Discrete and Algorithmic Geometry 2013(Part 2)

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This is the preliminary version of the lecture notes for the second part of *Discrete and Algorithmic Geometry* (Universitat Politècnica de Catalunya), held in the fall semester of 2013 by Ferran Hurtado and Julian Pfeifle.

These notes are fruit of the collaborative effort of all participating students, who have taken turns in assembling this text. The name of each scribe figures in each corresponding section.

Suggestions for improvements will always be gladly received by `julian.pfeifle@upc.edu`.

Contents

Lecture 1. Convex Polytopes 5
1. Faces 5

Lecture 2. Asymptotic f-vectors of families of polytopes 9
1. The Unimodality Conjecture 9
2. Operations on polytopes 11

Lecture 3. Cyclic polytopes 13
1. Notions of equivalence between polytopes $P, Q \subset \mathbb{R}^d$. 13
2. How many k-dimensional faces does $C_d(n)$ have? 13

Polytopality of a graph 15

Lecture 4. Cyclic Polytopes 17

Lecture 5. Kalai's simple way of telling a simple polytope from its graph 19

Lecture 6. Lattice points in multiples of politopes 21
1. Generating functions for rational cones 21

Lecture 7. Lattice Geometry 23

Lecture 8. Ehrhart-Macdonald Reciprocity 25
1. Statement of the theorem 25
2. Auxiliary results 25
3. Proof of Ehrhart-Macdonald Reciprocity 26
4. Degree of a lattice polytope 26
5. Reflexive polytopes 27

Lecture 9. Systems of sparse polynomial equations 29
1. Wronski polynomials 29

Scribes 2013 31
1. Alex Alvarez 31
2. Cecilia Girón Albert 31
3. Anna Somoza 31
4. Daniel Torres 31
5. Borja Elizalde 32
6. Xavier Tapia 32

Bibliography 33

Papers to referee 35
Bibliography 35

LECTURE 1

Convex Polytopes

Scribe: Cecilia Girón

A convex polytope can be defined in two different ways:

- V*-polytope* (discrete geometry): is the convex hull of the finite non-empty point set in $\mathbf{R}^d$.
- H*-polytope* (linear/integer optimization): An H-polyhedron is an intersection of a finite number of linear half spaces in some $\mathbf{R}^d$, if non-empty. And an H-polytope is a bounded H-polyhedron.

1. Faces

One of the properties studied about polytopes is their faces. A **face** F of a polytope P is a set of the form:

$$F = \{x \in \mathbb{R}^d :< a, x >= n\} \cap P$$

where $a \in (\mathbb{R}^d)^*$ (dual space), $b \in \mathbb{R}$ and $P \subseteq \{x \in \mathbb{R}^d :< a, x > \leq b\} \longleftrightarrow$ The inequality $< a, x > \leq b$ is valid for P. Notice that P is actually a face of itself.

We can also study the dimension $\dim F$ of a face. Let P be a d dimensional polytope, then if a face F of P has dimension:

i) $d - 1$, it is called a **facet**.
ii) $d - 2$, it is called a **ridge**.
iii) 1, it is called an **edge**.
iv) 0, it is called a **vertex**.
v) -1 then $F = \emptyset$.

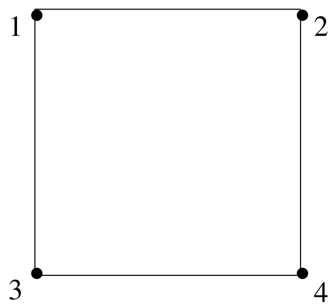
The partially ordered set of all faces $\mathcal{F}(P)$ of a convex polytope P forms a Eulerian lattice called **face lattice**. The face lattice can be used, for instance, to count the number of faces of same dimension:

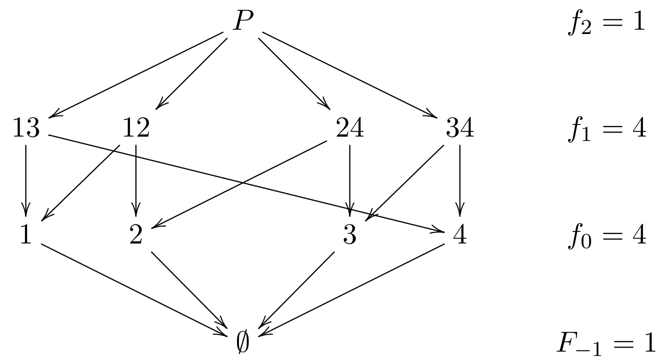
$$f_i = \#\{\text{ faces } F \text{ of } P \text{ with } \dim F = i\}$$

EXAMPLE. Let P be a cube in two dimensions, so a square. Notice that $\dim P = 2$, for every edge $ij \in P$ $\dim(ij) = 1$ and for every vertex i $\dim i = 0$, for $i, j = 1, 2, 3, 4$ and $i \neq j$. With this information we can construct the face lattice and study some properties of P.

♣

EXAMPLE.Let's study now the dimension of the faces of a hypercube of dimension d $\square^d = \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1, i = 1, \cdots, d\}$: $f_{-1}(\square^d) = 1, f_0(\square^d) = 2^d, f_{d-1}(\square^d) = 2d$ and $f_d(\square^d) = 1$. Notice that the radius r from the center of the cube to one of its vertices is $r = \sqrt{d} - 1$, thus the exterior circus of the polytope has radio r and the interior radio 1.





d	2	3	4	5	$\cdots$	100	$\cdots$	10^{100}
$r = \sqrt{d} - 1$	$\sqrt{2} - 1$	$\sqrt{3} - 1$	1	$\sqrt{5} - 1$	$\cdots$	9	$\cdots$	$10^{50} - 1$
f_0	4	8	16	32	$\cdots$	2^{100}	$\cdots$	$2^{10^{100}}$
f_1	4	6	8	10	$\cdots$	200	$\cdots$	$2 \cdot 10^{100}$

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An other property that can be studied about the faces of convex polytopes is whether they are a simplex or not. Let P be a polytope such that $\dim P = d$ and $\mathcal{F}(p) = k + 1$. It is said to be **simplicial** if it is k-simplex, i.e. if each of its faces is a simplex; and it is called **simple** if each of its vertices is contained in exactly d faces where $\dim P = d$.

