Reducing the Number of Elements in a Linear Antenna Array by the Matrix Pencil Method

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***Abstract*—The synthesis of a nonuniform antenna array with as few elements as possible has considerable practical applications. This paper introduces a new non-iterative method for linear array synthesis based on the matrix pencil method (MPM). The method can synthesize a nonuniform linear array with a reduced number of elements, and can be also used to reduce the number of elements for linear arrays designed by other synthesis techniques. In the proposed method, the desired radiation pattern is first sampled to form a discrete pattern data set. Then we organize the discrete data set in a form of Hankel matrix and perform the singular value decomposition (SVD) of the matrix. By discarding the non-principal singular values, we obtain an optimal lower-rank approximation of the Hankel matrix. The lower-rank matrix actually corresponds to fewer antenna elements. The matrix pencil method is then utilized to reconstruct the excitation and location distributions from the approximated matrix. Numerical examples show the effectiveness and advantages of the proposed synthesis method.**

***Index Terms*—Array synthesis, low rank approximation, matrix pencil method (MPM), nonuniform antenna arrays, singular value decomposition (SVD), sparse antenna array.**

I. INTRODUCTION

SYNTHESIS of antenna arrays with the minimum number of elements for a desired beam pattern is very significant in some applications (e.g., satellite communications) where the weight of antennas is extremely limited. Other practical advantages with fewer antenna elements include the reduction of cost and the simplification of antenna systems.

This paper focuses on the problem of reducing the total number of elements for linear antenna arrays. The problem can be described as follows. Let a linear array be composed of M identical antenna elements. The array factor is given by

$$F_M(\theta) = \sum_{i=1}^{M} R_i e^{jkd_i\cos\theta} \tag{1}$$

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where R_i' is the complex excitation coefficient of the ith element located at $x = d_i$ along the linear array direction x, and $k = (2\pi/\lambda)$ is the spatial wavenumber. The objective is to synthesize a new linear antenna array that has the minimum number of elements while maintaining the same desired pattern as $F_M(\theta)$ with a small tolerance. That is, we intend to find a solution to the following problem:

$$\begin{cases} \text{Min}\,\{Q\} \\ \text{Const.} \left\{ \underset{\{R_i',d_i'\}_{i=1,\ldots,Q}}{\text{Min}} \left\| F_M(\theta) - \sum_{i=1}^{Q} R_i' e^{jkd_i'\cos\theta} \right\|_L \right\} \leq \varepsilon \end{cases} \tag{2}$$

where R_i' and $d_i'(i = 1, \ldots, Q \leq M)$ are the complex excitations and locations for Q antenna elements, and $L = 2$ if the least square error (LSE) is used.

Before introducing our method for solving (2), it is worthwhile to comment briefly on the many other synthesis techniques that have been developed previously. These techniques can be generally classified as synthesis of (a) uniformly and (b) nonuniformly spaced arrays. For the synthesis of uniformly spaced array, many conventional synthesis techniques, such as Dolph-Chebyshev and Taylor methods [1], [2], allow the synthesis of narrow beam, low sidelobe or the optimization of an interesting parameter (e.g, radiation directivity). However, these traditional techniques sometimes require a large number of antenna elements for desired radiation characteristics because of the limitation of uniform element spacing. Naturally, the synthesis of nonuniformly spaced antenna arrays allows us to have much more freedom to achieve performance improvement [3]. Thus many practical techniques have been presented for the synthesis of nonuniform arrays. These techniques include optimization algorithms (such as dynamic programming [4], genetic algorithm (GA) [5], differential evolution algorithm (DEA) [6] and particle swarm optimization (PSO) method [7]), analytical methods [8], [9] and other synthesis techniques [10]–[13], and so on. Despite the success of these synthesis techniques, to our knowledge, synthesis of the array with the minimum number of elements for a desired pattern remains a challenging problem due to the following two reasons.

1) In order to reduce the number of antenna elements, the synthesis of a completely nonuniform array is desired. This is an inverse problem involving finding the solution of many unknowns (excitation amplitude, phase and position for each element). Many synthesis techniques do not guarantee a global optimum for all the variables. Some optimization algorithms (such as GA, DEA and PSO) capable of finding

the globally optimal solutions [5]–[7] are appropriate, but they can be time-consuming.

2) Most techniques [6]–[9], [11], [13] synthesize antenna arrays with a fixed number of elements or with prescribed lengths of arrays. They need to vary the prescribed element number or lengths of arrays to find all possible solutions with fewer array elements. This is not a computationally efficient way for synthesis of antenna arrays with a large number of elements.

This paper introduces a new non-iterative synthesis method as a complement to the existing synthesis methods. The method consists of two steps. In the first step, the singular value decomposition (SVD) technique is used to obtain the low rank approximation in LSE of the Hankel matrix constructed by the desired pattern samples. The lower-rank matrix data actually corresponds to the approximated pattern that consists of a smaller number of antenna elements. Hence, the first step allows us to determine at least how many elements are required in a given approximation tolerance for a desired pattern before the distributions of excitation elements are determined. After the required number of antenna elements is determined, the second step is to apply the matrix pencil method (MPM) [14], [15] to rearrange the excitation and location distributions for the new antenna array with the reduced number of elements.

