CHAPTER 17

YIELD CURVE FITTING

1. **Substitute the fitted function for $A(t;T)$, using the Ho & Lee model, back into the solution of the bond pricing equation for a zero-coupon bond,**

$$Z(r,t;T) = e^{A(t;T)-r(T-t)}.$$

What do you notice?

With a Ho & Lee model, the form of the fitted function for $A(t;T)$ is

$$A(t;T) = \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - (T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) \\ - \tfrac{1}{2}c^2(t-t^*)(T-t)^2.$$

Then

$$Z(t;T) = e^{\log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right)-(T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;t))-\frac{1}{2}c^2(t-t^*)(T-t)^2-r(T-t)} \\ = \frac{Z_M(t^*;T)}{Z_M(t^*;t)}e^{-(T-t)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;t))+\frac{1}{2}c^2(t-t^*)(T-t)+r\right)}.$$

We note that when $t = t^*$,

$$Z(t^*;T) = \frac{Z_M(t^*;T)}{Z_M(t^*;t^*)}e^{-(T-t^*)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;t))+\frac{1}{2}c^2(t^*-t^*)(T-t^*)+r\right)} \\ = Z_M(t^*;T).$$

2. **Differentiate Equation (17.2) twice to solve for the value of $\eta^*(t)$. What is the value of a zero-coupon bond with a fitted Vasicek model for the interest rate?**

We have

$$-\int_{t^*}^{T}\eta^*(s)B(s;T)\,ds \\ + \frac{c^2}{2\gamma^2}\left(T-t^*+\frac{2}{\gamma}e^{-\gamma(T-t^*)}-\frac{1}{2\gamma}e^{-2\gamma(T-t^*)}-\frac{3}{2\gamma}\right) \\ = \log(Z_M(t^*;T)) + r^*B(t^*,T).$$

Differentiating with respect to T,

$$-\int_{t^*}^{T}\eta^*(s)\frac{\partial B(s;T)}{\partial T}ds$$
$$-\eta^*(T)B(T,T)+\frac{c^2}{2\gamma^2}\left(1-2e^{-\gamma(T-t^*)}+e^{-2\gamma(T-t^*)}\right)$$
$$=\frac{\partial}{\partial T}\log(Z_M(t^*;T))+r^*\frac{\partial B(t^*,T)}{\partial T}.$$

Now

$$B(t;T)=\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right),$$

so

$$B(T,T)=0,$$

and

$$\frac{\partial B(t,T)}{\partial T}=e^{-\gamma(T-t)}.$$

Substituting back into the partial differential equation

$$-\int_{t^*}^{T}\eta^*(s)e^{-\gamma(T-s)}ds+\frac{c^2}{2\gamma^2}\left(1-2e^{-\gamma(T-t^*)}+e^{-2\gamma(T-t^*)}\right)$$
$$=\frac{\partial}{\partial T}\log(Z_M(t^*;T))+r^*e^{-\gamma(T-t^*)}.$$

Differentiating again with respect to T,

$$-\eta^*(T)+\gamma\int_{t^*}^{T}\eta^*(s)e^{-\gamma(T-s)}ds$$
$$+\frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)}-2\gamma e^{-2\gamma(T-t^*)}\right)$$
$$=\frac{\partial^2}{\partial T^2}\log(Z_M(t^*;T))-\gamma r^*e^{-\gamma(T-t^*)}.$$

Substituting for the integral from the previous equation, we find

$$-\eta^*(T)+\gamma\left(\frac{c^2}{2\gamma^2}\left(1-2e^{-\gamma(T-t^*)}+e^{-2\gamma(T-t^*)}\right)\right.$$
$$\left.-\frac{\partial}{\partial T}\log(Z_M(t^*;T))-r^*e^{-\gamma(T-t^*)}\right)$$

$$+\frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)}-2\gamma e^{-2\gamma(T-t^*)}\right)$$
$$=\frac{\partial^2}{\partial T^2}\log(Z_M(t^*;T))-\gamma r^* e^{-\gamma(T-t^*)}.$$

This simplifies to

$$\eta^*(T)=-\frac{\partial^2}{\partial T^2}\log(Z_M(t^*;T))+\frac{c^2}{2\gamma}-\gamma\frac{\partial}{\partial T}\log(Z_M(t^*;T))$$
$$-\frac{c^2}{2\gamma}e^{-2\gamma(T-t^*)},$$

and

$$\eta^*(t)=-\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t))-\gamma\frac{\partial}{\partial t}\log(Z_M(t^*;t))$$
$$+\frac{c^2}{2\gamma}\left(1-e^{-2\gamma(t-t^*)}\right).$$

We then have

$$A(t;T)=-\int_t^T \eta^*(s)B(s;T)\,ds$$
$$+\frac{c^2}{2\gamma^2}\left(T-t+\frac{2}{\gamma}e^{-\gamma(T-t)}-\frac{1}{2\gamma}e^{-2\gamma(T-t)}-\frac{3}{2\gamma}\right)$$

and substituting for η^* and integrating, we find

$$=\log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right)-B(t;T)\frac{\partial}{\partial t}\log(Z_M(t^*;t))$$
$$-\frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)}-e^{-\gamma(t-t^*)}\right)\left(e^{2\gamma(t-t^*)}-1\right).$$

The value of a zero-coupon bond is

$$Z(r,t;T)=e^{A(t;T)-rB(t;T)},$$

with $A(t;T)$ given by the above, and

$$B(t;T)=\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right).$$

3. Use market data for the price of zero-coupon bonds to fit a Ho & Lee model. Refit the model with data for a week later in time. Compare the two curves for $\eta^*(t)$.

