Sheet 2
Discrete Geometry

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Exercise 1. Show that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F. In particular, P has only finitely many faces.

By definition, a face F is a bounded intersection of the polytope P and a hyperplane, i.e. $F = P \cap \{x \in \mathbb{R}^n : \mathbf{a}x = a_0\}$. Therefore, F is a polytope and by the V-polytope definition it is the convex hull of a finite set of $\mathbb{R}^n$. The vertices of F are:

$$V(F) = \{G \leq F : dimG = 0\} = \{G \leq P : dimG = 0, G \leq G\} = \{G \leq P : dimG = 0\} \cap F = V(P) \cap F$$

Since the number of vertices of the polytope is finite, there are a finite number of subsets of $V(P)$. Therefore, for any face F we have $V(F) \in \mathcal{P}(V(P))$, and since $\mathcal{P}(V(P))$ is finite, so does the number of faces.

Exercise 2. Let $P \subset R^d$, $Q \subset R^e$ be two non-empty polytopes. Prove that the set of faces of the cartesian product polytope $P \times Q = \{(p, q) \in R^{d+e} : p \in P,\ q \in Q\}$ exactly equals $\{F \times G : F \text{ is face of } P,\ G \text{ is face of } Q\}$. Conclude that

$$f_k(P \times Q) \ = \sum_{i+j=k,\ i,j \geq 0} f_i(P) f_j(Q) \qquad \text{for } k \geq 0.$$

Let $S = \{F \times G : F \leq P, G \leq Q\}$, $F = P \cap \{x \in R^d : ax = a_0\}$ and $F = Q \cap \{y \in R^e : by = b_0\}$.

To prove that the cartesian product of faces is a face, i.e. $S \subset \mathcal{F}(P \times Q)$, we define the cartesian prodruct of each of the faces F and G with the polytope that are not contain in, that is:

$$\begin{aligned} F \times Q &= (P \times Q) \cap \{(x, y) \in R^{d+e} : (f, 0)(x, y) = f_0\} = \{(f, q) \in R^{d+e} : f \in F, q \in Q\} \\ P \times G &= (P \times Q) \cap \{(x, y) \in R^{d+e} : (0, g)(x, y) = g_0\} = \{(p, g) \in R^{d+e} : p \in P, g \in G\} \end{aligned}$$

Since $F \subset P$ and $G \subset Q$, the intersection $F \times Q \cap P \times G = F \times G$. Hence, S is a subset of $\mathcal{F}(P \times Q)$.

To prove the other inclusion consider the face $H \leq P \times Q$. By definition

$$H = P \times Q \cap \{(x, y) \in R^{d+e} : (a, b)(x, y) = c_0\}$$

Taking $a_0 = \max_{(x,y) \in H}\{ax\}$ and $b_0 = \max_{(x,y) \in H}\{by\}$. Both maximums happen at the same time because if not there would exists a pair $(x', y') \in R^{d+e}$ such that $ax' + by' \leq c_0$ and then the face H would not be well defined. Then $a_0 + b_0 = c_0$ and the face H can be written as follow:

$$H = P \times Q \cap \{x \in R^d : ax = a_0\} \times \{y \in R^e : by = b_0\} = F \times Q \in S$$

Now, for the last part of the exercise, notice that every $k - face$ of $P \times Q$ is the cartesian product of a $F \leq P$ and $G \leq Q$, the dimension of $F \times G$ is the sum of the dimension of each face, this is:

$$f_k(P \times Q) \ = \sum_{i+j=k,\ i,j \geq 0} f_i(P) f_j(Q) \qquad \text{for } k \geq 0.$$

Exercise 3. Show that all induced cycles of length 3, 4 and 5 in the graph of a simple d-polytope P are graphs of 2-faces of P. Conclude that the Petersen graph is not the graph of any polytope of any dimension. (*Hint for* 5*-cycles:* First show this for $d = 3$. Then prove that any 5-cycle in a simple polytope is contained in some 3-face, and use that faces of simple polytopes are simple.)

For the 3-cycle, notice that every two edges chosen they are incident and therefore the halfspace that formed intersected with the polytope is a 2-face, which is a triangle because the graph is induced.

