Discrete and Algorithmic Geometry

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1. **Show that a face F of a polytope P is exactly the convex hull of all vertices of P contained in F. In particular, P has only finitely many faces.**

   We first prove that F is also a polytope. As P is a polytope, it is the bounded intersection of halfspaces and as F is a face, there exists some linear inequality $\mathbf{cx} \leq c_0$ valid for P such that the hyperplane $H = \{x \in \mathbb{R}^d : \mathbf{cx} = c_0\}$ satisfies $F = H \cap P$. Therefore, F can be seen as the intersection of the halfspaces that define P and the halfspace defined by H.

   We will continue proving $F \cap V \subseteq \text{vert}(F)$, where V is the set of vertices of P. Take $\mathbf{x} \in F \cap V$, so there exists some hyperplane H' such that $H' \cap P = \{x\}$ and the inequality associated to H' is valid for P. Then the inequality is also valid for F and $H' \cap F = \{x\}$. Therefore, $\text{conv}(F \cap V) \subseteq \text{conv}(\text{vert}(F))$.

   Take $\mathbf{x} \in F$. $\mathbf{x}$ is also in P, so we can put $\mathbf{x} = \sum \lambda_i \mathbf{v_i}$, where $V = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$, $\lambda_i \geq 0$ for all i and $\sum \lambda_i = 1$. Using now the hyperplane H defined before,

   $$c_0 = \mathbf{cx} = \mathbf{c} \sum \lambda_i \mathbf{v_i} = \sum \lambda_i \mathbf{c v_i} \leq \sum \lambda_i c_0 = c_0$$

   Then, we have $\sum \lambda_i(\mathbf{cv_i} - c_0) = 0$ and as every term has the same sign, it has to be $\lambda_i(\mathbf{cv_i} - c_0) = 0$ for every i. That means $\lambda_i = 0$ for every $v_i \notin F \cap V$. Therefore, $\mathbf{x} \in \text{conv}(F \cap V)$, which means that $F \subseteq \text{conv}(F \cap V) \subseteq \text{conv}(\text{vert}(F)) = F$. Therefore, $F = \text{conv}(F \cap P)$ and as the vertices of P are finite, there are finitely many faces.

2. **Let $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ be two non-empty polytopes. Prove that the set of faces of the cartesian product polytope $P \times Q = \{(p, q) \in \mathbb{R}^{d+e} : p \in P, q \in Q\}$ exactly equals $\{F \times G : F \text{ is face of } P, G \text{ is face of } Q\}$. Conclude that**

   $$f_k(P \times Q) = \sum_{i+j=k,\ i,j \geq 0} f_i(P) f_j(Q) \qquad \textbf{for } k \geq 0$$

   Consider first a face F of P defined by a hyperplane $H_P = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{c_P x} = c_{0P}\}$ and a face G of Q defined by a hyperplane $H_Q = \{\mathbf{x} \in \mathbb{R}^e : \mathbf{c_Q x} = c_{0Q}\}$. Define $H = \{\mathbf{x} \in \mathbb{R}^{d+e} : (\mathbf{c_P}, \mathbf{c_Q})\mathbf{x} = c_{0P} + c_{0Q}\}$. Then the associated inequality is valid for $P \times Q$ and $H \cap (P \times Q) = F \times G$.

   Let us see now the other inclusion. Consider a face A of $P \times Q$, defined by the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^{d+e} : (\mathbf{c_P}, \mathbf{c_Q})\mathbf{x} = c_0\}$. Let $\alpha_P = \max\{\mathbf{c_P x} : \mathbf{x} \in P\}$ and $\alpha_Q = \max\{\mathbf{c_Q x} : \mathbf{x} \in Q\}$, that are well-defined. Let $(\mathbf{p_0}, \mathbf{q_0})$ be some pair satisfying $\mathbf{c_P p_0} + \mathbf{c_Q q_0} = \alpha_P + \alpha_Q = c_0$ (it exists because otherwise the face would be empty). But by definition of those α values this equality must hold for each $(\mathbf{p}, \mathbf{q}) \in A$. Then, taking $F = \{\mathbf{x} \in P : \mathbf{c_P x} = \alpha_P\}$, $G = \{\mathbf{x} \in Q : \mathbf{c_Q x} = \alpha_Q\}$ we have faces in P and Q, which completes the proof.

