ΠΕΤΣΑ ΓΕΩΡΓΙΑ
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5η ΟΜΑΔΑ ΑΣΚΗΣΕΩΝ

$\boxed{\text{ασκ. 33}}$

• $f(x) = \tan x$

$$f'(x) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{\tan x - \tan x_0}{x-x_0} = \lim_{x\to x_0}\left(\frac{\frac{\sin x}{\cos x}}{x-x_0} - \frac{\frac{\sin x_0}{\cos x_0}}{x-x_0}\right)$$

$$= \lim_{x\to x_0}\left(\frac{\sin x}{\cos x(x-x_0)} - \frac{\sin x_0}{\cos x_0(x-x_0)}\right) = \lim_{x\to x_0}\left(\frac{\sin x\cdot\cos x_0 - \sin x_0\cdot\cos x}{\cos x\cdot\cos x_0(x-x_0)}\right)$$

$$\overset{(1)}{=} \lim_{x\to x_0}\frac{\sin(x-x_0)}{(x-x_0)\cdot\cos x\cdot\cos x_0} = \lim_{x\to x_0}\frac{\sin(x-x_0)}{x-x_0}\cdot\lim_{x\to x_0}\frac{1}{\cos x\cdot\cos x_0} \overset{(2)}{=}$$

$$= \frac{1\cdot 1}{\cos^2 x_0} = \frac{1}{\cos^2 x_0}.$$

(1) Γνωρίζουμε ότι $\sin(x-y) = \sin x\cos y - \sin y\cos x$

(2) $\lim_{x\to x_0}\frac{\sin(x-x_0)}{x-x_0}$ θέτω $h = x-x_0$ άρα $h\to 0$

οπότε $\lim_{h\to 0}\frac{\sin h}{h} = 1$

• $f(x) = \cot x$

$$f'(x) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\cot x - \cot x_0}{x-x_0} = \lim_{x\to x_0}\left(\frac{\frac{\cos x}{\sin x}}{x-x_0} - \frac{\frac{\cos x_0}{\sin x_0}}{x-x_0}\right)$$

$$= \lim_{x\to x_0}\left(\frac{\cos x}{\sin x(x-x_0)} - \frac{\cos x_0}{\sin x_0(x-x_0)}\right) = \lim_{x\to x_0}\left(\frac{\cos x\cdot\sin x_0 - \cos x_0\cdot\sin x}{\sin x\ (x-x_0)\sin x_0}\right)$$

$$= \lim_{x\to x_0}\left(-\frac{\sin(x_0-x)}{x_0-x}\right)\cdot\lim_{x\to x_0}\frac{1}{\sin x\sin x_0} = -1\cdot\frac{1}{\sin^2 x_0} = \frac{-1}{\sin^2 x_0}$$

$\boxed{\text{ασκ. 34}}$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^2 e^{\cos(1/x)} - 0}{x-0} = \lim_{x\to 0} x e^{\cos(1/x)}$$

$-1 \le \cos(1/x) \le 1 \Rightarrow e^{-1} \le e^{\cos(1/x)} \le e^{1}$

Αν $x>0$:

$\frac{x}{e} \le x \cdot e^{\cos(1/x)} \le x \cdot e$

$\lim_{x\to 0^+} \frac{x}{e} = 0 = \lim_{x\to 0^+} x\cdot e$ άρα $\lim_{x\to 0^+}\left(x e^{\cos(1/x)}\right) = 0$ ①

Αν $x<0$:

$\frac{x}{e} \ge x e^{\cos(1/x)} \ge x\cdot e$

$\lim_{x\to 0^-} \frac{x}{e} = 0 = \lim_{x\to 0^-} (x\cdot e)$ άρα $\lim_{x\to 0^-}\left(x\cdot e^{\cos(1/x)}\right) = 0$ ②

Άρα από ① και ② έχουμε ότι

$\lim_{x\to 0} x e^{\cos(1/x)} = 0$

$\boxed{\text{ασκ. 35}}$

$f(x) = \cot x$, $x \in \mathbb{R}$

$f^{-1}(y) = \cot^{-1} y$, $y \in \mathbb{R}$

$(f^{-1}(y))' = \frac{1}{f'(x)} = \frac{1}{-\frac{1}{\sin^2 x}} = -\frac{\sin^2 x}{1} = -\frac{\sin^2 x}{\sin^2 x + \cos^2 x}$

$= -\frac{1}{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} = \frac{-1}{1 + \frac{\cos^2 x}{\sin^2 x}} = \frac{-1}{1 + \left(\frac{\cos x}{\sin x}\right)^2} =$

$= -\frac{1}{1+\cot^2 x} = \frac{-1}{1+y^2}$, $y \in \mathbb{R}$

$\boxed{\text{ασκ. 36}}$

$f(a) = g(a)$ ① , $f'(x) \ge g'(x)$ στο I

Θέτουμε την $h(x) = f(x) - g(x)$ στο I. Η h είναι παραγωγίσιμη ως διαφορά ~~...~~ παραγωγίσιμων συναρτήσεων.

