Value-at-Risk and ES Solutions

CQF

1. **Portfolio Risk warm up.** Consider a position of £5 million in a single asset X with daily volatility of 1%. What are the annualised and 10-day standard deviations? Using the Normal factor calculate 99%/10day VaR in money terms.

   Solution: In order to annualise volatility we use the additivity of variance,

   $$\sigma_{1Y} = \sqrt{\sigma_{1D}^2 \times 252} = \sigma_{1D}\sqrt{252} = 0.01 \times \sqrt{252} \approx 0.16$$

   Notice 1% daily volatility equates to about 16% volatility per annum, and the asset can reasonably move within the range of $\pm 2\sigma$.

   For Value at Risk we need Factor value, which corresponds to confidence $c = 99\%$. In statistics the following is known as confidence interval: $\mu \pm \text{Factor} \times \sigma$. Using tables for Normal Distribution, the factor value that cuts 1% on the left tail as $\Phi(-2.33) \approx 0.01$.

   $$\text{VaR}_{99\%/10D} = \Phi^{-1}(1 - 0.99) \times \sigma_{10D} \times \Pi = 2.33 \times 0.01 \times \sqrt{10} \times £5\,\text{million} = £368,405$$

   where Π is portfolio value (for one asset).

2. Now, consider a portfolio of two assets X and Y, £100,000 investment each. The daily volatilities of both assets are 1% and correlation between their returns $\rho_{XY} = 0.3$. Calculate 99%/5day Analytical VaR (in money terms) for this portfolio.

   Solution:
   It is convenient to operate *in money terms* $\sigma = £1000$, which is 1% from £100,000. Operating via weights involves multiplying $\sigma^2 w^2 \Pi^2 = 0.01^2 \times 0.5^2 \times 200,000^2 = 1000^2$.

   $$\begin{aligned} \sigma_\Pi^2 &= \sigma_X^2 + 2\rho_{XY}\,\sigma_X\sigma_Y + \sigma_Y^2 \\ \sigma_\Pi^2 &= 1000^2 + 2 \times 0.3 \times 1000 \times 1000 + 1000^2 = 2.6 \times 10^6 \end{aligned}$$

   which gives Portfolio standard deviation $\sigma_\Pi = £1,612.45$, for daily change in value.

   Scaling to 5 days and using the factor value for $c = 99\%$ confidence, the result is

   $$\text{VaR}_{99\%/5D} = 2.33 \times 1612.45 \times \sqrt{5} = £8,401.$$

   Question 1 and 2 calculations assume portfolio value (same as cumulative P&L) follows Normal Distribution. 99% VaR risk measure represents any move beyond 2.33σ standard deviations, however we do not know how worse loss can be.

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3. **Fully Analytical VaR for Efficient Markets** – Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile* in order to derive the formula for VaR calculation,

$$\Pr\left(x \leq \text{VaR}(X)\right) = 1 - c.$$

$\boxed{\textbf{Solution:}}$ We start with probability for the P&L (loss) X exceeding VaR(X) threshold: specific loss amount $x < 0$ being worse than VaR < 0:

$$\begin{aligned}\Pr\left[x \leq \text{VaR}(X)\right] &= 1 - c \\ \text{VaR}_c(X) &= \inf\{x \,|\, \Pr(X > x)\ \leq 1 - c\} = \inf\{x \,|\, F_X(x) \geq c\}\end{aligned}$$

Infimum notation represents the greatest lower bound of the set of values x. For 99% confidence, the probability of loss being observed below the infimum (VaR value) is equal to $(1 - 0.99) = 0.01$.

$$\begin{aligned}\Pr(\phi \leq \frac{\text{VaR}(X) - \mu}{\sigma}) &= 1 - c \\ \Phi^{-1}\left[\Phi(\frac{\text{VaR}(X) - \mu}{\sigma})\right] &= \Phi^{-1}\left[1 - c\right] \\ \text{VaR}(X) &= \mu + \Phi^{-1}(1 - c) \times \sigma\end{aligned}$$

and convert X to a Standard Normal variable ϕ

P&L X can be ANY random variable but we work with returns, which we *assume* to be Normally distributed.. If we standardise that random variable (make it Z-score), distribution is made explicit. Applying $\Phi^{-1}[\ldots]$ to both sides gets rid of probability notation because $\Pr(x \leq X)$ is CDF by definition.

