Discrete and Algorithmic Geometry

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Maximal periods of (Ehrhart) quasi-polynomials

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The main goal of this paper is to study the minimum period of the coefficients of the *Ehrhart quasi-polynomial function.* For this aim, the writers construct the polytope

$$(1)\Delta \quad = \quad conv\left\{\left(\frac{1}{p_0},0,\cdots,0\right),\left(0,\frac{1}{p_1},0,\cdots,0\right),\cdots,\left(0,\cdots,0,\frac{1}{p_d}\right)\right\}$$

where $p_d|p_{d-1}|\cdots|p_0$, and define the *integer-point counting function* $\mathcal{L}_\Delta(k)$ as an Ehrhart quasi-polynomial. They prove that the function coefficients c_i of $\mathcal{L}_\Delta(k)$ have minimum period p_j.

In the other hand, the paper also studies the minimum period of the coefficient functions c_i of

$$C(k) = \sum_{l=0}^{d+e+1} A(k-l)B(l) = c_{d+e+1}(k)k^{d+e+1} + \cdots + c_0(k)$$

where $A(k) = \sum_{i=0}^{d} a_i(k)k^i$ and $B(k) = \sum_{j=0}^{e} b_j(k)k^j$ are two given *quasi-polynomials* functions whose coefficient functions a_i and b_i have minimum period α_i and β_i respectively. The convolution $C(k)$ is another quasi-polynomial function where its c_j have minimum period γ_j.

The article also provides an alternative proof of *Zaslavsky theorem*, which states that the period of γ_{i+j} divides $\mathrm{lcm}\{\alpha_{i+1},\cdots,\alpha_d,\beta_{j+1},\cdots,\beta_e,g_j\}$ where $g_j = \mathrm{lcm}\{\mathrm{gdc}(\alpha_i,\beta_{j-i}) : 0 \le i \le d, 0 \le j-e \le e\}$. Further, the authors face the problem of constructing a quasi-polynomial whose convolution satisfies Zaslavsky theorem with equality. Under the conditions $d \ge e$ and $\alpha_d|\alpha_{d-1}|\cdots|\alpha_e|\beta_e|\alpha_{e-1}|\beta_{e-1}|\cdots|\alpha_0|\beta_0$, they define the polytopes Δ_A and Δ_B under the same construction that in (1). Then, they prove that the convolution of the integer-point counting functions $\mathcal{L}_{\Delta_A}$ and $\mathcal{L}_{\Delta_B}$ satisfies Zaslasvky theorem with equality.

In addition, after proving this last result they conjecture that it is still true for the condition $\alpha_d|\alpha_{d-1}|\cdots|\alpha_0$ and $|\beta_e|\beta_{e-1}|\cdots|\beta_0$, which seems to be more natural.

Personal opinion

In my opinion, the paper is very well written. The introduction provides a schema of its general idea with some definitions that will be needed in further sections. That helps the reader to hook on and keep reading to find out how all the results are developed among the article.

Every proof is detailed described, although some of the “constructions” seem easy to be done but in practice they actually take quite some time and painkillers. For instance, the induction step of the proof of Theorem 4, or the construction of the Ehrhart function before mentioned in (1).

As a suggestion, I think it would be nice to include some pictures showing the idea behind some of the proves. For example, in the proof of Proposition 8 they could include a drawing referring to the positive and negative lattice points and the relation between them.

In general I think the paper is wonderful, with some concepts hard to understand but explained in a really intuitively way. Besides that, I like the idea of including some furthers studies that can be done to improve the given results.