Further Mathematical Methods: II

In this lecture ...

- Double Integration
  - Review and examples
  - Applications to joint probability distributions
  - The gamma function
- Fourier Transforms
  - Definition and standard results

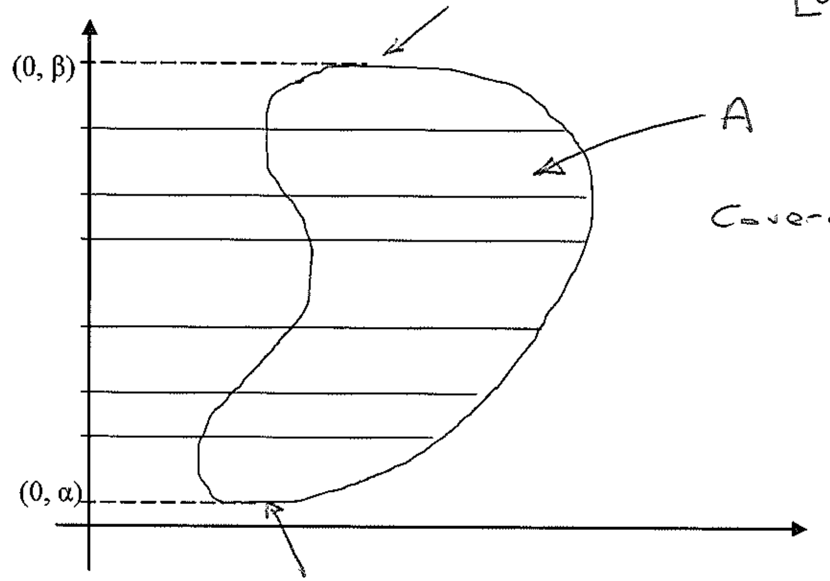
  - Applications to differential equations

- Power series solutions of Ordinary Differential Equations

1 Double Integration

1.1 Evaluation of $\iint_A f(x,\ y)\, dx dy$

$[\alpha, \beta]$ range for y

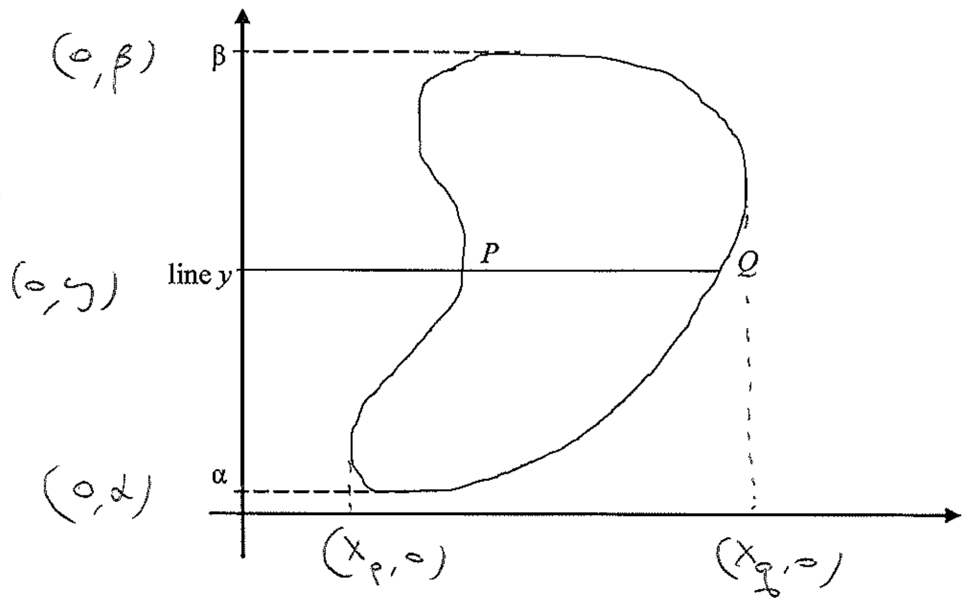


Covering A in x direction

$$\iint_A f\,(x,\ y)\ \ dx\ dy$$

$$= \underbrace{\int_\alpha^\beta \iint \{ \int f\,(x,\ y)\}_{x_P(y)}^{x_Q(y)}\ \ dx\}\ dy}_{}$$

<u>So</u> limits are given by:



$\left.\begin{matrix} x_p \\ x_q \end{matrix}\right\}$ fns. of y

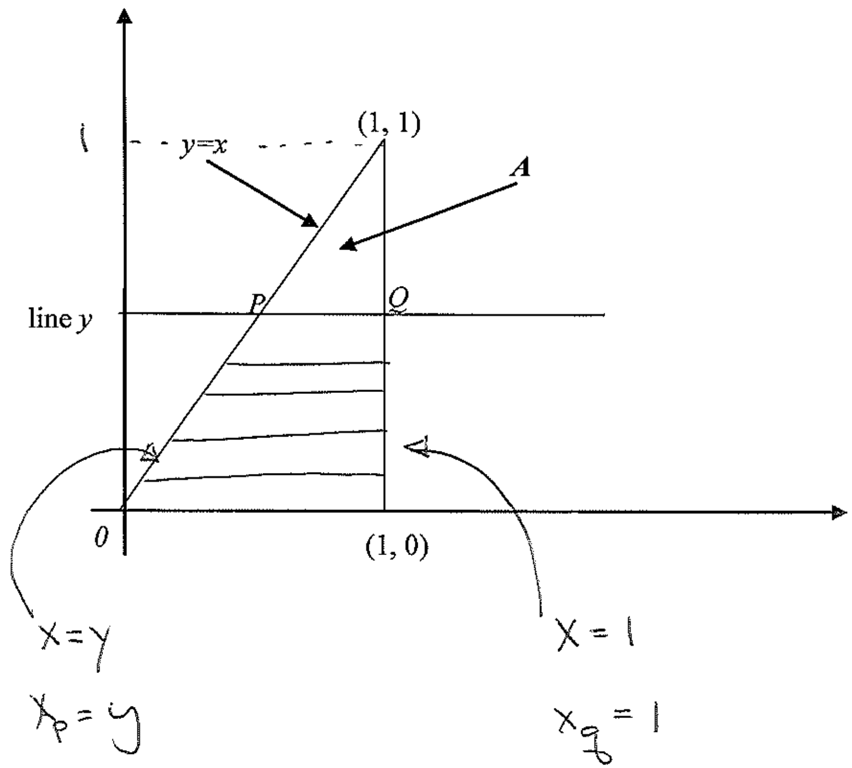
Example: Evaluate

$$\iint_A (x+y)\ dx\ dy$$

where A is the Δ in the following diagram:

$\alpha = 0$

$\beta = 1$



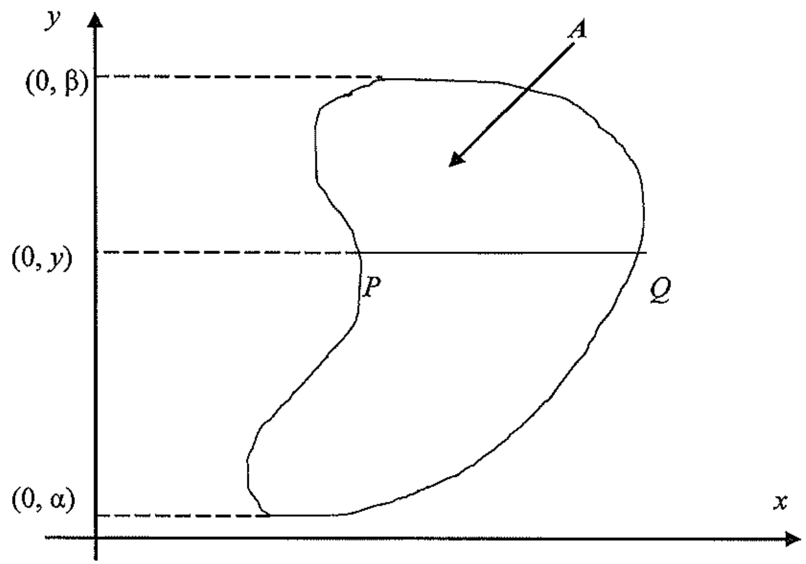
$$
\begin{aligned}
x_P &= y \quad P(y,\ y) \\
x_Q &= 1 \quad Q(1,\ y)
\end{aligned}
$$

$$
\begin{aligned}
I &= \int_{y=0}^{y=1} \left\{ \int_{x_P=y}^{x_Q=1} x + y \ dx \right\} dy \\
\int_y^1 (x+y)\,dx &= \left[\frac{x^2}{2} + xy \right]_y^1 = \left(\frac{1}{2} + y \right) - \left(\frac{y^2}{2} + y^2 \right) && \text{inner integral} \\
I &= \int_0^1 \left(\frac{1}{2} + y - \frac{3y^2}{2} \right) dy = \left(\frac{y}{2} + \frac{y^2}{2} - \frac{y^3}{2} \right)_0^1 && \text{outer integral} \\
&= \frac{1}{2}
\end{aligned}
$$

So generally

$$\iint_A f(x,\ y)dx\ dy$$

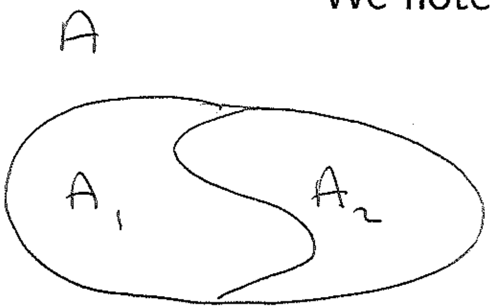
where A is defined as



$x_P,\ x_Q$ function of y

$$= \underbrace{\int_{\alpha}^{\beta} \left\{ \int_{x_P}^{x_Q} f\,(x,\ y) \right\} dy}_{\text{repeated integral}}$$

We note in passing that

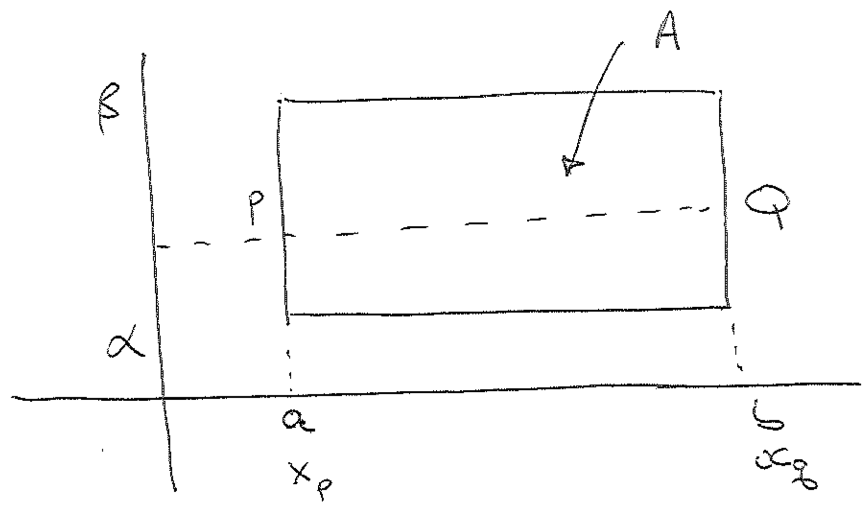


$$\iint_A f\ dx\ dy = \iint_{A_1} f\ dx\ dy + \iint_{A_2} f\ dx\ dy$$

A: $A_1 + A_2$

The main problem lies in the limits. We consider the following examples –
(Main work)

Examples:



1. <u>A Rectangle</u>

$a \leq x \leq b$

$\alpha \leq y \leq \beta$

<u>Here</u> $x_P = a, \; x_Q = b$

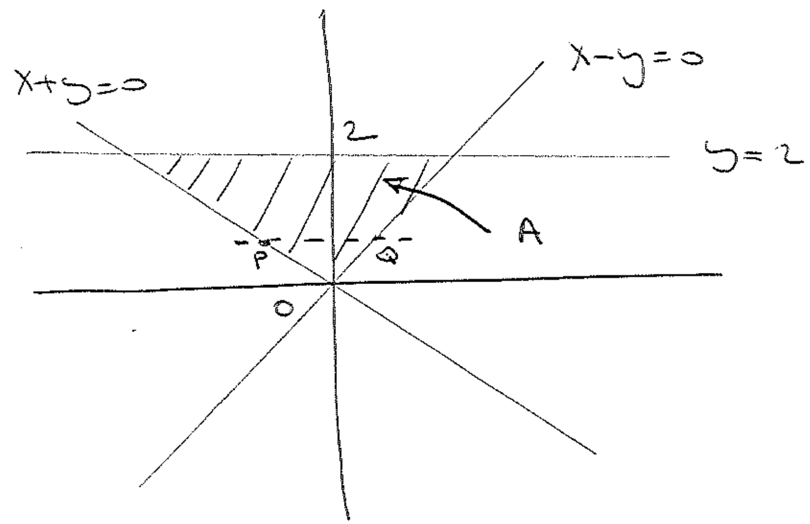
$\alpha \leq y \leq \beta$

$\therefore$

$$\iint_A f \; dx \; dy = \int_\alpha^\beta \left\{ \int_a^b f \; dx \right\} dy$$