Inverse CDF is a percentile function.

$\text{VaR}_c(X)$ calculation is an inverse task: we are given the answer $(1 - c)$ 'chunk of probability' cut on the tail, and want to know which value cuts it (eg, HistSim column of sorted returns). Standard Percentile is always known $\Phi^{-1}(1 - 0.99) \approx -2.32635$. Excel function is *=NORM.S.INV(0.01)*.

The standard percentile value ≈ -2.32635 is referred to as **Factor**. This is the value of Analytical VaR when when all returns x_t are Standarised into **Z-scores** $z_t = \frac{x_t - \mu}{\sigma}$.

4. What about Expected Shortfall? The universal definition of ES in terms of expectations algebra is given as follows:

$$\begin{aligned}\text{ES}_c(X) &= \mathbb{E}[\,X \,|\, X \leq \text{VaR}_c(X)] \\ \text{ES}_c(X) &= \frac{1}{1 - c}\int_0^{1-c} \text{VaR}_u(X) du\end{aligned}$$

The actual ES calculation formula will vary depending on the distribution of P&L X, a random variable. Derive ES calculation formula for the case of Normal Distribution using the result $\mathrm{VaR}(X) = \mu + \Phi^{-1}(1-c) \times \sigma$.

Solution: ES universal definition (via integral above) means averaging over all VaR values, e.g., $99\%, 99.1\%, 99.2\%, \ldots$ – the tail percentile values.

$$\begin{aligned}
\mathrm{ES}_c(X) &= \frac{1}{1-c}\int_c^1 \mathrm{VaR}_u(X)du \qquad \text{percentile changed to upper} \\
&= \frac{1}{1-c}\int_c^1 \left(\mu + \sigma\Phi^{-1}(1-u)\right)du \\
&= \mu + \frac{1}{1-c}\int_c^1 \sigma\Phi^{-1}(1-u)du
\end{aligned}$$

To cancel out $\sigma\Phi^{-1}()$, it makes sense to choose $u = \Phi_Z(z)$. Respectively, $du = \phi_Z(z)dz$ and integration limits re-map $[c, 1] \mapsto [\Phi^{-1}(c), \infty]$

$$\begin{aligned}
&= \mu + \frac{1}{1-c}\int_{\Phi^{-1}(c)}^{\infty} \sigma\Phi^{-1}\left(1-\Phi_Z\right)\phi_Z dz \\
&= \mu - \frac{1}{1-c}\int_{\Phi^{-1}(c)}^{\infty} \sigma\, z\, \phi(z)\, dz \\
&= \text{using} \qquad \int ze^{-z^2/2} = -e^{-z^2/2} \\
&= \mu - \frac{1}{1-c}\sigma\left[-e^{-\infty^2/2} - \left(-e^{[\Phi^{-1}(c)]^2/2}\right)\right] \\
&= \mu - \sigma\,\frac{\phi(\Phi^{-1}(c))}{1-c}.
\end{aligned}$$

The result has a quirk of ICDF being inside PDF but this is simply $\phi(-2.32635)$ for our 99^{th} Standard Percentile.

5. Let's figure out a few numbers for these efficient, 'elliptical' markets – the term means asset (market index) returns are Normally distributed or close. Consider the left tail.

   - What percentage of returns are outside 2σ from the mean?
   - What is an average tail loss? To compute mean of the tail, divide the first moment by the total probability mass of the tail (computed by CDF).

   Derive formula solutions and provide numerical answers for Standard Normal.

