ΠΕΤΣΑ ΓΕΩΡΓΙΑ
ΑΜ: 3200155
7η ΟΜΑΔΑ ΑΣΚΗΣΕΩΝ

άσκ. 43

$$f(x) = \begin{cases} c + x/4, & 0 \le x < 2 \\ 0, & \text{αλλού} \end{cases}$$

(α) Πρέπει $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Άρα: $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 c + \frac{x}{4}\,dx = \left[cx + \frac{x^2}{8}\right]_0^2$

$= 2c + \frac{4}{8} - 0 = 2c + \frac{1}{2}$. Πρέπει: $2c + \frac{1}{2} = 1 \Rightarrow 2c = \frac{1}{2} \Rightarrow \boxed{c = 1/4}$

(β) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 \frac{x^2 + x}{4}\,dx = \frac{1}{4}\int_0^2 x^2 + x\,dx = \frac{1}{4}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2$

$= \frac{1}{4}\left(\frac{8}{3} + \frac{4}{2}\right) = \frac{1}{4}\left(\frac{16+12}{6}\right) = \frac{28}{24} = \frac{7}{6} \Rightarrow \boxed{E(X) = \frac{7}{6}}$

(γ) Επειδή $Y = \lfloor X \rfloor$, και $0 \le X < 2$, σημαίνει ότι η Y μπορεί να λάβει τις τιμές 0 και 1. Άρα

- $P(Y=0) = P(0 \le X \le 1) = \int_0^1 f(x)\,dx = \frac{1}{4}\int_0^1 x + 1\,dx$
  $= \frac{1}{4}\left[\frac{x^2}{2} + x\right]_0^1 = \frac{1}{4}\left(\frac{1}{2} + 1\right) = \frac{3}{8} \Rightarrow \boxed{P(Y=0) = 3/8}$
- $P(Y=1) = 1 - P(Y=0) = 1 - \frac{3}{8} = \frac{5}{8} \Rightarrow \boxed{P(Y=1) = 5/8}$

άσκ. 44

(α) $E(X) = 1$ και $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 (ax + bx^2)\,dx$

$= \left[a\frac{x^2}{2} + b\frac{x^3}{3}\right]_0^2 = \frac{4a}{2} + \frac{8b}{3} - 0 = 2a + \frac{8b}{3}$.

Άρα $2a + \frac{8b}{3} = 1 \Rightarrow \boxed{a = \frac{1}{2} - \frac{8b}{6}}$ (1). Επίσης πρέπει

$E(X) = 1$ και $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 ax^2 + bx^3\,dx =$

$= \int_0^2 \left(\frac{3 - 8b}{6}x^2 + bx^3\,dx\right)dx = \int_0^2 \frac{6bx^3 - 8bx^2 + 3x^2}{6}\,dx$

$$= \frac{1}{6}\int_0^2 6bx^3 - 8bx^2 + 3x^2\,dx = \frac{1}{6}\left[\frac{3bx^4}{2} - \frac{8bx^3}{3} + x^3\right]_0^2$$

$$= \frac{1}{6}\left(\frac{3b\cdot 16}{2} - \frac{8b\cdot 8}{3} + 8\right) = \left(24b - \frac{64b}{3} + 8\right)\frac{1}{6}$$

$$= \frac{1}{6}\left(\frac{72b - 64b + 24}{3}\right) = \frac{1}{6}\left(\frac{8b+24}{3}\right) = \frac{8b+24}{18}$$

Άρα $1 = \frac{8b+24}{18} \Rightarrow 18 = 8b+24 \Rightarrow 8b = -6 \Rightarrow b = -\frac{6}{8} \Rightarrow \boxed{b = -\frac{3}{4}}$

Άρα α (1) έχουμε $\alpha = \frac{1}{2} - \frac{8b}{6} \Rightarrow \alpha = \frac{1}{2} + \frac{6}{6} \Rightarrow \boxed{\alpha = \frac{3}{2}}$

(β) Από (α) έχουμε $f(x) = \begin{cases} \frac{3x}{2} - \frac{3x^2}{4}, & 0<x<2 \\ 0, & \text{αλλού.} \end{cases}$

