B_i \subset C_i^l \subset B_l$, so $\mathcal{C}^*$ is a countable basis for a topology on X.

Because $\mathcal{C}^*$ is contained in $\mathcal{C}$, it suffices to prove that the topology generated by $\mathcal{C}^*$ is finer than the topology on X, which is clear by the above construction: Given a point x of X with a basis member $B_n \in \mathcal{B}$ containing x, there is a basis member $C_l^n \in \mathcal{C}$ containing x for some $l \in \mathbb{N}$. $\qquad\square$

Definition 3.1.3 (Separation axioms)**.** A topological space X is called a Fréchet space (or a T_1 space) if every finite subset of X is closed in X. In the rest definitions, assume X is a Fréchet space.

(a) X is called a Hausdorff space (or a T_2 space) if given two distinct points a and b in X, there are disjoint neighborhoods of a and b.

(b) X is called a regular space (or a T_3 space) if given a point $a \in X$ and a nonempty closed subset $B \subset X$ not containing a, there are disjoint neighborhoods of a and B.

(c) X is called a **completely regular space** (or a $T_{3\frac{1}{2}}$ space) if, given a point $a \in X$ and a nonempty closed subset $B \subset X$ not containing a, there is a continuous function $f : X \to [0,1]$ such that $f(a) = 1$ and $f(B) = \{0\}$.

(d) X is called a **normal space** (or a T_4 space) if, given two nonempty disjoint subsets in X there are disjoint neighborhoods of those closed subsets.

Remark (Alternative definitions for some separabilities). Assume that X is a T_1 space.

(a) X is a regular space if and only if given a point $a \in X$ with its neighborhood U in X, there is a neighborhood of a in X whose closure in X is contained in U.

(b) X is a completely regular space if and only if given a point p of X and its neighborhood U in X, there is a continuous function $f : X \to [0,1]$ such that $f(p) = 1$ and $f(X \setminus U) = \{0\}$.

(c) X is a normal space if and only if given a nonempty closed subset $B \subset X$ with its neighborhood U in X, there is a neighborhood of B in X whose closure in X is contained in U.

For their proof, see Problem 3.2.1.

3.1.2 Basic properties of countabilities and separabilities

As one can expect from the fact that a basis of the topology plays a major role in theory, second-countability is a strong condition.

Proposition 3.1.4. Suppose X is a second-countable space.

(a) X is a Lindelöf space, i.e., every open cover of X contains a countable subcover.

(b) X is a separable space, i.e., X has a countable dense subset.

Proof. Left as an exercise. $\square$

Remark. A space X is said to be hereditarily Lindelöf space if every subspace of X is a Lindelöf space.

The converse of Proposition 3.1.4 are valid when the space is metrizable.

Proposition 3.1.5. For metrizable spaces, second countability, separability, and being a Lindelöf space coincide. In other words,

(a) every metrizable Lindelöf space is second countable.

(b) every metrizable separable space is second countable.

Proof. Let X be a metrizable space, and let d be a metric on X which induces the topology on X.
(a) For each $n \in \mathbb{N}$, the open cover $\{B_d\left(x, n^{-1}\right) : x \in X\}$ has a countable subcover; let each member of a countable subcover be denoted by $B_k^n = B_d(x_k^n, n^{-1})$ with $k \in \mathbb{N}$. We want to show that the collection $\mathcal{B} := \{B_k^n\}_{n,k \in \mathbb{N}}$ is a countable basis of the topology on X. Clearly, $\mathcal{B}$ is an open cover of X. If a point $p \in X$ is contained in two members $B_{k_1}^{n_1}$ and $B_{k_2}^{n_2}$, let r be the positive number which is the minimum among the following four positive numbers:

$$d(p, x_{k_1}^{n_1}), \quad \frac{1}{n_1} - d(p, x_{k_1}^{n_1}), \quad d(p, x_{k_2}^{n_2}), \quad \frac{1}{n_2} - d(p, x_{k_2}^{n_2}).$$

Suppose n is large so that $2/n < r$, and let j be an index such that B_j^n contains p. Then B_j^n is a member of $\mathcal{B}$ such that $p \in B_j^n \subset B_{k_1}^{n_1} \cap B_{k_2}^{n_2}$. Thus, $\mathcal{B}$ is a basis of a topology on X, and it is easy to show that $\mathcal{B}$ generates the metric topology on X induced by d.
(b) If $D = \{a_n\}_{n \in \mathbb{N}}$ is a countable dense subset of X, then $\{B_d(a_n, 1/k) : k \in \mathbb{N}\}$ is a basis of the topology on X. $\square$

We now introduce some inheritance properties.

Proposition 3.1.6 (Inheritance of countabilities)**.** Some countabilities are preserved as follow:

  (a) First and second countabilities are inherited to subspaces and countable product spaces.

  (b) Every closed subspace of a Lindelöf space is a Lindelöf space.

  (c) Every open subspace of a separable space is a separable space.

Proof. See Problem 3.2.2. □

Proposition 3.1.7 (Inheritance of separabilities)**.** Some separabilities are preserved as follow:

  (a) A subspace of a Hausdorff (regular, completely regular) space is a Hausdorff (regular, completely regular) spaces. (For a normal space X, a *closed* subspace of X is normal.)

  (b) A product of Hausdorff (regular, completely regular) spaces is a Hausdorff (regular, completely regular) spaces.

  (c) If the product space $\prod_{\alpha \in \mathcal{A}} X_\alpha$ is a Hausdorff (regular, normal) space, then so is each X_α for $\alpha \in \mathcal{A}$.

Proof. See Problems 3.2.3 to 3.2.6. □

3.1.3 Examples of regular spaces

Example 3.1.8 (Ordered spaces are regular)**.** Let X be an ordered space, and let $U = (a, b)$ be a basis member in X containing $p \in X$.

  (i) Suppose U consists of only one point p. Then, U is both open and closed in X, so we may choose $V = U$.

  (ii) Suppose that (a, p) or (p, b) is empty and the other is nonempty. Without loss of generality, we assume that (p, b) is empty and (a, p) is nonempty. Choose a point $a' \in (a, p)$ and let $V = (a', b) = (a', p]$. Any point $x \in X$ with $x > p$ does not belong to the closure of V in X; (p, ∞) does not intersect U. Any point $x \in X$ with $x < a'$ does not belong to the closure of V in X; $(-\infty, a')$ does not intersect V. Hence, the closure of V in X is contained in $[a', p] \subset U$.

  (iii) Suppose (a, p) and (p, b) are nonempty. Let a' and b' be points of X in (a, p) and (p, b), respectively, and let $V = (a', b')$. Clearly, V is a neighborhood of p in X, and it is easy to check that the closure of V in X is contained in U.

Therefore, every ordered space is a regular space. In fact, it is known that every ordered space is a normal space, whose proof will not be introduced in this note.

Example 3.1.9 (Locally compact Hausdorff spaces are (completely) regular)**.** Suppose X is a locally compact Hausdorff space, and let Y be a compact Hausdorff space containing X such that $Y \setminus X$ is a singleton. Then Y is normal, (completely) regular (illustrated in the following subsection), so X is also (completely) regular.

3.1.4 Examples of normal spaces

Theorem 3.1.10. Lindelöf regular spaces are normal.

Proof. Let A and B be disjoint closed subspaces of X. For each $a \in A$, let S_a be a neighborhood of a in X contained in $X \setminus B$. Using regularity, let U_a be a neighborhood of a in X whose closure in X is contained in S_a. And construct V_b for each $b \in B$ as we constructed U_a for each $a \in A$.

Even if $\{U_a\}_{a \in A}$ and $\{V_b\}_{b \in B}$ are open covers of A and B, respectively, the unions of the members in each collection need not be disjoint. (After drawing pictures on a sketchbook) one may wish to set

$$F_n := U_n \setminus \bigcup_{k=1}^{n} \overline{V_k}, \quad G_n := V_n \setminus \bigcup_{k=1}^{n} \overline{U_k}$$

for each $n \in \mathbb{N}$, and

$$F := \bigcup_{n=1}^{\infty} F_n, \quad G := \bigcup_{n=1}^{\infty} G_n.$$

We wish F and G to be disjoint neighborhoods of A and B, respectively. Clearly, F and G are neighborhoods of A and B, since

$$F = \bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} \overline{V_n} \supset A, \quad G = \bigcup_{n=1}^{\infty} V_n \setminus \bigcup_{n=1}^{\infty} \overline{U_n} \supset B.$$

Also, $F \cap G = \varnothing$; otherwise, $F_n \cap G_k \neq \varnothing$ for some $n, l \in \mathbb{N}$. Without loss of generality, we may assume $n \leq k$. If $p \in F_n \cap G_k$, then $p \in F_n \subset U_n$; on the other hand, G_k does not intersect $\overline{U_n}$, hence $p \notin G_k$.

Therefore, F and G separate A and B and X is a normal space. $\qquad \square$

Theorem 3.1.11. Metrizable spaces are normal.

Proof. Let X be a metrizable space and d be a metric on X inducing the topology on X. Let A and B be nonempty and disjoint closed subspaces of X. For each $a \in A$ and $b \in B$, let s_a and t_b be positive real numbers such that $B_d(a, s_a) \subset X \setminus B$ and $B_d(b, t_b) \subset X \setminus A$. Define

$$U := \bigcup_{a \in A} B_d(a, s_a/3), \quad V := \bigcup_{b \in B} B_d(b, t_b/3).$$

Then U and V are neighborhoods of A and B in X. If $U \cap V$ is nonempty, then $B_d(a, s_a/3) \cap B_d(b, t_b/3)$ contains a point $z \in X$. We then have the following inequality:

$$d(a, b) \leq d(a, z) + d(z, b) < \frac{s_a}{3} + \frac{t_b}{3} < \max\{s_a, t_b\},$$

so $b \in B_d(a, s_a)$ or $a \in B_d(b, t_b)$, a contradiction. Therefore, $U \cap V$ is empty, proving that X is normal. $\quad \square$

Theorem 3.1.12. Well-ordered sets are normal. (In fact, every ordered space is normal.)

Proof. Let X be a well-orderd set.

Step 1: Proving that every subspace of X of the form $(x, y]$ is open in X.
Let $A = (x, y]$ be a subset of X. Then $X \setminus A = (-\infty, x] \sqcup (y, \infty)$; because (y, ∞) has the least element $y' \in X$, $X \setminus A = (-\infty, x] \sqcup [y', \infty)$ is closed in X.

Step 2: Proving that X is normal.
Let A and B be nonempty disjoint closed subspace of X.

(i) Assume that neither of them contains the least element m of X. For each $a \in A$, there is a neighborhood $(x_a, a]$ of a in X not intersecting B; for each $b \in B$, there is a neighborhood $(y_b, b]$ of b in X not intersecting A. Define

$$U := \bigcup_{a \in A} (x_a, a], \quad V := \bigcup_{b \in B} (y_b, b].$$

They are open in X and U covers A, and V covers B. If U and V are not disjoint, $(x_a, a] \cap (y_b, b]$ is nonempty for some $a \in A$ and $b \in B$, Without loss of generality, we may assume $a < b$; then we have $a \in (y_b, b]$, a contradicton.

(ii) Suppose, without loss of generality, A contains m. Because the singletone $\{m\} = [m, m] = (-\infty, m]$ is not only closed but also open in X, $A \setminus \{m\}$ is also closed in X. By the preceeding part, there are disjoint neighborhood U and V of $A \setminus \{m\}$ and B, respectively. Then, $U \cup \{m\}$ and $V \setminus \{m\}$ are disjoint neighborhoods of A and B, respectively.

Therefore, every well-ordered space is normal. $\qquad \square$

Example 3.1.13 ($\mathbb{R}_l$ is normal). Suppose A and B are nonempty disjoint closed subsets of $\mathbb{R}_l$. For each $a \in A$, let $[a, x_a)$ be a neighborhood of a in $\mathbb{R}_l$ contained in $\mathbb{R}_l \setminus B$, and for each $b \in B$, let $[b, y_b)$ be a neighborhood of b in $\mathbb{R}_l$ contained in $\mathbb{R}_l \setminus A$. Define

$$U := \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad V := \bigcup_{b \in B} [b, y_b),$$

and assume $U \cap V \neq \varnothing$. Then $[a, x_a) \cap [b, y_b)$ is nonempty for some $a \in A$ and $b \in B$. Since $a \neq b$, we may assume $a < b$, which implies that $a < b < x_a < y_b$ and that b is contained in $[a, x_a)$, a contradiction. Therefore, $U \cap V = \varnothing$, proving that $\mathbb{R}_l$ is a normal space.

Example 3.1.14 (Compact Hausdorff spaces are normal). Because closedness and compactness coincide in compact Hausdorff spaces and Hausdorff spaces are compactly normal, compact Hausdorff spaces are normal.

3.2 Countability and separation axioms (Problems)

Problem 3.2.1. Show that the suggested alternative definition for regular, completely regular, and normal spaces are equivalent to the original definitions.

Solution. We use the overline notation to denote the closure in X.

(a) Assume $B = \{x\}$ for a given point $x \in X$ in the proof of (c).

(b) (In this case, the proof is quite simple, for we do not have to consider larger subsets.) Let X be a completely regular space and suppose that a point x of X and its neighborhood U in X is given. Then $B = X \setminus U$ is closed in X, so there is a continuous map $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(B) = \{0\}$.

Assume conversely, and let x be a point of X and B be a closed subset of X not containing x. Then $U = X \setminus B$ is a neighborhood of x in X, so there is a continuous map $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) = \{0\}$.

(c) Let X be a normal space and suppose that a closed subset B of X and its neighborhood U in X are given. Then $C = X \setminus U$ is closed in X, so there are disjoint neighborhoods V of B and W of C in X. Then $B \subset V \subset X \setminus W$ and $\overline{V} \subset \overline{X \setminus W} = X \setminus W \subset X \setminus C = U$.

Assume conversely, and let B and C be disjoint closed subsets of X. If V is a neighborhood of B in X such that $\overline{V} \subset X \setminus C$, then V and $X \setminus \overline{V}$ are disjoint neighborhood of B and C in X.

Problem 3.2.2. Prove Proposition 3.1.6.

Solution. (a) Let Y be a subspace of X. If $\{B_n\}_{n \in \mathbb{N}}$ is a countable base at $p \in Y$ in X, then $\{B_n \cap Y\}_{n \in \mathbb{N}}$ is a countable base at p in Y; if $\{B_n\}_{n \in \mathbb{N}}$ is a countable basis of the topology on X, then $\{B_n \cap Y\}_{n \in \mathbb{N}}$ is a countable basis of the topology on Y.

