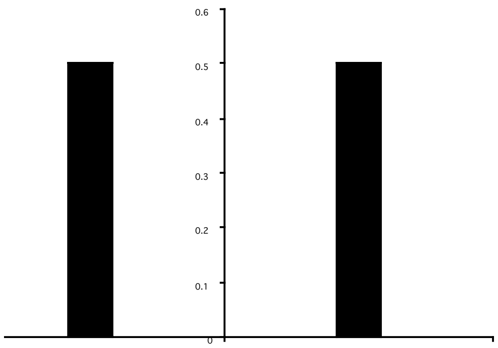
Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

The Central Limit Theorem

Fundamental to the whole subject of finance is the simple concept known as "The Central Limit Theorem." Although not always acknowledged, this is how money is made. Contrary to the popular belief that quants slave away at sophisticated math models to guarantee every trade is profitable, sometimes they make money, sometimes they lose it but...on average they come out ahead. Let's see how this works.

Distributions

A one-dollar bet on the toss of a fair coin gives you a 50% chance of making a dollar, for a head, say, and a 50% chance of losing one dollar, for a tail. We could represent this outcome by the simple bar chart shown below. The heights of the two bars are both 0.5 = 50%. This is an example, a rather simple example, of a probability density function.



Mean

The average amount you'll make on this bet is zero. The average of +1 and -1 each having 50% probability of occurring is just zero. It's a fair coin.

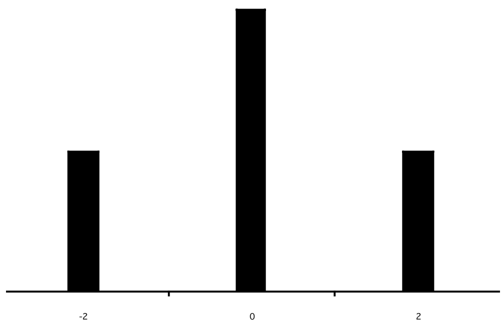
Standard Deviation

The standard deviation, measuring the spread of possible outcomes, is 1.

Adding Up the Random Numbers

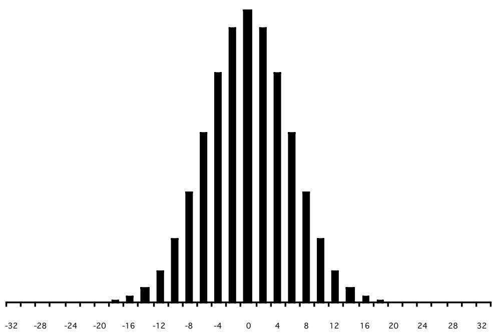
Now suppose we play a little game where we toss the coin twice. Each time there is $1 riding on the outcome. How much money might we have at the end? It will be $2, $0 or -$2. Unless one of the coin lands on an edge, payoffs of $1 and -$1 aren't possible.

And the probabilities of these outcomes? If the chance of getting one head is 50%, the probability of getting one head and then another is 50% of 50% or 25%. There's the same chance of getting two tails, leaving 50% chance of a head and a tail or a tail and a head. Again, we can plot the probability density function.



The average is still zero. But the standard deviation is now the square root of 2.

Let's toss the coin 32 times. The distribution looks like this



The mean is still zero, but the standard deviation is now square root of 32.

We are getting close to a statement of the Central Limit Theorem.

As the number of tosses gets larger and larger, so the shape of the probability density function gets closer and closer to the famous bell-shaped curve of the Normal or Gaussian distribution. The mean is always zero and the standard deviation is the square root of the number of coin tosses.

If this limiting behavior only worked for the tossing of coins then it wouldn't be that much use. The amazing thing is that the Normal distribution is the limiting distribution if you add up many random numbers from any basic building-block distribution...provided a few simple criteria are satisfied:

1. The mean of the individual distribution must be finite, and constant (it can't vary from one toss to the next for example)
2. The standard deviation of the individual distribution must be finite and constant (you mustn't bet different amounts on each toss)
3. Each random number must be independent of previous ones

Now let's see the result.

The Central Limit Theorem

Add up N independent, identically distributed random numbers, each draw having mean of m and standard deviation of s, then the sum will tend to a Normal distribution as N tends to infinity with mean of Nm and standard deviation sN1/2.

Example

You cut a deck of playing cards. If you get a court card or Ace you lose one dollar. If you get a 2 to 10 inclusive you win one dollar. Your average is

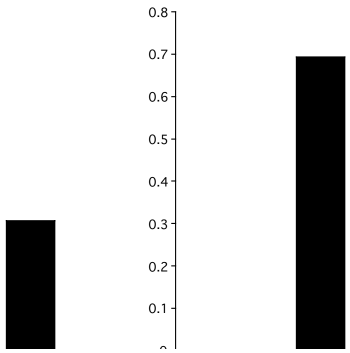
$$\frac{9}{13} \times 1 + \frac{4}{13} \times (-1) = \frac{5}{13}.$$

Your variance is

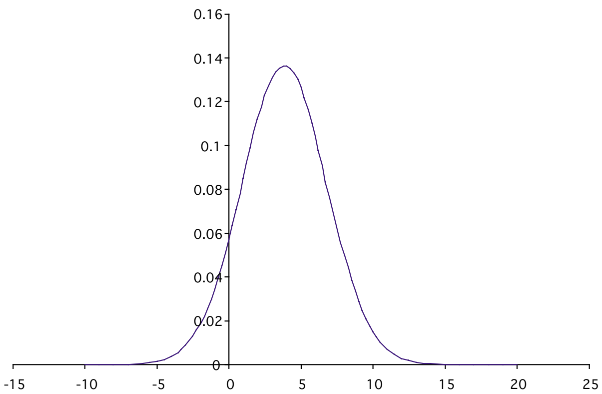
$$\frac{9}{13} \times \left(1 - \frac{5}{13}\right)^2 + \frac{4}{13} \times \left(-1 - \frac{5}{13}\right)^2$$
$$= \frac{144}{169}$$

so that the standard deviation is 12/13.

Here's your distribution after one cut.



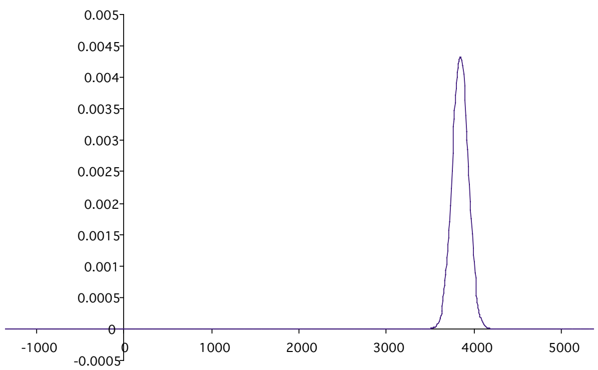
After 10 cuts your profit distribution looks something like this.



Despite heavy odds in your favor you may still lose money.

After 10,000 cuts of the deck (a very heavy session!) the distribution of your winnings would look like this (see right)

The mean is 10,000 x 5/13 and the standard deviation is 100 x 12/13. The mean is now enormously greater than the standard deviation.



Although it is possible to lose money, the chances are microscopically small.

The Central Limit Theorem is important because it tells you what happens after a large number of bets or investments. It shows that the two most important factors in an investment are its mean and its standard deviation. The best investment is one that has the largest mean (and msut definitely be positive!) and the smallest standard deviation.