Further Mathematical Methods: II

In this lecture ...

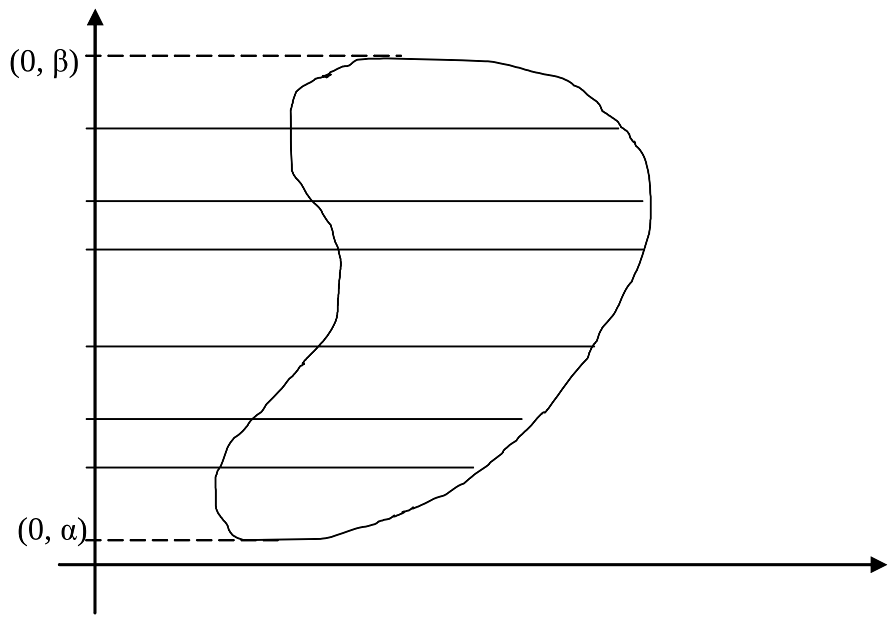
- Double Integration
  - Review and examples
  - Applications to joint probability distributions
  - The gamma function
- Fourier Transforms
  - Definition and standard results

  - Applications to differential equations

- Power series solutions of Ordinary Differential Equations

1 Double Integration

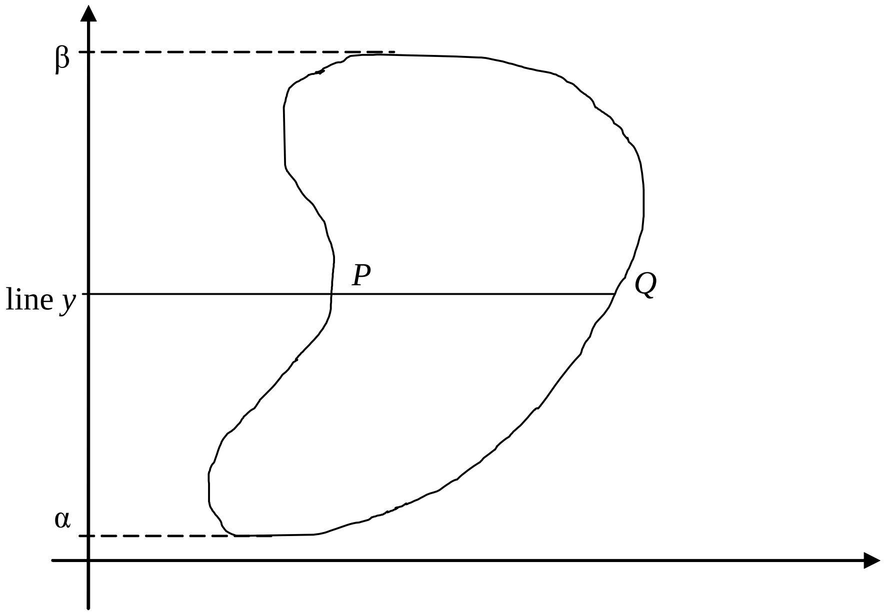
1.1 Evaluation of $\iint_A f\,(x,\ y)\,dxdy$



$$\iint_A f\,(x,\ y)\ \,dx\ dy$$

$$= \int_{\alpha}^{\beta} \{f\,(x,\ y)\}_{x_P(y)}^{x_Q(y)}\ \,dx\}\ dy$$

<u>So</u> limits are given by:

