Further Mathematical Methods: 1

In this lecture ...

- Further first order differential equations
  - Exact equation
  - Bernouilli equation
  - Homogeneous equations
- Further Complex Numbers
  - De Moivres Theorem and applications

Function of a complex variable

Exact Equation

$$-\frac{M}{N} \equiv f(x,y)$$

$$\frac{dy}{dx} = f(x,y)$$

We start by stating a result from calculus: Given a function $G(x, y)$ the total change (or *differential*) denoted dG is defined as

$$\boxed{dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy}$$

$$\frac{dy}{dx} = f(x,y)$$

An equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

$$M\,dx = -N\,dy$$

$$\boxed{-\frac{M}{N} = dy/dx}$$

is called an **Exact equation**.

Any 1^{st} order equation can be written in the form (1), where M, N are functions of x & y.

Mdx+Ndy=0

For example $\dfrac{dy}{dx} = x$ becomes $x\,dx - dy = 0$, so $M(x,y) = x$ and $N(x,y) = -1$.

Theorem

Definition: The equation $M dx + N dy = 0$ is <u>exact</u> (or **Perfect**) if $\exists$ a function $G(x,y)$ s.t. (such that) the differential $dG = M dx + N dy$

The definition above is also a theorem, but we are not interested in the proof.

However, another result which follows from this (called a Corollary) is

Corollary: If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact then $\exists$ $G(x,y)$ s.t. $M(x,y)\,dx + N(x,y)\,dy = dG = 0 \;\therefore\; G(x,y) = \text{constant}$ and this is the solution of the original equation (1).

This is now used to solve equations of type (1).

Example: $\underbrace{(2x+3y)}_{M_y}\,dx + \underbrace{(3x-y)}_{N_x}dy = 0$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

So $M = 2x + 3y$ $\quad N = 3x - y$. Is this equation exact?

$dG = M\,dx + N\,dy$

$$\frac{\partial M}{\partial y} = 3 = \frac{\partial N}{\partial x}$$ so equation is exact.

So $\boxed{\exists\ G(x,y)}$ s.t. $dG = \dfrac{\partial G}{\partial x}dx + \dfrac{\partial G}{\partial y}dy \equiv (2x+3y)\,dx + (3x-y)dy$

$\therefore$ equate coeffs of dx, dy in turn

$$\left.\begin{aligned}\frac{\partial G}{\partial x} &= 2x + 3y \quad \text{(A)}\\ \frac{\partial G}{\partial y} &= 3x - y \quad \text{(B)}\end{aligned}\right\}$$

(A) → integrate wrt x

(B) → integ. wrt y

Integrate (A) wrt x keeping y fixed. Similarly Integrate (B) wrt y keeping x fixed.

const in y

$$\int \frac{\partial G}{\partial x} dx = \boxed{G = x^2 + 3xy + \varphi(y)} \tag{2}$$

const in x

$$\int \frac{\partial G}{\partial y} dy = \boxed{G = 3xy - \frac{1}{2}y^2 + \psi(x)} \tag{3}$$

$\varphi(y) = C - \frac{1}{2}y^2$

$$(2) \equiv (3)$$

$$\therefore x^2 + 3xy + \varphi(y) \equiv 3xy - \frac{1}{2}y^2 + \psi(x)$$

$\Rightarrow \varphi(y)$

$\Rightarrow \psi(x)$

These are identical if $\varphi(y) + \frac{1}{2}y^2 = \psi(x) - x^2 = c$ (recall $F(x) = H(y) \Rightarrow$ each side constant)

indep. of x

indep. of y

$\therefore \psi(x) = c + x^2$ (we have a choice of choosing either)

$\phi(y) = c - \frac{1}{2}y^2$

$\therefore G(x,y) = x^2 + 3xy - \frac{1}{2}y^2 + c$

$\phi(y)$

Solution is $\boxed{G = \text{constant}}$ (from earlier corollary)

$\Rightarrow$ GS is $\boxed{x^2 + 3xy - \frac{1}{2}y^2 = c}$

$G \equiv$ const.

Test: diff.

$x^2 + 3xy - \frac{1}{2}y^2 = c$

Reducible To Exact Form

Unless we are fairly lucky or the problem is particularly straight forward, most equations will not be exact. That is equations of type (1) will have

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.}$$

We can use an *integrating factor* (I.F) approach to convert the equation to exact form. If

$$\frac{M_y - N_x}{N} = f(x)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

then we multiply (1) by the I.F $\mu(x)$, where

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\, dx\right).$$

If

$$\frac{N_x - M_y}{M} = g(y)$$

then the I.F $\mu(y)$, is

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\, dy\right).$$

$Mdx + Ndy = 0$

Example: Consider the IVP $\underbrace{xdx}_{=0} + \underbrace{(x^2y + 4y)}dy = 0$, $y(4) = 0$ $*$