Άρα $h'(x) = f'(x) - g'(x) \ge 0$ (από υπόθεση)

άρα $h \uparrow$ στο I. Για $x \in I$ έχουμε ότι $a \le x \overset{h\uparrow}{\Rightarrow}$

$h(a) \le h(x) \Rightarrow f(a) - g(a) \le f(x) - g(x) \overset{①}{\Rightarrow} 0 \le f(x) - g(x)$

$\Rightarrow g(x) \le f(x) \Rightarrow f(x) \ge g(x)$, το ζητούμενο

$\boxed{\text{ασκ. 37}}$

(γ') $\int \arcsin x\, dx = \int \sin^{-1} x\, dx$ θέτω $u = \sin^{-1}(x)$ άρα $du = \frac{1}{\sqrt{1-x^2}} dx$

$= \int u\, dx = \int u(x)' dx = u \cdot x - \int x\, du$

$= \sin^{-1}(x) \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}} dx$ (A)

θέτω $u = 1 - x^2$ άρα $du = -2x dx \Rightarrow \frac{du}{-2x} = dx$

(A) $= x\sin^{-1}(x) - \int \frac{\cancel{x}}{\sqrt{u}} \frac{du}{-2\cancel{x}} = x \sin^{-1}(x) + \frac{1}{2}\int u^{-1/2} du$

$= x\sin^{-1}(x) + \frac{1}{2} \frac{u^{-1/2+1}}{-1/2+1} + c = x\sin^{-1}(x) + u^{1/2} + c$

$= x\sin^{-1} x + \sqrt{u} + c = x\arcsin x + \sqrt{1-x^2} + c$

(δ') $\int \cos^{-1} x\, dx$. θέτω $u = \cos^{-1} x$ άρα $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$= x\cos^{-1} x - \int -x \frac{1}{\sqrt{1-x^2}} dx$ θέτω $w = 1 - x^2$ άρα $dw = -2x dx \Rightarrow -\frac{dw}{2x} = dx$

$= x\cos^{-1} x - \frac{1}{2}\int x \frac{1}{\sqrt{w}} \frac{1}{2x} dw = x\cos^{-1} x - \frac{1}{2}\left(w^{-1/2}\right)$

$= x\cos^{-1} x - \frac{1}{2} \cdot 2 \frac{w^{-1/2+2/2}}{1/2} = x\cos^{-1} x - w^{1/2} + c$

$= x \arccos x - \sqrt{1-x^2} + c$

(ε') Ξέρουμε ότι $\cos 2x = \cos^2 x - \sin^2 x$, $\cos^2 x = 1 - \sin^2 x$ άρα

$\cos 2x = 1 - \sin^2 x \Rightarrow \boxed{\sin^2 x = \frac{1 - \cos 2x}{2}}$ άρα

$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2} dx = \frac{x}{2} - \frac{\cos 2x}{2} \cdot \frac{1}{2} + c$

$= \frac{x}{2} - \frac{\sin 2x}{4} + c$

(ζ') $\int \cos^2 x\,dx = \int 1-\sin^2 x = \int 1\,dx - \int \sin^2 x\,dx$

$= x - \frac{x}{2} + \frac{\sin 2x}{4} + c = \frac{x}{2} + \frac{\sin 2x}{4} + c$

(ι') $\int \tan^2 x = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}\right) dx$

$= \int \sec^2 x - 1\,dx = \tan x - x + c$

(η') $\int \sqrt{1+\sqrt{x}}\,dx$ Θέτω $u = 1+\sqrt{x} \Rightarrow \sqrt{x} = u-1$

$= \int \sqrt{u}\,dx = \int \sqrt{u}(2\sqrt{x})\,du$ $\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,du$

$= \int \sqrt{u}\,(2u-2)\,du = \int \left(2u^{3/2} - 2u^{1/2}\right) du =$

$= \frac{2u^{5/2}}{5/2} - \frac{2u^{3/2}}{3/2} + c = \frac{4}{5}u^{5/2} - \frac{4u^{3/2}}{3} + c$

$= \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + c$

(α') $\int x^2 \sin x\,dx = \int x^2(-\cos x)'\,dx = -x^2\cos x + \int 2x\cos x\,dx$

$= -x^2\cos x + 2\int x\cos x\,dx = -x^2\cos x + 2\int x(\sin x)'\,dx =$

$= -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right) =$

$= -x^2\cos x + 2x\sin x - 2\cos x + c$

(β') $\int x^2\cos x\,dx = \int x^2(\sin x)'\,dx = x^2\sin x - \int 2x\sin x\,dx$

$= x^2\sin x - 2\int x\sin x\,dx = x^2\sin x - 2\int x(-\cos x)'\,dx$

$= x^2\sin x - 2\left(-x\cos x + \int \cos x\,dx\right) = x^2\sin x + 2x\cos x - 2\sin x\,dx + c$