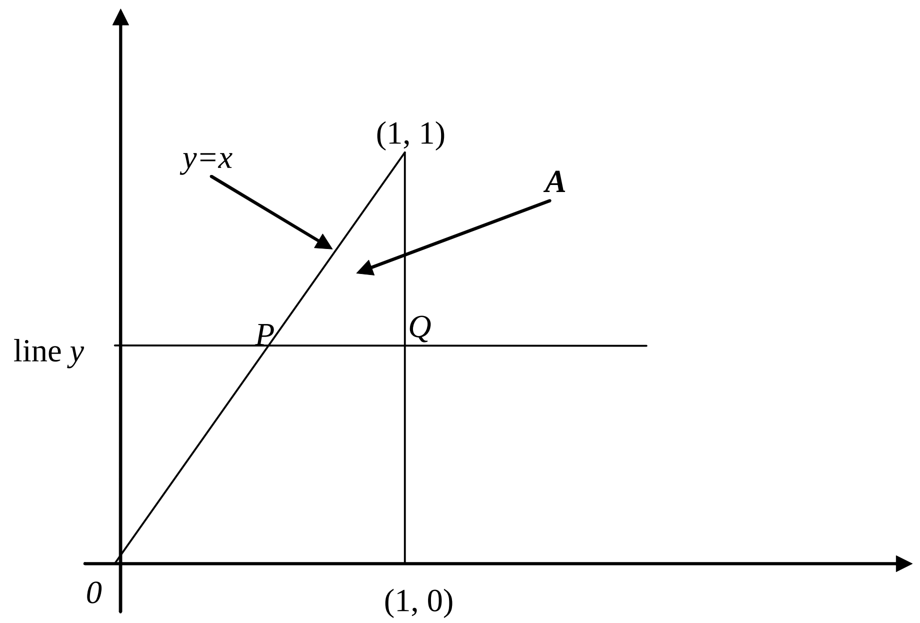


β
line y
P
Q
α

Example: Evaluate

$$\iint_A (x+y)\ dx\ dy$$

where A is the Δ in the following diagram:



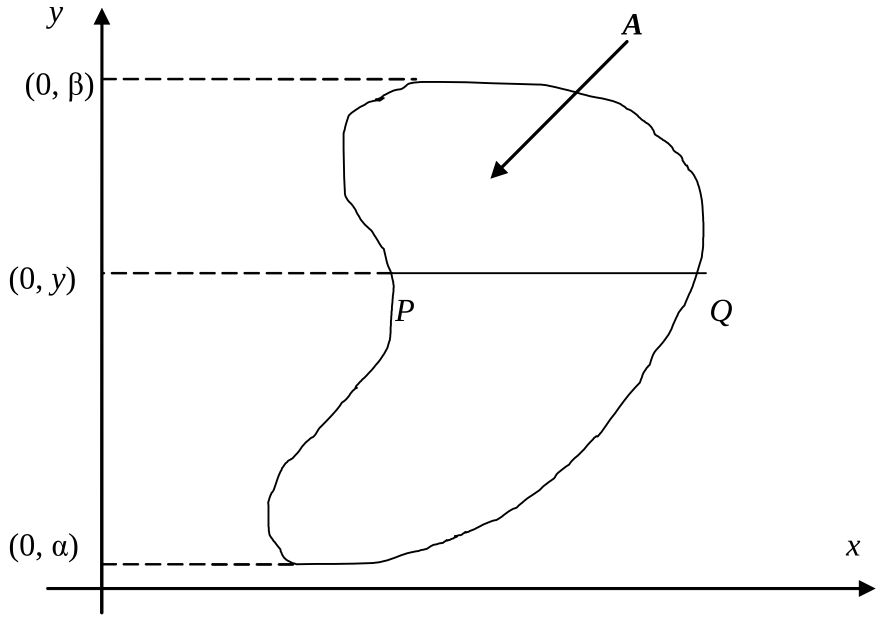
$$
\begin{aligned}
x_P &= y \quad P(y,\ y) \\
x_Q &= 1 \quad Q(1,\ y)
\end{aligned}
$$

$$
\begin{aligned}
I &= \int_{y=0}^{y=1} \{_{x_P=y}^{x_Q=1} x + y \ dx\} dy \\
\int_y^1 (x+y)\,dx &= \left[\frac{x^2}{2} + xy\right]_y^1 = \left(\frac{1}{2} + y\right) - \left(\frac{y^2}{2} + y^2\right) \\
I &= \int_0^1 \left(\frac{1}{2} + y - \frac{3y^2}{2}\right) dy = \left(\frac{y}{2} + \frac{y^2}{2} - \frac{y^3}{2}\right)_0^1 \\
&= \frac{1}{2}
\end{aligned}
$$

