Mnemetic Lax-Idempotent Monads and Compactness

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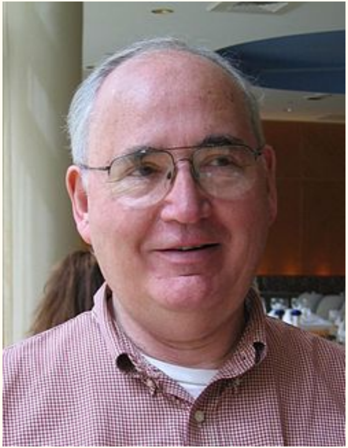
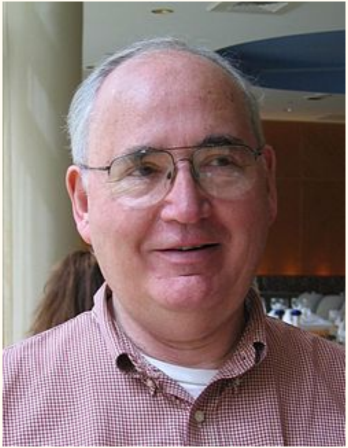
Overview

1. Scott's Story (No Categories)
    - Domain Theory: Ideals, Finite Elements



Overview

1. Scott's Story (No Categories)
   - Domain Theory: Ideals, Finite Elements
2. Our Story (Yes Categories)
   - Generalization: Lax-Idempotent Monads, Compact Objects



Scott DCPO

Definition
A subset S of a poset $(D, \leqslant)$ is directed if $S \neq \varnothing$, and for any $x, y \in S$ there is some $z \in S$ such that $x \leqslant z$ and $y \leqslant z$.

Scott DCPO

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A subset S of a poset $(D, \leqslant)$ is directed if $S \neq \emptyset$, and for any $x, y \in S$ there is some $z \in S$ such that $x \leqslant z$ and $y \leqslant z$.

Definition
A poset $(D, \leqslant)$ is a directed join complete poset (DCPO) if each directed subset $S \subseteq D$ has a join, i.e., there is an element $\bigsqcup S \in D$ such that for any $y \in D$:

$$\forall s \in S : s \leqslant y \iff \bigsqcup S \leqslant y$$

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Example

- $\mathbb{N} \rightharpoonup \mathbb{N}$ (partial functions $\mathbb{N}$ to $\mathbb{N}$)
- $\mathcal{P}(X)$ (powerset of a set X ordered by inclusion)
- Cliques of a coherence space

Algebraic Domains (Part 1)

Definition
An element x of a DCPO $(D, \leqslant)$ is finite (or compact) if for any directed subset $S \subseteq D$:

$$x \leqslant \bigsqcup S \iff \exists s \in S : x \leqslant s$$

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Example

- For $\mathbb{N} \rightharpoonup \mathbb{N}$ the finite elements are finite partial functions
- If X is a set, an element of $\mathcal{P}(X)$ is finite if it is a finite subset of X.

Notation: For a DCPO $(D, \leqslant)$, denote $\mathsf{K}(D) := \{x \in D \mid x \text{ is finite}\}$.

Algebraic Domains (Part 2)

Definition
A DCPO $(D, \leqslant)$ is algebraic if for every $x \in D$, the set $S_x := \{y \in D \mid y \text{ is finite and } y \leqslant x\}$ is directed and $x = \bigsqcup S_x$.

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Slogan
In Algebraic DCPO's, we can approximate everything through finite data.

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Example

- $\mathbb{N} \rightharpoonup \mathbb{N}$ is algebraic.
- Any coherence space is defined by its finite cliques.
- $\mathcal{P}(X)$ is algebraic.

The Idl-Construction

Definition
For any poset $(X, \leqslant)$, let

$$\mathsf{Idl}(X) := \{S \subseteq X \mid S \text{ is directed and down-closed}\}.$$

Lemma
Any monotone map $f : X \to D$ (where D is a DCPO) uniquely extends to $\tilde{f} : \mathsf{Idl}(X) \to D$ by $\tilde{f}(S) = \bigsqcup_{s \in S} f(s)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & D \\
\mathsf{Idl}(X) \arrow[ur, dotted, "\tilde{f}"'] &
\end{tikzcd}.$$

Mnemetic Monads: Why is Idl special

Cool Fact!
Every $\mathsf{Idl}(X)$ is algebraic! The finite elements of $\mathsf{Idl}(X)$ are all the principal ideals $\eta_X(x) = \downarrow x := \{y \in X | y \leqslant x\}$.