Επομένως: $P(X<1) = \int_0^1 \left(\frac{3x}{2} - \frac{3x^2}{4}\right)dx =$

$$\left[\frac{3x^2}{4} - \frac{3x^3}{12}\right]_0^1 = \frac{3}{4} - \frac{3}{12} = \frac{6}{12} = \frac{1}{2} \Rightarrow \boxed{P(X<1) = 1/2}$$

$VAR(X) = E(X^2) - E(X) = E(X^2) - 1 \quad (2)$

$$E(X^2) = \int_0^2 x^2\left(\frac{3}{2}x - \frac{3}{4}x^2\right)dx = \int_0^2 \left(\frac{3x^3}{2} - \frac{3}{4}x^4\right)dx$$

$$= \left[\frac{3x^4}{8} - \frac{3x^5}{20}\right]_0^2 = \frac{3\cdot 16}{8} - \frac{3\cdot 32}{20} = \frac{6}{5} \Rightarrow \boxed{E(X^2) = 6/5}$$

Από (2) έχουμε: $VAR(X) = \frac{6}{5} - 1 = \frac{1}{5} \Rightarrow \boxed{VAR(X) = 1/5}$

$\boxed{\text{άσκ. } 45}$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 2\sqrt{x}, & 0 < x < 1/4 \\ 1, & x \geq 1/4 \end{cases}$$

(α) $P(X > 1/9 \mid X > 1/16) = \dfrac{P(X > 1/9,\ X > 1/16)}{P(X > 1/16)} = \dfrac{P(X > 1/9)}{P(X > 1/16)}$

$= \dfrac{1 - P(X \leq 1/9)}{1 - P(X \leq 1/16)} = \dfrac{1 - F(1/9)}{1 - F(1/16)} = \dfrac{1 - 2\sqrt{1/9}}{1 - 2\sqrt{1/16}} = \dfrac{1 - 2/3}{1 - 2/4} = \dfrac{4}{6} = \dfrac{2}{3}$

$\Rightarrow \boxed{P(X > 1/9 \mid X > 1/16) = \dfrac{2}{3}}$

(β) $E(X) = \displaystyle\int_{-\infty}^{\infty} x f(x)\, dx.$ $\qquad f(x) = F'(x) = \begin{cases} 0, & x \leq 0 \\ \dfrac{1}{\sqrt{x}}, & 0 < x < 1/4 \\ 0, & x \geq 1/4 \end{cases}$

$E(X) = \displaystyle\int_0^{1/4} \frac{x}{\sqrt{x}}\, dx = \int_0^{1/4} \frac{x^1}{x^{1/2}} = \int_0^{1/4} x^{1/2}\, dx = \left[\frac{2x^{3/2}}{3}\right]_0^{1/4}$

$= \dfrac{2 \cdot \sqrt{(1/4)^3}}{3} - 0 = \dfrac{2 \cdot 1/8}{3} = \dfrac{1}{12} \Rightarrow \boxed{E(X) = \dfrac{1}{12}}$

(γ) $E\left(\left|X - \frac{1}{8}\right|\right) = \displaystyle\int_{-\infty}^{\infty} \left|x - \frac{1}{8}\right| f(x)\, dx = \int_0^{1/4} \left|x - \frac{1}{8}\right| \frac{1}{\sqrt{x}}\, dx \quad (*)$

- $x - \frac{1}{8} > 0 \Rightarrow x > 1/8$ και $x - \frac{1}{8} < 0 \Rightarrow x < 1/8$, άρα

$(*) \Rightarrow \displaystyle\int_0^{1/8} \left(-x + \frac{1}{8}\right) \frac{1}{\sqrt{x}}\, dx + \int_{1/8}^{1/4} \left(x - \frac{1}{8}\right) \frac{1}{\sqrt{x}}\, dx = \int_0^{1/8} \left(\frac{1}{8\sqrt{x}} - \frac{x}{\sqrt{x}}\right) dx + \int_{1/8}^{1/4} \frac{x}{\sqrt{x}} - \frac{1}{8\sqrt{x}}\, dx$

$= \left[\dfrac{\sqrt{x}}{4} - \dfrac{2x^{3/2}}{3}\right]_0^{1/8} + \left[\dfrac{2x^{3/2}}{3} - \dfrac{\sqrt{x}}{4}\right]_{1/8}^{1/4} =$

