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RECURSIVE GAUSSIAN FILTERS

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Abstract – The paper discuss the general methodology for the design of recursive digital filters with phase correction. The method for the approximation of the Gaussian characteristic presented below ensures the smallest possible error for a given filter order. It has been shown that by selecting appropriate initial values of the filter we can eliminate the so-called edge effect and evaluate the whole profile being registered. To describe the filter transfer function, the discrete incremental difference operator (delta operator) is applied in place of the usual shift operator *z*. This increases the accuracy of representation of filter parameters and decreases the number rounding errors.

Keywords: Gaussian filters, recursive filters, edge effect.

1. INTRODUCTION

It has been assumed that roughness, waviness and form constitute the geometrical surface structure. These features differ in the range of components wavelengths or, in other words, range of components spatial frequencies. The components with high frequencies or short wavelengths are referred to as roughness, those with medium frequencies are called waviness and, finally, those with small frequencies are considered as form. To separate the above features, analog and digital filters are used in surface metrology. Previously, analog filters called 2RS filters were applied mainly because the whole measuring path had been realized in the analog technique. However, 2RC filters cause a phase shift and in consequence a considerable profile distortion. Recently, analog filters were replaced by various digital filters [1], the most popular of which at the moment is one with a frequency transmission characteristic described by the Gaussian curve [2].

The most natural is to implement a Gaussian filter as a filter with a finite impulse response. This approach requires a great deal of calculation; yet the increase in the microprocessor speed makes the disadvantage less troublesome. A more important drawback is the difficulty in the determination of the mean line for the first and last profile sections (hereafter referred to as the *edge effect*). The methods applying FFT and convolution algorithms are also used, see [1] and the references within. In [3], a fast recursive algorithm is derived using the central limit theorem for the Gaussian transmission characteristic approximation.

This paper deals with the general methodology for the design of recursive digital filters with phase correction. Though the considerations concern only filters with a characteristic described by the Gaussian function, a similar technique can be applied to the design of filters with any desired transmission characteristic. The method for the approximation of a Gaussian characteristic presented below assures the smallest possible error for a given filter order. It has been established that, by selecting an appropriate initial filter value, we are able to eliminate the edge effect and, therefore, evaluate the whole registered profile. Furthermore, to describe the filter transfer function, the discrete incremental difference operator (delta operator) is applied in place of the usual shift operator *z*. This increases the accuracy of the representation of filter parameters and decreases the rounding errors. It is especially important when the filter is implemented by means of the reduced word length arithmetic.

2. APPROXIMATION OF THE TRANSMISSION CHARACTERISTIC OF THE GAUSSIAN FILTER

There are two stages in the design of recursive digital filters. First, an appropriate transfer function of an analog filter is selected. Then, a digital equivalent of the analog filter is determined. When designing analog filters, we try to select a filter transfer function in the form of a proper rational function in order to obtain appropriate accuracy of the approximation of the desired filter characteristic. The characteristic of a low-pass Gaussian filter defining the damping of the wave with the length λ is described by the following equation

$$H(\lambda) = e^{-\pi\left(\frac{\alpha\lambda_c}{\lambda}\right)^2}, \tag{1}$$

where λ_c is the filter cutoff length. The coefficient α is usually selected in order to assure filter damping for $\lambda = \lambda_c$ equal to $1/2$. This yields

$$\alpha = \sqrt{\ln 2/\pi}\ . \tag{2}$$

Defining the normalized frequency $\Omega = \lambda_c/\lambda$, the filter characteristic can be expressed in the form of

$$H(\Omega) = e^{-\pi(\alpha\Omega)^2} \tag{3}$$

One of the methods for the approximation of the function $H(\Omega)$ by means of a rational function is to apply the well known property

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n . \tag{4}$$

By selecting an appropriate value of the filter order $2n$, we obtain the following approximation of the Gaussian filter characteristic

$$H_n(\Omega) = \frac{1}{\left(1+\frac{\beta\pi\alpha^2}{n}\Omega^2\right)^n} . \tag{5}$$

The additional coefficient β is introduced to rescale the frequency slightly so that for the cutoff frequency $\Omega_c = 1$, the filter damping equals exactly $1/2$. Hence $\beta = n(2^{1/n}-1)/\ln 2$. For instance, for $n=4$, we get $\beta = 1{,}09187$. The approximation method presented here was proposed in [3], though its derivation was slightly different, i.e. the central limit theorem was applied in place of property (4).