Clearly this equation is not exact because $\dfrac{\partial M}{\partial y} = 0 \neq \dfrac{\partial N}{\partial x} = 2xy.$

Look at (first)

$$\begin{aligned}\frac{M_y - N_x}{N} &= \frac{-2xy}{x^2y + 4y} \\ &= \frac{-2x}{x^2 + 4} = f(x)\end{aligned}$$

which is a function of x alone. So I.F is

$$\mu(x) = \exp\left(-\int \frac{2x}{x^2+4}dx\right) = \frac{1}{x^2+4}$$

which we multiply the differential equation with to get the exact equation

$$\left(\frac{x}{x^2+4}\right)dx + ydy = 0$$

So $\exists\ G(x,y)$ s.t. $dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy \equiv \left(\frac{x}{x^2+4}\right)dx + ydy$

$\therefore$

$$\frac{\partial G}{\partial x} = \frac{x}{x^2+4} \qquad \text{(C)}$$
$$\frac{\partial G}{\partial y} = y \qquad \text{(D)}$$

As with the previous example integrate (C) wrt x keeping y fixed, and integrate (D) wrt y keeping x fixed.

$$G = \frac{1}{2}\ln\left|x^2+4\right| + \varphi(y) \tag{4a}$$

const in y

$$G = \frac{1}{2}y^2 + \psi(x) \tag{4b}$$

const in x

$$(4a) \equiv (4b)$$

$$\therefore \underbrace{\frac{1}{2}\ln\left|x^2+4\right| + \varphi(y)}_{} \equiv \underbrace{\frac{1}{2}y^2 + \psi(x)}_{}$$

$\varphi(y) - \frac{1}{2}y^2 =$

$\psi(x) - \frac{1}{2}\ln|x^2+4|$

$= c$

$F(x) = c$

$H(y) = c$

$F(x) = H(y) \quad \forall x, y$

$= \text{const.}$

Identical if $\varphi(y) - \frac{1}{2}y^2 = \psi(x) - \frac{1}{2}\ln\left|x^2+4\right| = c$

$\therefore$Let us choose $\psi(x) = \frac{1}{2}\ln\left|x^2+4\right| + c$

$\therefore \boxed{G(x,y) = \frac{1}{2}y^2 + \frac{1}{2}\ln\left|x^2+4\right| + c}$

Solution is $G =$constant

$$\Rightarrow \underbrace{\frac{1}{2}y^2 + \frac{1}{2}\ln\left|x^2+4\right|}_{G} = c. \quad = \text{const.}$$

We can tidy this up multiplying through by 2 and taking exponentials

$$\exp\left(y^2 + \ln\left|x^2+4\right|\right) = C$$

$$\boxed{\exp\left(y^2\right)\left(x^2+4\right) = K}$$

which is the general solution. Now use initial condition to determine K. When $x = 4$, $y = 0$ gives $K = 20$. Hence the particular solution becomes

$$\boxed{e^{y^2}\left(x^2 + 4\right) = 20.}$$

Particular solution to $(*)$

Bernoulli Equation

$y' + P(x)y = Q(x)$

This an ODE of the form

$n = 0$

$n \geq 2$

Linear Eq$^{\underline{n}}$

$$y' + P(x)\, y = Q(x)\, y^n \tag{5}$$

$\div$ thro' by y^n

and is nonlinear due to the term y^n, but for $n = 0, 1$ (5) is linear. In the case $n \geq 2$, divide (5) through by y^n, to obtain

$$\boxed{\frac{1}{y^n} y' + P(x) \frac{1}{y^{n-1}} = Q(x)} \tag{6}$$

subst.

Now let $z = \frac{1}{y^{n-1}}$ then

$$\frac{dz}{dx} = \frac{d}{dx}\left(y^{-n+1}\right) = \frac{d}{\cancel{dy}}\left(y^{-n+1}\right)\frac{\cancel{dy}}{dx}$$

$z = \frac{1}{y^{n-1}}$

$z' = -\frac{(n-1)}{y^3} \frac{dy}{dx}$

$$\frac{dz}{dx} = \frac{-(n-1)}{y^n} \frac{dy}{dx} \qquad (7)$$

Rearranging (7) gives $\frac{1}{y^n} y' = \frac{-1}{(n-1)} z'$ so (6) becomes

$\frac{1}{y^{n-1}} = z$

$$\frac{-1}{(n-1)} z' + P(x)\, z = Q(x)$$

p

Then multiplying through by $-(n-1)$ gives

$$z'(x) + \widehat{P}(x) z = \widehat{Q}(x)$$

Linear Eq$^{\underline{n}}$

where $\widehat{P}(x) = -(n-1)\, P(x)$, $\widehat{Q}(x) = -(n-1)\, Q(x)$.

