Discrete and Algorithmic Geometry 2011 (Part 2)

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This is the preliminary version of the lecture notes for the second part of *Discrete and Algorithmic Geometry* (Universitat Politècnica de Catalunya), held in the fall semester of 2011 by Vera Sacristan and Julian Pfeifle.

These notes are fruit of the collaborative effort of all participating students, who have taken turns in assembling this text. The name of each scribe figures in each corresponding section.

The main literature for this course consists of [**CS99**], [**CBGS08**] and [**Sen95**].

Suggestions for improvements will always be gladly received by `julian.pfeifle@upc.edu`.

Contents

LECTURE 1

Introduction to Packings

Scribe: Ferran Dachs Cadefau

The general content of the lectures.

1. Packings

Definition 1.1. A family $\{K_i\}_{i\in I}$ of compact convex sets $K_i \subseteq \mathbb{R}^d$ with non-empty interior (this implies that K_i are full-dimensional) is a *packing* if:

$$\text{int}(K_i \cap K_j) = \emptyset \qquad \text{for } i \neq j$$

It is possible that the boundaries of two different K_i overlap, but not the interior. If we are working in a Hausdorff space, subsets are compact if and only if they are closed and bounded. More generally, we can work with non-convex packings, but they are harder to work with. For example the next example due to M.C. Escher:

FIGURE 1. M.C. Escher, Plane Filling II, Lithograph 1957

Definition 1.2. If there exists $C \in \mathbb{R}^d$ such that $\bigcup_{i\in I} K_i \subseteq C$ then C is called a *container* of the packing. These always exist: take $C = \bigcup_{i\in I} K_i$. The *natural container* of the packing is

$$C_{nat} = \text{conv} \bigcup_{i\in I} K_i$$

We will pack repetitions of the same figure, that is, K_i for all $i \in I$ is the same set. Another thing that we can consider is a fixed container: For example, we can pack squares in squares as in [**Fri09**], or circles in squares, as in `http://hydra.nat.uni-magdeburg.de/packing/csq/csq.html`, or regular polyhedra [**ea10**]. As we can see in the second example, if we have a fixed container it is hard to find a optimum solution, and moreover, the optimum solution can have no regularity!

Definition 1.3. We can speak about the quality of the packings using their *density*

$$\delta_{bin} = \frac{\sum_{i=1} V(K_i)}{V(C)}$$

and *natural density*

$$\delta_{Nat} = \frac{\sum_{i=1} V(K_i)}{V(C_{Nat})}.$$

From now on, the K_i will be congruent spheres.

2. Density of disk packings in the plane

Lemma 1.4 (Thue in 1892)**.**

$$\delta_{Nat}(n \text{ disks in } \mathbb{R}^2) \underset{n\to\infty}{\longrightarrow} \delta_{Nat}(\text{hexagonal packing})$$

$$\delta_{Nat}(n \text{ thin disks in } \mathbb{R}^3) = 1$$

where thin disks are: $D^2 \times \square^1$ and the ideal packing is a cylinder. For bigger dimensions (thin disks are: $D^2 \times \square^{d-1}$) the ideal packing is again a cylinder.

3. Packings of Spheres

Observation 1.5. We defined: $S^d = \left\{x \in \mathbb{R}^{d+1} : \|x\| = 1\right\}$. Except for S^0 all spheres are connected, and all S^i for $i > 1$ are simply connected.

Definition 1.6. Let $Z = \operatorname{conv}\{\text{centers of } K_i : i \in I\}$. We say that the associated packing is a

(1) *Sausage* if $\dim Z = 1$;
(2) *Pizza* if $2 \leqslant \dim Z \leqslant d-1$;
(3) *Pile* if $\dim Z = d$.

For example, in $\mathbb{R}^2$ a Sausage is composed of n circles with their centers on a line. In $\mathbb{R}^3$, we get a Pizza for example by thinking of n spheres with their centers on a plane.

$$\delta_{Nat}(\text{Sausage of } n \text{ spheres in } \mathbb{R}^d) = \frac{\sum_{i=1} V\left(K_i\right)}{V\left(\operatorname{conv}\bigcup_{i\in I} K_i\right)} = \frac{n\beta(d)}{\beta(d) + 2(n-1)\beta(d-1)},$$

where $\beta(d)$ are the volume of the unit ball in $\dim d$. To calculate the volume of $\operatorname{conv}\bigcup_{i\in I} K_i$ we have used that the convex hull is a cylinder of height $n-1$ and two halves of a sphere.

$$\delta_{Nat}(\text{Sausage of } n \text{ spheres in } \mathbb{R}^d) \underset{n\to\infty}{\longrightarrow} \frac{\beta(d)}{2\beta(d-1)}.$$

For example the case $n = 4$ and $d = 3$ the best packing is a Sausage instead of for example the Tetrahedral packing as we can see in The paper of J.M.Wills.

Exercise 1.7. *Calculate the δ_{Nat} of the tetrahedral packing.*

In dimension 3 the best packings are shown in Table 1.

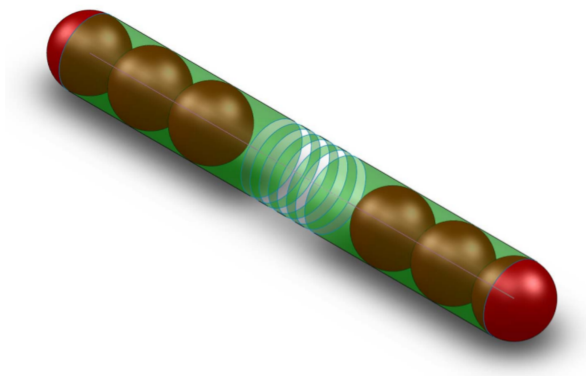


FIGURE 2. Sausage in $\mathbb{R}^3$ with its natural density C_{Nat}.

n (number of balls)	4	...	55	56	57	58	59	60	61	62	63	64	$\geqslant 65$
Type of best packing	S	...	S	P	S	S	P	P	P	P	S	S	P
Verified or Conjectured	V	C	C	V	C	C	V	V	V	V	C	C	V

TABLE 1. Best packings in dimension 3. Here S stands for Sausage, P for Pile, C is Conjectured and V is Verified.

Conjecture 1.8 (Sausage Conjecture (Lászlo Fejes Tóth))**.**

$$\delta(d,n) = \delta_{Nat}(W_n^d) \qquad \forall n \in \mathbb{N}, \quad d \geqslant 5$$

Where W_n^d is the sausage packing ("Wurst" in German).

Theorem 1.9 (Martin Henk, Jörg Wills, Ulrich Betke 1986; see [**BH98**])**.**

$$\delta(d,n) = \delta_{Nat}(W_n^d) \qquad \forall n \geqslant 42, \quad d \geqslant 5$$

4. The Unit cube

Now, we can consider $\square^d$, the unit cube in $\mathbb{R}^d$:

$$\square^d \;=\; \mathrm{conv}\{(a_1, ...a_d) | a_i = 0 \text{ or } 1, \text{ for } 0 \leqslant i \leqslant d\} \subseteq \mathbb{R}^d$$

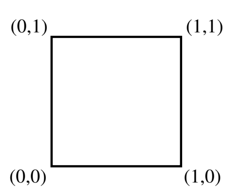


FIGURE 3. $\square^2$

Observation 1.10. The number of vertices of $\square^d$ is 2^d.

Definition 1.11.

$$\square^d = \{(a_1, ...a_d) | 0 \leqslant a_i \leqslant 1 \quad \forall 0 \leqslant i \leqslant d\} \subseteq \mathbb{R}^d$$

We can consider the faces of $\square^d$, and his *dimension* are the dimension of his affine span.

- If dimension are 0 we talk about *vertices*.
- If dimension are 1 we talk about *edges*.
- If dimension are $d-1$ we talk about *facet*.

Observation 1.12. The number of facets of $\square^d$ is $2d$, one for each inequality.

Exercise 1.13. *Calculate all the number of dimension i subspaces.*

Observation 1.14. The distance between a vertex and the barycenter is the radius of the *circumscribed sphere*. If V is a vertex, and B the barycenter, we have:

$$\|V_i - B\| = \|(0, ..., 0) - (1/2, ..., 1/2)\| = \|(1/2, ..., 1/2)\| = \sqrt{d}\frac{1}{2}$$

We can choose $V = (0, ..., 0)$ because all vertices are at the same distance from the barycenter.

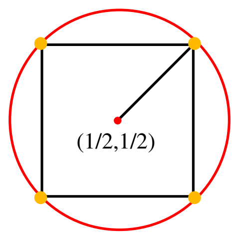


FIGURE 4. The distance between a vertex and the barycenter is the radius of the circumscribed sphere.

