Mass conservation for a steady flow

$$\nabla\cdot(\rho \underline{v}) = 0$$

For a potential flow: $\underline{v} = \nabla\phi$

$$\nabla\cdot(\rho\nabla\phi) = 0$$

From Bernoulli's equation we have

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\underline{v}|^2 + \int\frac{dp}{\rho(p)} = C(t)$$

For a perfect gas:

$$p = K\rho^{\gamma}\ , \quad c^2 = \gamma\frac{p}{\rho} \quad \text{and} \quad c^2 = \gamma K\rho^{\gamma-1}$$

we get $\displaystyle\int\frac{dp}{\rho(p)} = \frac{c^2}{\gamma-1}$

For a steady flow Bernoulli's equation becomes:

$$\frac{1}{2}|\underline{v}|^2 + \frac{c^2}{\gamma-1} = \frac{1}{2}|\underline{v}|^2_\infty + \frac{c_\infty^2}{\gamma-1}$$

where $v_\infty$ and $c_\infty$ are properties of the flow at some reference point, typically at infinity.

We can get the sound speed:

$$c = c_\infty\left\{1 + \frac{\gamma-1}{2}\left[\frac{v_\infty^2 - v^2}{c_\infty^2}\right]\right\}^{1/2}$$

for density:

$$\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = 1 + \frac{\gamma-1}{2}\,\frac{v_\infty^2 - v^2}{c_\infty^2}$$

$$\rho = \rho_\infty\left\{1 + \frac{\gamma-1}{2}\,\frac{v_\infty^2 - v^2}{c_\infty^2}\right\}^{\frac{1}{\gamma-1}}$$

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we write

$$\int \varphi \nabla \cdot (\rho \nabla \phi) \, d\Omega = 0 \qquad \text{with } \rho \text{ a function of } \phi .$$

integrate by parts:

$$-\int \rho \nabla \varphi \cdot \nabla \phi \, d\Omega + \int_{\Gamma} \rho \varphi \frac{\partial \phi}{\partial n} \, d\Gamma = 0$$

on $\Gamma_v$ the velocity is imposed

$$\frac{\partial \phi}{\partial n} = V$$

on $\Gamma_p$ the potential is imposed

$$\phi = g$$

we get:

$$-\int \rho \nabla \varphi \cdot \nabla \phi \, d\Omega + \int_{\Gamma_v} \rho \varphi V \, d\Gamma = 0 \qquad \text{with } \phi = g \text{ on } \Gamma_p$$

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to solve the non-linear problem we need to find $\delta\phi$ such that $\phi+\delta\phi$ is solution of the problem.

$$-\int(\rho+\delta\rho)\nabla\varphi\cdot\nabla(\phi+\delta\phi)\,d\Omega+\int(\rho+\delta\rho)\varphi V\,d\Gamma=0 \text{ with } \phi+\delta\phi=g \text{ on } \Gamma_p.$$

if we linearize with respect to $\delta\phi$ and $\delta\rho$:

$$-\int\rho\nabla\varphi\cdot\nabla\phi+\delta\rho\nabla\varphi\cdot\nabla\phi+\rho\nabla\varphi\cdot\nabla\delta\phi\,d\Omega$$
$$+\int\rho\varphi V+\delta\rho\varphi V\,d\Gamma=0 \qquad \text{with } \delta\phi=g-\phi \text{ on } \Gamma_p$$

or

$$-\int\delta\rho\nabla\varphi\cdot\nabla\phi+\rho\nabla\varphi\cdot\nabla\delta\phi\,d\Omega+\int\delta\rho\varphi V\,d\Gamma$$
$$=\int\rho\nabla\varphi\cdot\nabla\phi\,d\Omega-\int\rho\varphi V\,d\Gamma \qquad \text{with } \delta\phi=g-\phi \text{ on } \Gamma_p.$$

by writing $\delta\rho$ in terms of $\delta\phi$:

$$-\int-\frac{\rho}{c^2}(\nabla\phi\cdot\nabla\delta\phi)(\nabla\varphi\cdot\nabla\phi)+\rho\nabla\varphi\cdot\nabla\delta\phi\,d\Omega-\int\frac{\rho}{c^2}(\nabla\phi\cdot\nabla\delta\phi)\varphi V\,d\Gamma$$
$$=\int\rho\nabla\varphi\cdot\nabla\phi\,d\Omega-\int\rho\varphi V\,d\Gamma \quad \text{with } \delta\phi=g-\phi \text{ on } \Gamma_p$$

on $\Gamma_v$ we write $\nabla\phi\cdot\nabla\delta\phi=\frac{\partial\phi}{\partial n}\frac{\partial\delta\phi}{\partial n}+\frac{\partial\phi}{\partial\tau}\frac{\partial\delta\phi}{\partial\tau}=\frac{\partial\phi}{\partial n}\left(V-\frac{\partial\phi}{\partial n}\right)+\frac{\partial\phi}{\partial\tau}\frac{\partial\delta\phi}{\partial\tau}$

we get

$$\int_\Omega\frac{\rho}{c^2}(\nabla\varphi\cdot\nabla\phi)(\nabla\phi\cdot\nabla\delta\phi)-\rho\nabla\varphi\cdot\nabla\delta\phi\,d\Omega-\int_{\Gamma_v}\frac{\rho}{c^2}V\varphi\frac{\partial\phi}{\partial\tau}\frac{\partial\delta\phi}{\partial\tau}\,d\Gamma$$
$$=\int_\Omega\rho\nabla\varphi\cdot\nabla\phi\,d\Omega-\int_{\Gamma_v}\rho\varphi V\,d\Gamma+\int_{\Gamma_v}\frac{\rho}{c^2}V\varphi\frac{\partial\phi}{\partial n}\left(V-\frac{\partial\phi}{\partial n}\right)d\Gamma$$

with $\delta\phi=g-\phi$ on $\Gamma_p$

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$$v^2 = \sum_i \left(\frac{\partial \phi}{\partial x_i}\right)^2$$

$$v^2 + \delta(v^2) = \sum_i \left[\frac{\partial}{\partial x_i}(\phi + \delta\phi)\right]^2 = \sum_i \left(\frac{\partial \phi}{\partial x_i} + \frac{\partial \delta\phi}{\partial x_i}\right)^2$$

$$= \sum_i \left(\frac{\partial \phi}{\partial x_i}\right)^2 + 2\sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial \delta\phi}{\partial x_i} + h.o.t.$$

so $\delta(v^2) = 2\,\nabla\phi \cdot \nabla\delta\phi$

for the variation of density $\delta\rho$:

$$\left(\frac{\rho + \delta\rho}{\rho_\infty}\right)^{\gamma-1} = 1 + \frac{\gamma-1}{2}\,\frac{v_\infty^2 - v^2 - \delta(v^2)}{c_\infty^2}$$

$$\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} + \delta\rho\,(\gamma-1)\,\frac{\rho^{\gamma-2}}{\rho_\infty^{\gamma-1}} = 1 + \frac{\gamma-1}{2}\,\frac{v_\infty^2 - v^2 - \delta(v^2)}{c_\infty^2}$$

so we get

$$\delta\rho = \left(\frac{\rho_\infty}{\rho}\right)^{\gamma-1}\rho\,\frac{1}{2}\,\frac{-\delta(v^2)}{c_\infty^2} = \frac{c_\infty^2}{c^2}\,\rho\,\frac{1}{2}\,\frac{-2\nabla\phi\cdot\nabla\delta\phi}{c_\infty^2}$$

$$\delta\rho = -\frac{\rho}{c^2}\,\nabla\phi\cdot\nabla\delta\phi$$