Example:

$z = \frac{1}{y^{n-1}} = \frac{1}{y^2}$ $n = 3$

Solve the equation

$$y' + 2xy = xy^3$$

$\equiv y' + P(x)y = Q(x)y^n$

z

This can be written as $\boxed{\frac{1}{y^3}y' + 2x\frac{1}{y^2} = x}$, i.e. $n = 3$, therefore put $z = \frac{1}{y^2}$,so

$$z' = -\frac{2}{y^3}y'$$

which can be re-written as $\frac{1}{y^3}y' = -\frac{1}{2}z' \therefore -\frac{1}{2}z' + 2xz = x$, or

$$\boxed{z' - 4xz = -2x} \quad (8)$$

Lin. Eq.

$P = -4x$

$Q = -2x$

$y' + P(x)y = Q(x)$

which is linear with $P = -4x$; $Q = -2x$.

$$\text{I.F} = R(x) = \exp\left(-4\int x\,dx\right) = \exp\left(-2x^2\right)$$

and multiply through (8) by $\exp\left(-2x^2\right)$

$$\therefore \quad \exp\left(-2x^2\right)\left(z' - 4xz\right) = -2x\exp\left(-2x^2\right)$$

$$\text{Then } \frac{d}{dx}\left(z\exp\left(-2x^2\right)\right) = -2x\exp\left(-2x^2\right)$$

$$z \exp\left(-2x^2\right) = -2\int x \exp\left(-2x^2\right)\, dx + c,$$

we integrate rhs by substitution : put $u = 2x^2$

$$z \exp\left(-2x^2\right) = \frac{1}{2}\exp\left(-2x^2\right) + c$$

$\boxed{z = \frac{1}{2} + c\exp\left(2x^2\right)}$ and we know $z = \frac{1}{y^2}$, so the GS becomes

$$\boxed{\frac{1}{y^2} = \frac{1}{2} + c\exp\left(2x^2\right).}$$

$$\frac{dy}{dx} = \underbrace{f(x,y)}$$

homogeneous

Homogeneous Equation

Definition: A function $f(x,y)$ is **homogeneous of degree** k if

$$f(tx,ty) = t^k f(x,y)$$

homog. of degree K

Example $f(x,y) = \underbrace{\sqrt{(x^2+y^2)}}$

$$\begin{aligned} f(tx,ty) &= \sqrt{\left[(tx)^2+(ty)^2\right]} \\ &= t\sqrt{\left[\left(x^2+y^2\right)\right]} \\ &= tf(x,y) \end{aligned}$$

K = 1

$= t^1 f(x,y)$

So f is homogeneous of degree one.

Example $f(x,y) = \dfrac{x+y}{x-y}$ then

$$\begin{aligned} f(tx,ty) &= \frac{tx+ty}{tx-ty} \\ &= t^0\left(\frac{x+y}{x-y}\right) \\ &= t^0 f(x,y) \longrightarrow \quad k=0 \end{aligned}$$

So f is homogeneous of degree zero.

Example $f(x,y) = x^2 + y^3$

$$\begin{aligned} f(tx,ty) &= (tx)^2 + (ty)^3 \\ &= t^2x^2 + t^3y^3 \quad t^2\left[x^2 + ty^3\right) \\ &\neq t^k \underbrace{\left(x^2+y^3\right)}_{=f(x,y)} \end{aligned}$$

for any k. So f is not homogeneous.

Definition The differential equation $\boxed{\dfrac{dy}{dx} = f(x,y)}$ is said to be *homogeneous* when $f(x,y)$ is homogeneous of degree k for some k.

Method of Solution

Put $y = vx$ where v is some (as yet) unknown function. Hence we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(vx) = x\frac{dv}{dx} + v\frac{dx}{dx} \\ &= \boxed{v'x + v}\end{aligned}$$

$y' = v'x + v$

Hence

$$f(x,y) = f(x,vx) = \boxed{x^k f(1,v)}$$

(ξ, η) generalised variable

Now f is homogeneous of degree k – so

$$\underbrace{f(t\xi, t\eta)}_{} = \underbrace{t^k}_{} f(\xi,\eta)\ \forall\ \xi,\ \eta$$

homog. degree k

homog. degree k

so

$$f\left(x\xi, x\eta\right) = x^k f\left(\xi, \eta\right) \quad \forall\ \xi,\ \eta$$

put $\boxed{\xi = 1, \quad \eta = v}$

$$f\left(x.1, x.v\right) = \boxed{x^k f\left(1, v\right)}$$

x

y

$f(x, y)$

1 v

The differential equation now becomes

$$\underbrace{v'x + v}_{dy/dx} = x^k f\left(1, v\right) = f(x, y)$$

which is not always solvable - the method may not work. But when $k = 0$ (homogeneous of degree zero) then $x^k = 1$.