Observation 1.15. The distance between a facet and the barycenter is the radius of the *inscribed sphere*, $\frac{1}{2}$.

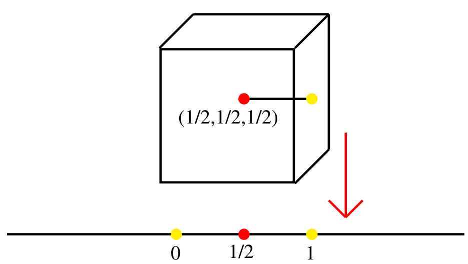


FIGURE 5. The distance between a facet and the barycenter is the radius of the inscribed sphere, $\frac{1}{2}$

Here we show the radii of the circumscribed and the inscribed spheres in some dimensions:

d	1	2	100	10^4
ρ_{circ}	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	5	50
ρ_{in}	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

It's difficult to think in high dimensions. For more, see [**Bal97**].

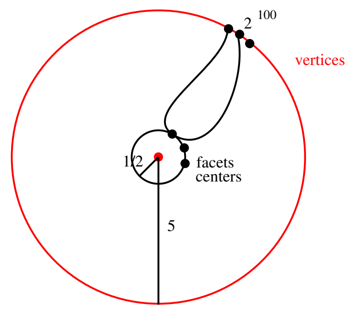


FIGURE 6. Representation of the vertices and the facets in dimension 100.

Observation 1.16. If we draw 2^d spheres centered in the vertices with radius $\frac{1}{2}$. Which is the radius of the maximum sphere that we can draw centered in the barycenter tangent to the others (as we can see in Figure 7)? $\frac{1}{2}\left(\sqrt{d}-1\right)$

d	2	3	4	5	100
$\frac{1}{2}\left(\sqrt{d}-1\right)$	0.2	$<\frac{1}{2}$	$\frac{1}{2}$	$>\frac{1}{2}$	$\frac{9}{2}$

In the table we can see that in dimensions over 5 the sphere goes out the facets!

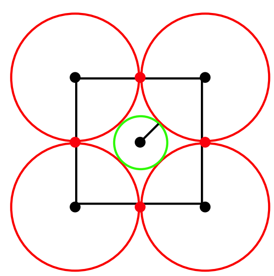


FIGURE 7. Representation of the vertices and the facets in dimension 100.

LECTURE 2

Volumes of balls and cubes; Lattice Polytopes

Scribe: Victor Bravo

1. Comparing the volumes of balls and cubes

Given an n-dimensional ball of radius r, we have that $\mathrm{vol}(B_r^n) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} r^n$, where Γ is the Gamma Function, which is defined in the following way:

(1) $\Gamma(m+1) = m!$, for $m \in \mathbb{N}_0$.
(2) $\Gamma(m+\frac{1}{2}) = \dfrac{(2m)!}{m!4^m}\sqrt{\pi}$.

Example 2.1. $vol(B_r^1) = \dfrac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2})} r = \dfrac{1!\sqrt{\pi}4^1}{2!\sqrt{\pi}} r = 2r.$

Example 2.2. $vol(B_r^2) = \dfrac{\pi}{1!} r^2 = \pi r^2.$

Now, we want to know the asymptotic behaviour, i.e., having a cube with a ball inside, we want to know how evolves the volume of the cube compared with the volume of the ball. Using the unit cube, in dimension 1, we have the same volume for the cube and the ball because they are the same thing. In dimension 2 (see figure 1), we have a square with every edge of length 1 and then, the ball has radius $1/2$. In dimension 3 (see figure 2), we have a cube with every edge of length 1 and then, the ball also has radius $1/2$, etc.

Then, the general case is, $\dfrac{\mathrm{vol}(B_{1/2}^n)}{\mathrm{vol}({}_0\Box_1^n)} = \mathrm{vol}(B_{1/2}^n) =$ fraction of unit cube taken up by largest ball contained inside.

Now, using Stirling's approximation, $\Gamma(x+1) \approx \sqrt{2\pi x}\left(\dfrac{x}{e}\right)^x$, $x \in \mathbb{R}_{\geq 0}$, we have that asymptotically,

$$\mathrm{vol}(B_{1/2}^n) \overset{n\to\infty}{\longrightarrow} \frac{\pi^{\frac{n}{2}}(\frac{1}{2})^n}{\sqrt{2\pi\frac{n}{2}}(\frac{n/2}{e})^{\frac{n}{2}}} = \frac{\pi^{\frac{n}{2}} 2^{\frac{n}{2}} e^{\frac{n}{2}}}{\sqrt{\pi n}\, n^{\frac{n}{2}} 2^n} = \frac{1}{\sqrt{\pi n}}\left(\frac{\pi e}{2n}\right)^{\frac{n}{2}} \overset{n\to\infty}{\longrightarrow} 0.$$

Example 2.3. $\dfrac{vol(B_{1/2}^{100})}{vol(\Box^{100})} \approx 10^{-67}.$

Then, we have bad news for numerical integration (for example in the case of Monte Carlo integration) when it is used in physics or in financial mathematics because, by the example above, we will be not able to count from 1 to 10^{-67}. This is too long. So, this works worst as the dimension increases. In conclusion, we will not use Monte Carlo Integration to calculate volumes in high dimensions because the volumes of the balls will be so tiny.

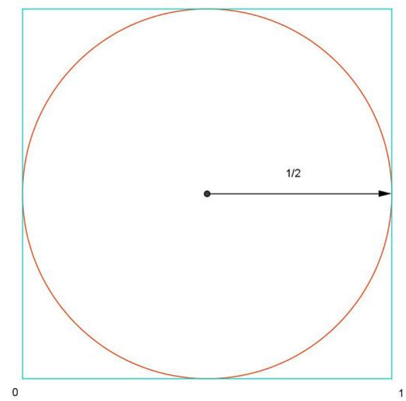


FIGURE 1. Example in dimension 2.

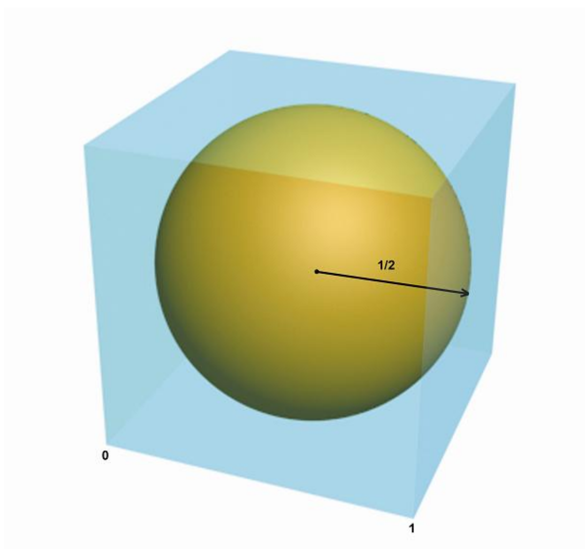


FIGURE 2. Example in dimension 3.

Remark 2.4. An example of Monte Carlo Integration in physics consists in throw random points into our space and count how many points fall inside and how many points fall outside. Then, do the fraction which divides the number of points inside and the number of points thrown and this fraction approximates the volume (it is used at CERN). In the other hand, Monte Carlo Integration is used in financial mathematics, for example if we have a portfolio with many variables and we have to integrate, one way to integrate by all this variables is using Monte Carlo Integration.

2. Lattices and lattice polytopes

Now, we will talk about lattice packings of spheres. A lattice has two different meanings in mathematics: a partially ordered set or a group. We are gonna talk about the group.

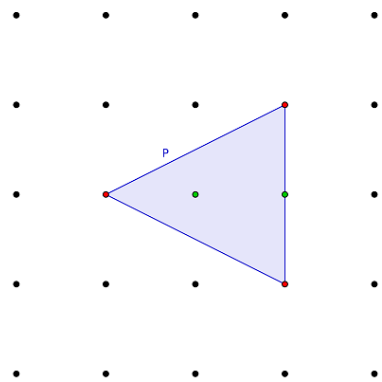


FIGURE 3. A lattice triangle.

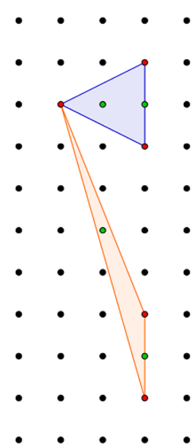


FIGURE 4. Two lattice triangles.

The most important lattice is $\mathbb{Z}^d$, and it's called the integer lattice. This is an abelian group with the sum: $x, y \in \mathbb{Z}^d \Rightarrow -x \in \mathbb{Z}^d, x + y \in \mathbb{Z}^d$, and the sum is commutative.