Therefore, as every k-face of $P \times Q$ is the cartesian product of a face of P and a face of Q, that we know that they are also polytopes, the dimension k is exactly the sum of the dimensions of these faces, so

$$f_k(P \times Q) = \sum_{i+j=k,\ i,j\geq 0} f_i(P) f_j(Q) \qquad \textbf{for } k \geq 0$$

3. **Show that all induced cycles of length 3, 4 and 5 in the graph of a simple d-polytope P are graphs of 2-faces of P . Conclude that the Petersen graph is not the graph of any polytope of any dimension.**

   For 3-cycles the result is trivial, because any two incident edges induce a 2-face and it is a triangle because the graph is induced. In the case of 4-cycles, if we pick two non-adjacent vertices (one of the two opposed pairs) and consider for each of these two vertices their two incident edges in the cycle, that induces two 2-faces. But if these faces are different, their intersection is not proper, so that is a contradiction and the induced faces are equal.

   For 5-cycles, let us prove it first for 3-polytopes. If it is not a face, it divides the polytope in two non-empty components, because the polytope is planar. But as the graph is 3-connected, each of those components is connected to the cycle through 3 edges. As we have only five vertices, that contradicts the fact that the polytope is simple.

   In general dimension, if we consider two adjacent adjacent edges they form a 2-face, so if we consider three consecutive edges in the cycle, they form a 3-face. So we now have four vertices of the cycle in a 3-face. If the 5th vertex is in the same face, then we are in the previous case. Otherwise, the two edges of the cycle which are incident to this vertex form a 2-face that intersects improperly the 3-face, so we have a contradiction.

   Finally, as the Petersen graph has two induced 5-cycles that share 3 points and that is not possible.

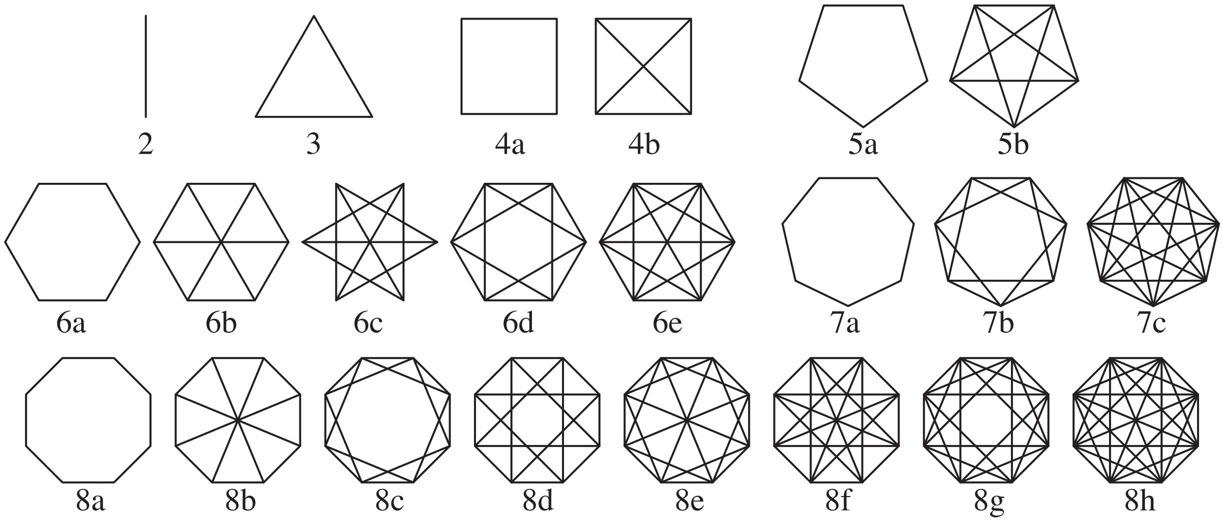
4. **Let $n \in \mathbb{N}$ be an integer and S denote a subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $\Gamma_n(S)$ is the graph whose vertex set is $\mathbb{Z}_n$, and whose edge set is the set of pairs of vertices whose difference lies in $S \cup (-S)$.**

   The following figure collects all connected circulant graphs on up to 8 vertices. Determine the *polytopality range* for as many of these graphs as you can, i.e., the set of integers d such that the graph in question is the graph of a d-dimensional polytope.