For spaces $X_1, X_2, \cdots$, let $\mathcal{B}_n = \{B_{n,k}\}_{k \in \mathbb{N}}$ be a countable base at $x_n \in X_n$ in X_n. Then

$$\left\{ \prod_{j \in \mathbb{N}} U_j : \begin{array}{l} U_j \in \mathcal{B}_j \text{ for each } j \in \mathbb{N} \text{ and} \\ U_j = X_j \text{ for all but finitely many values of } j \end{array} \right\}$$

is a countable base at $x \in \prod_{n \in \mathbb{N}} X_n$, where $\pi_n(x) = x_n$ for all $n \in \mathbb{N}$. When each $\mathcal{B}_n$ is a countable basis of the topolgy on X_n, then the above collection is a countable basis of the topology on $\prod_{n \in \mathbb{N}} X_n$.

(b) Let Y be a closed subspace of a Lindelöf space X, and let $\{A_\alpha\}_\alpha$ be a covering of Y by sets open in Y (then, for each α, $A_\alpha = Y \cap O_\alpha$ for an open subset O_α of X). Since $X \setminus Y$ is open in X, countably many members A_α, together with $X \setminus Y$, cover X, as desired.

(c) Let X be a separable space and Y be an open subspace of X. And let D be a countable dense subset of X, and use the overline notation to denote the closure in X. We wish to show that $D \cap Y$ is a (countable) dense subset of Y, i.e., $\overline{D \cap Y} \cap Y = Y$. For this, it suffices to show $Y \subset \overline{D \cap Y}$. In fact, if $p \in Y$ and whenever V is a neighborhood of p in X contained in Y, then V contains a point of $D \cap Y$. Hence, every point of Y belongs to $\overline{D \cap Y}$, as desired.

Problem 3.2.3. Prove Proposition 3.1.7 for Hausdorff spaces.

Solution. All (a), (b), and (c) are obvious.

Problem 3.2.4. Prove Proposition 3.1.7 for regular spaces.

Solution.   (a) Let X be a regular space and Y be a subspace of X. Assume that $p \in Y$ and C is a closed subset of Y not containing p. Because $C = Y \cap A$ for some subset A closed in X and A does not contain p, by regularity of X, there are disjoint neighborhoods U of p and V of A in X. Then $Y \cap U$ and $Y \cap V$ are disjoint neighborhoods of p and C in Y.

(b) Let X_α be a regular space for each $\alpha \in \mathcal{A}$ (here, $\mathcal{A}$ is an index set), and let $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$. To justify the regularity of X, we make use of the alternative definition. Let x be a point of X and U be a neighborhood of x in X. Here, we may assume that $U = \prod_{\alpha \in \mathcal{A}} B_\alpha$, where B_α is open in X_α for each $\alpha \in \mathcal{A}$ and $B_\alpha = X_\alpha$ for all but finitely many values of $\alpha \in \mathcal{A}$. For each $\alpha \in \mathcal{A}$, use the regularity of X_α to find a neighborhood V_α of x_α in X_α whose closure in X_α is contained in B_α (when $B_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$). Then $V = \prod_{\alpha \in \mathcal{A}} V_\alpha$ is a neighborhood of x in X whose closure is contained in U.

(c) Given $\alpha \in \mathcal{A}$, let p_α be a point of X_α and C_α be a closed subset of X_α not containing p_α. Let x be any point of X whose α-component is p_α and let $K = \pi_\alpha^{-1}(C_\alpha)$. Because K is closed in X and does not contain x, by the regularity of X, there are disjoint neighborhoods U of x and V of K in X. Applying the result of the following remark (together with the homeomorphism), we can conclude that X_α is regular.

Remark (An embedding of a space into a product space). Let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of topological spaces, and assume that X_α is a singleton whenever $\alpha \neq \beta$. Then X_β is homeomorphic to $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$. Indeed, one can impose an explicit homeomorphism. Let s_α be the unique element of S_α for each $\alpha \neq \beta$, and let $\imath : X_\beta \to X$ be the map defined by

$$(\pi_\alpha \circ \imath)(x) = \begin{cases} x & \text{(if } \alpha = \beta) \\ s_\alpha & \text{(otherwise)} \end{cases} \quad \text{for all } x \in X_\beta.$$

Then $\imath$ is a desired homeomorphism. As an example, $\mathbb{R}^2$ and the plane $\{(x,y,z) \in \mathbb{R}^3 : y = 17\} \approx \mathbb{R}^2 \times \{17\}$ are homeomorphic.

Problem 3.2.5. Prove Proposition 3.1.7 for completely regular spaces.

Solution.   (a) Let Y be a subspace of a completely regular space X, and let p be a point of Y and A be a closed subset of Y. Then $A = Y \cap B$ for some closed subset B of X, and B does not contain p. By the complete regularity of X, there is a continuous map $f : X \to [0,1]$ separating p and B. The restriction of f to Y separates p and A.

(b) Let X_α be a completely regular space for each $\alpha \in \mathcal{A}$ (here, $\mathcal{A}$ is an index set), and let $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$. Let p be a point of X and U be a neighborhood of p in X. We may assume that $U = \prod_{\alpha \in \mathcal{A}} B_\alpha$, with B_α being an open subset of X_α for each $\alpha \in \mathcal{A}$ and $B_\alpha = X_\alpha$ for all but $\alpha_1, \cdots, \alpha_n \in \mathcal{A}$. For each $i = 1, \cdots, n$, let $f_i : X_{\alpha_i} \to [0,1]$ be a continuous map such that $f_i(p_{\alpha_i}) = 1$ and $f_i(X_{\alpha_i} \setminus B_{\alpha_i}) = \{0\}$. Then, the map $f : X \to [0,1]$ defined by $f(x) = f_1(x_{\alpha_1}) \times \cdots \times f_n(x_{\alpha_n})$ for $x \in X$ is a continuous map with the properties that $f(p) = 1$ and $f(X \setminus U) = \{0\}$.

Problem 3.2.6. Prove Proposition 3.1.7 for normal spaces.

Solution. (a) Suppose Y is a closed subspace of a normal space X. If A and B are disjoint closed subspaces of Y, then they are closed in X, too. By normality, there are disjoint open subspaces U and V in X containing A and B, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y containing A and B.

(c) Given $\alpha \in \mathcal{A}$, let B_α and C_α be disjoint closed subsets of X_α, and let $B = \pi_\alpha^{-1}(B_\alpha)$ and $C = \pi_\alpha^{-1}(C_\alpha)$. Because $B \cap C = \varnothing$, by the regularity of X, there are disjoint neighborhoods U of B and V of B in X. Again, applying the natural homeomorphism given in the previous remark, one can conclude that X_α is normal.

Problem 3.2.7. Let $f, g : X \to Y$ be continuous maps, where Y is a Hausdorff space. Show that $\{x \in X : f(x) = g(x)\}$ is a closed subspace of X.

Solution. We prove the closedness of the given subset by proving that $A := \{x \in X : f(x) \neq g(x)\}$ is open in X. Note that $A = \bigcup_{a,b \in Y, \, a \neq b}(f^{-1}(\{a\}) \cap g^{-1}(\{b\}))$. If, for each pair (a, b) of distinct points of Y, U_a and V_b are disjoint neighborhoods of a and b in Y, we have

$$A = \bigcup_{a,b \in Y, \, a \neq b} (f^{-1}(U_a) \cap g^{-a}(V_b)),$$

because $f^{-1}(U_a) \cap g^{-1}(V_b) = \bigcup_{s \in U_a} \left(f^{-1}(\{s\}) \cap g^{-1}(V_b)\right) = \bigcup_{s \in U_a, \, t \in V_b}(f^{-1}(\{s\}) \cap g^{-1}(\{t\}))$ for each pair (a, b). It easily follows that A is open in X.

Problem 3.2.8. For a topological space X and its subset A, we say that A is *discrete* if every point of A is an isolated point of A in X. Show that an uncountable subset of a second countable space is indiscrete, i.e., every uncountable subset of a second countable space has a limit point.

Solution. Let $\mathcal{B}$ be a countable basis of the second countable space X, and suppose A is an uncountable discrete subset of X. For each point x in A, let $B_x \in \mathcal{B}$ be chosen so that $B_x \cap A = \{x\}$. Since the collection of B_x for $x \in A$ is countable, there are at most countably many distinct singletons $\{x\}$ for $x \in A$, a contradiction.

3.3 The Urysohn lemma

Theorem 3.3.1 (Urysohn lemma). If X is a normal space, then any two (nonempty) disjoint closed subspaces of X can be separated by a continuous function on X. To be precise, if A and B are (nonempty) disjoint closed subspaces of X, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Remark. The Urysohn lemma implies that if every pair of closed sets in X can be separated by disjoint open sets, i.e., X is normal, then each such pair can be separated by a continuous function. The converse is obvious. (Why?)

Proof. Let A and B be nonempty and disjoint closed subsets of X.

Step 1. Let P be the set of rational numbers in $[0, 1]$ and write $P = \{x_1, x_2, x_3, \cdots\}$ with $x_1 = 1, x_2 = 0$. Now define an open subset U_p for $p \in P$ as follows:

(1) Fitst, let $U_1 := X \setminus B$, which contains A. Second, by normality, we can find a neighborhood U_0 of A in X whose closure is in U_1.

(2) Suppose we have constructed U_p for $p \in P_n$, where $P_n = \{x_1, \cdots, x_n\}$ for $n \geq 3$ such that $p, q \in P_n$ with $p < q$ implies

$$\overline{U_p} \subset U_q. \tag{3.1}$$

For convinience, denote $r = p_{n+1}$. We want U_r to satisfy eq. (3.1) with all U_p with indices in P_{n+1}. For this, it suffices to care the immediate successor s and predecessor p of r in P_{n+1}; if U_r satisfies eq. (3.1) with U_p and U_s, then U_r satisfies eq. (3.1) with all the other U_p with $p \in P_{n+1}$.

Since U_p and U_s satisfies $\overline{U_p} \subset U_s$, by normality, we can find a neighborhoof U_r of the closed subspace $\overline{U_p}$ in X such that $\overline{U_r} \subset U_s$. For such U_r, eq. (3.1) is satisfied for every pair of U_p with $p \in P_{n+1}$.

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By induction, we have U_p defined for all $p \in P$.

Step 2. We now define U_p for all rational numbers p as follows: just let $U_p = \varnothing$ whenever $p < 0$ and $U_p = X$ whenever $p > 1$. For such U_p with $p \in \mathbb{Q}$, eq. (3.1) is satisfied.

Step 3. For each point $x \in X$, define the set

$$C(x) := \{p \in \mathbb{Q} : x \in U_p\}.$$

For each x, the set $C(x)$ is bounded (below by a nonnegative real number, and above by 1). Thus, we may define the function $f : X \to [0, 1]$ by $f(x) = \inf C(x)$ for $x \in X$.

Step 4. We now show that the function f is a continuous map mapping A onto $\{0\}$ and B onto $\{1\}$. First, since every point $a \in A$ belongs to U_0, $f(a) = 0$; since every point $b \in B$ does not belong to U_1, $f(b) = 1$. Before proving continuity, we note the following lemma:

$$x \in \overline{U_r} \text{ implies } f(x) \le r, \text{ and } x \notin U_r \text{ implies } f(x) \ge r.$$

To check the continuity, fix a point $x \in X$ and let (a, b) be a neighborhood of $f(x)$ in $\mathbb{R}$. We want to find a neighborhood W of x in X whose image under f is contained in (a, b).[1] Using the density of $\mathbb{Q}$, we can find rational numbers c, d such that $a < c < f(x) < d < b$. If we can find a neighborhood of x in X whose image under f is a subset of $[c, d]$, the proof ends. By the above lemma, if $c < f(x) < d$, then $x \in U_d \setminus \overline{U_c}$. Hence, $W := U_d \setminus \overline{U_c}$ is a neighborhood of x in X. Furthermore, if $y \in W$, then $c \le f(y) \le d$, hence W is a neighborhood of x in X with the image in (a, b). Therefore, f is a continuous function from X into $[0, 1]$ mapping A into $\{0\}$ and B onto $\{1\}$. $\qquad\square$

Problem 3.3.1. Prove the lemma introduced in the fourth step of the proof of Theorem 3.3.1.

Solution. Suppose first that $x \in \overline{U_r}$. Whenever $r < p$, since $x \in \overline{U_r} \subset U_p$, $C(x)$ contains $p \in \mathbb{Q}$ with $p > r$; $f(x) = \inf C(x) \le r$. Conversely, if $x \notin U_r$, whenever $p < r$, since $\overline{U_p} \subset U_r$, $p \notin C(x)$ for all $p < r$ and it implies that $f(x) = \inf C(x) \ge r$.

The following problem deals with a property of completely regular spaces, and its solution does not require the Urysohn lemma.

Problem 3.3.2. Let X be a completely regular space and A, B be disjoint closed subspaces of X. Show that if A is compact, then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Solution. For each point $a \in A$, let $f_a : X \to [0, 1]$ be a continuous function such that $f_a(a) = 0$ and $f_a(B) = \{1\}$. Note that the collection $\{f_a^{-1}([0, r)) : a \in A\}$ is an open cover of A by sets open in X, where r is a real number such that $0 < r < 1/2$. Thus, A can be covered by finitely many sets $f_{a_i}^{-1}([0, r))$ for $i = 1, \cdots, n$. Define the function $f : X \to [0, 1]$ by $f(x) = f_1(x) \times \cdots \times f_n(x)$ for $x \in X$. Then $0 \le f(a) < r$ for all $a \in A$ and $f(B) = \{1\}$. If $g : [0, 1] \to [0, 1]$ is the function defined by

$$g(x) = \begin{cases} 0 & \text{if } 0 \le x < r \\ \dfrac{x - r}{1 - r} & \text{otherwise} \end{cases},$$

then the composite $g \circ f$ is a desired continuous map from X into $[0, 1]$.

In Problem 2.1.2, we have proved that every connected metrizable space having more than one point is uncountable. As we have studied in the preceeding section that every metrizable space is normal, one might want to generalize the result to a broader category of spaces. In the following problem, we will prove that every connected regular space with more than one point is uncountable.

Problem 3.3.3 (Nontrivial connected regular spaces are uncountable)**.** Prove that a connected normal space having more than one point is uncountable. And deduce that a connected regular space having more than one point is uncountable.

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[1] When trying to prove continuuity with the original definition of continuity, you might encounter some technical problem.

Solution. Suppose X is a connected normal space having more than one point. Since every singletone is closed, there is a continuous map $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$, where a, b are disjoint points of X. Since X is connected, $f(X)$ is connected, so $f(X) = [0, 1]$. Therefore, X is necessarily uncountable.

Assume there is a connected regular space X with more than one point which is countable. It is clear that X is Lindelöf, so X is a Lindelöf regular space; a normal space. It contradicts to the former result.