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Remembering structure
We have that

$$\mathsf{K}(\mathsf{Idl}(X)) = \{S \in \mathsf{Idl}(X) \mid S \text{ is finite}\} \cong X.$$

So Idl remembers the structure of the generators.

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Goal
Can we axiomatize this?

Part 2: Our Story

Plan: Generalize the Scott Picture!

1. Idl is very special: Every DCPO X has maps $\bigsqcup : \mathsf{Idl}(X) \rightleftarrows X : \eta_X$ forming a Galois connection ($=$ adjunction)

$$\bigsqcup \dashv \eta_X .$$

[1]Anders Kock. **"Monads for which structures are adjoint to units".** In: *J. Pure Appl. Algebra* (1995), Volker Zöberlein. **"Doctrines on 2-Categories.".** In: *Mathematische Zeitschrift* (1976).

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2. Idl acts on monotone maps and preserves their comparisons
$f \leqslant g \implies \mathsf{Idl}(f) \leqslant \mathsf{Idl}(g)$ (for monotone maps $f, g : X \to Y$).

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This means that Idl is a lax-idempotent monad[1]!

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Lax-Idempotent Monads

A pseudomonad (T, μ, η) on a 2-category $\mathcal{K}$ is lax-idempotent if $\mathsf{T}\eta \dashv \mu$ such that $\epsilon : \mu\mathsf{T}\eta \Rightarrow \mathsf{id}$ is given by $\lambda : \mu\mathsf{T}\eta \cong \mathsf{id}$.

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What
A pseudomonad T is lax-idempotent if it is kind of like a free cocompletion. And the algebras should be cocomplete in a certain sense.

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Example
Idl freely adds directed joins. And Idl-$Alg \cong$ DCPO!

The General Picture

Ingredients
What we used: Pos, Idl, DCPO and Finite Elements

The General Picture

Ingredients
What we will use: $\mathcal{K}$, T, T-Alg and Compact Objects

Translation

- $\mathcal{K}$ is a 2-category (think posets, monotone functions and pointwise comparison)
- T is a lax-idempotent pseudomonad, think adding a class of joins to a poset
- T-Alg is the category of algebras of T, think DCPO, sup-lattices
- Compact objects generalize finite elements and join-prime elements

See Blackboard!

More instances of this

This recipe works for the down-set construction

$$\mathcal{D}(X) := \{S \subseteq X | S \text{ down-closed}\}.$$

1. $\mathcal{K} = \text{Pos}$
2. $\mathsf{T} = \mathcal{D}$
3. T-Alg = SUP, these are posets admitting all joins
4. Compact objects correspond to completely join prime elements

More instances of this

It also works for ex/lex construction $X_{ex/lex}$[2]!

1. $\mathcal{K} =$ Lex, categories with finite limits
2. T $=$ (-)$_{\text{ex/lex}}$
3. T-$Alg =$ ExCat, these are Barr Exact Categories
4. Compact objects correspond to regular projective objects

[2] A. Carboni and E.M. Vitale. **"Regular and exact completions"**. In: *Journal of Pure and Applied Algebra* (1998).

The abstract definition of Mnemetic Monad

The canonical comparison 2-cell
A lax-idempotent monad T on a 2-category $\mathcal{K}$ has a canonical 2-cell $\theta_{\mathsf{T}X} : \mathsf{T}\eta_X \Rightarrow \eta_{\mathsf{T}X}$ for each $X \in \mathcal{K}$

$$\begin{tikzcd} \mathsf{T}X \arrow[r, bend left, "\mathsf{T}\eta_X"] \arrow[r, bend right, "\eta_{\mathsf{T}X}"'] \arrow[r, phantom, "\Downarrow\theta_{\mathsf{T}X}"] & \mathsf{T}^2X \end{tikzcd} .$$

Definition
A lax-idempotent monad T is mnemetic exactly when for each $X \in \mathcal{K}$, η_X is the inverter of $\theta_{\mathsf{T}X}$

$$\begin{tikzcd} X \arrow[r, "\eta_X"'] & \mathsf{T}X \arrow[r, bend left, "\mathsf{T}\eta_X"] \arrow[r, bend right, "\eta_{\mathsf{T}X}"'] \arrow[r, phantom, "\Downarrow\theta_{\mathsf{T}X}"] & \mathsf{T}^2X \end{tikzcd} .$$

Continuous Algebras

Definition

A DCPO D is continuous if $\bigsqcup : \mathsf{Idl}(X) \to X$ has a left adjoint λ.