Now, if we have $v_1, \ldots, v_n \in \mathbb{Z}^d$, and we have a look to $P = \text{conv}\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$, we define a lattice polytope as the convex hull of a finite set of points with integer coordinates.

Now, we can do the next question: When two lattice triangles "the same"? The first observation is that we have to answer is: When two polytopes are "the same"? In Figure 4, we can say that the two lattice triangles are "the same" because they share all properties respect to the lattice.

Now, forgetting lattices, the answer to the question for polytopes in general is: Klein's Erlangen Program. In this program, Klein identifies the geometry with the groups of automorphisms, i.e., what Klein makes is to say what the geometry is, by seeing which group of automorphisms leaves certain object invariant.

Some groups that we must have in mind are $O(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : A^{-1} = A^T\}$ and $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap \{A \in \mathbb{R}^{n\times n} : \det A = 1\}$. In the other hand, we can also have in mind the set of translations in $\mathbb{R}^{n\times n}$, $T(n, \mathbb{R}^{n\times n})$, which satisfies $SO(n, \mathbb{R}^{n\times n}) \rtimes T(n, \mathbb{R})$, where $\rtimes$ is the semi-direct product, which means: two subsets, $P, Q \subseteq \mathbb{R}^n$, are "the same" if $\exists A \in O(n)$ and $\exists t \in \mathbb{R}^n : Q = A \cdot P + t$, i.e., I can obtain Q from P through a rotation A and a translation t (i.e., P and Q are congruent), and this is what we know as Euclidean Geometry.

Now, remembering lattices, we have to change the euclidean geometry by lattice geometry, i.e., we want bijective homomorphisms that preserves the lattices. So, we want to determine $\text{Aut}(\mathbb{Z}^d) = \{\text{affine transformations that leave } \mathbb{Z}^d \text{ invariant}\}$, and this is to find conditions on $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}^n$ such that $Ax + t \in \mathbb{Z}^d$, $\forall x \in \mathbb{Z}^d$. These conditions are:

- $x = 0$, want $A \cdot 0 + t \in \mathbb{Z}^d \iff t \in \mathbb{Z}^d$.
- $x = e_i$, with e_i a generating vector of our lattice, want $A \cdot e_i \in \mathbb{Z}^d \iff$ every column of $A \in \mathbb{Z}^d \iff A \in \mathbb{Z}^{d\times d}$.

Now, for A to be an automorphism, it must be invertible, and A^{-1} must belong to $\mathbb{Z}^{d\times d}$.

Example 2.5. *Suppose that $d = 2$. We have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$. Then, $A^{-1} = \frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \frac{1}{\det A}[(c_{ij})]$, where (c_{ij}) represents the cofactors of A. And $A^{-1} \in \mathbb{Z}^{2\times 2}$ because $ad - bc$ never divides the entries of $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. (This has to be proved.)*

Then, $\text{Aut}(\mathbb{Z}^d) = \left\{A \in \mathbb{Z}^{d\times d} : \det A = \pm 1\right\} \rtimes \mathbb{Z}$.

Observe that the set of orientation-preserving linear (not affine) automorphisms of $\mathbb{Z}^d$ is $\text{Sl}_d(\mathbb{Z}) = \left\{A \in \mathbb{Z}^{d\times d} : \det A = 1\right\}$, the special linear group with integer coefficients. On the other hand, $\left\{A \in \mathbb{Z}^{d\times d} : \det A = -1\right\}$ is not a group.

Then, what lattice geometry means is that geometry with group automorphisms: $Sl_d(\mathbb{Z}) \rtimes \mathbb{Z}^d$, and this is mapping $x \longmapsto Ax + t$, with $t \in \mathbb{Z}^d$, $A \in \mathbb{Z}^{d\times d}$, and $\det A = \pm 1$. Then, any two lattice polytopes in correspondence by any of this automorphisms will be the same polytope.

Now, observe that in Figure 4, using that the image of the vectors are the same than the columns of the matrix A, we have that $A = \left[\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$, with $A \cdot e_1 = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$ and $A \cdot e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Also, observe that the following transforms (called *shears*) are typical lattice transforms in $\mathbb{Z}^2$: $\left[\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]$. This can be used in exercise 3 of list 1.

After seen this, we are going to see some definitions:

Let $P \subseteq \mathbb{R}^d$ be a polytope ($\sim$ convex hull of finitely many points). A linear inequality of the form $ax \leq b$ with $a \in (\mathbb{R}^d)^*$, $x \in \mathbb{R}^d$, $b \in \mathbb{R}$ is valid for P if all points of P satisfy it. (observe that $(\mathbb{R}^d)^*$ represents the dual space of $\mathbb{R}^d$).

A face of P is $P \cap \{x \in \mathbb{R}^d : ax = b\}$, where $ax \leqslant b$ is a valid linear inequality for P. In particular, $\emptyset$ is always a face of P (example: $0x \leqslant 1$), and P is always a face of P (example: $0x \leqslant 0$). This, bring us to a second meaning of lattice:

The face lattice of P is the poset (partially ordered set) of faces of P with the inclusion.

Example 2.6. *If we have the polytope of figure 5, this polytope will have the face lattice of figure 6 (where O represents the $\emptyset$).*

If anybody wants to read about this, then read "Lectures on Polytopes" by Ziegler.

This can be applied to cubes, for example, as follows:

$$(100011)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \leqslant k,$$

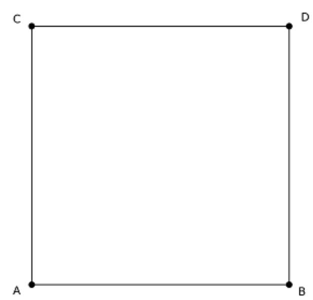


FIGURE 5. Square ABCD.

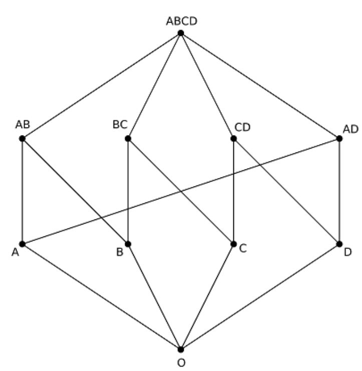


FIGURE 6. Its face lattice.

where k represents the non-zero entries (in this case $k = 3$), and using this, we can calculate the barycenters.

LECTURE 3

Pick's Theorem; Lattice packings of spheres

Scribe: Miquel Raïch

1. Pick's Theorem

Theorem 3.1 (Pick). *Let P be a lattice polygon in the plane (P is closed, convex, simple and its vertices lie in $\mathbb{Z}^2$). The area of P is*

$$A(P) = \operatorname{vol}_2 P = I + \frac{1}{2}B - 1$$

where:
$I =$ *number of interior lattice points of* $P = \#\left\{(\textit{int } P) \cap \mathbb{Z}^2\right\}$,
$B =$ *number of boundary lattice points of* $P = \#\left\{\partial P \cap \mathbb{Z}^2\right\}$

PROOF. [part]

(1) Show that Pick's formula is additive: if $P = P_1 \cup P_2$, then

$$I + \frac{1}{2}B - 1 = \left(I_1 + \frac{1}{2}b_1 - 1\right) + \left(I_2 + \frac{1}{2}B_2 - 1\right)$$

$$\begin{aligned} A(P) =& A(P_1) + A(P_2) \\ I = I(P) =& I_1 + I_2 + L - 2 \\ B = B(P) =& B_1 + B_2 - 2L + 2 \end{aligned}$$

(Principle of Inclusion-Exlusion $\to$ Möbius function)

$\lceil$ This proves:
   (a) Pick$(P_1 \cup P_2) \Leftarrow$ Pick(P_1), Pick(P_2)

   (b) Pick$(P_1) \Leftarrow$ Pick$(P_1 \cup P_2)$, Pick(P_2) $\rfloor$

(2) Prove it for lattice triangles.

□

2. Lattice packings of spheres

A **lattice-packing** of congruent spheres ($\equiv$ same radius) in $\mathbb{R}^d$ is a packing such that the set $Z = \{\text{centers of the spheres}\}$ is a lattice L (free abelian group).