Hence

$$\boxed{v'x + v = f\left(1, v\right)}$$

or

$$x\frac{dv}{dx} = f(1, v) - v$$

$$\frac{dy}{dx} = \underbrace{f(x,y)}_{\text{homog. of degree } 0}$$

which is separable, i.e.

$$\int \frac{dv}{f(1,v) - v} = \int \frac{dx}{x} + c$$

$$y = vx$$

$$v = y/x$$

and the method is guaranteed to work.

Example

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

$$f(x,y)$$

homog. of deg. 0

replace $x \to tx$

$y \to ty$

First we check:

$$\frac{ty - tx}{ty + tx} = t^0 \left(\frac{y - x}{y + x}\right)$$

which is homogeneous of degree zero. So put $y = vx$

$x = 1$

$y = v$

$$v'x + v = \underbrace{f(x, \cancel{y}x)}_{} = \frac{vx - x}{vx + x} = \boxed{\frac{v - 1}{v + 1}} = f(1, v)$$

dy/dx

therefore

$$v'x = \overbrace{\frac{v - 1}{v + 1}}^{f(1,v)} - v$$

$$x \frac{dv}{dx} = \frac{-\left(1 + v^2\right)}{v + 1}$$

Take x's on one side & v terms on the other -

and the D.E is now separable

$$
\begin{aligned}
\int \frac{v+1}{v^2+1} dv &= -\int \frac{1}{x} dx \\
\int \frac{v}{v^2+1} dv + \int \frac{1}{v^2+1} dv &= -\int \frac{1}{x} dx \\
\frac{1}{2} \ln\left(1+v^2\right) + \arctan v &= -\ln x + c \\
\frac{1}{2} \ln x^2 \left(1+v^2\right) + \arctan v &= c
\end{aligned}
$$

$y = vx$

replace v by y/x

Now we turn to the original problem, so put $v = \boxed{\frac{y}{x}}$

$$\frac{1}{2} \ln x^2 \left(1 + \frac{y^2}{x^2}\right) + \arctan\left(\frac{y}{x}\right) = c$$

which simplifies to

$$\frac{1}{2} \ln\left(x^2+y^2\right) + \arctan\left(\frac{y}{x}\right) = c.$$

Equation Reducible to Homogeneous Form

i.e. if c, C eqn is not homog.

The equation

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

$X + h$

$Y + k$

is not homogeneous in its current form.

Method: Put

Substitution

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

(h, k) is the soln. of

where $h,\ k$ are solutions of

(h, k)

$$\left.\boxed{\begin{aligned} ah + bk + c &= 0 \\ Ah + Bk + C &= 0 \end{aligned}}\right\}$$

simultaneous eqn

1

$Y = y - k \qquad \longleftarrow \qquad y = Y + k \qquad x = X + h$

i.e. the geometric interpretation of the above is that (h, k) is the intersection of the lines $ah + bk + c = 0$ and $Ah + Bk + C = 0$. Obviously (h, k) exists provided the lines are not parallel. Then

$X = x - h$

$$\frac{dy}{dx} = \frac{d(Y + k)}{d(X + h)} = \frac{dY}{dX}$$

so

$$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{A(X + h) + B(Y + k) + C}$$
$$= \frac{aX + bY + \boxed{(ah + bk + c)}}{AX + BY + \boxed{(Ah + Bk + C)}}$$

$= 0$

$= 0$

second of

(7)

which becomes (from using the earlier expressions)

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$$

$\rightarrow$ homog. of degree zero

$\rightarrow Y(X)$

and is homogeneous of degree zero. Now set $Y = VX$ and proceed as outlined earlier.

Example

$$y' = \frac{2x + y - 1}{x + 2y + 1}$$

put $x = X + h, \quad y = Y + k$ where

$$\boxed{\left.\begin{array}{l} 2h + k - 1 = 0 \\ h + 2k + 1 = 0 \end{array}\right\}}$$

to solve for h & k

hence $h = 1, \ k = -1$ and $x = X + 1, \ y = Y - 1$

$$\frac{dY}{dX} = \frac{2X + Y}{X + 2} \equiv \frac{aX + bY}{AX + BY}$$

making the equation homogeneous of degree zero, so we put $\boxed{Y = VX}$

$$\frac{dY}{dX}$$

$$
\begin{aligned}
V'X + V &= \frac{2X + VX}{X + 2VX} = \frac{2+V}{1+2V} \\
V'X &= \frac{2+V}{1+2V} - V \\
X\frac{dV}{dX} &= \frac{2\left(1 - V^2\right)}{1+2V}
\end{aligned}
$$

$F(1,V)$

which is a separable equation.

