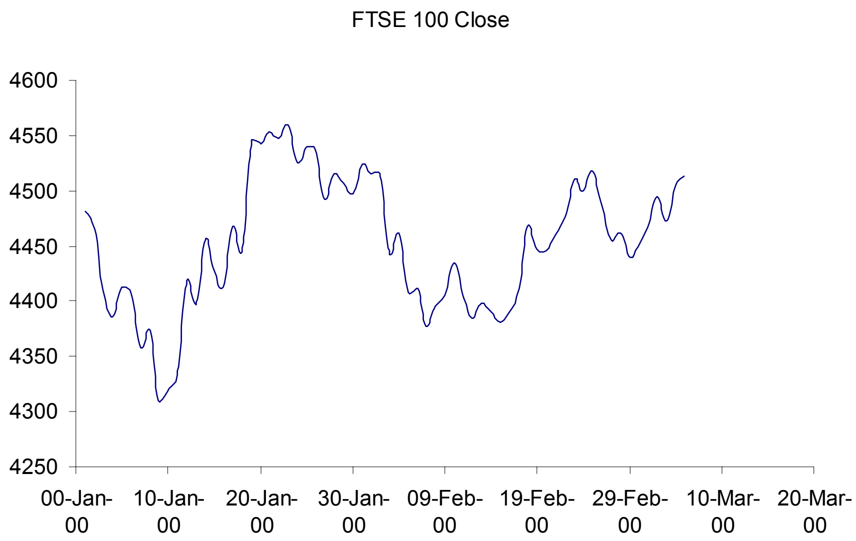
Stochastics 1 Review of Calculus & Differential Equations

All asset prices (stocks, indices, interest rates) are stochastic processes.

They change randomly over time, but the manner in which they change can be modelled.



It is possible to divide these changes in to parts, the first a non-random, deterministic component called the drift of the process, and the second is a ’noise’ term - the random part we call the volatility of the process.

Stochastic Differential Equations allow us to model the behaviour of randomly occurring processes.

Brownian Motion

A stochastic process $\{ W(t) : t \in \mathbb{R}_+\}$ is defined to be Brownian motion (or a Wiener process) if

- for each $t > 0$ and $s > 0$, $W(t) - W(s)$ is normal with mean 0 and variance $|t-s|$, i.e. $(W(t) - W(s)) \sim N(0, |t-s|)$
- $W(t+s) - W(t)$ is indpendent of $W(t)$
- $W(0) = 0$ and the path $t \mapsto W(t)$ is continuous.

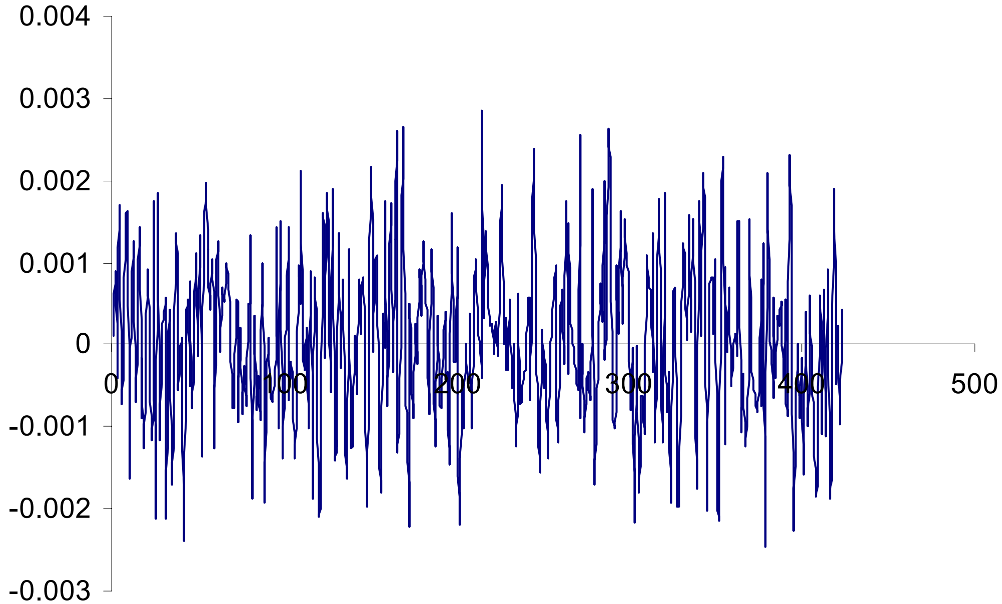
Also called *standard Brownian motion.*

Brownian motion is important because :

- It is the canonical driving noise process that many other models can be built from

- It has many nice properties as it a Markov process, a Gaussian process and a martingale.

