Discrete and Algorithmic Geometry

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Sheet 5

UNDER CONSTRUCTION

(1) Let $P \subset \mathbb{R}^d$ be a convex polytope and $v \in \mathbb{R}^d$. Then $\operatorname{stk} P = \operatorname{conv}(P \cup \{v\})$ is obtained from P by *stacking* on the facet F if v is *beyond* exactly F and *beneath* all other facets: If $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ where the rows of A are $a_1, \dots, a_m$, it should happen that $\langle a_i, v\rangle > b_i$ for exactly one i, while $\langle a_j, v\rangle \leq b_j$ for $j \neq i$.

(a) For simplicial d-polytopes P, derive a formula for $f_k(\operatorname{stk} P)$, $0 \leq k < d-1$, in terms of $f_k(P)$ and $f_{k-1}(\Delta^{d-1})$, where Δ^{d-1} is the $(d-1)$-dimensional simplex.

(b) Do the same for $\operatorname{stk}^N(P)$, the polytope obtained from P by N stackings.

(c) Prove Danzer's result from 1964 that for large enough d and suitable N, the Unimodality Conjecture for f-vectors fails for N-fold stacked cross-polytopes.

(d) Do better, for example by using cyclic polytopes. What is the lowest dimension for which you can make the Unimodality Conjecture fail? Can you beat 20?

Software

(1) To test the code in `face_selector.cc` we wrote in class, complete the skeleton file `selected_face.cc` to a `polymake` client that outputs the (indices of the vertices on the) minimal face selected by a linear function on a given polytope.

(a) Test your two programs. For example, using `$p=cube(3);`, the command

```
print selected_face($p, face_selector($p, new Set([0,1])));
```

should return $\{0,1\}$; while

```
print selected_face($p, face_selector($p, new Set([0,7])));
```

should return $\{0,1,2,3,4,5,6,7\}$. Can you think of more, meaningful tests?

(b) Use your new client to calculate `selected_face($p,$a)`, where `$p=cube($d)`, `$a=face_selector($p,new Set([0,1]))`, and `$d` ≥ 3 varies. How large can you make `$d` and still get an answer in 10 minutes of computation? What (if anything) changes if instead you use `$p=polarize(center(cyclic($d,2*$d)))`?

Turning in your work

Put your answers into a .pdf file. To turn it in, use `gpg` and the public key `julian.gpg.pub` in the `github` repository to create an encrypted copy that is only readable by me. Then commit and push this encrypted file to the repository.