CHAPTER 8

THE BLACK–SCHOLES FORMULAE AND THE 'GREEKS'

1. **Find the explicit solution for the value of a European option with payoff $\Lambda(S)$ and expiry at time T, where**

$$\Lambda(S) = \begin{cases} S & \textbf{if } S > E \\ 0 & \textbf{if } S < E. \end{cases}$$

We can rewrite the payoff as

$$\Lambda(S) = \max(E - S, 0) - E\mathcal{H}(E - S) + S.$$

We can therefore synthesise the option with a portfolio of long a put, short a binary put and long a share, where both options are European with exercise price E and expiry at time T. Since we have explicit solutions for the value today of the constituent parts of this portfolio, we can write down the value of our option:

$$\begin{aligned} V(S,t) &= \left(Ee^{-r(T-t)}N(-d_2) - SN(-d_1)\right) \\ &\quad - \left(Ee^{-r(T-t)}(1 - N(d_2))\right) + S \\ &= S(1 - N(-d_1)) = SN(d_1), \end{aligned}$$

since $N(x) + N(-x) = 1$, where

$$d_1 = \frac{\log(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

2. **Find the explicit solution for the value of a European supershare option, with expiry at time T and payoff**

$$\Lambda(S) = \mathcal{H}(S - E_1) - \mathcal{H}(S - E_2),$$

where $E_1 < E_2$.

We can synthesise the supershare as two binary calls. We have already found the present value of a binary call, so we can just write down the value of the supershare as

$$\begin{aligned} V(S,t) &= e^{-r(T-t)}N\left(\overline{d_2}\right) - e^{-r(T-t)}N\left(\overline{\overline{d_2}}\right) \\ &= e^{-r(T-t)}\left(N\left(\overline{d_2}\right) - N\left(\overline{\overline{d_2}}\right)\right), \end{aligned}$$

where

$$\overline{d_2} = \frac{\log(S/E_1) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$\overline{\overline{d_2}} = \frac{\log(S/E_2) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

3. **Consider the pay-later call option. This has payoff $\Lambda(S) = \max(S - E, 0)$ at time T. The holder of the option does not pay a premium when the contract is set up, but must pay Q to the writer at expiry, only if $S \geq E$. What is the value of Q?**

The value of the pay-later call at expiry can be written as

$$V(S,T) = \max(S - E, 0) - Q\mathcal{H}(S - E).$$

The option is therefore equivalent to a portfolio of long a call and short a binary call, both with exercise price E and expiry at time T. The value of this portfolio today is

$$V(S,t) = \left(SN(d_1) - Ee^{-r(T-t)}N(d_2)\right) - Qe^{-r(T-t)}N(d_2).$$

Initially, the contract has no value, so $V(S,t) = 0$ and

$$Q = \frac{Se^{r(T-t)}N(d_1)}{N(d_2)} - E.$$

4. **Find the implied volatility of the following European call. The call has four months until expiry and an exercise price of \$100. The call is worth \$6.51 and the underlying trades at \$101.5, discount using a short-term risk-free continuously compounding interest rate of 8% per annum.**

Substituting into the Black–Scholes formula for the value of a call option:

$$6.51 = 101.5 \times N(d_1) - 100 \times e^{-.08\times 0.25} \times N(d_2),$$

where

$$d_1 = \frac{\log(101.5/100) + \left(0.08 + \frac{1}{2}\sigma^2\right) \times 0.25}{\sqrt{0.25}\sigma},$$

and

$$d_2 = d_1 - \sqrt{0.25}\,\sigma.$$

We use an iterative method to solve this equation for σ (or use Solver in Excel to find the σ that equates the left and right hand sides of the equation) and find that the implied volatility is $\sigma = 0.23$.

5. Consider a European call, currently at the money. Why is delta hedging self financing in the following situations?

(a) The share price rises until expiry,

(b) The share price falls until expiry.

We consider the position of the writer of the option. The initial delta of the call is positive and somewhere between 0 and 1, at a value Δ_0 say. The writer of the option buys Δ_0 of a share to hedge his position.