However, the convergence of the sequence $(1+x/n)^n$ is relatively slow, so, in order to obtain good accuracy of the approximation of the Gaussian filter characteristic, it is necessary to apply an appropriately high filter order. For example, if we assume that $n=16$ [3], the approximation error will be less than 0.5%. A much better approximation for a given filter order will be obtained if we apply the expansion of the function e^x into the Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{6}$$

Taking into account the first $n+1$ of the expansion components, we get the following approximation of the Gaussian filter characteristic

$$H_n(\Omega) = \frac{1}{1+\sum_{i=1}^{n}\frac{\beta\pi\alpha^2}{n!}\Omega^{2n}} . \tag{7}$$

Similarly, the coefficient β is introduced to assure $H_n(1) = 1/2$. For instance, for $n=4$ we need to assume that $\beta = 1{,}00109$. Still, better approximation accuracy can be achieved with the optimization method. Let us formulate the following problem: for a certain number n, let us find the parameters $a_1, a_2, \ldots, a_n$, which minimize the index

$$Q(a_1,a_2,\ldots,a_n) = \int_0^m \left(e^{-x} - \frac{1}{1+a_1x+\ldots+a_nx^n}\right)^2 dx , \tag{8}$$

where m is a number several times greater than the cutoff frequency. Thus, the index Q represents the quality of the approximation function e^{-x} by means of the rational function. It is most reasonable to assume that the initial values in the optimization procedure are $a_i = 1/i!$ For example, for $n=4$, we obtained the following optimal parameter values

$$a_1 = 0{,}9239 ,\ a_2 = 0{,}8319 ,\ a_3 = -0{,}2271 ,\ a_4 = 0{,}1905 .$$

As a result of the optimization the value of the objective function decreased about twenty times. An appropriate filter characteristic is obtained assuming that

$$H_n(\Omega) = \left.\frac{1}{1+a_1x+\ldots+a_nx^n}\right|_{x=\pi(\alpha\Omega)^2} . \tag{9}$$

Fig. (1) shows a deviation of three transmission characteristics (5), (7), (9) from the Gaussian filter characteristic. Fig. (2) presents a deviation of the transmission characteristic (9) from the Gaussian filter characteristic for the three values $n = 3,4,5$.

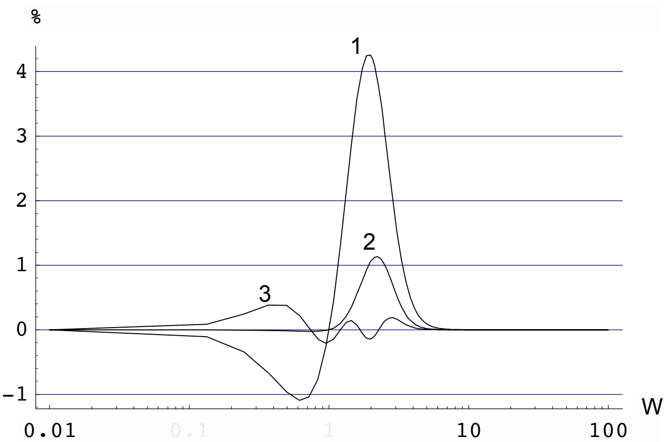

Fig. 1. Deviation of three transmission characteristics (5), (7), (9) from the Gaussian filter characteristic (lines 1,2,3 respectively).