Let $\{v_1, \ldots, v_n\} \in \mathbb{R}^d$ be a generating set for

$$M = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}, \text{ then } L = \{M\lambda \colon \lambda \in \mathbb{Z}^n\}$$

$$M = \overbrace{\left[\begin{array}{cc} 1 & \frac{1}{2} \\ 0 & \frac{1}{2}\sqrt{3} \end{array}\right]}^{n}\Big\}d \qquad\qquad M\lambda = \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]\left[\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array}\right] = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

Sphere packing $B_0 + L = \{B_0 + v \colon v \in L\} = \{B_0 + M\lambda \colon \lambda \in \mathbb{Z}^n\}$

$\forall p, q \in P,\ \exists D_p, D_q \colon D_p \cap Dq = \emptyset$ $\qquad$ $p \in P \subset \mathbb{R}^d$ discrete set of points

Voroni cell of p w.r.t. P is

$$\mathrm{Vor}(P) = \{y \in \mathbb{R}^d \colon \|y - p\| \leq \|y - q\| \ \forall q \in P\}$$

⌈ Georges Voronoi (s. XIX) ⌋

Voronoi cells are intersections of half spaces

$$\mathrm{Vor}(P) = \bigcap_{q \in P} H_q \qquad \text{where } H_q = \{y \in \mathbb{R}^d \colon \|y - p\| \leq \|y - q\|\}$$

Definition 3.2.
polyhedron $\equiv^{def}$ intersection of half-spaces
polytope $\equiv^{def}$ convex hull of a finite point set $\overset{\text{FTPT}}{\equiv}$ bounded polyhedron
(FTPT: Fundamental theorem of polytope theory)

(1) any convex hull of a finite point set is an intersection of half-spaces [easy by calculating convex hull].

(2) any bounded intersection of half-spaces is the convex hull of a finite set of points, unless the intersection is empty.

Any lattice is isomorphic to some $\mathbb{Z}^n$, as abelian groups, by the map $v \in L \leftrightarrow \lambda \in \mathbb{Z}^n \colon v = M\lambda$

[I will put images another day :P]

LECTURE 4

The hexagonal lattice and laminated lattices

Scribe: Ane Santos

1. The hexagonal lattice

Definition 4.1. Let $v_1, \dots, v_n \in \mathbb{Z}^d$ and the lattice $L = {}_{\mathbb{Z}}\langle v_1, \dots, v_n\rangle = \{\sum \lambda_i v_i : \lambda_i \in \mathbb{Z}\} = \{M\lambda : \lambda \in \mathbb{Z}^n\}$ with $M = [v_1, \dots, v_n] \in \mathbb{Z}^{d\times n}$. M is called the *generator matrix* and ${}_{\mathbb{Z}}\langle v_1, \dots, v_n\rangle$ the *integer hull* of the lattice.

We will study two variants of the hexagonal lattice:

$$A_{2,\mathbb{R}^2} = \left\{ \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix} \lambda : \lambda \in \mathbb{Z}^2 \right\}, \qquad A_{2,\mathbb{R}^3} = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \lambda : \lambda \in \mathbb{Z}^2 \right\},$$

with respective generating matrices $M = \left[\begin{smallmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{smallmatrix}\right]$ and $M' = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$.

Firstly, we study $A_{2,\mathbb{R}^3}$:

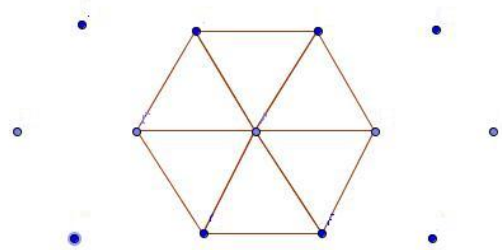


FIGURE 1. $A_{2,\mathbb{R}^3}$

$$A_{2,\mathbb{R}^3} = \left\{M'\lambda : \lambda \in \mathbb{Z}^2\right\} = \left\{ \begin{bmatrix} \lambda_1 \\ -\lambda_1 + \lambda_2 \\ -\lambda_2 \end{bmatrix} : \lambda_1, \lambda_2 \in \mathbb{Z} \right\}$$

We want to find a hyperplane that contains $A_{2,\mathbb{R}^3}$. We are in $\mathbb{R}^3$, so this hyperplane is of the form $\left\{x \in \mathbb{R}^3 : \langle a, x\rangle = a_0\right\}$. But we know 0 is in $A_{2,\mathbb{R}^3}$ so $a_0 = 0$ and

$$H = \left\{x \in \mathbb{R}^3 : \langle a, x\rangle = 0\right\} = \left\{\omega \in \mathbb{R}^3 : \langle \omega, x\rangle, \forall \in \operatorname{colspan} M\right\},$$

where colspan $M = {}_{\mathbb{R}}\langle v_1, \ldots, v_n\rangle = \operatorname{Im} M$ (it is an abelian group and it is also a vector space). So, $H = \left\{x \in \mathbb{R}^3 : \langle a, x\rangle = 0\right\} = \ker M$

$\dim \operatorname{Im} M = 2$ and $\dim A_{2,\mathbb{R}^3} = 3 \Longrightarrow \dim A_{2,\mathbb{R}^3} = \dim\ker M + \dim \operatorname{Im} M \Longrightarrow \dim\ker M = 1$

A generator of $\ker M$ will be $\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$

$[111]\left[\begin{smallmatrix}1 & 0\\ -1 & 1\\ 0 & -1\end{smallmatrix}\right] = [00] \Rightarrow \ker M = {}_{\mathbb{R}}\langle[111]\rangle$

So $A_{2,\mathbb{R}^3} \subset \left\{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\right\} = \left\{x \in \mathbb{R}^3 : \mathbb{1}x = 0\right\}$

Definition 4.2. The *Gram matrix* of a lattice L with generator matrix M is $G_L = M^T M$.

Definition 4.3. The *determinant of a lattice* L with generator matrix M is the determinant of the Gram matrix. $\det L = \det M^T \cdot \det M = (\det M)^2$

Observation 4.4. G_L is always a symmetric matrix because $G_L^T = \left(M^T M\right)^T = M^T M$.

We calculate the determinants of $A_{2,\mathbb{R}^2}$ and $A_{2,\mathbb{R}^3}$:

$$\begin{aligned}
\det A_{2,\mathbb{R}^2} &= \det\begin{bmatrix}1 & 0\\ \frac{1}{2} & \frac{1}{2}\sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}1 & \frac{1}{2}\\ 0 & \frac{1}{2}\sqrt{3}\end{bmatrix} = (\tfrac{1}{2}\sqrt{3})(\tfrac{1}{2}\sqrt{3}) = \frac{3}{4},\\
\det A_{2,\mathbb{R}^3} &= \det\begin{bmatrix}0 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ -1 & 1\\ 0 & -1\end{bmatrix} = \det\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} = 3.
\end{aligned}$$

Definition 4.5. The *minimum norm* of a lattice L is $\mu_L = \min\left\{\|v\|^2 : v \in L\setminus\{0\}\right\}$

From the minimum norms $\mu_{A_{2,\mathbb{R}^2}} = 1$, $\mu_{A_{2,\mathbb{R}^3}} = \sqrt{2}$, we conclude that both the determinants and the minimum norms of $A_{2,\mathbb{R}^2}$ and $A_{2,\mathbb{R}^3}$ are different. However, we should not conclude that these lattices are really different:

Definition 4.6. Two lattices are *isomorphic* if one is obtained from the other by rotation, reflection, translation and scaling.

The most general map between isomorphic lattices is therefore of the form

$$x \longmapsto \alpha A + t, \qquad \text{where } t \in \mathbb{R}^n,\ A \in O(n),\ \alpha \in \mathbb{R}^*.$$

Note that negative α correspond to reflections.

Definition 4.7 (Packing density of L)**.** $\Delta_L = \frac{\text{vol(sphere in packing)}}{\text{vol}(\Pi_L)=\sqrt{detL}}$, where $\Pi_L = \{\sum \lambda_i v_i : \lambda_i \in [0,1)\}$ is the fundamental parallelopiped.

To calculate the packing density of $A_{2,\mathbb{R}^2}$ and $A_{2,\mathbb{R}^3}$, note that in $A_{2,\mathbb{R}^2}$ the radius of the sphere is $\frac{1}{2}$ so the volume is $\left(\frac{1}{2}\right)^2\pi$. We obtain the same density, which is as it should be for isomorphic lattices:

$$\begin{aligned}
\Delta_{A_{2,\mathbb{R}^2}} &= \frac{\left(\frac{1}{2}\right)^2\pi}{\sqrt{3/4}} = \frac{\pi}{2\sqrt{3}},\\
\Delta_{A_{2,\mathbb{R}^3}} &= \frac{\left(\frac{1}{2}\sqrt{2}\right)^2\pi}{\sqrt{3}} = \frac{\pi}{2\sqrt{3}}.
\end{aligned}$$

The connection between these two representations is via the map

$$\begin{bmatrix} 1 & \frac{-1}{\sqrt{3}} \\ -1 & \sqrt{3} \\ 0 & \frac{-2}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So, we have two different ways to write the same lattice. The advantages of M' over M are that the coordinates are nicer and the symmetries of the lattice are more easily seen.

