Discrete and Algorithmic Geometry 2013(Part 2)

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Version of November 30, 2013

This is the preliminary version of the lecture notes for the second part of *Discrete and Algorithmic Geometry* (Universitat Politècnica de Catalunya), held in the fall semester of 2013 by Ferran Hurtado and Julian Pfeifle.

These notes are fruit of the collaborative effort of all participating students, who have taken turns in assembling this text. The name of each scribe figures in each corresponding section.

Suggestions for improvements will always be gladly received by `julian.pfeifle@upc.edu`.

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LECTURE 1

Convex Polytopes

Scribe: Cecilia Girón

A convex polytope can be defined in two different ways:

- V*-polytope* (discrete geometry): is the convex hull of the finite non-empty point set in $\mathbf{R}^d$.
- H*-polytope* (linear/integer optimization): An H-polyhedron is an intersection of a finite number of linear half spaces in some $\mathbf{R}^d$, if non-empty. And an H-polytope is a bounded H-polyhedron.

1. Faces

One of the properties studied about polytopes is their faces. A **face** F of a polytope P is a set of the form:

$$F = \{x \in \mathbb{R}^d :< a, x >= n\} \cap P$$

where $a \in (\mathbb{R}^d)^*$ (dual space), $b \in \mathbb{R}$ and $P \subseteq \{x \in \mathbb{R}^d :< a, x > \leq b\} \longleftrightarrow$ The inequality $< a, x > \leq b$ is valid for P. Notice that P is actually a face of itself.

We can also study the dimension $\dim F$ of a face. Let P be a d dimensional polytope, then if a face F of P has dimension:

i) $d-1$, it is called a **facet**.
ii) $d-2$, it is called a **ridge**.
iii) 1, it is called an **edge**.
iv) 0, it is called a **vertex**.
v) -1 then $F = \emptyset$.

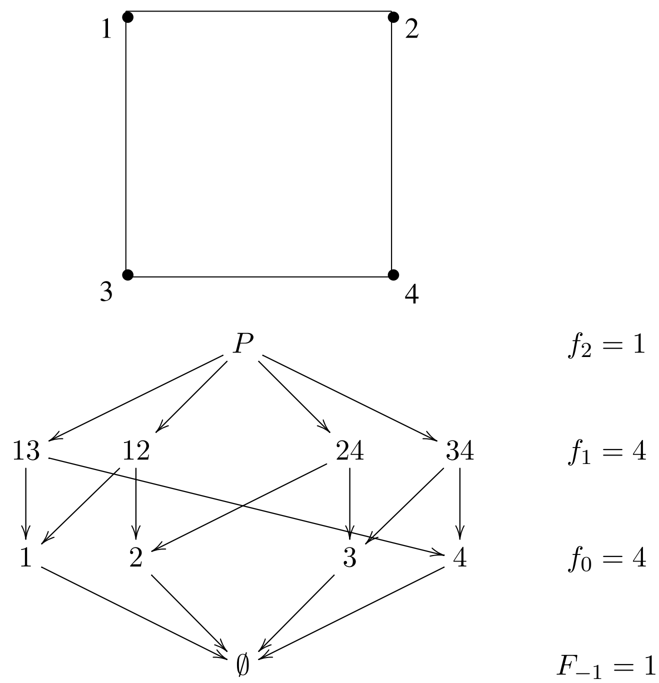
The partially ordered set of all faces $\mathcal{F}(P)$ of a convex polytope P forms a Eulerian lattice called **face lattice**. The face lattice can be used, for instance, to count the number of faces of same dimension:

$$f_i = \#\{ \text{ faces } F \text{ of } P \text{ with } \dim F = i\}$$

EXAMPLE. Let P be a cube in two dimensions, so a square. Notice that $\dim P = 2$, for every edge $ij \in P$ $\dim(ij) = 1$ and for every vertex i $\dim i = 0$, for $i, j = 1, 2, 3, 4$ and $i \neq j$. With this information we can construct the face lattice and study some properties of P.

♣

EXAMPLE.Let's study now the dimension of the faces of a hypercube of dimension d $\square^d = \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1, i = 1, \cdots, d\}$: $f_{-1}(\square^d) = 1, f_0(\square^d) = 2^d, f_{d-1}(\square^d) = 2d$ and $f_d(\square^d) = 1$. Notice that the radius r from the center of the cube to one of its vertices is $r = \sqrt{d} - 1$, thus the exterior circus of the polytope has radio r and the interior radio 1.



d	2	3	4	5	$\cdots$	100	$\cdots$	10^{100}
$r = \sqrt{d} - 1$	$\sqrt{2} - 1$	$\sqrt{3} - 1$	1	$\sqrt{5} - 1$	$\cdots$	9	$\cdots$	$10^{50} - 1$
f_0	4	8	16	32	$\cdots$	2^{100}	$\cdots$	$2^{10^{100}}$
f_1	4	6	8	10	$\cdots$	200	$\cdots$	$2 \cdot 10^{100}$

♣

An other property that can be studied about the faces of convex polytopes is whether they are a simplex or not. Let P be a polytope such that $\dim P = d$ and $\mathcal{F}(p) = k + 1$. It is said to be **simplicial** if it is k-simplex, i.e. if each of its faces is a simplex; and it is called **simple** if each of its vertices is contained in exactly d faces where $\dim P = d$.

