Discrete and Algorithmic Geometry

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Sheet 2

(1) **Show that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F. In particular, P has only finitely many faces.**

By the paper, F is also a polytope and $F \cap Vert(P) = Vert(F)$, now $F = conv(vert(F)) = conv(vert(P) \cap F)$, so F is the convex hull of all vertices of P contained in F.

Now, the polytope P can be seen as the convex hull of a finite number of points, suppose that $P =< v_1, v_2, ..., v_n >$, exists $v_{i_1}, ..., v_{i_m}$ such that $Vert(P) = \{v_{i_1}, ..., v_{i_m}\}$ and how every subset of $v_{i_1}, ..., v_{i_m}$ could be a face or not, the number of faces is less or equal than $2^{|Vert(P)|} = 2^d$.

(2) **Let $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ be two non-empty polytopes. Prove that the set of faces of the cartesian product polytope $P \times Q = \{(p, q) \in \mathbb{R}^{d+e} : p \in P,\ q \in Q\}$ exactly equals $\{F \times G : F$ is face of P, G is face of $Q\}$. Conclude that**

$$f_k(P \times Q) \;=\; \sum_{i+j=k,\ i,j\geq 0} f_i(P) f_j(Q) \qquad \textbf{for } k \geq 0.$$

A face F of a polytope P can be defined by an hyperplane as follows: if F is a face of P, then exist a hyperplane $H = \{x \in \mathbb{R}^d : vx = b\}$ such that $P \subseteq \{x \in \mathbb{R}^d : vx \leq b\}$ and $F = P \cap H$. In what follows, lets say that a face F of a polytope is defined by an hyperplane H_F.

We want to prove that $P \times Q = \{F \times G : F$ is face of P, G is face of $Q\}$, lets proof first that $P \times Q \supseteq \{F \times G : F$ is face of P, G is face of $Q\}$:

Take $H_P = \{x \in \mathbb{R}^d : v_P x = b_P\}$ a hyperplane which defines a face F of P and $H_Q = \{x \in \mathbb{R}^e : v_Q x = b_Q\}$ a hyperplane which defines a face G of Q. Now define $H_{(P,Q)} = \{x \in \mathbb{R}^{d+e} : (v_P, v_Q)x = b_P + b_Q\}$, then the inequality $(v_P, v_Q)x \leq b_P + b_Q$ holds for $P \times Q$ and $H \cap (P \times Q) = F \times G$.

Now, lets proof that $P \times Q \subseteq \{F \times G : F$ is face of P, G is face of $Q\}$, suppose K is a face of $P \times Q$ defined by a hyperplane $H = \{x \in \mathbb{R}^{d+e} : vx = b\}$ where $v \in \mathbb{R}^{d+e} \setminus \{0\}$ and $b \in \mathbb{R} \setminus \{0\}$, $K = H \cap (P \times Q)$ and $P \times Q \subseteq \{x \in \mathbb{R}^{d+e} : vx \leq b\}$. Now, take $v = (v_P, v_Q)$ where, $v_P \in \mathbb{R}^d \setminus \{0\}$ and $v_Q \in \mathbb{R}^e \setminus \{0\}$,m now we are going to build to faces of P and Q respectively such that K be the direct product of these two faces. To do that, let's take $b_P \in \mathbb{R} \setminus \{0\}$ such that $P \subseteq \{x \in \mathbb{R}^d : v_P x \leq b_P\}$ and take $F = P \cap \{x \in \mathbb{R}^d : v_P x = b_P\}$, $G = Q \cap \{x \in \mathbb{R}^e : v_Q x = b - b_P\}$, F is a face of P by it's definition, let's see that G is a face of Q. We have that, $(v_P, v_Q)x \leq b$ for $x \in \mathbb{R}^{d+e}$ and $v_P x \leq b_P$ for $x \in \mathbb{R}^d$, so, we take $x = x_d + x_e$ where, $x_d \in \mathbb{R}^d$ and $x_e \in \mathbb{R}^d$, so, $v_P x_d + v_Q x_e \leq b = b_P + b - b_P$, how $v_P x \leq b_P$ for $x \in \mathbb{R}^d$ that implies $v_Q x_e \leq b - b_P$ so G is a face of Q and $K = F \times G$.

Finally, we have seen that every face of dimension k of $P \times Q$ is the cartesian product of a face of P and a face of Q and k is the sum of the dimension of these two faces so the formula it's true.

(3) **Show that all induced cycles of length 3, 4 and 5 in the graph of a simple d-polytope P are graphs of 2-faces of P. Conclude that the Petersen graph is not the graph of any polytope of any dimension. (*Hint for* 5*-cycles:* First show this for $d = 3$. Then prove that any 5-cycle in a simple polytope is contained in some 3-face, and use that faces of simple polytopes are simple.)**

(4) **Let $n \in \mathbb{N}$ be an integer and S denote a subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $\Gamma_n(S)$ is the graph whose vertex set is $\mathbb{Z}_n$, and whose edge set is the set of pairs of vertices whose difference lies in $S \cup (-S)$.**

The following figure collects all connected circulant graphs on up to 8 vertices. Determine the *polytopality range* for as many of these graphs as you can, i.e., the set of integers d such that the graph in question is the graph of a d-dimensional polytope.

(5) **Let $\square^d$ be the d-dimensional ± 1-cube. How large can the volume of a simplex in $\square^d$ become? (*Hint:* `en.wikipedia.org/wiki/Hadamard_inequality`. Write a C++ program to attain explicit bounds for $d \geq 2$ as large as you can.)**