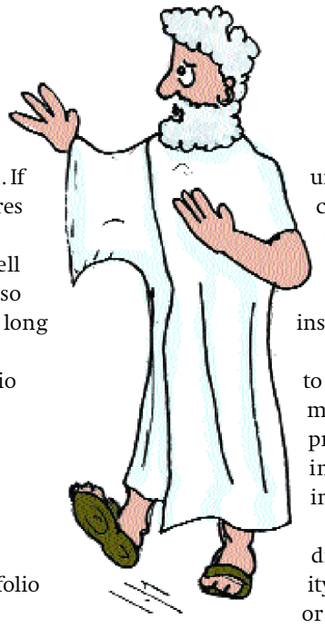
Lyceum

Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a centre of study and research in both science and philosophy.

The Binomial Model

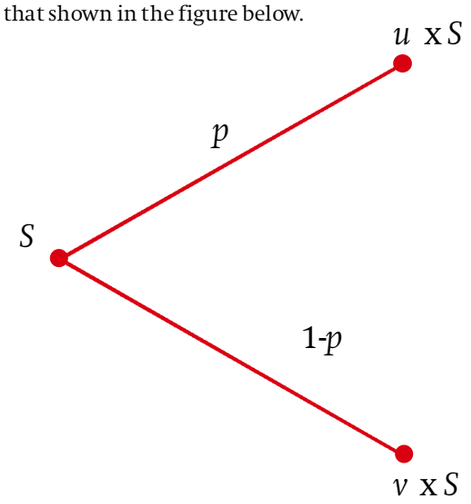


Quick test

It is one day before the expiry of a call option struck at 100. The stock is currently valued at 100. You are reliably informed that there is a 60 per cent chance of the stock rising to 101 and a 40 per cent chance of it falling to 99. You are in a world with a zero interest rate. What value would you give to the call option? Make a note of your answer; we'll come back to this later.

Branches

In the above question we had a stock price that was to take one of two values a day later. Using more general notation we have a situation like that shown in the figure below.



The stock price is currently S, can rise to uS or fall to vS. The probability of the rise is p. In our example $S = 100$, $u = 1.01$, $v = 0.99$ and $p = 0.6$.

Now let's assume that we hold a call option on this asset that is going to expire tomorrow. This option has a strike of 100.

Holding just the stock or the option is risky:

● If the asset rises we have 101, a profit of 1. If it falls we have 99, a loss of 1

● If the asset rises to 101 we get a payoff of 1. If it falls to 99 we get no payoff, the asset expires out of the money

Here's a trick, it's called hedging. Let's sell short a quantity 1/2 of the underlying asset so that now we have a portfolio consisting of a long option position and a short stock position.

If the asset rises to 101 we have a portfolio worth

$$\max(101-100, 0) - 0.5 \times 101$$
$$= 1-0.5 \times 101 = -99/2.$$

If the asset falls we have

$$\max(99 - 100, 0) - 0.5 \times 99$$
$$= 0-0.5 \times 99 = -99/2.$$

So, whether the asset rises or falls, the portfolio takes the same value, −99/2.

If this portfolio takes the same value whatever happens to the underlying asset then it is a riskless portfolio. If it is riskless we can justifiably discount this guaranteed cashflow to the present, one day before expiry. In the simple case where the interest rate is zero we can say that the value of the portfolio the day before expiry is −99/2.

But what is this portfolio made up of today? The option and a short stock position. If we call the option value V, then we have

$$V-0.5 \times 100 = -99/2.$$

Thus the option value is 0.5.

Our opening question's solution is therefore 0.5. If you answered 0.6 to the question you were probably looking at the expected payoff and discounting that. The expected payoff can be calculated from

$$0.6 \times 1 + 0.4 \times 0 = 0.6.$$

Do not despair if this was your answer.

This is a question that we often ask. First we check that people know about the binomial model, the answer is invariably yes. Then we hit them with the pricing question. If they truly understand the binomial concept they should get the answer 0.5. However, from our experiences 95 per cent of people say 0.6 instead.

The most important idea to take away from the binomial model is that the real probability of the stock rising is irrelevant in the pricing of options.

The price of an option is dictated not by the probability of the underlying rising or falling but by the process of risk elimination. What this boils down to is that the option value depends not on the direction of the stock, but on its amount of randomness, here measured by u and v, on the timestep, and on the risk-free interest rate.

Discounting at the risk-free rate is strictly only permitted for risk-free cashflows!

Generalization

In order to generalize this option pricing methodology there are several key steps.

1. Pick u and v. The choice of these is governed by the volatility of the underlying asset and the timestep.
2. Set up the hedged portfolio. At this stage you won't know how much of the underlying asset to hedge with, so call the quantity held short Δ, delta.
3. Choose Δ such that the portfolio values at expiry are the same, whether the asset moves up or down.
4. Discount this portfolio value to the present and calculate the current option value.

We will see some more of the details in another Lyceum meeting.