

## Exercise 1, Question 1a

$$\frac{\frac{}{A \vdash A} A_x}{\vdash A \Rightarrow A} \Rightarrow_I$$

$$\frac{\frac{}{a:A \vdash a:A} A_x}{\vdash \lambda a. a : A \Rightarrow A} \Rightarrow_I$$

# Exercise 1, Question 1b

$\lambda a. \lambda b. (\pi_2(a)) ((\pi_1(a)) b)$

$$\begin{array}{c}
 \frac{}{\Gamma \vdash a : (A \Rightarrow B) \wedge (B \Rightarrow C)} Ax \quad \frac{}{\Gamma \vdash \alpha : (A \Rightarrow B) \wedge (B \Rightarrow C)} Ax \quad \frac{}{\Gamma \vdash \pi_1(\alpha) : A \Rightarrow B} \wedge^1_E \quad \frac{}{\Gamma \vdash b : A} Ax \\
 \frac{}{\Gamma \vdash \pi_2(\alpha) : B \Rightarrow C} \wedge^2_E \quad \frac{}{\Gamma \vdash \pi_1(\alpha) b : B} \Rightarrow_E \\
 \frac{}{\alpha : (A \Rightarrow B) \wedge (B \Rightarrow C), b : A \vdash p_1 p_2 : C} \Rightarrow_I \\
 \frac{}{\alpha : (A \Rightarrow B) \wedge (B \Rightarrow C) \vdash \lambda b. p' : A \Rightarrow C} \Rightarrow_I \\
 \frac{}{\vdash \lambda a. p : (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \Rightarrow_I
 \end{array}$$

## Exercise 1, Question 2

$$\begin{array}{c}
\frac{\frac{\frac{}{A_x} \quad \frac{}{A_x}}{\Gamma \vdash a:A \quad \Gamma \vdash b:B} \wedge_I \quad \frac{}{\Gamma \vdash \lambda c: (A \wedge B) \Rightarrow C}}{\Gamma \vdash \lambda c: (A \wedge B) \Rightarrow C} \Rightarrow_E \quad \frac{\frac{\frac{}{A_x} \quad \frac{}{A_x}}{\Gamma \vdash y:A \wedge B} \wedge'_E \quad \frac{}{\Gamma \vdash \pi_1(y):A} \wedge''_E \quad \frac{}{\Gamma \vdash \pi_2(y):B} \wedge''_E}{\Gamma \vdash \lambda c: A \Rightarrow B \Rightarrow C} \Rightarrow_E (x2) \\
\frac{\frac{\lambda c: (A \wedge B) \Rightarrow C, a:A, b:B \vdash c(a,b) : C}{\lambda c: (A \wedge B) \Rightarrow C \vdash \lambda a. \lambda b. p' : A \Rightarrow B \Rightarrow C} \Rightarrow_I (x2) \quad \frac{\lambda c: A \Rightarrow B \Rightarrow C, y:A \wedge B \vdash c(\pi_1(y), \pi_2(y)) : C}{\lambda c: A \Rightarrow B \Rightarrow C \vdash \lambda y. q' : A \wedge B \Rightarrow C} \Rightarrow_I \\
\frac{\lambda c: (A \wedge B) \Rightarrow C \vdash \lambda a. \lambda b. p' : A \Rightarrow B \Rightarrow C}{\vdash \lambda x. p : ((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow B \Rightarrow C)} \Rightarrow_I \quad \frac{\lambda c: A \Rightarrow B \Rightarrow C \vdash \lambda y. q' : A \wedge B \Rightarrow C}{\vdash \lambda x. q : (A \Rightarrow B \Rightarrow C) \Rightarrow (A \wedge B) \Rightarrow C} \Rightarrow_I \\
\frac{\vdash \lambda x. p : ((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow B \Rightarrow C) \quad \vdash \lambda x. q : (A \Rightarrow B \Rightarrow C) \Rightarrow (A \wedge B) \Rightarrow C}{\vdash (\lambda x. p, \lambda x. q) : ((A \wedge B) \Rightarrow C) \Leftrightarrow (A \Rightarrow B \Rightarrow C)} \wedge_I
\end{array}$$

$$(\lambda x. \lambda a. \lambda b. \neg c(a, b), \lambda x c. \lambda y. \neg c(\pi_1(y), \pi_2(y)))$$