   Solution: The percentage of returns outside n standard deviations on the left tail is given by CDF, the function argument is **limit of integration**,

$$\Phi(\mu - n\sigma) = \int_{-\infty}^{\mu-n\sigma} f(x)dx = \int_{-\infty}^{\mu-n\sigma} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

For Standard Normal, the percentage of returns on the left tail cut by 2σ is

$$\Phi(-2) = \int_{-\infty}^{-2} \phi(z) = \int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = 0.02275$$

Making a step ahead, using the ready Expected Shortfall formula (coming next),

$$\begin{aligned}
\mathrm{ES} &= \frac{1}{\Phi(-2)} \int_{-\infty}^{-2} \frac{z}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = -\frac{1}{0.02275}\frac{1}{\sqrt{2\pi}} e^{-2} \approx -2.37 \\
& \text{using} \qquad \int z e^{-z^2/2} = -e^{-z^2/2} \\
& -2.37 < -2, \qquad \mathrm{ES}_c < \mathrm{VaR}_c
\end{aligned}$$

Expected Shortfall $\mathrm{ES}_c(X) = \frac{1}{1-c}\int_0^{1-c} \mathrm{VaR}_u(X) du$ is characterised as

- the average tail loss: average loss but conditional on loss being below VaR.

Consider 'mean of the tail' BEFORE conditioning, to compute **the mean** we have to compute the first moment $x\, f(x)$ but **within** $-\infty$ to $\mu - n\sigma$ limits

$$\int_{-\infty}^{\mu - n\sigma} x\, f(x) dx \;\equiv\; \int_0^{1-c} \mathrm{VaR}_u(X) du$$

Then, we need to divide by the probability on which we conditioned, which is probability mass of the tail $\frac{1}{\int_{-\infty}^{\mu-n\sigma} f(x)dx} = \frac{1}{1-c}$.

We form a general expression for **the average on the tail** as follows:

$$\begin{aligned}
\mathrm{ES}_c(X) &= \frac{\int_{-\infty}^{\mu-n\sigma} x\, f(x) dx}{\int_{-\infty}^{\mu-n\sigma} f(x) dx} \\
&= \frac{1}{\Phi(\mu - n\sigma)} \int_{-\infty}^{\mu-n\sigma} \frac{x}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx
\end{aligned}$$

Make change of variable $-z = (x-\mu)/\sigma$. We exercise intuition that things must be simpler in terms of Standard Normal

$$x = \mu - z\sigma, \qquad dx = -\sigma dz$$

$$-\frac{(x-\mu)^2}{2\sigma^2} = -z^2/2$$

$$= \quad \frac{1}{\Phi(x)} \int_{-\infty}^{x} (\mu - z\sigma) \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2} (-\cancel{\sigma} dz) \qquad \text{new variable under integral}$$

$$= \quad \text{cdf for Standard Normal} \qquad \Phi(\mathbf{x}) = \int_{-\infty}^{\mathbf{x}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \quad -\frac{1}{\Phi(x)} \left(\mu\,\Phi(x) - \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{x} z\, e^{-z^2/2}\, dz \right)$$

$$= \quad \text{using} \qquad \int z e^{-z^2/2} = -e^{-z^2/2} \qquad \text{and} \qquad e^{-\infty} = 1/e^{\infty} = 0$$

$$= \quad -\left(\mu - \sigma \frac{1}{\Phi(x)} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}} \right)$$

Underbrace highlights Standard Normal PDF $\phi(x)$ and the result can be re-expressed as follows, it gives a positive value but one has to remember that **ES is a loss**,

$$\mathrm{ES}_c = -\left(\mu - \sigma \frac{\phi(\Phi^{-1}(1-c))}{1-c} \right)$$

where for Standard Normal.

$$\begin{aligned} \mathrm{ES}_{99\%} &= -\left(0 - \frac{\phi\left(\Phi^{-1}(0.01)\right)}{0.01} \right) \\ &= \frac{\phi\left(-2.32635\right)}{0.01} = 0.026652/0.01 = 2.6652. \end{aligned}$$

$$-2.6652 < -2.32635, \qquad \text{loss } \mathrm{ES}_{99\%} < \mathrm{VaR}_{99\%}$$

