CHAPTER 14

FIXED-INCOME PRODUCTS AND ANALYSIS: YIELD, DURATION AND CONVEXITY

1. **A coupon bond pays out 3% every year, with a principal of $1 and a maturity of five years. Decompose the coupon bond into a set of zero-coupon bonds.**

   The coupon bond can be expressed as

   $$0.03 \sum_{i=1}^{5} Z(t; i) + Z(t; 5),$$

   where $Z(t; T)$ is the value of a zero-coupon bond at time t, with maturity at time T and a principal of 1.

2. **Construct a spreadsheet to examine how $1 grows when it is invested at a continuously compounded rate of 7%. Redo the calculation for a discretely compounded rate of 7%, paid once per annum. Which rate is more profitable?**

   After T years, $1 invested at a continuously compounded rate of 7% is worth

   $$e^{0.07T}.$$

   $1 invested at a discretely compounded rate of 7% is worth

   $$(1 + 0.07)^T.$$

   The continuously compounded rate is clearly more profitable.

3. **A zero-coupon bond has a principal of $100 and matures in 4 years. The market price for the bond is $72. Calculate the yield to maturity, duration and convexity for the bond.**

   For a zero-coupon bond,

   $$V = Pe^{-y(T-t)},$$

   and so the yield to maturity is

   $$y = -\frac{\log(V/P)}{T-t} = -\frac{\log(72/100)}{4} = -\tfrac{1}{4}\log(0.72) = 0.082$$

Then

$$\frac{dV}{dy} = -(T-t)Pe^{-y(T-t)},$$

and the duration is

$$-\frac{1}{V}\frac{dV}{dy} = \frac{1}{V}(T-t)Pe^{-y(T-t)} = \frac{1}{72}(4)(100)e^{-4y} = 4.$$

The convexity is

$$\frac{1}{V}\frac{d^2V}{dy^2} = \frac{1}{V}(T-t)^2Pe^{-y(T-t)} = \frac{1}{72}(4)^2(100)e^{-4y} = 16.$$

4. A coupon bond pays out 2% every year on a principal of \$100. The bond matures in 6 years and has market value \$92. Calculate the yield to maturity, duration and convexity for the bond.

$$V = Pe^{-y(T-t)} + \sum_{i=1}^{N} C_i e^{-y(t_i-t)},$$

and so

$$92 = 100e^{-6y} + \sum_{i=1}^{6} 2e^{-y(i)}.$$

We must solve this equation for y to find the yield to maturity of the coupon bond. This can be easily achieved via iteration or the Goalseek or Solver method in Excel. We find that the yield to maturity is

$$y = 0.034.$$

The duration is given by

$$-\frac{1}{V}\frac{dV}{dy},$$

where

$$\frac{dV}{dy} = -(T-t)Pe^{-y(T-t)} - \sum_{i=1}^{N} C_i(t_i-t)e^{-y(t_i-t)}.$$

The duration is therefore

$$\frac{1}{92}\left((6)(100)e^{-6y} + \sum_{i=1}^{6} 2(i)e^{-y(i)}\right) = 5.699.$$

The convexity is given by

$$\frac{1}{V}\frac{d^2V}{dy^2} = \frac{1}{V}\left((T-t)^2Pe^{-y(T-t)} + \sum_{i=1}^{N} C_i(t_i-t)^2e^{-y(t_i-t)}\right)$$

$$= \frac{1}{92}\left((6)^2(100)e^{-6y} + \sum_{i=1}^{6} 2(i)^2 e^{-y(i)}\right) = 33.506.$$

5. **Zero-coupon bonds are available with a principal of \$1 and the following maturities:**

   1 year (market price \$0.93),

   2 years (market price \$0.82),

   3 years (market price \$0.74).

   Calculate the yield to maturities for the three bonds. Use a bootstrapping method to work out the forward rates that apply between 1–2 years and 2–3 years.

   The yield to maturity for the 1 year bond is

$$y_1 = -\frac{\log 0.93}{1} = 0.073.$$

   The yield to maturity for the 2 year bond is

$$y_2 = -\frac{\log 0.82}{2} = 0.099.$$

   The yield to maturity for the 3 year bond is

$$y_3 = -\frac{\log 0.74}{3} = 0.100.$$

   The 1–2 year forward rate satisfies

$$2y_2 = y_1 + F_{1-2},$$

   therefore

$$F_{1-2} = 2y_2 - y_1 = 0.126.$$

   The 2–3 year forward rate satisfies

$$3y_3 = 2y_2 + F_{2-3},$$

   therefore

$$F_{2-3} = (3y_3 - 2y_2) = 0.103.$$

6. **What assumption do we make when we duration hedge? Is this a reasonable assumption to make?**

   We assume that the only movements that occur are parallel shifts in the yield curve. For instance, consider two bonds:

   Bond A has a yield of 6.12%,

   Bond B has a yield of 6.5%.