2. <u>A Triangle</u>

with sides



A is Region bounded by these 3 lines

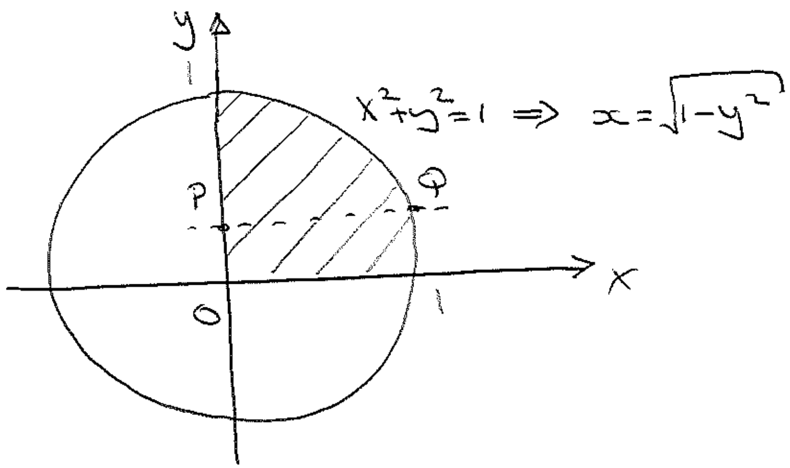
$$\begin{cases} x+y = 0 \\ x-y = 0 \\ y = 2 \end{cases}$$

In this case

$$x_P = -y;\ x_Q = y$$
$$\alpha = 0;\ \beta = 2$$

we go from x=−y to x=y bounded by 0 & 2 in the y-axis

$$\iint_A f\ dx\ dy = \int_0^2 \left\{ \int_{-y}^{y} f\ dx \right\} dy$$



3 A is the region defined by

$$x^2 + y^2 \leq 1, \; x, \; y \geq 0$$

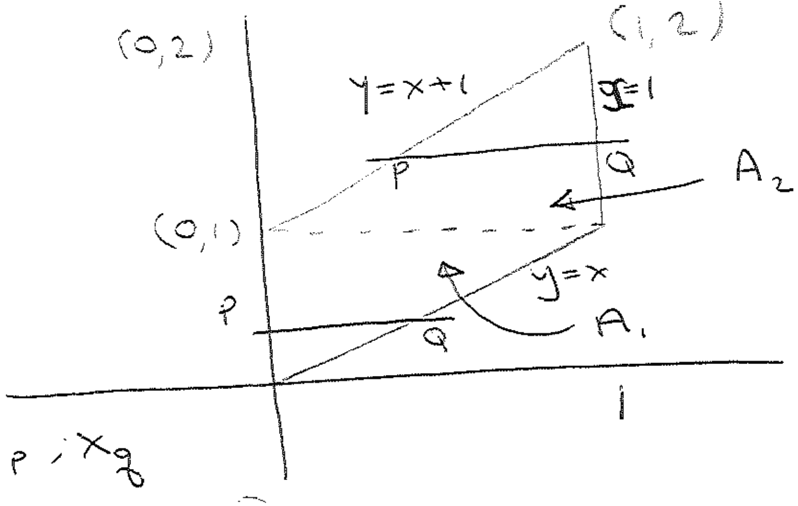
unit circle first quadrant

$y \in [0,1]$

$$\iint_A f \; dx \; dy = \int_0^1 \left\{ \int_0^{\sqrt{1-y^2}} f \; dx \right\} dy$$

Difficulty: A parallelogram

For this A we do not have a simple value for x_P (or x_Q)   2 stages



$A = A_1 + A_2$

For A_1 $\quad x_P = 0, \; x_P = y$

For A_2 $\quad x_P = y - 1, \; x_Q = 1$

So

we don't have a unique (simple) set x_p, x_q

$$\iint_A f\ dx\ dy = \iint_{A_1} f\ dx\ dy + \iint_{A_2} f\ dx\ dy$$

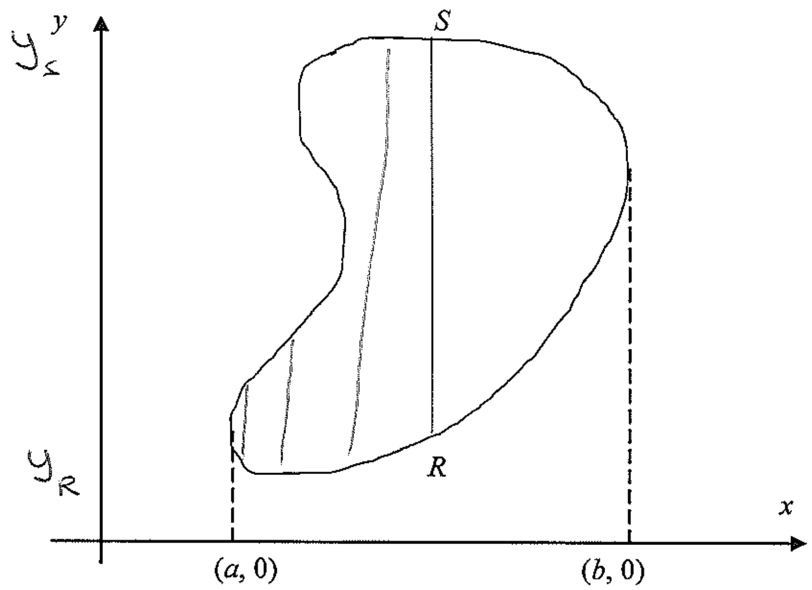
$$\iint_{A_1} = \int_0^1 \left\{ \int_0^y f\ dx \right\} dy \quad (0 \leq y \leq 1 \text{ in } A_1)$$

$$\iint_{A_2} = \int_1^2 \left\{ \int_{y-1}^1 f\ dx \right\} dy \quad (1 \leq y \leq 2 \text{ in } A_2)$$

Sometimes, then, we want to do the y—integration first:

$$\iint_A f\ dx\ dy$$

$$= \int_a^b \left\{ \int_{y_R}^{y_S} f \; dy \right\} dx$$



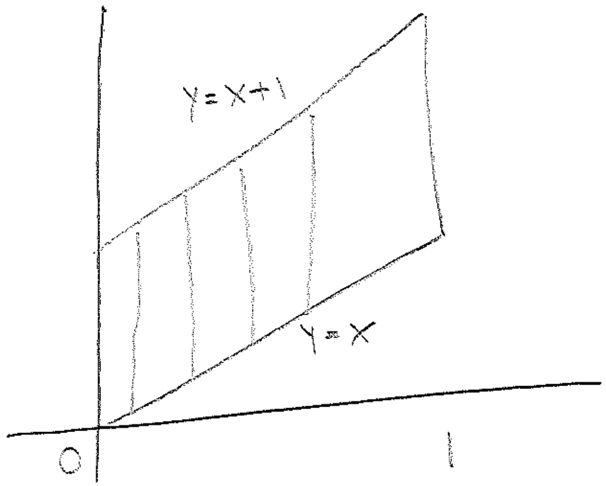
Here $y_R,\ y_S$ depend on x

Example:

A is the parallelogram discussed earlier

$y_R = x \quad a = 0$

$y_S = x + 1 \quad b = 1$



$\int_{y=x}^{y=x+1} dy$

$\int_0^1 dx$

$$\iint_A f \, dx \, dy = \int_0^1 \left\{ \int_x^{x+1} f \, dy \right\} dx$$

1.2 Uses of Double Integration

AREAS

Theorem

integrate over 1

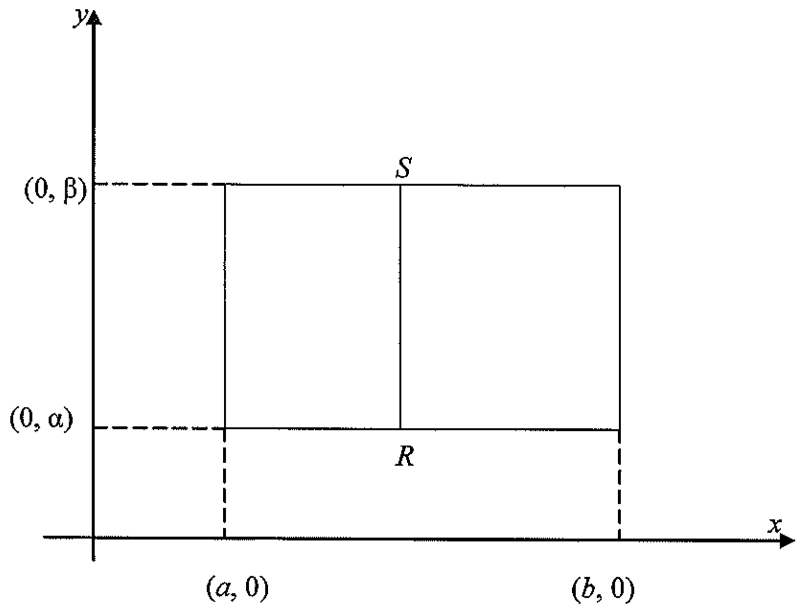
$$\iint_A 1 \, dx \, dy = \text{ area of } A$$

Here we have $f(x,\ y) = 1 \ \forall (x,\ y)$ in A

Example

A rectangle $a \leq x \leq b$, $\alpha \leq y \leq \beta$

$$\text{area} = \iint_A \mathbf{1} \cdot dx\,dy = \int_{\alpha}^{\beta} \left\{ \int_{a}^{b} \mathbf{1}\; dy \right\} dx$$



$$
\begin{aligned}
&= \int_a^b [y]_\alpha^\beta \, dx = \int_a^b (\beta - \alpha) \, dx = (\beta - \alpha) \, [x]_a^b \\
&= (\beta - \alpha) \, (b - a)
\end{aligned}
$$

1.3 Changing to Plane Polars

If

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

then

$$\iint_A f(x,y)\,dxdy = \iint_{A'} F(r,\theta)\,r dr d\theta$$

where

1. $F(r,\theta) = f(r\cos\theta, r\sin\theta)$

2. A' is the region A described in (r,θ) coordinates.

Ex: 1

1.4 Joint PDF for Continuous Random Variables

Recall that the cumulative distribution function $F(x)$ of a RV X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(s)\, ds$$

$F(x)$ is related to the PDF $p(x)$ by

$$p(x) = \frac{dF}{dx}.$$

R.V.

Consider the pair (X, Y) with joint pdf $p_{XY}(x, y)$ and cdf $F_{XY}(x, y)$. They are related through a similar fashion

$$p_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

Integrating this (as before) gives the cdf as

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p_{XY}(s, t)\, dt ds$$

which allows to calculate the probability

$$\mathbb{P}\left(X \leq x,\ Y \leq y\right).$$

We can extend the simple properties of $p_{XY}\left(x,y\right)$ to two dimensions:

- $p_{XY}\left(x,y\right) \geq 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}\left(x,y\right) dxdy = 1$
- $\iint_{R} p_{X,\,Y}\left(x,y\right) dxdy = \mathbb{P}\left(\left(X,Y\right) \in R\right).$
- $\mathbb{P}\left(a < X \leq b,\ c < Y \leq d\right) = \int_{c}^{d}\int_{a}^{b} f_{XY}\left(x,y\right) dxdy$

If X and Y are independent random variables the cdf can be expressed in separable form

$$F_{XY}(x,y) = F_X(x)\,F_Y(y)\,.$$

Then differentiating gives

$$\begin{aligned}\frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} &= \frac{\partial F_X}{\partial x}\frac{\partial F_Y}{\partial y} \\ p_{XY}(x,y) &= p_X(x)\,p_Y(y)\,.\end{aligned}$$

$$\left.\begin{aligned} p_X(x) &= \int_{\mathbb{R}} p_{XY}(x,y)\,dy \\ p_Y(y) &= \int_{\mathbb{R}} p_{XY}(x,y)\,dx \end{aligned}\right\}\text{Marginal pdfs}$$

Ex: 2

1.5 The Gamma Function Revisited

The Gamma Function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt \quad (x > 0)$$

The condition on x is a convergent criterion.

If $a > 0$

$$\int_0^a x^p dx \quad \text{exists for } p > -1 \qquad \text{to avoid } 1/0 \text{ or } \ln 0$$

$$\int_a^\infty x^p dx \quad \text{exists for } p < -1 \qquad \text{to avoid } \infty^{p+1}$$

Integration by parts gives us

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^{x} dt = x \underbrace{\int_0^\infty e^{-t}\, t^{x-1} dt}_{\Gamma(x)} = x(x-1)\int_0^\infty e^{-t}\, t^{x-2} dt =$$

$$\ldots\ldots = x! \qquad (\ddagger)$$

already seen

Important results:

$$\begin{aligned}\Gamma(n+1) &= n! \;\; (n \geq 0) \\ \Gamma(1) &= 1\end{aligned}$$

and also from (‡)

$$\Gamma(x+1) = x\Gamma(x).$$

Theorem

$$\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \; d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Proof Start with the definition of the gamma function

$$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t} dt$$

and make the substitution $t = x^2$ which gives $\quad dt = 2x\,dx$

$$\begin{aligned}\Gamma(m) &= \int_0^\infty \left(x^2\right)^{m-1} \exp\left(-x^2\right) \underbrace{2x dx}_{dt} \\ &= 2\int_0^\infty x^{2m-1} \exp\left(-x^2\right) dx\end{aligned}$$