For continuous algebras (X, α, λ) one always gets a 2-cell, $\theta_X : \lambda \Rightarrow \eta_X$. Thus we can take the inverter

$$\begin{tikzcd} \mathsf{K}_{\mathsf{T}}X \arrow[r, hook] & X \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\eta_X"'] \arrow[r, phantom, "\Downarrow\theta_X"] & \mathsf{T}X \end{tikzcd} .$$

Continuous Algebras

Definition
A T-algebra (X, α) is continuous if $\alpha : \mathsf{T}(X) \to X$ has a left adjoint λ. We denote the 2-category of continuous T-algebras by T-$Cont$.

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Example
In domain theory, this inverter selects the finite elements of a continuous DCPO.

What is K?

Definition
The functor $\mathsf{K}_\mathsf{T} : \mathsf{T}\text{-}Cont \to \mathcal{K}$ associates to any continuous T-algebra (X, α, λ) the maximal compact subobject $\mathsf{K}_\mathsf{T} X \hookrightarrow X$.

What is K?

Definition
The functor $\mathsf{K}_\mathsf{T} : \mathsf{T}\text{-}\mathcal{C}ont \to \mathcal{K}$ associates to any continuous T-algebra (X, α, λ) the maximal compact subobject $\mathsf{K}_\mathsf{T} X \hookrightarrow X$.

Example
In the case for Idl, $\mathsf{K}_{\mathsf{Idl}}(D) = \{x \in D \mid x \text{ is finite}\}$ (where D is a continuous Idl-algebra, i.e. a continuous DCPO).

Mnemetic Monads

Lemma
A lax-idempotent monad T *is mnemetic exactly when* $\mathsf{K}_{\mathsf{T}}(\mathsf{T}X) \cong X$.

Continuous Algebras as Coalgebras!

Another cool fact!!
We can recover T-$Cont$ as coalgebras for FU

$$\begin{tikzcd} \mathcal{K} \arrow[loop above, "\mathsf{T}"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{T}\text{-}Alg \arrow[loop above, "FU"] \arrow[l, bend left, "U"] \end{tikzcd} .$$

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\end{tikzcd}
.
$$

There is an induced functor Free : $\mathcal{K} \to$ T-$Cont$

A theorem about compact stuff

Theorem
If $\mathcal{K}$ has enough limits, the induced functor $\mathsf{Free} : \mathcal{K} \to \mathsf{T}\text{-}\mathcal{C}ont$ *has as a right adjoint* K.

Fun Diagram
Blackboard!

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Moreover, T is mnemetic iff Free is a local equivalence.

The Idl picture on Pos

$$\begin{tikzcd}
& \mathsf{DCPO} \arrow[dl, shift left, "U"] \arrow[dr, shift left, "R"] \arrow[dl, phantom, "\perp"] \arrow[dr, phantom, "\top"] & \\
\mathsf{Pos} \arrow[ur, shift left, "\mathsf{Idl}"] \arrow[rr, hook, shift left, "\mathsf{Free}"] \arrow[rr, phantom, "\perp"] & & \mathsf{ContPos} \arrow[ll, shift left, "\mathsf{K}"] \arrow[ul, shift left, "L"]
\end{tikzcd}$$

The Idl **picture on** Pos

$$
\begin{tikzcd}
 & \mathsf{DCPO} \arrow[dl, shift left, "U"] \arrow[dr, shift left, "R"] & \\
\mathsf{Pos} \arrow[ur, shift left, "\mathsf{Idl}"] \arrow[ur, phantom, "\perp"] \arrow[rr, hook, shift left, "\mathsf{Free}"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "\sim"] & & \mathsf{ContPos} \arrow[ul, shift left, "L"] \arrow[ul, phantom, "\top"] \arrow[ll, shift left, "\mathsf{K}"] \arrow[dl, shift left] \\
 & \mathsf{AlgDCPO} \arrow[ul, shift left] \arrow[ur, hook, shift left] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}
$$

The Idl picture on Dlat

$$
\begin{tikzcd}
 & \mathsf{Frm} \arrow[dl, shift left=1.5, "U"] \arrow[dr, shift left=1.5, "R"] & \\
\mathsf{Dlat} \arrow[ur, shift left=1.5, "\mathsf{Idl}"] \arrow[ur, phantom, "\perp"] \arrow[rr, hook, shift left=1.5, "\mathsf{Free}"] \arrow[rr, phantom, "\perp"] & & \mathsf{StabFrm}^{3} \arrow[ul, shift left=1.5, "L"] \arrow[ul, phantom, "\top"] \arrow[ll, shift left=1.5, "\mathsf{K}"]
\end{tikzcd}
$$

[3] Christopher Townsend. **“Stably locally compact locales are dual to continuous posets”.** In: *Journal of Pure and Applied Algebra* (2022).