Note: The second curve for η^* starts a week after the first curve. They should not be plotted starting at the same point in time.

For a Ho & Lee model, we fit

$$\eta^*(t) = c^2(t - t^*) - \frac{\partial^2}{\partial t^2} \log(Z_M(t^*; t)).$$

Figure 17.1 shows the fitted function, and the function re-fitted a week later.

For the two functions to be consistent, we would need the refitted curve to lie on top of the first curve. In fact, the refitted data is similar to a translation of the first curve, and the two functions are clearly inconsistent.

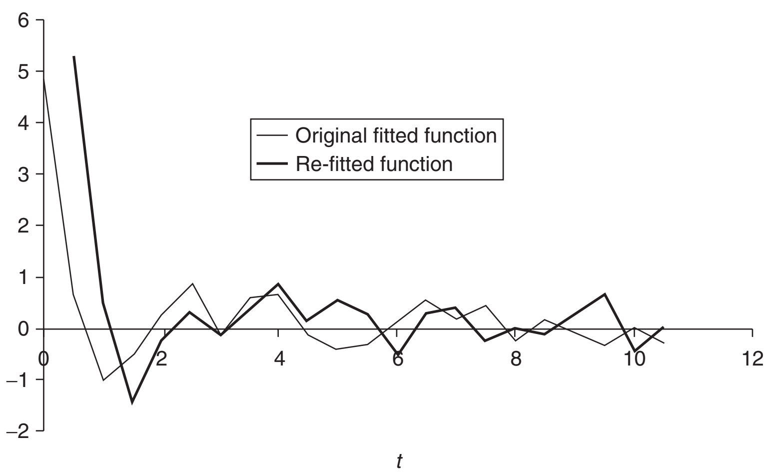

Figure 17.1 The fitted and re-fitted $\eta^*(t)$ for the Ho & Lee model.

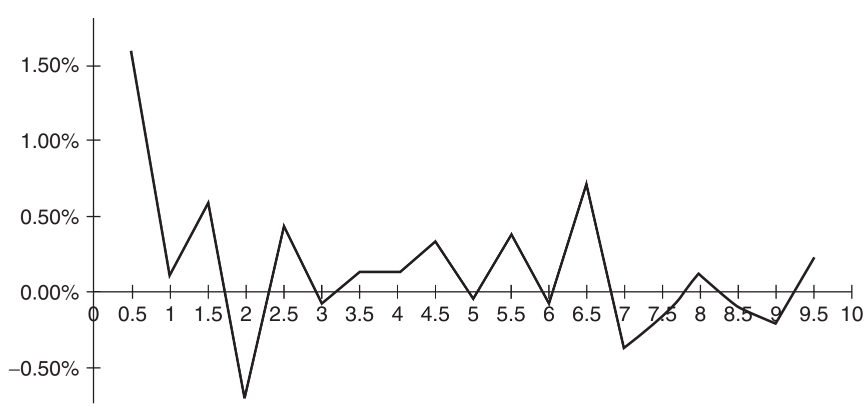

Figure 17.2 A fitted $\eta^*(t)$ for the Vasicek model.

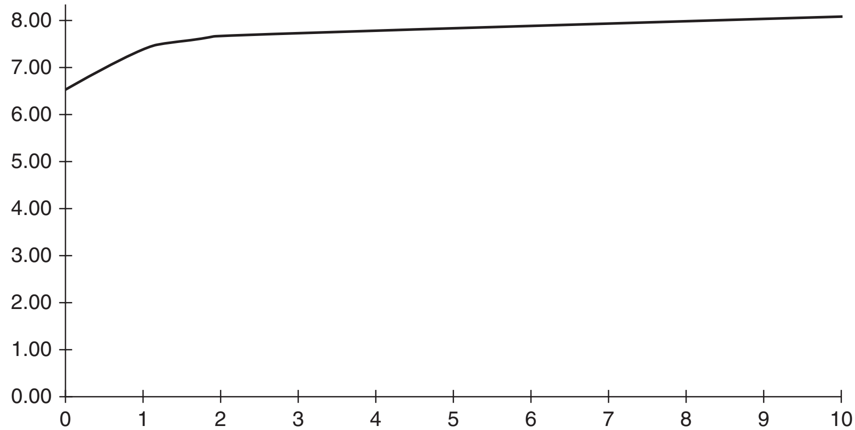

Figure 17.3 The yield curve.

4. **Use market data for zero-coupon bond prices to fit a Vasicek model for the interest rate.**

For a Vasicek model, we fit

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)) - \gamma\frac{\partial}{\partial t}\log(Z_M(t^*;t)) + \frac{c^2}{2\gamma}(1 - e^{-2\gamma(t-t^*)}).$$

Figure 17.2 shows the fitted function for the yield curve shown in Figure 17.3.