Portfolio VaR and ES calculation. Inputs are vectors of expected returns and volatilities (annualised) as well as vector of allocations $\boldsymbol{w}$. Question remains about usefulness of 252D VaR (it is neither a robust prediction, not possible to backtest verifiably) but formulae are available,

$$\begin{aligned} \mathrm{VaR}_c(\Pi) &= \boldsymbol{w'\mu} + \text{Factor} \times \sqrt{\boldsymbol{w'\Sigma w}} \\ &= \mu_\Pi + \text{Factor} \times \sigma_\Pi \end{aligned}$$

$$\mathrm{ES}_c(\Pi) = -\mu_\Pi + \sigma_\Pi \frac{\phi(\text{Factor})}{1-c}$$

where Factor $= \Phi^{-1}(1-c) = \Phi^{-1}(0.01) \approx -2.32635$ but for ES we have to impose the negative sign. Portfolio return $\mu_\Pi = \boldsymbol{w'\mu}$ and portfolio risk $\sigma_\Pi = \sqrt{\boldsymbol{w'\Sigma w}}$ must be recognisable.

6. **The Tail Story...** Improvement to risk measures made by choosing a distribution more adequate to characterise the tail data. $\mathrm{GPD}_{\xi,\beta}$ is a proven approximation for the losses $X = v$ being above the threshold u_0 (for the right tail). How much above? That is regulated by another variable y.

$$\Pr\left[(\text{Loss} - u_0) \leq y \,|\, \text{Loss} > u_0\right] = \mathrm{GPD}_{\xi,\beta}(y)$$

ξ is a tail index, and β is scale parameter with the role similar to standard deviation – estimated from the tail data by Maximum Likelihood (optimising the joint probability from all observations). See EVT Example in computational Lab files.

Extreme Value Theory shows that tails of the wide range of distribution share common properties. Analytical VaR is known – it has been derived as inversion of $F(\mathrm{VaR}) = c$

$$\mathrm{VaR}_c = u + \frac{\beta}{\xi}\left[\left(\frac{N}{N_u}(1-c)\right)^{-\xi} - 1\right].$$

where $c = 99\%$ confidence, N_u is the number of exceedances among N observations, and $u = u_0$ is chosen threshold for loss, expressed as DV or percentage.

The useful expectation result below holds for exceedances of ANY random variable and is known as **mean excess function**,

$$e(u) = \mathbb{E}[X - u|X > u] = \frac{\beta + \xi(u - u_0)}{1 - \xi}.$$

Question: use the VaR formula and excess function result in order to derive the formula for Expected Shortfall under EVT. Start with definition of ES as Conditional VaR, and seek to insert the excess function.

$$\mathrm{ES}_c = \mathbb{E}[X|X \geq \mathrm{VaR}_c].$$

$\boxed{\textbf{Solution:}}$

$$\begin{aligned}
\mathrm{ES}_c = \mathbb{E}[X|X \geq \mathrm{VaR}_c] \quad &= \quad \mathrm{VaR}_c + \mathbb{E}[X - \mathrm{VaR}_c|X \geq \mathrm{VaR}_c] \\
&\quad\quad \text{substitute excess function } \mathrm{VaR}_c > u_0 \text{ (right tail) and make } u_0 \to u \\
&= \quad \mathrm{VaR}_c + \frac{\beta + \xi\left(\mathrm{VaR}_c - u\right)}{1 - \xi} \\
&= \quad \left(1 + \frac{\xi}{1-\xi}\right)\mathrm{VaR}_c + \frac{\beta - \xi u}{1 - \xi} \\
&= \quad \frac{\mathrm{VaR}_c + \beta - \xi u}{1 - \xi}
\end{aligned}$$

$$\mathrm{ES}_c = \frac{1}{1-\xi}\left(u + \frac{\beta}{\xi}\left[\left(\frac{N}{N_u}(1-c)\right)^{-\xi} - 1\right]\right) + \frac{\beta - \xi u}{1-\xi} \quad \dagger$$

More on **underpinnings of EVT**, coming from Balkema-de Haan-Pickands theorem:

$$\begin{aligned}
\Pr\left[v \le (u+y)\,|v > u\right] \quad &= \quad F_u(y) = G_{\xi,\beta}(y) \\
F_u(y) \quad &= \quad \frac{F(u+y) - F(u)}{1 - F(u)} \\
\Pr\left[(v-u) \le y\,|v > u\right] \quad &\text{is our} \quad \Pr\left[(\text{Loss} - u_0) \le y \,|\, \text{Loss} > u_0\right]
\end{aligned}$$

$F_u(y)$ notation means $F_u(v - u \le y) = F_u(\text{Excess} \le y)$.