- Not so nice as it trajectories are nowhere differentiable.


0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
0
100
200
300
400
500

dW : the increment of Brownian motion

Over any non-infinitessimal time interval, Brownian motion is a random variable.

Over any such interval $(t, t + dt)$ we can think of its increment as

$$dW = W(t + dt) - W(t)$$

$$\mathsf{E}[dW] = \mathsf{E}[W_{t+dt} - W_t] = 0$$

therefore

$$\mathsf{E}(dW) = 0$$

and

$$\lim_{dt \to 0} dW^2 = dt.$$

It follows that

$$\mathrm{E}\left(dW^2\right) = dt$$

and

$$dW = \sqrt{dt}\,\phi$$

where $\phi \sim N(0, 1)$

Diffusion Process

G is called a diffusion process if

$$dG(t) = A(G,t)dt + B(G,t)dW(t) \qquad [\Im]$$

This is also an example of a Stochastic Differential Equation (SDE) for the process G and consists of two components:

1. $A(G, t)dt$ is deterministic – coefficient of dt is known as the drift

2. $B(G,t)dW$ is random – coefficient of dW is known as the diffusion or volatility.

We say G evolves according to (or follows) this process.

For example

$$dG(t) = (G(t) + G(\tfrac{t}{2}))dt + dW(t)$$

is not a diffusion (although it is a SDE)

- $A \equiv 0$ and $B \equiv 1$ reverts the process back to Brownian motion
- Called time-homogeneous if A and B are not dependent on t.
- $dW^2 = B^2 dt$

The diffusion $[\Im]$ can be written in integral form as

$$G(t) = G(0) + \int_0^t A(G,\tau)d\tau + \int_0^t B(G,\tau)dW(\tau)$$

Remark: A diffusion G is a Markov process if - once the present state $G(t) = g$ is given, the past $\{G(\tau),\ \tau < t\}$ is irrelevant to the future dynamics.

So if for example we have a diffusion $G(t)$

$$dG = \mu G dt + \sigma G dW \tag{1}$$

then the drift is $A(G,t) = \mu G$ and diffusion is $B(G,t) = \sigma G$.

The process (1) is also called Geometric Brownian Motion (GBM) or Exponential Brownian motion (EBM).

Brownian motion $X(t)$ is used as a basis for a wide variety of models. Consider a pricing process $\{ S(t) : t \in \mathbb{R}_+ \}$: we can model its instantaneous change dS by a SDE

$$dS = a(S,t)dt + b(S,t)dX$$

By choosing different coefficients a and b we can have different properties for the diffusion process.

A very popular finance model for generating asset prices is the GBM given by (1). The instantaneous return on a stock $S(t)$ is a constant coefficient SDE

$$\frac{dS}{S} = \mu dt + \sigma dX \tag{2}$$

where μ and σ are the return's drift and volatility, respectively.

Functions of a stochastic variable: Itô's Lemma

In continuous time models, changes are small (infinitessimal). Calculus is used to analyse small changes: so we a require an extension of 'ordinary' calculus to variables governed by diffusion processes.

Recall taylor's theorem: if the function $f = f(S,t)$, for small $dS,\ dt$

$$V(S+dS,t+dt) = V(S,t) + \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2$$
$$+ O(dS^3, dt^3)$$

Suppose S follows

$$dS = a(S,t)dt + b(S,t)dX$$

Remember that

$$\mathsf{E}(dX) = 0, \;\; dX^2 = dt$$

we only work to $O(dt)$ - anything smaller we ignore and we also know that

$$dS^2 = \sigma^2 S^2 dt$$

So the change dV when $V(S,t) \rightarrow V(S + dS, t + dt)$ is given by

$$dV = V(S + dS, t + dt) - V(S,t) =$$

$$\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}[S\mu dt + S\sigma dX] + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt$$

Re-arranging to have the standard form of a SDE $dG = a(G,t)dt + b(G,t)dX$ gives

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dX. \tag{3}$$

This is Itô's Formula - the Taylor theorem for stochastic variables.

