1 Sheet 2

1. Show that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F. In particular, P has only finitely many faces.

   By proposition 2.3, F is a polytope (2.3.i) with $vert(F) = vert(P) \cap F$ (2.3.iii). By proposition 2.2, $F = conv(vert(F)) = conv(vert(P) \cap F)$, so we have showed that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F.

   By definition, every polytope P can be written as the convex hull of a finite set of points V. By proposition 2.2.ii, $vert(P) \subset V$ for all possible V defining P. Since $vert(F) \subset vert(P) \quad \forall F \subset P$ face, and every subset of $vert(P)$ determines at most 1 face, the number of possible faces is bounded by $2^{\# vert(P)}$, which is finite.

2. Let $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ be two non-empty polytopes. Prove that the set of faces of the cartesian product polytope $P \times Q = \{(p,q) \in \mathbb{R}^{d+e} : p \in P,\ q \in Q\}$ exactly equals $\{F \times G : F \text{ is face of } P,\ G \text{ is face of } Q\}$. Conclude that

$$f_k(P \times Q) \;=\; \sum_{i+j=k,\ i,j \geq 0} f_i(P) f_j(Q) \qquad \text{for } k \geq 0.$$

   Let us show that $F \times G$ is a face of $P \times Q$, where F and G are faces of P and Q, respectively. Write

$$F = \left\{y \in \mathbb{R}^d : ay = b\right\} \cap P, \text{ where } a \in (\mathbb{R}^d)^*, b \in \mathbb{R} \text{ and } P \subset \left\{y \in \mathbb{R}^d : ay \leq b\right\}$$
$$G = \{z \in \mathbb{R}^e : cz = d\} \cap Q, \text{ where } c \in (\mathbb{R}^e)^*, d \in \mathbb{R} \text{ and } Q \subset \{z \in \mathbb{R}^e : cz \leq d\}$$

   Note that

$$F \times G = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{R}^{d+e} : ay = b, \quad cz = d, \quad y \in P, \quad z \in Q \right\} \subset$$
$$\subset \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{R}^{d+e} : \begin{pmatrix} a \\ c \end{pmatrix}^{\top} \begin{pmatrix} y \\ z \end{pmatrix} = b + d \quad y \in P, \quad z \in Q \right\}$$

   so $F \times G$ is a face of $P \times Q$ if, and only if, $\begin{pmatrix} a \\ c \end{pmatrix}^{\top} \begin{pmatrix} y \\ z \end{pmatrix} \leq b + d \quad \forall y \in P, z \in Q$ and the inclusion can be turned into an equality.

   It is easy to see that $\begin{pmatrix} a \\ c \end{pmatrix}^{\top} \begin{pmatrix} y \\ z \end{pmatrix} \leq b + d \quad \forall y \in P, z \in Q$, since $ay \leq b \quad \forall y \in P$ and $cz \leq d \quad \forall z \in Q$ by hypothesis.

   About the inclusion, a priori we cannot ensure that it is an equality, since there could exist, y, z such that $ay = \tilde{b}, cz = \tilde{d}$, with $\tilde{b} + \tilde{d} = b + d$.

   However, suppose $\tilde{b} > b$. Then we would have $y \in P$, with $ay > b$, which is a contradiction. So $\tilde{b} \leq b$. Analogously, $\tilde{d} \leq d$. Thus, $\tilde{b} = b$ and $\tilde{d} = d$, so the inclusion is actually an equality and $F \times G$ is a face of $P \times Q$.

   (Falta la otra inclusión)

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\begin{aligned}
f_k(P \times Q) = & \\
&= \#\{\text{faces } H \neq \varnothing \text{ of } P \times Q : dim(H) = k\} = \\
&= \#\{\text{faces } F \times G : F \neq \varnothing \text{ face of } P, G \neq \varnothing \text{ face of } Q, dim(F) + dim(G) = k\} = \\
&= \sum_{i+j=k;i,j\geq 0} \#\{\text{faces } F \text{ of } P : dim(F) = i\} \cdot \#\{\text{faces } G \text{ of } Q : dim(G) = j\} = \\
&= \sum_{i+j=k;i,j\geq 0} f_i(P) f_j(Q)
\end{aligned}
$$

3. Show that all induced cycles of length 3, 4 and 5 in the graph of a simple d-polytope P are graphs of 2-faces of P. Conclude that the Petersen graph is not the graph of any polytope of any dimension. (*Hint for* 5*-cycles:* First show this for $d = 3$. Then prove that any 5-cycle in a simple polytope is contained in some 3-face, and use that faces of simple polytopes are simple.)

4. Let $n \in \mathbb{N}$ be an integer and S denote a subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $\Gamma_n(S)$ is the graph whose vertex set is $\mathbb{Z}_n$, and whose edge set is the set of pairs of vertices whose difference lies in $S \cup (-S)$.

   The following figure collects all connected circulant graphs on up to 8 vertices. Determine the *polytopality range* for as many of these graphs as you can, i.e., the set of integers d such that the graph in question is the graph of a d-dimensional polytope.

5. Let $\square^d$ be the d-dimensional ± 1-cube. How large can the volume of a simplex in $\square^d$ become? (*Hint:* `en.wikipedia.org/wiki/Hadamard_inequality`. Write a C++ program to attain explicit bounds for $d \geq 2$ as large as you can.)