Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

Monte Carlo Simulation for Pricing

The random walk of assets

One of the foundations of quantitative finance theory is the random walk for asset prices. Modeling financial instruments as random walks has been one of the great success stories of finance and economics. The random walk theory leads on to the modern portfolio theory of Markowitz, the continuous-time asset allocation models of Merton and the Black-Scholes theory of option pricing The models for the random evolution of stock prices require knowledge of just two quantities, the expected return on the asset and the asset's volatility. Figure 1 shows several simulations of the path of an asset.

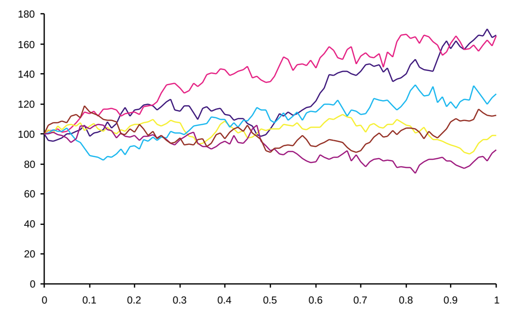

Figure 1: Simulations of an asset. Each path has the same expected return and volatility, but eachis different.

Real and risk-neutral worlds

Because the theory for the behavior of assets is founded upon randomness and probability, it is natural for simulations to play a key role in the application of the theories to practice. These simulation methods are usually called Monte Carlo, for obvious reasons. Phelim Boyle is one of the famous names in the use of Monte Carlo methods to price derivatives. He showed how to find the fair value of an option by simulating many, many possible asset price paths. It's an interesting but subtle point that the model for assets assumes random price paths yet the Black-Scholes theory uses dynamic delta hedging to eliminate all randomness and hence risk from an option portfolio. A result of this is that when you price an option you don't actually simulate what the asset might do in practice but what it would do in an artificial world. This world is called the *risk-neutral world*. The difference between the real world and the risk-neutral world is in the value of the asset's expected return. So, when you come to simulate asset price paths you have to ensure that the expected return on the asset is set to the risk-free rate. In words, the theoretical value of an option is "the expected value of the present value of the payoff under a risk-neutral random walk." Whether you price by Monte Carlo simulation, analytical solution of the Black-Scholes equation or solve the Black-Scholes equation numerically, you should get the same answer.

Advantages of MC

The main advantage of Monte Carlo simulations for pricing derivatives is in the ease of programming. It's a technique that is very simple to program, and can even be done on a spreadsheet. It is straightforward to price quite exotic contracts with fancy path dependency. You can even price many path-dependent contracts simultaneously for relatively little extra computational effort. The disadvantages are in speed of computation and in the pricing of derivatives with in-built decisions, such as American options. As far as speed is concerned, the finite difference solution of a Black-Scholes-type equation will be faster than a Monte Carlo simulation as long as there are four or fewer underlying assets. But, with five or more underlyings, such as you get in basket options, you'd be better off using Monte Carlo simulations.

Quasi MC

Very occasionally you will find yourself having to price a basket option with many, many underlyings. If the contract is European, independent of price paths, on lognormal assets (asset movements scale with asset price) with constant volatilities and constant correlations (some big 'ifs'!) then there is a nice formula for the option's fair value. This formula takes the form of an expectation, or a multi-dimensional integral. This expectation can be estimated using Monte Carlo simulations. However, there's a faster technique, called Quasi-Monte Carlo (QMC), that is even better. Instead of choosing random numbers in the simulation you choose carefully distributed but completely deterministic numbers. This technique has been around since the 1960s but has been used in finance for less than a decade. The beauty of the various Monte Carlo methods is that the computational time is virtually independent of the number of dimensions, the number of underlying assets. The basic Monte Carlo method gives an accuracy of order $1/\sqrt{N}$ where N is the number of simulations. So for an extra decimal place in accuracy you need 100 times as many paths, and the program will take 100 times as long to run. However, if you can use one of the QMC methods for pricing then you get a much better order of accuracy of $1/N$. The extra decimal place accuracy will only take ten times as long to compute.

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