1.1. Exercises done during the lecture 8/11/2013. Each one includes one.

Alex Alvarez.: The set of vertices is the following:

```
(1 : -1 : -1 : -1 : 0 : 0)
(1 : 1 : -1 : -1 : 0 : 0)
(1 : -1 : 1 : -1 : 0 : 0)
(1 : 1 : 1 : -1 : 0 : 0)
(1 : 0 : 0 : 1 : 0 : 0)
(1 : 0 : 0 : 1 : 1 : 0)
(1 : 0 : 0 : 1 : 0 : 1)
```

And this can be obtained as the join of a 2-cube with a 2-simplex. If we construct that in Polymake:

```
polytope > $p = join_polytopes(cube(2), simplex(2));

polytope > print($p->VERTICES);
1 -1 -1 -1 0 0
1 1 -1 -1 0 0
1 -1 1 -1 0 0
1 1 1 -1 0 0
1 0 0 1 0 0
1 0 0 1 1 0
1 0 0 1 0 1
```

Thus, we can use the program to see the number of facets and the vertices in each facet:

```
polytope > print $p->N_FACETS;
polymake: used package cddlib
  Implementation of the double description method of Motzkin et al.
  Copyright by Komei Fukuda.
  http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html

polymake: used package lrslib
  Implementation of the reverse search algorithm of Avis and Fukuda.
  Copyright by David Avis.
  http://cgm.cs.mcgill.ca/~avis/lrs.html

7
polytope > print $p->POINTS_IN_FACETS;
{0 2 4 5 6}
{0 1 4 5 6}
{1 3 4 5 6}
{2 3 4 5 6}
{0 1 2 3 5 6}
{0 1 2 3 4 6}
{0 1 2 3 4 5}
```

If we focus now in the graph of the polytope, we can check the number of edges and we can also see it:

```
polytope > print $p->GRAPH->N_NODES;
7
polytope > print $p->GRAPH->N_EDGES;
19
polytope > print $p->GRAPH->VISUAL;
```

Therefore, we can see that the last two vertices are not connected, but they are connected to all the other vertices.

Exercise ?? (team members):

LECTURE 2

Asymptotic f-vectors of families of polytopes

Scribe: Cecilia Girón

1. The Unimodality Conjecture

In this section we are going to study the *unimodality conjecture* which says that there exists an $l = P(L) \in \mathbb{N}$ such that $f_0 \leq f_1 \leq \cdots \leq f_l \leq f_{l+1} \leq \cdots \leq f_{d-1} \leq f_d$. We woould like to know if it is true.

First, we define the f**-vector** as the vector of the form $(f_0, f_1, \cdots, f_{d-1})$ where f_i is as defined before in (1). We will say that it is a **flag** f**-vector** $(f_s)_s = [d]$ such that f_s count the number of flags $F_{i_1} \subset F_{i_2} \subset \cdots \subset F_{i_k}$ where $s = \{i_1, i_2, \cdots, i_k\}$ and $\dim F_{i_k} = i_k$ [1].

The unimodal conjecture described before is known to be false for simplical polytopes of dimension $d \geq 19$ and for non-simplicial polytopes of dimension $d \geq 8$. The following conjecture is not known to be false.

Restricted unimodal conjecture (Anders Bjorner):

$$f_0 \leq f_1 \leq \cdots \leq f_{\lfloor \frac{d-1}{4} \rfloor}$$
$$f_{\lfloor \frac{3(d-1)}{4} \rfloor} \geq \cdots \geq f_{d-1}$$

Intuitively we are sure that there is no way this conjecture could be false, but there is not proof of this. We don't even know if $f_k \geq \frac{1}{10000} \min\{f_0, f_{d-1}\}$ is true.

1.1. Exercises worked on during the lecture 11/11/2013. Each team includes one.

Exercise 1b (Alex Alvarez and Ivan Geffner): *Using the simple form $n! \approx \left(\frac{n}{e}\right)^n$ of Stirling's formula, show that $\phi_d(x) := \log \binom{d}{xd}$ is asymptotically proportional to $-x \log x - (1 - x)\log(1-x)$, for $x \in (0,1)$ and $d \to \infty$. Discuss the real function ϕ_d on $[0,1]$.*

Using the simple form of Stirling's formula, we obtain the following form:

$$\binom{d}{xd} \approx \frac{d^d}{xd^{xd}(d-xd)^{d-xd}}$$

So applying the logarithm, we get

$$\begin{aligned} \phi_d(x) &= d \log d - xd \log xd - (d - xd)\log(d - xd) \\ &= d \log d - xd \log x - xd \log d - (d - xd)(\log d + \log(1-x)) \\ &= d(-x \log x - (1-x)\log(1-x)) \end{aligned}$$

[1] You can also read about cd-index

Thus, the first part of the exercise is proven. Let us study now the shape of the function.

The function clearly vanishes when x tends to 0 and 1 and in this interval is non-negative. The first derivative of the function is

$$\frac{d}{dx}(-(1-x)\log(1-x) - x\log(x)) = \log(1-x) - \log(x)$$

and therefore, there is only one point in which derivative vanishes, that is $x = \frac{1}{2}$. Now, the second derivative is

$$\frac{d^2}{dx^2}\log(1-x) - \log(x) = \frac{1}{x(x-1)}$$

We can conclude then that the point is a maximum, so we have characterized the shape of the function.

