

CHAPTER 6

THE BLACK–SCHOLES MODEL

1. Check that the following are solutions of the Black–Scholes equation:

- (a) $V(S, t) = S$,
(b) $V(S, t) = e^{rt}$.

Why are these solutions of particular note?

The Black–Scholes equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

It is simple to substitute for (a) and (b) and show that they do satisfy the equation. These solutions are important as they have financial significance. (a) represents a share, and (b) cash. It shows that we can discount shares and cash using Black–Scholes, in the same way as we would options, and still be consistent with simpler methods for discounting these items.

2. What is the most general solution of the Black–Scholes equation with each of the following forms?

- (a) $V(S, t) = A(S)$,
(b) $V(S, t) = B(S)C(t)$.

(a) Substitute $V = A(S)$ into the Black–Scholes equation to get

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 A}{dS^2} + rS \frac{dA}{dS} - rA = 0.$$

Since the powers of S in the equation match the order of the derivatives, we are motivated to try a solution of the form $A(S) = S^n$. We then find

$$\frac{1}{2}\sigma^2 n(n-1) + r(n-1) = 0.$$

This has roots $n = 1$ and $n = -2r/\sigma^2$. The general solution for $V(S, t)$ is therefore

$$V(S, t) = CS + DS^{-2r/\sigma^2},$$

where C and D are arbitrary constants.

- (b) Substitute $V = B(S)C(t)$ into the Black-Scholes equation to find, on rearranging,

$$\frac{1}{C} \frac{dC}{dt} = -\frac{1}{B} \left(\frac{1}{2} \sigma^2 S^2 \frac{d^2 B}{dS^2} + rS \frac{dB}{dS} - rB \right).$$

The left hand side is a function of t only and the right hand side is a function of S only. Therefore both sides must be equal to a constant, k say. We then solve the left hand side to find

$$C = C_0 e^{kt},$$

and the right hand side gives us

$$\frac{1}{2} \sigma^2 S^2 \frac{d^2 B}{dS^2} + rS \frac{dB}{dS} + (k - r)B = 0.$$

We try $B(S) = S^n$ to find

$$\frac{1}{2} \sigma^2 n^2 + (r - \frac{1}{2} \sigma^2) n + k - r = 0.$$

This quadratic equation for n has roots n_1 and n_2 , where

$$n_1 = \frac{1}{\sigma^2} \left(\frac{1}{2} \sigma^2 - r + \sqrt{(r - \frac{1}{2} \sigma^2)^2 - 2\sigma^2(k - r)} \right),$$

and

$$n_2 = \frac{1}{\sigma^2} \left(\frac{1}{2} \sigma^2 - r - \sqrt{(r - \frac{1}{2} \sigma^2)^2 - 2\sigma^2(k - r)} \right).$$

The general solution for V is then

$$V(S, t) = (ES^{n_1} + FS^{n_2}) e^{kt},$$

where E and F are arbitrary constants.

- 3. Prove the following bounds on European call options $C(S, t)$, with expiry at time T , on an underlying share price S , with no dividends:**

(a)

$$C \leq S,$$

(b)

$$C \geq \max(S - Ee^{-r(T-t)}, 0),$$

(c)

$$0 \leq C_1 - C_2 \leq (E_2 - E_1)e^{-r(T-t)},$$

where C_1 and C_2 are calls with exercise prices E_1 and E_2 respectively, and $E_1 < E_2$.

We use the following arbitrage argument for various portfolios, Π :

If the payoff from a portfolio is greater than or equal to an amount of cash, M , at time T , then, in the absence of arbitrage opportunities, the present value of the portfolio is greater than or equal to the discounted value of the cash, $Me^{-r(T-t)}$.

If this were not true, then we could borrow an amount $Me^{-r(T-t)}$ from the bank, at time t , and purchase the portfolio. At time T , we would payoff the loan with the payoff from the portfolio and make a risk-free profit. But this is a contradiction of our no-arbitrage assumption.