Claim 4.8. *Any permutation of the coordinate axes in $\mathbb{R}^3$ is a symmetry of $A_{2,\mathbb{R}^3}$.*

PROOF. Let $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a symmetry of $L = A_{2,\mathbb{R}^3}$, so that $P(L) = L$. This means that for all $x \in L$, we should have $P(x) \in L$, which is in turn equivalent to the condition that for all $\lambda \in \mathbb{Z}^2$, there must exist $\beta \in \mathbb{Z}^2$ such that

$$M\beta = PM\lambda. \tag{4.1}$$

(In particular, this coordinatizes $x \in L$ as $x = M\lambda$).

We want to prove that if P is a permutation, then for any $\lambda \in \mathbb{Z}^2$ we can always find a $\beta \in \mathbb{Z}^2$ that makes equation (4.1) true. We know P, M and λ, so we have to find β. This is a linear equation for β. We must show that the linear equation $M\beta = b$ has a unique solution for any $b = b_\lambda = PM\lambda$. The solution is unique if $\operatorname{rank} M$ is maximal, i.e. $\operatorname{rank} M = 2$. By inspection, M really has rank 2, so we only have to see if it always has a solution. From the Fundamental Theorem of Linear Algebra (part 2) [**Str80**], [**Str93**], the system (4.1) has a solution if and only if

$$\begin{aligned} & b \in \operatorname{colspan} M = \operatorname{Im} M \\ \iff \;& b \perp (\operatorname{colspan} M)^\perp \\ \iff \;& b^T y = 0 \text{ whenever } y \perp \operatorname{colspan} M \\ \iff \;& b^T y = 0 \text{ whenever } y^T M = 0. \end{aligned}$$

Since $[y_1 y_2 y_3] \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{smallmatrix}\right] = [y_1 - y_2, y_2 - y_3]$, we conclude that $y^T M = 0$ if and only if

$$0y = \alpha \mathbb{1} b^T y = \lambda^T M^T P^T \alpha \mathbb{1} = \alpha \lambda^T M^T P^T \mathbb{1} = \alpha \lambda^T M^T \mathbb{1};$$

but $M^T \mathbb{1} = 0$ because $\mathbb{1}$ is in the $\ker$ of M. $\square$

2. Laminated lattices

Define $\mathbb{L}_0 = \{L^0\}$, $L^0 = \{0\} = \mathbb{R}^0$ the zero dimensional lattice and $m := 4$ (usually m is 4 because then the spheres in the corresponding lattice packing have radius 1).

For $n > 0$, $\mathbb{L}_{n+1} = \{L_1^{n+1}, \ldots, L_{a_n}^{n+1}\}$ is the collection of $n+1$-dimensional lattices such that

(1) each L_i^{n+1} has constant minimal norm m
(2) each L_i^{n+1} contains at least one L_j^n as a sublattice
(3) each L_i^{n+1} has minimal determinant subject to (1), (2)

We will see which are these lattices:

$\mathbb{L}_1$: This lattice must be of the form $k\mathbb{Z}$. It needs minimal norm $m = 4$, so we must take $2\mathbb{Z}$, which satisfies (2) and (3). So the unique laminated lattice of rank 1 is $2\mathbb{Z}$.

$\mathbb{L}_2$: Taking $M = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ satisfies (1),(2) and (3), and yields $2\mathbb{Z}^2$. However, it is not necessary that our laminated lattice contain only integer points, the only condition is that it must contain $2\mathbb{Z}$. Thus, we have the two candidates $2\mathbb{Z}^2$ and A^2. We decide between the two by calculating the determinant corresponding to the generator matrices $M_1 = \left[\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right]$ and $M_2 = \left[\begin{smallmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{smallmatrix}\right]$:

$$\begin{aligned} \det 2\mathbb{Z}^2 &= \det M_1^T M_1 = \det \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 16; \\ \det A_2 &= \det M_2^T M_2 = \det \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = 12. \end{aligned}$$

We see that $\det A_2 < \det 2\mathbb{Z}^2$, which comes about because the area of the fundamental parallelopiped for A_2 is less than that of the square. So $\mathbb{L}_2$ is A_2.

We will see now how can we do the sphere packing:

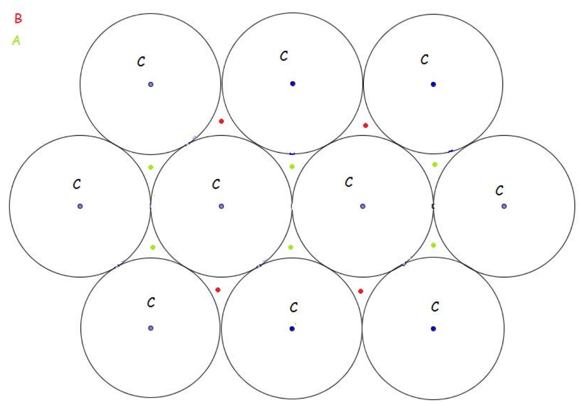


FIGURE 2. sphere packing

For the next layer we have two options, put them (the centers of the sphere) over the deep holes A or over the deep holes B. For the second layer we put, we will have the possibility of putting them over C (the centers of the spheres of the first layer). Each lattice obtained by snugly packing copies of A_2 is determied by the sequences ABAB.... (this is the hexagonal close packing(He attoms)) ABCABC....(this is the face-centered cubic lattice (A_3)) of equivalence classes of deep holes.

In each step there are two options to choose from, which makes uncountably many possibilities in total.

LECTURE 6

Some basic concepts in geometry and software; Introduction to Packings

Scribe: Ferran Dachs Cadefau

For the first part see [**Pfe11**].

1. Big numbers

Using computers, you can represent big numbers, you can do:

$\mathbb{N}, \mathbb{Z}$	`int, long int, long long int` or work with arbitrary precision (slower)	$-9.2 \cdot 10^{18}...9.2 \cdot 10^{18}$ `gmplib.org`
$\mathbb{Q}$	$\{(n,d) : n \in \mathbb{Z}, d \in \mathbb{N}_{>0}\}/\sim$ where $(n,d) \sim (n',d')$ iff $nd' = n'd$	`gmplib.org`
$\mathbb{R}$	algebraic numbers ok ($\sqrt{2}$, $\sqrt[4]{3}$), transcendental ones not (π, e)	`cgal.org, mpfr.org`
$\mathbb{C}, \mathbb{H}$	pairs or quadruples of reals	`#include <complex>`

But how to represent very big numbers? For example using 4 symbols what is the biggest number that you can represent?

$$10^{99} < 9^{999} < 9^{9^{9^{9}}} = 9 \uparrow 4 < 9 \uparrow 9 \uparrow 9 \uparrow 9 = 9 \uparrow\uparrow 4 < 9 \Uparrow 9 \Uparrow 9 \Uparrow 9$$

This is known as Knuth's up-arrow notation, to represent bigger numbers there are the Ackerman Function:

$$A(k,n) = n \Uparrow^{(k)} n$$
$$A(n,n) = n \Uparrow^{(n)} n$$

For example:

$$A(2) = 2 \Uparrow 2 = 2 \uparrow 2 \uparrow 2 = 2 \uparrow \left(2^2\right) = 2^{2^{2^{2}}}$$
$$A(3) = 3 \Uparrow^{(3)} 3 = 3 \Uparrow 3 \Uparrow 3 \Uparrow 3$$

Using this we can define $\alpha(x)$ as:

$$\alpha(x) = \max\{n \in \mathbb{N}_{\geqslant 1} : A(n) < x\}$$

It appears, for example, in the complexity of the lower envelope of a segment arrangement. In the plane is $\Omega(n\alpha(n))$

2. Projective geometry and Polarity

2.1. Projective geometry. We define the projective space as:

$$\begin{aligned} \mathbb{P}^n &= \{\text{lines through } 0 \in \mathbb{R}^{n+1}\} \\ &= S^n/\mathbb{Z}_2 \end{aligned}$$

We are identifying the antipodes: x and $-x$. But we have a problem: $\mathbb{P}^1$, $\mathbb{P}^3$ are orientable, but for example $\mathbb{P}^2$ not: we can't define interior (inside/outside), therefore we can't define convex.

In $\mathbb{P}^2$ The line through points p_1 and p_2:

- p_1 lies on l: $p_1 \perp l$
- p_2 lies on l: $p_2 \perp l$

Therefore: $l = \lambda \cdot p_1 \times p_2$ ($\times$ is the cross-product of vectors in $\mathbb{R}^3$, in higher dimensions is the determinant).