Exercise 2b (Borja and Cecilia): *Using the simple form $n! \approx \left(\frac{n}{e}\right)^n$ of Stirling's formula, show that $\psi_d(x) := d(1-x) + \log\binom{d}{xd}$ is asymptotically proportional to $1 - x - x\log x - (1-x)\log(1-x)$, where $\log = \log_2$ denotes the binary logarithm. Find an approximation to the maximum of this function on $(0,1)$.*

For the first part of the exercise, by applying the Stirling's formula, in the binomial for of the given function:

$$\binom{d}{xd} = \left(\frac{d}{xd^x(d(1-x))^{1-x}}\right)^d$$

Then, substituting in $\psi_d(x)$:

$$\begin{aligned}
\psi_d(x) &= d(1-x) + \log\left(\frac{d}{xd^x(d(1-x))^{1-x}}\right)^d \\
&= d(1-x) + d\left(\log d - x\log x - x\log d - (1-x)\log d - (1-x)\log(1-x)\right) \\
&= d(1 - x - x\log x - (1-x)\log(1-x))
\end{aligned}$$

Hence, $\psi_d(x)$ is asymptotically proportional to $1 - x - x\log x - (1-x)\log(1-x)$

For the second part of the exercise, in order to find the maximum of the function

$$f(x) = 1 - x - x\log(x) - (1-x)\log(1-x)$$

in (0,1) we will the points that have first derivative equal to 0, that is x such that $f'(x) = 0$, and computing $f'(x)$ we get:

$$f'(x) = -1 - (\log x + 1) - (-\log(1-x) - 1) = \log\left(\frac{1-x}{x}\right) - 1$$

Now the points that make $f'(x) = 0$ are the ones that make $\log(\frac{1-x}{x}) = 1$, which is the same as x such that $\frac{1-x}{x} = e$, which translates into:

$$x_{\max} = \frac{1}{e+1}$$

The shape of this function is a growing function from $x = 0$ starting at $f(0) = 1$ to $x = x_{\max}$, where it gets it's maximum, that is approximated by $f(x_max) \approx 1.0414$ and then decreases to 0 at $x = 1$.

2. Operations on polytopes

- **Cartesian (direct) product** $P \times Q$.
- **Direct sum** $P^d \oplus q^e \subset \mathbb{R}^{d+e}$.
- $P * Q \subset \mathbb{R}^{d+e+1}$. It is like $\oplus$ but the subspaces are skew (i.e. affine and they have no point on common). For example $\square^1 * \square^1 = Pyr(P)$.

EXAMPLE: Given $f_k(P)$, calculate the k-th entry of $Pyr(P)$:

$$\begin{aligned} f(P) &= (f_0, f_1, \cdots, f_{d-1} \\ f_k(Pyr(P)) &= (f_0 + 1, f_1 + f_0, f_2 + f_1, \cdots, f_{d-2} + f_{d-3}, f_{d-1} + f_{d-2}, 1 + f_{d-1}) \end{aligned}$$

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- **Connected sum** $P^d \# Q^d$ where P has as simplicial face f and Q has a simplicial face G.

This last operation is used to join the asymptotic function $f(\square^d)$ and its dual $f(\diamondsuit^d)$. To make it work, since $\square^d$ has no triangulations in its faces, it is enough to cut away one vertex and, this way, get a simplex. Merging both functions using the connected sum gives place to a new function which is a non-unimodal function.

LECTURE 3

Cyclic polytopes

Scribe: Anna Somoza

Remark 3.1. We say that a real algebraic curve $t \mapsto (\varphi_1(t), \dots, \varphi_d(t)) \in \mathbb{R}^d$ has degree d if no $d+1$ points on C lie on a hyperplane.

Theorem 3.2. *The convex hull of $n \geq d+1$ points on an algebraic curve of degree* d *is called the* cyclic polytope $C_d(n)$*, which is a simplicial polytope.*

Note that we are talking about *the* cyclic polytope, denoting unicity. This is in terms of equivalence between polytopes, a topic that we will now go through.

1. Notions of equivalence between polytopes $P, Q \subset \mathbb{R}^d$.

When we talk about P and Q being equivalent polytopes we usually refer to the existence of some kind of application $T : \mathbb{R}^d \to \mathbb{R}^d$ that takes P to Q. In this sense, there are different kinds of equivalence:

- Congruent: $\exists T \in \mathrm{SO}_d(\mathbb{R}), t \in \mathbb{R}^d$ such that $TP + t = Q$;
- Linearly equivalent or isomorphic: is the case when $T \in \mathrm{GL}_d(\mathbb{R})$ and $TP = Q$.
- Affine equivalent or isomorphic: for $T \in \mathrm{GL}_d(\mathbb{R}), t \in \mathbb{R}^d$ and $TP + t = Q$.
- Projectively equivalent or isomorphic: if there exists an admissible projective transformation T that sends P to Q. We need that $T^{-1}(H_\infty) \cap P = \emptyset$[1].
- Combinatorially equivalent or isomorphic: this equivalence refers to its structure, $\mathcal{F}(P) \cong \mathcal{F}(Q)$, so that the face lattice are equivalent as graded posets.

In the case of cyclic polytopes we are in the situation that the convex hull of $d+1$ points are always combinatorially equivalent.

Then we can reformulate the theorem as follows:

Theorem 3.3. *The convex hull of $n \geq d+1$ points on an algebraic curve of degree* d *is called the* cyclic polytope $C_d(n)$ *(up to combinatorial equivalence). The cyclic polytope is a simplicial polytope.*

2. How many k-dimensional faces does $C_d(n)$ have?

Every k-face of a cyclic polytope can be defined as the intersection of $d-k$ facets. We have to take into account that every facet is defined by a set of d points with the consideration that they have to come in $\lfloor \frac{d}{2} \rfloor$ adjacent pairs so that the hyperplane that they define keep the remaining $n-d$ points at the same side of it.