(a) $\Pi = S - C$:

At time T , the portfolio is worth

$$\Pi(T) = S - \max(S - E, 0) \geq 0.$$

In the absence of arbitrage opportunities, we must therefore have

$$\Pi(t) = S - C \geq 0,$$

hence

$$S \geq C.$$

(b) same portfolio as (a):

We can also see that

$$\Pi(T) \leq E.$$

In the absence of arbitrage opportunities, we must therefore have

$$\Pi(t) = S - C \leq Ee^{-r(T-t)},$$

hence

$$C \geq S - Ee^{-r(T-t)}.$$

Now $C \geq 0$ as C has a non-negative payoff, so

$$C \geq \max(S - Ee^{-r(T-t)}, 0).$$

(c) $\Pi = C_1 - C_2$:

At time T , the portfolio is worth

$$\Pi(T) = \max(S - E_1, 0) - \max(S - E_2, 0),$$

Which gives us that

$$0 \leq \Pi(T) \leq E_2 - E_1.$$

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In the absence of arbitrage opportunities, we must therefore have

$$0 \leq \Pi(t) \leq (E_2 - E_1)e^{-r(T-t)},$$

hence

$$0 \leq C_1 - C_2 \leq (E_2 - E_1)e^{-r(T-t)}.$$

- 4. Prove the following bounds on European put options $P(S, t)$, with expiry at time T , on an underlying share price S , with no dividends:**

(a)

$$P \leq Ee^{-r(T-t)},$$

(b)

$$P \geq Ee^{-r(T-t)} - S,$$

(c)

$$0 \leq P_2 - P_1 \leq (E_2 - E_1)e^{-r(T-t)},$$

where P_1 and P_2 are calls with exercise prices E_1 and E_2 respectively, and $E_1 < E_2$.

We use the same form of arbitrage argument as in Question 3.

(a) $\Pi = P - E$:

At time T , the portfolio is worth

$$\Pi(T) = \max(E - S, 0) - E \leq 0.$$

In the absence of arbitrage opportunities, we must therefore have

$$\Pi(t) = P - Ee^{-r(T-t)} \leq 0,$$

hence

$$P \leq Ee^{-r(T-t)}.$$

(b) $\Pi = S + P$:

At time T , the portfolio is worth

$$\Pi(T) = S + \max(E - S, 0) \geq E.$$

In the absence of arbitrage opportunities, we must therefore have

$$\Pi(t) = S + P \geq Ee^{-r(T-t)},$$

hence

$$P \geq Ee^{-r(T-t)} - S.$$

(c) $\Pi = P_2 - P_1$: At time T , the portfolio is worth

$$\Pi(T) = \max(E_2 - S, 0) - \max(E_1 - S, 0),$$

which gives us that

$$0 \leq \Pi(T) \leq E_2 - E_1.$$

In the absence of arbitrage opportunities, we must therefore have

$$0 \leq \Pi(t) \leq (E_2 - E_1)e^{-r(T-t)},$$

hence

$$0 \leq P_2 - P_1 \leq (E_2 - E_1)e^{-r(T-t)}.$$

5. Prove the following bounds on European call options $C(S, t)$, on an underlying share price S , with no dividends:

(a)

$$C_A \geq C_B,$$

where C_A and C_B are calls with the same exercise price, E and expiry dates T_A and T_B respectively, and $T_A > T_B$.

(b)

$$C_2 \leq \frac{E_3 - E_2}{E_3 - E_1} C_1 + \frac{E_2 - E_1}{E_3 - E_1} C_3,$$

where C_1 , C_2 and C_3 are calls with the same expiry, T , and have exercise prices E_1 , E_2 and E_3 respectively, where $E_1 < E_2 < E_3$.

Hint: Consider $E_2 = \lambda E_1 + (1 - \lambda)E_3$.

We use the same form of arbitrage argument as in Question 3.

(a) $\Pi = C_A - C_B$: At time T_B , the portfolio is worth

$$\Pi(T_B) = C_A(S, T_B) - \max(S - E, 0).$$

Now Question 3, part (b) gives us that

$$C_A(S, T_B) \geq \max(S - Ee^{-r(T_A-T_B)}, 0) \geq \max(S - E, 0),$$

so

$$\Pi(T_B) \geq 0.$$

In the absence of arbitrage opportunities, we must therefore have

$$\Pi(t) = C_A - C_B \geq 0,$$

hence

$$C_A \geq C_B.$$

- (b) $\Pi = -C_2 + \lambda C_1 + (1 - \lambda)C_3$:
 (where we have chosen λ such that $E_2 = \lambda E_1 + (1 - \lambda)E_3$).
 Then

$$\lambda = \frac{E_3 - E_2}{E_3 - E_1} \text{ and } (1 - \lambda) = \frac{E_2 - E_1}{E_3 - E_1}.$$