$$\int \frac{1+2V}{1-V^2} = 2\int \frac{dX}{X}$$

For the left hand side using a partial fraction approach gives

$$\frac{1+2V}{(1-V)(1+V)} \equiv \frac{3/2}{1-v} + \frac{-1/2}{1+V}$$

hence

$$
\begin{aligned}
\int \left(\frac{3/2}{1-V} + \frac{-1/2}{1+V} \right) dV &= 2\int \frac{dX}{X} \\
-\frac{3}{2}\ln(1-V) - \frac{1}{2}\ln(1+V) &= 2\ln X + c \\
\frac{3}{2}\ln(1-V) + \frac{1}{2}\ln(1+V) + 2\ln X &= k \\
\ln(1-V)^{3/2}(1+V)^{1/2}X^2 &= k \\
(1-V)^{3/2}(1+V)^{1/2}X^2 &= C
\end{aligned}
$$

Now use $V = \frac{Y}{X}$:

$$
\begin{aligned}
\left(1-\frac{Y}{X}\right)^{3/2}\left(1+\frac{Y}{X}\right)^{1/2} X^2 &= C \\
(X-Y)^{3/2}(X+Y)^{1/2} &= C \\
(X-Y)^3(X+Y) &= K
\end{aligned}
$$

and we know $X = x - 1$, $Y = y + 1$ so the general solution becomes

$$(x - y - 2)^3 (x + y) = \text{constant}$$

$x = X + 1$

$y = Y - 1$

Special Case

$$\frac{u-3}{2u-1}$$

The lines

$$\begin{aligned} ah + bk + c &= 0 \\ Ah + Bk + C &= 0 \end{aligned}$$

Same gradient

are parallel.

Example:

$$\frac{dy}{dx} = \frac{2x+y-3}{4x+2y-1}$$

$m = -2$

$\rightarrow 2(2x+y)-1$

lines here are parallel with slope of -2. The denominator of the right hand side can be written as $2\,(2x+y)-1$ so try a substitution of the form $u = 2x+y$, i.e. $y = u - 2x \longrightarrow$

$$\frac{dy}{dx} = \frac{du}{dx} - 2 = \frac{u-3}{2u-1}$$

and the differential equation becomes

$$y' = u' - 2 = \frac{u-3}{2u-1}$$

which in terms of the new variable becomes

$$\begin{aligned} u' &= \frac{u-3}{2u-1} + 2 \\ \frac{du}{dx} &= \frac{5u-5}{2u-1} \end{aligned}$$

which is separable. We present the working in full to show the integration step

$$\begin{aligned}
\int \frac{2u-1}{5u-5}\,du &= \int dx \\
\boxed{\frac{1}{5}\int \left(2+\frac{1}{u-1}\right)du = x+c} \quad &\circledast \quad \text{important.} \\
\frac{1}{5}\left(2u+\ln\left(u-1\right)\right) &= x+c
\end{aligned}$$

Now to return to original variables, put $u = y + 2x$ to get the final form

$$\boxed{\frac{1}{5}\left(2y+4x+\ln\left(y+2x-1\right)\right) = x+c}$$

which is the general solution.

Runge Kutta

$z = x + iy$

Complex Numbers

De Moivre, Theorem

For any $z \in \mathbb{C}$, the expression

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \tan z = \frac{\sin z}{\cos z}$$

defines the generalised circular functions, and

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \quad \cosh z = \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

the generalised hyperbolic function.

Using Euler's formula with positive and negative components we have

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta \cancel{+ i \sin\theta} \\ e^{-i\theta} &= \cos\theta \cancel{- i \sin\theta} \end{aligned}\right\}$$

Adding gives

$$2\cos\theta = e^{i\theta} + e^{-i\theta} \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and subtracting gives

$$2i\sin\theta = e^{i\theta} - e^{-i\theta} \Rightarrow \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.}$$

We can extend these results to consider other functions:

$$\begin{aligned}
\operatorname{cosec} z &= \frac{1}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \cot z = \frac{1}{\tan z} \\
\cosh\operatorname{ec} z &= \frac{1}{\sinh z}, \quad \operatorname{sec} h z = \frac{1}{\cosh z}, \quad \operatorname{cot} h z = \frac{1}{\tanh z}
\end{aligned}$$

We can also obtain a relationship between circular and hyperbolic functions:

$$\sin(iz) = \frac{1}{2i}\left(e^{-z} - e^{z}\right)$$

$$\frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$$

we know $\boxed{1/i = -i}$ hence

$$\sin(iz) = -i.\frac{1}{2}\left(e^{-z} - e^{z}\right) = i.\underbrace{\frac{1}{2}\left(e^{z} - e^{-z}\right)}_{\sinh z}$$

Relationships between trig & hyperbolic fns.

so

$$\boxed{\sin(iz) = i \sinh z.}$$

Similarly it can be shown that

$$\begin{aligned} \sinh(iz) &= i \sin z \\ \cos(iz) &= \cosh z \\ \cosh(iz) &= \cos z \\ \cancel{\sinh(iz)} &\cancel{=} \cancel{i \sin z} \end{aligned}$$

Verify each one

Example:

Let $z = x + iy$ be any complex number, find all the values for which $\cosh z = 0$.

