CHAPTER 16

ONE-FACTOR INTEREST RATE MODELING

1. **Substitute**

$$Z(r, t; T) = e^{A(t;T)-rB(t;T)},$$

into the bond pricing equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0.$$

What are the explicit dependencies of the functions in the resulting equation?

If

$$Z(r, t; T) = e^{A(t;T)-rB(t;T)},$$

(with final data $Z(r, T; T) = 1$), then

$$\frac{\partial Z}{\partial t} = \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)Z,$$

$$\frac{\partial Z}{\partial r} = -BZ,$$

and

$$\frac{\partial^2 Z}{\partial r^2} = B^2 Z.$$

Substituting into the bond pricing equation,

$$\left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)Z + \tfrac{1}{2}w^2B^2Z - (u - \lambda w)BZ - rZ = 0,$$

which simplifies to

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \tfrac{1}{2}w^2B^2 - (u - \lambda w)B - r = 0,$$

with final data

$$A(T; T) = B(T; T) = 0.$$

u, w and λ depend on r and t. A and B depend on t and T.

2. **Simulate random walks for the interest rate to compare the different named models suggested in this chapter.**

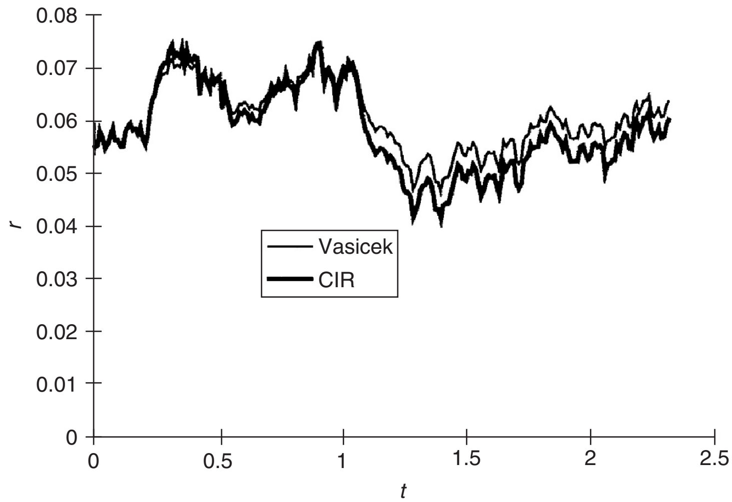

Figure 16.1 A simulation of the Vasicek and CIR models.

The Vasicek model takes the form

$$dr = (\eta - \gamma r)\,dt + \beta^{1/2}\,dX.$$

The Cox, Ingersoll and Ross model takes the form

$$dr = (\eta - \gamma r)\,dt + \sqrt{\alpha r}\,dX.$$

Figure 16.1 shows a simulation of these two models using the same random numbers.

3. **What final condition (payoff) should be applied to the bond pricing equation for a swap, cap, floor, zero-coupon bond, coupon bond and a bond option?**

Final condition for a swap:

$$V(r, T) = (r - r_s)P,$$

where r_s is the fixed rate and P is the principal.

Final condition for a cap:

$$V(r, T) = \max(r - r_c, 0)P,$$

where r_c is the cap rate and P is the principal.

Final condition for a floor:

$$V(r, T) = \max(r_f - r, 0)P,$$

where r_f is the floor rate and P is the principal.

Final condition for a zero-coupon bond:

$$V(r, T) = P,$$

where P is the principal.

Final condition for a coupon bond:

$$V(r, T) = (1 + c)P,$$

where c is the (discrete) coupon rate and P is the principal.

Final condition for a bond option:

$$V(r, T) = \max(Z(r, T) - E, 0),$$

where E is the exercise price and $Z(r, t)$ is the value of the underlying bond at time t.

