ΠΕΤΣΑ ΓΕΩΡΓΙΑ
ΑΜ: 3200155
9^{η} ΟΜΑΔΑ ΑΣΚΗΣΕΩΝ

$\boxed{\text{ασκ. 55}}$

(α) $\int_0^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{h\to\infty}\int_0^h \frac{1}{\sqrt{x}}\,dx$. Είναι: $2\int_0^h \frac{1}{2\sqrt{x}}\,dx = 2\int_0^h (\sqrt{x})'\,dx$

$= 2\left[\sqrt{x}\right]_0^h = 2\sqrt{h} - 2\sqrt{0} = 2\sqrt{h}$. Άρα $\lim_{h\to\infty}\int_0^h \frac{1}{\sqrt{x}}\,dx =$

$\lim_{h\to\infty}(2\sqrt{h}) = \infty$

(γ) $\int_0^{\pi} |\tan x|\,dx = \int_0^{\pi/2} \tan x\,dx + \int_{\pi/2}^{\pi} -\tan x\,dx$. Είναι $\int_0^h \tan x\,dx =$

$\int_0^h \frac{\sin x}{\cos x}\,dx$ θέτω $u=\cos x$, $du=-\sin x\,dx$ $= \int_0^h -\frac{1}{u}\,du = \int_0^h (-\ln u)'\,du = \left[-\ln(\cos x)\right]_0^h = -\ln(\cosh) + 0$.

$\lim_{h\to\pi/2^-} -\ln(\cosh) = -(-\infty) = +\infty$

Είναι: $\int_h^{\pi} -\tan(x)\,dx = \int_h^{\pi} -\frac{\sin x}{\cos(x)}\,dx$ (θέτω $u=\cos(x)$, $du = -\sin x\,dx$) $= \int_h^{\pi} \frac{1}{u}\,du = \left[\ln(\cos x)\right]_h^{\pi} = 0 + \ln(\cosh)$

$\lim_{h\to\pi/2^+} -\ln(\cosh) = \infty$

Άρα $\int_0^{\pi} |\tan x|\,dx = \infty + \infty = \infty$

$\boxed{\text{ασκ. 58}}$

(α) $\lim_{x\to 0} \frac{\tan x}{10 + 5e^{1/x^3}}$,

• $\lim_{x\to 0^-} \frac{\tan x}{5e^{1/x^3}+10} = \lim_{x\to 0^-}\left(\tan x \cdot \frac{1}{5e^{1/x^3}+10}\right) = \frac{0}{0+10} = 0$

Αφού $\lim_{x\to 0^-} e^{1/x^3} = 0$, $u = 1/x^3$, γιατί $\lim_{x\to 0^-} 1/x^3 = -\infty$, άρα $\lim_{u\to -\infty} e^u = 0$

$\lim_{x\to 0^+} \frac{\tan x}{5e^{1/x^3}+10} = \lim_{x\to 0^+}\left(\tan x \cdot \frac{1}{5e^{1/x^3}+10}\right) = 0 \cdot 0 = 0$

$u = 1/x^3$ $\quad \lim_{x\to 0^+} \frac{1}{x^3} = +\infty$, και $\lim_{u\to\infty} e^u = \infty$

Άρα και $\lim_{x\to 0} \frac{\tan x}{10 + 5e^{1/x^3}} = 0$

$\boxed{\text{ασκ. 55}}$

(β) $$\int_{-\infty}^{\infty} \frac{\log|x|}{x}\,dx = \int_{-\infty}^{-1}\frac{\log|x|}{x}\,dx + \int_{-1}^{0}\frac{\log|x|}{x}\,dx + \int_{0}^{1}\frac{\log|x|}{x} + \int_{1}^{\infty}\frac{\log|x|}{x}\,dx$$

• $$\int_{-\infty}^{-1}\frac{\log|x|}{x}\,dx = \lim_{h\to\infty}\int_{h}^{-1}\frac{\log|x|}{x}\,dx = \lim_{h\to-\infty}\int_{h}^{-1}\left(\left(\frac{\log|x|}{2}\right)^2\right)' =$$

$$= \lim_{h\to\infty}\left(0-\left(\frac{\log|h|}{2}\right)^2\right) = -\infty$$

• $$\int_{1}^{\infty}\frac{\log|x|}{x}\,dx = \lim_{h\to\infty}\int_{1}^{h}\frac{\log|x|}{x}\,dx = \lim_{h\to\infty}\int_{1}^{h}\left(\left(\frac{\log|x|}{2}\right)^2\right)' =$$

$$= \lim_{h\to\infty}\left(\left(\frac{\log|h|}{2}\right)^2-0\right) = \infty$$

Επειδή έχουμε ένα όριο $+\infty$ και ένα $-\infty$ δεν μπορούμε να υπολογίσουμε το $\int_{-\infty}^{\infty}\frac{\log|x|}{x}\,dx$