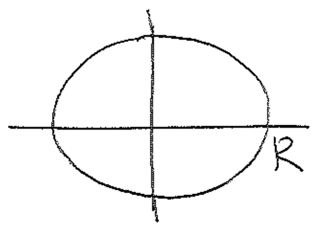
Similarly

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} \exp\left(-y^2\right) dy$$

therefore

$$\begin{aligned}\Gamma(m)\Gamma(n) &= 4\left(\int_0^\infty x^{2m-1} \exp\left(-x^2\right) dx\right)\left(\int_0^\infty y^{2n-1} \exp\left(-y^2\right) dy\right) \\ &= 4\int\int_A x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx dy\end{aligned}$$

where A is the region of integration defined by the first (positive) quadrant. Introduce polar coordinates



$\lim_{R\to\infty}$

$$\begin{aligned}x &= r\cos\theta \\ y &= r\sin\theta\end{aligned}$$

$$4\int\int_A r^{2m-1} \cos^{2m-1}\theta \; r^{2n-1} \sin^{2n-1}\theta \cdot e^{-r^2} \underbrace{r\,dr\,d\theta}_{dx\,dy}$$

26

to transform the integrand to

$$r^{2m+2n-2}\cos^{2m-1}\theta\sin^{2n-1}\theta\exp\left(-r^2\right)$$

and $\boxed{dxdy \longrightarrow rdrd\theta}$ group r & θ terms separately

$$\Gamma(m)\Gamma(n) = 4\underbrace{\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta}_{\text{integral we want}}\underbrace{\int_0^{\infty} r^{2(m+n)-1}e^{-(r^2)}dr}_{\frac{1}{2}\Gamma(m+n)}$$

$$\int_0^\infty (r^2)^{m+n-1} e^{-r^2} . r\,dr$$

Put $t = r^2$

$dt = 2r\,dr$

$$= \int_0^\infty (t)^{(m+n)-1} e^{-t} \tfrac{1}{2}\,dt$$

$$= \frac{1}{2}\int_0^\infty t^{(m+n)-1} e^{-t}\,dt$$

$$= \frac{1}{2}\Gamma(m+n)$$

so rearranging gives the result

$$\boxed{\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}}$$

Example Calculate $\int_0^{\pi/2}\cos^4\theta\sin^3\theta\ d\theta \equiv \int_0^{\pi/2} c^{2m-1} s^{2n-1} d\theta$

Hence

$$\begin{aligned} 2m-1 &= 4 \longrightarrow m = 5/2 \\ 2n-1 &= 3 \longrightarrow n = 2 \end{aligned}$$

so integral equals

Verify

$$\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(2\right)}{2\Gamma\left(\frac{9}{2}\right)} = \frac{\frac{3}{2}.\frac{\sqrt{\pi}}{2}.1}{2\left(\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{\sqrt{\pi}}{2}\right)} = \frac{2}{35}$$

Example $I = \int_0^{\pi/2} \cos^6\theta \, d\theta$ No $\sin^{2n-1}\theta$ term

$$\begin{aligned} 2m-1 &= 6 \longrightarrow m = 7/2 \\ 2n-1 &= 0 \longrightarrow n = 1/2 \end{aligned}$$

Hence $I =$

$$\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(4\right)} = \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\frac{\sqrt{\pi}}{2}.\sqrt{\pi}}{2\left(3.2\right)} = \frac{5\pi}{32}$$

2 The Fourier Transform

If $f = f(x)$ then consider

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx. \qquad \text{or } \frac{1}{2\pi} \quad \text{or none}$$

If this special integral converges, it is called the *Fourier Transform* of $f(x)$.
Similar to the case of Laplace Transforms, it is denoted as $\mathcal{F}(f)$, i.e.

$\int_0^\infty f(t) e^{-st} dt$

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx = \widehat{f}(\omega).$$

The *Inverse Fourier Transform* is then   i.e $\mathbb{C}$-plane to $\mathbb{R}$-plane

$$\mathcal{F}^{-1}\left(\widehat{f}(\omega)\right) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-ix\omega} d\omega = f(x).$$

The convergent property means that $\widehat{f}(\omega)$ is bounded and we have

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Functions of this type $f(x) \in L_1(-\infty, \infty)$ and are called *square integrable.*

We know from integration that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$ standard result

Hence

$$\begin{aligned} \left|\widehat{f}(\omega)\right| &= \left| \int_{\mathbb{R}} f(x)\, e^{ix\omega} dx \right| \\ &\leq \int_{\mathbb{R}} \left| f(x)\, e^{ix\omega} \right| dx \end{aligned}$$ from above result.

and Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ implies that $\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, therefore

$$\left|\widehat{f}(\omega)\right| \leq \int_{\mathbb{R}} |f(x)|\, dx < \infty.$$

In addition to the boundedness of $\widehat{f}(\omega)$, it is also continuous (requires a $\delta - \epsilon$ proof).

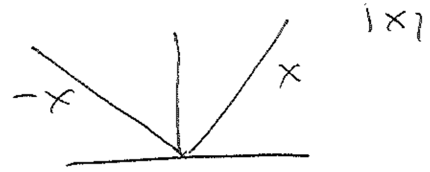
Ex: Fourier transform of $e^x = f(x)$

$$\hat{f}(\omega) = \mathcal{F}(f) = \int_{\mathbb{R}} f(x)\, e^{ix\omega}\, dx = \int_{-\infty}^{\infty} e^x e^{ix\omega}\, dx$$

$$= \int_{-\infty}^{\infty} e^{(1+i\omega)x}\, dx = \frac{1}{(i\omega+1)} e^{(1+i\omega)x}\Big|_{-\infty}^{\infty} \longrightarrow \infty \quad \text{ie integral diverges.}$$

$\therefore$ Fourier transform of e^x does not exist.

Example: Obtain the Fourier transform of $f\ (x) = e^{-|x|}$



$$
\begin{aligned}
\widehat{f}(\omega) &= \mathcal{F}(f) = \int_{-\infty}^{\infty} f\ (x)\, e^{ix\omega} dx \\
&= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega} dx \\
&= \int_{-\infty}^{0} e^{-|x|} e^{ix\omega} dx + \int_{0}^{\infty} e^{-|x|} e^{ix\omega} dx \\
&= \int_{-\infty}^{0} e^{x} e^{ix\omega} dx + \int_{0}^{\infty} e^{-x} e^{ix\omega} dx = \\
& \int_{-\infty}^{0} \exp\left[(1+i\omega)\, x\right] dx + \int_{0}^{\infty} \exp\left[-\left(1-i\omega\right) x\right] dx \\
&= \frac{1}{(1+i\omega)} \exp\left[(1+i\omega)\, x\right]\Big|_{-\infty}^{0} \overset{-}{+} \frac{1}{(1-i\omega)} \exp\left[-\left(1-i\omega\right) x\right]\Big|_{0}^{\infty} \\
&= \frac{1}{(1+i\omega)} + \frac{1}{(1-i\omega)} = \frac{2}{\left(1+\omega^2\right)}
\end{aligned}
$$