   In order to discard some of the possibilities for each graph, a program in C++ has been used. The main idea of this code is testing every graph against known properties that a d-dimensional polytope must satisfy.

   For $d = 2$, polygons must satisfy that the number of edges is equal to the number of vertices. For $d = 3$, polytopes are completely characterized using Steinitz's Theorem (planar and 3-connected). Finally, for $d > 3$, Balinski's Theorem says that the graph has to be d-connected and the Principal Subdivision Property says that every vertex of the graph is the principal vertex of a principal subdivision of K_{d+1}. Those properties are the ones checked by the program.

   The following results are given by the program. Notice that every possible dimension with $d < 4$ is also sure to be correct because the properties checked are enough to claim that, but the rest are only candidates that are valid so far.



2	**3**	**4a**	**4b**	**5a**	**5b**	**6a**	**6b**	**6c**	**6d**	**6e**
1	2	2	3	2	4	2	-	3	3	4, 5

7a	**7b**	**7c**	**8a**	**8b**	**8c**	**8d**	**8e**	**8f**	**8g**	**8h**
2	-	4, 5, 6	2	-	3	-	4	4	4, 5	4, 5, 6, 7

So now, we need to check each dimension candidate with dimension higher than three. First let us discuss about complete graphs. Recall the following result (Ziegler, Lectures on Polytopes):

Corollary. *Let $n > d \geq 2$. The cyclic polytope $C_d(n)$ is $\lfloor \frac{d}{2} \rfloor$-neighborly, that is, any subset $S \subseteq [n]$ of $|S| \leq \frac{d}{2}$ vertices forms a face.*

That means that for $d \geq 4$, every pair of vertices are connected by edges in $C_d(n)$, so the graph of this polytope is K_n. In particular, K_n is the graph of some polytope in dimensions from 4 to $n-1$.

Now consider one by one each of the dimension candidates:

- **5b:** The graph is complete, so we know that 4 is a valid dimension.
- **6e:** The graph is K_6, so we know that both candidates are valid.
- **7c:** The graph is K_7 and so every candidate is valid.
- **8f:** Using an adapted version of the previous program, we can search for the 3-faces of the graph. The only ones are those induced by vertices $\{0,1,4,5\}$, $\{1,2,5,6\}$, $\{0,3,4,7\}$ and $\{2,3,6,7\}$, so we only have 4 valid tetrahedra and therefore the dimension is not valid.
- **8g:** For dimension 4, it can be the graph of the crosspolytope C_4^{Δ} and for dimension 5, the join of two 2-cubes.
- **8h:** The graph is K_8, hence every candidate is valid.

Therefore, the final result is

2	**3**	**4a**	**4b**	**5a**	**5b**	**6a**	**6b**	**6c**	**6d**	**6e**
1	2	2	3	2	4	2	-	3	3	4, 5

7a	**7b**	**7c**	**8a**	**8b**	**8c**	**8d**	**8e**	**8f**	**8g**	**8h**
2	-	4, 5, 6	2	-	3	-	(4)	-	4, 5	4, 5, 6, 7

Notice that for 8e, the candidate has not been discarded nor verified, that is why it appears between parenthesis.

5. **Let $\square^d$ be the d-dimensional ± 1-cube. How large can the volume of a simplex in $\square^d$ become?**

   This can be reduced to finding the biggest determinant (in absolute value) of $d+1$ points in $\square^d$ in homogeneous coordinates. Applying Hadamard's inequality, we can bound that determinant by $(d+1)^{\frac{d+1}{2}}$. In order to achieve better results, a program that selects random matrices with points in $\square^d$ and computes their determinants has been used.

   The following table shows the best result found for each dimension after 10^6 random matrices:

Dimension	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Determinant	16	48	160	576	2304	8192	27648	98304

Looking at these values, we can conjecture that the values grow a little faster than 2^d. The program also offers a report on the number of matrices found for each value. Exact results can be seen running the program, but the fact is that the biggest values are very difficult to "catch" choosing the matrices randomly.