$F(u+y) - F(u)$ is probability that $u \le v \le (u+y)$. Same as $\emptyset \le (v-u) \le y$.

$1 - F(u)$ is simple unconditional probability $\Pr(v > u)$ as expected in the denominator. This is Pr(Loss $\ge$ Threshold). Quantity $1 - F(u)$ is estimated by $\frac{N_u}{N}$, inverse of which appears in the VaR formula.

To obtain conditional probability, $\Pr\left[v \le (u+y)\,|v > u\right] = \dfrac{\Pr\left[v \le (u+y)\right]}{\Pr\left[v > u\right]}$.

EVT VaR derived by applying CDF of Generalised Pareto Distribution to exeedances $(u+y)-u$. Preparation work to obtain $\Pr[v>x]$ which is "loss over threshold" $\Pr[\text{Loss} > (u+y)]$.

$$\begin{aligned}
\Pr[v \geq (u+y)\,|v>u] &= 1 - G_{\xi,\beta}(y) \\
&\quad \text{introduce new arbitrary quantity } x > u \\
&\quad u+y=x,\ y=x-u \\
\Pr[v>x] &= (1-F(u))\,(1-G_{\xi,\beta}(x-u)) \\
&= \frac{N_u}{N}\left(1+\xi\frac{x-u}{\beta}\right)^{-1/\xi}
\end{aligned}$$

This result can be used to estimate the probability of any DV Loss x above value u.

$$\begin{aligned}
F(\text{VaR}) &= 1 - \Pr[v>x] = c \\
&\quad \text{confidence that we will not exceed threshold} \\
F(\text{VaR}) &= 1 - \frac{N_u}{N}\left(1+\xi\frac{\text{VaR}-u}{\beta}\right)^{-1/\xi} = c \\
&\frac{N_u}{N}\left(1+\xi\frac{\text{VaR}-u}{\beta}\right)^{-1/\xi} = 1-c \\
\text{VaR}_c &= u + \frac{\beta}{\xi}\left[\left(\frac{N}{N_u}(1-c)\right)^{-\xi} - 1\right].
\end{aligned}$$

Practical Considerations

1. Choice of threshold u_0 is approximately the next value after 95th percentile empirically (sorted column losses or returns), even in the age of Basel III of 99% VaR. Using 5 observations would not allow to investigate the shape of the tail in robust manner.[1]

2. Financial data (outright Loss amounts) typically generates $\xi = 0.1\ldots0.4$.

3. When $\xi = 0.25$ the first three moments of $G_{\xi,\beta}(y)$ are defined and finite, as it increases towards $\xi = 0.5$, only the first moment is finite, rendering theoretical variance (and therefore, potential risk) undefined.

4. EVT can be safely used to estimate Analytical VaR and ES when the confidence level is very high (eg, 99.9%).

[1] 5 is 1% from 504 days which is regulatory two-year period. It gives a very small subset as the tail data.

7. **Back to ABC.** Assume three bonds A,B and C, each has a face value of £$1,000$ payable at maturity. The the independent probability of default is 0.5%.

   (a) For the portfolio equally invested in bonds A, B and C, the 99% VaR is £$1,000$. Explain this result.

   (b) Calculate the Expected Shortfall for the bond A only. Assume 1% tail and partial loss possible.

   (c) Calculate the Expected Shortfall of a portfolio *equally invested* in bonds A, B and C. Assume 1% tail and continuous distribution.

   (d) Compare results (b) and (c) to conclude whether ES is *sub-additive.*

   Solution:

   (a) Remember that VaR is value (loss) at the percentile. The probability of no default is $1 - 0.005 * 3 = 0.985$ and so 98.5th percentile loss is £0. Adding the probability of the next outcome (1 default) gives us $0.985 + 0.01485 = 0.99985$th percentile, so the minimum loss at 99*th* percentile is £$1,000$.