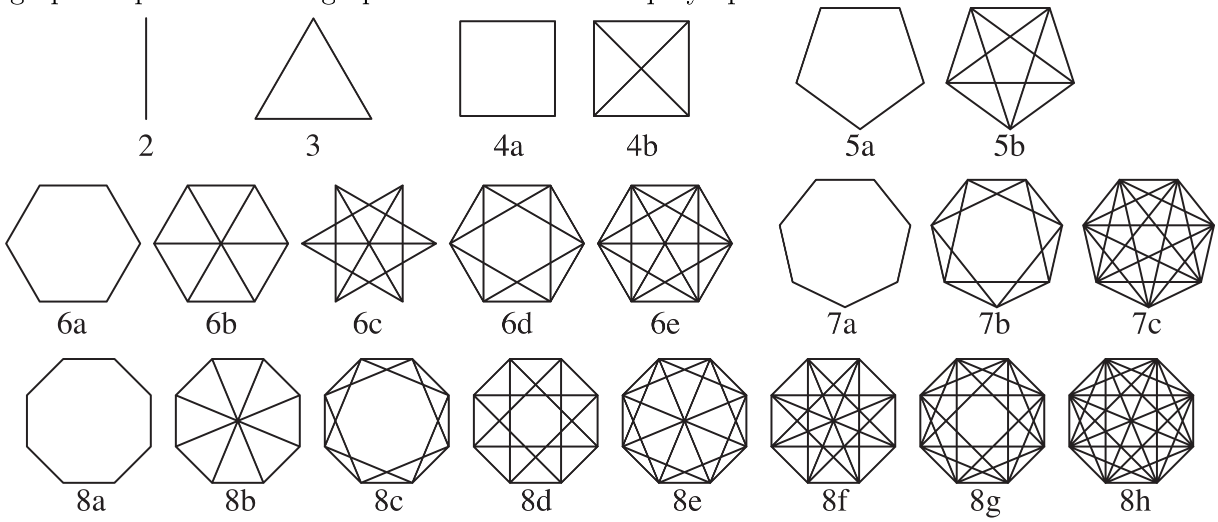
For the 4-cycle, let $\{w, x, y, z\}$ be the vertices of the induced cycle. Then the edges $\{yx, wx\}$ and $\{wz, yz\}$ induce two 2-faces like before. The intersection of this two faces is not empty because the vertices y and w are contained in it, so does the edge yw. Since the intersection of two faces is a face, both faces are the same, otherwise its intersection wouldn't be proper. Thus, an induced 4-cycle is a graph of a 2-face of P.

For 5-cycle we first show the case $d = 3$, so every vertex of the polytope has degree 3.Suppose that the induced 5-cycle doesn't induce a 2-face. Then the cycle separates the polytope in two different parts A and B, the outside and the inside part respectively. Since every vertex of the polytope must have degree two we need to connect the cycle by three edges with the outside part, and the same for the inside part. That is, we need to add six more edges to the cycle, but then, since every vertex of C_5 has degree two, by the Pigeon-hole Principle, there is a vertex with degree four, which gives us a contradiction. Therefore, the induced 5-cycle induces a 2-face. Now, we want to see that for any d every induced 5-cycle lies in a 3-face and therefore, by the first part, it induces a 2-face. Let $\{v, w, x, y, z\}$ be the vertex of the $5 - cycle$. We know that two adjacent edges form a $2 - face$ and together with another adjacent edge form a 3-face. So let F_1 be the 3-face defined by the intersection of P with the halfspace induced by the edges $\{vw, wx, xy\}$, and let F_2 be the 2-face defined by the intersection of P and the halfspace induced by the edges $\{yz, zv\}$. Now, if $z \in F_1$ the we conclude that C_5 is contained in a 3-face of P and we are done. Suppose that $z \notin F_1$. Then, since $y, v \in F_1 \cap F_2$ the edge joining this two vertices is also in the intersection and then $F_1 \cap F_2 \notin \mathcal{F}(P)$ and we have a contradiction, so $z \in F_1$.

Using this last part of the exercise, we prove that the Petersen graph is not the graph of any polytope of any dimension. Notice that Petersen graph has two 5-cicles such that they intersect in three vertices, but as showed before two 2-faces of P don't separate cycles (neither more than two), and its intersection is either an edge or the emptyset.Thus, we reach the desired result.

