# CHAPTER 21

## PORTFOLIO MANAGEMENT

1. **Work out the efficient frontier for the following set of assets:**

| Asset | $\mu$ | $\sigma$ |
|---|---|---|
| A | 0.08 | 0.12 |
| B | 0.10 | 0.12 |
| C | 0.10 | 0.15 |
| D | 0.14 | 0.20 |

**The correlation coefficients between the four assets are given by**

$$\rho = \begin{pmatrix} \mathbf{1} & \mathbf{0.2} & \mathbf{0.5} & \mathbf{0.3} \\ \mathbf{0.2} & \mathbf{1} & \mathbf{0.7} & \mathbf{0.4} \\ \mathbf{0.5} & \mathbf{0.7} & \mathbf{1} & \mathbf{0.9} \\ \mathbf{0.3} & \mathbf{0.4} & \mathbf{0.9} & \mathbf{1} \end{pmatrix}.$$

The efficient portfolio will be made up of assets B and D only. This is because asset A has the same level of risk as asset B, but a lower reward and asset C has a higher level of risk than asset B, but the same reward.

The risk and reward for the efficient portfolio are then given by

$$\mu_\Pi = W\mu_B + (1 - W)\mu_D,$$

and

$$\sigma_\Pi^2 = W^2\sigma_B^2 + 2W(1 - W)\rho\sigma_B\sigma_D + (1 - W)^2\sigma_D^2,$$

where $\rho = 0.4$.

Substituting for the relevant numbers, we find

$$\mu_\Pi = 0.1W + 0.14(1 - W),$$

and

$$\sigma_\Pi^2 = 0.12^2W^2 + 0.4 \times 0.12 \times 0.2 \times 2W(1 - W) + 0.2^2(1 - W)^2.$$

To find the efficient frontier, we vary $W$. The efficient frontier is shown in Figure 21.1.

2. **Find the efficient frontier for the assets in the table above, when asset D is replaced by a risk-free asset, E, which has a mean of**

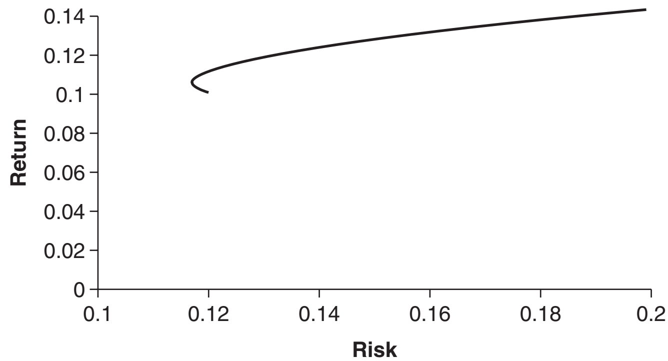


**Figure 21.1** The efficient frontier for assets A, B, C and D.

**0.10 over our time horizon of two years. Asset E is uncorrelated with the other assets.**

When we replace asset D by the risk-free asset E, the efficient portfolio is then made up of assets B and E. E has a mean of 0.1 ($= \mu_E T$), therefore $\mu_E = 0.05$. Since $\sigma_E = 0$ and assets B and E are uncorrelated, the efficient portfolio risk and reward are given by

$$\mu_\Pi = W\mu_B + (1 - W)\mu_E,$$

and

$$\sigma_\Pi = W\sigma_B.$$

Substituting for the relevant numbers, we find

$$\mu_\Pi = 0.1W + 0.05(1 - W),$$

and

$$\sigma_\Pi = 0.12\ W.$$

When we vary $W$, we find that the efficient frontier is a straight line. The efficient frontier is shown in Figure 21.2.

