Chapter 19

Heath, Jarrow & Morton and Brace, Gatarek & Musiela Models

1. **Derive the equation for the evolution of the forward rate:**

$$dF(t;T) = \frac{\partial}{\partial T}\left(\tfrac{1}{2}\sigma^2(t,T) - \mu(t,T)\right)dt - \frac{\partial}{\partial T}\sigma(t,T)\,dX,$$

from

$$dZ(t;T) = \mu(t,T)Z(t;T)\,dt + \sigma(t,T)Z(t;T)\,dX,$$

and

$$F(t;T) = -\frac{\partial}{\partial T}\log Z(t;T).$$

If

$$dZ = \mu Z dt + \sigma Z dX,$$

then Itô's Lemma gives us that

$$d\log Z = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma dX.$$

Differentiating with respect to T,

$$\frac{\partial}{\partial T}d\log Z = \frac{\partial}{\partial T}\left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \frac{\partial}{\partial T}\sigma dX,$$

and

$$dF = \frac{\partial}{\partial T}\left(\tfrac{1}{2}\sigma^2 - \mu\right)dt - \frac{\partial}{\partial T}\sigma dX.$$

2. **Perform a simulation, using the method of Section 19.7, to value an option on a zero-coupon bond. You will need to decide upon a suitable form for $\nu(t,T)$, the forward rate volatility. Does your choice have a standard representation as a model for the spot rate?**

To price a derivative using a Monte Carlo simulation perform the following steps. Today is t^* when we know the forward rate curve $F(t^*;T)$:

(a) Simulate a realized evolution of the whole risk-neutral forward rate curve for the necessary length of time, until T^*, say. This requires a simulation of

$$dF(t;T) = m(t,T)\,dt + \nu(t,T)\,dX,$$

where

$$m(t,T) = \nu(t,T)\int_t^T \nu(t,s)\,ds.$$

After this simulation we will have a realization of $F(t;T)$ for $t^* \le t \le T^*$ and $T \ge t$. This is a realization of the whole forward rate path.

(b) Using this forward rate path calculate the value of all the cashflows that would have occured.

(c) Using the realized path for the spot interest rate $r(t)$ calculate the present value of these cashflows.

(d) Return to Step 1 to perform another realization, and continue until you have a sufficiently large number of realizations to calculate the expected present value as accurately as required.

A suitable choice for $\nu(t,T)$ would be

$$\nu(t,T) = c.$$

Since

$$\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T),$$

we find

$$\sigma(t,T) = -c(T-t),$$

and this is an HJM representation of the Ho & Lee model.

3. **Use forward rate data to perform a Principal Component Analysis. What are the three main components in the forward price movements and what are their weights?**

For a principal component analysis, perform the following steps:

(a) Calculate the changes in the forward rates. Find the covariance matrix ($\mathbf{M}$) for the covariances between the changes in the rates of different maturities. This step is shown if Figure 19.1

(b) Use the power method to find the normalized principal eigenvector ($\mathbf{v}$) and associated eigenvalue (λ) of the covariance matrix. This is the first principal component.