ασκ. 59

α) $\int_1^{\infty} \frac{2+\cos x}{x} dx$. Ξέρουμε ότι $-1 \le \cos x \le 1 \Rightarrow 1 \le \cos x + 2 \le 3 \overset{x>0}{\Rightarrow} \underbrace{\frac{1}{x} \le \frac{\cos x + 2}{x}} \le 3/x$

$$\int_1^h \frac{1}{x} \le \int_1^h \frac{\cos x + 2}{x} \Rightarrow [\ln x]_1^h \le \int_1^h \frac{\cos x + 2}{x} dx \Rightarrow$$

$$\Rightarrow \lim_{h\to\infty} \ln h \le \lim_{h\to\infty} \int_1^h \frac{\cos x + 2}{x} dx \Rightarrow \infty \le \lim_{h\to\infty} \int_1^h \frac{\cos x + 2}{x} dx$$, άρα

$$\int_1^{\infty} \frac{2+\cos x}{x} = \infty$$

β) $$\lim_{h\to\infty} \frac{\int_0^h \frac{2+\cos x}{x}}{h} \underset{DLH}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{h\to\infty} \frac{\left(\int_0^h \frac{2+\cos x}{x}\right)'}{1} = \lim_{h\to\infty} \frac{2+\cos h}{h} =$$

$$= \lim_{h\to\infty}\left(\frac{2}{h} + \frac{\cos h}{h}\right) = 0 + 0 = 0$$

$$|\cos h| \le 1 \overset{h>0}{\Rightarrow} -\frac{1}{h} \le \frac{\cos h}{h} \le \frac{1}{h} \Rightarrow \lim_{h\to\infty} -1/h \le \lim_{h\to\infty} \frac{\cos h}{h} \le \lim_{h\to\infty} 1/h$$

$\Rightarrow -0 \le \lim_{h\to\infty} \frac{\cos h}{h} \le 0$ Άρα από το θεώρημα της παρεμβολής

θα είναι $\lim_{h\to\infty} \frac{\cos h}{h} = 0$

ασκ. 61

$$\int_0^3 \frac{x+5}{\sqrt{9-x^2}} dx$$

θέτω $x = 3\sin u$, $dx = 3\cos u\, du$

Για $x=0$: $0 = 3\sin u \Rightarrow u = 0$

Για $x=3$: $3 = 3\sin u \Rightarrow u = \pi/2$

$$= \int_0^{\pi/2} \frac{(3\sin u + 5)\, 3\cos u}{\sqrt{9 - 9\sin^2 u}} du = \int_0^{\pi/2} \frac{9\sin u \cos u + 15\cos u}{\sqrt{9 - 9\sin^2 u}} du$$

$$= \int_0^{\pi/2} \left(\frac{9\sin u \cos u}{3\sqrt{1-\sin^2 u}} + \frac{15\cos u}{3\sqrt{1-\sin^2 u}} \right) du = \int_0^{\pi/2} \left(3\frac{\sin u \cos u}{|\cos u|} + \frac{5\cos u}{|\cos u|} \right) du =$$

$$\int_0^{\pi/2} 3\frac{\sin u \cos u}{\cos u} + \frac{5\cos u}{\cos u} du = \int_0^{\pi/2} (3\sin u + 5) du = \left[3(-\cos u) + 5u \right]_0^{\pi/2}$$

$$= 3(-\cos(\pi/2)) + \frac{5\pi}{2} - 3(\cos 0) - 0 = -3\cdot 0 + 5\pi/2 - 3(-1) - 0 = \boxed{\frac{5\pi}{2} + 3}$$

άσκ. 60

(α) $\int_{-e}^{e} \frac{x}{1+x^2} dx = \frac{1}{2}\int_{-e}^{e} \frac{2x}{1+x^2} dx$