4. **What form does the bond pricing equation take when the interest rate satisfies the Vasicek model**

$$dr = (\eta - \gamma r)\, dt + \beta^{1/2}\, dX?$$

Solve the resulting equations for A and B in this case, to find

$$A = \frac{1}{\gamma^2}\left((B - T + t)\left(\eta\gamma - \tfrac{1}{2}\beta\right)\right) - \frac{\beta B^2}{4\gamma},$$

and

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)}).$$

The bond pricing equation is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\beta\frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r)\frac{\partial V}{\partial r} - rV = 0.$$

For a zero-coupon bond, we have

$$V = e^{A(t;T) - rB(t;T)},$$

and final data

$$V(r, T) = 1.$$

Substituting into the partial differential equation (see Question 1),

$$\frac{dA}{dt} - r\frac{dB}{dt} + \tfrac{1}{2}\beta B^2 - (\eta - \gamma r)B - r = 0.$$

We can rearrange this to

$$\frac{dA}{dt} + \tfrac{1}{2}\beta B^2 - \eta B = r\left(\frac{dB}{dt} - \gamma B + 1\right).$$

The left-hand side is a function of t only and the right-hand side is r times a function of t, we must therefore have

$$\frac{dA}{dt} + \tfrac{1}{2}\beta B^2 - \eta B = 0,$$

and

$$\frac{dB}{dt} - \gamma B + 1 = 0.$$

The final data $V(r, T) = 1$ gives us

$$A(T; T) = B(T; T) = 0.$$

We first solve the second equation:

The homogeneous problem

$$\frac{dB}{dt} - \gamma B = 0,$$

has general solution

$$B = C_1 e^{\gamma t}.$$

To solve the inhomogeneous problem we note that

$$B = \frac{1}{\gamma},$$

is a particular solution.

The general solution for B is then

$$B(t; T) = C_1 e^{\gamma t} + \frac{1}{\gamma},$$

for some arbitrary constant, C_1.

The final condition $B(T; T) = 0$ then gives us

$$C_1 = -\frac{1}{\gamma} e^{-\gamma T},$$

and

$$B(t; T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right).$$

Substituting back into the ordinary differential equation for A,

$$\begin{aligned}\frac{dA}{dt} &= \eta B - \tfrac{1}{2}\beta B^2 = \eta\frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) - \tfrac{1}{2}\beta\frac{1}{\gamma^2}\left(1 - e^{-\gamma(T-t)}\right)^2 \\ &= -\tfrac{1}{2}\frac{\beta}{\gamma^2} e^{-2\gamma(T-t)} + \left(\frac{\beta}{\gamma^2} - \frac{\eta}{\gamma}\right) e^{-\gamma(T-t)} + \left(\frac{\eta}{\gamma} - \tfrac{1}{2}\frac{\beta}{\gamma^2}\right).\end{aligned}$$

Integrating, we find

$$A = -\tfrac{1}{4}\frac{\beta}{\gamma^3}e^{-2\gamma(T-t)} + \left(\frac{\beta}{\gamma^3} - \frac{\eta}{\gamma^2}\right)e^{-\gamma(T-t)} + \left(\frac{\eta}{\gamma} - \tfrac{1}{2}\frac{\beta}{\gamma^2}\right)t$$
$$+ C_2,$$

for some arbitrary constant, C_2.

The final condition $A(T;T) = 0$ then gives us

$$C_2 = -\frac{3}{4}\frac{\beta}{\gamma^3} + \frac{\eta}{\gamma^2} + \left(\tfrac{1}{2}\frac{\beta}{\gamma^2} - \frac{\eta}{\gamma}\right)T.$$

Substituting for

$$e^{-\gamma(T-t)} = 1 - \gamma B,$$

and

$$e^{-2\gamma(T-t)} = (1 - \gamma B)^2,$$

we find

$$\begin{aligned}
A = &-\frac{\beta}{4\gamma^3}(1 + \gamma^2 B^2 - 2\gamma B) + \frac{1}{\gamma^3}(\beta - \eta\gamma)(1 - \gamma B) \\
&+ \frac{1}{\gamma^2}(\eta\gamma - \tfrac{1}{2}\beta)t + C_2 \\
= &-\frac{\beta}{4\gamma^3}(1 + \gamma^2 B^2 - 2\gamma B) + \frac{1}{\gamma^3}(\beta - \eta\gamma)(1 - \gamma B) \\
&+ \frac{1}{\gamma^2}(\eta\gamma - \tfrac{1}{2}\beta)t \\
&- \frac{3}{4}\frac{\beta}{\gamma^3} + \frac{\eta}{\gamma^2} + \left(\tfrac{1}{2}\frac{\beta}{\gamma^2} - \frac{\eta}{\gamma}\right)T,
\end{aligned}$$

which simplifies to

$$A(t;T) = \frac{1}{\gamma^2}(B - T + t)(\eta\gamma - \tfrac{1}{2}\beta) - \frac{\beta B^2}{4\gamma}.$$