[1]We need this condition because some projective spaces are non orientable, so what we want to keep infinity away from our polytope

Then, we claim that $f_k(C_d(n)) = \binom{n}{k}$ if $k \leq \frac{d}{2}$, because once we fix k points we can complete the set of points with adjacent pairs to find a facet containing all, and by this procedure we can find $d-k$ facets so that our k face is defined by the intersection of the found facets.

For the $k > \frac{d}{2}$ see Lecture 4.

2.1. How many n-gons exist in $\mathbb{R}^2$? Consider the 4-gon consisting on the square of side 1, $\square^2$.

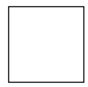
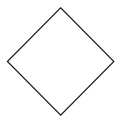


FIGURE 1. Square of side 1.

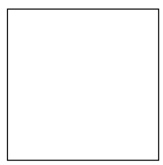
Then,

It is congruent to



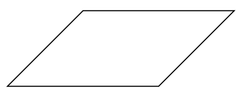
because it includes rotations,

It is affinely equivalent to



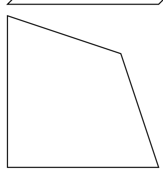
because it includes homotecies.

It is linearly equivalent to



because it accepts transformations in the basis.

It is projectively equivalent to



because to define a projection you must fix $d+2$ points.

Definition 3.4. The *realization space* of a convex polytope $P \subset \mathbb{R}^d$ with n vertices is

$$R(P) = \{M \in \mathcal{M}_{n\times d}(\mathbb{R}) \mid P \text{ is projectively equivalent to } \operatorname{conv}\{\operatorname{col} A\}\}.$$

FIGURE 2. The realization space $R(5-\text{gon})$ so that it is convex.

Polytopality of a graph

The *graph* of a polytope is its 1-skeleton:

$$\mathrm{sk}^k(P) = \{F \le P \mid \dim(F) \le k\},$$

$$G(P) = \mathrm{sk}^1(P) = \{\text{vertices and edges in } P\}.$$

We may wonder, given a graph, if it can be the graph for any polytope. That is what we call the polytopality range:

Definition 3.5 (Polytopality range)**.** Let G be a graph. Then, its polytopality range is

$$\mathrm{PR}(G) = \{d \in \mathbb{N} \mid \exists P^d \text{ with } \mathrm{sk}^1(P) = G\}$$

To decide for a certain graph whether it is the graph for a d-polytope or not, we can use the following criteria among others:

Theorem 3.6 (Steinitz)**.** *G is the graph of a 3-dimensional polytope if and only if it is simple, planar and 3-connected.*

Theorem 3.7 (Balinski)**.** *The graph $G(P)$ is d-connected for every d-polytope P.*

Definition 3.8. A *principal subdivision* of a graph G is a subdivision of the edges of G into paths, such that there exist a so-called principal vertex with the property that edges incident to it are not subdivided.

Proposition 3.9 (d-PSP property)**.** *Let P be a d-polytope. Then, every vertex in $G(P)$ is the principal vertex of a principal subdivision K_{d+1}.*

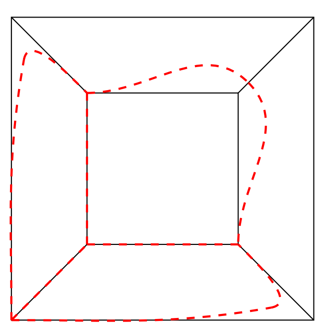


FIGURE 3. This graph has the 3-PSP property because we have fount a K_4 minor.

Corollary 3.10. *No graph of a 3-polytope is the graph of a d-polytope with $d \ge 4$. Therefore, graphs of 3-polytopes are dimensionally unambiguous, i.e. if it is a 3-polytope graph its polytopality range is $\{3\}$.*

We get this corollary as a consequence of the d-PSP property, because if there was $d^* \geq 4$ such that it was a d^*-polytope, it would contain a principal subdivision of K_{d^*+1}, $d^* + 1 \geq 5$, so it would in particular contain a K_5-minor.

LECTURE 4

Cyclic Polytopes

Scribe: Daniel Torres

Let $x\colon \mathbb{R} \to \mathbb{R}^d$ an algebraic curve. Then we can choose $n > d$ points of the curve, for example, for $t_1 < \cdots < t_n$ choosing $x(t_1), \dots, x(t_n)$, in order to construct a polytope resulting convex hull of these points. An example of this could be the degree d *moment curve* $\mu_d(t) = (t, t^2, \dots, t^d)$.

At first, one can think that this way to construct a polytope it's just silly example, since choosing random points in a algebraic curve seems to be the same that choosing random points in $\mathbb{R}^d$. However, an algebraic curve of degree d relating vertices let us obtain an interesting result.

Theorem 4.1. *Let μ be a degree d algebraic curve in $\mathbb{R}^d$. Then, the combinatorial type of the polytope $\operatorname{conv}\{\mu(t_1), \dots, \mu(t_n)\}$ is independent of the choice of $t_1, \dots, t_n$.*

LECTURE 5

Kalai's simple way of telling a simple polytope from its graph

Scribe: Alex Alvarez

Conjecture 5.1 (Micha Perles, ∼1970)**.** *Any d-polytope is determined by its graph, i.e., $sk^1(P)$ determines $\mathcal{F}(P)$.*

The conjecture is false, e.g. for simplicial polytopes. On the other hand, K_n is dimensionally ambiguous, e.g. $sk^1(C_d(n)) = sk^1(C_e(n))$ for $d, e < n$. Besides there are neighborly graphs that are not combinatorially isomorphic to cyclic ones. However, the result is true for simple polytopes:

Theorem 5.2 (Roswitha Blind, Peter Mani, 1987)**.** *Any simple d-polytope is determined by its graph.*

Their proof is not constructive, but in 1988 Kalai found an algorithmic approach. Let us see the details of that approach.

Definition 5.3 (Acyclic orientation)**.** An acyclic orientation of a graph G is an orientation of $E(G)$ without directed cycles.

Definition 5.4 (Abstract Objective Function, AOF)**.** An AOF of $G = sk^1(P)$ is an acyclic orientation that has a unique sink on each face of P.

