$$\mathbb{E}[S_i] = \mathbb{E}\left[\sum_{j=1}^{i} R_j\right]$$

$$= \sum_{j=1}^{i} \mathbb{E}[R_j] = 0$$

$$\mathbb{V}[S_i] = \mathbb{V}\left[\sum_{j=1}^{i} R_j\right]$$

$$= \sum_{j=1}^{i} \mathbb{V}[R_j] = i \times 1 = i$$

$X(t)$ B.M

X_t $\|$

$W(t)$ Wiener process

W_t

$$f(x)$$

$$\frac{df}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$F(X) = \lim_{\delta X \to 0} \frac{F(X+dX) - F(X)}{\delta X}$$

$$\delta X \text{ or } dX \sim O(\sqrt{dt})$$

$$\frac{dX}{dt} = \lim_{dt \to 0} \frac{X(t+dt) - X(t)}{\delta t}$$

$$= \lim_{dt \to 0} \frac{dX}{\delta t} \sim \frac{\sqrt{\delta t}}{\delta t}$$

$$= \frac{1}{\sqrt{\delta t}}$$

X is a D.M

$F = F(X)$ is a fn. of a D.M

Let $X \to X + dX$; what happens to $F(X+dx)$?

1D T.S.E

$$F(X+dX) = F(X) + \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dX^2$$

$$+ \text{H.O.T.}$$

$$dF = F(X+dX) - F(X) = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dt$$

Revisiting $F = x^2$

$dF = 2x\,dx$

$f(x)$

$$f(x+dx) = f(x) + f'(x)\,dx \cancel{+ \tfrac{1}{2} f''(x)\,dx^2}$$

$$f(x+dx) - f(x) = f'(x)\,dx \cancel{+ \dots}$$

$$f'(x) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$

$x + e^t \sin x \qquad t^2 + x$

$$dF = \frac{dF}{dx}\,dx + \frac{1}{2}\frac{d^2F}{dx^2}\,dt$$

<u>Example</u>: $F = e^x$ $\qquad \frac{dF}{dx} = e^x = \frac{d^2F}{dx^2}$

$$dF = \underbrace{e^x}_{\text{diffusion}}\,dx + \underbrace{\frac{1}{2}e^x}_{\text{drift}}\,dt$$

2D HS : $f = f(t, X)$ $\quad t \to t + dt$

$X \to X + dX$

dF

2D T.J.E $F(t+dt, X+dX) = F(t, X) + \frac{\partial F}{\partial t} dt$

$+ \frac{\partial F}{\partial x} dx + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \cancel{dx^2} + - - -$

$\to dt$

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \right) dt + \frac{\partial F}{\partial x} dX \quad 2$$

$$dF = \frac{dF}{dx} dx + \frac{1}{2} \frac{d^2 F}{dx^2} dt \quad 1$$

Stochastic Integration Formula

We just obtained Itô 2

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)dt + \boxed{\frac{\partial F}{\partial X}\,dX}$$

$$\frac{\partial F}{\partial X}\,dX = dF - \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)dt$$

Now integrate both sides over $[0, t]$

$$\int_0^t \frac{\partial F}{\partial X_s}\,dX_s = \int_0^t dF - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)ds$$

$$= F(t, X_t) - F(0, X_0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial X_s^2}\right)ds$$

If $F = F(X_t)$ then the integration formula on the previous slide becomes the simpler version

$$\int_0^t \frac{dF}{dX_s}\,dX_s = F(X_t) - F(X_0) - \frac{1}{2}\int_0^t \frac{d^2F}{dX_s^2}\,ds$$