It is worthwhile noting that among existing synthesis techniques, the Prony-based synthesis method [16] is the closest to the proposed method. It is well known that the matrix pencil method is less sensitive to noise and more computationally efficient than Prony's method [15]. Furthermore, the MPM-based antenna synthesis approach introduced here is different from the Prony-based synthesis method, in the following three aspects: 1) the proposed method automatically finds the required number of elements for producing the desired pattern within a given tolerance by performing low rank approximation of the pattern samples matrix, while the Prony-based synthesis needs to set the number of elements before performing the array synthesis; 2) the proposed method avoids the requirement in the Prony-based method to use a high-accuracy computer routine for obtaining the polynomial roots. Note that this requirement makes the Prony-based method difficult to synthesize arrays with a large number of elements; and 3) the SVD used in the proposed method overcomes the problem of ill-conditioned linear equations, while in the Prony-based method the linear equations must be well conditioned, as pointed out by [8]. Some synthesis examples show the effectiveness of the proposed method.

It is noted that the approach of minimizing the number of elements using SVD technique can be also viewed as one specialization of more general mathematical theory widely used for nonuniform and redundancy-reduced sampling of signals. Some early studies in relevant applications include [17], [18] in field synthesis and [19] in optimal beamforming for MIMO systems.

II. Estimation of the Minimum Number of Antenna Elements

Let a linear array be composed of M identical elements. The array factor is described by (1). Let $u = \cos\theta$ and $\omega_i = kd_i$, thus (1) can be written as

$$F_M(\cos^{-1} u) = \sum_{i=1}^{M} R_i e^{j\omega_i u}. \tag{3}$$

Equation (3) is in the form of a sum of exponentials. The problem described in (2) is to use as few exponentials (or antenna elements) as possible to approximate the original pattern function F_M within a desired tolerance. In the signal processing community, the matrix pencil method has been proven to be useful for dealing with the problem related to the exponentials [14], [15]. The method of determining the minimum value of Q in (2) is described as follows.

First, we sample the pattern function in uniform steps of u from $u = -1$ to $u = +1$. Let

$$u_n = n\Delta = \frac{n}{N}, \quad n = -N, \ldots, 0, \ldots, N \tag{4}$$

where the number of samples is $(2N+1)$. We have

$$f_M(n) = F_M(\cos^{-1} n\Delta) = \sum_{i=1}^{M} R_i z_i^n \tag{5}$$

where $z_i = e^{j\omega_i \Delta}$. According to the Nyquist sampling theorem, the condition that $\Delta \le \lambda/(2d_{\max})$ must be satisfied, where $d_{\max} = \max\{d_i\}$. For instance, $\Delta \le 1/(M-1)$ for the M-element array with a uniform spacing of $\lambda/2$. In other words, $(2M-1)$ sampling points are adequate to describe the pattern of the M-element uniformly spaced array.

Then, we construct the Hankel matrix from the sampled pattern data to arrive at (6), shown at the bottom of the page, where $y(n) = f_M(n-N)$. The parameter $\{N, L\}$ are chosen such that $2N - L \ge M$, and $L + 1 \ge M$. For instance, we can set $N = L = M$, as the minimum number of samples while maintaining the best performance [14], [15]. Then the singular value decomposition (SVD) of the matrix $[\mathbf{Y}]$ is carried out as

$$[\mathbf{Y}] = [\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^H \tag{7}$$

$$[\mathbf{Y}] = \begin{bmatrix} y(0) & y(1) & \ldots & y(L) \\ y(1) & y(2) & \ldots & y(L+1) \\ \vdots & \vdots & & \vdots \\ y(2N-L) & y(2N-L+1) & \ldots & y(2N) \end{bmatrix}_{(2N-L+1)\times(L+1)} \tag{6}$$

where $[\mathbf{U}] \in C^{(2N-L+1)\times(2N-L+1)}$ and $[\mathbf{V}] \in C^{(L+1)\times(L+1)}$ are unitary matrices. $[\mathbf{\Sigma}] = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_M, \ldots, \sigma_P; \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_P\}$ with $\{\sigma_i\}$ being the ordered singular values of $[\mathbf{Y}]$, and $P = \min\{2N-L+1, L+1\}$.

