CHAPTER 23

CREDIT RISK

1. **A risk-free government, zero-coupon bond, with a principal of \$100, maturing in two years has a value of \$91.75. A risky corporate zero-coupon bond with the same principal and maturity is worth \$88.25. What is the market's estimate of the risk of default assuming zero recovery?**

   If we use y_1 to denote the yield on the risk-free bond then

   $$100\, e^{-2y_1} = 91.25$$

   from which

   $$y_1 = \frac{1}{2} \ln\left(\frac{100}{91.75}\right) = 4.305\%.$$

   The '2' in the above is because it is a two-year bond.

   The same calculation for the risky bond, with yield being $y_1 + p_1$ where p_1 is the credit spread gives

   $$100\, e^{-2(y_1+p_1)} = 88.25$$

   from which

   $$y_1 + p_1 = \frac{1}{2} \ln\left(\frac{100}{88.25}\right) = 6.250\%,$$

   so

   $$p_1 = 6.250 - 4.305 = 1.945\%.$$

2. **The same government and company as in the above question now issue five-year bonds with \$100 principal and prices for government bond and corporate bond of \$82.15 and \$78.89 respectively. What does this now say about the market's view on the probability of the company defaulting?**

   The credit spread/hazard rate for the first two years is given by the above calculation. But now we have to find the interest rate and credit spread for the next three years. Now use y_2 to denote the forward rate over years three to five.

   $$91.25\, e^{-3y_2} = 82.15$$

from which

$$y_2 = \frac{1}{3}\ln\left(\frac{91.75}{82.15}\right) = 3.684\%.$$

Now if p_2 is the credit spread for years three to five we have

$$100\ \exp(-2(y_1 + p_1) - 3(y_2 + p_2)) = 78.89.$$

Therefore

$$\begin{aligned}-3p_2 &= \ln\left(\frac{78.89}{100}\right) + \ln\left(\frac{100}{88.25}\right) + \ln\left(\frac{91.75}{82.15}\right)\\ &= \ln\left(\frac{78.89 \times 91.75}{88.25 \times 82.15}\right).\end{aligned}$$

So

$$p = 0.053\%$$

The market thinks that the risk of default is much less over years three to five than years one and two.

3. **Repeat the analyses of the above two problems but with the extra assumption that on default their is a recovery rate of 10%, i.e. you will receive \$10 at maturity if there has been default.**

The above calculation changes to allow for \$10 with probability $1 - e^{-p_1' T}$. Here the $'$ denotes probabilities when there is 10% recovery. So for years one and two

$$\begin{aligned}100\ e^{-2p_1'} + 10\ (1 - e^{-2p_1'}) &= \frac{88.25 \times 100}{91.75},\\ 90\ e^{-2p_1'} &= \frac{100 \times 88.25}{91.75} - 10,\end{aligned}$$

so

$$p_1' = 2.166\%.$$

Notice that since the bond value is unchanged but it is now likely than you get some money even in default, there must be a higher probability of default.

The calculation for years three to five is just

$$\frac{78.89}{82.15} = 0.1 \times (1 - e^{-2p_1}) + e^{-2p_1'}\left(e^{-3p_2'} + 0.1 \times \left(1 - e^{-3p_2'}\right)\right),$$

from which

$$p_2' = 0.060\%.$$

4. **Construct the intermediate steps in the derivation of the equation for the value of a risky bond (when default is governed by a**

Poisson process):

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - (r + p)V = 0.$$

We construct a hedged portfolio:

$$\Pi = V(r, p, t) - \Delta Z(r, t),$$

where Z is a riskless zero-coupon bond paying \$1 at time T.

Over a small time step, dt, there is a probability of $(1 - p\,dt)$ that the bond does not default. In this case, the change in the value of the portfolio during a time step is, from Itô's Lemma,

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2}\right) dt + \frac{\partial V}{\partial r} dr$$
$$- \Delta \left(\left(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2}\right) dt + \frac{\partial Z}{\partial r} dr\right).$$

We choose Δ to eliminate the risky dr term, i.e.

$$\Delta = \frac{\partial V}{\partial r} \Big/ \frac{\partial Z}{\partial r},$$

and find

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} - \frac{\partial V}{\partial r} \Big/ \frac{\partial Z}{\partial r} \left(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2}\right)\right) dt.$$

There is a probability of $p\,dt$ that the bond will default. In this case, the change in the value of the portfolio is

$$d\Pi = -V + O(dt^{1/2}).$$

In absence of arbitrage,

$$E[d\Pi] = r\Pi dt,$$

and taking expectations, we find

$$(1 - pdt)\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} - \frac{\partial V}{\partial r} \Big/ \frac{\partial Z}{\partial r} \left(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2}\right)\right) dt$$
$$+ pdt(-V) = r\left(V - \frac{\partial V}{\partial r} \Big/ \frac{\partial Z}{\partial r} Z\right) dt.$$

Then

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} - \frac{\partial V}{\partial r}\left(\frac{\frac{\partial Z}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2} - rZ}{\frac{\partial Z}{\partial r}}\right) - (r + p)V = 0.$$

Now Z satisfies the riskless bond pricing equation,

$$\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0,$$

so

$$\frac{\frac{\partial Z}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2} - Z}{\frac{\partial Z}{\partial r}} = -(u - \lambda w).$$

Substituting back into our partial differential equation for V, we find

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - (r + p)V = 0.$$