CHAPTER 7

PARTIAL DIFFERENTIAL EQUATIONS

1. **Consider an option with value $V(S,t)$, which has payoff at time T. Reduce the Black–Scholes equation, with final and boundary conditions, to the diffusion equation, using the following transformations:**

(a)

$$S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V(S,t) = Ev(x,\tau),$$

(b)

$$v = e^{\alpha x + \beta \tau} u(x,\tau),$$

for some α and β. What is the transformed payoff? What are the new initial and boundary conditions? Illustrate with a vanilla European call option.

We use a European call option which satisfies the following problem:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with boundary conditions

$$V(0,t) = 0 \text{ and } V(S,t) \sim S \text{ as } S \to \infty,$$

and final data

$$V(S,T) = \max(S - E, 0).$$

The first transformation gives us the following problem for $v(x,\tau)$:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv,$$

where $k = 2r/\sigma^2$, with boundary conditions

$$v(x,\tau) \to 0 \text{ as } x \to -\infty \text{ and } v(x,\tau) \sim e^x \text{ as } e^x \to \infty,$$

and initial data

$$v(x,0) = \max(e^x - 1, 0).$$

The second transformation gives us

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

We can eliminate the $\partial u/\partial x$ term by choosing α such that

$$0 = 2\alpha + (k-1),$$

and we can eliminate the u term by choosing β such that

$$\beta = \alpha^2 + (k-1)\alpha - k.$$

Solving these equations for α and β, we find

$$\alpha = -\tfrac{1}{2}(k-1) \text{ and } \beta = -\tfrac{1}{4}(k+1)^2.$$

These choices give us the following problem for u:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions

$$u(x,\tau) \to 0 \text{ as } x \to -\infty \text{ and } u(x,\tau) \sim e^{\frac{1}{2}(k+1)x} \text{ as } e^x \to \infty,$$

and initial data

$$u(x,0) = \max\left(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right).$$

2. The solution to the initial value problem for the diffusion equation is unique (given certain constraints on the behavior, it must be sufficiently smooth and decay sufficiently fast at infinity). This can be shown as follows.

Suppose that there are two solutions $u_1(x,\tau)$ and $u_2(x,\tau)$ to the problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \textbf{on} \ -\infty < x < \infty,$$

with

$$u(x,0) = u_0(x).$$

Set $v(x,\tau) = u_1 - u_2$. This is a solution of the equation with $v(x,0) = 0$. Consider

$$E(\tau) = \int_{-\infty}^{\infty} v^2(x,\tau)\,dx.$$

Show that

$$E(\tau) \geq 0, \quad E(0) = 0,$$

and integrate by parts to find that

$$\frac{dE}{d\tau} \leq 0.$$

Hence show that $E(\tau) \equiv 0$ and, consequently, $u_1(x, \tau) \equiv u_2(x, \tau)$.

Now $v^2 \geq 0$, so

$$E(\tau) = \int_{-\infty}^{\infty} v^2(x, \tau)\, dx \geq 0,$$

and

$$E(0) = \int_{-\infty}^{\infty} v^2(x, 0)\, dx = 0.$$

Differentiating with respect to τ,

$$\frac{dE}{d\tau} = \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} v^2(x, \tau)\, dx = \int_{-\infty}^{\infty} \frac{\partial v^2}{\partial \tau} dx$$
$$= \int_{-\infty}^{\infty} 2v \frac{\partial v}{\partial \tau} dx = \int_{-\infty}^{\infty} 2v \frac{\partial^2 v}{\partial x^2} dx,$$

as v satisfies the diffusion equation. Integrating by parts,

$$\frac{dE}{d\tau} = \left[2v \frac{\partial v}{\partial x} \right]_{-\infty}^{\infty} - 2 \int_{-\infty}^{\infty} \left(\frac{\partial v}{\partial x} \right)^2 dx$$
$$= -2 \int_{-\infty}^{\infty} \left(\frac{\partial v}{\partial x} \right)^2 dx \leq 0$$

as $(\partial v / \partial x)^2 \geq 0$.

We now have $E(\tau) \geq 0$, $E(0) = 0$ and $dE/d\tau \leq 0$. This can only be possible if $E(\tau) \equiv 0$. Then $v^2 \equiv 0$ and so $u_1 - u_2 \equiv 0$, hence $u_1 \equiv u_2$.