   The bonds have different maturities and durations but we assume that a move of $x\%$ in A's yield is accompanied by a move of $x\%$ in

B's yield. If this is the case, then if we hold A bonds and B bonds in the inverse ratio of their durations (with one long position and one short) we will be leading-order hedged:

$$\Pi = V_A(y_A) - \Delta V_B(y_B).$$

The change in the value of this portfolio is

$$\delta\Pi = \frac{\partial V_A}{\partial y_A}x - \Delta\frac{\partial V_B}{\partial y_B}x + \text{higher-order terms.}$$

If we choose

$$\Delta = \frac{\partial V_A}{\partial y_A}\Big/\frac{\partial V_B}{\partial y_B}$$

then we can eliminate the leading-order risk.

Unfortunately, we can observe other shifts in the yield curve, for instance twisting and skewing, although a principal component analysis may reveal that the largest contribution to yield curve movement does come from a parallel shift.

7. Solve the equation

$$\frac{dV}{dt} + K(t) = r(t)V,$$

with final data $V(T) = 1$. This is the value of a coupon bond when there is a *known* interest rate, $r(t)$. What must we do if interest rates are not known in advance?

We first solve the homogeneous equation

$$\frac{dW}{dt} = r(t)W,$$

to find

$$W = Ae^{-\int_t^T r(\tau)\,d\tau}.$$

To solve the inhomogeneous equation, we try a particular solution of the form

$$V = f(t)e^{-\int_t^T r(\tau)\,d\tau}.$$

Substituting into the original equation for V,

$$\frac{df}{dt}e^{-\int_t^T r(\tau)\,d\tau} + rfe^{-\int_t^T r(\tau)\,d\tau} + K(t) = rV,$$

which simplifies to

$$\frac{df}{dt} + K(t)e^{\int_t^T r(\tau)\,d\tau} = 0.$$

This has solution

$$f = \int_t^T K(t')e^{\int_{t'}^T r(\tau)\,d\tau}dt'.$$

The general solution for V is then

$$V(t) = e^{-\int_t^T r(\tau)\,d\tau}\left(A + \int_t^T K(t')e^{\int_{t'}^T r(\tau)\,d\tau}dt'\right).$$

We choose A such that $V(T) = 1$, to find $A = 1$ and

$$V(t) = e^{-\int_t^T r(\tau)\,d\tau}\left(1 + \int_t^T K(t')e^{\int_{t'}^T r(\tau)\,d\tau}dt'\right).$$

If interest rates are not known in advance, we must create a stochastic model for their possible future values.

8. **The following is a term sheet for a step-up note paying a fixed rate that changes during the life of the contract. Plot the price/yield curve for this product today, ignoring the call feature.**

PTE 6 Year Non-Call 2 Year Fixed Rate Step-up Note

Aggregate Principal Amount	PTE 10,000,000,000
Trade Date	4 November 1997
Issue Date	25 November 1997
Settlement Date	25 November 1997
Maturity Date	25 November 2003
Issue Price	100%
Coupon	Years 1-2: 5.75% Years 3-6: 6.25%
Issuer Optional Redemption	The issuer has the right, but not the obligation, to redeem the Notes at 100% of Nominal, in whole but not in part, on 25 November 1999 with 10 Business Days Prior notice.
Payment Frequency	Annual
Daycount Convention	30/360
Governing Law	English

This indicative termsheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 14.1 A PTE six-year non-call two year fixed rate step-up note.

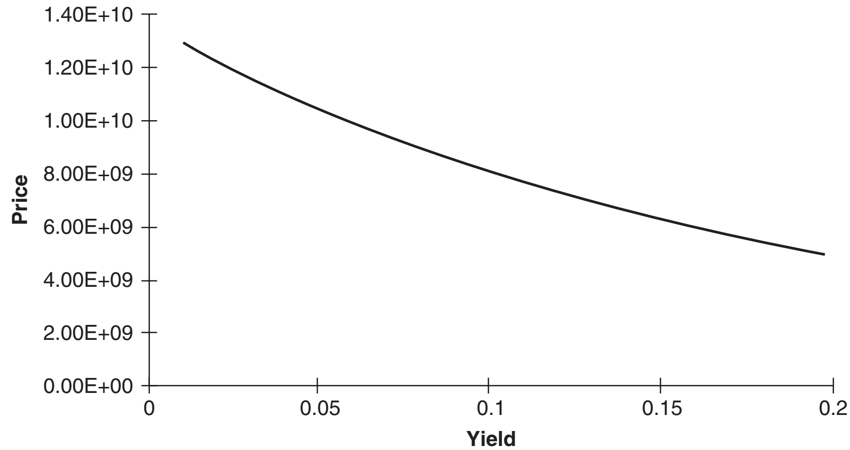

Figure 14.2 Price vs yield curve for the step-up note.

What effect will the call feature have on the price of this contract?

The relationship between price (V) and yield (y) for this product is

$$V = 10^{10}\left(e^{-6y} + 0.0575\left(e^{-y} + e^{-2y}\right)\right.$$
$$\left. + 0.0625\left(e^{-3y} + e^{-4y} + e^{-5y} + e^{-6y}\right)\right).$$

A price/yield curve plot is shown in Figure 14.2.

The call feature gives extra rights to the issuer and consequently lowers the price of the contract.