3. **Where should we be on the efficient frontier in Question 1 if we wish to minimize the chance of a return less than 0.05?**

If $C(\cdot)$ is the cumulative distribution function for the standardized Normal distribution then

$$C\left(\frac{\mu_\Pi - r^*}{\sigma_\Pi}\right)$$

is the probability that the return exceeds $r^*$. If we want to minimise the chance of a return of less than $r^*$ we should choose the

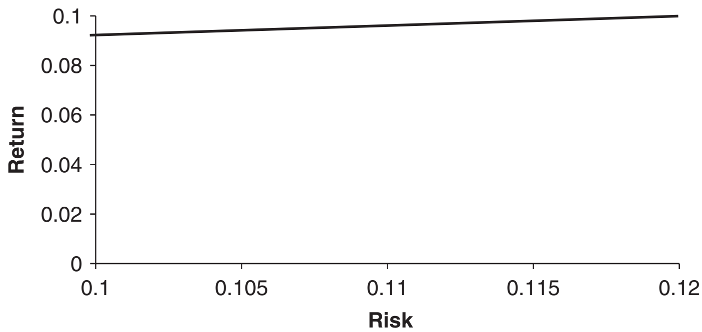


**Figure 21.2** The efficient frontier for assets A, B, C and E.

portfolio from the efficient frontier set $\Pi_{\text{eff}}$ with the largest value of the slope

$$\frac{\mu_{\Pi_{\text{eff}}} - r^*}{\sigma_{\Pi_{\text{eff}}}}.$$

To minimise the chance of a return less than 0.05, we must therefore maximize

$$\frac{\mu_\Pi - 0.05}{\sigma_\Pi},$$

where

$$\mu_\Pi = 0.1W + 0.14(1 - W),$$

and

$$\sigma_\Pi^2 = 0.12^2 W^2 + 0.4 \times 0.12 \times 0.2 \times 2W(1 - W) + 0.2^2(1 - W)^2.$$

We can simplify this problem to maximise

$$\frac{0.09 - 0.04W}{(0.0352W^2 - 0.0608W + 0.04)^{1/2}},$$

with respect to $W$. The way this slope varies with the weight, $W$, is shown in Figure 21.3.

To find the maximum, we differentiate the equation for the slope with respect to $W$ and set the answer equal to zero. We find

$$\begin{aligned}
&(0.0352W^2 - 0.0608W + 0.04)^{1/2}(-0.04) \\
&\quad -(0.09 - 0.04W)\tfrac{1}{2}(0.0352W^2 - 0.0608W + 0.04)^{-1/2} \\
&\quad\quad \times(0.0704W - 0.0608) = 0,
\end{aligned}$$

which simplifies to

$$1.952W = 1.136.$$

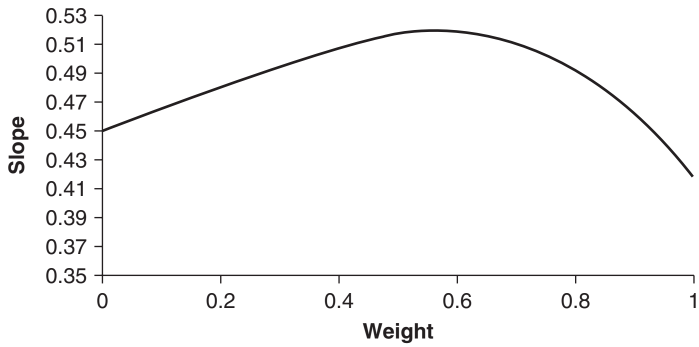


**Figure 21.3** Slope vs Weight for the efficient portfolio of Question 1.

The solution of this problem is $W = 0.582$, at which value,

$$\mu_\Pi = 0.117 \text{ and } \sigma_\Pi = 0.129.$$

**4. What are the economic significances of $\alpha$ and $\beta$ in the Capital Asset Pricing Model and how are they measured or estimated in practice?**

In the Capital Asset Pricing Model, we relate the return on an asset to the return on a representative index, $M$. We write the return on the $i$th asset as

$$R_i = \alpha_i + \beta_i R_M + \epsilon_i.$$

$\alpha$ is a measure of the return of the stock independent of the market as a whole. $\beta$ is the ratio of the expected risk premium of the asset to the expected risk premium of the market. It is a measure of the asset's sensitivity to changes in the market index.

We use historical data to measure these parameters, and perform a linear regression on a plot of $R_i$ vs $R_M$ to find $\beta$. Since betas have a tendency to revert to their means, high betas will over-predict (and low betas will under-predict) future true betas. We must make an adjustment to take account of this mean reversion. We may also wish to take account of the changing underlying characteristics of the company whose asset returns we are trying to predict.