Raman Sanyal's "Topolofical obstructions for vertex numbers of Minkowski sums" resume

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Basing on the paper by Raman Sanyal "Topolofical obstructions for vertex numbers of Minkowski sums", I construct a resume of the main goal of the paper. I don't include proofs and just make an easy collection of what he wants to show. The paper wants to show that for polytopes $P_1, P_2, ..., P_r \subset \mathbb{R}^d$, each having $n_i \geq d+1$ vertices, the Minkowski sum $P_1 + P_2 + ... + P_r$ cannot achieve the maximum of $\prod_i n_i$ vertices if $r \geq d$.

1. Introduction

For two polytopes $P, Q \subset \mathbb{R}^d$, their Minkouski sum is the convex polytope:

$$P + Q = \{p + q : p \in P, q \in Q\} \subset \mathbb{R}^d$$

Minkowski Sums are used in several fields of Maths, so it's important to know about it's behaviour, and this can be deduced by knowing the facial structure of $P+Q$. But this is not trivial (even knowing P and Q). So his paper tries to explain the combinatorial structure of Minkowski sums. It begins with three theorems related to bounds for f-vector shapes.

Theorem 1.1. *Let $P_1, P_2, ..., P_r \subset \mathbb{R}^d$ be polytopes. Then:*

$$f_0(P_1 + ... + P_r) \leq \prod_{i=1}^{r} f_0(P_i)$$

Theorem 1.2 (Fukuda and Weibel)**.** *For every $r < d$ there are d-polytopes $P_1, ..., P_r \subset \mathbb{R}^d$, each with arbitrary large number of vertices, such that the Minkowski sum $P_1 + ... + P_R$ attains the trivial upper bod on the number of vertices.*

Theorem 1.3. *Let $r \geq d$ and let $P_1, ..., P_r \subset \mathbb{R}^d$ be polytopes with $F_0(P_i) \geq d+1$ vertices for all $i = 1, 2, ..., r$. Then:*

$$f_0(P_! + ... + P_r) \leq \left(1 - \frac{1}{(d+1)^r}\right) \prod_{i=1}^{r} f_0(P_i)$$

Then, the last part of the section is a explanation of every section of the paper.

2. The problem, some reductions and a reformulation

In this section, the author shows some observations that reduces the statement to one special case (per dimension), particularly he gives a reformulation of the problem that casts it into a stronger question concerning projections of polytopes.

For a sum $P+Q$ of two polytopes, it is easy to see that if $F \subset P+Q$ is a proper face, then F is of the form $F = G+H$ with $G \subset P$ and $H \subset Q$ being faces. this yields that the set of vertices of a Minkowski sum is a subset of the pairwise sums of vertices of the polytopes involved.

Proposition 1. *Let P and Q be two polygons in the plane. Then:*

$$f_0(P+Q) \leq f_0(P) + f_0 = (Q)$$

This elementary geometrical reasoning fails in higher dimensions and the author employs topological machinery for the general case, so he gives some observations that will simplify the general case:

Observation (Dimension of summands)**.** *Let P_1, ...,$P_r \subset \mathbb{R}^d$ be polytopes each having at least $d+1$ vertices. Then there are full.dimensional polytopes $P'_1, ..., P'_r$ with $F_0(P'_i) = F_0(P_i)$ and $F_0(P'_1 + ... + P'_r) \geq f_0(P_1 + ... + P_r)$*

Observation (Number of summands)**.** *P_1, ...,$P_r \subset \mathbb{R}^d$ be d-polytopes such that $P_1 + ... + P_r$ attains the trivial upper bound, then so does every subsum $P_{i_1} + ... + P_{i_k}$ with $\{i_1, ..., i_k\} \subseteq [r]$.*

Observation (Combinatorial type of summands)**.** *Let P_1, ...,$P_r \subset \mathbb{R}^d$ be d-polytopes such that $P_1 + ... + P_r$ attains the trivial upper bound. For every $i \in [r]$ let $P'_i \subset P_i$ be a vertex induced, full-dimensional subpolytope, then $P'_1 + ... + P'_r$ attains the trivial upper bound.*

Observation. *The Minkowski sum $P+Q$ is the projection of the product $P \times Q$ under the map $\pi : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ with $\pi(x,y) = x+y$.*

Theorem 2.1. *Let P be a polytope combinatorially equivalent to a d-fold product of d-simplices and let $\pi : P \longrightarrow \mathbb{R}^d$ be a linear projection. Then:*

$$f_0(\pi P) \leq f_0(P) - 1 = (d+1)^d - 1$$

This theorem touches upon propierties of the realization space of products of simplices. While, in general, realization spaces are rather delicate objects, the statement at hand is on par with the fact that positively spanning vector configurations with prescribed sign patterns of linear dependencies don't exist.