At time T , the portfolio is worth

$$\begin{aligned} \Pi(T) = & -\max(S - E_2, 0) + \lambda \max(S - E_1, 0) \\ & + (1 - \lambda) \max(S - E_3, 0). \end{aligned}$$

On inspection, we see that

$$\Pi(T) \geq 0.$$

In the absence of arbitrage opportunities, we must therefore have

$$\Pi(t) = -C_2 + \lambda C_1 + (1 - \lambda)C_3 \geq 0.$$

Substituting for λ , we find

$$C_2 \leq \frac{E_3 - E_2}{E_3 - E_1} C_1 + \frac{E_2 - E_1}{E_3 - E_1} C_3.$$

- 6. $C(S, t)$ and $P(S, t)$ are the values of European call and put options, with exercise price E and expiry at time T . Show that a portfolio of long the call and short the put satisfies the Black-Scholes equation. What boundary and final conditions hold for this portfolio?**

Call and put options satisfy Black-Scholes, so we have that

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0,$$

and

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0.$$

Subtracting the second equation from the first, we find that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$

where $V(S, t) = C(S, t) - P(S, t)$. Hence a portfolio of long a call and short a put satisfies Black-Scholes.

The Boundary conditions for the portfolio are

$$V(0, t) = C(0, t) - P(0, t) = -Ee^{-r(T-t)},$$

and

$$V(S, t) = C(S, t) - P(S, t) \sim S \text{ as } S \rightarrow \infty.$$

The final condition is

$$\begin{aligned} V(S, T) &= C(S, T) - P(S, T) = \max(S - E, 0) - \max(E - S, 0) \\ &= S - E. \end{aligned}$$

7. Consider an option that expires at time T . The current value of the option is $V(S, t)$. It is possible to synthesise the option using vanilla European calls, all with expiry at time T . We assume that calls with all exercise prices are available and buy $f(E)$ of the call with exercise price E , which has value $C(S, t; E)$. The value of the synthesised option is then

$$V(S, t) = \int_0^\infty f(E') C(S, t; E') dE'.$$

Find the density of call options, $f(E)$, that we must use to synthesise the option.

Hint: Synthesise the option payoff to find $f(E)$.

We synthesise the option payoff, $V(S, T) = \Lambda(S)$:

At time T ,

$$\Lambda(S) = \int_0^\infty f(E') \max(S - E', 0) dE',$$

This simplifies to

$$\Lambda(S) = \int_0^S f(E')(S - E') dE'.$$

Differentiating, we find

$$\frac{d\Lambda(S)}{dS} = \int_0^S f(E) dE,$$

and differentiating again,

$$\frac{d^2\Lambda(S)}{dS^2} = f(S).$$

The density of call options that we need is therefore

$$f(E) = \frac{d^2\Lambda(E)}{dE^2}.$$

The current price of the option is then

$$V(S, t) = \int_0^\infty \frac{d^2 \Lambda(E')}{dE'^2} C(S, t; E') dE'.$$

- 8. Find the random walk followed by a European option, $V(S, t)$. Use Black-Scholes to simplify the equation for dV .**

We use Itô's Lemma for a function $V(S, t)$, where $dS = \sigma S dX + \mu S dt$:

$$dV = \frac{\partial V}{\partial S} dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

We substitute for the dt term from Black-Scholes equation,

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$

to find

$$dV = \frac{\partial V}{\partial S} dS + r \left(V - S \frac{\partial V}{\partial S} \right) dt.$$

- 9. Compare the equation for futures to Black-Scholes with a constant, continuous dividend yield. How might we price options on futures if we know the value of an option with the same payoff with the asset as underlying?**

Hint: Consider Black-Scholes with a constant, continuous dividend yield $D = r$.

The equation for an option, $W(F, t)$, with the futures price, F , as underlying is

$$\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 W}{\partial F^2} - rW = 0.$$

Black-Scholes for an option, $V(S, t)$, on an asset, S , with a constant, continuous dividend yield, D , is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0.$$

If we choose $D = r$ in the latter, then the equations are identical (assuming the options have the same boundary and final conditions). Therefore, if we know how to price the option on the asset, we can use this to value the option on the forward, by giving the asset a dividend yield of r .