Έστω

$f(x) = \frac{x}{1+x^2}$ $\quad f(-x) = \frac{-x}{1+(-x)^2} = -\frac{x}{1+x^2}$

$= -f(x).$

Άρα η συνάρτηση είναι περιττή, επομένως το $\int_{-e}^{e} f(x)\,dx$ θα ισούται με $2\int_0^e f(x)\,dx$ άρα

$\frac{2}{2}\int_0^e \frac{2x}{1+x^2} dx = \int_0^e (\ln|1+x^2|)' dx$

$\left(\text{αφού } (\ln(1+x^2))' = \frac{1\cdot(1+x^2)'}{1+x^2} = \frac{2x}{1+x^2}\right)$ Άρα είναι

$\int_0^e (\ln(1+x^2))' dx = \left[\ln|1+x^2|\right]_0^e = \ln(1+e^2) - \cancel{\ln(1+0)} = \boxed{\ln(e^2+1)}$

(β) $\int_{-1}^{1} \log\left(\frac{x+10}{x+100}\right) dx = \int_{-1}^{1}\left(\log(x+10) - \log(x+100)\right) dx$

$= \int_{-1}^{1} \log(x+10)\,dx - \int_{-1}^{1} \log(x+100)\,dx$

Παρατηρώ ότι:

$\left((x+10)\log(x+10) - x\right)' = \log(x+10) + \frac{x+10}{x+10} - 1 = \log(x+10)$

$\left((x+100)\log(x+100) - x\right)' = \log(x+100) + \frac{x+100}{x+100} - 1 = \log(x+100)$

Άρα:

$= \int_{-1}^{1}\left((x+10)\log(x+10) - x\right)' - \int_{-1}^{1}\left((x+100)\log(x+100) - x\right)' dx$

$= \left[(x+10)\log(x+10) - x\right]_{-1}^{1} - \left[(x+100)\log(x+100) - x\right]_{-1}^{1} =$

$\boxed{11\log(11) - 9\log(9) - 101\log(101) + 99\log(99)}$

(γ)'

$$\lim_{h\to\infty}\int_0^h \frac{1}{1+x^2}\,dx \, . \quad \text{Είναι} \quad \int_0^h \frac{1}{1+x^2}\,dx = \int_0^h (\tan^{-1}x)'\,dx$$

$$= \left[(\tan^{-1}x)\right]_0^h = \tan^{-1}(h) - \cancelto{0}{\tan^{-1}(0)} = \tan^{-1}(h)$$

Άρα $\lim_{h\to\infty} \tan^{-1}(h) = \frac{\pi}{2}$ $\left(\text{αφού } \lim_{x\to\pi/2^{+}} \tan x = \infty \right)$

ασκ. 58

(β) $\lim\limits_{n\to\infty} \frac{(\log n)^7}{\sqrt{n}} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{DLH}{=}} \lim\limits_{n\to\infty} \frac{\frac{7(\log n)^6}{n}}{\frac{1}{2\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{14\, n^{1/2}(\log n)^6}{n}$

$= 14 \lim\limits_{n\to\infty} \frac{(\log n)^6}{\sqrt{n}} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{DLH}{=}} 14\cdot 6 \lim\limits_{n\to\infty} \frac{(\log n)^5/n}{\frac{1}{2\sqrt{n}}} = 168 \lim\limits_{n\to\infty} \frac{\sqrt{n}(\log n)^5}{n}$

$= 168 \lim\limits_{n\to\infty} \frac{(\log n)^5}{\sqrt{n}} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{DLH}{=}} 168 \lim\limits_{n\to\infty} \frac{\frac{5(\log n)^4}{n}}{\frac{1}{2\sqrt{n}}} = 1680 \lim\limits_{n\to\infty} \frac{(\log n)^4}{\sqrt{n}} \overset{\frac{\infty}{\infty}}{\underset{DLH}{=}}$

$= 6720 \lim\limits_{n\to\infty} \frac{\frac{(\log n)^3}{n}}{\frac{1}{2\sqrt{n}}} = 13440 \lim\limits_{n\to\infty} \frac{(\log n)^3}{\sqrt{n}} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{DLH}{=}} 40320 \lim\limits_{n\to\infty} \frac{\frac{(\log n)^2}{n}}{\frac{1}{2\sqrt{n}}}$