$= \dfrac{1}{4\sqrt{8}} - \dfrac{2}{3\sqrt{8^3}} + \dfrac{2}{3\sqrt{4^3}} - \dfrac{1}{8} - \dfrac{2}{3\sqrt{8^3}} + \dfrac{1}{4\sqrt{8}} = \dfrac{1}{6\sqrt{2}} - \dfrac{1}{24}$

$= \dfrac{\sqrt{2}}{12} - \dfrac{1}{24} = \dfrac{2\sqrt{2} - 1}{24} \Rightarrow \boxed{E(|X - 1/8|) = \dfrac{2\sqrt{2} - 1}{24}}$

$\boxed{\text{ασκ. 46}}$

$$f_X(x) = \begin{cases} c(1-x^2), & x \in (-1,1) \\ 0, & x \notin (-1,1) \end{cases}$$

(α) Πρέπει $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-1}^{1} c(1-x^2)dx = \int_{-1}^{1} (c - x^2 c)dx$

$$= \left[cx - \frac{cx^3}{3} \right]_{-1}^{1} = c - \frac{c}{3} + c - \frac{c}{3} = 2c - \frac{2c}{3}$$

Είναι: $2c - \frac{2c}{3} = 1 \Rightarrow \frac{6c - 2c}{3} = 1 \Rightarrow 4c = 3 \Rightarrow \boxed{c = 3/4}$

(β) $E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^{1} x \cdot \frac{3}{4}(1-x^2)\, dx = \int_{-1}^{1} \left(\frac{3x}{4} - \frac{3x^3}{4} \right) dx$

$$= \left[\frac{3x^2}{8} - \frac{3x^4}{16} \right]_{-1}^{1} = \frac{3}{8} - \frac{3}{16} - \frac{3}{8} + \frac{3}{16} = 0$$

(γ) $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-1}^{1} \left(\frac{3x^2}{4} - \frac{3x^4}{4} \right) dx$

$$= \left[\frac{3x^3}{12} - \frac{3x^5}{20} \right]_{-1}^{1} = \frac{3}{12} - \frac{3}{20} + \frac{3}{12} - \frac{3}{20} = \frac{6}{12} - \frac{6}{20} =$$

$$= \frac{2}{10} \Rightarrow \boxed{E(x^2) = \frac{1}{5}}$$

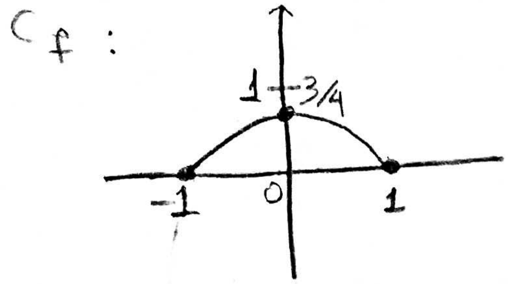
Άρα $VAR(x) = E(x^2) - E(x) = \frac{1}{5} - 0 \Rightarrow \boxed{VAR(x) = 1/5}$

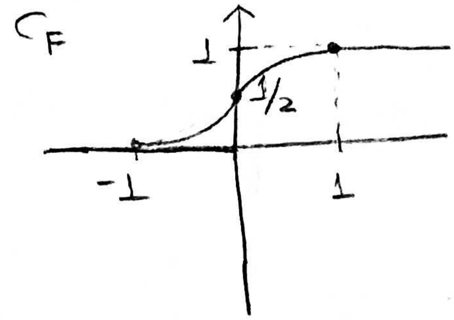
(δ) Για $x \in (-1,1)$ έχουμε

$$F_X(x) = \int_{-1}^{x} \left(\frac{3}{4} - \frac{3t^2}{4} \right) dt = \left[\frac{3t}{4} - \frac{3t^3}{12} \right]_{-1}^{x} = \frac{3x}{4} - \frac{3x^3}{12} + \frac{3}{4} - \frac{3}{12}$$

$$= \frac{9x - 3x^3 + 6}{12} \Rightarrow F_X(x) = \frac{-3x^3 + 9x + 6}{12} = \frac{-x^3 + 3x + 2}{4}$$