	Loss	Cumulative Density
No defaults	£0	$\approx 98.51\%$ $\quad 0.995 \times 0.995 \times 0.995 \quad$ for (not ABC)
1 default	£$1,000$	$\approx 1.485\%$ $\quad 3 \times 0.005 \times 0.995^2 \quad$ for A (not BC) + B (not AC) + C (not AB)
2 defaults	£$2,000$	0.0074625% $\quad 3 \times 0.005^2 \times 0.995 \quad$ for AB (not C) + BC (not A) + AC (not B)
3 defaults	£$3,000$	0.0000125% $\quad 0.005 \times 0.005 \times 0.005 \quad$ for ABC

Table 1: Loss Distribution (*cdf*)

   Loss Distribution takes account of combinatorial outcome of all cases: 1 or 2 or 3 defaults have respectively, 2, 1, and 0 survivals. The large chunk of loss distribution's density assigned zero loss value, at $\Phi^{-1}(1 - 0.9851) = -2.17274$ Standard Percentile. Computing $\Phi^{-1}(1 - 0.9851 - 0.01485) = -3.8906$ Standard Percentile is too far to define a tail.
   Here is our problem of discrete loss mapped onto continuous analytical Normal distribution.

   (b) We are in the tail situation (loss has to occur) but we retain the probability of default. Conditional probability of loss is $\frac{\Pr}{1-c}$ and and we multiply it by Loss as usual (probability $\times$ expected value),

$$\text{ES} = \frac{\Pr}{1-c} \times \text{Loss} = \frac{0.5\%}{1\%} \times 1000 = £500.$$

   (c) We already established in (a) that 99% VaR for the portfolio is £$1,000$. Because ES (CVaR) is an average of VaRs on the tail, it can't be less than this value.

$$\text{ES}_{1-c} = \frac{1}{1-c} \int_0^{1-c} \text{VaR}_\gamma(X) d\gamma$$

For ES for a continuously distributed variable we integrate, where VaR_γ is loss which in our case £3,000, £2,000 £1,000 and $d\gamma$ are 'chunks' of density that are unequal (see Loss Distribution).

ES ≡ Average loss on the tail ≡ EV over tail of loss distribution

Discrete case has summation instead of integration. Form Conditional Loss Distribution from the tail of Loss Distribution:

$$\mathbf{ES} = 0.992525 \times 1000 + 0.0074625 \times 2000 + 0.0000125 \times 3000 = \pounds\mathbf{1,007.49}$$

How did we obtained those densities?

	Loss	Likelihood
3 defaults	£3,000	0.0000125%/1%
2 defaults	£2,000	0.0074625%/1%
1 default	£1,000	≈ 0.9925 †

Table 2: Conditional Loss Distribution

where Pr(1 default) = 1 - Pr(2 defaults) - Pr(3 defaults) so,

$$1 - 0.0074625 - 0.0000125 = 0.992525. \quad \dagger$$

We take this opportunity to illustrate a side of Bayesian rule: to recover the unconditional distribution, you have to integrate as follows:

$$f(y) = \int f(y|x) f(x) dx$$

Discrete implementation: take those conditional probabilities (Table 2) and multiply them by the respective marginal probabilities (Table 1, rounded).

$$\begin{aligned} 0.9925 \times 0.015 + 0.0074625 \times 0.0075 + 0.0000125 \times 0.00001 &\approx 0.0149 \\ 1 - \text{PrSurv} = 1 - \times 0.995^3 &\approx 0.0149. \end{aligned}$$

0.0149 is the total density allocated to the conditional tail from **Total** Loss Distribution. At least one default Pr= 0.0149, no defaults Pr= 0.9851 sum up to 1.

There is one disclaimer to make: given the first discrete loss occurs at 0.9851 it makes sense to use this percentile. Then numerical results will be different for 98.51*th* VaR and ES where the first default serves as 'a natural boundary'.

(d) ES for the portfolio of ABC is noticeably less than 3 × ES of each bond = £1,500 and so it is sub-additive. This holds for most cases, except a few that are only of academic interest.

8. **The practice** – What are two main numerical methods that support VaR Backtesting and Stress-testing in terms of generating rather than merely sampling asset returns? What are their drawbacks?