$\cosh z = 0$

$\sin(x+y) = \sin x \cos y + \sin y \cos x$

We use the hyperbolic identity

$$\boxed{\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b}$$

to give

$$\begin{aligned} \cosh z &= \cosh\,(\overbrace{x+iy}^{z}) = \cosh x \cosh iy + \sinh x \sinh iy \\ &= \cosh x \cos y + i \sinh x \sin y \end{aligned}$$

$i \sin y$

Now use these identities

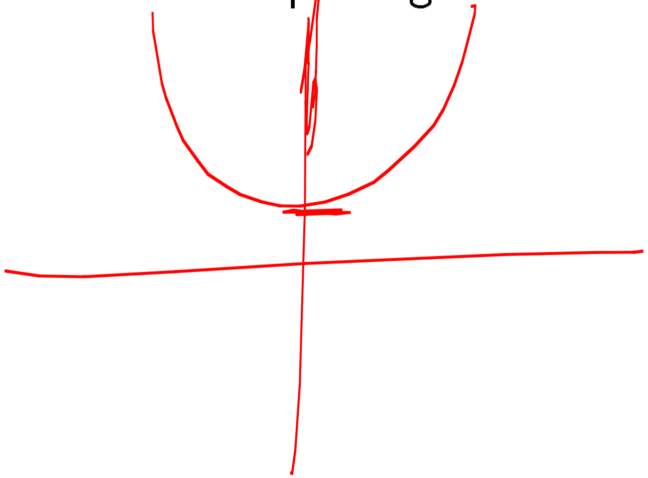
i.e.

$$\cosh x \cos y + i \sinh x \sin y = 0 + i\,0$$

Re part

Im part

so equating real and imaginary parts we have two equations



$$\left.\begin{aligned} \cosh x \cos y &= 0 \\ \sinh x \sin y &= 0 \end{aligned}\right\}$$

$\sin\left(\frac{\pi}{2} + n\pi\right) = \boxed{\cos n\pi}$

$= (-1)^n$

$(2k+1)\frac{\pi}{2}$

From the first we know that $\cosh x \neq 0$ so we require $\cos y = 0 \Rightarrow y = \frac{\pi}{2} + n\pi \ \forall n \in \mathbb{Z}$.

Putting this in the second equation gives

$$\sinh x \underbrace{\sin (2n+1)\frac{\pi}{2}} = 0$$

where

$$\sin (2n+1)\frac{\pi}{2} = \cos n\pi = (-1)^n$$

so

$$\sinh x = 0$$

$x = 0$

$y = \frac{\pi}{2} + n\pi$

which has the solution $x = 0$. Therefore the solution to our equation $\cosh z = 0$ is

$$z_n = i\,(2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z}$$

$z = x + iy$

De Moivres Theorem

Proof by induction

$z + \bar{z}$

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \left(e^{i\theta}\right)^n\\ &= e^{in\theta}\\ &= \cos n\theta + i\sin n\theta\end{aligned}$$

D.M.T.

Similarly

$$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta.$$

It is quite common to write $\cos\theta + i\sin\theta$ as cis.

If

$$z = e^{i\theta} = \cos\theta + i\sin\theta \quad \text{then} \quad \frac{1}{z} = e^{-i\theta} = \bar{z} = \cos\theta - i\sin\theta.$$

$z^{-1} = \frac{1}{z}$ $=$

Re $(\cos\theta + i\sin\theta)$

So

$$\begin{aligned}\cos\theta &= \boxed{\text{Re}\, z = \frac{1}{2}(z+\overline{z})} = \boxed{\frac{1}{2}\left(z+\frac{1}{z}\right)} \\ \sin\theta &= \text{Im}\, z = \frac{1}{2i}(z-\overline{z}) = \frac{1}{2i}\left(z-\frac{1}{z}\right).\end{aligned}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$
$$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

Also $z^n = e^{in\theta} \longrightarrow$

$$\begin{aligned}z^n + z^{-n} &= \underbrace{(\cos n\theta + i\sin n\theta)}_{z^n} + \underbrace{(\cos n\theta - i\sin n\theta)}_{z^{-n}} \\ &= 2\cos n\theta\end{aligned}$$

used D.M.T.

