ΠΕΤΣΑ ΓΕΩΡΓΙΑ
ΑΜ: 3200155
8η ΟΜΑΔΑ ΑΣΚΗΣΕΩΝ

1

Άσκηση 51

(α) $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

Θέτω $x = 3\sin\theta \Leftrightarrow \theta = \sin^{-1}(x/3)$

$\frac{dx}{d\theta} = 3\cos\theta \Rightarrow dx = 3\cos\theta\, d\theta$

Για $x=3$:

$3 = 3\sin\theta \Leftrightarrow 1 = \sin\theta \Rightarrow \boxed{\theta = \pi/2}$

Για $x=-3$

$-3 = 3\sin\theta \Leftrightarrow -1 = \sin\theta \Rightarrow \boxed{\theta = -\pi/2}$

Άρα: $\int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta = \int_{-\pi/2}^{\pi/2} 9\sqrt{1-\sin^2\theta}\; 3\cos\theta\, d\theta$

$= \int_{-\pi/2}^{\pi/2} 9\, |\cos\theta| \cdot \cos\theta\, d\theta = \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\, d\theta$

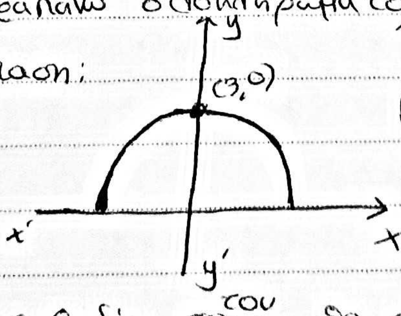
$= 9\int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2}\, d\theta = 9\int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta}{2} d\theta + 9\int_{-\pi/2}^{\pi/2} \frac{1}{2}\, d\theta$

⊛ γιατί: $\left(\frac{\sin 2\theta}{4}\right)' = \frac{\cos 2\theta}{4}\cdot 2 = \frac{\cos 2\theta}{2}$

$\overset{⊛}{=} 9\int_{-\pi/2}^{\pi/2} \left(\frac{\sin(2\theta)}{4}\right)' d\theta + 9\left[\frac{\theta}{2}\right]_{-\pi/2}^{\pi/2} =$

$\left[\frac{\sin 2\theta}{4}\right]_{-\pi/2}^{\pi/2} + 9\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = 9\,\frac{\sin \pi}{4} - 9\,\frac{\sin(-\pi)}{4} + 9\,\frac{2\pi}{4} = \boxed{\frac{9\pi}{2}}$

(β) ο τύπος του παραπάνω ολοκληρώματος θα έχει την εξής γρ. παράσταση: με ακτίνα $r=3$, άρα



Για $x=0$:

$y = \sqrt{9-x^2} \Rightarrow y = \sqrt{9} \Rightarrow y = 3$

Άρα το εμβαδόν του θα είναι: $\frac{\pi r^2}{2} = \boxed{\frac{\pi \cdot 9}{2}}$

~~άσκ 53~~

(α) $\lim_{n\to\infty}\frac{\int_1^n \sqrt{x}\log x\,dx}{n^2}$

Υπολογίζουμε το $\int_1^n \sqrt{x}\ln x\,dx = \int_1^n x^{1/2}\ln x\,dx = \int_1^n \left(\frac{2x^{3/2}}{3}\right)'\ln x\,dx$

$= \left[\frac{2x^{3/2}}{3}\ln x\right]_1^n - \int_1^n \frac{2x^{3/2}}{3}\cdot\frac{1}{x}\,dx$

$= \frac{2n^{3/2}}{3} - \frac{2}{3} - \int_1^n \frac{2x^{1/2}}{3}\,dx = \frac{2n^{3/2}}{3} - \frac{2}{3} - \frac{2}{3}\int_1^n\left(\frac{2x^{3/2}}{3}\right)'dx$

$= \frac{2n^{3/2}}{3} - \frac{2}{3} - \frac{2}{3}\left[\frac{2x^{3/2}}{3}\right]_1^n = \frac{2n^{3/2}}{3} - \frac{2}{3} - \frac{2}{3}\left(\frac{2n^{3/2}}{3} - \frac{2}{3}\right)$