If we now consider S which follows a lognormal random walk, ie $V = \log(S)$ then (3) gives

$$d((\log S)) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dX$$

Integrating both sides over a given time horizon (between t_0

and T)

$$\int_{t_0}^{T} d((\log S)) = \int_{t_0}^{T} \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_{t_0}^{T} \sigma dX \qquad (T > t_0)$$

we obtain

$$\log \frac{S(T)}{S(t_0)} = \left(\mu - \frac{1}{2}\sigma^2\right)(T - t_0) + \sigma(X(T) - X(t_0))$$

Assuming at $t_0 = 0$, $X(0) = 0$ and $S(0) = S_0$ the exact solution becomes

$$S_T = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\phi\sqrt{T}\right\}. \tag{4}$$

(4) is of particular interest when considering the pricing of a simple European option. Stock prices cannot become negative,

so we allow S, a non-dividend paying stock to evolve according to the lognormal process given above - and acts as the starting point of the Black-Scholes framework.

However μ is replaced by the risk-free interest rate r in (2) and the introduction of the risk-neutral measure - in particular the Monte Carlo method for option pricing.

Another important example of a SDE was put forward by Vasicek in 1977. This model has a mean reverting Ornstein-Uhlenbeck process for the short rate and is used for generating interest rates, given by

$$dr_t = (\eta - \gamma r_t)dt + \sigma dX_t. \tag{5}$$

So drift $= (\eta - \gamma r_t)$ and volatility $= \sigma$.

γ refers to the reversion rate and $\frac{\eta}{\gamma}(=\overline{r})$ denotes the mean rate, and we can rewrite this random walk (5) for dr as

$$dr_t = -\gamma(r_t - \overline{r})dt + \sigma dX_t.$$

By setting $W_t = r_t - \overline{r}$, W_t is a solution of

$$dW_t = -\gamma W_t dt + \sigma dX_t, \tag{6}$$

hence it follows that W_t is an Ornstein-Uhlenbeck process and an analytic solution for this equation exists, by using an integrating factor $I_t = \exp(\gamma t)$. We use the product rule by writing

$$\begin{aligned} d(I_t W_t) &= I_t dW_t + W_t dI_t \\ &= \exp(\gamma t)(-\gamma W_t dt + \sigma dX_t) + \gamma W_t \exp(\gamma t)dt \\ &= \sigma \exp(\gamma t) dX_t. \end{aligned}$$

and integrating over $[0,t]$ gives

$$W_t = \alpha \exp(-\gamma t) + \sigma \int_0^t \exp[\gamma(s-t)]dX_s \qquad (7)$$

where $W(0) = \alpha = W_0$.

By using integration by parts, i.e. $\int v \; du = uv - \int u \; dv$ we can simplify (7).

$$u = X_s$$

$$v = \exp(\gamma(s-t)) \rightarrow dv = \gamma \exp(\gamma(s-t))ds$$

Therefore

$$\int_0^t \exp(\gamma(s-t))dX_s = X_t - \gamma \int_0^t \exp(\gamma(s-t))X_s \, ds$$

and we can write (7) as

$$W_t = \alpha \exp(-\gamma t) + \sigma \left(X_t - \gamma \int_0^t \exp(\gamma (s - t)) X_s \, ds \right)$$

allowing numerical treatment for the integral term.

Itô Integrals

Consider the usual (Riemann) definition of a definite integral

$$\int_0^T f(t)dt$$

as the area under the curve where the curve is the graph of $f(t)$ plotted against t.

Assuming f is a "well behaved" function on $[0, T]$, we can define this in many different ways (which all lead to the same value for the defnite integral).

Divide the interval $[0, T]$ into N intervals with end points $t_0 = 0 < t_1 < t_2 < \;\ldots\ldots\; < t_{N-1} < t_N = T$, where the length of an interval $t_i - t_{i+1}$ tends to zero as $N \to \infty$.