The rank of the Hankel matrix $[\mathbf{Y}]$ and the number of nonzero singular values would be equal to the number of exponentials. In general, there should be M nonzero singular values for the M -element array antenna. However, observations show that for many designed antenna arrays in literature, the number of principal singular values is less than the number of antenna elements. This means that the contributions of some elements corresponding to non-principal singular values can be replaced by the combination of other elements. Thus we can discard the non-principal values to obtain a low rank approximation of $[\mathbf{Y}]$, which corresponds to a new antenna array with fewer elements. A typical method is to set these non-principal singular values equal to zero. That is

$$[\mathbf{Y}_Q] = [\mathbf{U}][\mathbf{\Sigma}_Q][\mathbf{V}]^H \tag{8}$$

where $[\mathbf{\Sigma}_Q] = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_Q, 0, \ldots, 0\}$ and $Q \leq M$. It has been proven that among all the matrices with rank of Q, $[\mathbf{Y}_Q]$ has the minimum approximation error in Frobenius norm [20], i.e., mathematically

$$\|[\mathbf{Y}] - [\mathbf{Y}_Q]\|_F = \min_{\mathrm{rank}([\mathbf{X}])=Q} \|[\mathbf{Y}] - [\mathbf{X}]\|_F = \sqrt{\sum_{i=Q+1}^{M} \sigma_i^2}. \tag{9}$$

From (9), the approximation error decreases monotonously as Q, the number of antenna elements, increases. The final error goes to zero when $Q = M$. This means the radiation pattern of the new array can always achieve a good approximation of the original pattern. Hence, the problem of approximating the pattern data is in fact different from noise filtering for signal estimation [15] even though they both use a low rank approximation technique.

In a practical synthesis problem, the minimum value of Q is chosen as follows:

$$Q = \min\left\{ q; \left| \frac{\sqrt{\sum_{i=q+1}^{P} \sigma_i^2}}{\sqrt{\sum_{i=1}^{q} \sigma_i^2 < \varepsilon}} \right. \right\} \tag{10}$$

where ε is a small positive number. The choice of ε depends on how accurately the reconstructed pattern approximates the original radiation pattern. Some synthesis examples will be presented to show how to choose the value of ε.

III. Rearrangement of Excitations and Locations of Antenna Elements

Once the low rank matrix $[\mathbf{Y}_Q]$ is available, the parameters $\{z_i'\}$ corresponding to the locations of the new Q array elements can be obtained by solving the following generalized eigenvalue problem:

$$([\mathbf{Y}_{Q,f}] - z'[\mathbf{Y}_{Q,l}])\,\bar{v}' = 0 \tag{11}$$

where $[\mathbf{Y}_{Q,f}] \in C^{(2N-L+1)\times L}$ is obtained from $[\mathbf{Y}_Q]$ by deleting the first column, and $[\mathbf{Y}_{Q,l}] \in C^{(2N-L+1)\times L}$ is obtained from $[\mathbf{Y}_Q]$ by deleting the last column. The parameters z_i' would be equal to the nonzero eigenvalues z'. However, a more computationally effective method than using (11) is to find the eigenvalues of the following matrix:

$$\left\{ \left([\mathbf{V}_{Q,b}]^H[\mathbf{V}_{Q,b}]\right)^{-1} \left([\mathbf{V}_{Q,t}]^H[\mathbf{V}_{Q,b}]\right) - z' \right\} \tag{12}$$

where $[\mathbf{V}_{Q,t}] \in C^{L\times Q}$ (resp., $[\mathbf{V}_{Q,b}] \in C^{L\times Q}$) is obtained by removing the top (resp., bottom) row of $[\mathbf{V}_Q] \in C^{(L+1)\times Q}$ which contains only Q dominant left-singular vectors of $[\mathbf{V}]$ in (8).

From (12), only the inverse of a $(Q\times Q)$ matrix and the eigenvalues of a $(Q \times Q)$ matrix are required in this method. Thus, (12) is more computationally effective than directly using (11) above or using [15, Eq. (24)]. It is worth noting that using (12) for obtaining the parameters z_i' s avoids the high accuracy polynomial rooting and has also removed the ill condition problem, both arising in the Prony based synthesis technique.

Once the z_i' s are obtained, the locations of antenna elements are given by (7) in [16]. That is,

$$\hat{d}_i' = \frac{1}{jk\Delta}\ln(\hat{z}_i'). \tag{13}$$

As in the Prony based synthesis technique, the source locations d_i' calculated by (13) or (7) in [16] may turn out to be complex if $|z_i'| \neq 1$. Only the real parts of d_i' s are physically realizable. However, through numerous synthesis examples with a variety of patterns, we found that exactly as pointed out in [16], the magnitudes of imaginary parts of d_i' s were very insignificant for the most patterns having broadside maxima. For instance, the imaginary parts of d_i' s in (13) were found to be ten orders of magnitude smaller than the real parts for synthesis of conventional broadside patterns (such as Chebyshev pattern and Taylor array factor [1]).

Hence, an approximate method is to take only the real parts as the estimates of d_i' s [16]. Thus the estimated excitations and locations are given as

$$\hat{d}_i' = \frac{1}{jk\Delta}\ln(\hat{z}_i') \tag{14}$$

$$\hat{R}_i' = \left([\hat{\mathbf{Z}}]^H[\hat{\mathbf{Z}}]\right)^{-1}[\hat{\mathbf{Z}}]\bar{f}_M \tag{15}$$