$= \frac{2n^{3/2}}{3} - \frac{2}{3} - \frac{4n^{3/2}}{9} + \frac{4}{9} = \frac{6n^{3/2} - 6 - 4n^{3/2} + 4}{9}$

$= \frac{2n^{3/2} - 2}{9}$. Άρα $\lim_{n\to\infty}\frac{2n^{3/2}-2}{9n^2} = \lim_{n\to\infty}\frac{2n^{3/2}}{9n^2} = \boxed{0}$

άσκ 53

(β) $\lim_{x\to0^+}(x^2e^{1/x}) = \lim_{x\to0^+}\frac{e^{1/x}}{1/x^2}$

Θέτω $u = 1/x$, $\lim_{x\to0^+}\frac{1}{x} = \infty$, άρα $u\to\infty$.

Άρα: $\lim_{u\to\infty}\frac{e^u}{u^2} \underset{DLH}{\overset{\frac{\infty}{\infty}}{=}} \lim_{u\to\infty}\frac{e^u}{2u} \underset{DLH}{\overset{\frac{\infty}{\infty}}{=}} \lim_{u\to\infty} e^u = \boxed{\infty}$

(γ) $\lim_{x\to0^-} x^2e^{1/x} = \lim_{x\to0^-}\frac{e^{1/x}}{1/x^2}$. Θέτω $u = 1/x$, $\lim_{x\to0^-}\frac{1}{x} = -\infty$ άρα $u\to-\infty$

άρα $\lim_{u\to-\infty}\frac{e^u}{u^2} = \lim_{u\to-\infty}\frac{e^u}{2u} = \lim_{u\to-\infty}\frac{e^u}{2} = \frac{0}{2} = \boxed{0}$

(δ) $\lim_{n\to\infty}\frac{e^n+\cos n+\log n}{e^n} = \lim_{n\to\infty}\frac{e^n}{e^n} + \lim_{n\to\infty}\frac{\cos n}{e^n} + \lim_{n\to\infty}\frac{\log n}{e^n}$

$= 1 + 0 + \lim_{n\to\infty}\frac{1/n}{e^n} = 1 + \lim_{n\to\infty}\frac{1}{ne^n} = 1 + 0 = \boxed{1}$

Είναι: $|\cos n| \le 1 \Rightarrow \left|\frac{\cos n}{e^n}\right| \le \frac{1}{e^n} \Rightarrow -\frac{1}{e^n} \le \frac{\cos n}{e^n} \le \frac{1}{e^n}$

$\lim_{n\to\infty} -\frac{1}{e^n} = 0 = \lim_{n\to\infty}\frac{1}{e^n}$, άρα $\lim_{n\to\infty}\frac{\cos n}{e^n} = 0$

(ε)′ $\lim_{x\to 0^+} \sqrt{x}^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln\sqrt{x}^{\sqrt{x}}} = \lim_{x\to 0^+} e^{\sqrt{x}\ln\sqrt{x}}$

Θέτω $u = \sqrt{x}\ln\sqrt{x}$

$\lim_{x\to 0^+} u = \lim_{x\to 0^+} \sqrt{x}\ln\sqrt{x}$ θέτω $h = \sqrt{x}$, $\lim_{x\to 0^+} h = 0$

$\lim_{h\to 0^+} h\ln h = \lim_{h\to 0^+} \frac{\ln h}{1/h} \overset{\frac{-\infty}{\infty}}{\underset{DLH}{=}} \lim_{h\to 0^+} \frac{1/h}{-1/h^2} = \lim_{h\to 0^+} -\frac{h^2}{h}$

$= \lim_{h\to 0^+}(-h) = 0$. άρα $u \to 0$. Άρα $\lim_{x\to 0^+} e^{\sqrt{x}\ln\sqrt{x}} = \lim_{u\to 0} e^u = 1$.

(ς)′ $\lim_{h\to\infty} \int_1^h \frac{1+|\cos t|}{t}\,dt$.

Ισχύει ότι $\frac{1+|\cos t|}{t} \geq \frac{1}{t} \Rightarrow \int_1^h \frac{1+|\cos t|}{t}\,dt \geq \int_1^h \frac{1}{t}\,dt$. (1)

Είναι $\int_1^h \frac{1}{t}\,dt = [\ln|t|]_1^h = \ln h - \cancel{\ln 1}^{\,0} = \ln h$.