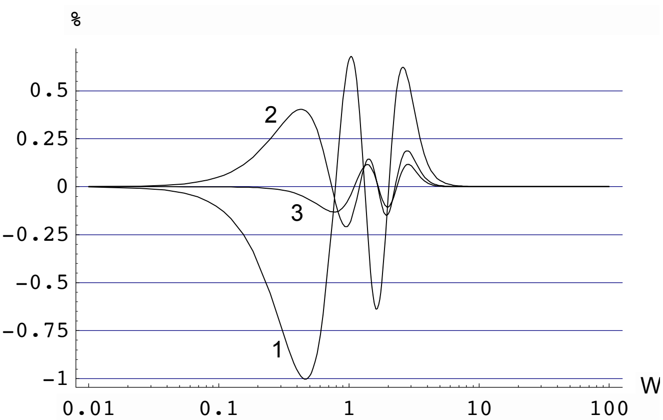

Fig. 2. Deviation of the transmission characteristic (9) from the Gaussian filter characteristic for the three values $n = 3,4,5$ (lines 1,2,3 respectively)..

We can see that the maximum deviation of the filter characteristic (9) from the Gaussian filter characteristic is less than 0,5% for the value n equal to 4.

3. DISCRETE FILTERS

Details of the digital implementation of analog filters can be found in the literature concerning the digital signal processing. That is why we shall discuss the problem briefly only to make the presentation complete. Substituting $\Omega = s/j$, where j is an imaginary unit, we obtain a transfer function of the Laplace filter $H_n(s)$. It should be noted that $H_n(s)$ is a function of s^2. Thus, if s_o is one of the filter poles then $-s_o$ is the filter pole too. In consequence, the filter transfer function can be represented in the form of $H_n(s) = H_{on}(s)H_{on}(-s)$, where $H_{on}(s)$ is a stable transfer function, i.e. with all the poles placed in the left half-plane $\Re(s) < 0$. Applying the bilinear transformation

$$s = \frac{1}{\tan(\pi / N_c)} \frac{z-1}{z+1}, \tag{10}$$

where N_c is the number of samples within the cutoff length, and z is a variable of the Z transform (or the forward shift operator), we can transform an analog filter into a digital one $H_{on}(z)$. The filtration process will consist of two stages: the forward signal filtration, in which the filtered signal depends on the previous values of the input signal, and again the backward refiltration, in which the output filter depends on the future values of the input signal. The filter does not cause any signal phase shift, because the backward filter corrects the phase shift caused by the forward filter.

When implementing a digital filter, we should pay particular attention to the accuracy of representation of the filter coefficients. If the sampling length is small, then the filter poles are located close to the point $z = 1 + 0j$. This makes it necessary to take into account many significant digits of the filter coefficients. Let us consider the example a fourth order filter, $n = 4$, and the number of samples within cutoff $N_c = 2000$. If we take into account the values of the coefficients with an accuracy of eight significant digits (27 digits in the binary representation) we will observe the filter instability. Only if 14 significant digits are considered (45 digits in the binary representation) will the accuracy of the Gaussian filter characteristic representation be satisfactory. It is particularly essential in the case of filter implementation by means of a processor with reduced word length arithmetic. Moreover, a small sampling length causes a considerable increase of rounding errors of filtering algorithm [4]. To overcame these disadvantages, the operator z is replaced with the delta operator defined by [4]

$$\delta = \frac{z-1}{\Delta}, \tag{11}$$

which in place of (10) gives the transformation

$$s = \frac{1}{\tan(\pi / N_c)} \frac{\Delta\delta}{\Delta\delta + 2}, \tag{12}$$

where $\Delta = \lambda_c / N_c$ is the sampling length. In this way we obtain a transfer function expressed in terms of the delta operator, $H_{on}(\delta)$. One of the interesting properties of the delta operator is the fact that the coefficients of the transfer function $H_{on}(\delta)$ converge to the coefficients of the transfer function of an appropriate analog filter, when $\Delta \to 0$ (with the cutoff frequency $\Omega_c = 2\pi/\lambda_c$). The method of the implementation of the filter defined by means of the delta operator is as follows. Let the quadruple ($A \in R^{n \times n}, b, c \in R^n, d \in R^1$) represents the state space realization of the transfer function $H_{on}(\delta)$. The equations describing the forward and backward part of the are the following