where

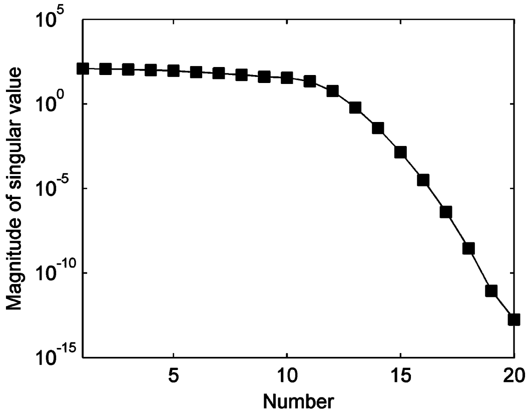
$$\hat{z}_i' = \frac{z_i'}{|z_i'|} \tag{16}$$

$$\bar{f}_M = (f_M(-N), f_M(-N+1)\ldots, f_M(N))^T \tag{17}$$

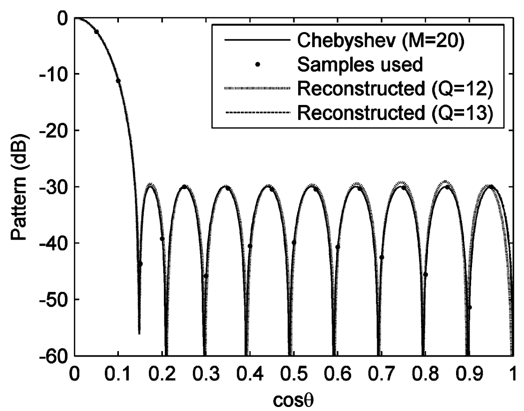
$$[\hat{\mathbf{Z}}] = \begin{bmatrix} (\hat{z}_1')^{-N} & (\hat{z}_2')^{-N} & \cdots & (\hat{z}_Q')^{-N} \\ (\hat{z}_1')^{-N+1} & (\hat{z}_2')^{-N+1} & \cdots & (\hat{z}_Q')^{-N+1} \\ \vdots & \vdots & & \vdots \\ (\hat{z}_1')^{N} & (\hat{z}_2')^{N} & \cdots & (\hat{z}_Q')^{N} \end{bmatrix}_{(2N+1)\times Q} \tag{18}$$

(15) finds the least squares solution of excitations, which makes up the effect of discarding the imaginary parts of d_i' s to some degree.

In our synthesis experimentations, (12)–(18) allowed excellent synthesis of all interesting broadside patterns with the number of antenna elements determined by (10). For the shaped



(a)



(b)

Fig. 1. (a) Singular value spectrum of the broadside Chebyshev pattern $T19(x)$. (b) Reconstruction of the desired pattern by nonuniform arrays with $Q = 12$ and 13.

beam patterns, the method may fail to obtain the realizable synthesis with the minimum number of elements because the imaginary parts of d_i' s in (13) are usually not negligible. Nevertheless, the MPM based method can be still used to reduce the number of elements for a given antenna array with the shaped beam pattern by varying Q between the minimum value obtained by (10) and M, the original number of antenna elements. For generalized applications of the MPM based synthesis technique, a method for constraining all the parameters z_i' locating on a unit circle (that is, $|z_i'| = 1$) is being investigated.

IV. Numerical Results

A. Example 1: Synthesis of a Chebyshev Pattern

As a first example, the design of a nonuniform array with fewer antenna elements than that required by a uniformly spaced Chebyshev array will be given. It is supposed that a broadside $T_{19}(x)$ pattern with a side-lobe level $SLL = -30\,\mathrm{dB}$ is desired. Twenty antenna elements are spaced at equidistance $d = 0.5\lambda$ in the original Chebyshev array. Note that we set the sampling parameter $N = M$ for all the synthesis examples except in Example 3, and no accuracy improvement is observed by further increasing the number of samples. Fig. 1(a) represents the singular value spectrum of the Chebyshev pattern samples. The

TABLE I
Positions and Amplitudes of the Reconstructed Nonuniformly Spaced Arrays and the Uniformly Spaced Chebyshev Array

Chebyshev ($M = 20$)		Nonuniform ($Q = 12$)		Nonuniform ($Q = 13$)	
i	R_i	d_i/λ	R_i	d_i/λ	R_i
1	1	0.4254	1	0	1
2	0.97010	1.2755	0.91407	0.8206	0.95818
3	0.91243	2.1236	0.75974	1.6381	0.84113
4	0.83102	2.9671	0.56719	2.4481	0.67176
5	0.73147	3.8011	0.37122	3.2432	0.48115
6	0.62034	4.6371	0.26841	4.0071	0.30046
7	0.50461			4.7145	0.23345
8	0.39104				
9	0.28558				
10	0.32561				