$\lim_{h\to\infty} \ln h = \infty$. Άρα από την (1) έχουμε

ότι $\lim_{h\to\infty} \int_1^h \frac{1+|\cos t|}{t}\,dt = \infty$

ασκ. 54

(a) $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$

$= -\int (\ln(\sin x + \cos x))'\,dx$

$= \boxed{-\ln(|\sin x + \cos x|) + C}$

αφού $(\ln(\sin x + \cos x))' = \frac{1}{\sin x + \cos x}(\sin x + \cos x)'$

$= \frac{\cos x - \sin x}{\sin x + \cos x}$

(β)' $\int \frac{e^x}{\cos^2 e^x}\,dx$ Θέτω $u=e^x$, $du=e^x\,dx$

$= \int \frac{1}{\cos^2 u}\,du = \int (\tan u)'\,du = \tan u + C = \tan e^x + C$

$\boxed{\text{ασκ. 51}}$

γ' $\int \frac{dx}{x(1+x^2)}$ (1)

$x^2+1>0$ άρα και το x μη δεν μηδενίζεται στο 0.

Άρα: $\frac{Ax+B}{x^2+1} + \frac{\Gamma}{x}$

$\frac{Ax^2+Bx+\Gamma(x^2+1)}{(x^2+1)\,x}$

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+\Gamma}{x^2+1}$ άρα

$1 = A(x^2+1) + Bx^2 + \Gamma x$

$1 = Ax^2 + A + Bx^2 + \Gamma x \Rightarrow 1 = x^2(A+B) + A + \Gamma x$

$\left.\begin{array}{l} A+B=0 \\ A=1 \\ \Gamma=0 \end{array}\right\}$ $\begin{array}{l} B=-1 \\ A=1 \\ \Gamma=0 \end{array}$ άρα $\frac{1}{x} + \frac{-x}{x^2+1} = \frac{1}{x(x^2+1)}$

Άρα το (1) γίνεται $\int \frac{1}{x} - \frac{x}{x^2+1}\,dx = \int \frac{1}{x} - \int \frac{x}{x^2+1}\,dx$

$= \ln|x| + C - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$ θέτω $u=x^2$ $du=2x\,dx$

$= \ln|x| + C - \frac{1}{2}\int \frac{1}{u+1}\,du = \ln|x| + C - \frac{1}{2}\ln|u+1| + C$

$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C$

ασκ 52

(α')

Θέτω

$h(n) = \frac{f(n)}{g(n)}$, αρκεί νδο $h(x) = \frac{f(x)}{g(x)} \to 0$

$$\lim_{x\to\infty} h(x) = \frac{a_0 n^v_v + b n^{v-1}_{v-1} + \dots + \gamma}{e^n} \overset{\text{n φορές DLH}}{=} \lim_{x\to\infty} \frac{a_0 v!}{e^x} = 0$$

Άρα και $\lim_{n\to\infty} h(n) = 0$

επειδή το όριο μιας συνάρτησης θα ισούται με το όριο της αντίστοιχης ακολουθίας

(β') Αν $f_1(n) = o(g(n))$ και $f_2(n) = o(g(n))$, τότε $f_1(n) f_2(n) = o(g(n))$

Έστω ότι ισχύει. άρα $\lim_{n\to\infty} \frac{f_1(n)}{g(n)} = 0$, $\lim_{n\to\infty} \frac{f_2(n)}{g(n)} = 0$

Αν $f_1(n) f_2(n) = o(g(n))$, τότε $\lim_{n\to\infty} \frac{f_1(n) f_2(n)}{g(n)} = 0$

Έστω π.χ. $f_1(n) = n$

$f_2(n) = n^2 + 1$

$g(n) = n^3$

$$\lim_{n\to\infty}\left(\frac{f_1(n)}{g(n)}\right) = \lim_{n\to\infty}\left(\frac{n}{n^3}\right) = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\lim_{n\to\infty}\left(\frac{f_2(n)}{g(n)}\right) \quad \lim_{n\to\infty} \frac{n^2+1}{n^3} = \lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \frac{f_1(n) f_2(n)}{g(n)} = \lim_{n\to\infty} \left(\frac{(n^2+1)\, n}{n^3}\right) = \lim_{n\to\infty}\left(\frac{n^3+n}{n^3}\right) = \lim_{n\to\infty}\left(\frac{n^3}{n^3}\right) = 1 \neq 0$$

