CHAPTER 25

CRASHMETRICS

1. **Extend the example CrashMetrics spreadsheet to incorporate many underlyings, all related via an index. How would interest rate products be incorporated?**

   In the following I am assuming that the relevant option formulas can be approximated by Taylor series expansions. In practice it is more likely that you will use the full formulas because we are considering large moves in the underlying.

   In the single-index, multi-asset model we can write the change in the value of the portfolio as

$$\delta\Pi = \sum_{i=1}^{N} \Delta_i \, \delta S_i + \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma_{ij} \, \delta S_i \, \delta S_j,$$

   assuming that the Taylor series expansion is valid.

   We assume that the percentage change in each asset can be related to the percentage change in the benchmark, x, when there is an extreme move, by

$$\delta S_i = \kappa_i x S_i.$$

   In this case,

$$\begin{aligned}\delta\Pi &= x \sum_{i=1}^{N} \Delta_i \kappa_i S_i + \tfrac{1}{2} x^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma_{ij} \kappa_i S_i \kappa_j S_j \\ &= xD + \tfrac{1}{2} x^2 G.\end{aligned}$$

   The first-order coefficient D is the crash delta and the second-order coefficient G is the crash gamma.

   We constrain the change in the benchmark by

$$-x^- \le x \le x^+.$$

   The worst-case portfolio change occurs at one of the end points of this range or at the internal point

$$x = -\frac{D}{G}.$$

In this last case the extreme portfolio change is

$$\delta\Pi_{\text{worst}} = -\frac{D^2}{2G}.$$

Suppose that there are M contracts available with which to hedge our portfolio. Let us call the deltas of the kth hedging contract Δ_i^k, meaning the sensitivity of the contract to the ith asset, $k = 1, \dots, M$. The gammas are similarly Γ_{ij}^k. Denote the bid-offer spread by $C_k > 0$, meaning that if we buy (sell) the contract and immediately sell (buy) it back we lose this amount.

We add a number λ_k of each of the available hedging contracts to our original position. Our portfolio now has a first-order exposure to the crash of

$$x\left(D + \sum_{k=1}^{M}\lambda_k \sum_{i=1}^{N}\Delta_i^k \kappa_i S_i\right),$$

and a second-order exposure of

$$\tfrac{1}{2}x^2\left(G + \sum_{k=1}^{M}\lambda_k \sum_{i=1}^{N}\sum_{j=1}^{N}\Gamma_{ij}^k \kappa_i S_i \kappa_j S_j\right).$$

The portfolio now loses a guaranteed amount

$$\sum_{k=1}^{M}|\lambda_k|\,C_k$$

because we cannot close our new positions without losing out on the bid-offer spread.

The total change in the portfolio with the static hedge in place is therefore

$$\begin{aligned}\delta\Pi = {} & x\left(D + \sum_{k=1}^{M}\lambda_k \sum_{i=1}^{N}\Delta_i^k \kappa_i S_i\right) \\ & + \tfrac{1}{2}x^2\left(G + \sum_{k=1}^{M}\lambda_k \sum_{i=1}^{N}\sum_{j=1}^{N}\Gamma_{ij}^k \kappa_i S_i \kappa_j S_j\right) - \sum_{k=1}^{M}|\lambda_k|\,C_k.\end{aligned}$$

We can also apply CrashMetrics to interest rate products, using a yield as the benchmark. A Taylor series expansion of an interest rate product, V, gives us

$$\begin{aligned}\frac{\delta V}{V} &= \frac{1}{V}\frac{dV}{dy}\delta y + \frac{1}{2V}\frac{d^2V}{dy^2}\delta y^2 + \dots \\ &= -D\delta y + \tfrac{1}{2}C\delta y^2,\end{aligned}$$

where D is the duration and C is the convexity. We can then follow the same methodology as for the asset price dependent world. If we were to value a portfolio consisting of asset price and interest rate products, we would have to construct a multi-index model.

2. **Sudden, large movements in a stock are usually accompanied by an increase in implied volatilities. Incorporate a vega term into the one-asset CrashMetrics model. What is the role of actual volatility in CrashMetrics?**

To incorporate a vega term into the one-asset model, we note that the change in the value of our portfolio is

$$\delta\Pi = \Delta\delta S + \tfrac{1}{2}\Gamma\delta S^2 + \nu\delta\sigma,$$

where ν is the vega, when a crash is accompanied by a rise in volatility. Again this is using the simple Taylor series approximation which in practice might not be sufficiently accurate.

We must then choose a model for $\delta\sigma$. We can make δS and $\delta\sigma$ independent and use a two-index model or we could make

$$\delta\sigma = f(\delta S),$$

for some function f and use a single index. In this case, we would have to analyse historical data to decide upon a sensible form for the function.

There is no significant role for actual volatility in CrashMetrics, although you may wish to use the actual volatility in your calculations of delta and gamma.