singular values beyond the 12th value decay rapidly. Some very small singular values can be discarded. Thus the desired pattern can be reconstructed by fewer antenna elements. The criterion (10) gives the minimum value, $Q = 12$ at $\varepsilon = 10^{-2}$, or $Q = 13$ at $\varepsilon = 10^{-3}$. Note that although (10) defines the approximation error not in the "minmax" sense but in the sense of the least squares, a fair amount of synthesis examples show that $\varepsilon = 10^{-2}$ (resp. $\varepsilon = 10^{-3}$) allowed excellent reconstructions (generally, the SLL error is within 0.5 dB) of desired patterns with $SLL \geq -20$ dB (resp. $SLL \geq -40$ dB). For the following cases, $\varepsilon = 10^{-3}$ is used for the accurate reconstruction of low SLL patterns. Fig. 1(b) shows the comparison between reconstructed patterns (using $Q = 12$ and $Q = 13$, respectively) and the desired pattern. As can be seen, 13 nonuniformly spaced antenna elements almost exactly reproduce the desired pattern produced by the 20 uniformly spaced Chebyshev elements. Therefore, the saving in the number of antenna elements is 40% for $Q = 12$ and 35% for $Q = 13$. Note that the synthesized excitations in this case are symmetrical, although our procedure does not constrain the excitation distribution to be of any form. Table I shows the corresponding element positions and excitation amplitudes (for $Q = 12$ and $Q = 13$, resp.) and amplitudes of the uniformly spaced Chebyshev array. It is observed that the both new arrays have a slightly smaller aperture than the original array. Similarly, most of the following synthesis examples also yield a smaller aperture. The whole synthesis process takes only about 12 milliseconds on an IBM-T61 PC in this example. The CPU time for the other examples except for the last one is less than 50 ms.

B. Example 2: Synthesis of a Taylor-Kaiser Array Pattern

Next, the 29-element Taylor-Kaiser array reported in [2] with $SLL = -25$ dB is to be optimized in this example by the proposed technique. As a result, a nonuniform array with only 17 elements ($\varepsilon = 10^{-3}$) is found to produce a satisfactory approximation of the desired pattern. The saving of the number of antenna elements is 41.4%. Both the reconstructed pattern and the desired pattern are shown in Fig. 2. The reconstructed element

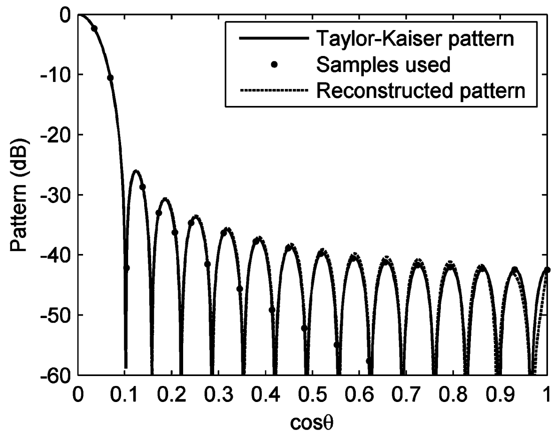


Fig. 2. Patterns of the constructed nonuniform array with 17 elements and the Taylor-Kaiser array with 29 elements.

TABLE II
POSITIONS AND AMPLITUDES OF THE RECONSTRUCTED 17-ELEMENT ARRAY AND AMPLITUDES OF THE 29-ELEMENT TAYLOR-KAISER ARRAY

Taylor-Kaiser		Position/Excitation	
i	R_i	d_i/λ	R_i
1	1	0	1
2	0.99328	0.8831	0.97859
3	0.97329	1.7652	0.91634
4	0.94063	2.6451	0.81903
5	0.89622	3.5211	0.69547
6	0.84132	4.3905	0.55651
7	0.77748	5.2485	0.41370
8	0.70645	6.0842	0.27782
9	0.63017	6.8661	0.15704
10	0.55065		
11	0.46994		
12	0.39004		
13	0.31282		
14	0.24001		
15	0.17309		

positions and amplitudes compared with the amplitudes of the Taylor-Kaiser array are shown in Table II.

C. Example 3: Synthesis of a Desired Pattern Used by Kumar and Branner in [8]

The desired pattern is chosen to be the same as that shown in Fig. 4(c) presented by Kumar and Branner in [8]. In this example, we use 80 samples to characterize the pattern; this number is larger than M because the pattern cannot be exactly produced by the original array. The current method optimizes both the element excitations and positions in order to reduce the array element number. As a result, a pattern of only 13 antenna elements with $SLL = -\,28.2$ dB is synthesized (in Fig. 4(c) presented in [8], only element positions are optimized and the synthesized pattern with $SLL = -19.3$ dB requires 17 elements). The comparison of the synthesized patterns is shown in Fig. 3. Clearly the current method produces a more accurate approximation of the desired pattern. Table III shows the synthesized positions and amplitudes by the proposed method and the optimized positions given in [8]. In this case, we save 23.5% antenna elements.

D. Example 4: Synthesis of a Pattern Presented By Kurup

This example will show that simultaneous optimization of excitation amplitudes, phases and positions further reduces the number of elements compared with the case where only phases and positions are optimized. For comparison, we use the pattern function synthesized by [6] as an input to our procedure. As a result, the proposed method ($\varepsilon = 10^{-3}$) obtains a 24-element array with nonuniform excitation amplitudes, phases and element positions, while in [6], 32 elements with nonuniform excitation phases and element positions are required for the desired pattern. Fig. 4 shows the reconstructed pattern compared with the original pattern in [6]. The synthesized excitations and phases are shown in Table IV. The saving in the number of elements is 25% for this case. Although maintaining uniform excitation amplitudes is important in some applications [6], synthesis of arrays with fewer elements is also significant in other situations. Which one is more desirable depends on the situations.