, άρα η αρχική υπόθεση δεν ισχύει

ασκ 52

γ' $f(n) = o(g(n))$, $g(n) = o(h(n))$, τότε $f(n) = o(h(n))$

νδο $\boxed{\lim_{n\to\infty} \frac{f(n)}{h(n)} = 0}$

Ξέρουμε ότι:
$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, $\lim_{n\to\infty} \frac{g(n)}{h(n)} = 0$

Είναι: $\lim_{n\to\infty} \frac{f(n)}{h(n)} = \lim_{n\to\infty} \frac{f(n)\cdot g(n)}{h(n)\, g(n)} = \lim_{n\to\infty} \left(\frac{f(n)}{g(n)} \cdot \frac{g(n)}{h(n)}\right) = 0\cdot 0$
$= 0.$

Άρα ισχύει.

άσκ. 53

(a)

$$\int_1^h \sqrt{x}\ln x\, dx = \int_1^h x^{1/2}\ln x\, dx = \int_1^h \left(\frac{2x^{3/2}}{3}\right)'\ln x\, dx =$$

$$= \left[\frac{2x^{3/2}}{3}\ln x\right]_1^h - \int_1^h \frac{2x^{3/2}}{3x}dx = \frac{2h^{3/2}}{3}\ln h - \cancel{\frac{2\cdot 1^{3/2}}{3}\ln 1} - \frac{2}{3}\int_1^h x^{1/2}dx$$

$$= \frac{2h^{3/2}}{3}\ln h - \frac{2}{3}\int_1^h\left(\frac{2x^{3/2}}{3}\right)'dx = \frac{2h^{3/2}}{3}\ln h - \frac{2}{3}\left[\frac{2x^{3/2}}{3}\right]_1^h =$$

$$= \frac{2h^{3/2}\ln h}{3} - \frac{4h^{3/2}}{9} + \frac{4}{9} = \frac{6h^{3/2}\ln h - 2h^{3/2} + 2}{9} = \frac{2\left(3h^{3/2}\ln h - h^{3/2} + 1\right)}{9}$$

Για: $\lim\limits_{h\to\infty}\left[\frac{2}{9}h^{3/2}\left(3\ln h - \frac{1}{2} + \frac{1}{2h^{3/2}}\right)\right] = \infty\left(\infty - \frac{1}{2} + 0\right) = \infty$

Άρα $\lim\limits_{h\to\infty}\frac{\int_1^h \sqrt{x}\log x\, dx}{h^2} = \lim\limits_{h\to\infty}\frac{\frac{2}{9}h^{3/2}\left(3\ln h - 1/2 + 1/2h^{3/2}\right)}{h^2} = \lim\limits_{h\to\infty}\frac{2}{9}h^{-1/2}\left(\ln h - 1/3 \cdot 1/h^{3/2}\right)$

$$= \frac{2}{9}\lim_{h\to\infty}\frac{3\ln h - 1/2 + 1/2h^{3/2}}{h^{1/2}} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{DLH}{=}} \frac{2}{9}\lim_{h\to\infty}\frac{3/h - 0 + \frac{-3}{4}h^{-5/2}}{-\frac{1}{2}h^{-1/2}}$$

$$= \frac{2}{9}\lim_{h\to\infty}\frac{3/h - 3/4h^{5/2}}{-1/2h^{1/2}} = -\frac{2}{9}\lim_{h\to\infty}\frac{\frac{1}{h}\left(3 - \frac{3}{4h^{3/2}}\right)}{\frac{1}{2\sqrt{h}}} = \frac{2}{9}\lim_{h\to\infty}\frac{2h^{1/2}}{h}\left(3 - \frac{3}{4h^{3/2}}\right)$$

$$= -\frac{2}{9}\lim_{h\to\infty} 2h^{-1/2}\left(3 - 3/4h^{3/2}\right) = -\frac{2}{9}\lim_{h\to\infty}\frac{2\left(3 - \frac{3}{4h^{3/2}}\right)}{\sqrt{h}}$$

$$= -\frac{12}{9}\cdot 0 = 0$$