3.4 The embedding theorem

In this section, we introduce the embedding theorem, stating that every completely regular space can be embedded into a (possibly uncountable dimensional) Euclidean space $\mathbb{R}^I$ for some I. As an application of the embedding theorem, we will prove Urysohn metrization theorem, stating that every second-countable regular space is metrizable, using the fact that a countable dimensional Euclidean space $\mathbb{R}^{\mathbb{N}}$ is metrizable.

Theorem 3.4.1 (The embedding theorem)**.** Let X be a completely regular space and let $\{f_\alpha\}_{\alpha \in I}$ be a collection of continuous functions from X into $\mathbb{R}$ separating points and closed subspaces of X. Define the function $F : X \to \mathbb{R}^I$ by $F = (f_\alpha)_{\alpha \in I}$. Then F denotes an embedding of X into $\mathbb{R}^I$. Furthermore, if each f_α maps X into $[0, 1]$, then F denotes an embedding of X into $[0, 1]^I$.

Proof. It is clear that F is continuous, since $\mathbb{R}^I$ (or $[0, 1]^I$) equips the product topology. If a and b are disjoint points of X, there are indices $i, j \in I$ such that $f_i(a) = 1$, $f_i(b) = 0$ and $f_j(a) = 0$, $f_j(b) = 1$, hence F is injective. Therefore, to show that F is an embedding of X, it remains to show that F is a homeomorphism from X into its image $F(X)$. For this, we need to show that F maps an open set in X onto an open set in $F(X)$.

Suppose U is an open subset of X, and let z be a point of $F(U)$, and let x be a point of U such that $F(x) = z$. We seek to find a neighborhood of z in $F(X)$ contained in $F(U)$. Let $i \in I$ be an index such that $f_i(x) > 0$ and $f_i(X \setminus U) = \{0\}$, and set $W = \pi_i^{-1}((0, \infty)) \cap F(X)$.

(a) Clearly, W is open in $F(X)$ and W contains z.

(b) If $y \in W$, there is a point $a \in X$ such that $F(a) = y$. Since $y \in \pi_i^{-1}((0, \infty))$, we have $f_i(a) = \pi_i(y) > 0$, so $a \notin X \setminus U$, implying $y \in F(U)$. Thus, W is a neighborhood of z in $F(X)$ which is contained in $F(U)$.

Therefore, F denotes an embedding of X into $\mathbb{R}^I$ (or into $[0, 1]^I$). $\qquad\square$

We now introduce the Urysohn metrization theorem, whose statement is stronger than every second countable regular space is normal.

Theorem 3.4.2 (Urysohn metrization theorem)**.** Every second countable regular space is metrizable.

To prove this theorem, we first prove the following countability lemma.

Lemma 3.4.3. Let X be a second countable regular space. Then there is a countable family of continuous functions from X into $[0, 1]$ separating points of X from closed subsets of X.

Proof of Lemma 3.4.3. Let $\{B_n\}$ be a countable basis of the topology on X. Whenever possible, let $f_{m,n}$ be a continuous function from X into $[0, 1]$ such that $f(\overline{B_m}) = \{1\}$ and $f(X \setminus B_n) = \{0\}$, where $\overline{B_m} \subset B_n$ (this could be done by applying Urysohn lemma). The collection of such continuous function will separate points of X from closed subsets of X, and the collection is countable. $\qquad\square$

Proof of Theorem 3.4.2. By the embedding theorem, X can be embedded into $\mathbb{R}^{\mathbb{N}}$. Since $\mathbb{R}^{\mathbb{N}}$ is a countable product of metrizable spaces, $\mathbb{R}^{\mathbb{N}}$ is metrizable, and its subspaces are also metrizable. $\qquad\square$

In the following chapter, the embedding theorem is applied when studying the Stone-Čech compactification of a completely regular space, to which evert continuous function into a compact Hausdorff space extends continuously (and uniquely[2]).

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[2]The uniqueness follows from the assumption that the codomain is a Hausdorff space.

Chapter 4

Compactification of completely regular spaces

4.1 Compactification

In this section, we introduce a general definition of a compactification, which, of cource, covers a one-point compactification.

Definition 4.1.1 (Compactification)**.** Given a space X, a space Y is called a compactification of X if the following are satisfied:

(i) Y is a compact Hausdorff space containing X as a subspace.

(ii) The closure of X in Y is Y.

Two compactifications Y_1 and Y_2 of X is said to be equivalent if there is a homeomorphism from Y_1 into Y_2 extending the identity map on X.

Remark. If the space X has a compactification Y, because X is a subspace of Y, X is necessarily completely regular.

It is natural to wonder if every completely regular space has a compactification. Hopefully, every completely regular space has a compactification, as justified by the lemma which will be introduced soon. The lemma states first that every space which embeds into a compact Hausdorff space has a compactification, and second that if such compactification is required to satisfy an "extension property" given as in the lemma, then such compactificaiton is unique up to equivalence.

Lemma 4.1.2 (Compactification extending an embedding)**.** Let X be a space such that there is an embedding h of X into a compact Hausdorff space Z. Then, there is a unique (up to equivalence) compactification Y of X with the following property: There is an embedding $H : Y \hookrightarrow Z$ extending h.[1]

$$
\begin{array}{ccc}
X & \xrightarrow{\text{ inclusion }} & Y \\
 & \searrow_{h} & \downarrow_{H} \\
 & & Z
\end{array}
$$
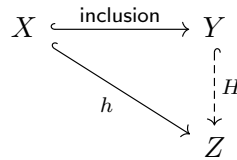
Such compactification Y is called the compactificaiton of X induced by h.

Proof. We first prove the existence of a compactification Y of X induced by $h : X \hookrightarrow Z$, and then we prove its uniqueness.

(a) Let $X_0 := h(X) \approx X$ and Y_0 be the closure $\overline{X_0}$ of X_0 in Z. To argue that Y_0 is a compactification of X_0, we check the axioms of compactificaitions. Because Y_0 is a closed subset of Z, Y_0 is a

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[1]Here, Y is unique up to equivalence, but H need not be unique.

compact Hausdorff space. Also, Y_0 contains X_0 as a subspace; every open subset of X_0 is of the form $X_0 \cap O = X_0 \cap (Y_0 \cap O)$, where O is an open subset of Z. Finally, the closure of X_0 in Y_0 is clearly $Y_0 \cap \overline{X_0} = Y_0$. Therefore, Y_0 is a compactification of X_0.

To find a compactification Y of X, we seek to find a space Y such that (X, Y) and (X_0, Y_0) are homeomorphic. Let A be any set disjoint from X which is in bijection with $Y_0 \setminus X_0$ (say $k : A \to Y_0 \setminus X_0$ is such a bijection), and define $Y = X \sqcup A$. Define a bijective map $H : Y \to Y_0$ by

$$H(x) = h(x) \quad \text{if } x \in X,$$
$$H(a) = k(a) \quad \text{if } a \in A.$$

Topologize Y by declaring that $U \subset Y$ is open in Y if and only if $H(U)$ is open in Y_0. (Indeed, the collection induced by such declaration is a topology on Y.) This topologization makes H a homeomorphism. It is easy to justify that Y is a compactification of X; it is because H extends h and $h : X \to X_0$ and $H : Y \to Y_0$ are homeomorphisms. To be brief, it is because the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow[\approx]{h} & X_0 \\
\imath \downarrow & & \downarrow \imath_0 \\
Y & \xrightarrow[H]{\approx} & Y_0
\end{array}
\; ,
$$

where $\imath$ is the inclusion embedding of X into Y and $\imath_0$ is the inclusion embedding of X_0 into Y_0.

(b) Suppose Y_1 and Y_2 are compactifications of X which extends h to embeddings from Y_1 and Y_2 into Z, respectively. Denote such embeddings by $H_1 : Y_1 \hookrightarrow Z$ and $H_2 : Y_2 \hookrightarrow Z$, respectively. By restricting the codomains of H_1 and H_2, we find that $H_2 \circ H_1^{-1}$ denotes a homeomorphism between Y_1 and Y_2 extending the identity map on X. Therefore, Y_1 and Y_2 are equivalent.

Therefore, if X is a space with an embedding h into a compact Hausdorff space Z, then X has a compactification Y such that there is an embedding of Y into Z extending h, and such a compactification exist uniquely up to equivalence. $\qquad \square$

Remark. We now jestify that every completely regular space has a compactification. If X is a completely regular space, by the embedding theorem, X could be embedded into $[0, 1]^I$ for some I, which is a compact Hausdorff space. Applying Lemma 4.1.2 to this situation gives a compactification of X. Therefore, a space X has a compactification if and only if X is completely regular.

4.2 The Stone-Čech compactification

It is highly recommended to review the embedding theorem before studying this section. Also, since this section covers a specific type of compactification, throughout this section we assume that X is a completely regular space, even if there is no mention about it.

It is known that finding a compactification Y of a (completely regular) space X to which every continuous map from X into $\mathbb{R}$ is continuously extended is a basic problem. Regarding this, if a given real-valued function on X were to be extended, then the function must have been bounded.

The idea we use to find such a compactification of X is to apply Lemma 4.1.2. To be precise, we first find an appropriate embedding of X in terms of all the bounded continuous functions on X. Lemma 4.1.2 then asserts the existence of an extended embedding of a compactification of X, which surely is in terms of the "extended" bounded continuous functions on Y. This idea is applicable, since we know from the embedding theorem that a completely regular space can be embedded into $[0, 1]^I$ for some I, which is a compact space.

For convinience, write $I = C_b^0(X, \mathbb{R})$ and for each $\alpha \in I$, let $B_\alpha = [\inf(\alpha), \sup(\alpha)] \subset \mathbb{R}$, and define

$$B = \prod_{\alpha \in I} B_\alpha.$$

Since I separates points of X from closed subsets of X, the map $F : X \to B$ defined by $F = (f)_{f \in I}$ is an embedding of X into the compact Hausdorff space B. By Lemma 4.1.2, there is a (unique) compactification Y of X such that F extends to an embedding F_* of Y into B, and the f-component f_* of F_* is a desired extension of f, since $f_* = \pi_f \circ F_*$ is continuous. The uniqueness of an extension of f easily follows from the assumption that the codomain $\mathbb{R}$ is a Hausdorff space.

To sum up, by considering an embedding F with regard to $C_b^0(X, \mathbb{R})$ and considering the compactification Y of X induced by F, we could show that every function in $C_b^0(X, \mathbb{R})$ can be uniquely extended to a continuois function on Y. The above observation is summarized as the following proposition:

Proposition 4.2.1 (Real-version of Stone-Čech compactification)**.** Let X be a completely regular space. There is a compactification Y of X satisfying the following property: Any function $f \in C_0^b(X, \mathbb{R})$ extends continuously to Y uniquely.

What we have shown is the existence of a compactification Y of X to which every bounded real-valued continuous function on X extends continuously to Y uniquely. Our goal is to find a compactification satisfying a more general property, which turns out to be a universal property:

Theorem 4.2.2 (Universal property of the Stone-Čech compactification)**.** Let X be a completely regular space. Then, there is a unique (up to equivalence) compactification $\beta(X)$ of X satisfying the following universal property: For any compact Hausdorff space K and a continuous map $f : X \to K$, there is a unique continuous map $f_* : \beta(X) \to K$ extending f. In other words, the pair $(\beta(X), \imath)$ ($\imath$ is the inclusion embedding of X into $\beta(X)$) makes the following diagram commutes:

We call such an extension $\beta(X)$ (or the pair $(\beta(X), \imath)$) the Stone-Čech compactification of X.

If there exists such Y for each X, then the Stone-Čech compactification can be defined by the above universal property. For this, one needs to show the existence part, rather than the uniqueness (up to equivalence) part.[2] The following proposition states that a compactification Y of X in Proposition 4.2.1 is, in fact, a compactification of X to which every continuous map from X to a compact Hausdorff space extends continuously and uniquely.

Proposition 4.2.3 (Existence part)**.** Let X be a completely regular space and Y be the compactification of X satisfying the property in Proposition 4.2.1. Then every continuous map from X to a compact Hausdorff space extends to Y continuously and uniquely. To be precise, for any compact Hausdorff space K and a continuous map $f : X \to K$, there is a unique continuous map $f_* : Y \to K$ extending f.

Proof. Remark that a compact Hausdorff space is completely regular, so we can embed K into $[0,1]^I$ for some I (let $e : K \hookrightarrow [0,1]^I$ be such an embedding). So we consider the composition $g := e \circ f : X \to e(K) \subset [0,1]^I$ rather than f.

For each $\alpha \in I$, find the unique extension $(g_\alpha)_* : Y \to [0,1]$ of $g_\alpha : X \to [0,1]$. Then, the map $g_* : Y \to [0,1]^I$ defined by $g_* = ((g_\alpha)_*)_{\alpha \in I}$ is the unique continuous extension of $g : X \to [0,1]^I$.

Since we have composited e at the left of f to obtain g, to assert that $f_* := e^{-1} \circ g_*$ is a desired extension of f, we need to show that f_* is well-defined. Then, it naturally follows that f_* is a unique continuous extension of f. Because the closure of X in Y is Y, we have

$$g_*(Y) \subset \overline{g_*(X)} = \overline{g(X)} \subset \overline{e(K)} = e(K).$$

Therefore, f_* is well-defined, so f_* is a desired extension of $f : X \to K$. The uniqueness follows from the assumption that K is a Hausdorff space. $\qquad\square$

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[2] Any object defined by a universal property exists uniquely up to some sense (for example, isomorphism or equivalence).

Problem 4.2.1. Show that a Stone-Cěch compactification of a completely regular space X is unique up to equivalence.

Solution. Let Y_1 and Y_2 be two Stone-Cěch compactifications of X, and let $\imath_k$ denote the inclusion embedding of X into Y_k for $k = 1, 2$. Clearly, each embedding is continuous. Hence, the universal property of Y_1 gives the continuous extension $(\imath_2)_\star : Y_1 \to Y_2$ of $\imath_2$ and the universal property of Y_2 gives the continuous extension $(\imath_1)_* : Y_2 \to Y_1$, and these extensions surely extends the identity map on X. Since the universal property of Y_1 gives a unique extension, the extension $(\imath_1)_* : Y_1 \to Y_1$ of $\imath_1$ is forced to be the identity map on Y_1. Therefore, $(\imath_1)_\star \circ (\imath_2)_*$ is the identity map on Y_1. Similarly, $(\imath_2)_* \circ (\imath_1)_\star$ is the identity map on Y_2. Therefore, $(\imath_1)_\star$ and $(\imath_2)_*$ are homeomorphisms, proving that Y_1 and Y_2 are equivalent. See the following commutative diagrams.
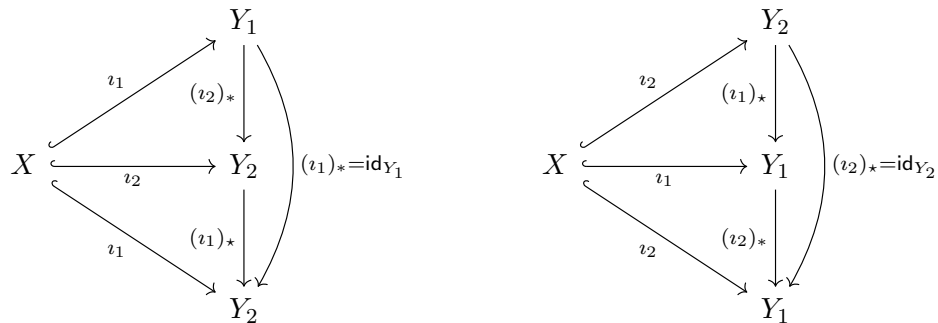
Chapter 5

Complete metric spaces and the space of continuous maps

5.1 Complete metric spaces

5.1.1 Complete metric spaces and uniform metrics

Definition 5.1.1 (Complete metric space). Let (X, d) be a metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ of points in X is called a Cauchy sequence, if given $\epsilon > 0$, there is an integer $N > 0$ such that $n, m > N$ implies $d(x_n, x_m) < \epsilon$. The space X is called a complete metric space if every Cauchy sequence in X is convergent.