∴ rearranging gives

$$\boxed{\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right).}$$

Similarly

$$\boxed{\sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)}$$

Finding Roots of Complex Numbers

Consider a number w, which is an n^{th} root of the complex number z. That is, if $w^n = z$, and hence we can write

$$\boxed{w = z^{1/n}.}$$

$z = re^{i\theta}$

We begin by writing in polar/mod-arg form

$$z = r\,(\cos\theta + i\sin\theta)\,.$$

hence

$$z^{1/n} = r^{1/n}(\cos\theta + i\sin\theta)^{1/n}$$

DMT

and then by DMT we have

$$z^{1/n} = r^{1/n}\left(\cos\frac{\boxed{\theta + 2k\pi}}{n} + i\sin\frac{\boxed{\theta + 2k\pi}}{n}\right) \quad k = 0, 1, \ldots.., n-1.$$

Any other values of k would lead to repetition.

nth root

This method is particularly useful for obtaining the $n-$ roots of unity. This requires solving the equation

$$z^n = 1.$$

There are only two real solutions here, $z = \pm 1$, which corresponds to the case of even values of n. If n is odd, then there exists one real solution, $z = 1$. Any other solutions will be complex. Unity can be expressed as

$$\boxed{1 = \cos 2k\pi + i \sin 2k\pi}$$

which is true for all $k \in \mathbb{Z}$. So $z^n = 1$ becomes

$$r^n \left(\cos n\theta + i \sin \left(n\theta \right) \right) = \cos 2k\pi + i \sin 2k\pi.$$

The modulus and argument for $z = 1$ is one and zero, in turn. Equating the modulus and argument of both sides gives the following equations

$$r^n = 1 \quad \text{and} \quad n\theta = 2k\pi$$

$$1^{1/n} = 1^{1/n}\left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}\right) \quad k = 0, \ldots, n-1$$

Therefore

$$\begin{aligned} z &= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 1 \\ &= \exp\left(\frac{2k\pi i}{n}\right) \quad k = 0, ..., n-1 \end{aligned}$$

$$\omega_n = e^{i\frac{2k\pi}{n}}$$

$k=0 \to 1$

$k=1 \quad \omega$

$k=2 \quad \omega^2$

$\vdots$

ω^{n-1}

If we set $\omega = \exp\left(\frac{2k\pi i}{n}\right)$ then the $n-$ roots of unity are $1, \omega, \omega^2, \ldots.., \omega^{n-1}$.

These roots can be represented geometrically as the vertices of an $n-$ sided regular polygon which is inscribed in a circle of radius 1 and centred at the origin. Such a circle which has equation given by $\boxed{|z| = 1}$ and is called the *unit disk.*

More generally the equation

$$|z - z_0| = R$$

represents a circle centred at z_0 of radius R. If $z_0 = \boxed{a + ib}$, then

$$\begin{aligned} |z - z_0| &= |(x, y) - (a, b)| \\ &= |(x - a) + i(y - b)| \end{aligned}$$