3. Geometric and Combinatorial Properties of Projections

The paper introduces first of all a few properties of faces under projections that are nice and important.

Definition 3.1. *Let P be a polytope, $F \subseteq P$ a proper face and $\pi : P \longrightarrow \pi(P)$ a linear projection of polytopes. The face F is strictly preserved under π if:*

1. *$H = \pi(F)$ is a face of $\pi(P)$*

2. F and H are combinatorially isomorphic, and

3. $\pi^{-1}(H)$ is equal to F.

what makes strictly preserved faces so nice is the fact that the above confitions can be checked prior to the projection by purely linear algebraic means, he gives a variant of it which requires some new notions:

Definition 3.2. *Let $P \subset \mathbb{R}^n$ be a full-dimensional polytope with 0 in the interior. The dual polytope is:*

$$P^{\Delta} = \{l \in (\mathbb{R}^n)^* : l(x) \leq 1 \forall x \in P\} = conv\{l_1, ..., l_m\} \subset (\mathbb{R}^n)^*$$

and for every face $F \subseteq P$ we denote by $F^0 = \{l \in P^{\Delta} : l|_F = 1\}$ the corresponding face of P^{Δ}. Furthermore, we define $I : \{faces\ of\ P\} \longrightarrow 2^{[m]}$ to be the map satisfying:

$$conv\{l_i : i \in I(F)\} = F^0$$

for every face $F \subseteq P$.

Let $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a linear projection and let q be a map fitting into the short exact sequence:

$$0 \longrightarrow \mathbb{R}^{n-d} \longrightarrow^q \mathbb{R}^n \longrightarrow^{\pi} \mathbb{R}^d \longrightarrow 0$$

Dualizing gives rise to a dual exact sequence:

$$0 \longleftarrow (\mathbb{R}^{n-d})^* \longleftarrow^{q^*} (\mathbb{R}^n)^* \longleftarrow^{\pi^*} (\mathbb{R}^d)^* \longleftarrow 0$$

The characterization of strictly preserved faces will be in terms of the dual map q^* and the dual to the face under consideration.

Lema 1 (Projection Lemma). *Let P be a polytope and $F \subset P$ a face. Then F is strictly preserved iff $0 \in$ int $q^*(F^0)$ =int conv$\{q^*(l_i) : i \in I(F)\}$*

To prove this, the paper proves first that $\pi(F)$ is a face of $\pi(P)$ if and only if $o \in q^*(F^0)$, and so, $\pi(F)$ is combinatorially equivalent to F if and only if $q^*(F^0)$ is full-dimensional.

3.1. Geometric side

Here, the author wants to expose when the convex hull of some vertices is a face, that's is knowing by the Gale duality theorem.

Theorem 3.1 (Gale duality). *Let $G = \{g_1, ..., g_m\}$ be a Gale transform in $(m-d-1)$ dimensional space, then there is a d-polytope Q with vertices $V = \{v_1, ..., v_m\}$ such that for every $I \subset [m]$:*

conv $\{v_i : i \in [m] - I\}$ is a face of Q if and only if $\{g_j : j \in I\}$ are positively dependent. Furthermore, Q is up to affine isomorphisms.

A Gale transform in dimension $m - d + 1$ is a finite vector configuration of m elements such that the subconfiguration $G - g_i$ is positively spanning (ie the cone of these vectors is equal to $\mathbb{R}^d$. This transformations are well-known notion from discrete geometry, and the autor facilities some references to know more about this transformations.

In this case, the importance of Gale transform is that there is a combinatorially unique polytope $A(P, \pi)$ associated to (P, π) which we call the associated polytope and the main property of $A(P, \pi)$ is that it sort of witnesses the survival of the vertices.

We have the following consequence of the theorem:

Collorary 1. *Let $P \subset \mathbb{R}^{d^2}$ be a polytope combinatorially equivalent to a d-fold product of d-simplices such that a projection to d-space preserves all the vertices. Then there is a $(2d-1)$-dimensional, simplicial polytope $A(P, \pi)$ with $d(d+1)$ vertices associated to the projection.*

3.2. Combinatorial side

Here the author deduces that the polytopal complex is a simplicial complex whose combinatorics is determined by the sole knowledge of the combinatorics of P and he give a rather general description of this complex.