Definition 5.5 (Unique Sink Orientation)**.** A Unique Sink Orientation is an orientation such that every face has a unique sink. The orientation is not necessarily acyclic.

Notice that AOFs exist because of the existence of generic linear objective functions. Consider the set of all AOFs and define the following function:

$$f(O) = h_0(O) + 2h_1(O) + \ldots + 2^k h_k(O) + \ldots + 2^d h_d(O) \tag{5.1}$$

where O is an AOF and $h_k(O) \in \mathbb{N}$ counts the vertices of in-degree k in O.

Let f denote the number of non-empty faces of P. Given an acyclic orientation O, then

(1) $f(O) \geq f$, because as O is acyclic each face has at least one sink.
(2) $f(O) = f$ if O is an AOF.

To determine all AOFs of P just from its graph $G = sk^1(P)$ do the following steps:

(1) Enumerate all acyclic orientations O of G.
(2) Calculate $f(O)$ for each such orientation O.
(3) Keep the orientations O with minimal $f(O)$.

The only remaining step to complete Kalai's method is showing a way to identify the faces of P from the knowledge of all the AOFs. Notice that the subgraphs of G that are graphs of faces are connected, k-regular and induced. With that, the following proposition is enough to characterize the faces:

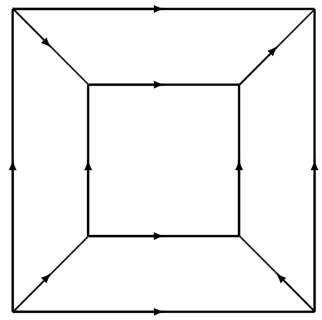


FIGURE 1. An example of AOF. For this orientation, $h_0 = 1$, $h_1 = 3$, $h_2 = 3$ and $h_3 = 1$. Therefore, $f(O) = 1 + 6 + 12 + 8 = 27 = 3^3$.

Proposition 5.6. *An induced connected k-regular subgraph H is the graph of a face of $P \Leftrightarrow H$ is an initial set for some AOF O.*

PROOF. $\Rightarrow$: Perturb a face-defining inequality to obtain a linear function with respect to which the vertices of F lie below all other vertices.

$\Leftarrow$: Let H be a connected k-regular subgraph of $sk^1(P)$ and let O be an AOF such that H is an initial set of O. As H has a unique sink v and is k-regular, the sink v has k incoming edges and as the polytope is simple, those k edges determine a k-face F and v is a sink in that face.

As v is the unique sink of O in F, all vertices of F are $\leq v$ with respect to O. But the initial set of v in G is H, so vert$(F) \subseteq$ vert(H). As both H and F are connected and k-regular, we have that vert$(F) =$ vert(H). $\square$

Notice that this algorithm is exponential, but in 1995 Friedman gave a polynomial algorithm to reconstruct the 2-faces, which had been proven before to be enough to reconstruct the whole polytope.

LECTURE 6

Lattice points in multiples of politopes

Scribe: Xavier Tapia

Let's define $L_P(t) = \sharp\{tP^d \cap \mathbb{Z}^d\}$, this number is equal to $\sharp\{P^d \cap \frac{1}{t}\mathbb{Z}^d\}$ where P is a polytope. Take now $P^o = intP$ as a topological space, and consider $L_{P^o}(t) = \sharp\{t(P^o)^d \cap \mathbb{Z}^d\}$.

Let's considerer first some examples of this:

If we take $d = 2$ and $t = 2$ we will have 9 points

Considerer some tables.

	0	1	2	3	4	...	t
$L_{\square^2}(t)$	1	4	9	16	25	...	$(t+1)^2$
$L_{(\square^2)^0}(t)$	0	0	1	4	9	...	$(t-1)^2$
$L_{\square^1}(t)$	1	2	3	4	5	...	$t+1$
$L_{(\square^1)^0}(t)$	0	0	1	2	3	...	$t-1$

It's possible to see that, in general:

Theorem 6.1. $L_{\square^d}(t) = (t+1)^d$, $L_{(\square^d)^0}(t) = (t-1)^d = (-1)^d L_{\square^d}(-t)$

In fact, this result is true for every polytope P not only for $\square^d$, this theorem is known as the Ehrart-Macdonald Reciprocity and it's the principal result we want to prove.

Theorem 6.2 (Ehrhart-Macdonald Reciprocity)**.**

$$L_P(-t) = (-1)^d L_{P^0}(t)$$

To proof this theorem we need some previous results,

1. Generating functions for rational cones

Take S a rational cone or polytope (i.e. the generators of 1-d rays with rational vertices coordinates). We define:

$$G_S(z) = G_S(z_1, z_2, ..., z_d) = \sum_{m \in S \cap \mathbb{Z}^d} z^m = \sum_{m \in S \cap \mathbb{Z}^d} z_1^{m_1} ... z_d^{m_d}$$

$G(z) = 1 + x + y + xy = (1+x)(1+y)$

Goal: Find $G_c(z)$ where $C = \{v_1, ..., v_d\} \subset$
R^d

Assume that C is simplicial (that0s mean C is a cone over a simplex), so the paralelopiped $\Pi_C = \{\lambda_1 v_1 + ... + \lambda_d v_d : 0 \leq \lambda_i \leq 1\}$ till C disjointlly.