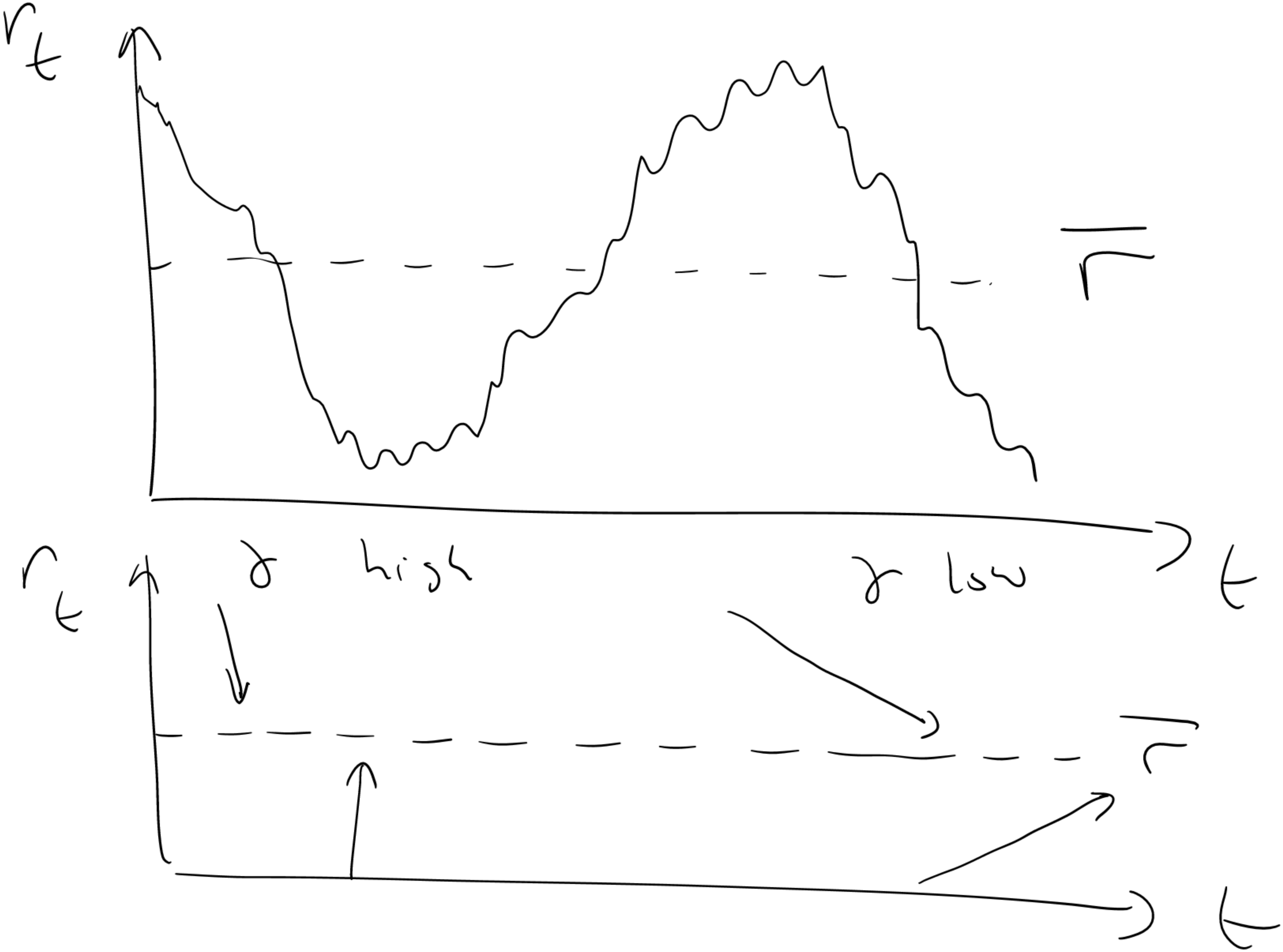
$$dr = (\eta - \gamma r)\,dt + \sigma\,dX$$

η, γ, σ consts.

$$dr = \gamma\left(\eta/\gamma - r\right)dt + \sigma\,dX$$

where $\eta/\gamma = \bar{r}$ mean rate.