We could approximate the definite integral by

1. left hand rectangle rule;

$$\int_0^T f(t)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)(t_{i+1} - t_i)$$

or

2. right hand rectangle rule;

$$\int_0^T f(t)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1})(t_{i+1} - t_i)$$

or

3. trapezium rule;

$$\int_0^T f(t)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \tfrac{1}{2} \left(f(t_i) + f(t_{i+1})\right)(t_{i+1} - t_i)$$

or

4. midpoint rule

$$\int_0^T f(t)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(\tfrac{1}{2}(t_i + t_{i+1})\right)(t_{i+1} - t_i)$$

or any other reasonable approximations to the area under the curve.

In the limit $N \to \infty$ $f(t)$ we get the same value for the definite integral.

Now consider the stochastic integral of the form

$$\int_0^T f(t, X)dt = \int_0^T f(t, X(t))dX(t)$$

where $X(t)$ is a Brownian motion. We can define this integral as

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

where $X_i = X(t_i)$, or as

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

or as

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{i+\frac{1}{2}}\right)(X_{i+1} - X_i),$$

where $t_{i+\frac{1}{2}} = \frac{1}{2}(t_i + t_{i+1})$ and $X_{i+\frac{1}{2}} = X\left(t_{i+\frac{1}{2}}\right)$ or in many other ways.

In the case of a stochastic variable $dX(t)$ ***the value of the stochatic integral*** **does** *depend on which definition we choose.*

In the case of a stochastic integral, the definition

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

is special. This definition results in the **Itô Integral**.

It is special because it is **non-anticipatory**; given that we are at time t_i we know $X_i = X(t_i)$ and therefore we know $f(t_i, X_i)$. The only uncertainty is in the $X_{i+1} - X_i$ term.

Compare this to a definition such as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

which is **anticipatory**; given that at time t_i we know X_i but are uncertain about the future value of X_{i+1}. Thus we are uncertain about *both* the value of

$$f(t_{i+1}, X_{i+1})$$

and the value of $(X_{i+1} - X_i)$ - there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to anticipate the future value of X_{i+1} so that we may evaluate $f(t_{i+1}, X_{i+1})$.

The Itô integral is consistent with Itô's lemma whereas the others are not.

Example: Consider $f(X) = X^2$, then

$$df = 2X\,dX + dt$$

so

$$\int_0^T 2X(t)\,dX(t) = \int_0^T df - dt$$
$$= X(T)^2 - X(0)^2 - T.$$

The Itô integral is defined as

$$\int_0^T 2X(t)\,dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} 2X_i(X_{i+1} - X_i)$$

Now note that

$$2b(a-b) = a^2 - b^2 - (a-b)^2$$

hence

$$2X_i(X_{i+1} - X_i) = X_{i+1}^2 - X_i^2 - (X_{i+1} - X_i)^2,$$

so that

$$\sum_{i=0}^{N-1} 2X_i(X_{i+1} - X_i) =$$

$$\sum_{i=0}^{N-1} X_{i+1}^2 - \sum_{i=0}^{N-1} X_i^2 - \sum_{i=0}^{N-1} (X_{i+1} - X_i)^2.$$

Now the first two expressions above give

$$\sum_{i=0}^{N-1} X_{i+1}^2 - \sum_{i=0}^{N-1} X_i^2 = X_N^2 - X_0^2$$

$$= X(T)^2 - X(0)^2.$$

In the limit $N \to \infty$, i.e. $dt \to 0$, $(X_{i+1} - X_i)^2 \to dt$, so

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} (X_{i+1} - X_i)^2 = T$$

<u>Exercise</u>: Show that Itô’s lemma implies that

$$3\int_0^t X^2\, dX = X(t)^3 - X(0)^3 - 3\int_0^t X(\tau)\, d\tau.$$

Show that the result also can be found by writing the integral

$$3\int_0^t X^2\, dX = \lim_{N\to\infty} 3 \sum_{i=0}^{N-1} X_i^2 (X_{i+1} - X_i)$$

use $3b^2(a - b) = a^3 - b^3 - 3b(a - b)^2 - (a - b)^3$

Applications of Stochastic Calculus

1. Use Itô's lemma to deduce the following stochastic integral

$$\int_0^t X(\tau)\ln\left[X^2(\tau)\right] dX(\tau) = F(X(t)) - \int_0^t 1 + \frac{1}{2}\ln\left[X^2(\tau)\right] d\tau$$

and find an expression for $F(X(t))$.

