ΠΕΤΣΑ ΓΕΩΡΓΙΑ
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ΟΜΑΔΑ ΑΣΚΗΣΕΩΝ 9

άσκ. 52

$$f_{XY}(x,y)=\begin{cases}\frac{1}{5}(x+2y), & (x,y)\in[0,1]\times[0,2]\\ 0, & \text{αλλιώς.}\end{cases}$$

(α) $$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy=\frac{1}{5}\int_0^1\left(x\int_0^2(x\cdot y+2y^2)dy\right)dx$$

$$=\frac{1}{5}\int_0^1\left(x\left[\frac{xy^2}{2}+\frac{2y^3}{3}\right]_0^2\right)dx=\frac{1}{5}\int_0^1 x\left(2x+\frac{16}{3}-0\right)dx$$

$$=\frac{1}{5}\int_0^1\left(2x^2+\frac{16x}{3}\right)dx=\frac{1}{5}\left[\frac{2x^3}{3}+\frac{16x^2}{6}\right]_0^1=$$

$$=\frac{1}{5}\left(\frac{2}{3}+\frac{8}{3}-0\right)=\frac{1}{5}\cdot\frac{10}{3}=\frac{10}{15}=\frac{2}{3}$$

(β) • $$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy=\int_0^2\frac{1}{5}(x+2y)\,dy=\frac{1}{5}\int_0^2(x+2y)\,dy$$

$$=\frac{1}{5}\left[xy+y^2\right]_0^2=\frac{1}{5}(x\cdot 2+4-0)=\boxed{\frac{2x+4}{5},\ x\in[0,1]}$$

• $$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx=\int_0^1\left[\frac{1}{5}(x+2y)\right]dx=\frac{1}{5}\int_0^1(x+2y)\,dx$$

$$=\frac{1}{5}\left(\left[\frac{x^2}{2}+2xy\right]_0^1\right)=\frac{1}{5}\left(\frac{1}{2}+2y-0\right)=$$

$$=\frac{1}{5}\left(\frac{1+4y}{2}\right)=\boxed{\frac{1+4y}{10},\ y\in[0,2]}$$

Συνολικά: $f_X(x)=\begin{cases}\frac{2x+4}{5}, & x\in[0,1]\\ 0, & \text{αλλιώς.}\end{cases}$ και $f_Y(y)=\begin{cases}\frac{1+4y}{10}, & y\in[0,2]\\ 0, & \text{αλλιώς.}\end{cases}$

άσκ. 53

$$f_{XY}(x,y)=\begin{cases} 6x^c y, & x,y\in[0,1]\times[0,1] \\ 0, & \text{αλλιώς} \end{cases}$$

(α) Το ολοκλήρωμα της πυκνότητας σ'όλο το επίπεδο πρέπει να ισούται με την μονάδα, άρα:

$$\iint_{R\times R} f_{XY}(x,y)\,dA = 6\int_0^1\left(\int_0^1 x^c\cdot y\,dy\right)dx = 6\int_0^1 x^c\left(\int_0^1 y\,dy\right)dx$$

$$= 6\int_0^1 x^c\left[\frac{y^2}{2}\right]_0^1 dx = 6\int_0^1 \frac{x^c}{2}\,dx = 3\int_0^1 x^c\,dx$$

$$= 3\left[\frac{x^{c+1}}{c+1}\right]_0^1 = 3\left(\frac{1}{c+1}-0\right) = \frac{3}{c+1}$$

Πρέπει $\frac{3}{c+1}=1 \Rightarrow 3 = c+1 \Rightarrow \boxed{c=2}$

Άρα $f_{XY}(x,y)=\begin{cases} 6yx^2, & x,y\in[0,1]\times[0,1] \\ 0, & \text{αλλού} \end{cases}$

(β) • $f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 6yx^2\,dy = \left[\frac{6y^2x^2}{2}\right]_0^1$

$$= \left[3y^2x^2\right]_0^1 = 3x^2-0 = \boxed{3x^2 \quad x\in[0,1]}$$

• $f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^1 (6yx^2)\,dx = \left[2x^3y\right]_0^1$

$$= 2y-0 = \boxed{2y,\ y\in[0,1]}$$

Συνολικά: $f_X(x)=\begin{cases} 3x^2, & x\in[0,1] \\ 0, & \text{αλλιώς} \end{cases}$ και $f_Y(y)=\begin{cases} 2y, & y\in[0,1] \\ 0, & \text{αλλιώς.} \end{cases}$