E. Example 5. Synthesis of a Scanned Array Pattern

A scanned pattern with no grating lobes at all directions used in [12] by Marchaud *et al.* is chosen as the desired pattern for checking if our method can produce fewer elements than those required in the literature where the generalized Gaussian quadrature is employed to produce nonuniform element locations and weights. Using a tolerance $\varepsilon = 10^{-3}$, we obtain $Q = 7$. The reconstructed pattern compared with the desired pattern is shown in Fig. 5. As can be seen, the reconstructed pattern is almost exactly the same as the desired pattern, while the reconstructed array saves one element compared with the array synthesized by [12]. Table V shows the reconstructed excitation amplitude distribution (excitation phases for all elements are zero at $\theta_m = 90°$).

F. Example 6: Optimization of an Electrically Large Linear Array

As the last example, we consider the optimization of an electrically large linear array to determine the efficiency of the proposed method in large array synthesis. A pattern with $HPBW = 0.4°$ (half power beam width) and $SLL = -37$ dB is desired. We choose the corresponding Chebyshev pattern samples as an input for our synthesis procedure. The entire synthesis process takes only 11.5 s. The results are shown in Fig. 6. In this case, we still use the tolerance $\varepsilon = 10^{-3}$. The minimum element number is 170 and the array length is 165.97λ. The amplitude and position distributions are nonuniform while all excitation phases are zeros. The result can be compared with that of [11, Fig.7], which synthesizes a pattern with the same HPBW and maximum SLL with 241 nonuniform (amplitude and phase) elements at an equidistance of 0.755λ. This example illustrates the reliability and efficiency of the proposed method for the optimization of electrically large linear arrays.

Furthermore, here we study the pattern sensitivity of the proposed synthesis to possible realization errors. We simulate the array pattern by randomly perturbing the element positions,

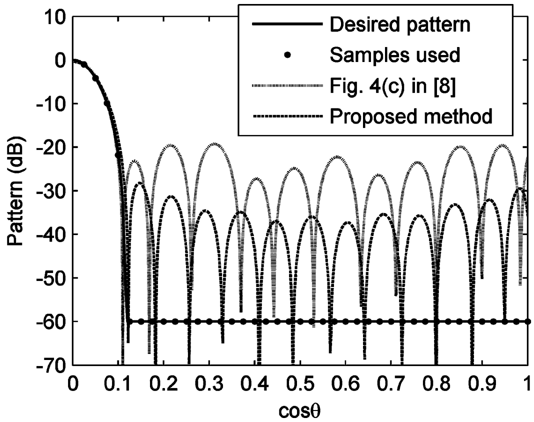


Fig. 3. Synthesized pattern by the proposed method compared with the pattern presented by Kumar and Branner in [8].

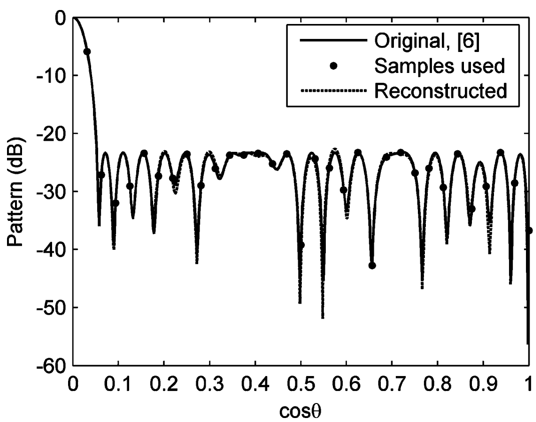


Fig. 4. Patterns of the reconstructed 24-element array by the proposed method and the 32-element array synthesized by [6].

TABLE III
THE POSITIONS AND AMPLITUDES SYNTHESIZED BY THE PROPOSED METHOD AND POSITIONS-ONLY OPTIMIZATION, [8]

position-only, [8]		position/excitation	
i	d_i/λ	d_i/λ	R_i
1	0	0	1
2	0.5	0.885	0.959
3	1.0	1.770	0.869
4	1.5	2.654	0.733
5	2.0	3.537	0.569
6	2.683	4.418	0.399
7	3.464	5.303	0.247
8	4.297		
9	5.163		

amplitudes and phases from the obtained solution one thousand times. The perturbations were set according to zero-mean normal distributions, with the standard deviations of 0.01λ for the element positions, 0.125 dB for the amplitudes, and 1° for the phases. The RMSEs (relative mean squared errors) of the perturbed patterns are 0.011, 0.0145, and 0.0175 for the variations in element positions, amplitudes and phases, respectively. The total RMSE is 0.0251 if all perturbations happen simultaneously. We also check the maximum and minimum values of the pattern envelope at each angle for all these perturbations. The upper and lower bounds of the variation in the vicinity of the mainlobe are shown in the inset of Fig. 6. It can be observed the upper bound in the sidelobe-region is almost within -35 dB. The perturbed pattern is still acceptable within the uncertainty range in the above parameters.