$\boxed{\textbf{Discussion:}}$

- **Monte Carlo** method requires generation of quasi-random numbers and relies on their low latency (evenness). Fractals reveal the problem of correlated random numbers occurs if number of asset paths (dimensions) is high.

  The assumption of a log-normal random walk is not as innocent because asset log-returns might not be Normal, and in times of stress are likely to be autocorrelated. Correlation among assets might give humongous covariance matrices that require factorisation (eg, by Cholesky decomposition) and simulation via the multi-factor PDEs. Monte-Carlo can be slow computationally.

- **Bootstrapping** (or Historic Simulation) method uses actual asset price movements taken from historical data. Common practice for VaR calculation is consider the last two years (this is still an arbitrary choice). We refer to bootstrapping as sampling from *the standardised historic residuals* – that is, we convert each log-return u_t into corresponding Standard Normal variable,

  $$Z_t^* = \frac{u_{t,Hist}}{\sqrt{\sigma^2_{t,GARCH}}}$$

  Instead the sample standard deviation σ_t, RiskMetrics methodology suggests $\sigma_{t+1,GARCH}$, a prediction of volatility from a pre-calibrated GARCH model (or EWMA if predicting the short-term volatility, for the periods of 10-60 days it might not be feasible to expect reversion of variance to the average long-term level $\bar{\sigma}^2$).

  Bootstrapping method requires a large amount of clean data, including the periods of past crises, such as credit crunch 2008. It happens that exactly the data needed is not available due to absence of liquidity in markets – the data suffers from 'structural breaks', such as missing market prices.

9. **A useful risk modelling technique...** Covariance matrix can be decomposed as $\boldsymbol{\Sigma} = \boldsymbol{AA'}$ by Cholesky method (presented in Credit Risk module). The result is a lower triangular matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Let $X_1(t)$ and $X_2(t)$ are two uncorrelated Wiener processes (orthogonal). How would you use the Cholesky result in order to construct two correlated processes?

$\boxed{\textbf{Solution:}}$

In order to perform Cholesky decomposition, the matrix $\boldsymbol{\Sigma}$ must be symmetric and **positive definite**, this is a numerical condition.

$$\boldsymbol{x'\Sigma x} > 0 \qquad \text{for any vector} \qquad \boldsymbol{x} \in \mathbb{R}$$

If matrix eigenvalues are positive $\lambda_1 > \cdots > \lambda_n > 0$, then this condition is satisfied.

Correlation is imposed by $\boldsymbol{Y} = \boldsymbol{AX}$ (result below), so $Y_1(t)$ and $Y_2(t)$ are correlated.

$$\begin{aligned} Y_1 &= \sigma_1 X_1 \\ Y_2 &= \rho\,\sigma_2 X_1 + \sqrt{1-\rho^2}\,\sigma_2 X_2 \end{aligned}$$

$Y_1(t)$ and $Y_2(t)$ remain Brownian Motions albeit non-standardised. In particular, as a linear combination of two Brownian Motions, $Y_2(t)$ **keeps the properties of BM**.

Simplify $\sigma_1 = \sigma_2 = 1$, the increment of such standardised $Y_t - Y_s, \forall s < t$ follows $N(0, \tau)$, which is the Normal distribution closed under sum

$$N(0, \tau\rho^2) \qquad \text{and} \qquad N(0, \tau(1-\rho^2))$$

Alternatively, consider the variance of any random variable

$$\begin{aligned} \mathbb{V}\text{ar}[Y_2(t)] &= \mathbb{V}\text{ar}\left[\rho\, X_1(t) + \sqrt{1-\rho^2}\, X_2(t)\right] \\ &= \rho^2\, \mathbb{V}\text{ar}[X_1(t)] + (1-\rho^2)\, \mathbb{V}\text{ar}[X_2(t)] \\ &= \rho^2\tau + (1-\rho^2)\tau = \tau \quad \text{variance scales with time; Standard BM has } \sigma^2 = 1 \end{aligned}$$

It follows that the increment $Y_2(t) - Y_2(s)$ is distributed as $\sim N(0, t-s) \equiv N(0, \tau)$.