The Idl picture on Dlat

$$
\begin{tikzcd}
 & \mathsf{Frm} \arrow[dl, shift left, "U"] \arrow[dr, shift left, "R"] & \\
\mathsf{Dlat} \arrow[ur, shift left, "\mathsf{Idl}"] \arrow[ur, phantom, "\perp"] \arrow[rr, hook, shift left, "\mathsf{Free}"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "\sim"] & & \mathsf{StabFrm}[3] \arrow[ul, shift left, "L"] \arrow[ul, phantom, "\top"] \arrow[ll, shift left, "\mathsf{K}"] \arrow[dl, shift left] \\
 & \mathsf{CohFrm} \arrow[ul, shift left] \arrow[ur, hook, shift left] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}
$$

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Small Presheaves

Cocompletetion under all small colimits
Adding all small colimits corresponds to taking small presheaves $\mathcal{P}(C)$[4] on a locally small category C.

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We get a notion of generator: atomic objects

Factoid
The atomic objects of $\mathcal{P}(C)$ are all retracts of representables $\rightarrow$ Cauchy completion of C. So $\mathcal{P}$ is not mnemetic!

[4] Paolo Perrone and Walter Tholen. **"Kan Extensions are Partial Colimits".** In: *Appl. Categ. Structures* (Aug. 2022).

Small Presheaves

$$
\begin{tikzcd}
\mathsf{Cat} \arrow[d, shift right=1.5ex] \arrow[d, phantom, "\dashv"] \\
\mathsf{Cat}_{cc} \arrow[u, hook, shift right=1.5ex] \arrow[r, "mnemetic"] & \mathsf{Cocomp}
\end{tikzcd}
$$

The Dream

Wow!

Maybe every lax-idempotent monad factors as an idempotent "Cauchy completion" followed by a mnemetic monad!

Our Hopes and Dreams

$$
\begin{tikzcd}
\mathsf{SUP} \arrow[r, shift left] \arrow[d, shift left] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\dashv"] & \vee\text{-}\mathsf{Lat} \arrow[l, shift left, "\mathsf{Idl}"] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] \\
\mathsf{DCPO} \arrow[u, dotted, shift left, "\mathsf{cl}"] \arrow[r, shift left] \arrow[r, phantom, "\top"] & \mathsf{Pos} \arrow[l, shift left, "\mathsf{Idl}"] \arrow[u, shift left, "\mathcal{D}_{fg}"]
\end{tikzcd}
$$

The Counterexample

Theorem
There is a lax-idempotent monad on DCPO*:*

$$\mathsf{cl} : \mathsf{DCPO} \to \mathsf{DCPO}$$

such that the associated notion of compactness K_{cl} *has* $\mathsf{cl}(\mathsf{K}_{\mathsf{cl}}(\mathsf{cl}(\mathcal{J}))) \ncong \mathsf{cl}(\mathcal{J})$ *(where* $\mathcal{J}$ *is a counterexample of Peter Johnstone*[5]*).*

[5] Peter T. Johnstone. **"Scott is not always sober".** In: 2006.

Conclusion & Thoughts

1. We found a way of associating a notion of generator to each lax-idempotent monad.
   - Leads to Stone Duality
   - Categorical Dualities (Gabriel-Ulmer)

   This extends the methods of domain theory to new domains.
2. We found that not every lax-idempotent monad easily gives rise to a duality: Basic idea Mnemetic $\implies$ Duality.

Future Work

- We have more exotic examples of lax-idempotent monads we would like to consider (ask me later).
- Strengthen framework (Virtual Double Categories).

Thank You!

I finished too early ...

Sorry!

Forms of compactness

For a Sup-Lattice X, $x \in X$ is completely join prime if for any $S \subseteq X$:

$$x \leqslant \bigvee S \Leftrightarrow \exists s \in S : x \leqslant s$$

When S is downwards closed we can restate this as:

$$x \leqslant \bigvee S \Leftrightarrow \downarrow x \subseteq S$$

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So the "compact" elements with respect to the down-set monad are the ones where the unit η_X behaves like a left adjoint to the algebra α.