(Συνέχεια στην τελευταία σελίδα)





$\boxed{\text{ασκ. 47}}$

$$f_X(x) = \begin{cases} cx + e^{-x}, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Πρέπει $\int_{-\infty}^{\infty} f(x)\,dx = 1$ άρα

(α) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 (cx + e^{-x})\,dx = \left[\frac{cx^2}{2} - e^{-x}\right]_0^1 = \frac{c}{2} - \frac{1}{e} - 0 + 1$$

$$\Rightarrow \frac{c}{2} - \frac{1}{e} + \cancel{1} = \cancel{1} \Rightarrow \frac{c}{2} = \frac{1}{e} \Rightarrow \boxed{c = \frac{2}{e}}$$

$$f(x) = \begin{cases} \frac{2x}{e} + e^{-x}, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

(β) $$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 \frac{2x^2}{e} + xe^{-x}\,dx = \int_0^1 \frac{2x^2}{e}\,dx + \int_0^1 x(-e^{-x})'\,dx$$

$$= \left[\frac{2x^3}{3e}\right]_0^1 - \left[xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx =$$

$$= \frac{2}{3e} - \frac{1}{e} + \left[-e^{-x}\right]_0^1 = -\frac{2}{3e} - \frac{2}{e} + 1 = -\frac{2}{e} + 1$$

$$\Rightarrow \boxed{E(X) = -\frac{2}{e} + 1}$$

(γ) $$P(X < 0{,}5) = \int_{-\infty}^{0{,}5} f(x)\,dx = \int_0^{0{,}5} \left(\frac{2x}{e} + e^{-x}\right)dx = \left[\frac{x^2}{e} - e^{-x}\right]_0^{0{,}5}$$

$$= \frac{1}{4e} - e^{-0{,}5} + 1$$

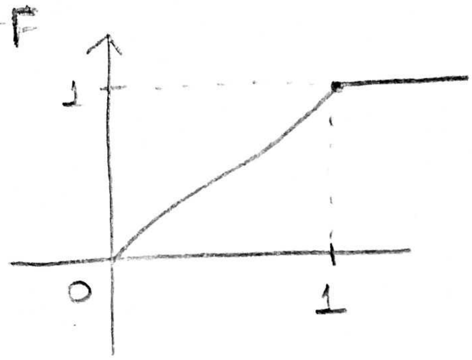
Για $x \in [0,1]$ έχουμε:

$$F_X(x) = \int_0^x \left(\frac{2t}{e} + e^{-t}\right)dt = \left[\frac{t^2}{e} - e^{-t}\right]_0^x = \frac{x^2}{e} - e^{-x} + 1$$

Για $x < 0$

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = 0$$

C_F



Για $x > 1$

$$F_X(x) = \overbrace{\int_{-\infty}^{0} f(t)\,dt}^{0} + \int_0^1 f(t)\,dt + \overbrace{\int_1^x f(t)\,dt}^{0} =$$

$$= \int_0^1 \left(\frac{2t}{e} + e^{-t}\right)dt = \left[\frac{t^2}{e} - e^{-t}\right]_0^1$$

$$= \frac{1}{e} - \frac{1}{e} - 0 + 1 = 1$$

Άρα $F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{e} - e^{-x} + 1, & x \in [0,1] \\ 1, & x > 1 \end{cases}$

$\boxed{\text{άσκ. 46 (β)}}$

Για $x > 1$

$$F(x) = \overbrace{\int_{-\infty}^{-1} f(t)\,dt}^{0} + \int_{-1}^{1} f(t)\,dt + \overbrace{\int_1^x f(t)\,dt}^{0} = \int_{-1}^{1}\left(\frac{3}{4} - \frac{3t^2}{4}\right)dt = \left[\frac{3t}{4} - \frac{3t^3}{12}\right]_{-1}^{1}$$

$$= \frac{3}{4} - \frac{3}{12} + \frac{3}{4} - \frac{3}{12} = \frac{18}{12} - \frac{6}{12} = \frac{12}{12} = 1$$

Για $x < -1$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

Άρα $F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{-x^3 + 3x + 2}{4}, & x \in (-1,1) \\ 1, & x > 1 \end{cases}$