So generally

$$\iint_A f(x,\ y)dx\ dy$$

where A is defined as



$x_P,\ x_Q$ function of y

$$= \underbrace{\int_{\alpha}^{\beta} \left\{ \int_{x_P}^{x_Q} f\,(x,\ y) \right\}}_{\text{repeated integral}} dy$$

We note in passing that

$$\iint_A f\ dx\ dy = \iint_{A_1} f\ dx\ dy + \iint_{A_2} f\ dx\ dy$$

A: $A_1 + A_2$

The main problem lies in the limits. We consider the following examples –

Examples:

1. A Rectangle

   $a \leq x \leq b$

   $\alpha \leq y \leq \beta$

   Here $x_P = a, \; x_Q = b$

   $\alpha \leq y \leq \beta$

   $\therefore$

$$\iint_A f \; dx \; dy = \int_\alpha^\beta \left\{ \int_a^b f \; dx \right\} dy$$

2. A Triangle

with sides

$$\begin{aligned} x+y &= 0 \\ x-y &= 0 \\ y &= 2 \end{aligned}$$

In this case

$$\begin{aligned} x_P &= -y;\ x_Q = y \\ \alpha &= 0;\ \beta = 2 \end{aligned}$$

$$\iint_A f\ dx\ dy = \int_0^2 \left\{ \int_{-y}^{y} f\ dx \right\} dy$$

3 A is the region defined by

$$x^2 + y^2 \leq 1, \; x, \; y \geq 0$$

$$\iint_A f \; dx \; dy = \int_0^1 \left\{ \int_0^{\sqrt{1-y^2}} f \; dx \right\} dy$$

Difficulty: A parallelogram

For this A we do not have a simple value for x_P (or x_Q)

<u>For A_1</u> $\qquad x_P = 0, \; x_P = y$

<u>For A_2</u> $\qquad x_P = y - 1, \; x_Q = 1$

So

$$\iint_A f\ dx\ dy = \iint_{A_1} f\ dx\ dy + \iint_{A_2} f\ dx\ dy$$

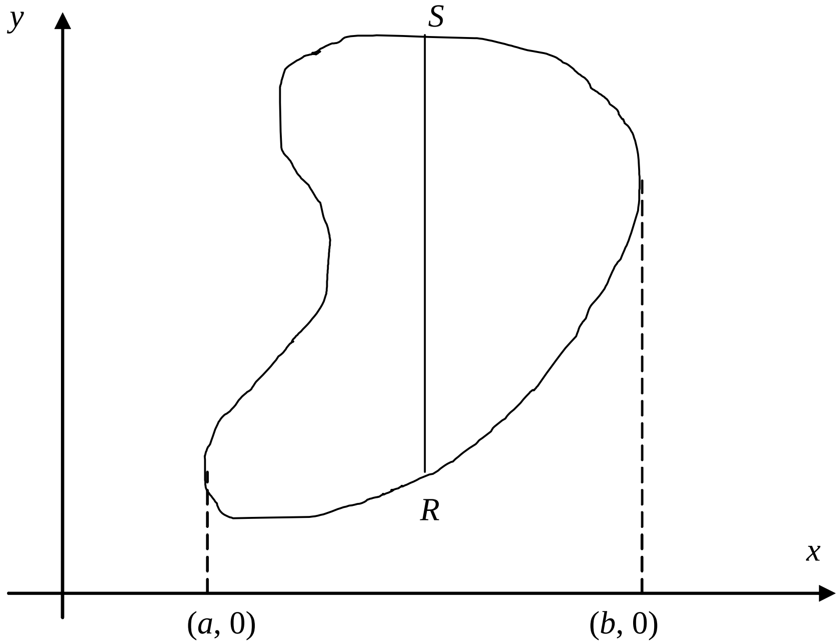
$$\iint_{A_1} = \int_0^1 \left\{ \int_0^y f\ dx \right\} dy \quad (0 \le y \le 1 \text{ in } A_1)$$

$$\iint_{A_2} = \int_1^2 \left\{ \int_{y-1}^1 f\ dx \right\} dy \quad (1 \le y \le 2 \text{ in } A_2)$$