TABLE IV
SYNTHESIS OF NONUNIFORM POSITION-AMPLITUDE-PHASE BY THE PROPOSED METHOD AND OPTIMIZATION OF POSITION-PHASE ONLY, [6]

position/amplitude, [6]			position/amplitude/phase		
i	d_i/λ	$\angle R_i$ (°)	d_i/λ	$\lvert R_i \rvert$	$\angle R_i$ (°)
1	0.26	26.1	0.454	1	26.05
2	0.76	23.8	1.334	0.932	27.92
3	1.26	28.2	2.230	0.941	27.12
4	1.83	31.1	3.168	0.915	29.49
5	2.33	26.5	4.043	0.796	26.09
6	2.94	30.7	4.969	0.877	27.86
7	3.47	28.7	5.871	0.658	27.45
8	4.06	24.4	6.817	0.590	37.66
9	4.73	26.6	7.738	0.575	26.93
10	5.27	28.4	8.628	0.570	0.43
11	6.01	26.6	9.616	0.567	61.27
12	6.88	38.3	10.498	0.565	28.81
13	7.77	27.7			
14	8.64	0			
15	9.62	61.3			
16	10.5	28.8			

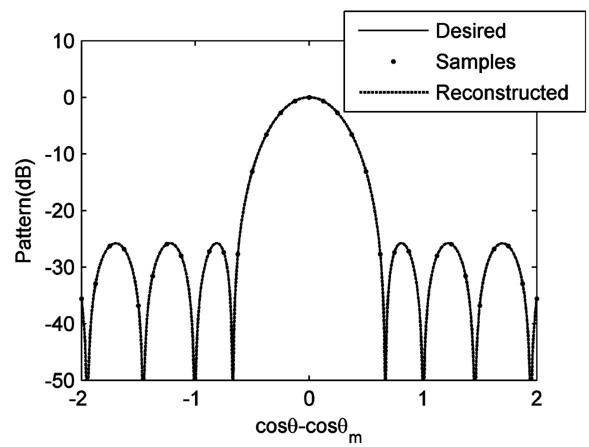


Fig. 5. Patterns of the reconstructed 7-element array by the proposed method and the 8-element array synthesized by [12].

TABLE V
POSITIONS AND EXCITATIONS OF THE RECONSTRUCTED SCANNED ARRAY ($\theta_m = 90^\circ$)

i	1	2	3	4
d_i/λ	0	0. 32080	0. 63028	0. 90845
R_i	1	0. 86068	0. 53513	0. 27815

V. CONCLUSION AND DISCUSSION

An effective approach for reducing the number of elements in linear antenna arrays has been presented. By performing the

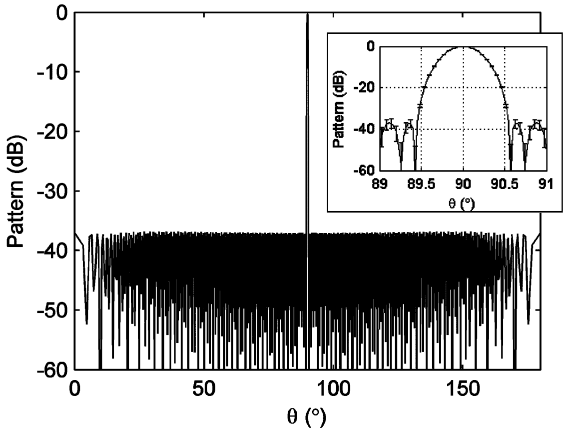


Fig. 6. Synthesized pattern with $HBPW = 0.4^\circ$ and $SLL = -37$ dB (an enlarged view of the main beam is shown in the inset). The error bars in the inset show the upper and lower bounds of the perturbed pattern due to the realization errors with standard deviation of 0.01λ for the element positions, 0.125 dB for the amplitudes, and 1° for the phases.

singular value decomposition of radiation pattern samples, we can determine how many array elements are required for producing the desired patterns within a given tolerance before the excitations and element positions are solved. Then the matrix pencil method is employed to simultaneously find the optimal excitation amplitude, phase and position for each new antenna element. By using the proposed method, conventional broadside patterns are reproduced by nonuniformly spaced linear arrays with fewer elements than that required in uniformly spaced linear arrays. The saving in the number of elements can be more than 40%. The proposed method is also used to further reduce the number of elements for nonuniform linear arrays of which partial parameters have been optimized by other synthesis methods. In addition, most of the synthesis results yield a smaller aperture than the original array. However, there exists a limitation in the proposed method for the synthesis of shaped beam patterns where the imaginary parts of synthesized element positions at the estimated minimum element number are not always negligible. Under these circumstances, we can search a realizable implementation with negligible imaginary positions by increasing the element number. Further research is under way to remove this limitation.

The proposed method is computationally efficient (e.g., the CPU time is just 11.5 s even for the synthesis of a 170-element nonuniform array, including the process of finding the minimum element number). This procedure is therefore also particularly suited for the design of large linear arrays with narrow pattern and very low SLL. In addition, the proposed method can be extended to the synthesis of planar antenna arrays.