3. **Suppose that $u(x, \tau)$ satisfies the following initial value problem:**

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \text{on } -\pi < x < \pi, \quad \tau > 0,$$

with

$$u(-\pi, \tau) = u(\pi, \tau) = 0, \quad u(x, 0) = u_0(x).$$

Solve for u using a Fourier Sine series in x, with coefficients depending on τ.

We set

$$u(x,\tau) = \sum_1^\infty c_n(\tau) \sin nx,$$

(note that this satisfies the boundary conditions at $x = -\pi, \pi$). Substituting into the diffusion equation, we find

$$\sum_1^\infty -n^2 c_n(\tau) \sin nx = \sum_1^\infty \frac{dc_n(\tau)}{d\tau} \sin nx.$$

Now the sin terms are orthogonal, so

$$-n^2 c_n = \frac{dc_n(\tau)}{d\tau},$$

for all n. We solve this to find

$$c_n = a_n e^{-n^2\tau},$$

for some arbitrary constants, a_n. The solution for u is then

$$u(x,\tau) = \sum_1^\infty a_n e^{-n^2\tau} \sin nx.$$

We now use the initial data for u to find the a_n. At time $t = 0$,

$$u_0(x) = u(x,0) = \sum_1^\infty a_n \sin nx.$$

Multiplying both sides by $\sin mx$ and integrating from $-\pi$ to π, we find

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} u_0(y) \sin ny \, dy.$$

The Fourier Sine series solution for u is therefore

$$u(x,\tau) = \sum_1^\infty \frac{1}{\pi} \left(\int_{-\pi}^{\pi} u_0(y) \sin ny \, dy \right) e^{-n^2\tau} \sin nx.$$

4. Check that u_δ satisfies the diffusion equation, where

$$u_\delta = \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{x^2}{4\tau}}.$$

u_δ does satisfy the diffusion equation, with

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} = \left(\frac{x^2}{8\sqrt{\pi}\tau^{5/2}} - \frac{1}{4\sqrt{\pi}\tau^{3/2}} \right) e^{-x^2/4\tau}.$$

5. **Solve the following initial value problem for $u(x, \tau)$ on a semi-infinite interval, using a Green's function:**

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \textbf{on } x > 0, \quad \tau > 0,$$

with

$$u(x, 0) = u_0(x) \textbf{ for } x > 0, \quad u(0, \tau) = 0 \textbf{ for } \tau > 0.$$

Hint: Define $v(x, \tau)$ as

$$v(x, \tau) = u(x, \tau) \textbf{ if } x > 0,$$
$$v(x, \tau) = -u(-x, \tau) \textbf{ if } x < 0.$$

Then we can show that $v(0, \tau) = 0$ and

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_0^\infty u_0(s)(e^{-(x-s)^2/4\tau} - e^{-(x+s)^2/4\tau})\, ds.$$

With v defined as above, consider a small distance δx:

$$v(\delta x, \tau) = u(\delta x, \tau) \text{ and } v(-\delta x, \tau) = -u(\delta x, \tau).$$

As $\delta x \to 0$,

$$v(-\delta x, \tau) = -v(\delta x, \tau)$$

and we must have that $v(0, \tau) = 0$. Using the fundamental solution to the diffusion equation (or by solving directly), we find that

$$\begin{aligned}
v(x, \tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} v(\xi, 0) e^{-(x-\xi)^2/4\tau}\, d\xi \\
&= \frac{1}{2\sqrt{\pi\tau}} \left(\int_0^\infty v(\xi, 0) e^{-(x-\xi)^2/4\tau}\, d\xi \right. \\
&\qquad \left. + \int_{-\infty}^0 v(\xi, 0) e^{-(x-\xi)^2/4\tau}\, d\xi \right) \\
&= \frac{1}{2\sqrt{\pi\tau}} \left(\int_0^\infty v(\xi, 0) e^{-(x-\xi)^2/4\tau}\, d\xi \right. \\
&\qquad \left. + \int_0^\infty v(-\eta, 0) e^{-(x+\eta)^2/4\tau} d\eta \right),
\end{aligned}$$

where we have substituted $\eta = -\xi$,

$$= \frac{1}{2\sqrt{\pi\tau}} \left(\int_0^\infty u_0(\xi) e^{-(x-\xi)^2/4\tau}\, d\xi - \int_0^\infty u_0(\eta) e^{-(x+\eta)^2/4\tau} d\eta \right)$$

$$= \frac{1}{2\sqrt{\pi\tau}} \int_0^\infty u_0(s) \left(e^{-(x-s)^2/4\tau} - e^{-(x+s)^2/4\tau} \right) ds.$$

6. Reduce the following parabolic equation to the diffusion equation.

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu,$$

where a and b are constants.