In situation (a), the share price then rises until expiry. This means that at expiry the share price is greater than the exercise price and the writer must deliver a share for an amount E. As the share price rises, the delta rises and the writer must buy more fractions of a share to hedge his position. At expiry, the delta will have risen to 1 and the writer will own one share which he then gives to the buyer of the option. The writer has had to buy fractions of a share at an increasing price and then sold this share for an amount E. The cost of buying the share in this fashion should be equal to the initial premium for the option plus the exercise price, E. In this case, the hedging is self-financing.

In situation (b), the share price then falls until expiry. This means that at expiry the share price is less than the exercise price and the option is not exercised. As the share price falls, the delta falls and the writer must sell some of his stock to hedge his position. At expiry, the delta will have fallen to 0 and the writer will have no shares left. The writer bought Δ_0 of a share and has then sold this in fractions at decreasing share prices. The loss he has made in doing this should be equal to the initial premium for the option. In this case, the hedging is self-financing.

6. Using the explicit solutions for the European call and put options, check that put-call parity holds.

The explicit values for the European call and put are

$$C_{BS}(S, t; E, T) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

and

$$P_{BS}(S, t; E, T) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1).$$

Noting that $N(x) + N(-x) = 1$, consider

$$\begin{aligned} C - P = &\left(SN(d_1) - Ee^{-r(T-t)}N(d_2)\right) \\ &- \left(Ee^{-r(T-t)}N(-d_2) - SN(-d_1)\right) \end{aligned}$$

$$= S\left(N(d_1) - N(-d_1)\right) - Ee^{-r(T-t)}\left(N(d_2) - N(-d_2)\right)$$
$$= S - Ee^{-r(T-t)}.$$

7. **Consider an asset with zero volatility. We can explicitly calculate the future value of the asset and hence that of a call option with the asset as the underlying. The value of the call option will then depend on the growth rate of the asset, μ. On the other hand, we can use the explicit formula for the call option, in which μ does not appear. Explain this apparent contradiction.**

   If an asset has zero volatility, then it is a risk-free investment. In absence of arbitrage, it must grow at the risk-free rate, r. Hence $\mu = r$ and the two are interchangeable in all that follows.

8. **The range forward contract is specified as follows: At expiry, the holder must buy the asset for E_1 if $S < E_1$, for S if $E_1 \le S \le E_2$ and for E_2 if $S > E_2$. Find the relationship between E_1 and E_2 when the initial value of the contract is zero and $E_1 < E_2$.**

   The range forward has a payoff of $\Lambda(S)$ and expiry at time T, where

$$\Lambda(S) = \begin{cases} S - E_1 & \text{if } S < E_1 \\ 0 & \text{if } E_1 \le S \le E_2 \\ S - E_2 & \text{if } S > E_2. \end{cases}$$

   This payoff can be expressed as

$$\Lambda(S) = \max(S - E_2, 0) - \max(E_1 - S, 0).$$

   We can therefore synthesise the contract using a European call option with exercise price E_2 and a European put option with exercise price E_1, both options expiring at time T. Since we have explicit solutions for the values of these options, we can write down the value of the range forward contract today:

$$V(S,t) = C_{BS}(S,t;E_2,T) - P_{BS}(S,t;E_1,T).$$

   We must then choose E_1 and E_2 such that the initial value of the contract is zero. The relationship between E_1 and E_2 is therefore

$$C_{BS}(S,t;E_2,T) = P_{BS}(S,t;E_1,T).$$

9. **A forward start call option is specified as follows: At time T_1, the holder is given a European call option with exercise price $S(T_1)$ and expiry at time $T_1 + T_2$. What is the value of the option for $0 \le t \le T_1$?**

The value of the option at time T_1 is

$$\begin{aligned} V(S(T_1), T_1) &= C_{BS}(S(T_1), T_1; S(T_1), T_1 + T_2) \\ &= S(T_1)N(A) - S(T_1)e^{-rT_2}N(B) \\ &= S(T_1)\left(N(A) - e^{-rT_2}N(B)\right), \end{aligned}$$

where

$$A = \frac{\left(r + \frac{1}{2}\sigma^2\right)\sqrt{T_2}}{\sigma},$$

and

$$B = \frac{\left(r - \frac{1}{2}\sigma^2\right)\sqrt{T_2}}{\sigma}.$$

At time T_1, the option is therefore equivalent to a constant amount of the asset, kS say. In absence of arbitrage opportunities, the value of the option today must also be worth kS, hence

$$V(S, t) = S(t)\left(N(A) - e^{-rT_2}N(B)\right).$$

10. Consider a delta-neutral portfolio of derivatives, Π. For a small change in the price of the underlying asset, δS, over a short time interval, δt, show that the change in the portfolio value, $\delta \Pi$, satisfies

$$\delta\Pi = \Theta\delta t + \tfrac{1}{2}\Gamma\, \delta S^2,$$

where $\Theta = \frac{\partial \Pi}{\partial t}$ and $\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$.

