SHEET5

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1. Exercices

(1) Let $P \subset \mathbb{R}^d$ be a convex polytope and $v \in \mathbb{R}^d$. Then $\operatorname{stk} P = \operatorname{conv}(P \cup \{v\})$ is obtained from P by *stacking* on the facet F if v is *beyond* exactly F and *beneath* all other facets: If $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ where the rows of A are $a_1, \dots, a_m$, it should happen that $\langle a_i, v\rangle > b_i$ for exactly one i, while $\langle a_j, v\rangle \leq b_j$ for $j \neq i$.

   (a) For simplicial d-polytopes P, derive a formula for $f_k(\operatorname{stk} P)$, $0 \leq k < d-1$, in terms of $f_k(P)$ and $f_{k-1}(\Delta^{d-1})$, where Δ^{d-1} is the $(d-1)$-dimensional simplex.
   (b) Do the same for $\operatorname{stk}^N(P)$, the polytope obtained from P by N stackings.
   (c) Prove Danzer's result from 1964 that for large enough d and suitable N, the Unimodality Conjecture for f-vectors fails for N-fold stacked cross-polytopes.
   (d) Do better, for example by using cyclic polytopes, or connected sums of cyclic polytopes and their polars. What is the lowest dimension for which you can make the Unimodality Conjecture fail? Can you beat 8?
   (e) If you stack "too often" onto $C_{20}(200)$, then unimodality is restored. How often?

1.1. **Solution.** Observe that to do this operation ever is possible and "it's gluing a pyramid over a facet". Using same notation, it is:

Suppose that exists x such that $Ax \leq b$ with only equality in i-th row a_i and $a_i \neq 0$ (that is: row i defines a non-empty facet). Then, let be j such that $a_{i,j} \neq 0$. By finiteness of non-trivial rows of A (i.e. finiteness of facets of P), exists $\epsilon > 0$ such that $A(x \pm \epsilon e_j)$ still being strictly less than b_k for each row $k \neq i$ and strictly greater than b_i at row i.

Let P have dimension d, and let F be a facet, defined by row i. Then, k-faces can be defined as points x such that Ax are equal to b in exactly $d-k$ (linearly) independent rows. These k-faces are contained in F if and only if i-th row is one of these rows.

Let v be a point with a good conditions to be stacked on F. Then $P \cup \operatorname{conv}(F \cup \{v\}) \subset \operatorname{stk} P$ by definition. By "good conditions" of v, each row except the i-th still defining a closed half-space containing P. Thus, if $x \in \operatorname{stk} P$ but $x \notin P$, then $a_i x > b_i$.

(a) For