Solution:

Consider $G(X(t))$ for which we have

$$dG = \frac{dG}{dX}dX + \frac{1}{2}\frac{d^2G}{dX^2}dt$$

Take $\dfrac{dG}{dX} = X\ln X^2 \Rightarrow G = \dfrac{X^2}{2}\left(\ln X^2 - 1\right)$. So try $G = \frac{1}{2}X^2\left(\ln X^2 - 1\right)$ and substitute in Itô above to give

$$dG = X\ln X^2\,dX + \frac{1}{2}(2 + \ln X^2)dt$$

Integrating over $[0,t]$ and re-aranging gives

$$\int_0^t X(\tau)\ln\left[X^2(\tau)\right]dX(\tau) = \frac{X^2(t)}{2}\left(\ln\left[X^2(t)\right] - 1\right) - \int_0^t 1 + \frac{1}{2}\ln\left[X^2(\tau)\right]d\tau$$

So

$$F(X(t)) = \frac{X^2(t)}{2}\left(\ln\left[X^2(t)\right] - 1\right)$$

2. Show that the solution of the SDE

$$dy = (A + By)dt + (Cy)dX \qquad (\$)$$

$(A, B$ constants$)$ can be written in the form

$$y(t) = \exp(\alpha t + \beta X(t))\left\{y_0 + \gamma \int_0^t e^{(-\alpha\tau - \beta X(\tau))} d\tau\right\} \qquad \#$$

Determine α, β, γ given $y(0) = y_0$.

$\boxed{\text{Solution:}}$ It is sufficient here to perform Itô on $y(t)$

$$dy = \left[\frac{\partial y}{\partial t} + \frac{1}{2}\frac{\partial^2 y}{\partial X^2} \right] dt + \frac{\partial y}{\partial X} dX \Rightarrow$$

writing $y(t) = h(t,X)g(t,X)$ where

From (#) we have

$h(t,X) = e^{(\alpha t + \beta X(t))}$

$g(t,X) = \left\{ y_0 + \gamma \int_0^t e^{(-\alpha\tau - \beta X(\tau))} d\tau \right\}$

$$\begin{aligned} \frac{\partial y}{\partial t} &= hg_t + gh_t = e^{(\alpha t + \beta X(t))} \gamma e^{(-\alpha t - \beta X(t))} \\ &+ \alpha e^{(\alpha t + \beta X(t))} \left\{ y_0 + \gamma \int_0^t e^{(-\alpha\tau - \beta X(\tau))} d\tau \right\} \\ &= \gamma + \alpha y \end{aligned}$$

Similarly

$$
\begin{aligned}
\frac{\partial y}{\partial X} &= hg_X + gh_X = \\
& e^{(\alpha t+\beta X(t))} \frac{\partial}{\partial X} \left\{ y_0 + \gamma \int_0^t e^{(-\alpha\tau-\beta X(\tau))} d\tau \right\} + \\
& \left\{ y_0 + \gamma \int_0^t e^{(-\alpha\tau-\beta X(\tau))} d\tau \right\} \beta e^{(\alpha t+\beta X(t))} \\
&= \beta y
\end{aligned}
$$

Then it follows

$$
\frac{\partial^2 y}{\partial X^2} = \frac{\partial}{\partial X}(\beta y) = \beta(\beta y) = \beta^2 y
$$

Itô's lemma for dy then gives

$$dy = \left[\frac{\partial y}{\partial t} + \frac{1}{2} \frac{\partial^2 y}{\partial X^2} \right] dt + \frac{\partial y}{\partial X} dX \;\; \Rightarrow$$

$$dy = \left(\alpha y + \frac{1}{2} \beta^2 y + \gamma \right) dt + \beta y \, dX$$

$$\equiv (A + By)dt + (Cy)dX \text{ from (\$)}$$

So comparing coefficients gives

$$\left. \begin{aligned} \gamma &= A \\ \beta &= C \\ \alpha &= B - \tfrac{1}{2} C^2 \end{aligned} \right\}$$

Use Itô's lemma to deduce the following formulae for SDE's and stochastic (Itô) integrals

3. Show that

$$\int_0^t X(\tau)\left[1 - e^{-X^2(\tau)}\right] dX(\tau) = F_1(X(t)) +$$

$$\int_0^t F_2(X(\tau))d\tau$$

and determine the functions F_1 and F_2.

