Discrete and Algorithmic Geometry

Julian Pfeifle, UPC, 2013

Sheet 2

due on Monday, November 18, 2013

Reading

(1) Read Lectures 3, 4 from Ziegler's *Lectures on Polytopes*.

Writing

(1) Show that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F. In particular, P has only finitely many faces.

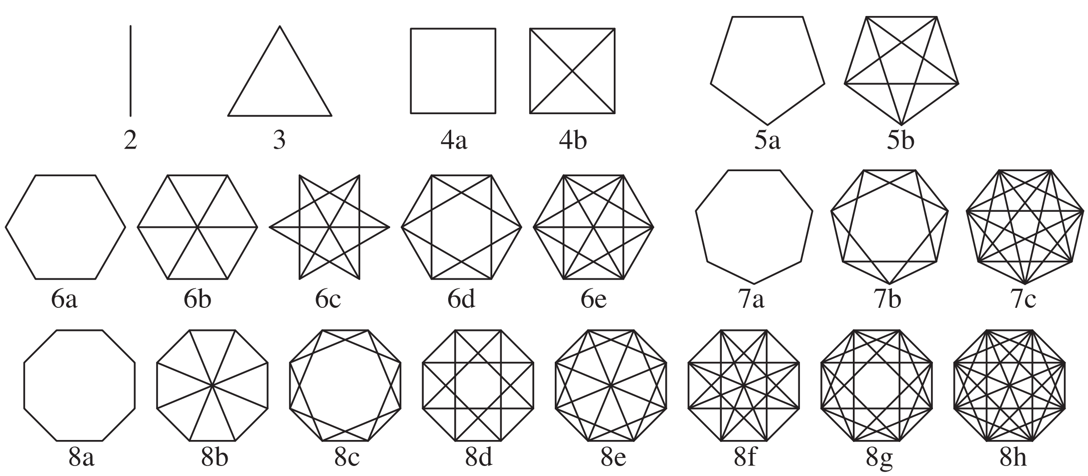
(2) Let $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ be two non-empty polytopes. Prove that the set of faces of the cartesian product polytope $P \times Q = \{(p, q) \in \mathbb{R}^{d+e} : p \in P,\ q \in Q\}$ exactly equals $\{F \times G : F \text{ is face of } P,\ G \text{ is face of } Q\}$. Conclude that

$$f_k(P \times Q) \ = \sum_{i+j=k,\ i,j \geq 0} f_i(P) f_j(Q) \qquad \text{for } k \geq 0.$$

(3) Show that all induced cycles of length 3, 4 and 5 in the graph of a simple d-polytope P are graphs of 2-faces of P. Conclude that the Petersen graph is not the graph of any polytope of any dimension. (*Hint for* 5-*cycles:* First show this for $d = 3$. Then prove that any 5-cycle in a simple polytope is contained in some 3-face, and use that faces of simple polytopes are simple.)

(4) Let $n \in \mathbb{N}$ be an integer and S denote a subset of $\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $\Gamma_n(S)$ is the graph whose vertex set is $\mathbb{Z}_n$, and whose edge set is the set of pairs of vertices whose difference lies in $S \cup (-S)$.

The following figure collects all connected circulant graphs on up to 8 vertices. Determine the *polytopality range* for as many of these graphs as you can, i.e., the set of integers d such that the graph in question is the graph of a d-dimensional polytope.



(5) Let $\square^d$ be the d-dimensional ± 1-cube. How large can the volume of a simplex in $\square^d$ become? (*Hint:* `en.wikipedia.org/wiki/Hadamard_inequality`. Write a C++ program to attain explicit bounds for $d \geq 2$ as large as you can.)