CHAPTER 11

AN INTRODUCTION TO EXOTIC AND PATH-DEPENDENT OPTIONS

1. **A chooser option has the following properties:**

   At time $T_C < T$, the option gives the holder the right to buy a European call or put option with exercise price E and expiry at time T, for an amount E_C. What is the value of this option when $E_C = 0$?

   Hint: Write down the payoff of the option and then use put-call parity to simplify the result.

   This chooser option has payoff

$$V(S, T_C) = \max\left(C_{BS}(S, T_C; E, T), P_{BS}(S, T_C; E, T)\right).$$

   We use put-call parity to substitute for the value of the put option, to find

$$\begin{aligned} V(S, T_C) &= \max\left(C_{BS}(S, T_C; E, T), C_{BS}(S, T_C; E, T) - S \right.\\ &\qquad\qquad \left. + Ee^{-r(T-T_C)}\right) \\ &= C_{BS}(S, T_C; E, T) + \max\left(Ee^{-r(T-T_C)} - S, 0\right) \\ &= C_{BS}(S, T_C; E, T) + P_{BS}\left(S, T_C; Ee^{-r(T-T_C)}, T_C\right). \end{aligned}$$

   The chooser payoff can therefore be synthesised as a call option with exercise price E and expiry at time T plus a put option with exercise price $Ee^{-r(T-T_C)}$ and expiry at time T_C. In absence of arbitrage opportunities, we must therefore have

$$V(S, t) = C_{BS}(S, t; E, T) + P_{BS}\left(S, t; Ee^{-r(T-T_C)}, T_C\right).$$

2. **How would we value the chooser option in the above question if E_C was non-zero?**

   When E_C is non-zero, the chooser option has payoff

$$\begin{aligned} V(S, T_C) = \max&\left(C_{BS}(S, T_C; E, T)\right. \\ &\qquad \left. - E_C, P_{BS}(S, T_C; E, T) - E_C, 0\right). \end{aligned}$$

We have explicit solutions for the values of $C_{BS}(S, T_C; E, T)$ and $P_{BS}(S, T_C; E, T)$, so we know the payoff $V(S, T_C)$ as a function of S. We must then solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with final data $V(S, T_C)$ and boundary conditions

$$V(0, t) = e^{-r(T_C - t)} \max(E - E_C, 0),$$

as

$$V(0, T_C) = \max(E - E_C, 0),$$

and

$$V(S, t) \sim S \text{ as } S \to \infty.$$

3. **Prove put-call parity for European compound options:**

$$C_C + P_P - C_P - P_C = S - E_2 e^{-r(T_2 - t)},$$

where C_C is a call on a call, C_P is a call on a put, P_C is a put on a call and P_P is a put on a put. The compound options have exercise price E_1 and expiry at time T_1 and the underlying calls and puts have exercise price E_2 and expiry at time T_2.

Consider the portfolio $\Pi_1 = C_C - P_C$, then

$$\begin{aligned}
\Pi_1(T_1) &= C_C(S, T_1) - P_C(S, T_1) \\
&= \max(C_{BS}(S, T_1; E_2, T_2) - E_1, 0) \\
&\quad - \max(E_1 - C_{BS}(S, T_1; E_2, T_2), 0) \\
&= C_{BS}(S, T_1; E_2, T_2) - E_1.
\end{aligned}$$

In the absence of arbitrage opportunities, we must have

$$\Pi_1(t) = C_C(S, t) - P_C(S, t) = C_{BS}(S, t; E_2, T_2) - E_1 e^{-r(T_1 - t)}.$$

Similarly, consider the portfolio $\Pi_2 = C_P - P_P$, then

$$\begin{aligned}
\Pi_2(T_1) &= C_P(S, T_1) - P_P(S, T_1) \\
&= \max(P_{BS}(S, T_1; E_2, T_2) - E_1, 0) \\
&\quad - \max(E_1 - P_{BS}(S, T_1; E_2, T_2), 0) \\
&= P_{BS}(S, T_1; E_2, T_2) - E_1.
\end{aligned}$$

In the absence of arbitrage opportunities, we must have

$$\Pi_2(t) = C_P(S, t) - P_P(S, t) = P_{BS}(S, t; E_2, T_2) - E_1 e^{-r(T_1 - t)}.$$

Then

$$
\begin{aligned}
C_C + P_P - C_P - P_C &= \left(C_{BS}(S,t;\,E_2,T_2) - E_1 e^{-r(T_1-t)}\right) \\
&\quad - \left(P_{BS}(S,t;\,E_2,T_2) - E_1 e^{-r(T_1-t)}\right) \\
&= C_{BS}(S,t;\,E_2,T_2) - P_{BS}(S,t;\,E_2,T_2) \\
&= S - E_2^{-r(T_2-t)},
\end{aligned}
$$

from put-call parity for vanilla call and put options.

4. **Find the value of the power European call option. This is an option with exercise price E, expiry at time T, when it has a payoff:**

$$\Lambda(S) = \max(S^2 - E, 0).$$

Hint: Note that if the underlying asset price is assumed to be lognormally distributed then the square of the price is also lognormally distributed.

We must solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with final data $\max(S^2 - E, 0)$.

If we substitute $R = S^2$, then we find

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 R\left((4R)\frac{\partial^2 V}{\partial R^2} + 2\frac{\partial V}{\partial R}\right) + 2rR\frac{\partial V}{\partial R} - rV = 0,$$

which gives us

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}(2\sigma)^2 R^2 \frac{\partial^2 V}{\partial R^2} + \left(\sigma^2 + 2r\right) R\frac{\partial V}{\partial R} - rV = 0,$$

with final data $\max(R - E, 0)$.

This is just the Black–Scholes equation for a European call option, with a volatility of 2σ, interest rate of r and dividend yield of $-(\sigma^2 + r)$. The value of the power option is therefore

$$C_{BS}(S^2, t;\,E, T),$$

with the above volatility, interest rate and dividend yield.