Sometimes, then, we want to do the $y-$integration first:

$$\iint_A f\ dx\ dy$$

$$= \int_a^b \left\{ \int_{y_R}^{y_S} f \ dy \right\} dx$$



Here $y_R, \ y_S$ depend on x

Example:

A is the parallelogram discussed earlier

$y_R = x \quad a = 0$

$y_S = x + 1 \quad b = 1$

$$\iint_A f \; dx \; dy = \int_0^1 \left\{ \int_x^{x+1} f \; dy \right\} dx$$

1.2 Uses of Double Integration

<u>AREAS</u>

Theorem

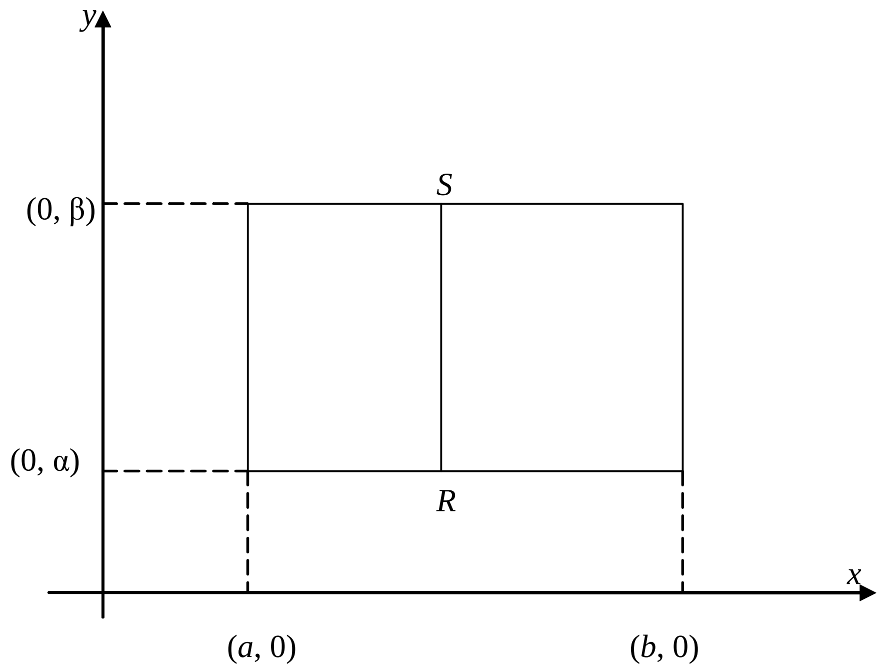
$$\iint_A 1 \; dx \; dy = \text{ area of } A$$

Here we have $f\,(x,\ y) = 1\ \forall\,(x,\ y)$ in A

Example

A rectangle $a \le x \le b$, $\alpha \le y \le \beta$

$$\text{area} = \iint_A 1 \cdot dx\,dy = \int \left\{ \int 1 \; dy \right\} dx$$



$$
\begin{aligned}
&= \int_a^b [y]_\alpha^\beta \, dx = \int_a^b (\beta - \alpha) \, dx = (\beta - \alpha) \, [x]_a^b \\
&= (\beta - \alpha) \, (b - a)
\end{aligned}
$$

1.3 Changing to Plane Polars

If

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

then

$$\iint_A f(x,y)\,dxdy = \iint_{A'} F(r,\theta)\,rdrd\theta$$

where

1. $F(r,\theta) = f(r\cos\theta, r\sin\theta)$
2. A' is the region A described in (r,θ) coordinates.

1.4 Joint PDF for Continuous Random Variables

Recall that the cumulative distribution function $F(x)$ of a RV X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(s)\, ds$$

$F(x)$ is related to the PDF $p(x)$ by

$$p(x) = \frac{dF}{dx}.$$

Consider the pair (X, Y) with joint pdf $p_{XY}(x, y)$ and cdf $F_{XY}(x, y)$. They are related through a similar fashion

$$p_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

Integrating this (as before) gives the cdf as

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p_{XY}(s, t)\, dt ds$$

which allows to calculate the probability

$$\mathbb{P}\left(X \leq x,\ Y \leq y\right).$$

We can extend the simple properties of $p_{XY}\left(x, y\right)$ to two dimensions:

- $p_{XY}\left(x, y\right) \geq 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}\left(x, y\right) dxdy = 1$
- $\iint_{R} p_{X,\ Y}\left(x, y\right) dxdy = \mathbb{P}\left(\left(X, Y\right) \in R\right).$
- $\mathbb{P}\left(a < X \leq b,\ c < Y \leq d\right) = \int_{c}^{d}\int_{a}^{b} f_{XY}\left(x, y\right) dxdy$

