1. Sheet 2

(1) Show that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F. In particular, P has only finitely many faces.

By proposition 2.3, F is a polytope (2.3.i) with $vert(F) = vert(P) \cap F$ (2.3.iii). By proposition 2.2, $F = conv(vert(F)) = conv(vert(P) \cap F)$, so we have showed that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F.

By definition, every polytope P can be written as the convex hull of a finite set of points V. By proposition 2.2.ii, $vert(P) \subset V$ for all possible V defining P. Since $vert(F) \subset vert(P) \quad \forall F \subset P$ face, and every subset of $vert(P)$ determines at most 1 face, the number of possible faces is bounded by $2^{\# vert(P)}$, which is finite.

(2) Let $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ be two non-empty polytopes. Prove that the set of faces of the cartesian product polytope $P \times Q = \{(p,q) \in \mathbb{R}^{d+e} : p \in P,\ q \in Q\}$ exactly equals $\{F \times G : F \text{ is face of } P,\ G \text{ is face of } Q\}$. Conclude that

$$f_k(P \times Q) \ = \sum_{i+j=k,\ i,j\geq 0} f_i(P) f_j(Q) \qquad \text{for } k \geq 0.$$

Let us show that $F \times G$ is a face of $P \times Q$, where F and G are faces of P and Q, respectively. Write

$$F = \left\{y \in \mathbb{R}^d : ay = b\right\} \cap P, \text{ where } a \in (\mathbb{R}^d)^*, b \in \mathbb{R} \text{ and } P \subset \left\{y \in \mathbb{R}^d : ay \leq b\right\}$$
$$G = \{z \in \mathbb{R}^e : cz = d\} \cap Q, \text{ where } c \in (\mathbb{R}^e)^*, d \in \mathbb{R} \text{ and } Q \subset \{z \in \mathbb{R}^e : cz \leq d\}$$

Note that

$$F \times G = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{R}^{d+e} : ay = b, \quad cz = d, \quad y \in P, \quad z \in Q \right\} \subset$$
$$\subset \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{R}^{d+e} : (a,c) \begin{pmatrix} y \\ z \end{pmatrix} = b + d \quad y \in P, \quad z \in Q \right\}$$

so $F \times G$ is a face of $P \times Q$ if, and only if, $(a,c)\begin{pmatrix} y \\ z \end{pmatrix} \leq b + d \quad \forall y \in P, z \in Q$ and the inclusion can be turned into an equality.

It is easy to see that $(a,c)\begin{pmatrix} y \\ z \end{pmatrix} \leq b + d \quad \forall y \in P, z \in Q$, since $ay \leq b \quad \forall y \in P$ and $cz \leq d \quad \forall z \in Q$ by hypothesis.

About the inclusion, a priori we cannot ensure that it is an equality, since there could exist, y, z such that $ay = \tilde{b}, cz = \tilde{d}$, with $\tilde{b} + \tilde{d} = b + d$.

However, suppose $\tilde{b} > b$. Then we would have $y \in P$, with $ay > b$, which is a contradiction. So $\tilde{b} \leq b$. Analogously, $\tilde{d} \leq d$. Thus, $\tilde{b} = b$ and $\tilde{d} = d$, so the inclusion is actually an equality and $F \times G$ is a face of $P \times Q$.

On the other hand, let us show that any face H of $P \times Q$ is the product of a face $F \subset P$ times a face $G \subset Q$.

Since H is a face of $P \times Q$,

$$H = \left\{x \in \mathbb{R}^{d+e} : ax = b\right\} \cap (P \times Q)$$

for some $a \in \left(\mathbb{R}^{d+e}\right)^*, b \in \mathbb{R}$ with $P \times Q \subset \{ax \leq b\}$.

Write $a = (a_P, a_Q)$, with $a_P \in (\mathbb{R}^d)^*, a_Q \in (\mathbb{R}^e)^*$. Consider the hyperplane of $\mathbb{R}^d$ defined by the equation $\{a_P y = b_P\}$, being b_P such that $P \subset \{a_P y \leq b_P\}$ and $F := P \cap \{a_P y = b_P\} \neq \varnothing$. Such b_P exists: imagine the hyperplane defined by $a_P y$ swipping from the *valid* side of the infinity until it hits some point of P, defining a face F of P.

Now let $G = \{z \in \mathbb{R}^e : a_Q z = b - b_P\} \cap Q$. It defines a face of Q, because:

$$\left.\begin{array}{l} ax \leq b \Leftrightarrow a_P y + a_Q z \leq b_P + b - b_P \\ a_P y \leq b_P \end{array}\right\} \Longrightarrow a_Q z \leq b - b_P$$

And we have proved that $H = F \times Q$.

Finally,

$$\begin{aligned} f_k(P \times Q) &= \\ &= \#\{\text{faces } H \neq \varnothing \text{ of } P \times Q : dim(H) = k\} = \\ &= \#\{\text{faces } F \times G : F \neq \varnothing \text{ face of } P, G \neq \varnothing \text{ face of } Q, dim(F) + dim(G) = k\} = \\ &= \sum_{i+j=k; i,j \geq 0} \#\{\text{faces } F \text{ of } P : dim(F) = i\} \cdot \#\{\text{faces } G \text{ of } Q : dim(G) = j\} = \\ &= \sum_{i+j=k; i,j \geq 0} f_i(P) f_j(Q) \end{aligned}$$

(3) Show that all induced cycles of length 3, 4 and 5 in the graph of a simple d-polytope P are graphs of 2-faces of P. Conclude that the Petersen graph is not the graph of any polytope of any dimension. (*Hint for* 5*-cycles:* First show this for $d = 3$. Then prove that any 5-cycle in a simple polytope is contained in some 3-face, and use that faces of simple polytopes are simple.)