1.1. Exercises done during the lecture 8/11/2013. Each one includes one.

Alex Alvarez.: The set of vertices is the following:

```
(1 : -1 : -1 : -1 : 0 : 0)
(1 : 1 : -1 : -1 : 0 : 0)
(1 : -1 : 1 : -1 : 0 : 0)
(1 : 1 : 1 : -1 : 0 : 0)
(1 : 0 : 0 : 1 : 0 : 0)
(1 : 0 : 0 : 1 : 1 : 0)
(1 : 0 : 0 : 1 : 0 : 1)
```

And this can be obtained as the join of a 2-cube with a 2-simplex. If we construct that in Polymake:

```
polytope > $p = join_polytopes(cube(2), simplex(2));

polytope > print($p->VERTICES);
1 -1 -1 -1 0 0
1 1 -1 -1 0 0
1 -1 1 -1 0 0
1 1 1 -1 0 0
1 0 0 1 0 0
1 0 0 1 1 0
1 0 0 1 0 1
```

Thus, we can use the program to see the number of facets and the vertices in each facet:

```
polytope > print $p->N_FACETS;
polymake: used package cddlib
  Implementation of the double description method of Motzkin et al.
  Copyright by Komei Fukuda.
  http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html

polymake: used package lrslib
  Implementation of the reverse search algorithm of Avis and Fukuda.
  Copyright by David Avis.
  http://cgm.cs.mcgill.ca/~avis/lrs.html

7
polytope > print $p->POINTS_IN_FACETS;
{0 2 4 5 6}
{0 1 4 5 6}
{1 3 4 5 6}
{2 3 4 5 6}
{0 1 2 3 5 6}
{0 1 2 3 4 6}
{0 1 2 3 4 5}
```

If we focus now in the graph of the polytope, we can check the number of edges and we can also see it:

```
polytope > print $p->GRAPH->N_NODES;
7
polytope > print $p->GRAPH->N_EDGES;
19
polytope > print $p->GRAPH->VISUAL;
```

Therefore, we can see that the last two vertices are not connected, but they are connected to all the other vertices.

Exercise ?? (team members):