If X and Y are independent random variables the cdf can be expressed in separable form

$$F_{XY}(x, y) = F_X(x)\, F_Y(y) \,.$$

Then differentiating gives

$$\begin{aligned} \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} &= \frac{\partial F_X}{\partial x} \frac{\partial F_Y}{\partial y} \\ p_{XY}(x, y) &= p_X(x)\, p_Y(y) \,. \end{aligned}$$

1.5 The Gamma Function Revisited

The Gamma Function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt \quad (x > 0)$$

The condition on x is a convergent criterion.

If $a > 0$

$$\int_0^a x^p dx \quad \text{exists for } p > -1$$
$$\int_a^\infty x^p dx \quad \text{exists for } p < -1$$

Integration by parts gives us

$$\begin{aligned} \int_0^\infty e^{-t}\, t^{x} dt &= x \int_0^\infty e^{-t}\, t^{x-1} dt = x(x-1) \int_0^\infty e^{-t}\, t^{x-2} dt = \\ \ldots\ldots\ldots &= x! \end{aligned} \tag{$\ddagger$}$$

Important results:

$$\begin{aligned} \Gamma(n+1) &= n! \ (n \geq 0) \\ \Gamma(1) &= 1 \end{aligned}$$

and also from (‡)

$$\Gamma(x+1) = x\Gamma(x).$$

Theorem

$$\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \ d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Proof Start with the definition of the gamma function

$$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t} dt$$

and make the substitution $t = x^2$ which gives

$$\begin{aligned}
\Gamma(m) &= \int_0^\infty \left(x^2\right)^{m-1} \exp\left(-x^2\right).2x\,dx \\
&= 2\int_0^\infty x^{2m-1} \exp\left(-x^2\right) dx
\end{aligned}$$

Similarly

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} \exp\left(-y^2\right) dy$$

therefore

$$\begin{aligned}
\Gamma(m)\,\Gamma(n) &= 4\left(\int_0^\infty x^{2m-1} \exp\left(-x^2\right) dx\right)\left(\int_0^\infty y^{2n-1} \exp\left(-y^2\right) dy\right) \\
&= 4\int\int_A x^{2m-1} y^{2n-1} e^{-\left(x^2+y^2\right)} dx\,dy
\end{aligned}$$

where A is the region of integration defined by the first (positive) quadrant. Introduce polar coordinates

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}$$

to transform the integrand to

$$r^{2m+2n-2}\cos^{2m-1}\theta\sin^{2n-1}\theta\exp\left(-r^2\right)$$

and $dxdy \longrightarrow rdrd\theta$

$$\Gamma(m)\Gamma(n) = 4\underbrace{\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta}_{\text{integral we want}}\underbrace{\int_0^{\infty} r^{2(m+n)-1}e^{-\left(r^2\right)}dr}_{\frac{1}{2}\Gamma(m+n)}$$

so rearranging gives the result

$$\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Example Calculate $\int_0^{\pi/2}\cos^4\theta\sin^3\theta\ d\theta$

Hence

$$\begin{aligned} 2m-1 &= 4 \longrightarrow m = 5/2 \\ 2n-1 &= 3 \longrightarrow n = 2 \end{aligned}$$

so integral equals

$$\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{2\Gamma\left(\frac{9}{2}\right)} = \frac{\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}.1}{2\left(\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}\right)} = \frac{2}{35}$$

Example $I = \int_0^{\pi/2} \cos^6\theta \; d\theta$

$$\begin{aligned} 2m-1 &= 6 \longrightarrow m = 7/2 \\ 2n-1 &= 0 \longrightarrow n = 1/2 \end{aligned}$$

Hence $I =$

$$\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}\cdot\sqrt{\pi}}{2(3.2)} = \frac{5\pi}{32}$$

2 The Fourier Transform

If $f = f(x)$ then consider

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx.$$

If this special integral converges, it is called the *Fourier Transform* of $f(x)$.
Similar to the case of Laplace Transforms, it is denoted as $\mathcal{F}(f)$, i.e.