Observation 1. Two edges incident to a vertex in a graph of a simple polytope P define a unique 2-face of P.

Proof. Let e_1, e_2 be edges indicent to a vertex v. Since P is simple, v can be seen as the intersection of d facets, and both e_1 and e_2 are the intersection of $d - 1$ of these facets. Thus, e_1 and e_2 share *exactly* $d - 2$ facets. The intersection of these $d - 2$ facets is, by proposition 2.3.ii, a face F. Since $e_1, e_2 \in F$, $\dim F \geq 2$. On the other hand, $\dim F \leq 2$, because F is the intersection of, at most, $d - \dim F$ facets.$\square$

Take a 3-cycle C_3 in the graph of a simple d-polytope P, $v_1 v_2 v_3$. Let e_{12}, e_{13}, e_{23} be the edges of the C_3, being e_{ij} the edge between v_i and v_j.

By Observation 1, e_{12} and e_{13} are incident to the same vertex v_1, so they define a 2-face F. Notice that $v_2 \in e_{12} \Rightarrow v_2 \in F$. Similarly, $v_3 \in F$.

Since F is convex, $\lambda u + (1-\lambda) v \in F \quad \forall u, v \in F, \lambda \in [0,1]$. Therefore, $e_{23} = \{\lambda v_2 + (1-\lambda) v_3\}_{\lambda \in [0,1]} \subset F$ and the triangle defined by e_{12}, e_{13}, e_{23} is actually a 2-face of P.

Take now a 4-cycle $C_4 = v_1 v_2 v_3 v_4$ in the graph of P, using the same notation as before, e_{ij}, for its edges.

Using Observation 1, let F be the 2-face defined by the edges e_{12} and e_{14}, incidents to v_1, and let G be the 2-face defined by e_{23} and e_{34}, incidents to v_3.

By convexity, doing the same argumentation that in the previous case, the segment $s_{24} = \{\lambda v_2 + (1-\lambda)v_4\}_{\lambda\in[0,1]}$ is contained in both F and G. In addition, the intersection of faces is a face, so $F \cap G$ is a face of dimension at least 1. Since C_4 is induced, the segment s_{24} is not in the graph of P and $F \cap G$ needs to be a 2-face. Therefore, $F = G = F \cap G$ and C_4 is its graph.

Take a 5-cycle C_5 in the graph of P, using the same notation as before.

Any 5-cycle in a simple polytope is contained in some 3-face: Using Observation 1, let F be the 2-face defined by the edges e_{12} and e_{23}, incidents to v_2. Notice that $F \cup e_{34}$ needs to be contained in some (at most) 3-face $\tilde{F}$, because the edge e_{34} is incident to a vertex $v_3 \in F$. Analogously, let G be the 2-face defined by e_{45} and e_{51}, incidents to v_5.

By convexity, doing the same argumentation that in the previous cases, the segment $s_{14} = \{\lambda v_1 + (1-\lambda)v_4\}_{\lambda\in[0,1]}$ is contained in both $\tilde{F}$ and G. In addition, the intersection of faces is a face, so $\tilde{F} \cap G$ is a face of dimension at least 1. Since C_5 is induced, the segment s_{14} is not in the graph of P and $\dim \tilde{F} \cap G > 1$. Hence, $G = \tilde{F} \cap G \subset \tilde{F}$ and C_5 is contained in a (at most) 3-face $\tilde{F}$ of P.

For $d = 3$, C_5 is the graph of a 2-face of P: Since $d = 3$, $G(P)$ is planar. Thus, any cycle subdivides $G(P)$ naturally into two subgraphs: the subgraph which is interior to the cycle, G_i, and the subgraph which is exterior to the cycle, G_e.

Take any planar embedding of $G(P)$ and assume C_5 is not the graph of any 2-face. Notice that both G_i and G_e are not empty: if one of them were empty, $G(P)$ could be embedded in the sphere in such a way that C_5 was at the bottom, with no vertex $\notin C_5$ below it, thus being a 2-face of P.

By Balinsky's Theorem, any $G(P)$ is 3-connected, so at least 3 edges must connect both G_i and G_e to C_5. Hence, one of the vertices of C_5 must have at least 4 edges incident to it, contradicting simpleness of P. Therefore, C_5 is the graph of a 2-face of P.

Finally, we can conclude that C_5 is the graph of a 2-face of P, for any dimension d of P. Indeed, C_5 is contained in some 3-face $\tilde{F}$ of P. Since faces of simple polytopes are simple, $\tilde{F}$ is a simple 3-polytope. Thus, C_5 is the graph of a 2-face of $\tilde{F} \subset P$.

(4) Let $n \in \mathbb{N}$ be an integer and S denote a subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $\Gamma_n(S)$ is the graph whose vertex set is $\mathbb{Z}_n$, and whose edge set is the set of pairs of vertices whose difference lies in $S \cup (-S)$.

The following figure collects all connected circulant graphs on up to 8 vertices. Determine the *polytopality range* for as many of these graphs as you can, i.e., the set of integers d such that the graph in question is the graph of a d-dimensional polytope.

(5) Let $\square^d$ be the d-dimensional ± 1-cube. How large can the volume of a simplex in $\square^d$ become? (*Hint:* `en.wikipedia.org/wiki/Hadamard_inequality`. Write a C++ program to attain explicit bounds for $d \geq 2$ as large as you can.)