General compactness

Definition. A generalized element $x : A \to X$ for an algebra (X, α) is T-compact if we have a natural bijection:

$$\frac{\eta_X x \Rightarrow u}{x \Rightarrow \alpha u}$$

stable under precomposition. Formally this means that:

$$\begin{tikzcd}
 & \mathsf{T}X \arrow[d, "\alpha"] \\
A \arrow[ur, "\eta_X x"] \arrow[r, "x"'] \arrow[ur, phantom, shift right=4ex, "\Leftarrow"] & X
\end{tikzcd}$$

is an absolute left lifting diagram, where the 2-cell is part of the iso $\alpha\eta_X \cong id$.

General compactness on Categories

- For $\mathfrak{X}$ cocomplete, $x : \mathbf{1} \to \mathfrak{X}$ is $\mathcal{P}$-compact iff it is atomic in the sense that

$$\hom(x, -) : \mathfrak{X} \to \mathsf{Set}$$

  preserves small colimits.
- For $\mathfrak{X}$ Ind-cocomplete, $x : \mathbf{1} \to \mathfrak{X}$ is Ind-compact when it is finitely presentable, i.e.

$$\hom(x, -) : \mathfrak{X} \to \mathsf{Set}$$

  preserves all filtered colimits.

Things that should exist!

$$
\begin{tikzcd}
\mathsf{Logos} & \mathsf{Lextensive} \arrow[l, "\mathsf{SInd}"'] \\
\mathsf{PreLogos} \arrow[u, dotted] & \mathsf{Lex} \arrow[l, "\mathsf{SInd}"] \arrow[u, "\coprod_{fin}"']
\end{tikzcd}
$$

A big cube of lax-idempotent stuff

$$
\begin{tikzcd}
\mathsf{Frm} & & \mathsf{Dlat} \arrow[ll] & \\
& \mathsf{SupLat} \arrow[ul] & & \vee - \mathsf{Lat} \arrow[ll] \arrow[ul] \\
\mathsf{PreFrm} \arrow[uu] & & \wedge - \mathsf{Lat} \arrow[ll] \arrow[uu] & \\
& \mathsf{DCPO} \arrow[ul] \arrow[uu] & & \mathsf{Pos} \arrow[ll] \arrow[ul] \arrow[uu]
\end{tikzcd}
$$

The same for categories 2

$$
\begin{tikzcd}
\mathrm{Logos} & & \mathrm{Lextensive} \arrow[ll] & \\
& \mathrm{Cocomp} \arrow[ul] & & \coprod_{\mathrm{fin}}\text{-}\mathrm{Cat} \arrow[ll] \arrow[ul] \\
\mathrm{PreLogos}_{\mathrm{SInd}} \arrow[uu] & & \mathrm{Lex} \arrow[ll] \arrow[uu] & \\
& \mathrm{SIndCat} \arrow[ul] \arrow[uu] & & \mathrm{Cat} \arrow[ll] \arrow[ul] \arrow[uu]
\end{tikzcd}
$$

Generalizing the Dlat situation

$$
\begin{tikzcd}
\mathsf{Logos} & \infty\text{-}\mathsf{Lextensive} \arrow[l] \\
\mathsf{ExCat} \arrow[u] & \mathsf{Lex} \arrow[l] \arrow[u]
\end{tikzcd}
$$

Generalizing the Dlat situation

$$\begin{tikzcd} \mathsf{Logos} & \mathsf{Lextensive} \arrow[l, "\mathsf{SInd}"'] \\ \mathsf{SindCat} \arrow[u, dotted] & \mathsf{Lex} \arrow[l, "\mathsf{SInd}"] \arrow[u, "\mathsf{Fam}_{\mathsf{fin}}"'] \end{tikzcd}$$

A pattern of these squares

$$\begin{tikzcd}
\mathcal{A}^{ST} & \mathcal{A}^{T} \arrow[l, "S" description] \\
\mathcal{A}^{S} \arrow[u, dotted] & \mathcal{A} \arrow[l, "S" description] \arrow[u]
\end{tikzcd}$$

The same for categories 1

$$
\begin{tikzcd}
\mathsf{Logos} & & \mathsf{Nope.} \arrow[ll] & \\
& \mathsf{Cocomp} \arrow[ul] & & \mathsf{Rex} \arrow[ll] \arrow[ul] \\
\mathsf{PreLogos} \arrow[uu] & & \mathsf{Lex} \arrow[uu] \arrow[ll] & \\
& \mathsf{IndCat} \arrow[ul] \arrow[uu] & & \mathsf{Cat} \arrow[ll] \arrow[ul] \arrow[uu]
\end{tikzcd}
$$