Our interest in differential equations continues, hence the reason for introducing this transform. We now look at obtaining Fourier transforms of derivative terms. We assume that $f\ (x)$ is continuous and $f(x) \to 0$ as $x \to \pm\infty$. Consider

$$\mathcal{F}\left\{f'(x)\right\} = \int_{\mathbb{R}} f'(x)\, e^{ix\omega} dx$$

$v = e^{ix\omega}$ $\quad u' = f'$

$v' = i\omega e^{ix\omega}$ $\quad u = f$

which is simplified using integration by parts

$$\underset{u}{f(x)}\, \underset{v}{e^{ix\omega}}\Big|_{-\infty}^{\infty} \underbrace{- i\omega \int_{\mathbb{R}} f\ (x)\, e^{ix\omega} dx}_{-\int uv'}$$

so

$$\mathcal{F}\left\{f\ '(x)\right\} = -i\omega \underbrace{\int_{\mathbb{R}} f\ (x)\, e^{ix\omega} dx}_{\hat{f}(\omega)} = -i\omega \hat{f}(\omega).$$

We can obtain the Fourier transform for the second derivative by performing integration by parts (twice) to give

$$\mathcal{F}\left\{f\ ''(x)\right\} = (-i\omega)^2\, \mathcal{F}\left\{f(x)\right\} = -\omega^2 \hat{f}(\omega).$$

Check

$v = e^{ix\omega}$ $\quad u' = f''$

$v' = i\omega e^{ix\omega}$ $\quad u = f'$

Example: Solve the diffusion equation problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= e^{-|x|}, \quad -\infty < x < \infty\end{aligned}$$

Here $u = u(x,t)$, so we begin by defining

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)\, e^{ix\omega} dx = \widehat{u}(\omega,t).$$

So F.T w.r.t x (not t)

Now take Fourier transforms of our PDE, i.e.

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

to obtain

$$\frac{d\widehat{u}}{dt} = -\omega^2 \widehat{u}(\omega,t).$$

$\mathcal{F}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t} \underbrace{\mathcal{F}(u)}_{\hat{u}}$

We note that the second order PDE has been reduced to a first order equation of type variable separable. This has solution

$$\widehat{u}(\omega,t) = Ce^{-\omega^2 t}.$$

$\int \frac{d\hat{u}}{\hat{u}} = -\omega^2 \int dt$

$\ln \hat{u} = -\omega^2 t + K$

$\hat{u} = Ce^{-\omega^2 t}$

We can find the constant of integration transforming the initial condition

$$\begin{aligned} \mathcal{F}\{u(x,0)\} &= \mathcal{F}\left\{e^{-|x|}\right\} \\ \hat{u}(\omega,0) &= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega}\, dx = \frac{2}{\left(1+\omega^2\right)}. \end{aligned}$$

from earlier example

Applying this to the solution $\hat{u}(\omega,t)$ gives

$\hat{u} = C e^{-\omega^2 t}$

at $t = 0$

$\frac{2}{1+\omega^2} = C e^{0}$

$$\hat{u}(\omega,0) = C = \frac{2}{\left(1+\omega^2\right)},$$

hence

$$\hat{u}(\omega,t) = \frac{2}{\left(1+\omega^2\right)} e^{-\omega^2 t}.$$

this is in $\omega - t$ plane

How do we get $u(x,t)$?

(5)

We now use the inverse transform to get $u(x,t) = \mathcal{F}^{-1}\left(\widehat{u}(\omega,t)\right)$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \widehat{u}(\omega,t)\, e^{-ix\omega} d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} e^{-ix\omega} d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \overset{\text{Euler}}{\left(\cos x\omega - i \sin x\right)} d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \cos x\omega \; d\omega - 2i \int_{-\infty}^{\infty} \overbrace{\frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t}}^{\text{even}} \underbrace{\sin x\omega}_{\text{odd}} \; d\omega.
\end{aligned}$$

This now simplifies nicely because $\frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \sin x\omega$ is an odd function,

hence

$$\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \sin x\omega \; d\omega = 0.$$

Therefore

$$u\left(x,t\right)=2\int_{-\infty}^{\infty}\frac{1}{\left(1+\omega^2\right)}e^{-\omega^2 t}\cos x\omega\ d\omega.$$

Now needs treatment of Residue Theory.

3 Power Series Solutions

3.1 Introduction

The Euler equation has a nice structure, i.e.

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

where the order of each derivative term and power of its coefficient in x is the same. The next step is to move away from this "nice pattern" and consider a more general equation of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0 \tag{1}$$

We look for solutions in the neighbourhood of $x = 0$.

We say that $x = 0$ is an *ordinary point* of the differential equation (1) if both $p(x)$ and $q(x)$ have Taylor expansions about $x = 0$.

i.e.

$$p(x) = p_0 + p_1 x + p_2 x^2 + O\left(x^3\right)$$
$$q(x) = q_0 + q_1 x + q_2 x^2 + O\left(x^3\right)$$

i.e. p & q are both diff'ble at x=0

with both p_i, $q_i \sim O(1)$ where $i = 0, 1, \ldots, n$.

If either or both $p(x)$, $q(x)$ do not have Taylor expansions about $x = 0$, then $x = 0$ is a *singular point* for the D.E.

Regular Singular Point: $xp(x)$ and $x^2 p(x)$ have Taylor expansions about $x = 0$.

Irregular Singular Point: all other points.

Examples:

1. $x\dfrac{d^2y}{dx^2} + x^3\dfrac{dy}{dx} + xy = 0$

This can written in standard form as $\dfrac{d^2y}{dx^2} + x^2\dfrac{dy}{dx} + y = 0 \Rightarrow p(x) = x^2$ & $q(x) = 1$ which both have Taylor expansions about $x = 0$.

Therefore $x = 0$ is an ordinary point of the differential equation.

2. $x^3\frac{d^2y}{dx^2} + 2x^2\frac{dy}{dx} + 5x^2y = 0$

which becomes $\frac{d^2y}{dx^2} + \underset{p}{\frac{2}{x}}\frac{dy}{dx} + \underset{q}{\frac{5}{x}}y = 0$ and $p(x) = \frac{2}{x}$ & $q(x) = \frac{5}{x}$ do not have a Taylor expansion about $x = 0$ - however $xp(x) = 2$ & $x^2q(x) = 5x$ do.

Therefore $x = 0$ is a regular singular point of the differential equation.