# Exercise 1, Question 3d

$$\begin{array}{c}
 \frac{\frac{\frac{}{Ax} \quad \frac{}{x:A \vee B, a:A \vdash a:A} \vee_2}{x:A \vee B \vdash x:A \vee B} \vee_1 \quad \frac{\frac{}{Ax} \quad \frac{}{x:A \vee B, b:B \vdash b:B} \vee_2}{x:A \vee B, b:B \vdash \iota_1(b):B \vee A} \vee_1}{x:A \vee B, a:A \vdash \iota_2(a):B \vee A} \vee_2}{x:A \vee B \vdash \text{case } x \text{ of } [\iota_1(a) \rightarrow \iota_2(a) \mid \iota_2(b) \rightarrow \iota_1(b)] : B \vee A} \vee_E \\
 \frac{x:A \vee B \vdash \text{case } x \text{ of } [\iota_1(a) \rightarrow \iota_2(a) \mid \iota_2(b) \rightarrow \iota_1(b)] : B \vee A}{\vdash \lambda x. \text{case } x \text{ of } [\iota_1(a) \rightarrow \iota_2(a) \mid \iota_2(b) \rightarrow \iota_1(b)] : A \vee B \Rightarrow B \vee A} \Rightarrow_I \\
 \frac{\vdash \lambda x. \text{case } x \text{ of } [\iota_1(a) \rightarrow \iota_2(a) \mid \iota_2(b) \rightarrow \iota_1(b)] : A \vee B \Rightarrow B \vee A \quad \vdash P : B \vee A \Rightarrow A \vee B}{\vdash (P, P) : (A \vee B) \Leftrightarrow (B \vee A)} \wedge_I
 \end{array}$$

# Exercise 1, Question 4a

$(\lambda x. \pi_1(x), \lambda x. (x, ()))$

$$\begin{array}{c}
 \frac{}{x : A \wedge T \vdash x : A \wedge T} A_x \\
 \hline
 \frac{}{x : A \wedge T \vdash \pi_1(x) : A} \wedge_E^1 \\
 \hline
 \frac{}{x : A \wedge T \vdash \pi_1(x) : A} \Rightarrow_I
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{x : A \vdash x : A} A_x \quad \frac{}{x : A \vdash () : T} T_I \\
 \hline
 \frac{}{x : A \vdash (x, ()) : A \wedge T} \wedge_I \\
 \hline
 \frac{}{x : A \vdash (x, ()) : A \wedge T} \Rightarrow_I
 \end{array}$$
  

$$\begin{array}{c}
 \frac{}{\vdash \lambda x. \pi_1(x) : A \wedge T \Rightarrow A} p \\
 \hline
 \frac{}{\vdash (\lambda x. \pi_1(x)) : A \wedge T \Leftrightarrow A} \wedge_I
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{\vdash \lambda x. (x, ()) : A \Rightarrow A \wedge T} q \\
 \hline
 \frac{}{\vdash (\lambda x. (x, ())) : A \Rightarrow A \wedge T} \wedge_I
 \end{array}$$
  

$$\frac{}{\vdash (p, q) : A \wedge T \Leftrightarrow A} \wedge_I$$

# Exercise 1, Question 4c

$$\begin{array}{c}
 \frac{
 \frac{
 \frac{A_x}{\vdash_E x:A \vee \perp \vdash x:A \vee \perp} \quad
 \frac{A_x}{x:A \vee \perp, a:A \vdash a:A} \quad
 \frac{
 \frac{A_x}{\perp_E x:A \vee \perp, f:\perp \vdash f:\perp} \quad
 \frac{A_x}{x:A \vee \perp, f:\perp \vdash q(f):A}
 }{\vdash_{V_I} x:A \vdash x:A}
 }{x:A \vee \perp \vdash \text{case } x \text{ of } [c_1(a) \rightarrow a \mid c_2(f) \rightarrow q(f)]:A} \quad
 \frac{A_x}{x:A \vdash c_1(x):A \vee \perp} \Rightarrow_I
 }{\vdash_{\Rightarrow_I} \text{case } x \text{ of } [c_1(a) \rightarrow a \mid c_2(f) \rightarrow q(f)]:A}
 \\
 \frac{
 \frac{
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 \frac{
 \frac{A_x}{\vdash \lambda x. p' : (A \vee \perp) \Rightarrow A}
 }{\vdash_{\wedge_I} \lambda x. p' : (A \vee \perp) \Rightarrow A}
 }{\vdash_{\wedge_I} \lambda x. p' : (A \vee \perp) \Rightarrow A} \quad
 \frac{
 \frac{
 \frac{A_x}{\vdash \lambda x. q' : A \Rightarrow A \vee \perp}
 }{\vdash_{\wedge_I} \lambda x. q' : A \Rightarrow A \vee \perp}
 }{\vdash_{\wedge_I} \lambda x. q' : A \Rightarrow A \vee \perp}
 }{\vdash_{\wedge_I} \lambda x. q' : A \Rightarrow A \vee \perp}
 }{\vdash_{\wedge_I} \lambda x. p' : (A \vee \perp) \Rightarrow A}
 }{\vdash_{\wedge_I} \lambda x. p' : (A \vee \perp) \Rightarrow A}
 }{\vdash_{\wedge_I} \lambda x. p' : (A \vee \perp) \Rightarrow A}
 \\
 \vdash (p, q) : (A \vee \perp) \Leftrightarrow A
 \end{array}$$