In particullar, $G_C(z) = \sum_{s_i \in \mathbb{N}} tr_{\sum s_i v_i} G_\Pi(z) = \sum_{s_i \in \mathbb{N}} z^{\sum s_i v_i} G_\Pi(z) = \frac{G_\Pi(z)}{(1-z^{v_1})...(1-z^{v_d})} \in \mathbb{Q}[[z_1, ..., z_d]]$

Theorem 6.3 (Beck-Robins theorem 3.5)**.** *Suppose* $C = cone\{v_1, ..., v_d\}$ *cone over a simplex, then for all* $v \in \mathbb{R}^d$ *the generating function satisfies:*

$$G_{v+C} = \frac{G_{v+\Pi_C(z)}}{(1-z^{v_1})...(1-z^{v_d})}$$

PROOF. In $G_{v+C}(z) = \sum_{m \in (v+C)\cap\mathbb{Z}^d} z^m$ write $m = w + \lambda_1 v_1 + ... + \lambda_d v_d$ for $\lambda_i \geq 0$, and take $\lambda_k = \lfloor \lambda_k \rfloor + \{\lambda_k\}$ so for that, we can write:

$$m = w + \lfloor \lambda_1 \rfloor v_1 + ... + \lfloor \lambda_d \rfloor v_d + \{\lambda_1\} v_1 + ... + \{\lambda_d\} v_d$$

The vector $P = w + \{\lambda_1\} v_1 + ... + \{\lambda_d\} v_d \in \mathbb{Z}^d$ is unique and any $u \in v + C \cap \mathbb{Z}^d$ can be written uniquely as $P + \sum n_i v_i$ with $P \in \Pi_C$ and $n_i \in \mathbb{Z}$.

We have:

$$\frac{\sigma_{w+\Pi_C}(z)}{(1-z^{v_1})...(1-z^{v_d})} = \left(\sum_{m\in(v+C)\cap\mathbb{Z}^d+w} z^m\right)\left(\sum_{k_i\in\mathbb{N}} z^{k_1 v_1}\right)...\left(\sum_{k_d\in\mathbb{N}} z^{k_d v_d}\right)$$

Note that if $\Delta C \cap \mathbb{Z}^d = \phi$ it suffices to take $\Pi_C^0 = \{\sum_{i=1}^d \lambda_i v_i, 0 < \lambda_i < 1\}$ the open fundamental paralelopiped. $\square$

Theorem 6.4. *For any pointed cone (it's maximum linear subspace contained in* C *is* $\{0\}$*)* $C = \{w + \sum \lambda_i v_i, \lambda_i \geq 0\}$ *with* $w \in \mathbb{R}^d$ *and* $v_i \in \mathbb{Z}^d$*, the generating function* $\sigma_C(z)$ *is a rational function in the* z_i*.*

LECTURE 7

Lattice Geometry

Scribe: Borja Elizale

A $\mathbb{Z}$-module is an abelian group of rank r, and we don't care about the torsion part, so it is isomorphic to $\mathbb{Z}^r$ for some r.

Definition 7.1. A lattice polytope is a convex polytope all whose vertices have integral coordinates.

A natural question that can arise is how can we know if the two lattice polytopes are equivalent. The answer to this question is that they are equivalent when they are related by an invertible mapping that leaves the lattice invariant. These mappings belong to the set of mappings:

$$\text{Isom}(\mathbb{Z}^d) = Sl_d(\mathbb{Z}) \times \text{ Translations}(\mathbb{Z})$$

And $Sl_d(\mathbb{Z})$ are the $d \times d$ matrices over the integer numbers that have determinant equal to 1 or -1. Let's consider, as an example, the lattice polytopes in the plane:

$$P_1 = conv(\{0, e_1, e_2\}) \tag{7.1}$$

$$P_2 = conv(\{0, e_1, 7e_1 + e_2\}) \tag{7.2}$$

$$P_3 = conv(\{0, e_1, 7e_1 + 2e_2\}) \tag{7.3}$$

We want to know if they are equivalent or not, and based on what we have said before, this happens if there exists a transformation in $Sl_d(\mathbb{Z})$ (in this case the dimension is 2) plus a translation that brings one to the other. Since the two first points are the same for the three polytopes, there is no translation and so we only have to find a transformation in $SL_d(\mathbb{Z})$ between them. Now for the first one this is equivalent to find a matrix $\mathbf{A}$ such that:

$$\mathbf{A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \tag{7.4}$$

And clearly $\mathbf{A}$ must be the matrix from the right side of the equation, which is in the set $SL_d(\mathbb{Z})$. For the second case this does not happen, the right side of the equation will be the matrix $\begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix}$, which is not in $SL_d(\mathbb{Z})$. So such a transformation does not exist and they are not equivalent. Computing the volume of a lattice polytope is computing the determinant of the square matrix built from the homogeneous coordinates of the points. This doesn't give properly the volume, but the volume of the smallest parallelepiped that contains the whole polytope. We say that a lattice simplex, Δ is:

Definition 7.2. *Unimodular* if $\det \Delta = \pm 1$

Definition 7.3. *Empty* if conv$\Delta \cap \mathbb{Z}^d =$vertices of Δ.

Definition 7.4. *Standart* if Δ is equivalent to conv$\{0, e_1, ..., e_d\}$

The relation between these concepts goes as follows:

Standart$\Leftrightarrow$Unimodular$\Rightarrow$ Empty

And if the dimension is equal to 2 we also have

Empty$\Rightarrow$Unimodular

But the other implication is not true in general (there is a counter example for dimension 3, and then for every dimension larger than 3). Let's proof the first relationship

PROOF. Δ Standart $\Rightarrow$ Unimodular. If Δ is such that $\mathbf{A}\Delta = \mathbf{A}_0$, for some $\mathbf{A}$ in $SL_d(\mathbb{Z})$, then it happens that $\det(\mathbf{A}\Delta) = \det(\mathbf{A}_0)$, and then $\pm 1 \det(\Delta) = \pm 1$ and this only happens if $\det(\Delta) = \pm 1$, which is the condition for it to be unimodular.

