CHAPTER 2

DERIVATIVES

1. **Express the value of the following portfolios of options at expiry as functions of the share price:**
   (a) Long one share, long one put with exercise price E,
   (b) Long one call and one put, both with exercise price E,
   (c) Long one call, exercise price E_1, short one call, exercise price E_2, where $E_1 < E_2$,
   (d) Long one call, exercise price E_1, long one put, exercise price E_2. There are three cases to consider,
   (e) Long two calls, one with exercise price E_1 and one with exercise price E_2, short two calls, both with exercise price E, where $E_1 < E < E_2$.

   Writing the value of the portfolio as a function, $\Lambda(S)$:

   (a)

$$\Lambda(S) = \begin{cases} E & \text{if } \ 0 \le S \le E \\ S & \text{if } \ S > E. \end{cases}$$

   (b)

$$\Lambda(S) = \begin{cases} E - S & \text{if } \ 0 \le S \le E \\ S - E & \text{if } \ S > E. \end{cases}$$

   (c)

$$\Lambda(S) = \begin{cases} 0 & \text{if } \ 0 \le S \le E_1 \\ S - E_1 & \text{if } \ E_1 < S \le E_2 \\ E_2 - E_1 & \text{if } \ S > E_2. \end{cases}$$

   (d) When $E_1 > E_2$,

$$\Lambda(S) = \begin{cases} E_2 - S & \text{if } \ 0 \le S \le E_2 \\ 0 & \text{if } \ E_2 < S \le E_1 \\ S - E_1 & \text{if } \ S > E_1. \end{cases}$$

   When $E_1 = E_2$,

$$\Lambda(S) = \begin{cases} E_1 - S & \text{if } \ 0 \le S \le E_1 \\ S - E_1 & \text{if } \ S > E_1. \end{cases}$$

When $E_1 < E_2$,

$$\Lambda(S) = \begin{cases} E_2 - S & \text{if} \quad 0 \le S \le E_1 \\ E_2 - E_1 & \text{if} \quad E_1 < S \le E_2 \\ S - E_1 & \text{if} \quad S > E_2. \end{cases}$$

(e)

$$\Lambda(S) = \begin{cases} 0 & \text{if} \quad 0 \le S \le E_1 \\ S - E_1 & \text{if} \quad E_1 < S \le E \\ 2E - E_1 - S & \text{if} \quad E < S \le E_2 \\ 2E - E_1 - E_2 & \text{if} \quad S > E_2. \end{cases}$$

2. **What is the difference between a payoff diagram and a profit diagram? Illustrate with a portfolio of short one share, long two calls with exercise price E.**

   A payoff diagram shows the value of a portfolio as a function of the share price at a specified time (usually the expiry date of an option in the portfolio). A profit diagram takes into account the initial cost of setting up the portfolio, by including the present value of this initial cost. If we wish to find the value of a portfolio using Black–Scholes, then our final condition should be represented by a payoff diagram.

   The example portfolio has payoff

$$\Lambda(S) = \begin{cases} -S & \text{if} \quad 0 \le S \le E \\ S - 2E & \text{if} \quad S > E. \end{cases}$$

   Figure 2.1 shows the payoff and profit diagrams for this portfolio. The payoff and profit diagrams differ significantly due to the large (negative) cost associated with going short the share.

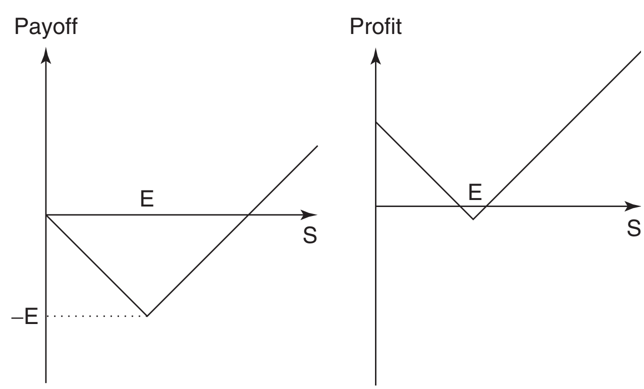

Figure 2.1 Payoff and profit diagrams.

3. **A share currently trades at \$60. A European call with exercise price \$58 and expiry in three months trades at \$3. The three month default-free discount rate is 5%. A put is offered on the market, with exercise price \$58 and expiry in three months, for \$1.50. Do any arbitrage opportunities now exist? If there is a possible arbitrage, then construct a portfolio that will take advantage of it. (This is an application of put-call parity.)**

   There is an arbitrage opportunity. If we buy a call and sell a put and a share then we will own a portfolio with payoff

   $$C - P - S = \max(S - E, 0) - \max(E - S, 0) - S = -E = -\$58.$$

   If we invest the money we make on setting up the portfolio ($-3 + 60 + 1.5 = \$58.5$) at the risk-free rate, then at expiry of the portfolio, we will make a risk-free profit of

   $$58.5e^{0.05(0.25)} - 58 = \$1.24.$$

4. **A three-month, 80 strike, European call option is worth \$11.91. The 90 call is \$4.52 and the 100 call is \$1.03. How much is the butterfly spread?**

   A butterfly call is made up of long one call of one strike, long one of another strike and short two calls with a strike in the middle. The butterfly spread therefore has a value of

   $$11.91 - 2 \times 4.52 + 1.03 = 3.90.$$

5. **Using the notation $V(E)$ to mean the value of a European call option with strike E, what can do say about $\frac{\partial V}{\partial E}$ and $\frac{\partial^2 V}{\partial E^2}$ for options having the same expiration?**

   Hint: Consider call and butterfly spreads and the absence of arbitrage.

   A call spread consisting of long one option and short another with a higher strike must have a positive value. If we write $V(E)$ to denote the value of a call option at a fixed S and t, i.e. a 'snapshot' of the price, then

   $$V(E_1) - V(E_2) > 0 \quad \text{if} \quad E_1 < E_2.$$

   If we write $E_1 = E$ and $E_2 = E + \delta E$ and let $\delta E \to 0$ we get

   $$\frac{\partial V}{\partial E} < 0.$$

Similarly, a butterfly spread must has positive value so

$$V(E - \delta E) - 2V(E) + V(E + \delta E) > 0$$

resulting in

$$\frac{\partial^2 V}{\partial E^2} > 0.$$