We try a substitution of the form

$$u(x, \tau) = e^{\alpha x + \beta \tau} v(x, \tau).$$

Then

$$\frac{\partial u}{\partial \tau} = \left(\beta v + \frac{\partial v}{\partial \tau} \right) e^{\alpha x + \beta \tau},$$

$$\frac{\partial u}{\partial x} = \left(\alpha v + \frac{\partial v}{\partial x} \right) e^{\alpha x + \beta \tau},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \left(\alpha^2 v + 2\alpha \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} \right) e^{\alpha x + \beta \tau}.$$

Substituting these into the partial differential equation for u, we find

$$\beta v + \frac{\partial v}{\partial \tau} = \left(\alpha^2 v + 2\alpha \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} \right) + a \left(\alpha v + \frac{\partial v}{\partial x} \right) + bv,$$

and on rearranging,

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (2\alpha + a)\frac{\partial v}{\partial x} + (\alpha^2 + a\alpha + b - \beta)v.$$

We choose α such that

$$2\alpha + a = 0,$$

to eliminate the $\partial v/\partial x$ term, and β such that

$$\alpha^2 + a\alpha + b - \beta = 0,$$

to eliminate the v term. We must therefore choose

$$\alpha = -\tfrac{1}{2}a \text{ and } \beta = b - \tfrac{1}{4}a^2.$$

This reduces the partial differential equation for v to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}.$$

7. Using a change of time variable, reduce

$$c(\tau)\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

to the diffusion equation when $c(\tau) > 0$.

Consider the Black–Scholes equation, when σ and r can be functions of time, but $k = 2r/\sigma^2$ is still a constant. Reduce the Black–Scholes equation to the diffusion equation in this case.

We try a change of variable of the form

$$u(x, \tau) = v(x, \tilde{\tau}),$$

where $\tilde{\tau} = F(\tau)$ is some function of τ. Then

$$\frac{\partial u}{\partial \tau} = \frac{\partial v}{\partial \tilde{\tau}}\frac{dF(\tau)}{d\tau},$$

and the partial differential equation becomes

$$c(\tau)\frac{dF(\tau)}{d\tau}\frac{\partial v(x, \tilde{\tau})}{\partial \tilde{\tau}} = \frac{\partial^2 v(x, \tilde{\tau})}{\partial x^2}.$$

We choose $F(\tau)$ such that

$$c(\tau)\frac{dF(\tau)}{d\tau} = 1,$$

i.e.

$$F(\tau) = \int^{\tau} \frac{ds}{c(s)},$$

and then the equation reduces to

$$\frac{\partial v}{\partial \tilde{\tau}} = \frac{\partial^2 v}{\partial x^2}.$$

We are now in a position to solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0,$$

where $k = 2r/\sigma^2$ is a constant. First of all, we transform

$$S = Ee^x \text{ and } V(S,t) = Ev(x,t),$$

to find

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\sigma^2(t)\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) + r(t)\frac{\partial v}{\partial x} - r(t)v = 0.$$

We can rearrange this into the form

$$\frac{2}{\sigma^2(t)}\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv = 0.$$

We now choose a new time variable, τ, such that

$$\frac{2}{\sigma^2(t)}\frac{\partial}{\partial t}v(x,t) = \frac{\partial}{\partial \tau}v(x,\tau),$$

i.e.

$$\tau = -\tfrac{1}{2}\int_0^t \sigma^2(s)\,ds.$$

This reduces the equation to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv.$$

Setting

$$v(x,\tau) = e^{\alpha x + \beta\tau}w(x,\tau),$$

and choosing α and β as usual,

$$\alpha = -\tfrac{1}{2}(k-1) \text{ and } \beta = -\tfrac{1}{4}(k+1)^2,$$

we find

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2}.$$

8. Show that if

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \textbf{on} \quad -\infty < x < \infty, \quad \tau > 0,$$

with

$$u(x,0) = u_0(x) > 0,$$

then $u(x,\tau) > 0$ **for all** τ**.**

Use this result to show that an option with positive payoff will always have a positive value.