$$dr = -\gamma(r - \bar{r})\,dt + \sigma\,dX$$



r_t
$\bar{r}$
t
r_t
γ high
γ low
$\bar{r}$
t

$$dr = -\gamma(r - \bar{r})\,dt$$

$r > \bar{r}$ — $-$ve trend

$r < \bar{r}$ — $+$ve trend

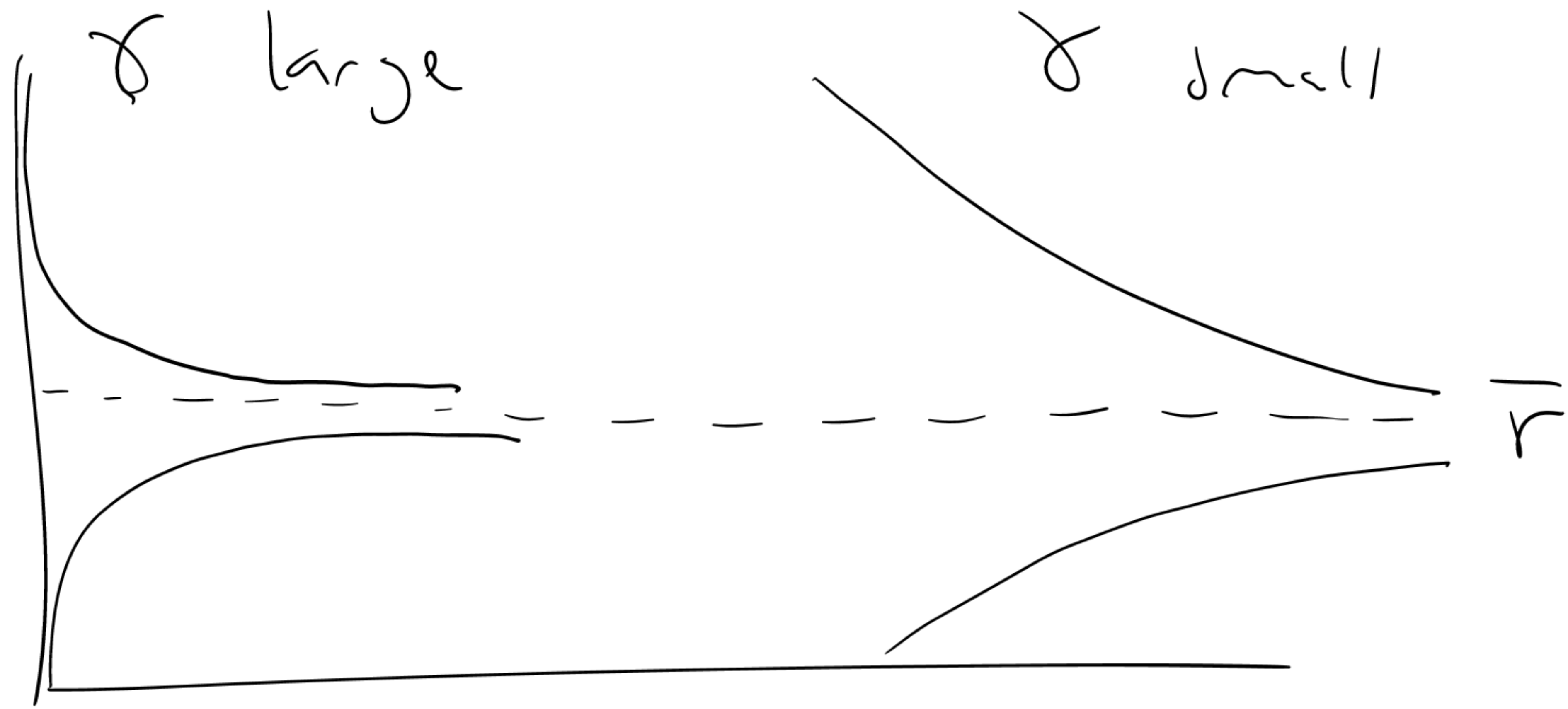
Vasicek in the absence of randomness

$$dr = -\gamma(r - \bar{r})\,dt$$

$$\int \frac{dr}{r - \bar{r}} = -\gamma \int dt$$

$$\log(r - \bar{r}) = -\gamma t + c$$

$$r = \bar{r} + A\boxed{e^{-\gamma t}}$$



γ large
γ small
$\bar{r}$

$$\mathbb{E}\left[\left(f(x)-l\right)^2\right] \longrightarrow 0$$

We say $f(x) \longrightarrow l$ in the mean square limit

$$(y_1 + y_2 + \cdots + y_n - t)(y_1 + y_2 + \cdots + y_n - t)$$

SDE for G_t

$$(*)\quad dG_t = a(G_t, t)\,dt + b(G_t, t)\,dX_t$$

Integrate both sides over $(0, t)$

$$\int_0^t dG_s = \int_0^t a(G_s, s)\,ds + \int_0^t b(G_s, s)\,dX_s$$

$$(\dagger)\quad G_t = G_0 + \int_0^t a(G_s, s)\,ds + \underbrace{\int_0^t b(G_s, s)\,dX_s}_{\text{Itô Integral}}$$

$$(*) \equiv (\dagger)$$