For the reciprocal, if Δ is unimodular, then $\det(\Delta) = \pm 1$, and if we want some $\mathbf{A}$ in $SL_d(\mathbb{Z})$ that satisfies $\mathbf{A}\Delta = \mathbf{A}_0 = Id(d)$, then $\mathbf{A} = \Delta^{-1}$ and Δ^{-1} belongs to $SL_d(\mathbb{Z})$ because it's determinant is ± 1. $\square$

PROOF. **Unimodular** $\Rightarrow$ **Empty**. By contradiction, if it was not empty, then we could divide the polytope into two subdivisions by taking the cones to the interior point that makes it non empty, so that each subdivision has area, at least, ± 1 and then whole area (which is the sum of the parts) would be greater than 1 contradicting the unimodularity. $\square$

Theorem 7.5. *Pick's theorem: if $P \subset \mathbb{R}^2$ is a reticular polygon, then area(P) $= I + \frac{B}{2} - 1$. Where $I =$ Int$P \cap \mathbb{Z}^2$ and $B = \partial P \cap \mathbb{Z}^2$.*

This theorem allows us to prove that for $d = 2$, **Empty$\Rightarrow$Unimodular**, because if it is empty, then $I = 0$ and then $\frac{A}{2} = 0 + \frac{3}{2} - 1$ and $A = 1$ so the polygon it is unimodular as well.

LECTURE 8

Ehrhart-Macdonald Reciprocity

Scribe: Albert Santiago

1. Statement of the theorem

Let P be a lattice polytope. Let $L_P : \mathbb{N} \longrightarrow \mathbb{N}$ be the lattice point counting function,

$$\begin{array}{rccl} L_P: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & t & \longmapsto & L_P(t) = \#\left\{tP \cap \mathbb{Z}^d\right\} \end{array}$$

and let $L_{P^0} : \mathbb{N} \longrightarrow \mathbb{N}$ be the lattice point counting function in the interior of P.

By Ehrhart's Theorem seen in the previous lecture, it is known that $L_P, L_{P^0} \in \mathbb{Q}[t]$.

Theorem 8.1 (Ehrhart-Macdonald Reciprocity)**.**

$$L_P(-t) = (-1)^d L_{P^0}(t)$$

Note that in previous lecture we have already seen an example for this EM reciprocity for the cube. Indeed,

$$L_{\square^d}(t) = (1+t)^d$$
$$L_{(\square^d)^0}(t) = (-1)^d(1-t)^d$$

2. Auxiliary results

Proposition 8.2. *Let $w_1, \ldots, w_d \in \mathbb{Z}^d$ be linearly independent. Let $C = \operatorname{cone} w_1, \ldots, w_d$ be cone over a simplex. Let $v \in \mathbb{R}^d$ such that $\partial(v + C) \cap \mathbb{Z}^d = \varnothing$. Then,*

$$\sigma_{v+C}\left(\frac{1}{z_1}, \cdots, \frac{1}{z_d}\right) = (-1)^d \sigma_{-v+C}(z_1, \ldots, z_d)$$

Notes:

- For the existence of such $v \in \mathbb{R}^d$, see exercise sheet #6.
- $\sigma_{v+C}\left(\frac{1}{z_1}, \cdots, \frac{1}{z_d}\right)$ is not a power series, so the theorem is meaningless at this level. At most, we can say it is an element from $\mathbb{Q}[z_1, \ldots, z_d]$, where the theorem holds.
- Try to see $\frac{1}{z_i}$ as the variables corresponding to $-P$.

PROOF. To be done. □

Proposition 8.3 (Stanley Reciprocity)**.** *Let C be a rational pointed cone (i. e. $C = \operatorname{cone} w_1, \ldots, w_d, w_i \in \mathbb{Q}^d$) with apex at the origin. Then,*

$$\sigma_C\left(\frac{1}{z}\right) = (-1)^d \sigma_{C^0}(z)$$

PROOF. To be done. □

Definition 8.4. We define the Ehrhart function for the interior lattice points of a polytope naturally as

$$\mathrm{Ehr}_{P^0}(z) := \sum_{t\geq 1} L_{P^0}(t)z^t$$

Note: We can start the sum at $t = 1$ rather than at $t = 0$ because the origin is not an interior point of P.

Observation 8.5. As it happened with the original Ehrhart function, the following equality holds:

$$\mathrm{Ehr}_{P^0}(z) = \sigma_{\mathrm{cone}(P^0)}(1, \ldots, 1, z)$$

Proposition 8.6. *Let P be a lattice polytope. Then,*

$$\mathrm{Ehr}_(P^0)(z) = (-1)^{\dim P+1}\,\mathrm{Ehr}_P\left(\frac{1}{z}\right)$$

Notes:

- The proposition is also valid for P any *rational* polytope, not only for lattice polytopes.
- Not every polytope can be perturbed into a rational one. There exist irrational polytopes.

PROOF. To be done. □

3. Proof of Ehrhart-Macdonald Reciprocity

To be done.

4. Degree of a lattice polytope

Observation 8.7. Let $p \in \mathbb{Q}[t]$ be a polynomial of degree d. If the following equality holds:

$$\sum_{t\geq 0} p(t)z^t = \frac{h_d z^d + \cdots h_1 z + h_0}{(1-z)^d}$$

then

$$\left.\begin{array}{r} h_d = \cdots = h_{k+1} = 0 \\ h_k \neq 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} p(-1) = p(-2) = \cdots = p(-(d-k)) = 0 \\ p(-(d-k+1)) \neq 0 \end{array}\right.$$

PROOF. See Theorem 3.18 at Beck-Robins' *Computing the continuous discretely*. □

Proposition 8.8. *Let $P \subset \mathbb{R}^d$ be a lattice polytope with Ehrhart function:*

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + \cdots h_1 z + h_0}{(1-z)^d}$$

Then,

$$\left.\begin{array}{r} h_d = \cdots = h_{k+1} = 0 \\ h_k \neq 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} L_P(-1) = L_P(-2) = \cdots = L_P(-(d-k)) = 0 \\ L_P(-(d-k+1)) \neq 0 \end{array}\right.$$

Definition 8.9. We call k the *degree of the lattice polytope P*.