3. $\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx} + \frac{4}{x^3}y = 0$

$p(x) = O\left(\frac{1}{x^2}\right)$ & $xp(x) = O\left(\frac{1}{x}\right)$; $\quad q(x) = O\left(\frac{1}{x^3}\right)$ & $x^2q(x) = O\left(\frac{1}{x}\right)$

None of these expressions have a Taylor expansion about $x = 0$.

Therefore $x = 0$ is an irregular singular point of the given differential equation.

Ex: Consider $(1-x^2)y'' - 2xy' + 6y = 0$ Legendre's eq^n of order 2

$x = 0$ is an ordinary point. $y'' - \frac{2x}{1-x^2} y' + \frac{6}{1-x^2} y = 0$

$$\left.\begin{array}{l} p = \frac{2x}{(1-x)(1+x)} \\ q = \frac{6}{(1-x)(1+x)} \end{array}\right\} \text{regular everywhere except } x = x_0 = \pm 1$$

Look at

$$\left.\begin{array}{l} (x-x_0)p(x) = \frac{2x(x-1)}{-(x-1)(1+x)} = -\frac{2x}{1+x} \\ \& \\ (x-x_0)^2 q(x) = \frac{6(x-1)^2}{(1-x)(1+x)} = \frac{6(1-x)}{1+x} \end{array}\right\} \begin{array}{l} \text{both have TSE at} \\ x=1 \quad \therefore \ x=1 \text{ is a} \\ \text{regular singular point} \end{array}$$

$x_0 = -1$

$$\left.\begin{array}{l} (x+1)p(x) = \frac{2x}{1-x} \\ (x+1)^2 q(x) = \frac{6(x+1)}{(1-x)} \end{array}\right\} \begin{array}{l} \text{both diff}^{\text{ble}} \text{ at } x=1 \\ \therefore \text{Reg. sing. pt.} \end{array}$$

3.2 Ordinary Point

Assume a solution of (1) of the form

$$y(x) = \sum_{n=0}^{\infty} A_n\, x^n \quad (A_0 \neq 0) \tag{2}$$

$A_{-1} = A_{-2} = \dots = 0$

with A_n constant.

Since no boundary conditions are imposed, the general solution involves two arbitrary constants - else the constants can be determined.

Substitute (2) into the equation given by (1) and equate to zero the coefficients of various powers of x.

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} A_n\, x^n \to q(x)\, y \sim \left(q_0 + q_1 x + q_2 x^2\right)\left(A_0 + A_1 x + A_2 x^2\right) \\
y'(x) &= \sum_{n=0}^{\infty} n\, A_n\, x^{n-1} \to p(x)\, y' \sim \left(p_0 + p_1 x + p_2 x^2\right)\left(A_1 + 2A_2 x + 3A_3 x^2\right) \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1)\, A_n\, x^{n-2} \to y'' \sim 2A_2 + 6A_3 x + 12A_4 x^2
\end{aligned}
$$

Go to first order want A_0, A_1 $\therefore$ 2nd order eq^n

$$\overbrace{2A_2 + 6A_3 x}^{y''} + (\overset{p}{p_0 + p_1 x})(\overset{y'}{A_1 + 2A_2 x}) + (\overset{q}{q_0 + q_1 x})(\overset{y}{A_0 + A_1 x}) = 0$$

$O(1)$: $\quad A_0 q_0 + A_1 p_0 + 2A_2 = 0$

$O(x)$: $\quad q_0 A_1 + 2p_0\, A_2 + p_1 A_1 + q_1 A_0 + 6A_3 = 0$

All coefficients can be expressed in terms of A_0 and A_1 which can be arbitrary.

Example

Obtain the general solution of

$$y'' - 2xy' + y = 0$$

$y' = \sum n A_n x^{n-1}$

$y'' = \sum n(n-1) A_n x^{n-2}$

about the ordinary point $x = 0$.

We assume a solution of the form $y(x) = \sum_{n=0}^{\infty} A_n x^n$ and substitute the expression and its derivatives into the ODE to yield

$$\overbrace{\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2}}^{y''} - 2x \overbrace{\sum_{n=0}^{\infty} n A_n x^{n-1}}^{y'} + \overbrace{\sum_{n=0}^{\infty} A_n x^n}^{y} = 0$$

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} + \underbrace{\sum_{n=0}^{\infty} (1-2n) A_n x^n}_{-2xy'+y} = 0$$

Want A_{n+2} in terms of A_n i.e $\sum \left(\quad \right) x^{n-2} = 0$

We require a recurrence relation for which a "trick" is used in the summation. In the second summation above, the n term is changed to $(n-2)$ to give $\sum_{n-2=0}^{\infty} (1-2(n-2)) A_{n-2}x^{n-2}$ which is equivalent to having $\sum_{n=2}^{\infty}$ ....

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} + \underbrace{\sum_{n=2}^{\infty} (1-2(n-2)) A_{n-2}x^{n-2}}_{\text{trick}} = 0$$

$n=2 \equiv n=0$
$n=3 \equiv n=1$ etc.

We are still unable to write the lhs of the expression above as one term of $O\left(x^{n-2}\right)$, because the lower limit of the first summation starts at $n=0$, whilst the other begins at $n=2$. This minor problem can be easily overcome by writing

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} + \sum_{n=0}^{\infty} (5-2n) A_{n-2}x^{n-2} = 0 \qquad (\dagger)$$

with $A_{-2} = A_{-1} = 0$

because $A_{-2} = A_{-1} = 0$ and $A_0 \neq 0$, and (†) can now be expressed as

$$\sum_{n=0}^{\infty} \{n(n-1)A_n + (5-2n)A_{n-2}\}\ x^{n-2} = 0.$$

equals zero rearrange

Collecting coefficients of x^{n-2} :

$$A_n = \frac{(2n-5)}{n(n-1)} A_{n-2} \quad (n \geq 2)$$

or

$$A_{n+2} = \frac{(2n-1)}{(n+2)(n+1)} A_n$$

which gives us the recurrence relationship which we sought.

$$n = 0: \quad A_2 = -\frac{1}{2} A_0; \quad n = 1: \quad A_3 = \frac{1}{6} A_1 = \frac{1}{3!} A_1$$

correction

So we see that all terms A_{2k} will be in terms of A_0 and odd ones $\boxed{A_{2k+1}}$ in terms of A_1.

$$
\begin{aligned}
n &= 2: \quad A_4 = \frac{3}{4.3}A_2 = -\frac{3}{4.3}\frac{1}{2}A_0 = -\frac{3}{4!}A_0 \\
n &= 3: \quad A_5 = \frac{5}{5.4}A_3 = \frac{5}{5.4}\frac{1}{3!}A_1 = \frac{5}{5!}A_1 \\
n &= 4: \quad A_6 = \frac{7}{6.5}A_4 = -\frac{7}{6.5}\frac{3}{4!}A_0 = -\frac{21}{6!}A_0 \\
n &= 5: \quad A_7 = \frac{9}{7.6}A_5 = \frac{9}{7.6}\frac{5}{5!}A_1 = \frac{45}{7!}A_1
\end{aligned}
$$