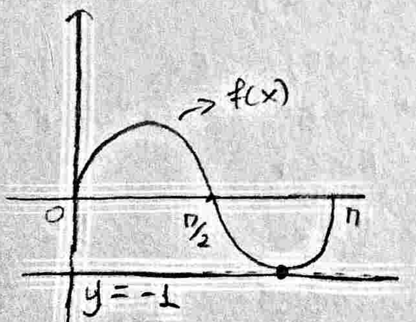
$= 40320 \lim\limits_{n\to\infty} \frac{(\log n)^2}{\sqrt{n}} \overset{\frac{\infty}{\infty}}{\underset{DLH}{=}} 80640 \lim\limits_{n\to\infty} \frac{\frac{\log n}{n}}{\frac{1}{2\sqrt{n}}} = 80640 \lim\limits_{n\to\infty} \frac{\log n}{\sqrt{n}}$

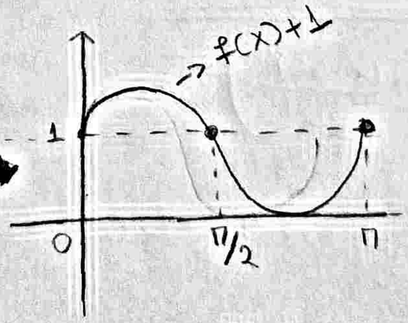
$\overset{\frac{\infty}{\infty}}{\underset{DLH}{=}} 80640 \lim\limits_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = 80640 \lim\limits_{n\to\infty} \frac{2\sqrt{n}}{n} = 80640 \lim\limits_{n\to\infty} \frac{2}{\sqrt{n}}$

$= 16128 \cdot 0 = \boxed{0}$

ασκ. 56

Ο όγκος που δημιουργείται θα ισούται με:





$V(S) = \pi \int_0^{\pi} (f(x)+1)^2\, dx$

$V(S) = \pi \int_0^{\pi} \sin^2 x + 2\sin x + 1\, dx = \pi\left(\int_0^{\pi} \sin^2 x + \int_0^{\pi} 2\sin x\right) + \pi[x]_0^{\pi} \quad (1)$

Υπολογίζω το: $\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \frac{1-\cos 2x}{2}\, dx = \int_0^{\pi} \frac{1}{2} - \frac{\cos 2x}{2}\, dx$

$= \left[\frac{x}{2}\right]_0^{\pi} - \left[\frac{\sin(2x)}{4}\right]_0^{\pi} = \frac{\pi}{2} - \frac{\sin 2\pi}{4} + 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$$(1) = \frac{\pi^2}{2} + \pi\left[-2\cos x\right]_0^{\pi} + \pi^2 = \frac{\pi^2}{2} - 2\pi\cos\pi + 2\pi\cos 0 + \pi^2$$

$$= \frac{\pi^2}{2} + 2\pi + 2\pi + \pi^2 = \boxed{\frac{3\pi^2}{2} + 4\pi}$$

$\boxed{\text{ασκ. 57}}$

α) $$\pi \int_0^{40} 300\left(1 - \frac{(x-20)^2}{20^4}\right) dx$$

β) $$\pi \int_0^{40} 300 - 300\,\frac{x^2 - 40x + 20^2}{20^4}\, dx = \pi \int_0^{40} 300 - \frac{3(x^2 - 40x + 400)}{1600}\, dx$$

$$= \pi\left[300x\right]_0^{40} - \pi \int_0^{40} \frac{3x^2 - 120x + 1200}{1600}\, dx = \pi(12000 - 0) -$$

$$-\pi\left[\frac{x^3 - 60x^2 + 1200x}{1600}\right]_0^{40} = 12.000\pi - \pi\left(\frac{40^3 - 60\cdot 40^2 + 1200\cdot 40}{1600} + 0\right)$$

$$= 12.000\pi - \pi\,\frac{16000}{1600} = 12.000\pi - 10\pi = 11.990\pi$$

γ) $$(f(x))' = \frac{300\left(-\frac{2(x-20)}{20^4}\right)}{2\sqrt{300\left(1 - \frac{(x-20)^2}{20^4}\right)}}$$

Ο παρονομαστής είναι πάντα θετικός, άρα αρκεί να δούμε το πρόσημο του αριθμητή

$$300\left(\frac{-2(x-20)}{20^4}\right) < 0 \Rightarrow -2x + 40 < 0 \Leftrightarrow 2x > 40 \Leftrightarrow x > 20$$ άρα

x	0	20	40
f'	+	0	−
f	↗		↘

Άρα οι μικρότερες διαστάσεις ενός ορθογωνίου κιβωτίου θα είναι 20, $f(20)$