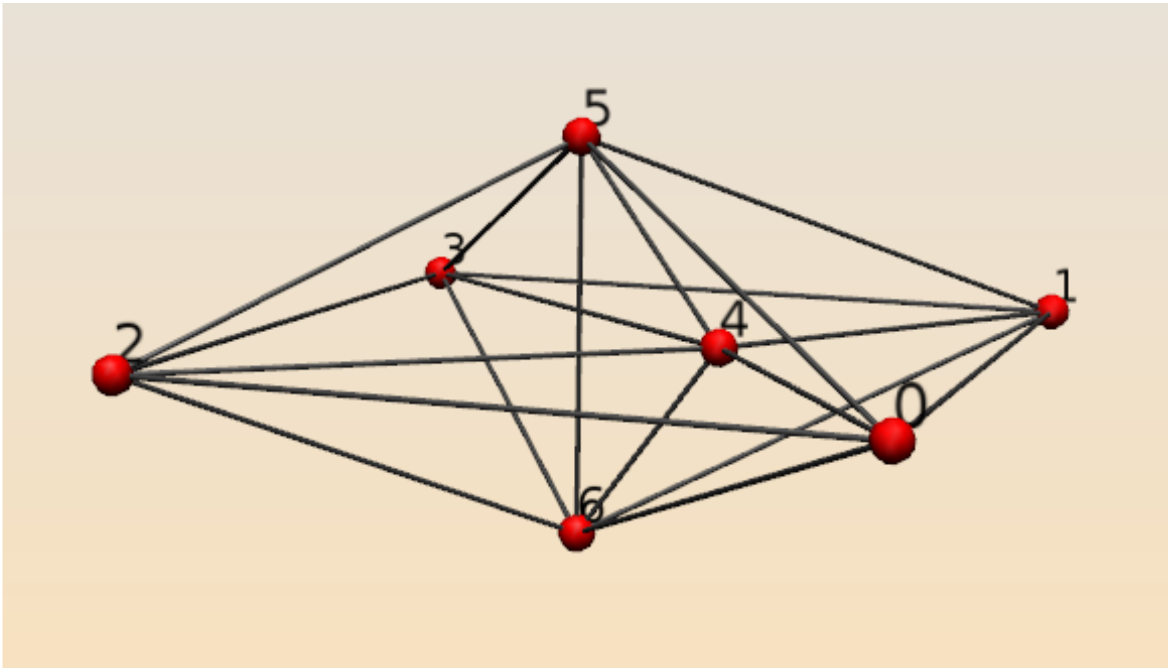


FIGURE 1. The graph of the polytope

LECTURE 2

Asymptotic f-vectors of families of polytopes

Scribe: Cecilia Girón

1. The Unimodality Conjecture

In this section we are going to study the *unimodality conjecture* which says that there exists an $l = P(L) \in \mathbb{N}$ such that $f_0 \leq f_1 \leq \cdots \leq f_l \leq f_{l+1} \leq \cdots \leq f_{d-1} \leq f_d$. We woould like to know if it is true.

First, we define the f**-vector** as the vector of the form $(f_0, f_1, \cdots, f_{d-1})$ where f_i is as defined before in (1). We will say that it is a **flag** f**-vector** $(f_s)_s = [d]$ such that f_s count the number of flags $F_{i_1} \subset F_{i_2} \subset \cdots \subset F_{i_k}$ where $s = \{i_1, i_2, \cdots, i_k\}$ and $\dim F_{i_k} = i_k$ [1].

The unimodal conjecture described before is known to be false for simplical polytopes of dimension $d \geq 19$ and for non-simplicial polytopes of dimension $d \geq 8$. The following conjecture is not known to be false.

Restricted unimodal conjecture (Anders Bjorner):

$$f_0 \leq f_1 \leq \cdots \leq f_{\lfloor \frac{d-1}{4} \rfloor}$$
$$f_{\lfloor \frac{3(d-1)}{4} \rfloor} \geq \cdots \geq f_{d-1}$$

Intuitively we are sure that there is no way this conjecture could be false, but there is not proof of this. We don't even know if $f_k \geq \frac{1}{10000} \min\{f_0, f_{d-1}\}$ is true.

1.1. Exercises worked on during the lecture 11/11/2013. Each team includes one.

Exercise 1b (Alex Alvarez and Ivan Geffner): *Using the simple form $n! \approx \left(\frac{n}{e}\right)^n$ of Stirling's formula, show that $\phi_d(x) := \log\binom{d}{xd}$ is asymptotically proportional to $-x\log x - (1-x)\log(1-x)$, for $x \in (0,1)$ and $d \to \infty$. Discuss the real function ϕ_d on $[0,1]$.*

Using the simple form of Stirling's formula, we obtain the following form:

$$\binom{d}{xd} \approx \frac{d^d}{xd^{xd}(d-xd)^{d-xd}}$$

So applying the logarithm, we get

$$\begin{aligned}
\phi_d(x) &= d\log d - xd\log xd - (d-xd)\log(d-xd) \\
&= d\log d - xd\log x - xd\log d - (d-xd)(\log d + \log(1-x)) \\
&= d(-x\log x - (1-x)\log(1-x))
\end{aligned}$$

[1] You can also read about cd-index

Thus, the first part of the exercise is proven. Let us study now the shape of the function.

The function clearly vanishes when x tends to 0 and 1 and in this interval is non-negative. The first derivative of the function is

$$\frac{d}{dx}(-(1-x)\log(1-x) - x\log(x)) = \log(1-x) - \log(x)$$

and therefore, there is only one point in which derivative vanishes, that is $x = \frac{1}{2}$. Now, the second derivative is

$$\frac{d^2}{dx^2}\log(1-x) - \log(x) = \frac{1}{x(x-1)}$$

We can conclude then that the point is a maximum, so we have characterized the shape of the function.

Exercise 2b (Borja and Cecilia): *Using the simple form* $n! \approx \left(\frac{n}{e}\right)^n$ *of Stirling's formula, show that* $\psi_d(x) := d(1-x) + \log\binom{d}{xd}$ *is asymptotically proportional to* $1 - x - x\log x - (1-x)\log(1-x)$*, where* $\log = \log_2$ *denotes the binary logarithm. Find an approximation to the maximum of this function on* $(0,1)$.