The point q on lines l_1 and l_2:

- q lies on l_1: $q \perp l_1$
- q lies on l_2: $q \perp l_2$

Therefore: $q = \lambda \cdot l_1 \times l_2$ (the cross-product of vectors in $\mathbb{R}^3$, in higher dimensions is the determinant).

Computationally, we can intersect lines and join points, in homogeneous coordinates or in Cartesian coordinates. In the first case we have:

```
void intersect_lines(const vec_t& l1, const vec_t& l2, vec_t& p)
{
  cross_product(l1, l2, p);
}
void join_points(const vec_t& p1, const vec_t& p2, vec_t& l)
{
  cross_product(p1, p2, l);
}
void cross_product(const vec_t& l1, const vec_t& l2, vec_t& p)
{
  p[0] = l1[1]*l2[2] - l1[2]*l2[1];
  p[1] = -l1[0]*l2[2] + l1[2]*l2[0];
  p[2] = l1[0]*l2[1] - l1[1]*l2[0];
}
```

This code is **correct** (calculates exactly what it should), **efficient** (No extraneous copying (&) and reuse of code), and **robust** (No influence of rounding errors and it handles all cases, even degenerate ones).

But, in Cartesian coordinates. A line is now $y = kx + d$, stored as a vector (k, d).

```
bool intersect_lines(const vec_t& l1, const vec_t& l2, vec_t& p)
 {
if (l1[0]==l2[0]) {
if (l1[1]==l2[1]) {
return COINCIDENT_LINES;
}
return PARALLEL_LINES;
```

```
}
p[0] = (l2[1]-l1[1])/(l1[0]-l2[0]);
p[1] = l1[0]*p[0] + l1[1];
}
```

This code is *somewhat* **efficient** (Again, no copying, but: no reuse of code for join_points), **not robust** (It's unstable numerically (==, /)), and *not even* **correct** (It doesn't handle parallel lines, but that's Euclidean geometry's fault).

2.2. Polarity. It's related to Projective duality, points are dual to hyperplanes:

$$\begin{array}{ccc} \mathbb{P}^n(\mathbb{R}) & \xrightarrow{d} & (\mathbb{P}^n(\mathbb{R}))^\star \\ p & \longmapsto & p^\star \end{array}$$

A duality is an involutory order-anti-isomorphism (Involution: $d^2 = \mathrm{id}$, isomorphism: bijection collineation, order-anti: $p \in l$ implies $p \supset l$).

The polarity is the same except by convexs.

Polarity send: points to half-spaces. Given $a \in \mathbb{R}^n$:

$$a \mapsto H_a = \{x \in \mathbb{R}^d : \langle a, x\rangle \leqslant 1\} \leftrightarrow x \supset (\mathbb{R}^d)^\star$$

Equivalently, they identify points with hyperplanes via homogeneous coordinates.

For example, if we take $p = (a : b : 1) \in \overrightarrow{\mathbb{P}^2}$ a point, then $p^\star = (a : b : 1) \in \left(\overrightarrow{\mathbb{P}^2}\right)^\star$ a line, and $p^{\star\star} = p$.

$P \in l \Leftrightarrow p \perp l \Leftrightarrow p^\star \perp l^\star \Leftrightarrow l^\star \perp p^\star \Leftrightarrow l^\star \in p^\star$

3. Polymake

Polymake is a free program in perl. For example we can define a cube in dimension 3 as:

```
polytope > $p=cube(3);
```

Another thing is to know properties about a polytope. For example the coordinates of the vertices:

```
polytope > print $p->VERTICES;
1 -1 -1 -1
1 1 -1 -1
1 -1 1 -1
1 1 1 -1
1 -1 -1 1
1 1 -1 1
1 -1 1 1
1 1 1 1
```

Or the number of faces of each dimension:

```
polytope > print $p->F_VECTOR;
8 12 6
```

Or the vertices in each facet:

```
polytope > print $p->VERTICES_IN_FACETS;
{0 2 4 6}
{1 3 5 7}
{0 1 4 5}
{2 3 6 7}
{0 1 2 3}
```

```
{4 5 6 7}
```

To see a representation of the polytope you can execute:

```
polytope > $p-> VISUAL;
```

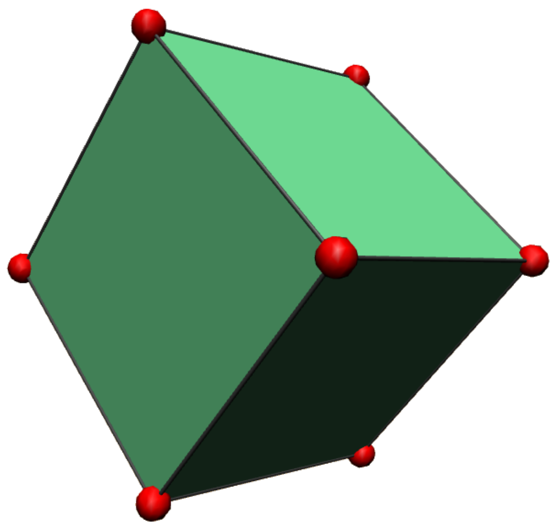


FIGURE 1. The representation of the cube in dimension 3 due with Polymake

To see a representation of graph of the polytope you can execute:

```
polytope > $p-> VISUAL_FACE_LATTICE;
```

To see the equations of the facets you can type:

```
polytope > print $p->FACETS;
1 1 0 0
1 -1 0 0
1 0 1 0
1 0 -1 0
1 0 0 1
1 0 0 -1
```

You must interpret as the following:

If you take $(1:1:0:0)$ gives you the facet:

$$\left\langle (1:1:0:0) \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} \right\rangle \geqslant 0$$

equivalently: $1+x \geqslant 0$ or $x \geqslant -1$. If you take $(1:-1:0:0)$ gives you the facet: $1-x \geqslant 0$ or $1 \geqslant x$.

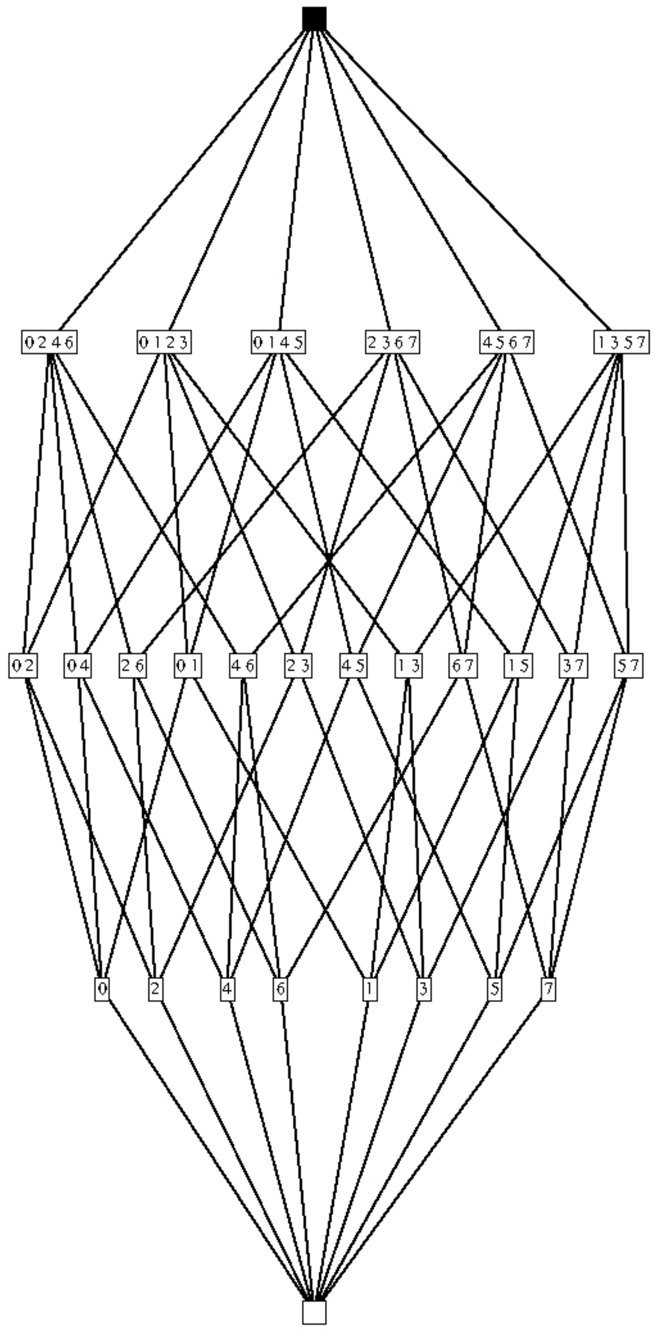


FIGURE 2. The representation of the graph of the cube due with Polymake.