$\boxed{\text{Solution:}}$ Start with Itô's lemma

$$dF_1 = \frac{\partial F_1}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F_1}{\partial X^2} dt$$

Integrating both sides over $[0, t]$

$$F_1(X(t)) - F_1(0) = \int_0^t \frac{\partial F_1}{\partial X} dX + \frac{1}{2}\int_0^t \frac{\partial^2 F_1}{\partial X^2} d\tau$$

so with

$$\frac{\partial F_1}{\partial X} = X\left[1 - e^{-X^2}\right] \Rightarrow F_1 = \frac{1}{2}\left(X^2 + e^{-X^2}\right) - \frac{1}{2}$$

$$\frac{\partial^2 F_1}{\partial X^2} = 1 - e^{-X^2} + 2X^2 e^{-X^2}$$

so

$$F_1(X(t)) = \frac{1}{2}X(t)^2 + \frac{1}{2}e^{-X(t)^2} - \frac{1}{2}$$

$$F_2(X) = -\frac{1}{2}\left(1 - e^{-X^2}\right) - X^2 e^{-X^2}$$

4. Show that

$$d\cos(X(t)) = \alpha\cos(X(t))dt + \beta\sin(X(t))dX$$

&

$$d\sin(X(t)) = \alpha\sin(X(t))dt - \beta\cos(X(t))dX$$

and find α & β.

$\boxed{\text{Solution:}}$ Put

$$\left.\begin{aligned} F &= \cos(X(t)) \\ G &= \sin(X(t)) \end{aligned}\right\} \Rightarrow \text{ Itô gives}$$

$$
\left.\begin{aligned}
dF &= \frac{\partial F}{\partial X} dX + \tfrac{1}{2} \frac{\partial^2 F}{\partial X^2} dt = -\sin(X) dX - \tfrac{1}{2} \cos(X) dt \\
dG &= \frac{\partial G}{\partial X} dX + \tfrac{1}{2} \frac{\partial^2 G}{\partial X^2} dt = \cos(X) dX - \tfrac{1}{2} \sin(X) dt
\end{aligned}\right\}
$$

comparing with earlier expressions gives

$$
\alpha = -\tfrac{1}{2}; \quad \beta = -1
$$

5. Given that

$$S = \exp\left(\alpha t + \beta \int_0^t dX(\tau) \right).$$

Find α and β such that S satisfies the SDE

$$dS = \mu S dt + \sigma S dX$$

where μ, σ are constants. Find the solution of this last equation when σ is still constant but $\mu = f(t)$ a given function of time.

$\boxed{\text{Solution:}}$ The equation for S can be written as

$$S = \exp(\alpha t + \beta \{X(t) - X(0)\}), \qquad (\widehat{1})$$

now

$$S_t = \alpha \exp(\alpha t + \beta\{X(t) - X(0)\}) = \alpha S$$

$$S_X = \beta \exp(\alpha t + \beta\{X(t) - X(0)\}) = \beta S \Rightarrow S_{XX} = \beta^2 S^2$$

So Itô's lemma then gives

$$dS = \alpha S dt + \beta S dX + \tfrac{1}{2}\beta^2 S^2 dt$$

$$= (\alpha + \tfrac{1}{2}\beta^2) S dt + \beta S dX \qquad (\hat{2})$$

and $(\hat{2})$ is equivalent to $dS = \mu S dt + \sigma S dX$

so $\boxed{\beta \equiv \sigma}$ and $\alpha + \tfrac{1}{2}\beta^2 \equiv \mu \Rightarrow \boxed{\alpha = \mu - \tfrac{1}{2}\sigma^2}$

Now we know $\mu = f(t)$, so try solution of form $(\hat{1})$ by replacing αt by $g(t)$ to give

$$dS = (g'(t) + \tfrac{1}{2}\beta^2)Sdt + \beta SdX$$

so we still have $\beta = \sigma$, but now $\mu = f\ (t) = g'(t) + \frac{1}{2}\sigma^2$. So integrating gives

$$g(t) = \int_0^t f\left(\hat{t}\right)d\hat{t} - \tfrac{1}{2}\sigma^2 t$$

and

$$S = \exp\left(\int_0^t f\left(\hat{t}\right)d\,\hat{t} - \tfrac{1}{2}\sigma^2 t + \sigma\int_0^t dX(\tau)\right)$$