Observation 8.10. If P is a d-polytope of degree k, then $L_P(-(d-k)) = (-1)^d L_{P^0}(-(d-k)) = 0$, so the $(d-k)$-th dilate of P does not contain any interior lattice point, and $(d-k+1)P$ is the smallest integer dilate that contains an interior lattice point.

5. Reflexive polytopes

Definition 8.11. A lattice polytope P with $0 \in P$ is *reflexive* if P can be expressed as:

$$P = \{x \in \mathbb{R}^d : Ax \leq \mathbb{1}\}$$

where $A \in \mathbb{Z}^{n \times d}$ is an integer matrix.

Observation 8.12. To be done.

To be ended.

LECTURE 9

Systems of sparse polynomial equations

Scribe: Anna Somoza

Goal: Our aim is to construct systems of sparse polynomial equations with non-trivial lower-bound on the number of real solutions.

Motivation: We think abount this problem inspired by Kouchironko bound: Given the polynomial system

$$\begin{cases} F_1(x_1, \dots, x_n) = 0 \\ \vdots \\ F_n(x_1, \dots, x_n) = 0 \end{cases}$$

such that they share the same Newton polynomial $N(F_i)$, then we get that the number of complex solutions for the system is $\operatorname{vol} N(F_i)$.

Tools: We will introduce:

(1) A generalization of Veronese embedding,
(2) replace system of polynomial equations in

$$(\mathbb{C}^*)^n = \mathbb{C}^n \backslash \{\bigcup_i \{x_i = 0\}\}$$

by a system of linear equations in $\mathbb{P}^{\Delta \cap \mathbb{Z}^n} = \mathbb{P}^{\Delta}$:

$$\begin{aligned} \varphi_\Delta : (\mathbb{C}^*)^n &\to \mathbb{P}^\Delta \\ (t_1, \dots, t_n) &\mapsto [t^m : m \in \Delta \cap \mathbb{Z}^n] \end{aligned}$$

Example 9.1. *For* $n = 1$

$$\begin{aligned} \varphi_\Delta : \mathbb{C}^* &\to \mathbb{P}^\Delta \\ t &\mapsto [1 : t : t^2 : t^3] \end{aligned}$$

- Degree of a map between orientable mainfolds:
$X_\Delta = \operatorname{cl}(\Im \varphi_\Delta)$ in a Zariski topology
toric ($(\mathbb{C}^*)^n$ acts on X_Δ) variety associated to Δ.

1. Wronski polynomials

We require a foldable (i.e. rainbow colored), full and regular triangulation of L.

Definition 9.2. A triangulation T is said to be foldable if there exists a graph homomorphism such that

$$\operatorname{sk}^1(T) = \operatorname{sk}^1(\operatorname{conv}\{0, e_1, \dots, e_n\}).$$

It is equivalent to say that it is foldable if and only if its dual graph is bipartite.

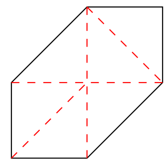


FIGURE 1. A non-foldable triangulation

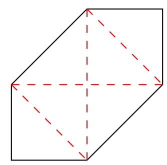


FIGURE 2. A non-foldable triangulation

Definition 9.3. We say that a triangulation is full if and only if

$$\mathrm{sk}^0\, T = \Delta \cap \mathbb{Z}.$$

Definition 9.4. We say that a triangulation is regular if and only if it is induced by a convex height function on $\Delta \cap \mathbb{Z}^n$

Scribes 2013

1. Alex Alvarez

I have done a double degree in Informatics Engineering and Mathematics at the UPC and during these years I have participated in programming contests both individually and representing the UPC. I am mainly interested in Algorithms and Data Structures and I would like to start a PhD in those topics next year, but I also enjoy learning about Discrete Mathematics, in particular Combinatorics and Graph Theory.

2. Cecilia Girón Albert

I've got my Degree in Mathematics at the Universidad Autónoma de Madrid, which I completed my fifth and sixth semesters at the University of Jyväskyla (Finland) as an Erasmus student. My degree has been mainly focused on subjects like analysis, statistics and numerical methods, although I am more interested in algebra and graph theory.

I decided to study the Master in Advanced Mathematics and Mathematical Engineering to keep developing my mathematics skills in some theoretical subjects that can be applied in real life problems. Therefore, I believe that this course is a great opportunity to learn more about algorithmic and computing science and even more importantly, it may help me to find out what field I would like to focus on in the future.

3. Anna Somoza

I have recently finished a degree in Mathematics at the Universitat Politècnica de Catalunya. During this degree I developed a great interest in Algebra fields. In particular, I took the optional subjects *Algebraic Geometry*, *Algebraic Topology* and *Galois Theory* and I wrote my Final Degree Thesis on a topic of Number Theory.

Now I'm taking the Master in Advanced Mathematics and Mathematical Engineering to develop my knowledge in these and other fields, and I my aim is to start a PhD in Number Theory next year. I enroled this subject because I have allways liked both computer science and geometry, and it seemed to be interesting. Therefore, I would be interested in the topic related to algebraic geometry.

4. Daniel Torres

My name is Daniel Torres and I am graduate in mathematics in UPC. Along the degree I have developed much interest in fields of topology, algebra and geometry, and some loathing to study (not to programming) numerical methods and modelling. I decided study this master, and particularly this subject, for expand my knowledge about my interests.

More concretely, I am doing this course with the hope it shows me about geometry.

5. Borja Elizalde

I have finished the degree in Mathematics at the UPC and I have also finished Industrial Engineering also at the UP (not CFIS).

I am interested in Number Theory and Algebraic Geometry in general, because the problems I like solving and thinking on usually belong to these fields. I am not sure at all about how I want to develop my professional career.

6. Xavier Tapia

I have finished a degree in Mathematics in UB, I spent 5 years to finish it because I have been living in Valencia for a year studiyng in UV. Along these years I have developed interest on Algebra and geometry, specially on finite groups and conmutative algebra. Now I'm working and studying at the same time this Master but I don't have any idea what is my professional future.

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