Remark. (a) (Reduction of criterion) Let (X, d) be a metric space. X is a complete metric space if and only if every Cauchy sequence in X has a convergent subsequence.

(b) Suppose that (X, d) is a complete metric space. Then a closed subspace of X is complete: every Cauchy sequence in a closed subspace C of X is a Cauchy sequence in X, hence it has the limit, which should belong to C. Conversely, a complete subspace of X is closed in X: If A is a complete subspace of X and p is a limit point of A, we can construct a Cauchy sequence of points in A which converges to p, and completeness implies that $p \in A$. Therefore, for a complete metric space, closedness and completeness of subspaces coincide.

(c) The product of countably many complete metric spaces is, under the induced D-metric, complete: Note that a sequence $(x_n)_n$ converges to x in the product space if and only if $(\pi_k(x_n))_n$ converges to x_n for all k. Because $(x_n)_n$ is a Cauchy sequenc in the product space, it can be easily shown that each $(\pi_k(x_n))_n$ is a Cauchy sequence in the k-th complete metric space.

Contrary to the part (c) in the preceeding remark, one cannot assert that $\mathbb{R}^I$ is complete for arbitrary I, when $\mathbb{R}^I$ equips the product topology.

Definition 5.1.2 (Uniform metric). Given a metric space (X, d) and an index set I, we define the uniform metric $\overline{\rho}$ corresponding to d on X^I as follows:

$$\overline{\rho} : X \times X \to [0, \infty), \quad (x, y) \mapsto \sup\{\overline{d}(x_\alpha, y_\alpha) : \alpha \in I\}.$$

One should remark that the topology induced by the uniform metric if finer than the product topology and coarser than the box topology.

In particular, if $f, g : I \to X$, then $\overline{\rho}(f, g) = \sup\{d(f(a), g(a)) : a \in I\}$.

Theorem 5.1.3. If (Y, d) is a complete metric space, then $(Y^X, \overline{\rho})$ is complete, where X is an index set.

Proof. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in Y^X. We first assert that $(f_n)_n$ is pointwise convergent; because $\overline{\rho}(f_n, f_m) \to 0$ as $n, m \to \infty$, for each $x \in X$, $(f_n(x))_n$ is a Cauchy sequence in Y, justifying the assertion. We now prove that $f_n \to f$ in $\overline{\rho}$, i.e., uniformly.[1] Let N be an integer such that $n, m \geq N$

---

[1]Indeed, the convergence in $\overline{\rho}$ and the uniform convergence coincide.

implies $\overline{\rho}(f_n, f_m) < \epsilon/2$; given $x \in X$, let p be an integer not less than N such that $d(f_p(x), f(x)) < \epsilon/2$. Then,

$$d(f_n(x), f(x)) \leq d(f_n(x), f_p(x)) + d(f_p(x), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon, \tag{5.1}$$

so $\overline{\rho}(f_n, f) \leq \epsilon$. Therefore, $f_n \to f$ uniformly. $\qquad\square$

Theorem 5.1.4 (Complete spaces of functions)**.** Let X be a topological space, (Y, d) be a complete metric space, and assume that Y^X equips the uniform metric $\overline{\rho}$ corresponding to d.

  (a) $C(X, Y)$ and $B(X, Y)$ are closed subspaces of Y^X.

  (b) Hence, if Y is complete, then $C(X, Y)$ and $B(X, Y)$ are complete.

Proof. Because part (b) follows directly from part (a), it suffices to prove part (a). Assume first that $f \in Y^X$ is a limit point of $C(X, Y)$. Then, there is a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous functions from X into Y, which converges to f uniformly. Because the convergence is uniform, f is necessarily continuous. The proof for $B(X, Y)$ is also easy, which is left as an exercise. $\qquad\square$

For example, when X is compact, we may impose the supremum(sup) metric

$$\|\cdot\| : C(X, Y) \times C(X, Y) \to [0, \infty), \quad (f, g) \mapsto \sup\{d(f(x), g(x)) : x \in X\}$$

on $C(X, Y)$, and such imposition is general.

5.1.2   Isometric embedding of a metric space

Recall that we call a map h from a metric space (X, d_X) into (Y, d_Y) such that $d_Y(h(a), h(b)) = d_X(a, b)$ for all $a, b \in X$ an isometric embedding of X into Y. If $h : X \to Y$ is an isometric embedding and Y is complete, the closure of $h(X)$ in Y is also a complete metric space, and we call this space the completion of X.

When trying to define objects, one should be interested in their existence, uniqueness, and exactness.

Theorem 5.1.5 (Existence of a completion of a metric space)**.** Let (X, d) is a metric space, and consider the map $\phi : X \to B(X, \mathbb{R})$ defined by

$$\phi(x) = l_x \text{ for } x \in X,$$

where $l_x : X \to \mathbb{R}$ maps $t \in X$ to $d(t, x) - d(t, a)$ (here, a is a given point of X, so $|l_x(t)| \leq d(x, a)$ for all $t \in X$). Then ϕ denotes an isometric embedding of X into $B(X, \mathbb{R})$. Because $\mathbb{R}$ is complete, the closure of $\phi(X)$ in $B(X, \mathbb{R})$ is complete, i.e., every metric space has a completion.

Theorem 5.1.6 (Uniqueness of a completion of a metric space)**.** Let $h_1 : X \hookrightarrow Y_1$ and $h_2 : X \hookrightarrow Y_2$ be isometric embeddings of X into complete metric spaces (Y_1, d_1) and (Y_2, d_2). Denote the closure of $h_i(X)$ in Y_i by $\overline{h_i(X)}$ $(i = 1, 2)$. Then, there is an isometry $H : \overline{h_1(X)} \to \overline{h_2(X)}$ which equals $h_2 \circ h_1^{-1}$ on $h_1(X)$.[2] Hence, the completion of X is unique up to an isometry.

---

[2]By an isometry we mean an isometric homeomorphism.

Proof of Theorem 5.1.5. Since it is checked that $l_x \in B(X, \mathbb{R})$, the map ϕ is well defined. Thus, it remains to check that ϕ is an isometric embedding of X into $B(X, \mathbb{R})$. For this, it suffices to show that $\overline{\rho}(l_x, l_y) = d(x, y)$ for all $x, y \in X$, where $\overline{\rho}$ is the uniform metric on $B(X, \mathbb{R})$. Observing that $l_x(t) - l_y(t) = d(t, x) - d(t, y)$, we can find that $\overline{\rho}(l_x, l_y) \leq d(x, y)$ and $l_x(x) - l_y(x) = -d(x, y)$, proving that $\overline{\rho}(l_x, l_y) = d(x, y)$. $\qquad\square$

Proof of Theorem 5.1.6. One should remark that $\overline{h_i(X)}$ is a complete metric space $(i = 1, 2)$. Define a map $H : \overline{h_1(X)} \to \overline{h_2(X)}$ as follows:

(1) When $x \in h_1(X)$, let $H(x) := h_2(h_1^{-1}(x))$. Since h_1 is a bijection between X and $h_1(X)$, the above setting is not ambiguous.

(2) When $x \in \overline{h_1(X)} \setminus h_1(X)$, set a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $h_1(X)$, converging to x (note that x is in a closure) and let $H(x) := \lim H(x_n)$.

   This setting seems to be ambiguous, because the resulting $H(x)$ might differ by the choice of $(x_n)_n$ (also, even the existence of the limit could be inquired).

   - As for the existence of the limit, note that $d_2(Hx_n, Hx_m) = d_1(x_n, x_m)$ so that the sequence $(Hx_n)_n$ is a Cauchy sequence in $\overline{h_2(X)}$.

   - As for the unambiguity, suppose $(a_j)_{j\in\mathbb{N}}$ is another sequence in $h_1(X)$ converging to x. Letting $X = \lim H(x_n)$ and $A = \lim H(a_j)$, we find that

$$
\begin{aligned}
d_2(X, A) &\leq d_2(X, H(x_n)) + d_2(H(x_n), H(a_j)) + d_2(H(a_j), A) \\
&\leq d_2(X, H(x_n)) + d_1(x_n, a_j) + d_2(H(a_j), A) \xrightarrow{n,j\to\infty} 0,
\end{aligned}
$$

   so $X = A$ and the above setting of $H(x)$ is not ambiguous.

We just established a map H from $\overline{h_1(X)}$ into $\overline{h_2(X)}$. It remains to show that H is an isometry. For this, it suffices to show that H is bijective and isometric, and H coincides $h_2 \circ h_1^{-1}$ on $h_1(X)$.

(a) Given $a, b \in \overline{h_1(X)}$, let $(a_n)_n$ and $(b_n)_n$ be seqences in $h_1(X)$ converging to a and b, respectively. Because $d_2(Ha, Hb) \leq d_2(Ha, Ha_n) + d_2(Ha_n, Hb_n) + d_2(Hb_n, Hb)$ and $d_2(Ha_n, Hb_n) \leq d_2(Ha_n, Ha) + d_2(Ha, Hb) + d_2(Hb, Hb_n)$, we have $\lim d_2(Ha_n, Hb_n) = d_2(Ha, Hb)$. Therefore, $d_2(Ha, Hb) = \lim d_2(Ha_n, Hb_n) = \lim d_1(a_n, b_n) = d_1(a, b)$. This implies that H is isometric and injective. The surjectivity is easily checked. (How?)

(b) Clearly, H extends $h_2 \circ h_1^{-1}$.

Therefore, there is an isometry between two completions of X. In other words, there is a unique completion of a metric space up to an isometry. $\qquad\square$

5.2 Pointwise equicontinuous collection of continuous maps

Definition 5.2.1 (Equicontinuity). Let X be a topological space, (Y, d) be a metric space, and let $\mathcal{F}$ be a subset of $C(X, Y)$, i.e., a collection of continuous functions from X into Y.

(a) (Pointwise equicontinuity) The collection $\mathcal{F}$ is said to be equicontinuous at a point $x_0 \in X$ if, for any $\epsilon > 0$, there is a neighborhood U of x_0 in X such that $x \in U$ and $f \in \mathcal{F}$ implies $d(f(x), f(x_0)) < \epsilon$. If $\mathcal{F}$ is equicontinuous at every point of X, then $\mathcal{F}$ is said to be pointwise equicontinuous.

(b) (Uniform equicontinuity) Suppose that X is a metric space and d_X be the metric on X inducing the topology on X. The collection $\mathcal{F}$ is said to be uniformly equicontinuous if, for every $\epsilon > 0$, there is $\delta > 0$ such that $a, b \in X$ with $d_X(a, b) < \delta$ and $f \in \mathcal{F}$ implies $d(f(a), f(b)) < \epsilon$.

Remark. In the course of introduction to mathematical analysis, we studied the following version of Ascoli's theorem:

Suppose K is a compact metric space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued continuous functions (continuous functions with values in a complete metric space, respectivley) on K. If $(f_n)_n$ is pointwise bounded (totally bounded under the sup metric) and uniformly equicontinuous, then $(f_n)_n$ is uniformly bounded and $(f_n)_n$ contains a uniformly convergent subsequence.

Lemma 5.2.2. Suppose that X is a topological space and (Y, d) is a metric space, and assume $C(X, Y)$ equips the uniform topology. If $\mathcal{F} \subset C(X, Y)$ is totally bounded under the uniform metric, then $\mathcal{F}$ is pointwise equicontinuous under d.

Proof. Fix a positive real number $0 < \epsilon < 1$ and write $\mathcal{F} = \bigcup_{i=1}^N B_{\overline{\rho}}(f_i, \epsilon)$, where $f_i \in \mathcal{F}$ for each i. To show pointwise equicontinuity, we fix a point $p \in X$ and choose a neighborhood U_i of p in X such that $x \in U_i$ implies $d(f_i(x), f_i(p)) < \epsilon$; when $U = \bigcap_{i=1}^N U_i$, then U is still a neighborhood of p in X and whenever $x \in U$ we have $d(f_i(x), f_i(p)) < \epsilon$ for all i. Given $f \in \mathcal{F}$, choose an index i such that $\overline{\rho}(f, f_i) < \epsilon$. Because

$$d(f(x), f(p)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(p)) + d(f_i(p), f(p)) < 3\epsilon$$

whenever $x \in U$. Therefore, every totally bounded subset of $C(X, Y)$ is pointwise equicontinuous. $\qquad\square$

The preceeding lemma states that a totally bounded (in the uniform metric) subset of continuous functions is pointwise equicontinuous. Its reverse is generally not true; for example, the countable collection $(f_n : \mathbb{R} \to \mathbb{R}, \ x \mapsto n)_{n\in\mathbb{N}}$. The following lemma states that the converse to the preceeding lemma is valid if both domain and codomain are compact.

Lemma 5.2.3. Let X be a topological space and (Y, d) be a metric space. Assume that both X and Y are compact. If the subset $\mathcal{F}$ of $C(X, Y)$ is pointwise equicontinuous, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.

Remark. Suppose X is a topological space and (Y, d) is a metric space. When we consider the space $C(X, Y)$, we first consider the uniform metric $\overline{\rho}$. If X is assumed to be compact, we can also consider the sup metric, which will be denoted by ρ.

(a) As the following properties are particularly related to points within arbitrarily small distance and both $\overline{\rho}$ and ρ coincide at every pair of continuous functions within distance less than 1, they are common properties of $C(X, Y)$ under each metric: completeness, total boundedness. (For example, if $C(X, Y)$ under the uniform metric is complete, then so is $C(X, Y)$ under the sup metric, and vice versa.)

(b) The boundedness of $C(X, Y)$ is not common; while the space may be unbounded under the sup metric, the space is clearly bounded under the uniform metric; $C(X, Y) = B_{\overline{\rho}}(0, 2)$, where 0 in the first argument is the zero map.

