Know your Itô - When and Where?

In this document W_t represents a Brownian Motion. This summarises all the results developed in class for obtaining SDEs/differentials

Itô multiplication table

$\times$	dt	dW_t
dt	$dt^2 = 0$	$dtdW_t = 0$
dW_t	$dW_tdt = 0$	$dW_t^2 = dt$

Itô I

$F = F\left(W_t\right).$

For example functions including W_t^n; $\exp\left(W_t\right)$; $\log W_t$; $W_t \cos W_t$

$$dF = \frac{dF}{dW_t} dW_t + \frac{1}{2} \frac{d^2 F}{dW_t^2} dt.$$

Itô II

$F = F(t, W_t)$.

For example functions including $W_t^n + t^2; t \exp(W_t)$; $\log W_t + \cos tW_t$

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2}\right) dt + \frac{\partial F}{\partial W_t} dW_t.$$

Itô III

$F = F(S_t)$ where the process S_t satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

For example functions including S_t^n; $\exp(S_t)$; $\log S_t$; $S_t \sin S_t$

$$dF = \left(\mu S_t \frac{dF}{dS_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{d^2 F}{dS^2} \right) dt + \sigma S_t \frac{dF}{dS_t} dW_t.$$

Itô IV

$F = F(t, S_t)$ where the process S_t satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

For example functions including $t^3 + S_t^n$; $\exp(tS_t)$; $\exp(S_t) + \log tS_t$; $t^2 S_t \sin S_t$

$$dF = \left(\frac{\partial F}{\partial t} + \mu S_t \frac{\partial F}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial F}{\partial S_t} dW_t.$$

Itô V

This is also a basic version of higher dimensional Itô for functions of the form $F = F\left(t, S_t^{(1)}, S_t^{(2)}\right)$ where the process $S_t^{(i)}$ for $i = 1, 2$ satisfies the two-factor model

$$dS_t^{(i)} = \mu_i S_t^{(i)} dt + \sigma_i S_t^{(i)} dW_t^{(i)};\ i = 1, 2;\ dW_1 dW_2 = \rho dt$$

for $|\rho| < 1$. It can be used for a function like $F\left(t, S_t^{(1)}, S_t^{(2)}\right) = t^3 + \left(S_t^{(1)}\right)^2 + S_t^{(1)} \cos S_t^{(2)}$

$$dF =$$

$$\left(\frac{\partial F}{\partial t} + \mu_1 S_t^{(1)} \frac{\partial F}{\partial S_t^{(1)}} + \mu_2 S_t^{(2)} \frac{\partial F}{\partial S_t^{(2)}} + + \frac{1}{2}\sigma_1^2 S_t^{(1)^2} \frac{\partial^2 F}{\partial S_t^{(1)^2}} + \frac{1}{2}\sigma_2^2 S_t^{(2)^2} \frac{\partial^2 F}{\partial S_t^{(2)^2}}\right) dt$$
$$+\sigma_1 S_t^{(1)} \frac{\partial F}{\partial S_t^{(1)}} dW_t^{(1)} + \sigma_2 S_t^{(2)} \frac{\partial F}{\partial S_t^{(2)}} dW_t^{(2)}$$

General Itô

Consider a process G_t satisfying the SDE

$$dG_t = A\left(G_t, t\right) dt + B\left(G_t, t\right) dW_t$$

for a general drift $A\left(G_t, t\right)$ and diffusion $B\left(G_t, t\right)$ respectively, and $F = F\left(t, G_t\right)$, then

$$dF = \left(\frac{\partial F}{\partial t} + A\left(G_t, t\right)\frac{\partial F}{\partial G_t} + \frac{1}{2}B^2\left(G_t, t\right)\frac{\partial^2 F}{\partial G_t^2}\right) dt + B\left(G_t, t\right)\frac{\partial F}{\partial G_t} dW_t.$$

Itô Product Rule

Let X_t, Y_t be two processes, where

$$\begin{aligned} dX_t &= a\left(t, X_t\right) dt + b\left(t, X_t\right) dW_t^{(1)}, \\ dY_t &= c\left(t, Y_t\right) dt + d\left(t, Y_t\right) dW_t^{(2)} \end{aligned}$$

Two-dimensional form of Taylor with $f\left(X_t, Y_t\right) = X_t Y_t$

$$df = \frac{\partial f}{\partial X_t} dX_t + \frac{\partial f}{\partial Y_t} dY_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} dX_t^2 + \frac{1}{2} \frac{\partial^2 f}{\partial Y_t^2} dY_t^2 + \frac{\partial^2 f}{\partial X_t \partial Y_t} dX_t dY_t$$

$$\begin{array}{ccc} \frac{\partial f}{\partial X_t} = Y_t & & \frac{\partial f}{\partial Y_t} = X_t \\ \frac{\partial^2 f}{\partial X_t^2} = 0 & \frac{\partial^2 f}{\partial Y_t^2} = 0 & \frac{\partial^2 f}{\partial X_t \partial Y_t} = 1 \end{array}$$

to give

$$d\left(X_t Y_t\right) = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

Itô Quotient Rule

By applying the earlier two-dimensional form of Taylor used in the product rule with $f\left(X_t, Y_t\right) = \dfrac{X_t}{Y_t}$

$$
\begin{array}{lll}
\dfrac{\partial f}{\partial X_t} = 1/Y_t & & \dfrac{\partial f}{\partial Y_t} = -X_t/Y_t^2 \\
\dfrac{\partial^2 f}{\partial X_t^2} = 0 & \dfrac{\partial^2 f}{\partial Y_t^2} = 2X_t/Y_t^3 & \dfrac{\partial^2 f}{\partial X_t \partial Y_t} = -1/Y_t^2
\end{array}
$$

which gives

$$d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left(\frac{dX_t}{X_t} - \frac{dY_t}{Y_t} - \frac{dX_t dY}{XY_t} + \left(\frac{dY_t}{Y_t}\right)^2\right)$$

Stochastic Integration Formula I

$$\int_0^t \frac{dF}{dW_t} dW_t = F(W_t) - F(W_0) - \frac{1}{2}\int_0^t \frac{d^2F}{dW_s^2} ds$$

Example:

$$\int_0^t W_t \sin W_t dW_t$$

Stochastic Integration Formula II

$$\int_0^t \frac{\partial F}{\partial W_t} dW_t = F(t, W_t) - F(0, W_0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial W_s^2} \right) ds$$

Example:

$$\int_0^t \left(tW_t + e^{W_t} \right) dW_t$$

Itô Integral - definition

$$\int_0^T f(t, W_t)\, dW_t = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(t_i, W_i)(W_{i+1} - W_i),$$

where $W_i \equiv W_{t_i}$