We can polarize a polytope, for example:

```
polytope > $q=polarize($p);
```

doing this, if we want to print the vertices of this new polytope, as we expected, gives the facets of the cube:

```
polytope > print $q->VERTICES;
1 -1 0 0
1 1 0 0
1 0 -1 0
1 0 1 0
1 0 0 -1
1 0 0 1
```

Another thing that we can do is make Voronoi Diagrams. For example if we want to know the Voronoi Diagram of the points $(1,1),(0,1),(-1,1),(1,-1),(0,-1)$ and $(-1,-1)$:

```
$VD = new VoronoiDiagram(SITES=>[[1,1,1],[1,0,1],[1,-1,1],
[1,1,-1],[1,0,-1],[1,-1,-1]]);
```

If we want to see a graphical representation:

```
$VD->VISUAL_VORONOI;
```

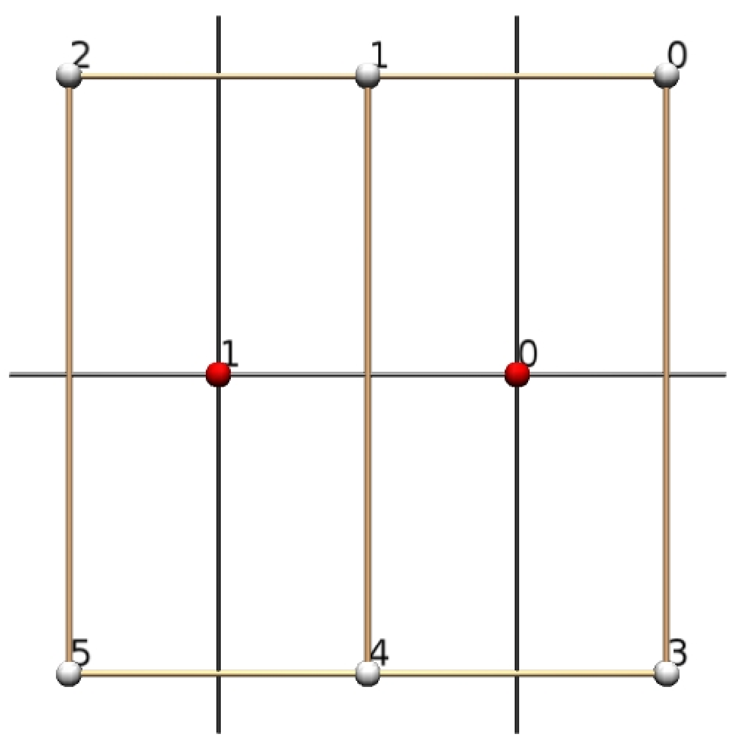


FIGURE 3. The representation of one Voronoi Diagram with Polymake.

Other properties are, for example the facets equation:
```
polytope > print $VD->FACETS;
2 -2 -2 1
1 0 -2 1
2 2 -2 1
2 -2 2 1
1 0 2 1
2 2 2 1
1 0 0 0
```
or the vertices of the diagram:
```
polytope > print $VD->VERTICES;
0 0 1 2
0 1 0 2
1 1/2 0 -1
0 -1 0 2
0 0 -1 2
1 -1/2 0 -1
```

4. Voronoi Cells in lattices

Given a lattice $L \subset \mathbb{R}^d$, find the facets of

$$\mathrm{Vor}(0) = \{x \in \mathbb{R}^d : \|x - 0\|^2 \leqslant \|x - v\|^2, \forall v \in L\}$$

Definition 6.1. $v \in L$ is *relevant* if the bisector of v and 0 contains a full dimensional face (i.e. a facet) of $\mathrm{Vor}(0)$.

In $\mathbb{Z}^3$ we have a cube and the polytope defined by the relevant points is an octahedron.

Observation 6.2. If the relevant vectors are precisely the minimal vector then $\mathrm{Vor}(0)$ is polar dual to the vertex figure of 0 in L.

Theorem 6.3 (Georges Voronoi, 1908)**.** *A non-zero vector $v \in L$ is relevant if and only if $\pm v$ are the only shortest vectors in the coset $v + 2L$.*

PROOF. $\Longrightarrow$
Suppose that $v, w \in L$ satisfy

$$w \in v + 2L \tag{6.1}$$

but

$$w \neq \pm v \tag{6.2}$$

and

$$\langle w, w \rangle \leqslant \langle v, v \rangle \tag{6.3}$$

Set:

$$t = \tfrac{1}{2}(v + w) \text{ and } u = \tfrac{1}{2}(v - w)$$

Using (6.1) we have that $t, u \neq 0$ and using (6.2), $t, u \in L$. We have:

$$H_v = \{x \in \mathbb{R}^d : \langle x, v\rangle \leqslant \frac{1}{2}\langle v, v\rangle\}$$

Let $x \in H_t \cap H_u$. Then:

$$\langle x, t\rangle \leqslant \frac{1}{2}\langle t, t\rangle$$

and

$$\langle x, u\rangle \leqslant \frac{1}{2}\langle u, u\rangle$$

Adding, we have:

$$\begin{array}{rcl} \langle x, v\rangle & \leqslant & \frac{1}{8}\left(\langle v, v\rangle + \langle w, w\rangle + 2\langle v, w\rangle + \langle v, v\rangle + \langle w, w\rangle - \langle v, w\rangle\right) \\ & = & \frac{1}{4}\left(\langle v, v\rangle + \langle w, w\rangle\right) \\ & \leqslant & \frac{1}{2}\langle v, v\rangle \end{array}$$

Were in the last one we have used (6.3). So $x \in H_v$, therefore v is not relevant.

$\Longleftarrow$ Suppose $v \in L$ is not relevant. Then there exists some $w \in L$. $w \neq 0$, $w \neq \pm v$ with:

$$\left\langle \frac{1}{2}v, w\right\rangle \geqslant \frac{1}{2}\langle w, w\rangle \tag{6.4}$$

and $\frac{1}{2}v \notin H_w$.Then

$$\begin{array}{rcl} \|v - 2w\|^2 & = & \langle v - 2w, v - 2w\rangle \\ & = & \langle v, v\rangle - 4\langle v, w\rangle + 4\langle w, w\rangle \\ & \leqslant & \langle v, v\rangle - 4\langle v, w\rangle + 4\langle v, w\rangle \\ & = & \langle v, v\rangle \end{array}$$

where in the inequality we have used (6.4), and is in $v + 2L$, not 0, $w \in L$.

□

LECTURE 8

Introduction to orbifolds

Scribe: Ane Santos

Definition 8.1. Informally, an *orbifold* is the quotient of a manifold (here, the Euclidean plane) by the action of a group.

torus	$\longleftrightarrow$	o
holes	$\longleftrightarrow$	*
non-orientability	$\longleftrightarrow$	×
boundary singularity	$\longleftrightarrow$	*n
cone point of order n	$\longleftrightarrow$	n

Theorem 8.2 (Magic theorem for the sphere)**.** *The total cost of the signature of any spherical group is* $\$2 - \frac{2}{g}$*, where g denotes the total number of symmetries.*

The Magic theorem in the plane is a special case because the number of symmetries in a plane is infinite, so the cost is always 2.

There are 14 spherical symmetry groups: $m, n \geq 1$

*532	*432	*332	*22n	*mn
		3*2	2*n	n*
				n×
532	432	332	22n	mn

If $n \to \infty$ and $m \to \infty$ in ***22n**, ***mn**, **2*n**, **n***, **n×**, **22n** and **mn** we get the 7 possible groups of friezes (cenefas).

We spent almost the entire lecture with scissors and tape, cutting out the orbifolds corresponding to the tesselations in [**CBGS08**].

LECTURE 9

Orbifolds

Scribe: Roger Ten

1. Defining an orbifold

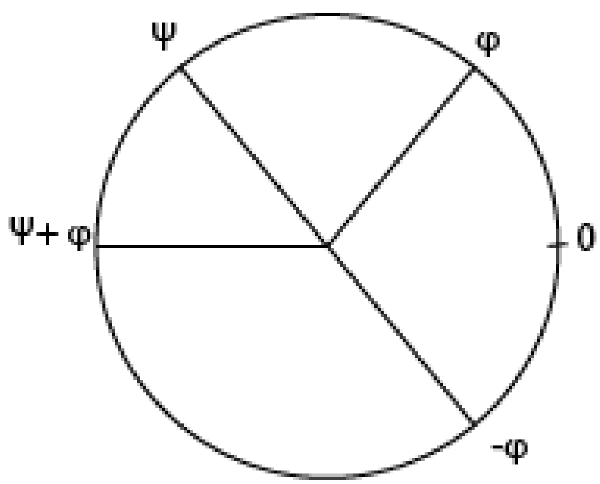
In this section X denotes a topological space and G denotes a topological group. So must begin by defining what is a topological group.