Proof. According to the preceeding remark, it suffices to justify the total boundedness under the sup metric ρ.

Using pointwise equicontinuity, for each point $a \in X$, find a neighborhood D_a of a in X such that $x \in D_a$ and $f \in \mathcal{F}$ implies $d(f(x), f(a)) < \epsilon$. Since X is compact, we can cover X with finitely many D_a, suppose $\{D_{a_i}\}_{i=1}^k$ covers X and write $U_i = D_{a_i}$. On the other hand, cover Y with finitely many open balls $V_1, \cdots, V_m$ of diameter less than ϵ.

Let J be the collection of all functions from $\{1, \cdots, k\}$ into $\{1, \cdots, m\}$. Given $\alpha \in J$, if there is a function $f \in \mathcal{F}$ such that $f(a_i) \in V_{\alpha(i)}$ for all $i = 1, \cdots, k$, choose one such function $f \in \mathcal{F}$ and label it f_α. The collection $\{f_\alpha\}$ is indexed by a subset I of J and is thus finite.

Choose a map $f \in \mathcal{F}$. Because V_j $(j = 1, \cdots, m)$ cover Y, for each i, $f(a_i)$ belings to some V_j; write $f(a_i) \in V_{\beta(i)}$, and consider the function f_β. Given a point $x \in X$, there is an index i such that $x \in U_i$, and

$$d(f(x), f_\beta(x)) \leq d(f(x), f(a_i)) + d(f(a_i), f_\beta(a_i)) + d(f_\beta(a_i), f_\beta(x)).$$

The first and the last term in the right-hand side are, respectivley, less than ϵ (pointwise equicontinuity) and the second term is also less than ϵ (the diameter of V_i is less than ϵ), so $d(f(x), f_\beta(x)) < 3\epsilon$. This implies that $f \in B_\rho(f_\beta, 4\epsilon)$. Therefore, $\mathcal{F}$ is covered by the open balls $B_\rho(f_\alpha, 4\epsilon)$ for $\alpha \in I$, so $\mathcal{F}$ is totally bounded under the sup metric. $\qquad\square$

The above theory, together with the theory developed in the following section, will be used in the last section when we prove Ascoli's theorem.

5.3 Topologies on the space of continuous functions

In this section, we first impose topologies on $C(X, Y)$ to which we can correspond pointwise convergence and uniform convergence. Throughout this section, unless stated otherwise, X and Y are assumed to be topological spaces.

5.3.1 Topology of pointwise convergence and uniform topology

Definition 5.3.1 (Topology of pointwise convergence)**.** The topology on Y^X generated by the collection of subsets

$$S(x, U) := \{f \in Y^X : f(x) \in U\} \quad (x \in X, U \text{ is open in } X)$$

is called the topology of pointwise convergence, or the point-open topology.

From now on, we say a sequence $(f_n)_{n \in \mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ pointwise when the sequence converges to f in the topology of pointwise convergence.

Remark. (a) Indeed, the topology on Y^X of pointwise convergence and the product topology on Y^X coincide; note that $S(x, U) = \pi_x^{-1}(U)$ for all $x \in X$ and open subspace U of X.

(b) The pointwise convergence coincides our old concept of pointwise convergence. To be specific, when $(f_n)_{n \in \mathbb{N}} \subset Y^X$, $f_n \to f$ pointwise if and only if $f_n(x) \to f(x)$ for each $x \in X$.

When considering uniform convergence, the codomain must be given as a metric space.

Definition 5.3.2 (Uniform topology)**.** Assume Y is a metric space with the metric d on Y. The uniform topology on Y is the topology on Y^X induced by the uniform metric $\overline{\rho}$ induced by d.

From now on, we say a sequence $(f_n)_{n \in \mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ uniformly when the sequence converges to f in the uniform topology.

Remark. (a) The uniform convergence coincides our old concept of uniform convergence.

(b) (Review of Theorem 5.1.4) Suppose $(f_n)_{n \in \mathbb{N}} \subset C(X, Y)$, where Y is a metric space. If $f_n \to f$ uniformly, then $f \in C(X, Y)$; it is because $C(X, Y)$ is a closed subspace of Y^X, when Y^X is assumed to equip the uniform metric.

(c) (Review of Lemma 5.2.2) If a subset of $C(X, Y)$ is totally bounded under the uniform metric, then the subset is pointwise equicontinuous under d.

Another remark: When X is a topological space and Y is a metric space, the uniform topology on Y^X is finer than the topology on Y^X of pointwise convergence.

5.3.2 Topology of compact convergence

Still, we assume that Y is a metric space.

Definition 5.3.3 (Topology of compact convergence)**.** The topology on Y^X generated by the collection of the subsets

$$B_C(f, \epsilon) := \left\{ g \in Y^X : \sup_{x \in C} d(f(x), g(x)) < \epsilon \right\} \quad \left(\begin{matrix} C \text{ is a compact subspace of } X, \\ f \in Y^X, \epsilon > 0 \end{matrix} \right) \tag{5.2}$$

as a basis is called the topology of compact convergence or the topology of uniform convergence on compact sets.[3]

Remark. The collection of sets in eq. (5.2) is indeed a basis of the topology on Y^X of compact convergence, and its justification is given here.

(1) It is clear that the collection covers Y^X.

(2) Given two basis members $B_C(f,a)$ and $B_K(g,b)$, where C and K are compact subspaces of X and $a, b > 0$ and a point $u \in B_C(f,a) \cap B_K(g,b)$, we wish to find a basis member B such that $u \in B \subset B_C(f,a) \cap B_K(g,b)$.

For a point $u \in B_C(f,a)$, if $\delta = a - \sup\{d(f(x), u(x)) : x \in C\} > 0$, then $B_C(u, \delta) \subset B_C(f,a)$.

Using the above observation, we can find a small positive real numbera α such that $B_C(u, \alpha) \subset B_C(f,a)$ and $B_K(u, \alpha) \subset B_K(g,b)$, and $B_{C \cup K}(u, \alpha)$ is a desired basis member.

From now on, we say a sequence $(f_n)_{n \in \mathbb{N}} \subset Y^X$ converges to $f \in Y^X$ compactly when the sequence converges to f in the topology of compact convergence, i.e., $f_n \to f$ uniformly on every compact subspace of X.

Theorem 5.3.4 (Inclusions regarding topologies on Y^X). Assume (Y, d) is a metric space. Then,

(uniform topology) $\supset$ (topology of compact convergence) $\supset$ (topology of pointwise convergence).

Furthermore, if X is compact, then the first two topologies coincide; if X is discrete, then the last two coincide.

Proof. We first show the inclusion.

(1) (The uniform topology is finer than the topology of compact convergence) Given a basis member $B_C(f, \epsilon)$ of the topology of compact convergence and its point g, we can find a positive real number $r < 1$ such that $B_C(g, r) \subset B_C(f, \epsilon)$. It is clear that $g \in B_{\overline{\rho}}(g, r) \subset B_C(g, r) \subset B_C(f, \epsilon)$.

(2) (The topology of compact convergence is finer than the topology of pointwise convergence) Given a basis member $S(x, B_d(p, \epsilon))$ of the topology of pointwise convergence and its point g, because $d(g(x), p) < \epsilon$, there is a positive real number r such that $B_d(g(x), r) \subset B_d(p, \epsilon)$. Because $\{x\}$ is compact, it is clear that $g \in B_{\{x\}}(g, r/2) \subset S(x, B_d(p, \epsilon))$.

We now show the coincidence under each case. When X is compact, then $B_\rho(f, \epsilon)$ is a basis member of the topology of compact convergence as well as a basis member of the uniform topology, proving the first coincidence. When X is discrete and $B_C(f, \epsilon)$ is a basis member of the topology of compact convergence, the compact subspace C of X is necessarily a finite subset of X; hence, writing $C = \{x_1, \cdots, x_n\}$, we have $B_C(f, \epsilon) = \bigcap_{i=1}^n S(x_i, B_d(f(x_i), \epsilon))$. $\qquad \square$

Before studying further properties of the topology of compact convergence, we introduce a topological space in which a subset is open if and only if its restriction to any compact subspace is open in the compact subspace.

Definition 5.3.5 (Compactly generated space). A topological space X is said to be compactly generated if a subspace A of X is open when the following property is satisfied:

$A \cap C$ is open in C whenever C is a compact subspace of X.

Remark. We may replace the word "open" by "closed," by considering set complements.

Example 5.3.6. Some examples of compactly generated spaces are introduced here.

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[3]The supremum of $d(f(x), g(x))$ for $x \in C$ is finite since C is compact.

(a) (Locally compact spaces) Suppose X is a locally compact space and let A be a subset of X such that $A \cap C$ is open in C whenever C is a compact subspace of X. We wish to justify that A is an open subspace of X.

Given $x \in A$, choose a neighborhood U of x in X that lies in a compact subspace C of X. Since $A \cap C$ is open in C, $A \cap U = (A \cap C) \cap U$ is open in U, and hence in X. Then $A \cap U$ is a neighborhood of x in X contained in A, so A is open in X.

(b) (First-countable spaces) Suppose X is a first-countable space and let B be a subset of X such that $B \cap C$ is closed in C whenever C is a compact subspace of X. We wish to justify that B is a closed subset of X.

Let x be a point of $\overline{B}$, where the overline notation means the closure in X. Since X has a countable base at x, there is a sequence $(x_n)_{n \in \mathbb{N}}$ of points in B converging to x. The subspace $C = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact, so $B \cap C$ is closed in C. Because $B \cap C$ contains all x_n, it also contains x, so $x \in B$. Therefore, B is closed.

For pointwise convergence, the limit map of continuous maps need not be continuous; this is valid for uniform convergence. For compact convergence, the assertion is valid when the domain is compactly generated, which is justified by the below theorem after a lemma.

Lemma 5.3.7. If X is compactly generated, then a map $f : X \to Y$ is continuous if and only if for each compact subspace C of X, the restriction $f|_C$ is continuous.

Proof. Only if part is obvious, and if part is also clear if one remarks that $(f|_C)^{-1}(U) = C \cap f^{-1}(U)$ whenever $U \subset Y$. $\square$

Theorem 5.3.8. Let X be a topological space, (Y, d) be a metric space. Then $C(X, Y)$ is closed in Y^X in the topology of compact convergence. (Hence, if a sequence $(f_n)_{n \in \mathbb{N}} \subset C(X, Y)$ converges to f compactly, then f is continuous.)

Proof. We need to show that a limit point f of $C(X, Y)$ in Y^X belongs to $C(X, Y)$. By the preceeding lemma, it suffices to show that $f|_C$ is continuous whenever C is a compact subspace of X. Given a compact subspace C of X, by considering the neighborhoods $B_C(f, 1/n)$ of f in Y^X, we can find a sequence $(g_n^C)_{n \in \mathbb{N}}$ of points in $C(X, Y)$ such that $g_n^C \in B_C(f, 1/n)$. In this case, $g_n^C|_C \to f|_C$ uniformly, so $f|_C$ is continuous. $\square$

Remark. The space $C(X, Y)$ (X is a topological space and (Y, d) is a metric space) is closed in Y^X in the uniform topology, or in the topology of compact convergence when it is further assumed that X is compactly generated.

5.3.3 Compact-open topology

When we introduced the uniform topology and the topology of compact convergence, we had to assume that Y is a metric space (and the topologies seem to depend on the choice of the metric on Y). It is natural to ask whether either of these topologies can be extended to the case where Y is an arbitrary topological space. It is known that there is no satisfactory answer to this question for the space Y^X; fortunately, one can prove something for its subspace $C(X, Y)$.

To study such case, in this subsection we assume X and Y are topological spaces.

Definition 5.3.9 (Compact-open topology)**.** The topology on $C(X, Y)$ generated by the collection of the subsets of the form

$$S(C, U) := \{f \in C(X, Y) : f(C) \subset U\} \quad (C \text{ is a compact subspace of } X \text{ and } U \text{ is open in } X)$$

as a subbasis is called the compact-open topology.

Remark. (a) The above definition of compact-open topology naturally extends to Y^X, which is not of our interest.

(b) Clearly, the compact-open topology is finer than the topology of pointwise convergence (the point-open topology).

Observation 5.3.10. When (Y, d) is a metric space, the compact-open topology on $C(X, Y)$ and the topology of compact convergence on $C(X, Y)$ coincide.

Proof. The following result helps in this proof:

> Suppose A is a compact subspace of X and V is an open subspace of X containing A. Then, there is $\epsilon > 0$ such that V contains the ϵ-neighborhood of A.

We first prove that the topology of compact convergence is finer than the compact-open topology. Given a subbasis member $S(C, U)$ of the compact-open topology (C is compact and U is open in X) and its point f, note that f is continuous so that $f(C)$ is compact. Because $f(C) \subset U$, there is $\epsilon > 0$ such that the ϵ-neighborhood of $f(C)$ is contained in U. Therefore, $f \in B_C(f, \epsilon) \subset S(C, U)$.

Now we prove that the compact-open topology is finer than the topology of compact convergence. Given a basis member $B_C(f, \epsilon)$ of the topology of compact convergence and its point g, without loss of generality, it suffices to find a member A of the topology of compact convergence such that $f \in A \subset B_C(f, \epsilon)$.[4] For each point $x \in C$, there is a neighborhood U_x of x in X such that

$$t \in \overline{U_x} \text{ implies } d(f(t), f(x)) < \epsilon/3.[5]$$

Using the compactness of C, choose finitely many $V_i = U_{x_i}$ $(i = 1, \cdots, n)$ covering C. If we write $C_i = \overline{V_i} \cap C$ for each i, the finite intersection $\bigcap_{i=1}^{n} S(C_i, B_d(f(x_i), \epsilon/3))$ contains f and is contained in $B_C(f, \epsilon)$. $\square$

In the beginning of this subsection, it is mentioned as a statement that the compact convergence topology (as well as the uniform topology) seems to depend on the metric on the codomain. By Observation 5.3.10, we can obtain the following coincidences:

Corollary 5.3.11. If Y is a metric space (so that the compact convergence topology and the compact-open topology coincide), the compact convergence topology on $C(X, Y)$ is independent of the metric on Y. Hence, if it is further assumed that X is compact (so that the uniform topology and the compact convergence topology coincide), the uniform topology on $C(X, Y)$ is independent of the metric on Y.

Continuity of evaluation maps under the compact-open topology on $C(X, Y)$

We end this section with some theory which will be helpful when studying Ascoli's theorem in the following section.

Definition 5.3.12 (Evaluation maps). Let A be a nonempty set and P be a collection of maps from A into a nonempty set B. The map $\mathrm{ev}_P : A \times P \to B$ defined by $\mathrm{ev}_P(a, f) = f(a)$ for $a \in A, f \in P$ is called the evaluation map with maps in P. If the context is clear, we may omit the subscript.

