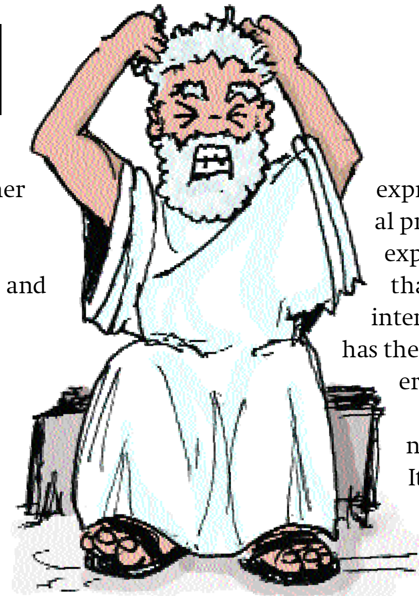
Lyceum

Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

The Binomial Model II



In an earlier Lyceum we saw some of the basic ideas behind the binomial pricing model. We'll go into those ideas in more depth now.

No numbers, just symbols

The three constants u and v are chosen to give the binomial model the same characteristics as the asset we are modeling. That means we want to capture the volatility of the underlying asset, in particular the volatility over the next timestep, dt. Although we won't be needing it, we'll also carry around the probability p for a while.

We choose

$$u = 1 + \sigma\sqrt{dt},$$

$$v = 1 - \sigma\sqrt{dt}$$

and

$$p = \frac{1}{2} + \frac{\mu\sqrt{dt}}{2\sigma}.$$

We have introduced two new parameters here: μ the drift of the asset and σ the volatility. We have chosen these parameters in such a way that (approximately)

- The expected return on the asset is $\mu\ dt$
- The standard deviation of returns is $\sigma\sqrt{dt}$

An equation for the value of an option

Suppose, for the moment, that we are one timestep away from expiry. Construct a portfolio consisting of one option and a short position in a quantity Δ of the underlying. This portfolio has value

$$\Pi = V - \Delta S$$

where the value of the option V is to be determined.

One timestep later, at expiry, the portfolio takes one of two values, depending on whether the asset rises or falls. These two values are

$V^+ - \Delta uS$ and $V^- - \Delta vS$,.

where V^+ is the option value if the asset rises and V^- the value if it falls.

The values of both of these expressions are known if we know Δ. And Δ is under our control.

Having the freedom to choose Δ we can make the value of this portfolio the same whether the asset rises or falls. This is ensured if we make

$$\Delta = \frac{V^+ - V^-}{(u - v)S}.$$

Then the new portfolio value is

$$\frac{uV^- - vV^+}{u - v}.$$

Since the value of the portfolio has been guaranteed, we can say that its value must coincide with the value of the original portfolio plus any interest earned at the risk-free rate, r. Thus

$$(1 + r\,dt)\left(V - \frac{V^+ - V^-}{u - v}\right) = \frac{uV^- - vV^+}{u - v}.$$

Rearranging we get

$$(1 + r\,dt)V = p'V^+ + (1 - p')V^-,$$

where

$$p' = \frac{1}{2} + \frac{r\sqrt{dt}}{2\sigma}.$$

The right-hand side of the equation for V is just like an expectation; it's the sum of probabilities multiplied by outcomes.

We see that the probability of a rise or fall is irrelevant as far as option pricing is concerned. But what if we interpret p' as a probability? Then we could say that the option price is the present value of an expectation. But not the real expectation.

Let's compare the expression for p' with the expression for the actual probability p. The two expressions differ in that where one has the interest rate r the other has the drift μ, but are otherwise the same.

We call p' the risk-neutral probability. It's like the real probability, but the real probability if the drift rate were r instead of μ.

Observe that the risk-free interest plays two roles in option valuation. It's used once for discounting to give present value, and it's used as the drift rate in the asset price random walk.

Risk-neutral pricing

Interpreting p' as a probability, the option pricing equation is the statement that

"the option value at any time is the present value of the risk-neutral expected value at any later time."

You can think of an option value as being the present value of an expectation, only it's not the real expectation.

The above model and analysis has introduced many key concepts.

- There is only one price for an option, and that doesn't depend on which direction you think the asset price is going, only on its volatility.
- That option price can be interpreted as an expectation. This is why we can actually value options by using simulation/Monte Carlo methodology.
- There is a simple algorithm for calculating an option price a timestep before expiry. As we'll see, this can be extended to value an option at any time before expiry.