Definition 9.1. (1) A *topological group* is a topological space that is simultaneously a group such that, the group operations are continuous.

(2) A topological space X is called *G-space* if a topological group G acts on X via a continuous map $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$ such that $g(hx) = (gh)x$ and $1_G x = x$.

Example 9.2.
- *Lie groups:* $O(n), SO(n)$
- S^1 *is a topological space and it is also a group via:*

$$\begin{array}{rccc} +: & S^1 \times S^1 & \to & S^1 \\ & (\varphi, \psi) & \to & \varphi + \psi \end{array}$$



Now is coming a list of definitions

Definition 9.3. Let $x \in X$ be a point.

(1) $G_x = \{g \in G : gx = x\} = Stab_G(x)$ is called the *stabilizer of* x or the isotropy subgroup of x. G_x is a subgroup of G ($G_x \leqslant G$).
(2) $G(x) = \{gx : g \in G\} \subset X$ is the *orbit of* x.
(3) The action of G on X is *free* if $G_x = \{1\} \forall x \in X$.

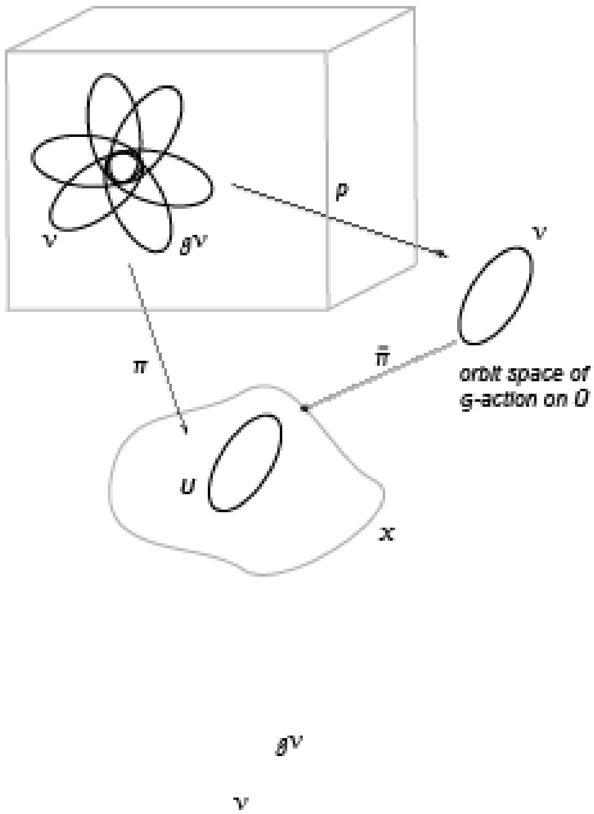
(4) The action of G on X is *transitive* if $G(x) = X \forall x \in X$, i.e., there exist only one orbit.
(5) The map $G/G_x \to G(x)$ is a continuous bijection
(6) The orbit space X/G is the set of all orbits in X. (It is a topological space with quocient topology).
(7) A map $f : X \to Y$ of G-spaces is *G-equivariant* (or G-map) if $f(gx) = g(f(x)); g \in G$. so the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Definition 9.4. An *orbifold chart* on a topological space X is tuple $(\tilde{\mathcal{U}}, G, \mathcal{U}, \pi)$, such that:

- $\mathcal{U} \subset X$ is an open subset (neighborhood).
- $\tilde{\mathcal{U}} \subset \mathbb{R}^n$ is an open subset.
- G is a finite group of homomorphisms of $\tilde{\mathcal{U}}$.
- $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$ can be factorized as $\pi = \tilde{\pi} \circ p$, where $p : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}/G$ is the orbit map and $\tilde{\pi} : \tilde{\mathcal{U}}/G \to \tilde{\mathcal{U}}$ is an homomorphism.

$$\begin{tikzcd} G & \underset{act\ on}{\circlearrowleft} & \tilde{\mathcal{U}} \arrow[dd, "\pi"] \arrow[dr, "p"] & \subset & \mathbb{R}^n \\ & & & \tilde{\mathcal{U}}/G \arrow[dl, "\tilde{\pi}"] & \\ & & \mathcal{U} & \subset & X \end{tikzcd}$$



Definition 9.5. Two orbifold charts are *compatible* if for all $\tilde{u_i} \in \tilde{\mathcal{U}}_i, i = 1, 2$ such that $\pi_1(\tilde{u_1}) = \pi_2(\tilde{u_2})$ there exists a homomorphism $h : \tilde{\mathcal{V}}_1 \to \tilde{\mathcal{V}}_2$, where $\tilde{\mathcal{V}}_i$ is a neighborhood of $\tilde{u_i}$ in $\tilde{\mathcal{U}}_i$, such that $\pi_1 = \pi_2 \circ h$ on $\tilde{\mathcal{V}}_1$

$$\begin{tikzcd}
\tilde{\mathcal{U}}_2 \arrow[d, "p_2"] & \supset & \tilde{\mathcal{V}}_2 \arrow[dr, "\pi_2"] & \circlearrowright & \tilde{\mathcal{V}}_1 \arrow[ll, "h"'] \arrow[dl, "\pi_1"] & \subset & \tilde{\mathcal{U}}_1 \arrow[d, "p_1"] \\
\tilde{\mathcal{U}}_2/G \arrow[rrr, "\tilde{\pi}_2"] & & & \tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2 & & & \tilde{\mathcal{U}}_1/G0 \arrow[lll, "\tilde{\pi}_1"']
\end{tikzcd}$$

Remark 9.6. $G_i = \{1\}$ yields manifolds.

Definition 9.7. An *orbifold atlas* is a collection of compatible orbifold charts that cover X.
An *orbifold* Q is an underlying Hausdorff topological space $|Q|$ with an orbifold atlas.

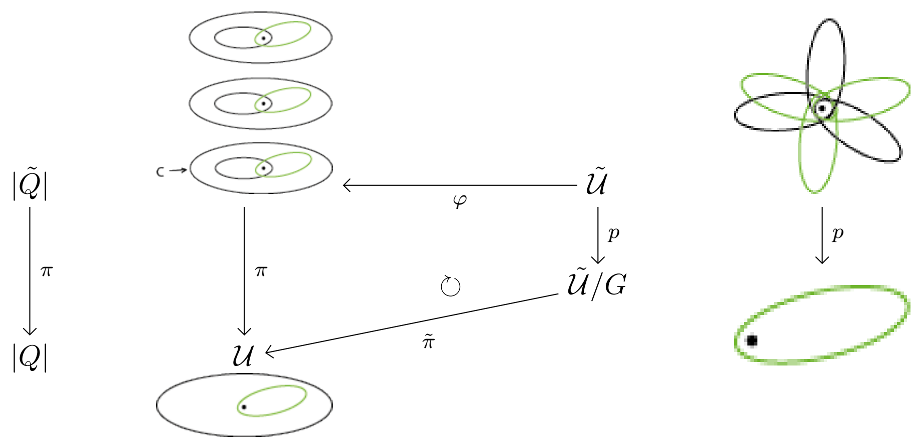
2. Covering orbifolds

A *covering* of a topological space X is a topological space Y, together with a continuous surjective projection $\pi : Y \to X$ such that for every $x \in X$ exist $\mathcal{U}_x$ open neighborhood of x such that the pre-image $\pi^{-1}(\mathcal{U}_x)$ is a disjoint union of copies of $\mathcal{U}_x$.

Definition 9.8. A *covering orbifold* of an orbifold Q is an orbifold $\tilde{Q}$ with a projection $\pi : |\tilde{Q}| \to |Q|$ between the underlying spaces with the following property:

For any $x \in |Q|$ there exist a neighborhood $\mathcal{U} \cong \tilde{\mathcal{U}}/G$; $\mathcal{U} \subset \mathbb{R}^n$ such that each connected component C of $\pi^{-1}(\mathcal{U})$ is homeomorphism to $\tilde{\mathcal{U}}/\Gamma_i$ for some subgroup $\Gamma_i \leqslant G$.

Remark 9.9. This homeomorphism φ must respect both projections, namely π and $p_i : \tilde{\mathcal{U}}/G \to \tilde{\mathcal{U}}/\Gamma_i$.



Definition 9.10. An orbifold is called *good* or developable if there exists some manifold that covers it. Otherwise, it is *bad* or not developable

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