The solution to the general problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \text{ on } -\infty < x < \infty, \tau > 0,$$

with

$$u(x,0) = u_0(x) \text{ and } u \to 0 \text{ as } |x| \to \infty,$$

is

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/4\tau} ds.$$

If $u_0(x) > 0$ then the contents of the integral are always positive and so $u(x,\tau) > 0$.

If an option has a positive payoff, then we solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with

$$V(S,T) > 0.$$

We apply the usual transformations to reduce the problem to the diffusion equation (as in Question 1). The first transformation gives us

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv,$$

where $k = 2r/\sigma^2$, with initial data

$$v(x,0) = \frac{1}{E} V(Ee^x, T) > 0.$$

The second transformation gives us (with the usual choice for α and β)

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

with initial data

$$u(x,0) = e^{-\alpha x} v(x,0) > 0.$$

The first part of the question then shows that $u(x,\tau) > 0$. Hence

$$v(x,\tau) = e^{\alpha x + \beta\tau} u(x,\tau) > 0,$$

and

$$V(S,t) = V(Ee^x, T - 2\tau/\sigma^2) = Ev(x,\tau) > 0,$$

and the option value, $V(S,t)$ is always positive.

9. If $f(x,\tau) \geq 0$ in the initial value problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + f(x,\tau), \quad \textbf{on} \quad -\infty < x < \infty, \quad \tau > 0,$$

with

$$u(x,0) = 0, \;\; \textbf{and } u \to 0 \textbf{ as } |x| \to \infty,$$

then $u(x,\tau) \geq 0$. Hence show that if C_1 and C_2 are European calls with volatilities σ_1 and σ_2 respectively, but are otherwise identical, then $C_1 > C_2$ if $\sigma_1 > \sigma_2$.

Use put-call parity to show that the same is true for European puts.

Note that here, f acts as a source of mass in the diffusion equation. C_1 and C_2 are call options and satisfy the following equations:

$$\frac{\partial C_1}{\partial t} + \tfrac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + rS\frac{\partial C_1}{\partial S} - rC_1 = 0,$$
$$\frac{\partial C_2}{\partial t} + \tfrac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 C_2}{\partial S^2} + rS\frac{\partial C_2}{\partial S} - rC_2 = 0.$$

Subtracting the second equation from the first, we find

$$\frac{\partial}{\partial t}(C_1 - C_2) + \tfrac{1}{2}\sigma_2^2 S^2 \frac{\partial^2}{\partial S^2}(C_1 - C_2) + rS\frac{\partial}{\partial S}(C_1 - C_2)$$

$$-r(C_1 - C_2) + \tfrac{1}{2}(\sigma_1^2 - \sigma_2^2)S^2\frac{\partial^2 C_1}{\partial S^2} = 0.$$

We set $V(S,t) = (C_1 - C_2)$ and note that $V(S,T) = 0$. We then use the transformation

$$V(S,t) = \frac{1}{E}v(x,\tau),$$

where

$$S = Ee^x \text{ and } t = T - 2\tau/\sigma_2^2,$$

to find

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv + k_1\left(\frac{\partial^2 C_1}{\partial x^2} - \frac{\partial C_1}{\partial x}\right),$$

with $v(x,0) = 0$, and where

$$k = \frac{2r}{\sigma_2^2} \text{ and } k_1 = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_2^2 E}.$$

We let

$$v = e^{\alpha x + \beta \tau} u(x, \tau),$$

and choose

$$\alpha = -\tfrac{1}{2}(k-1), \;\; \beta = -\tfrac{1}{4}(k+1)^2,$$

as usual to find

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + f(x, \tau),$$

where

$$f(x, \tau) = k_2 e^{-\alpha x - \beta \tau} \left(\frac{\partial^2 C_1}{\partial x^2} - \frac{\partial C_1}{\partial x} \right),$$

and $u(x, 0) = 0$. Since $f(x, \tau) > 0$ for all x, from the first part of the question we have $u(x, \tau) > 0$. Going back through the transformations, this means that $v(x, \tau) > 0$ and $V(S, t) > 0$, hence $C_1 > C_2$.

(Note that these are strict inequalities—we only have $C_1 = C_2$ when $S = 0$).

Put-Call parity gives us that

$$C_1 - P_1 = C_2 - P_2 = S - Ee^{-r(T-t)},$$

and so

$$C_1 - C_2 = P_1 - P_2 > 0,$$

therefore we also have $P_1 > P_2$ when $\sigma_1 > \sigma_2$.