For the first part of the exercise, by applying the Stirling's formula, in the binomial for of the given function:

$$\binom{d}{xd} = \left(\frac{d}{xd^x(d(1-x))^{1-x}}\right)^d$$

Then, substituting in $\psi_d(x)$:

$$\begin{aligned}
\psi_d(x) &= d(1-x) + \log\left(\frac{d}{xd^x(d(1-x))^{1-x}}\right)^d \\
&= d(1-x) + d\left(\log d - x\log x - x\log d - (1-x)\log d - (1-x)\log(1-x)\right) \\
&= d(1-x-x\log x - (1-x)\log(1-x))
\end{aligned}$$

Hence, $\psi_d(x)$ is asymptotically proportional to $1 - x - x\log x - (1-x)\log(1-x)$

For the second part of the exercise, in order to find the maximum of the function

$$f(x) = 1 - x - x\log(x) - (1-x)\log(1-x)$$

in (0,1) we will the points that have first derivative equal to 0, that is x such that $f'(x) = 0$, and computing $f'(x)$ we get:

$$f'(x) = -1 - (\log x + 1) - (-\log(1-x) - 1) = \log\left(\frac{1-x}{x}\right) - 1$$

Now the points that make $f'(x) = 0$ are the ones that make $\log(\frac{1-x}{x}) = 1$, which is the same as x such that $\frac{1-x}{x} = e$, which translates into:

$$x_{\max} = \frac{1}{e+1}$$

The shape of this function is a growing function from $x = 0$ starting at $f(0) = 1$ to $x = x_{\max}$, where it gets it's maximum, that is approximated by $f(x_max) \approx 1.0414$ and then decreases to 0 at $x = 1$.

2. Operations on polytopes

- **Cartesian (direct) product** $P \times Q$.
- **Direct sum** $P^d \oplus q^e \subset \mathbb{R}^{d+e}$.
- $P * Q \subset \mathbb{R}^{d+e+1}$. It is like $\oplus$ but the subspaces are skew (i.e. affine and they have no point on common). For example $\square^1 * \square^1 = Pyr(P)$.

EXAMPLE: Given $f_k(P)$, calculate the k-th entry of $Pyr(P)$:

$$
\begin{aligned}
f(P) &= (f_0, f_1, \cdots, f_{d-1} \\
f_k(Pyr(P)) &= (f_0 + 1, f_1 + f_0, f_2 + f_1, \cdots, f_{d-2} + f_{d-3}, f_{d-1} + f_{d-2}, 1 + f_{d-1})
\end{aligned}
$$

♣

- **Connected sum** $P^d \# Q^d$ where P has as simplicial face f and Q has a simplicial face G.

This last operation is used to join the asymptotic function $f(\square^d)$ and its dual $f(\diamondsuit^d)$. To make it work, since $\square^d$ has no triangulations in its faces, it is enough to cut away one vertex and, this way, get a simplex. Merging both functions using the connected sum gives place to a new function which is a non-unimodal function.

Scribes 2013

1. Alex Alvarez

I have done a double degree in Informatics Engineering and Mathematics at the UPC and during these years I have participated in programming contests both individually and representing the UPC. I am mainly interested in Algorithms and Data Structures and I would like to start a PhD in those topics next year, but I also enjoy learning about Discrete Mathematics, in particular Combinatorics and Graph Theory.

2. Cecilia Girón Albert

I've got my Degree in Mathematics at the Universidad Autónoma de Madrid, which I completed my fifth and sixth semesters at the University of Jyväskyla (Finland) as an Erasmus student. My degree has been mainly focused on subjects like analysis, statistics and numerical methods, although I am more interested in algebra and graph theory.

I decided to study the Master in Advanced Mathematics and Mathematical Engineering to keep developing my mathematics skills in some theoretical subjects that can be applied in real life problems. Therefore, I believe that this course is a great opportunity to learn more about algorithmic and computing science and even more importantly, it may help me to find out what field I would like to focus on in the future.

3. Anna Somoza

I have recently finished a degree in Mathematics at the Universitat Politècnica de Catalunya. During this degree I developed a great interest in Algebra fields. In particular, I took the optional subjects *Algebraic Geometry*, *Algebraic Topology* and *Galois Theory* and I wrote my Final Degree Thesis on a topic of Number Theory.

Now I'm taking the Master in Advanced Mathematics and Mathematical Engineering to develop my knowledge in these and other fields, and I my aim is to start a PhD in Number Theory next year. I enroled this subject because I have allways liked both computer science and geometry, and it seemed to be interesting. Therefore, I would be interested in the topic related to algebraic geometry.

4. Daniel Torres

My name is Daniel Torres and I am graduate in mathematics in UPC. Along the degree I have developed much interest in fields of topology, algebra and geometry, and some loathing to study (not to programming) numerical methods and modelling. I decided study this master, and particularly this subject, for expand my knowledge about my interests.

More concretely, I am doing this course with the hope it shows me about geometry.

5. Borja Elizalde

I have finished the degree in Mathematics at the UPC and I have also finished Industrial Engineering also at the UP (not CFIS).

I am interested in Number Theory and Algebraic Geometry in general, because the problems I like solving and thinking on usually belong to these fields. I am not sure at all about how I want to develop my professional career.

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