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx = \widehat{f}(\omega).$$

The *Inverse Fourier Transform* is then

$$\mathcal{F}^{-1}\left(\widehat{f}(\omega)\right) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-ix\omega} d\omega = f(x).$$

The convergent property means that $\widehat{f}(\omega)$ is bounded and we have

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Functions of this type $f(x) \in L_1(-\infty, \infty)$ and are called *square integrable.*

We know from integration that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Hence

$$\begin{aligned} \left|\widehat{f}(\omega)\right| &= \left| \int_{\mathbb{R}} f(x)\, e^{ix\omega} dx \right| \\ &\leq \int_{\mathbb{R}} \left| f(x)\, e^{ix\omega} \right| dx \end{aligned}$$

and Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ implies that $\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, therefore

$$\left|\widehat{f}(\omega)\right| \leq \int_{\mathbb{R}} |f\ (x)|\, dx < \infty.$$

In addition to the boundedness of $\widehat{f}(\omega)$, it is also continuous (requires a $\delta - \epsilon$ proof).

Example: Obtain the Fourier transform of $f\ (x) = e^{-|x|}$

$$
\begin{aligned}
\widehat{f}(\omega) &= \mathcal{F}(f) = \int_{-\infty}^{\infty} f\ (x)\, e^{ix\omega} dx \\
&= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega} dx \\
&= \int_{-\infty}^{0} e^{-|x|} e^{ix\omega} dx + \int_{0}^{\infty} e^{-|x|} e^{ix\omega} dx \\
&= \int_{-\infty}^{0} e^{x} e^{ix\omega} dx + \int_{0}^{\infty} e^{-x} e^{ix\omega} dx = \\
&\quad \int_{-\infty}^{0} \exp\left[(1+i\omega)\,x\right] dx + \int_{0}^{\infty} \exp\left[-(1-i\omega)\,x\right] dx \\
&= \frac{1}{(1+i\omega)} \exp\left[(1+i\omega)\,x\right]\Big|_{-\infty}^{0} + \frac{1}{(1-i\omega)} \exp\left[-(1-i\omega)\,x\right]\Big|_{0}^{\infty} \\
&= \frac{1}{(1+i\omega)} + \frac{1}{(1-i\omega)} = \frac{2}{\left(1+\omega^2\right)}
\end{aligned}
$$

Our interest in differential equations continues, hence the reason for introducing this transform. We now look at obtaining Fourier transforms of derivative terms. We assume that $f\ (x)$ is continuous and $f(x) \to 0$ as $x \to \pm\infty$. Consider

$$\mathcal{F}\left\{f'(x)\right\} = \int_{\mathbb{R}} f'(x)\, e^{ix\omega} dx$$

which is simplified using integration by parts

$$f(x)\, e^{ix\omega}\Big|_{-\infty}^{\infty} - i\omega \int_{\mathbb{R}} f\ (x)\, e^{ix\omega} dx$$

so

$$\mathcal{F}\left\{f\ '(x)\right\} = -i\omega \int_{\mathbb{R}} f\ (x)\, e^{ix\omega} dx = -i\omega \widehat{f}(\omega).$$

We can obtain the Fourier transform for the second derivative by performing integration by parts (twice) to give

$$\mathcal{F}\left\{f\ ''(x)\right\} = (-i\omega)^2 \mathcal{F}\{f(x)\} = -\omega^2 \widehat{f}(\omega).$$

Example: Solve the diffusion equation problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x, 0) &= e^{-|x|}, \quad -\infty < x < \infty \end{aligned}$$

Here $u = u(x, t)$, so we begin by defining

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)\, e^{ix\omega} dx = \widehat{u}(\omega, t).$$

Now take Fourier transforms of our PDE, i.e.

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

to obtain

$$\frac{d\widehat{u}}{dt} = -\omega^2 \widehat{u}(\omega, t).$$

We note that the second order PDE has been reduced to a first order equation of type variable separable. This has solution

$$\widehat{u}(\omega, t) = Ce^{-\omega^2 t}.$$

We can find the constant of integration transforming the initial condition

$$\begin{aligned}\mathcal{F}\left\{u\left(x,0\right)\right\} &= \mathcal{F}\left\{e^{-|x|}\right\} \\ \widehat{u}\left(\omega,0\right) &= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega} dx = \frac{2}{\left(1+\omega^2\right)}.\end{aligned}$$

Applying this to the solution $\widehat{u}\left(\omega,t\right)$ gives

$$\widehat{u}\left(\omega,0\right) = C = \frac{2}{\left(1+\omega^2\right)},$$

hence

$$\widehat{u}\left(\omega,t\right) = \frac{2}{\left(1+\omega^2\right)} e^{-\omega^2 t}.$$