Example 5.3.13. Let X and Y be topological spaces and consider the evaluation map $\mathrm{ev} : X \times C(X, Y) \to Y$. Suppose further that there is a metric d on Y inducing the topology on Y. Show that, under the compact-open topology on $C(X, Y)$, that the evaluation map is continuous.

Theorem 5.3.14. Let X be a locally compact Hausdorff space and suppose $C(X, Y)$ equips the compact-open topology. Then the evaluation map $\mathrm{ev} : X \times C(X, Y) \to Y$ is continuous.

Proof. To show the continuity of the evaluation map, it suffices to show that there is a neighborhood of $(a, f) \in X \times C(X, Y)$ which is mapped into an arbitrarily given neighborhood W of $f(a)$ in Y. Using the continuity of f, choose a neighborhood U of a in X such that $f(U) \subset W$. By hypothesis, there is a neighborhood V of a whose closure in X is a compact subspace of U. The box $U \times S(\overline{U}, W)$ is a desired neighborhood of (a, f). $\square$

---

[4]There is no loss of generality, since we can find a positive real number r such that $B_C(g, r) \subset B_C(f, \epsilon)$.

[5]By letting U_x be a neighborhood of x in X such that $t \in U_x$ implies $d(f(t), f(x)) < \epsilon/4$, the result follows from the inclusion $f(\overline{U_x}) \subset \overline{f(U_x)}$.

A consequence of this theorem is the theorem that follows.

Definition 5.3.15. Given a map $f : X \times Z \to Y$, there is a corresponding map $F : Z \to Y^X$, defined by $(F(z))(x) = f(x, z)$ for all $x \in X$ and $z \in Z$. Conversely, given $F : Z \to Y^X$, the above equation defines a corresponding map $f : X \times Z \to Y$. We say that F is the map of Z into $C(X, Y)$ which is induced by f.

Theorem 5.3.16. Let X and Y be topological spaces and give $C(X, Y)$ the compact-open topology. If $f : X \times Z \to Y$ is continuous, then so is its induced map $F : Z \to C(X, Y)$. The converse holds if X is a locally compact Hausdorff space.

Proof. We first prove that f is continuous when X is a locally compact Hausdorff space and F is continuous. Note that if $\alpha_i : A_i \to B_i$ denotes a continuous map for each $i = 1, \cdots, n$ then $\alpha_1 \times \cdot \alpha_n$ is also continuous, and remark the following commutative diagram:

$$X \times Z \xrightarrow[\text{id}_Z \times F]{} C(X, Y) \xrightarrow[\text{ev}]{} Y \ .$$

with f the composite arrow from $X \times Z$ to Y.

We now prove that F is continuous if f is continuous. For this, we fix a point $z_0 \in Z$ and seek to find a neighborhood W of z_0 in Z such that $F(W) \subset S(C, U)$, where $S(C, U)$ is any subbasis member of the compact-open topology on $C(X, Y)$ which contains $F(z_0)$, i.e., $f(C \times \{z_0\}) \subset U$. By definition, for each $c \in C$, $f(c, z_0) \in U$; by continuity, there is an open box $A_c \times B_c$ in $X \times Z$ containing (c, z_0) which is mapped into U under f. Since $C \times \{z_0\}$ is compact, this slice can be covered by finitely many boxes of the form $A_c \times B_c$; write $C \times \{z_0\} \subset \bigcup_{i=1}^{n}(A_{c_i} \times B_{c_i})$ and $B = \bigcap_{i=1}^{n} B_{c_i}$. Clearly, B is a neighborhood of z_0 in Z and $f(C \times B) \subset U$, i.e., $F(B) \in S(C, U)$, as desired. $\square$

5.4 Ascoli's theorem

Throughout this section, we assume that X is a topological space and (Y, d) is a metric space, unless stated otherwise. In this section, we study a generalized Ascoli's theorem, which characterizes the compact subspaces of $C(X, Y)$ in the topology of compact convergence. Also, some particular versions of Ascoli's theorem will be introduced.

5.4.1 Generalized Ascoli's theorem

Theorem 5.4.1 (Ascoli's theorem). Let X be a topological space and (Y, d) be a metric space. Give $C(X, Y)$ the topology of compact convergence, and let $\mathcal{F}$ be a subset of $C(X, Y)$.

  (a) Suppose that $\mathcal{F}$ is pointwise equicontinuous under d and pointwise relatively compact. Then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$; to say equivalently, $\mathcal{F}$ is relatively compact.

  (b) The converse of (a) is valid if X is a locally compact Hausdorff space.

Remark. For a Hausdorff space A, a subspace B of A is contained in a compact subspace of A if and only if A is relatively compact. Since if part is clear, we assume that B is contained in a compact subspsace K of A. Note that the closure $\overline{B}$ of B in X is contained in the closure $\overline{K}$ of K in X. Since A is a Hausdorff space, every compact subspace of A is closed, so $\overline{K} = K$. Hence, $\overline{A} \subset K$, implying that $\overline{A}$ is a closed subspace of K. Therefore, A is relatively compact.

Proof of (a). Impose Y^X the product topology, which equals to the topology of pointwise convergence. Note that Y^X is a Hausdorff space and that $C(X, Y)$ (which equips to topology of compact convergence) is not a subspace of Y^X. And let $\mathcal{G}$ denote the closure of $\mathcal{F}$ in Y^X. (It will be clear in Step 1 why we consider the closure of $\mathcal{C}$ in Y^X, not in $C(X, Y)$.)

Our proof then consists of the following four steps:

  • Step 1: Showing that $\mathcal{G}$ is a compact subspace of Y^X.

- Step 2: Showing that each member of $\mathcal{G}$ is continuous, and $\mathcal{G}$ is pointwise equicontinuous under d.

- Step 3: The topology on Y^X and the topology on $C(X,Y)$ coincide on $\mathcal{G}$.

- Step 4: Concluding the proof of (a).

Step 1. For each point $a \in X$, let C_a denote the closure of $\mathcal{F}(a)$ in Y. By assumption, each C_a is compact, hence the product $C = \prod_{a \in X} C_x$ is also compact. Because $\mathcal{F} \subset C \subset Y^X$, the closure of $\mathcal{F}$ in Y^X is also contained in C; because Y^X is a Hausdorff space, $\mathcal{G}$ is a compact subspace of Y^X.

Step 2. It suffices to check that $\mathcal{G}$ is pointwise equicontinuous under d. Given $p \in X$, let V_p be a neighborhood of p in X such that $f \in \mathcal{F}$ and $x \in V_p$ implies $d(f(x), f(p)) < \epsilon$. Given $g \in \mathcal{G}$ and $x \in V_p$, we can find $h \in \mathcal{F}$ such that $h \in \pi_p^{-1}(B_d(g(p), \epsilon)) \cap \pi_x^{-1}(B_d(g(x), \epsilon))$, i.e., $d(h(p), g(p)) < \epsilon$ and $d(h(x), h(x)) < \epsilon$. In this case,

$$d(g(x), g(p)) \leq d(g(x), h(x)) + d(h(x), h(p)) + d(h(p), g(p)) < 3\epsilon,$$

which proves that $\mathcal{G}$ is pointwise equicontinuous.

Step 3. It is clear that the topology on Y^X restricted to $\mathcal{G}$ is coarser than the topology on $C(X,Y)$ restricted to $\mathcal{G}$. Thus, it remains to show that the converse inclusion; for this, given a basis member $B_K(f, \epsilon) \cap \mathcal{G}$ of the latter topology (assume $f \in \mathcal{G}$), we need to find a basis member B of the pointwise convergence topology on Y^X such that $f \in B \cap \mathcal{G} \subset B_K(f, \epsilon) \cap \mathcal{G}$. (How could we skip some procedures?)

Using pointwise equicontinuity of $\mathcal{G}$, for each point $x \in X$, let U_x be a neighborhood of x in X such that $t \in U_x$ and $g \in \mathcal{G}$ implies $d(g(t), g(x)) < \epsilon/4$; using compactness of K, cover K with finitely many neighborhoods $U_{p_1}, \cdots, U_{p_n}$. Then, choose

$$B = \bigcap_{i=1}^{n} \pi_{p_i}^{-1}(B_d(f(p_i), \epsilon/4))$$

so that whenever $g \in B \cap \mathcal{G}$ we have $d(g(p_i), f(p_i)) < \epsilon/4$ for all i. If $g \in B \cap \mathcal{G}$ and $x \in K$, there is an index i such that $x \in U_{p_i}$, so

$$d(g(x), f(x)) \leq d(g(x), g(p_i)) + d(g(p_i), f(p_i)) + d(f(p_i), f(x)) < \frac{3}{4}\epsilon.$$

This implies that $g \in B_K(f, \epsilon) \cap \mathcal{G}$, as desired.

Step 4. Using the results from Step 1 to Step 3, we derive that $\mathcal{F}$ is contained in a compact subspace of $C(X,Y)$. By Step 1 and Step 2, the closure $\mathcal{G}$ of $\mathcal{F}$ in Y^X is a compact subspace of Y^X (in the product topology) which is contained in $C(X,Y)$; by Step 3, $\mathcal{G}$ is also a subspace of $C(X,Y)$ (in the topology of compact convergence). Therefore, $\mathcal{F}$ is contained in a compact subspace $\mathcal{G}$ of $C(X,Y)$. $\qquad\square$

Proof of (b). Assume that X is a locally compact Hausdorff space, and let $\mathcal{H}$ be a compact subspace of $C(X,Y)$ which contains $\mathcal{F}$. We show that $\mathcal{H}$ is pointwise equicontinuous under d and pointwise (relatively) compact, from which it follows that $\mathcal{F}$ is pointwise equicontinuous under d and pointwise relatively compact.

We first show that $\mathcal{H}$ is pointwise compact. For this, we fix a point $a \in X$, and consider the following composition:

$$
\begin{array}{ccccc}
C(X,Y) & \overset{\imath_a}{\hookrightarrow} & X \times C(X,Y) & \overset{\text{ev}}{\longrightarrow} & Y \\
\uparrow & & & & \uparrow \\
\mathcal{H} & \xrightarrow{\qquad\qquad\jmath\qquad\qquad} & & & \mathcal{H}(a)
\end{array} \; ,
$$

where $\imath_a$ maps $f \in C(X,Y)$ to $(a, f) \in X \times C(X,Y)$. Because $\imath_a$ is continuous for all $a \in X$ and ev is continuous (see Theorem 5.3.14, and note that the compact convergence topology on $C(X,Y)$ and the compact-open topology on the space coincide). Hence, $\jmath$ maps the compact subspace $\mathcal{H}$ of $C(X,Y)$ onto a compact $\mathcal{H}(a)$ of Y.

We now show that $\mathcal{H}$ is pointwise equicontinuous under d; for this, we fix a point $a \in X$ and let A be a compact subspace of X containing a neighborhood of a in X. Then it suffices to check that the following restriction

$$\mathcal{R} := \{f|_A : f \in \mathcal{H}\}$$

is equicontinuous at a. Give $C(A,Y)$ the compact convergence topology. Because A is compact, the compact convergence topology on $C(A,Y)$ and the uniform topology on $C(A,Y)$ coincide. Also, it can be easily verified that the restriction map $r : C(X,Y) \to C(A,Y)$ is continuous (see the following problem), which implies that $\mathcal{R} = r(\mathcal{H})$ is a compact subspace of $C(A,Y)$. Hence, $\mathcal{R}$ is totally bounded under the uniform metric on $C(A,Y)$ induced by d, so $\mathcal{R}$ is pointwise equicontinuous under d. $\qquad\square$

Problem 5.4.1. Show that the restriction map $r : C(X,Y) \to C(A,Y)$ in the proof of (b) above is continuous.

Solution. Given $f \in C(X,Y)$ and a basis member $B_2 := B_K(f,\epsilon)$ of the topology on $C(A,Y)$, because K is a compact subspace of A, K is a compact subspace of X. Hence, one can consider the basis member $B_1 := B_K(f,\epsilon)$ of the topology on $C(X,Y)$, and this basis member is mapped onto B_2 by the restriction map.

5.4.2 Some particular versions of Ascoli's theorem

We first introduce the following classical version of Ascoli's theorem. The following remark plays as a technically essential tool in this subsection, and it worths considering the remark as a lemma:

Remark. When (Y,d) is a complete metric space, relative compactness and total boundedness for subspaces of Y coincide.

Theorem 5.4.2 (Classical version of Ascoli's theorem)**.** Let X be a compact space; let $(\mathbb{R}^n, d)$ denote the Euclidean space in either the square metric or the Euclidean metric; give $C(X, \mathbb{R}^n)$ the corresponding uniform topology (which coincides the compact convergence topology). For a subset $\mathcal{F}$ of $C(X, \mathbb{R}^n)$, the following are equivalent:

(a) $\mathcal{F}$ is pointwise equicontinuous under d and pointwise bounded.

(b) $\mathcal{F}$ is relatively compact in $C(X, \mathbb{R}^n)$.

Proof. Since boundedness and total boundedness coincide in $\mathbb{R}$, it is clear that (a) implies (b). To prove the converse implication, assume that $\mathcal{F}$ is relatively compact in $C(X, \mathbb{R}^n)$ and let $\mathcal{G}$ be the closure of $\mathcal{F}$ in $C(X, \mathbb{R}^n)$. What we want to show is the following:

$$\mathcal{G} \text{ is pointwise equicontinuous under } d \text{ and pointwise bounded.}$$

We first prove that $\mathcal{G}$ is pointwise bounded. Since $\mathcal{G}$ is a compact subspace of $C(X,Y)$, given $\epsilon > 0$, there are finitely many maps $f_1, \cdots, f_n \in \mathcal{F}$ such that $\mathcal{G} \subset \bigcup_{i=1}^n B_{\overline{\rho}}(f_i, \epsilon)$, thus $\mathcal{G}$ is pointwise bounded. Now we prove pointwise equicontinuity. Suppose $g \in \mathcal{G}$ and let $f \in \mathcal{F}$ be a point such that $\overline{\rho}(g,f) < \epsilon/3$; given a point $p \in X$, let V be a neighborhood of p in X such that $x \in V$ and $h \in \mathcal{F}$ implied $d(h(x), h(p)) < \epsilon/3$. We then have

$$d(g(x), g(p)) \leq d(g(x), f(x)) + d(f(x), f(p)) + d(f(p), g(p)) < \epsilon$$

for all $x \in V$, proving pointwise equicontinuity of $\mathcal{G}$. $\qquad\square$

Corollary 5.4.3. Let X be a compact space; let d denote either the square metric or the Euclidean metric on $\mathbb{R}^n$; give $C(X, \mathbb{R}^n)$ the corresponding uniform topology (which coincides the compact convergence topology). For a subset $\mathcal{F}$ of $C(X, \mathbb{R}^n)$, the following are equivalent:

(a) $\mathcal{F}$ is compact.