Applying Itô's Lemma to the value of the portfolio, Π:

$$\delta\Pi = \frac{\partial \Pi}{\partial S}\delta S + \frac{\partial \Pi}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\delta S^2.$$

to order δt. If the portfolio is delta-neutral, then

$$\Delta = \frac{\partial \Pi}{\partial S} = 0,$$

and so

$$\delta\Pi = \frac{\partial \Pi}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\delta S^2 = \Theta\delta t + \tfrac{1}{2}\Gamma\, \delta S^2.$$

11. Show that for a delta-neutral portfolio of options on a non-dividend paying stock, Π,

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi.$$

The portfolio satisfies Black–Scholes equation,

$$\frac{\partial \Pi}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} + rS\frac{\partial \Pi}{\partial S} - r\Pi = 0,$$

and so

$$\Theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = \Pi.$$

If the portfolio is delta neutral, then $\Delta = 0$ and

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = \Pi.$$

12. Show that the vega of an option, v, satisfies the differential equation

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv + \sigma S^2 \Gamma = 0,$$

where $\Gamma = \frac{\partial^2 V}{\partial S^2}$. What is the final condition?

An option, V, satisfies Black–Scholes equation,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with final condition $V(S, T) = \Lambda(S)$. Differentiating with respect to σ,

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \sigma S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = 0,$$

with final condition $v(S, T) = 0$, where $v = \partial V/\partial \sigma$ is the vega of the option. Hence,

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv + \sigma S^2 \Gamma = 0.$$

13. Find the partial differential equation satisfied by ρ, the sensitivity of the option value to the interest rate.

An option, V, satisfies Black–Scholes equation,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Differentiating with respect to r,

$$\frac{\partial \rho}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \rho}{\partial S^2} + rS\frac{\partial \rho}{\partial S} - r\rho + S\frac{\partial V}{\partial S} - V = 0,$$

where $\rho = \partial V/\partial r$.

14. Use put-call parity to find the relationships between the deltas, gammas, vegas, thetas and rhos of European call and put options.

Put-call parity gives us

$$C(S,t) - P(S,t) = S - Ee^{-r(T-t)}.$$

Differentiating with respect to S,

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1,$$

i.e.

$$\Delta_C = 1 + \Delta_P.$$

Differentiating with respect to S for a second time,

$$\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0,$$

i.e.

$$\Gamma_C = \Gamma_P.$$

Similarly, differentiating the put-call parity result with respect to σ:

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma},$$

and with respect to t:

$$\frac{\partial C}{\partial t} - \frac{\partial P}{\partial t} = -rEe^{-r(T-t)},$$

i.e.

$$\Theta_C = \Theta_P - rEe^{-r(T-t)}.$$

Finally, differentiating the put-call parity result with respect to r:

$$\frac{\partial C}{\partial r} - \frac{\partial P}{\partial r} = E(T-t)e^{-r(T-t)},$$

i.e.

$$\rho_C = \rho_P + E(T-t)e^{-r(T-t)}.$$

15. The fundamental solution, u_δ, is the solution of the diffusion equation on $-\infty < x < \infty$ and $\tau > 0$ with $u(x,0) = \delta(x)$. Use this solution to solve the more general problem:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \textbf{on} \quad -\infty < x < \infty, \quad \tau > 0,$$

with $u(x,0) = u_0(x)$.

We can write the initial data for our problem in the form

$$u_0(x) = \int_{-\infty}^{\infty} u_0(s)\delta(x-s)\,ds.$$

Since we know the solution to the problem with initial data $\delta(x)$, we can superpose solutions of this form, weighted with our initial data, $u_0(x)$, to find

$$u(x,\tau) = \int_{-\infty}^{\infty} u_0(s)u_\delta(x-s,\tau)\,ds,$$

where

$$u_\delta(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}e^{-x^2/4\tau}.$$

Hence

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}\,ds.$$