We now use the inverse transform to get $u(x,t) = \mathcal{F}^{-1}(\widehat{u}(\omega,t))$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \widehat{u}(\omega,t)\, e^{-ix\omega} d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} e^{-ix\omega} d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \left(\cos x\omega - i \sin x\right) d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \cos x\omega \ d\omega - 2i\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \sin x\omega \ d\omega.
\end{aligned}
$$

This now simplifies nicely because $\dfrac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \sin x\omega$ is an odd function,

hence

$$
\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \sin x\omega \ d\omega = 0.
$$

Therefore

$$u\left(x,t\right)=2\int_{-\infty}^{\infty}\frac{1}{\left(1+\omega^{2}\right)}e^{-\omega^{2}t}\cos x\omega\ d\omega.$$

3 Power Series Solutions

3.1 Introduction

The Euler equation has a nice structure, i.e.

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

where the order of each derivative term and power of its coefficient in x is the same. The next step is to move away from this "nice pattern" and consider a more general equation of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0 \tag{1}$$

We look for solutions in the neighbourhood of $x = 0$.

We say that $x = 0$ is an *ordinary point* of the differential equation (1) if both $p(x)$ <u>and</u> $q(x)$ have Taylor expansions about $x = 0$.

i.e.

$$\begin{aligned} p(x) &= p_0 + p_1 x + p_2 x^2 + O\left(x^3\right) \\ q(x) &= q_0 + q_1 x + q_2 x^2 + O\left(x^3\right) \end{aligned}$$

with both $p_i, \; q_i \sim O(1)$ where $i = 0, 1, \ldots, n$.

If either or both $p(x), \; q(x)$ do not have Taylor expansions about $x = 0$, then $x = 0$ is a *singular point* for the D.E.

<u>*Regular Singular Point*</u>: $xp(x)$ and $x^2 p(x)$ have Taylor expansions about $x = 0$.

Irregular Singular Point: all other points.

Examples:

1. $x\dfrac{d^2y}{dx^2} + x^3\dfrac{dy}{dx} + xy = 0$

This can written in standard form as $\dfrac{d^2y}{dx^2} + x^2\dfrac{dy}{dx} + y = 0 \Rightarrow p(x) = x^2$ & $q(x) = 1$ which both have Taylor expansions about $x = 0$.

Therefore $x = 0$ is an ordinary point of the differential equation.

2. $x^3\frac{d^2y}{dx^2} + 2x^2\frac{dy}{dx} + 5x^2y = 0$

which becomes $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{5}{x}y = 0$ and $p(x) = \frac{2}{x}$ & $q(x) = \frac{5}{x}$ do not have a Taylor expansion about $x = 0$ - however $xp(x) = 2$ & $x^2q(x) = 5x$ do.

Therefore $x = 0$ is a regular singular point of the differential equation.

3. $\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx} + \frac{4}{x^3}y = 0$

$p(x) = O\left(\frac{1}{x^2}\right)$ & $xp(x) = O\left(\frac{1}{x}\right)$; $\quad q(x) = O\left(\frac{1}{x^3}\right)$ & $x^2q(x) = O\left(\frac{1}{x}\right)$

None of these expressions have a Taylor expansion about $x = 0$.

Therefore $x = 0$ is an irregular singular point of the given differential equation.

3.2 Ordinary Point

Assume a solution of (1) of the form

$$y\,(x) = \sum_{n=0}^{\infty} A_n\, x^n \quad (A_0 \neq 0) \tag{2}$$

with A_n constant.

Since no boundary conditions are imposed, the general solution involves two arbitrary constants - else the constants can be determined.

Substitute (2) into the equation given by (1) and equate to zero the coefficients of various powers of x.

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} A_n\, x^n \to q(x)\, y \sim \left(q_0 + q_1 x + q_2 x^2\right)\left(A_0 + A_1 x + A_2 x^2\right) \\
y'(x) &= \sum_{n=0}^{\infty} n\, A_n\, x^{n-1} \to p(x)\, y' \sim \left(p_0 + p_1 x + p_2 x^2\right)\left(A_1 + 2A_2 x + 3A_3 x^2\right) \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1)\, A_n\, x^{n-2} \to y'' \sim 2A_2 + 6A_3 x + 12A_4 x^2
\end{aligned}
$$

$$2A_2 + 6A_3 x + (p_0 + p_1 x)(A_1 + 2A_2 x) + (q_0 + q_1 x)(A_0 + A_1 x) = 0$$

$O(1)$: $\qquad A_0 q_0 + A_1 p_0 + 2A_2 = 0$

$O(x)$: $\qquad q_0 A_1 + 2p_0\, A_2 + p_1 A_1 + q_1 A_0 + 6A_3 = 0$

All coefficients can be expressed in terms of A_0 and A_1 which can be arbitrary.