(b) $\mathcal{F}$ is closed in $C(X, \mathbb{R}^n)$, pointwise equicontinuous under d, and bounded under the sup metric ρ.

Proof. Note the following observations:

- By Ascoli's theorem, $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is pointwise equicontinuous under d and pointwise totally bounded.

- By Heine-Borel theorem, $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is complete and totally bounded under the sup metric.

- Because $C(X, Y)$ is complete under the sup metric, closedness and completeness coincide for subspaces of $C(X, Y)$.

Hence, (a) implies (b). To show that (b) implies (a), it suffices to check that boundedness of $\mathcal{F}$ implies pointwise total boundedness of $\mathcal{F}$. Since $\mathcal{F} \subset B_{\overline{\rho}}(0, R)$ for some real number $R > 0$, $\mathcal{F}(a) \subset [-R, R]$ for all $a \in X$. Being a subspace of a totally bounded space, $\mathcal{F}(a)$ is also totally bounded for each $a \in X$, as desired. $\qquad \square$

As the second type, we assume that X is a compact space and (Y, d) is a complete metric spcae.

Theorem 5.4.4. Let X be a compact space, (Y, d) be a complete metric space, and $\mathcal{F}$ be a subset of $C(X, Y)$. Give $C(X, Y)$ the compact convergence topology (which coincides the uniform topology).

(a) If $\mathcal{F}$ is pointwise equicontinuous and pointwise totally bounded, then $\mathcal{F}$ is relatively compact.

(b) The converse of (a) holds if X is a Hausdorff space.

Remark. Suppose X is a compact Hausdorff space. Then $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is pointwise equicontinuous and pointwise totally bounded.

Proof. The results follow from Ascoli's theorem and the first remark. $\qquad \square$

As the third type, we assume that X is a locally compact Hausdorff space and (Y, d) is a complete metric space.

Theorem 5.4.5. Suppose X is a locally compact Hausdorff space and (Y, d) is a complete metric space. Give $C(X, Y)$ the compact convergence topology. Then a subset $\mathcal{F}$ of $C(X, Y)$ is relatively compact if and only if $\mathcal{F}$ is pointwise totally bounded and pointwise equicontinuous under d.

Proof. Note again that pointwise total boundedness and pointwise relative compactness coincide, since (Y, d) is complete. $\qquad \square$

Corollary 5.4.6 (Generalized Arzelà's theorem)**.** Let X be a σ-compact Hausdorff space and (Y, d) be a complete metric space. Give $C(X, Y)$ the compact convergence topology. If a sequence $(f_n)_{n \in \mathbb{N}} \subset C(X, Y)$ is pointwise equicontinuous and pointwise totally bounded under d, then the sequence $(f_n)_{n \in \mathbb{N}}$ has a subsequence which converges compactly in $C(X, Y)$.

Proof. It is not solved, yet. $\qquad \square$

Corollary 5.4.7 (Arzelà's theorem of undergraduate mathematical analysis)**.** Let X be a compact Hausdorff space and (Y, d) be a complete metric space. Give $C(X, Y)$ the compact convergence topology (which coincides the uniform topology). If a sequence $(f_n)_{n \in \mathbb{N}} \subset C(X, Y)$ is pointwise equicontinuous and pointwise totally bounded under d, then the sequence $(f_n)_{n \in \mathbb{N}}$ has a subsequence which converges compactly (or uniformly, since X is assumed to be compact) in $C(X, Y)$.

Proof. By the preceeding theorem, $(f_n)_{n \in \mathbb{N}}$ is relatively compact in $C(X, Y)$. Because $C(X, Y)$ is a Hausdorff space, the closure of $(f_n)_{n \in \mathbb{N}}$ is closed, which justifies the assertion. $\qquad \square$

Remark. When Y is given as $\mathbb{R}$ or $\mathbb{C}$, total boundedness reduces to boundedness, since they are equivalent on those metric spaces.

Part II

Further theory in general topology

Chapter 6

Quotient topology

6.1 Quotient topology

6.1.1 Quotient maps

Definition 6.1.1 (Quotient map). Let X and Y be topological spaces. A surjective map $p : X \to Y$ is called a quotient map if a subset U of Y is open if and only if $p^{-1}(U)$ is open in X.

Remark. Be noted that a quotient map need not be an open map.

Example 6.1.2. The following results will be helpful in practice, when one wishes to justify that a given surjective continuous map is a quotient map.

(a) A homeomorphism is an injective quotient map, and vice versa.

(b) An open (or closed) continuous surjection is a quotient map.

(c) For topological spaces X, Y Z, if $p : X \to Y$ and $q : Y \to Z$ are quotient maps, then $q \circ p : X \to Z$ is also a quotient map.

(d) Let $p : X \to Y$ be a continuous function. If there is a function continuous $f : Y \to X$ such that $p \circ f = id_Y$, then p is a quotient map. In short, a continuous map with a continuous right inverse is a quotient map.

(e) A retraction is a quotient map.[1]

(f) Cartesian product of quotient maps need not be a quotient map. See Problem 6.1.2.

For topological spaces X and Y, we say a subset A of X is saturated with respect to a surjective map $p : X \to Y$ when A contains every $p^{-1}(\{y\})$ that it intersects, i.e., A is the inverse image of a subset of Y. (In fact, because $A = p^{-1}(B)$ for some subset B of Y and $B = p(p^{-1}(B))$, we find that $A = p^{-1}(p(A))$.) To argue p is a quotient map is equivalent to argue that p is a continuous surjection mapping a saturated open (closed, respcetively) subsets of X onto an open (closed) subsets of Y.

A restriction of a quotient map to a subspace need not be a quotient map. One has, however, the following proposition:

Proposition 6.1.3. Let $p : X \to Y$ be a quotient map, and let A be a subspace of X that is saturated with respect to p. Let $q : A \to p(A)$ be the restriction of p to A.

(a) If A is open or closed in X, then q is a quotient map.

(b) If p is an open map or a closed map, then q is a quotient map.

Proof. Our goal in either case is to show that a subset V of $p(A)$ is open in $p(A)$ whenever $q^{-1}(V)$ is open in A.

---

[1] Given a topological space X and its subspace A, a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$ is called a retraction of X onto A.

(a) Assume A is an open subset of X. Because A is saturated, $A = p^{-1}(p(A))$, so $p^{-1}(V) \subset p^{-1}(p(A)) = A$, which implies that $q^{-1}(V) = A \cap p^{-1}(V) = p^{-1}(V)$. Therefore, V is open in Y and in $p(A)$.

(b) Suppose p is an open map. Because $q^{-1}(V)$ is open in A, $q^{-1}(V) = A \cap U$ for some open subset U of X, and $V = p(q^{-1}(V)) = p(A \cap U)$. Because $p(A \cap U) = p(A) \cap p(U)$ (see the following remark), we find that V is open in $p(A)$.

The same results hold for closed cases, which can be proved by replacing the words "open" by "closed." $\quad\square$

Remark. Let $p : X \to Y$ be a quotient map and A be a subset of X saturated with respect to p. If B is a subset of X, then it is clear that $p(A \cap B) \subset p(A) \cap p(B)$. If $y \in p(A) \cap p(B)$, then $y = p(b)$ for some $b \in B$. Because A is saturated and $y \in p(A)$, $b \in A$, so $y = p(b) \in p(A \cap B)$, as desired.

In practice, it is difficult to check if a given surjective continuous map is a quotient map by merely checking the defining condition. The examples given in Example 6.1.2 will be helpful when justifying that a given surjective continuous map is a quotient map. Here, we provide proofs of (d) and (e).

Proof. (d) We need to justify that a subset U of Y is open in Y whenever $p^{-1}(U)$ is open in X. If U be a subset of Y such that $p^{-1}(U)$ is open, then $U = id_Y^{-1}(U) = f^{-1}(p^{-1}(U))$, so U is open in Y.

(e) Let A be a subspace of X and $p : X \to A$ be a retraction of X onto A. It suffices to check that a subset U of A is open in A whenever $p^{-1}(A)$ is open in X. Because $U = p(p^{-1}(U)) = p^{-1}(U) \cap A$, U is open in A. $\quad\square$

6.1.2 Quotient topology

Definition 6.1.4 (Quotient topology)**.** Let X be a topological space and A be a nonempty set. Given a surjection $p : X \to A$, there is a unique topology $\mathcal{T}$ on A relative to which p is a quotient map, which is given as

$$\mathcal{T} = \{U \subset A : p^{-1}(U) \text{ is open in } X\}. \tag{6.1}$$

This topology is called the quotient topology induced by p.

Problem 6.1.1. Check if the collection $\mathcal{T}$ in eq. (6.1) is a topology on A.

Definition 6.1.5 (Quotient space)**.** Let X be a topological space and let X^* be a partition of X. Let $p : X \to X^*$ be the natural projection, i.e., the surjection that maps each point of X to the member of X^* containing the point. In the quotient topology induced by p, the space X^* is called a quotient space.

Remark. The typical open set of X^* is a collection of equivalence classes whose union is an open set of X.

Theorem 6.1.6. Let $p : X \to Y$ be a quotient map. Let Z be a topological space and $f : X \to Z$ be a map which is constant on each set $p^{-1}(\{y\})$ for $y \in Y$. Then f induces a unique map $\overline{f} : Y \to Z$ such that $\overline{f} \circ p = f$. Indeed, f is surjective if and only if $\overline{f}$ is surjective. Furthermore,

(a) $\overline{f}$ is continuous if and only if f is continuous.

(b) $\overline{f}$ is a quotient map if and only if f is a quotient map.
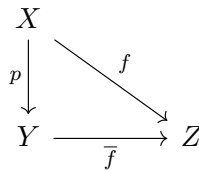
Proof. Well-definedness of $\overline{f}$ and its uniqueness are almost clear. Also, it is clear that f is surjective if and only if $\overline{f}$ is surjective. Note that $f^{-1} = p^{-1} \circ \overline{f}^{-1}$.

(a) It is clear that f is continuous if $\overline{f}$ is continuous. To show the converse, assume f is continuous and U is an open subset of Z. Since $f^{-1}(U)$ is open in X and p is a quotient map, $\overline{f}^{-1}(U)$ is also open in Y. Thus, $\overline{f}$ is a continuous map.

(b) It is clear that f is a quotient map if $\overline{f}$ is a quotient map. To show the converse, assume f is a quotient map. By (a), $\overline{f}$ is a continuous map, and it is clear that $\overline{f}$ is a surjection. Thus, it remains to show that a subset U of Z is open in Z if (and only if) $\overline{f}^{-1}(U)$ is open in Y. When $\overline{f}^{-1}(U)$ is open in Y, $f^{-1}(U) = (p^{-1} \circ \overline{f}^{-1})(U)$ is open in X, so U is open in Z, as desired. $\qquad\square$

Corollary 6.1.7. Let $f : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:
$$X^* = \{f^{-1}(\{z\}) : z \in Z\}.$$
Give X^* the quotient topology induced by the natural projection map $p : X \to X^*$.

(a) The map f induces a bijective continuous map $f^* : X^* \to Z$, which is a homeomorphism if and only if f is a quotient map.

(b) If Z is a Hausdorff space, then so is X^*.

Proof. (a) Note that the natural projection map $p : X \to X^*$ is a quotient map. By the preceeding theorem, there is a well-defined continuous bijection $f^* : X^* \to Z$. Furthermore, f^* is a quotient map (hence, a homeomorphism) if and only if f is a quotient map.

(b) Given two distinct points a^* and b^* of X^*, $\overline{f}(a^*)$ and $\overline{f}(b^*)$ are distinct points of Z, hence there are disjoint neighborhoods separating $\overline{f}(a^*)$ and $\overline{f}(b^*)$; their preimage under $\overline{f}$ are disjoint neighborhoods in X^* separating a^* and b^*. $\qquad\square$

6.1.3 Examples

Example 6.1.8. Define an equivalence relation $\sim$ on the plane $X = \mathbb{R}^2$ as follows:
$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_1 + y_1^2 = x_2 + y_2^2.$$
(It is left as an exercise to check that $\sim$ is indeed an equivalence relation on X.) Let X^* be the corresponding quotient space. In fact, X^* is the set which consists of the collections of the form $f^{-1}(\{c\})$ for $c \in \mathbb{R}$, where $f(x, y) = x + y^2$. Because f is a continuous surjection onto $\mathbb{R}$, there is a unique continuous bijection f^* such that $f^* \circ p = f$, where $p : X \to X^*$ is the natural projection. Furthermore, the map $\iota : \mathbb{R} \to \mathbb{R}^2$ defined by $\iota(x) = (x, 0)$ for all $x \in \mathbb{R}$ is a continuous right inverse of f, so f is a quotient map. Therefore, X^* and $\mathbb{R}$ are homeomorphic.

Let $\sim$ denote the equivalence relation on X defined as follows:
$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } x_1^2 + y_1^2 = x_2^2 + y_2^2.$$
(It is left as an exercise to check that $\sim$ is indeed an equivalence relation on X.) Now, X^* is the collection of the collections of the form $g^{-1}(\{c\})$ for $c \geq 0$, where $g(x, y) = x^2 + y^2$. Because g is surjective and continuous, there is a unique bijective continuous map $f^* : X^* \to [0, \infty)$. Because the map $r : [0, \infty) \to \mathbb{R}^2$ defined by $r(x) = (\sqrt{x}, 0)$ for all $x \in [0, \infty)$ is a continuous right inverse of g, g is a quotient map. Therefore, X^* and $[0, \infty)$ (equipped with the order topology) are homeomorphic.

Example 6.1.9. We will justify that D^2 and S^2 are homeomorphic. Define a map $f : D^2 \to S^2$ by
$$f \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} = \begin{pmatrix} \sin(\pi r)\cos(\theta) \\ \sin(\pi r)\sin(\theta) \\ \cos(\pi r) \end{pmatrix},$$

And let X^* be the partition of X defined by the collection of $f^{-1}(\{p\})$ for $p \in S^2$. Since f is surjective and continuous, the naturally induced map $f^* : X^* \to S^2$ is also surjective and continuous. To show that f^* is a homeomorphism, it suffices to show that f^* is a quotient map, which can be justified by showing that f is a quotient map. Indeed, f is a quotient map, since f is an open map.