Exercise 4. Let $n \in \mathbb{N}$ be an integer and S denote a subset of $\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $\Gamma_n(S)$ is the graph whose vertex set is $\mathbb{Z}_n$, and whose edge set is the set of pairs of vertices whose difference lies in $S \cup (-S)$.

The following figure collects all connected circulant graphs on up to 8 vertices. Determine the *polytopality range* for as many of these graphs as you can, i.e., the set of integers d such that the graph in question is the graph of a d-dimensional polytope.



In order to solve the problem we will use the following results:

(i) For $d = 2$ every polytope must verify that $f_0 = f_1$.

(ii) For $d = 3$, by *Swteinitz's theorem* a graph is the graph of a 3-dimesional polytope if and only if it is 3-connected and planar.

(iii) For $d \geq 3$

  (iii.a) *Balinski's theorem* hold that a graph of a d-dimensional polytope has to be d-connected.

  (iii.b) The *d-PSP Property* holds that a graph of a d-dimensional polytope contains a K_{d+1}.

Now, by using these results we can immediately see that for the case **2**, since it is 1-regular with two vertices and one edge, so $1 - connected$, it is the graph of a 1-dimensional polytope. Also, the cases where the graph is a cycle, so **3**, **4a**, **5a**, **6a**, **7a** and **8a**, by *(ii)*, they are graphs of 2-dimensional polytopes. Furthermore, since the graphs **4b** and **6c** are 3-connected and planar, they are graphs of a 3-dimensional polytope. Notice also that by *(iii)*, every complete graph K_d has range $\{4, \ldots, d-1\}$. Thus, **5b** has range $\{4\}$, **6e** has range $\{4, 5\}$, **7c** has range $\{4, 5, 6\}$ and **8h** has range $\{4, 5, 6, 7\}$.

For the other cases we need to work a little more:

- **6b**: It is 3-connected and no planar because it contains $K_{3,3}$, therefore by *(ii)* and *(iii.a)* there exists no plytope with such a graph.

- **6d**: By *(i)* and *(iii.a)* it can have dimensions four or three. But, it doesn't contain a K_5, therefore by *(iii.b)* there is no 4-dimensional polytope having this graph. Now, we observe that it is planar and 3-connected (by 4-connectivity), therefore by *(ii)* there range of this graph is $\{3\}$.

- **7b**: By *(i)* and *(iii.a)* it can have dimensions four or three. But since it does not contain a K_5 by *(iii.b)* it cannot be a 4-dimensional polytope. Notice also that it contains a $K_{3,3}$, therefore it is not planar and *(ii)* there is no 3- dimensional polytope o having such a graph. Therefore, it is not graph of a polytope of any dimension.

- **8b**: Since graph is no planar, by *(ii)* it is the graph of no polytope in any dimension.

- **8c**: By *(i)* and *(iii.a)* it can have dimensions four or three. Since it does not contain a K_5 by *(iii.b)* there is no 4-dimensional polytope of this graph. In the other hand the graph is planar and 3-connected, therefore by *(ii)* it has range $\{3\}$.

- **8d**: Once again, by *(i)* and *(iii.a)* it can have dimensions four or three. But it does not contain a K_5 and by *(iii.b)* there is no 4-dimensional polytope of this graph. Moreover, the graph is not planar and by *(ii)* its range is $\{\emptyset\}$.

- **8e**: By *(i)* and *(iii.a)* it can have dimensions five, four or three. Since it does not contain a K_6 by *(iii.b)* there is no polytope of dimensions 5 having this graph. Because it contains **8b** the graph is not planar and therefore is not of a polytope of dimension three. It could have dimension four but I didn't find any example or proof taht denies this.

- **8f**: By *(i)* and *(iii.a)* it can have dimensions five, four or three. It does not contain a K_6 and neither is planar, so the only possibility is that it is the graph of a 4-dimensional polytope. But if we look for the 3-faces of the polytope we only find four different tetrahedra and then, there is not such a graph of a 4-dimensional polytope.

- **8g**: By *(i)* and *(iii.a)* it can have dimension six, five, four or three. But since it doesn't contain a K_7 and it is not planar the only possibilities are dimension five and four. For dimension four we could think, for instance, in a crosspolytope C_4^{Δ}, and for dimension five we can do the join of two 2-cubes.