Example

Obtain the general solution of

$$y'' - 2xy' + y = 0$$

about the ordinary point $x = 0$.

We assume a solution of the form $y(x) = \sum_{n=0}^{\infty} A_n \, x^n$ and substitute the expression and its derivatives into the ODE to yield

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} - 2x \sum_{n=0}^{\infty} n A_n x^{n-1} + \sum_{n=0}^{\infty} A_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} + \sum_{n=0}^{\infty} (1-2n) A_n x^n = 0$$

We require a recurrence relation for which a "trick" is used in the summation. In the second summation above, the n term is changed to $(n-2)$ to give $\sum_{n-2=0}^{\infty} (1-2(n-2))\, A_{n-2} x^{n-2}$ which is equivalent to having $\sum_{n=2}^{\infty}$ ....

$$\sum_{n=0}^{\infty} n(n-1)\, A_n x^{n-2} + \sum_{n=2}^{\infty} (1-2(n-2))\, A_{n-2} x^{n-2} = 0$$

We are still unable to write the lhs of the expression above as one term of $O\left(x^{n-2}\right)$, because the lower limit of the first summation starts at $n=0$, whilst the other begins at $n=2$. This minor problem can be easily overcome by writing

$$\sum_{n=0}^{\infty} n(n-1)\, A_n x^{n-2} + \sum_{n=0}^{\infty} (5-2n)\, A_{n-2} x^{n-2} = 0 \qquad (\dagger)$$

because $A_{-2} = A_{-1} = 0$ and $A_0 \neq 0$, and (†) can now be expressed as

$$\sum_{n=0}^{\infty} \{n(n-1)A_n + (5-2n)A_{n-2}\} \; x^{n-2} = 0.$$

Collecting coefficients of x^{n-2} :

$$A_n = \frac{(2n-5)}{n(n-1)} A_{n-2} \quad (n \geq 2)$$

or

$$A_{n+2} = \frac{(2n-1)}{(n+2)(n+1)} A_n$$

which gives us the recurrence relationship which we sought.

$$n = 0 : \quad A_2 = -\frac{1}{2} A_0; \quad n = 1 : \quad A_3 = \frac{1}{6} A_1 = \frac{1}{3!} A_1$$

So we see that all terms A_{2k} will be in terms of A_0 and odd ones A_{2k} in terms of A_1.

$$
\begin{aligned}
n &= 2: \quad A_4 = \frac{3}{4.3}A_2 = -\frac{3}{4.3}\frac{1}{2}A_0 = -\frac{3}{4!}A_0 \\
n &= 3: \quad A_5 = \frac{5}{5.4}A_3 = \frac{5}{5.4}\frac{1}{3!}A_1 = \frac{5}{5!}A_1 \\
n &= 4: \quad A_6 = \frac{7}{6.5}A_4 = -\frac{7}{6.5}\frac{3}{4!}A_0 = -\frac{21}{6!}A_0 \\
n &= 5: \quad A_7 = \frac{9}{7.6}A_5 = \frac{9}{7.6}\frac{5}{5!}A_1 = \frac{45}{7!}A_1
\end{aligned}
$$

The solution is

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} A_n\, x^n = \sum_{k=0}^{\infty}\left(A_{2k}\, x^{2k} + A_{2k+1}\, x^{2k+1}\right) \\
&= A_0 \underbrace{\left[1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{21}{6!}x^6 + O\left(x^8\right)\right]}_{=y_1} + \\
&\quad A_1 \underbrace{\left[x + \frac{1}{3!}x^3 - \frac{5}{5!}x^5 + \frac{451}{7!}x^7 + O\left(x^9\right)\right]}_{=y_2} \\
&= A_0 y_1(x) + A_1 y_2.
\end{aligned}
$$

The linear combination $A_0 y_1(x) + A_1 y_2$ becomes the general solution of the equation. The terms A_0 , A_1 are arbitrary.