CHAPTER 22

VALUE AT RISK

1. **Assuming a Normal distribution, what percentage of returns are outside the negative two standard deviations from the mean? What is the mean of returns falling in this tail? (This is called the censored mean.)**

The probability density function for the Normal distribution with mean μ and volatility σ is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

The percentage of returns outside -2 standard deviations is then

$$\int_{-\infty}^{\mu-2} f(x)\,dx = \int_{-\infty}^{\mu-2} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}\,dx = N\left(-2/\sigma\right).$$

The mean of returns given that they fall in this tail (the censored mean) is

$$\begin{aligned}
\frac{\int_{-\infty}^{\mu-2} xf(x)\,dx}{\int_{-\infty}^{\mu-2} f(x)\,dx} &= \frac{1}{N\left(-2/\sigma\right)} \int_{-\infty}^{\mu-2} \frac{x}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}\,dx \\
&= \frac{1}{N\left(-2/\sigma\right)} \left(-\frac{\sigma}{\sqrt{2\pi}} e^{-2/\sigma^2} + \mu N\left(-2/\sigma\right) \right) \\
&= -\frac{a\sigma}{\sqrt{2\pi} N\left(-2/\sigma\right)} e^{-2/\sigma^2} + \mu.
\end{aligned}$$

By way of example, for a Normal distribution with mean 0 and variance 1, we find that the percentage of returns outside -2 standard deviations is

$$\int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = N(-2) = 0.0228,$$

i.e. 2.28%. The mean of returns falling in this tail is

$$\begin{aligned}
\frac{1}{N(-2)} \int_{-\infty}^{-2} \frac{x}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \frac{-1}{\sqrt{2\pi} N(-2)} e^{-2} &= -0.0540/0.0228 \\
&= -2.3684.
\end{aligned}$$

2. What criticisms of Value at Risk as described here can you think of? Consider distributions other than Normal, discontinuous paths and non-linear instruments.

This chapter essentially presents two methods for calculating value at risk. The first gives us a formula for the VaR, for instance

$$-\alpha(1-c)\delta t^{1/2}\sqrt{\sum_{j=1}^{M}\sum_{i=1}^{M}\Delta_i\Delta_j\sigma_i\sigma_j\rho_{ij}},$$

where we have used a delta approximation to value derivatives.

To derive this formula, we make many assumptions which can be criticized:

(a) We assume that assets follow the random walk

$$dS = \sigma S\,dX + \mu S\,dt,$$

where dX is drawn from a normal distribution. If we study asset price data, we see that the distribution for dX has fatter tails and a higher peak than a normal distribution.

(b) We assume that we know all the relevant volatilities and correlations. These two quantities are notoriously hard to measure in practice.

(c) We assume that we can approximate the sensitivity of the portfolio to the change in the underlying of an option using some linear model (for instance, dependent on the delta of the option). If we try to price a 'highly' nonlinear product, then this approach will not give accurate results.

The second method is the use of simulations. The two main approaches are Monte Carlo, based on the generation of Normally distributed random numbers, and bootstrapping, using actual asset price movements taken from historical data.

Criticisms of the Monte Carlo approach include that we again assume the asset follows a lognormal random walk and that the method can be very slow since we must run tens of thousands of simulations and for each one, we may have to solve a multi-factor partial differential equation.

The main criticism of the bootstrapping approach is that the method requires a large amount of historical data and this may entail including data that corresponds to completely different economic circumstances.