Example 6.1.10. Let X^* be the quotient space $(S^1 \times S^1)/\sim$, where $(z, w) \sim (u, v)$ if and only if $zw = uv$. Our goal is to justify that $X^* \approx S^1$. Consider the function $f : S^1 \times S^1 \to S^1$ defined by $f(u, v) = uv$. Then, the collection of all preimages of singletons in S^1 under f is precisely the collection of all equivalence classes with respect to $\sim$. Because f is continuous and surjective, there is a unique bijective continuous map $f^* : X^* \to S^1$. To show that f^* is a homeomorphism, it suffies to show that f is a quotient map. Considering the map $\iota : S^1 \to S^1 \times S^1$ defined by $\iota(z) = (z, 1)$, we can easily observe that ι is a continuous right inverse of f. Hence, f is a quotient map, and it follows that f^* is a homeomorphism.

Problem 6.1.2. Let $\mathbb{R}_K$ be the set $\mathbb{R}$ equipped with the K-topology, and let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point. And let $p : \mathbb{R}_K \to Y$ be the quotient map.

(a) Show that Y satisfies the T_1 axiom but Y is not a Hausdorff space.

(b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map.

Solution. (a) Remark that the K-topology on $\mathbb{R}$ is generated as a basis by $\{(a, b) : a, b \in \mathbb{R} \text{ and } a < b\} \cup \{(a, b) - K : a, b \in \mathbb{R} \text{ and } a < b\}$.

Every singleton in Y is of the form $\{a\}$ for $a \in \mathbb{R} \setminus K$ or $\{K\}$. With respect to the natural projection $p : \mathbb{R}_K \to Y$, the preimages of both types of singletons are closed in $\mathbb{R}_K$, so Y satisfies the T_1 axiom.

To argue that Y is not a Hausdorff space, observe 0^* and k^*, where $k \in K$ and note that a typical open set in Y is the collection of equivalence classes belonging to Y whose union is an open set in $\mathbb{R}_K$. Given a neighborhood of 0^*, there is a basis member $(a, b) - K$ of $\mathbb{R}_K$ containing 0. If k_1 is a point of K such that $0 < k_1 < b$, then $k^* = k_1^*$, so a neighborhood of k^* in Y necessarily contains the projection of a neighborhood of k_1 in $\mathbb{R}_K$. This justifies that there is no pair of disjoint neighborhoods of 0^* and k^* in Y.

(b) Because Y is not a Hausdorff space, the diagonal of Y is not closed in $Y \times Y$. The preimage of the diagonal of Y under $p \times p$ is

$$\{(x, x) : x \in \mathbb{R} \setminus K\} \cup (K \times K) = \{(x, x) : x \in \mathbb{R}\} \cup (K \times K).$$

Since the diagonal of $\mathbb{R}$ is closed in $\mathbb{R} \times \mathbb{R}$, it is also closed in a finer space $\mathbb{R}_K \times \mathbb{R}_K$; $K \times K$ is obviously closed in $\mathbb{R}_K \times \mathbb{R}_K$. Hence, the preimage of the diagonal of Y under $p \times p$ is closed under $\mathbb{R}_K \times \mathbb{R}_K$. This is invalid, if $p \times p$ is a quotient map.

6.2 Further properties regarding quotient maps

Chapter 7

Topological groups

7.1 Topological groups

Definition 7.1.1 (Topological groups)**.** A group G is called a topological group if G is also a topological space such that both multiplication and inversion are continuous. (The continuity axiom can be replaced by the following axiom: The map from $G \times G$ into G defined by $(a, b) \mapsto ab^{-1}$ is continuous.)

Remark. A continuous group homomorphism is called a topological group homomorphism.

Let G be a topological group. For any point a of G, the following maps

$$\rho_a : G \to G, \qquad g \mapsto ga$$
$$\lambda_a : G \to G, \qquad g \mapsto ag$$
$$\gamma_a : G \to G, \quad g \mapsto aga^{-1}$$

will be frequently used in this chapter to investigate properties of topological groups. Indeed, the above maps are group isomorphisms which are also homeomorphisms.

Observation 7.1.2. Some direct conclusions from the result that the above maps are group-isomorhphic homeomorphisms are listed here:

(a) G is a homogeneous space, i.e., for every pair x, y of points of G, there is a homeomorphism of G onto itself mapping x to y.

(b) Every neighborhood W of g in G can be written as $W = Ug = gV$, where $U = Wg^{-1}$ and $V = g^{-1}W$ are neighborhoods of the identity.[1]

(c) Let G, K be topological groups and $f : G \to K$ be a group homomorphism. Then f is continuous on G if f is continuous at a point of G.

While (a) is direct and (b) is clear, (c) seems to be explained. Assume that f is continuous at $p \in G$, and choose $x \in G$ and let W be a neighborhood of $y := f(x)$ in K. Then $f(p)y^{-1}W$ is a neighborhood of $f(p)$ in K, hence there is a neighborhood U of p in G such that $f(U) \subset f(p)y^{-1}W$. The neighborhood $xp^{-1}U$ of x in G, as desired, is mapped into W under f.

Example 7.1.3. (a) The (abelian) groups $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{R}^{>0}, \cdot)$ are topological groups. (As usual, impose each space the order topology.)

(b) $(S^1, \cdot)$ is a topological group. (As usual, let $A \subset S^1$ be a basis member if A is the intersection of an open ball in $\mathbb{C}$ and S^1.)

(c) The general linear group $GL_n(\mathbb{R})$ is a topological group, when it is topologized naturally as a subspace of $\mathbb{R}^{n^2}$.

Topological groups behave well in the context of products.

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[1] A neighborhood of the identity in a topological group is called an identity neighborhood.

Proposition 7.1.4. Suppose that $(G_\alpha)_{\alpha \in I}$ is a collection of topological groups.

(a) The product $G := \prod_\alpha G_\alpha$, endowed with the product topology, is a topological group.

(b) For each index $\alpha \in I$, the projection map π_α is an open topological group homomorphism.

(c) (A universal property) For every topological group H and topological group homomorphisms $f_\alpha : H \to G$ for $\alpha \in I$, there is a unique topological group homomorphism $f : H \to G$ such that $\pi_\alpha \circ f = f_\alpha$ for each $\alpha \in I$. In short, there is a unique topological group homomorphism $f : H \to G$ satisfying the following commutative diagram:

$$
\begin{array}{ccc}
 & \prod_{\alpha \in I} G_\alpha & \\
{\scriptstyle f} \nearrow & & \searrow {\scriptstyle \pi_\gamma} \\
H & \xrightarrow[\;\; f_\gamma \;\;]{} & G_\gamma
\end{array}
\quad .
$$

Proof. To show that the product of topological groups is a topological group, it suffices to check continuity of multiplication and inversion: being endowed with the product topology and each multiplication and inversion are continuous, the multiplication and inversion in G are also continuous. Because each projection is open and continuous, it is clear that each projection is an open topological group homomorphism. The existence and uniqueness of a "group homomorphism" f is due to a universal property of product groups; its continuity follows from the equation $\pi_\alpha \circ f = f_\alpha$ for $\alpha \in I$. $\qquad\square$

Topological groups also behave well in the context of subgroups and closures.

Proposition 7.1.5. A subgroup H of a topological groups G is a topological group. Also, the closure $\overline{H}$ of H in G is a subgroup of G, hence a topological group. In addition, if H is a normal subgroup of G, then $\overline{H}$ is also a normal subgroup of G.

Proof. It is easy to check that subgroups of a topological group is a topological group. To check that $\overline{H}$ is a subgroup of G, it suffices to check if $ab^{-1} \in \overline{H}$ for any $a, b \in \overline{H}$. Considering the following continuous map $\kappa : G \times G \to G$ defined by $\kappa(x, y) = xy^{-1}$, we have $\kappa(\overline{H} \times \overline{H}) = \kappa(\overline{H \times H}) \subset \overline{\kappa(H \times H)} = \overline{H \times H} = \overline{H} \times \overline{H}$, so H is a subgroup of G. Finally, assume H is a normal subgroup of G. To show that $\overline{H}$ is a normal subgroup of G, it suffices to check if $ghg^{-1} \in \overline{H}$ for all $g \in G$ and $h \in \overline{H}$; consider the continuous map γ_g for $g \in G$, and observe that $\gamma_g(\overline{H}) \subset \overline{\gamma_g(H)} = \overline{H}$. $\qquad\square$

Regarding subgroups, we introduce the following proposition.

Proposition 7.1.6. Let G be a topological group and H be a subgroup of G.

(a) The subgroup H is open in G if it contains a nonempty open set.

(b) If H is open in G, then H is closed in G.

(c) The subgroup H is closed in G if and only if there is an open subset U of G such that $H \cap U$ is a nonempty closed subspace of U.

Proof. Let H be a subgroup of G. To prove (a), suppose H contains an open subspace U of G. Then H is open in G, since $H = UH = \bigcup_{h \in H} Uh$. To prove (b), assume H is open in G. Because $G \setminus H = \bigcap_{a \in G \setminus H} aH$ is open, H is closed. When proving (c), note that one way is clear. Let U be an open subset of G such that $U \cap H$ is a nonempty closed subspace of U. Letting $V = U \cap \overline{H}$ where the overline is used to denote the closure in G, V is an open subset of $\overline{H}$ such that $V \cap H = U \cap H$ is a nonempty closed subspace of V, hence of $\overline{H}$. Because H is dense in $\overline{H}$, it can be easily checked that $V \cap H$ should be empty and $V \subset H$. Because V is an open subspace of $\overline{H}$, H turns out to be an open subspace of $\overline{H}$, hence a closed subspace of $\overline{H}$. Because the closure of H in $\overline{H}$ is $\overline{H}$, we have $\overline{H} = H$, i.e. H is closed in G, as desired. $\qquad\square$

Remark. As an undesired result obtained in the proof of (a) is the following:

For a topological group G and its open subset U, XU and UX are open for any subset X of G.

Using this result, we can derive that the maps $(x,y) \mapsto xy$ and $(x,y) \mapsto x^{-1}y$ defined on $G \times G$ are open (and continuous).

We end this section with the discovery that a T_1-topological space is necessarily a regular space.

Theorem 7.1.7. A topological space satisfying a T_1-axiom is a Hausdorff space and a regular space.

To prove this, we need the following lemma:

Lemma 7.1.8. If U is a neighborhood of $e \in G$ in G, there is a symmetric identity neighborhood V such that $VV \subset U$. (We say a subset A of G is symmetric if $A^{-1} = A$.)

Proof of Lemma 7.1.8. Because multiplication is continuous, there is an identity neighborhood V_1 such that $V_1 V_1 \subset U$; because inversion is continuous, there is an identity neighborhood W such that $V := WW^{-1} \subset V_1$. Therefore, $VV \subset U$, and it is easy to check that V is a symmetric identity neighborhood. $\square$

Proof of Theorem 7.1.7. Suppose that every singletone in G is closed and choose two distinct points $x, y \in G$.

Want: an identity neighborhood V in G such that $xy^{-1} \notin VV$.

If the above desire is satisfied, we then have $xy^{-1} \neq u^{-1}v$ for all $u, v \in V$, $ux \neq vy$, thus $Vx \cap Vy = \varnothing$. Using the preceeding lemma, one can find a symmetric identity neighborhood V such that $VV \subset G \setminus \{xy^{-1}\}$. This proves that G is a Hausdorff space.

It remains to prove that a T_1-topological group is regular. For this, it suffices to show the existence of a symmetric identity neighborhood $Vx \cap VA = \varnothing$, where A is a closed subspace of G and x is a point of G not contained in A. As in the first paragraph, our goal is to find an identity neighborhood V in G such that $ax^{-1} \notin VV$ for all $a \in A$, and for this we find such V satisfying $VV \subset G \setminus Ax^{-1}$ by using the preceeding lemma. This proves that G is a regular space. $\square$

7.2 Quotients of topological groups

Suppose that H is a subgroup of a topological group G. We give the quotient G/H the quotient topology induced by the projection map $p : G \to G/H$. Then a subset of G/H is open if and only if its preimage under p is open in G.

Proposition 7.2.1. Let G be a topological group and H be a subgroup of G. Then, the projection map $p : G \to G/H$ is an open quotient map, hence G/H is a quotient space of G. Moreover, the quotient space G/H is Hausdorff if and only if H is closed in G, and G/H is discrete if and only if H is open in G.

Proof. Since G/H is given the quotient topology induced by p, p is clearly a quotient map. If U is an open subspace of G, then $p(U)$ is open in G/H, since $p^{-1}(p(U)) = UH$ is open in G and p is a quotient map. Because p is chosen so that every point of G is mapped to an element of G/H that contains the point, G/H is a quotient space of G.

It is clear that H is closed in G if G/H is a Hausdorssf space, since $H = p^{-1}(\{H\})$ and $\{H\}$ is a closed singletone in G/H. Conversely, if H is closed in G, because $H = p^{-1}(\{H\})$, we find that $\{H\}$ is closed in G/H, hence every translation of $\{H\}$ is closed, i.e., G/H is a Hausdorff space.

Similarly, it is clear that H is open in G if G/H is discrete. Converse implication is now almost clear. $\square$

Theorem 7.2.2. Let G be a topological group.

(a) If N is a normal subgroup of G, then the group G/N is a topological group with respect to the quotient topology on G/N.

(b) The quotient map $p : G \to G/N$ is an open topological group homomorphism.

(c) Moreover, the topological group G/N is a Hausdorff space if and only if N is closed. Hence, in particular, $G/\overline{N}$ is a Hausdorff topological group.

Proof. To check that G/N is a topological group, it remains to check if multiplication and inversion are continuous, which are easy to check. The remaining assertions follow from the preceeding proposition. $\square$

Example 7.2.3. Consider the abelian group $(\mathbb{R}, +)$ and its normal subgroup $\mathbb{Z}$. The quotient group $(\mathbb{R}/\mathbb{Z}, +)$ is homeomorphic to $(S^1, \cdot)$

We end this section with a morphism theorem.

Theorem 7.2.4 (Morphism theorem for topological groups)**.** Let $f : G \to H$ be a topological group homomorphism, and suppose that the normal subgroup N of G is contained in $ker(f)$ Then f factors through the open topological group homomorphism $p : G \to G/N$ by a unique topological group homomorphism $\overline{f}$, satisfying the following commutative diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\ f\ } & K \\
{\scriptstyle p}\big\downarrow & \nearrow{\scriptstyle \overline{f}} & \\
G/N & &
\end{array}
$$

Furthermore, f is open if and only if $\overline{f}$ is open.

Proof. The existence and uniqueness of such group homomorphism $\overline{f}$ is already proved. Since $p^{-1} \circ \overline{f}^{-1} = f^{-1}$ and p is a quotient map, $\overline{f}$ is easily turned out to be continuous, so $\overline{f}$ is a topological group homomorphism. It can also be checked easily that f is open if and only if $\overline{f}$ is open. $\square$

7.3 Further properties of topological groups

Chapter 8

Topological vector spaces

8.1 Topological vector spaces

8.2 Further properties of topological vector spaces

Chapter 9

Tietze extension theorem

Chapter 10

Manifolds and partition of unity

Chapter 11

Fundamental groups