He begins by defining the complement complex of a simplicial complex K^c, that is the topological clousure of the set formed by all the vertices without every set of those that form a facet. And then, for P^Δ is a simplicial polytope he denotes by $B(P^\Delta)$ the simplicial complex of all proper faces of P^Δ.

Theorem 3.2. *Let P be a simple polytope whose vertices are preserved under π. Then the complex $B(P^\Delta)^c$ is realized in the boundary of the associated polytope $A(P, \pi)$*

4. Interlude: Embeddability of simplicial complexes

In this section, the author wants to show to the reader that he did not deal one difficult problem for another one. All the things have been token from Matousek's book, in which he presents means for dealing with embeddability questions in a combinatorial fashion.

Let's begin by trying to explain what is the main theorem of this section.

Theorem 4.1. *Let K be a simplicial complex. If:*

$$ind_{\mathbb{Z}_2}(K^*_\Delta)^2 > d$$

then K is not embeddable into the d-sphere.

First of all let's try to explain every new concept here.

- The category of free $\mathbb{Z}_2$ spaces consists of topological spaces X together with a free action of the group $\mathbb{Z}_2$, ie, a fixed point free involution on X. Over these spaces the $\mathbb{Z}_2$ index is a numerical invariant $ind_{\mathbb{Z}_2}$ which is the smallest integer d such that there is a $\mathbb{Z}_2$-equivariant map $X \longrightarrow_{\mathbb{Z}_2} \mathbb{S}^d$. For example $ind_{\mathbb{Z}_2}\mathbb{S}^d = d$, which is equivalent to the Borsuk-Ulam theorem.
- For a simplicial complex K we define the deleted join of K to be the complex

$$(K^*_\Delta)^2 = \{\sigma \uplus \tau : \sigma, \tau \in K, \sigma \cap \tau = \emptyset\}$$

The $\mathbb{Z}_2$ index is rather difficult to calculate, but there is a beutiful theorem due to Karanbir Sarkaria:

Theorem 4.2 (Sarkaria's coloring/embedding theorem)**.** *Let K be a simplicial complex with n vertices and let $F = F(K)$ the set of minimal non-faces. Then:*

$$ind_{\mathbb{Z}_2}(K^*_\Delta)^2 \geq n - \chi(KG(F)) - 1$$

Where $KG(F)$ is the generalized kneser grapf of F. This theorem can be used to prove that $K_\{3, 3\}$ is not planar.

5. Analysis of the complement complex

Determining upper bounds on the chromatic number of graphs is easier than finding equivariant maps. The idea is that the complexes are made up of (possibly) simpler ones, that is they are joins of complexes.

He applies these results to the complex $K = L^{*d}$ with $L = \binom{[d+1]}{\leq 1}$. From the definition of L he sees that the minimal non-faces are exactly the two element subsets of $[d+1]$, that is $\mathbf{F} = \binom{[d+1]}{2}$. The resulting Kneser graph $KG(\mathbf{F})$ is an instance of a famous family of graphs, the ordinary Kneser graphs $KG_{n,k} = \binom{KG[n]}{k}$. The determination of their chromatic numbers is one of the first success stories of topological combinatorics:

Theorem 5.1 (Lovasz-Kneser theorem)**.** *For $0 < 2k- \leq n$ the chromatic number of the Kneser graph $KG_{n,k}$ is $\chi(KG_{n,k}) = n - 2k + 2$.*

And now, by using this theorem, the author proves the theorem 2.1.

6. Remarks

Are there two triangles P and Q in the plane whose sum is a 9-gon? The polytope $P+Q$ is a 9-gon if its normal fan $N(P+Q)$ has nine extremal rays. The normal fan $N(P+Q)$ equals $N(P) \wedge N(Q)$, the common refinement of the fans $N(P)$ and $N(Q)$. Thus, the cones in $N(P+Q)$ are pairwise intersections of cones of $N(P)$ and $N(Q)$, it follows that the extremal rays, i.e. the 1-dimensional cones, of $N(P+Q)$ are just the extremal rays of P and of Q. If P and Q are triangles, then each one has only three extremal rays, therefore $N(P+Q)$ has at most six extremal rays and falls short of being a 9-gon.

Finally, he shows a polytope $P \subset \mathbb{R}^4$ given by a matrix inequality with a parameter ϵ such that when $\epsilon = 0$ this is just a Cartesian product of two triangles. So since is a simple polytope, is possible to choose an $\epsilon > 0$ sufficiently small without changing the combinatorial type.

Then, by what he exposes in section 4, the set $G = q^*(verP^{\Delta})$ is the Gale transform of a polytope A combinatorially equivalent to an octahedron when $0 < \epsilon \leq 1$.