forward part of the filter:

$$\begin{gathered} v_f(i+1) = v_f(i) + \Delta(Av_f(i) + bx(i)), \\ w(i) = c^T v(i) + dx(i), \quad i = 1,2,\ldots, N, \end{gathered} \tag{13}$$

backward part of the filter:

$$\begin{gathered} v_b(i-1) = v_b(i) + \Delta(Av_b(i) + bw(i)), \\ y_m(i) = c^T v_b(i) + dw(i), \; i = N, N-1, \ldots, 1, \end{gathered} \tag{14}$$

where x is the input of the filter, v_f, v_b are the states of the forward and backward parts of the filter respectively, y_m is the output of the filter and, finally, N is the total number of profile samples.

4. EDGE EFFECT ELIMINATION

The initial values of the filter state variables, $v_f(1)$ and $v_b(N)$, have direct influence on the determined values of the mean line at the ends of the registered profile. This may cause the occurrence of so-called edge effect consisting in a considerable deviation of the determined mean line from the expected value at both profile ends. Taking into account the speed of the filter transient response attenuation, we can conclude that the edge effect can influence about a half of the cutoff at both ends of the profile. Thus, this part of the profile should not be analyzed. This would be a considerable drawback of the filter, as the length of the registered profile is often too short to reject its fragments. It appears, however, that careful selection of the initial values of the filters states can eliminate the edge effect.

Let us note that the edge effect will cause an increase in the mean square difference between the profile and the determined mean line. Then, it seems reasonable to demand that the initial values of the filter $v_f(1)$ be appropriately selected in order to minimize the index

$$Q(v_1) = \sum_{i=1}^{N} \left(x(i) - w(i)\right)^2, \; v_f(1) = v_1. \tag{15}$$

Since the index is a square function of the variable v_1, the problem of the minimization of the index (15) can be solved analytically. Let us note that the filtered signal w can be represented as

$$w(i) = \eta(i) + w^o(i)\,, \tag{16}$$

where w^o is the filter response for initial value $v_f(1) = 0$, while η is the filter response for the zero input signal. From the filter equations we have

$$\eta(i) = c^T (I + \Delta A)^{i-1} v_1 \stackrel{\mathrm{df}}{=} c(i)^T v_1\,. \tag{17}$$

Equating the partial derivative $\partial Q / \partial v_1$ to zero, after some manipulations we obtain

$$v_1 = \left(\sum_{i=1}^{N} c(i) c^T(i) \right)^{-1} \left(\sum_{i=1}^{N} c(i) w^o(i) \right). \tag{18}$$

The filtration algorithm is thus divided into four steps:

1. determination of the filter response for the initial value $v_f(1) = 0$,
2. determination of the initial values of the filter v_1 from formula (18),
3. determination of the filter response for the zero input signal and $v_f(1) = v_1$,
4. determination of the filtered signal from $w(i) = \eta(i) + w^o(i)$.

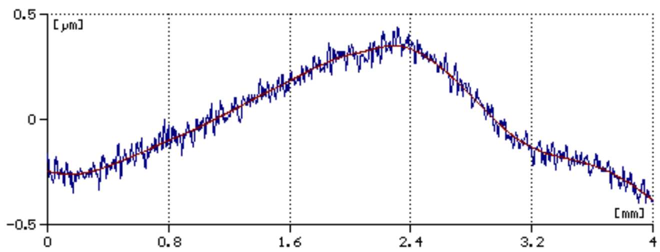

Fig. 3. Example of a roughness profile with a significant form component and the determined mean line.

The initial value of v_N of the backward filter part should be selected in a similar way. Fig. (3) shows a diagram of a roughness profile with the determined mean line. This is an example of a profile with a significant form component. We can see the described method of filter initial values selection prevents the occurrence of the edge effect.

5. CONCLUSION

The results show that even for an eighth order filter the optimization of the filter parameters assures a very good approximation of the Gaussian filter with an approximation error less than 0.5%. The analysis of the profiles for various surface types testifies that the edge effect does not occur if the initial filter values are selected by means of the described method.

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