(γ) • $P\left(X<\frac{1}{3}\right)=\int_0^{1/3} 3x^2\,dx=\left[x^3\right]_0^{1/3}=\frac{1}{27}-0=\boxed{\frac{1}{27}}$

• $P(Y>2X)=P(X\leq Y/2)=\int_0^{1/2}\left(\int_0^{y/2} 6yx^2\,dx\right)dy$

$=\int_0^{1/2}\left[2x^3y\right]_0^{y/2}dy=\int_0^{1/2}\left(2\frac{y^3\cdot y}{8}-0\right)dy$

$=\int_0^{1/2}\frac{y^4}{4}\,dy=\left[\frac{y^5}{20}\right]_0^{1/2}=1/2^5\cdot 20$

$\boxed{\text{ασκ. 55}}$

$P(A>0)=1-P(A\leq 0)$. (1)

• $P(A\leq 0)=P(X_1<3{,}5,\ X_2<3{,}5,\ X_3<3{,}5,\ X_4<3{,}5,\ X_5<3{,}5)$

$=P(X_1<3{,}5)\cdot P(X_2<3{,}5)\cdot P(X_3<3{,}5)\cdot P(X_4<3{,}5)\cdot P(X_5<3{,}5)$

$=\left(\int_0^{3,5}\frac{1}{12-0}\,dx\right)^5=\left(\int_0^{3,5}\frac{1}{12}\,dx\right)^5=\left(\left[\frac{x}{12}\right]_0^{3,5}\right)^5=\left(\frac{35}{120}\right)^5=0{,}0021.$

Άρα (1) $\Rightarrow P(A>0)=1-\left(\frac{35}{120}\right)^5=0{,}9978$

$\boxed{\text{ασκ. 54}}$

$E(X)=\int_0^{\infty} y f_X(y)\,dy=\int_0^{\infty}\left(\int_0^{y} d\cancel{x}\right)f_X(y)\,dy=\int_0^{\infty}\left(\int_0^{y} f_X(y)\,dx\right)dy=$

$=\int_0^{\infty}\left(\int_x^{\infty} f_X(y)\,dy\right)dx=\int_0^{\infty}P(X>x)\,dx=\int_0^{\infty}1-P(\cancel{X\leq x})\,dx=$

$=\int_0^{\infty}(1-F_X(x))\,dx.$

$\boxed{\text{ασκ. 56}}$

$X \sim$ γεωμ$(1/4)$ $\quad Y \sim$ ομοιόμορφη κατανομή στο $[-X, X]$

(α) $$P(Y > 3/2) = \sum_{k=1}^{\infty} P(Y > 3/2, X = k) = \sum_{k=1}^{\infty} P(Y > 3/2 \mid X = k) \cdot P(X = k)$$

$$= \cancel{0}(1-1) \cdot P(X=k) + \sum_{k=2}^{\infty} \left(\int_{3/2}^{k} \frac{1}{k-(-k)} dx \right) \frac{1}{4} \left(1 - \frac{1}{4}\right)^{k-1} =$$

$$= \sum_{k=2}^{\infty} \frac{k - 3/2}{2k} \cdot \frac{3^{k-1}}{4^k} = \sum_{k=2}^{\infty} \left(\frac{1}{6} \left(\frac{3}{4}\right)^k - \frac{1}{4k} \left(\frac{3}{4}\right)^k \right) = \frac{1}{6} \sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^k - \frac{1}{4} \sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{3}{4}\right)^k$$

$$= \frac{1}{6} \cdot \frac{(3/4)^2}{1 - 3/4} - \frac{1}{4} \left(-\log\left(1 - \frac{3}{4}\right) - \frac{3}{4} \right) = \frac{3}{8} - \frac{\log 4}{4} + \frac{3}{16}$$

(β) $$P(X = 2 \mid Y > 3/2) = \frac{P(X = 2, Y > 3/2)}{P(Y > 3/2)} = \frac{P(X=2) \cdot P(Y > 3/2 \mid X = 2)}{P(X > 3/2)} =$$

$$= \frac{\left(\frac{3}{16}\right) \int_{3/2}^{2} \frac{1}{4} dx}{\frac{9}{16} - \frac{\log 4}{4}}$$