The solution is

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} A_n\, x^n = \sum_{k=0}^{\infty}\left(A_{2k}\, x^{2k} + A_{2k+1}\, x^{2k+1}\right) \\
&= A_0 \underbrace{\left[1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{21}{6!}x^6 + O\left(x^8\right)\right]}_{=y_1} + \\
&\quad A_1 \underbrace{\left[x + \frac{1}{3!}x^3 - \frac{5}{5!}x^5 + \frac{451}{7!}x^7 + O\left(x^9\right)\right]}_{=y_2} \\
&= A_0 y_1(x) + A_1 y_2.
\end{aligned}
$$

even powers in terms of A_0

odd in terms of A_1

The linear combination $A_0 y_1(x) + A_1 y_2$ becomes the general solution of the equation. The terms A_0 , A_1 are arbitrary.

I.C / B.C $\longrightarrow$ A_0, A_1

Ex: 1

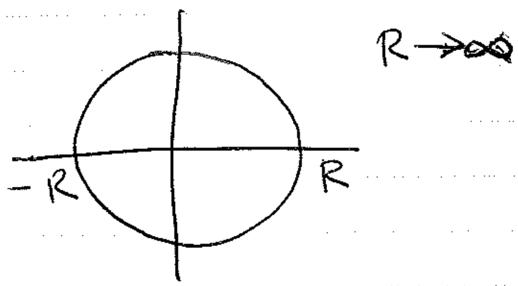
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \qquad \text{put } I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

then also $I = \displaystyle\int_{-\infty}^{\infty} e^{-y^2}\,dy \longrightarrow$

$$I^2 = \left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\left(\int_{\mathbb{R}} e^{-y^2}\,dy\right)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy$$

use plane polars $x = r\cos\theta$, $y = r\sin\theta$



$dx\,dy = r\,dr\,d\theta$

$e^{-(x^2+y^2)} = e^{-r^2}$ $\qquad 0 < r < \infty$, $\; 0 < \theta < 2\pi$

$$\underbrace{\int_0^{2\pi} \underbrace{\int_0^{\infty} e^{-r^2}\, r\,dr}_{①}\, d\theta}_{②}$$

15

① $\int_0^\infty e^{-r^2} r\,dr$ use subst. $u = r^2$

$= \frac{1}{2}$

$\frac{1}{2}\int_0^{2\pi} d\theta = \frac{\theta}{2}\Big|_0^{2\pi} = \pi$

$\therefore$ $I^2 = \pi \longrightarrow I = \sqrt{\pi}$

$\therefore$ $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

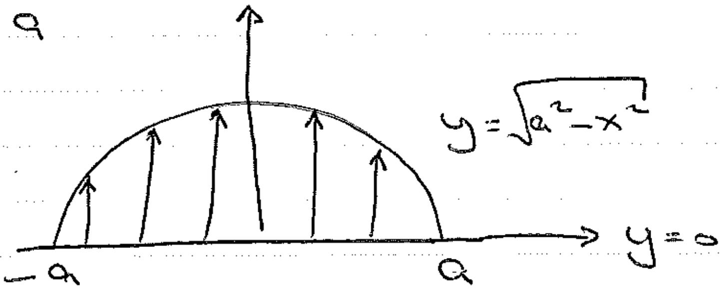
$\Longrightarrow$ $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

1c

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$$

$y=\sqrt{a^2-x^2} \implies x^2+y^2 = a^2 = r$

$-a \leq x \leq a$



$0 \leq r \leq a$ $\qquad dy\,dx = r\,dr\,d\theta$

$0 \leq \theta \leq \pi$

$$\int_{0}^{\pi}\int_{0}^{a} (r^2)^{3/2}\, r\,dr\,d\theta$$

$$\int_{0}^{\pi}\int_{0}^{a} r^4\,dr\,d\theta = \int_{0}^{\pi} \frac{r^5}{5}\Big|_{0}^{a} d\theta$$

$$= \frac{1}{5}\int_{0}^{\pi} a^5\,d\theta = \frac{a^5}{5}\theta\Big|_{0}^{\pi} = \frac{\pi a^5}{5}$$

Ex: 2 Joint pdf of (x, Y) is given by

$$f_{XY} = \begin{cases} k e^{-(x+2y)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

a) find k. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 1$

$$k\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x+2y)}\,dx\,dy = k\int_{0}^{\infty} -e^{-(x+2y)}\Big|_{0}^{\infty} dy$$

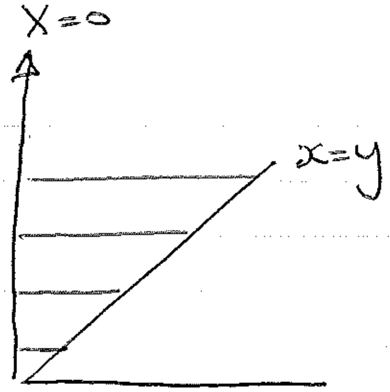
$$k\int_{0}^{\infty} +e^{-2y}\,dy = -\frac{k}{2}e^{-2y}\Big|_{0}^{\infty} = 1 \longrightarrow \boxed{k=2}$$

b) $$\mathbb{P}[1 < X < \infty,\ 0 < Y < 1] = 2\int_{0}^{1}\int_{1}^{\infty} e^{-(x+2y)}\,dx\,dy$$

$$= 2\int_{0}^{1} e^{-(1+2y)}\,dy =$$

$$= -e^{-(1+2y)}\Big|_{0}^{1} = -\left[e^{-3} - e^{-1}\right] = e^{-1} - e^{-3}$$

$\mathbb{P}[X<Y]$



$$\mathbb{P}[X<Y] = 2\int_0^\infty \int_{x=0}^{x=y} e^{-(x+2y)}\,dx\,dy$$

$$= -2\int_0^\infty \left(e^{-3y} - e^{-2y}\right) dy = \frac{1}{3}$$

To check Indep. Must show

$$p_{XY}(x,y) = p_X(x)\, p_Y(y)$$

$$p_X(x) = \int_0^\infty p_{XY}(x,y)\,dy = Ke^{-x}\int_0^\infty e^{-2y}\,dy = e^{-x}$$

$$p_Y(y) = \int_0^\infty p_{XY}(x,y)\,dx = Ke^{-2y}\int_0^\infty e^{-x}\,dx = 2e^{-2y}$$

i.e. $p_X(x)\, p_Y(y) = 2e^{-x}e^{-2y} = 2e^{-(x+2y)} = p_{XY}(x,y)$

$\therefore$ indep.