(c) Define a new matrix

$$\mathbf{N} = \mathbf{M} - \lambda \mathbf{v}\mathbf{v}^T.$$

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Forward rates:**				**Changes in rates:**								
2		**1month**	**3 month**	**6 month**	**1 month**	**3 month**	**6 month**						
3	22-Sep-88	8.25000	8.31250	8.56250									
4	23-Sep-88	8.25000	8.31250	8.56250	0.00000	0.00000	0.00000						
5	26-Sep-88	8.31250	8.37500	8.62500	0.06250	0.06250	0.06250						
6	27-Sep-88	8.31250	8.43750	8.62500	0.00000	0.06250	0.06250		**1 month**	**3 month**	**6 month**		
7	28-Sep-88	8.42188	8.50000	8.81250	0.10938	0.06250	0.12500	1 month	0.007501				
8	29-Sep-88	8.37500	8.68750	8.81250	–0.04688	–0.18750	0.00000	3 month	0.003831	0.004225			
9	30-Sep-88	8.31250	8.62500	8.75000	–0.06250	–0.06250	–0.06250	6 month	0.003628	0.004020	0.004997		
10	3-Oct-88	8.31250	8.62500	8.68750	0.00000	0.00000	–0.06250						
11	4-Oct-88	8.31250	8.56250	8.68750	0.00000	–0.06250	0.00000	**Scaled conariance matrix:**					
12	5-Oct-88	8.31250	8.56250	8.68750	0.00000	0.00000	0.00000		**1 month**	**3 month**	**6 month**		
13	6-Oct-88	8.31250	8.56250	8.68750	0.00000	0.00000	0.00000	1 month	0.119072				
14	7-Oct-88	8.31250	8.52500	8.75000	0.00000	0.06250	0.06250	3 month	0.060822	0.067075			
15	10-Oct-88	8.25000	8.56250	8.56250	–0.06250	–0.06250	–0.18750	6 month	0.057590	0.063822	0.079320		
16	11-Oct-88	8.25000	8.56250	8.62500	0.00000	0.00000	0.06250						
17	12-Oct-88	8.31250	8.62500	8.68750	0.06250	0.06250	0.06250						
18	13-Oct-88	8.31250	8.64063	8.68750	0.00000	0.01563	0.00000						
19	14-Oct-88	8.31250	8.62500	8.62500	0.00000	–0.01563	–0.06250						
20	17-Oct-88	8.31250	8.62500	8.62500	0.00000	0.00000	0.00000						
21	18-Oct-88	8.31250	8.62500	8.62500	0.00000	0.00000	0.00000						
22	19-Oct-88	8.31250	8.62500	8.62500	0.00000	0.00000	0.00000						
23	20-Oct-88	8.37500	8.68750	8.68750	0.06250	0.06250	0.06250						
24	21-Oct-88	8.37500	8.68750	8.68750	0.00000	0.00000	0.00000						
25	24-Oct-88	8.37500	8.68750	8.75000	0.00000	0.00000	0.06250						
26	25-Oct-88	8.37500	8.68750	8.75000	0.00000	0.00000	0.00000						
27	26-Oct-88	8.37500	8.68750	8.75000	0.00000	0.00000	0.00000						
28	27-Oct-88	8.37500	8.68750	8.68750	0.00000	0.00000	–0.06250						

= B4-B3

= COVAR(E4:E1721,F4:F1721)

= I8*SQRT(252)

Figure 19.1 One-, three- and six-month rates and the changes.

Return to step (b) to find the first principal component of **N** which will be the second principal component of **M**. Continue repeating these two steps until you have found as many principal components as you require. The magnitude of each eigenvalue corresponds to the weight of each component.

Figure 19.2 shows the first three principal components for US forward rate data. The first component is relatively flat, similar to a line

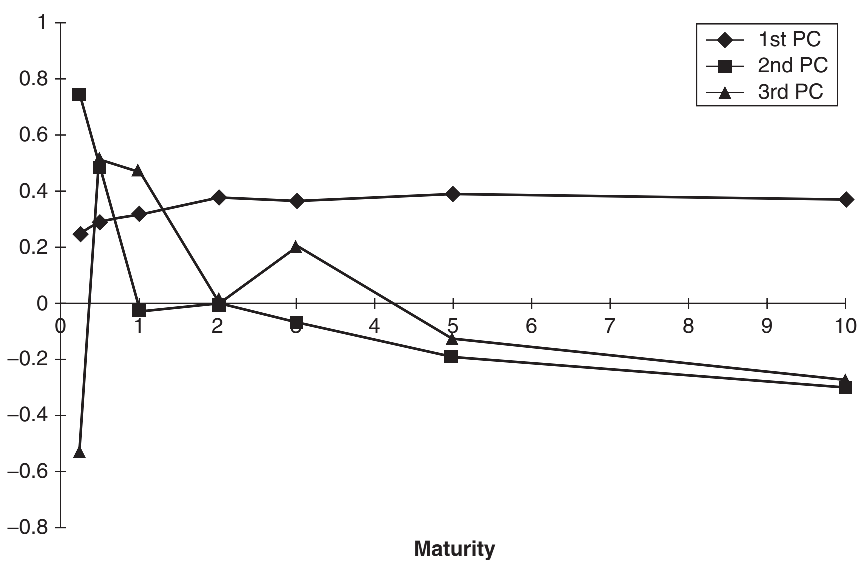

Figure 19.2 Principal components for the US forward rate curve.

of the form '$y = a$', and represents a parallel shift of the forward rate curve. The second is similar to a line of the form '$y = -ax$' and represents a tilt of the forward rate curve. The third is similar to a curve of the form '$y = -ax^2$' and represents a bend of the forward rate curve. Typical weights for these three components might be 70%, 20% and 5% respectively.