Finally, it should be noted that the proposed synthesis method deals solely with array factors, as in (3). It can be straightforwardly extended to deal with the total radiation pattern with the multiplication of the array factor and element antenna pattern. However, from the practical point of view, some physical array effects, such as mutual coupling and edge effects, are very important aspects but remain to be dealt with. From many synthesis results, the proposed method can obtain larger element spacings than those of the original arrays. For instance, each reconstructed element spacing in all the synthesis examples except the fifth example reported in our manuscript, is larger than half wavelength. This implies the reconstructed arrays would have smaller mutual coupling than the original arrays. Further study on the physical array effects with full-wave electromagnetic simulation is beyond the scope of this paper.

REFERENCES

[1] C. A. Balanis, *Antenna Theory: Analysis and Design*, 3rd ed. New York: Wiley, 2005.

[2] S. J. Orfanidis, Electromagnetic Waves and Antenna 2004 [Online]. Available: http://www.ece.rutgers.edu/~orfanidi/ewa/.

[3] Y. T. Lo and S. W. Lee, "A study of space-tapered arrays," *IEEE Trans. Antennas Propag.*, vol. 14, pp. 22–30, Jan. 1966.

[4] M. I. Skolnik, G. Nemhauser, and J. W. Sherman, III, "Dynamic programming applied to unequally spaced arrays," *IEEE Trans. Antennas Propag.*, vol. 12, pp. 35–43, Jan. 1964.

[5] K. K. Yan and Y. Lu, "Sidelobe reduction in array-pattern synthesis using genetic algorithm," *IEEE Trans. Antennas Propag.*, vol. 45, pp. 1117–1122, Jul. 1997.

[6] D. G. Kurup, M. Himdi, and A. Rydberg, "Synthesis of uniform amplitude unequally spaced antenna array using the differential evolution algorithm," *IEEE Trans. Antennas Propag.*, vol. 51, pp. 2210–2217, Sep. 2003.

[7] P. J. Bevelacqua and C. A. Balanis, "Minimum sidelobe levels for linear arrays," *IEEE Trans. Antennas Propag.*, vol. 55, pp. 2210–2217, Dec. 2007.

[8] B. P. Kumar and G. R. Branner, "Design of unequally spaced arrays for performance improvement," *IEEE Trans. Antennas Propag.*, vol. 47, pp. 511–523, Mar. 1999.

[9] B. P. Kumar and G. R. Branner, "Generalized analytical technique for the synthesis of unequally spaced arrays with linear, planar, cylindrical or spherical geometry," *IEEE Trans. Antennas Propag.*, vol. 53, pp. 621–633, Feb. 2005.

[10] R. J. Mailloux and E. Cohen, "Statistically thinned arrays with quantized element weights," *IEEE Trans. Antennas Propag.*, vol. 39, pp. 436–447, Apr. 1991.

[11] A. Monorchio and S. Genovesi, "An efficient interpolation scheme for the synthesis of linear arrays based on Schelkunoff polynomial method," *IEEE Trans. Antennas Wireless Propag. Lett.*, vol. 6, pp. 484–487, 2007.

[12] F. B. T. Marchaud, G. D. de Villiers, and E. R. Pike, "Element positioning for linear arrays using generalized Gaussian quadrature," *IEEE Trans. Antennas Wireless Propag. Lett.*, vol. 51, no. 6, pp. 1357–1363, Jun. 2003.

[13] P. Jarske, T. Sramaki, S. K. Mitra, and Y. Neuvo, "On the properties and design of nonuniformly spaced linear arrays," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 372–380, Mar. 1988.

[14] Y. Hua and T. K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, no. 5, pp. 814–824, May 1990.

[15] T. K. Sarkar and O. Pereira, "Using the matrix pencil method to estimate the parameters by a sum of complex exponentials," *IEEE Antennas Propag. Mag.*, vol. 37, pp. 48–55, Feb. 1995.

[16] E. K. Miller and D. M. Goodman, "A pole-zero modeling approach to linear array synthesis I: the unconstrained solution," *Radio Sci.*, vol. 18, pp. 57–69, Jan.–Feb. 1983.

[17] G. Caille, Y. Cailloce, C. Guiraud, D. Auroux, T. Touya, and M. Masmousdi, "Large multibeam array antennas with reduced number of active chains," presented at the 2nd Eur. Conf. on Antennas and Propagation, Edinburgh, U.K., Nov. 11-16, 2007.

[18] G. Mazzarella and G. Panariello, "Array thinng in conormal array synthesis," in *Proc. IEEE AP-S. Int. Symp.*, Seattle, WA, Jun. 1994, vol. 1, pp. 505–506.

[19] J. B. Anderson, "Array gain and capacity for known random channels with multiple element arrays at both ends," *IEEE J. Sel. Areas Commun.*, vol. 18, pp. 2172–2178, Nov. 2000.

[20] I. T. Jolliffe, *Principal Component Analysis*, 2nd ed. New York: Springer-Verlag, 2002.

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