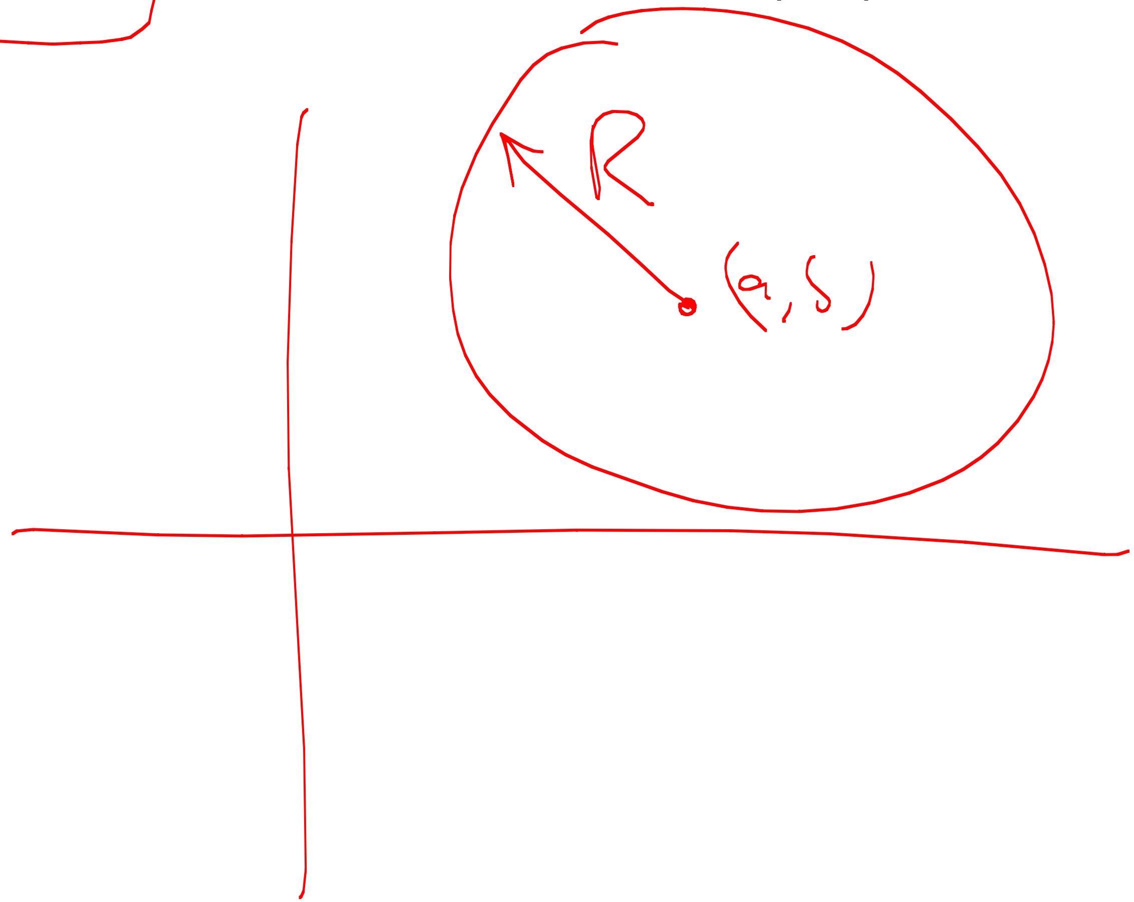
and

$|z - z_0| = R$

b

$$
\begin{aligned}
|(x-a) + i\,(y-b)|^2 &= R^2 \\
(x-a)^2 + (y-b)^2 &= R^2
\end{aligned}
$$

which is the cartesian form for a circle, centred at (a, b) with radius R.



$\boxed{|z| = 1}$

Applications

$$2\cos\theta = \left(z + \frac{1}{z}\right)$$

$$(2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4$$

Example 1

Calculate the indefinite integral $\int \cos^4\theta \, d\theta.$

$$2^4 \cos^4\theta =$$

We begin by expressing $\cos^4\theta$ in terms of $\cos n\theta$ (for different n).

$$\boxed{\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)} \Rightarrow 2^4\cos^4\theta = \left(z + \frac{1}{z}\right)^4 \therefore$$

$$\begin{aligned} 2^4\cos^4\theta &= z^4 + 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4} \text{ using Pascals triangle} \\ &= z^4 + 4z^2 + 6 + 4\frac{1}{z^2} + \frac{1}{z^4} \\ &= 2\frac{1}{2}\underbrace{\left(z^4 + \frac{1}{z^4}\right)} + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \end{aligned}$$

$$\left(z^n + \frac{1}{z^n}\right)$$

We know

$$\boxed{\tfrac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \cos n\theta}$$

$$2^4 \cos^4\theta = \underbrace{2.\tfrac{1}{2}\left(z^4 + \frac{1}{z^4}\right)}_{\cos 4\theta} + 4.\underbrace{2.\tfrac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}_{\cos 2\theta} + 6$$

hence

$$\begin{aligned} 2^4 \cos^4\theta &= 2\cos 4\theta + 8\cos 2\theta + 6 \\ \cos^4\theta &= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \therefore \end{aligned}$$

Now integrating

$$\begin{aligned} \int \cos^4\theta d\theta &= \frac{1}{8}\int (\cos 4\theta + 4\cos 2\theta + 3)\ d\theta \\ &= \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + K \end{aligned}$$

Example 2

As another application , express $\cos 4\theta$ in terms of $\cos^n \theta$.

We know from De Moivres theorem that

$$\cos 4\theta = \boxed{\text{Re}\left(\cos 4\theta + i\sin 4\theta\right)}$$

So

$$\cos 4\theta = \text{Re}\,\boxed{\left(\cos\theta + i\sin\theta\right)^4},$$

expand to power 4

and put $c \equiv \cos\theta$, $\quad is \equiv i\sin\theta$, to give

$$\begin{aligned}
\cos 4\theta &= \text{Re}\left(c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4\right)\\
\cos 4\theta &= \text{Re}\left(c^4 + i4c^3 s - 6c^2 s^2 - i4cs^3 + s^4\right)\\
\cos 4\theta &= \boxed{c^4 - 6c^2 s^2 + s^4}
\end{aligned}$$

$c^2 + s^2 = 1$

$s^2 = 1 - c^2$

Now $s^2 = 1 - c^2$, $\therefore$

$$\begin{aligned}\cos 4\theta &= c^4 - 6c^2\left(1 - c^2\right) + \left(1 - c^2\right)^2 = 8c^4 - 8c^2 + 1 \Rightarrow \\ \cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1.\end{aligned}$$

Example 3

Before doing the next example, consider the geometric series $\sum_{k=0}^{n} ar^k = a + ar + ar^2 + ... + ar^n$. The term r is called the *common ratio* and has a sum

$$a\frac{1 - r^{n+1}}{1 - r}$$

As this is a power series it will only converge if $|r| < 1$. As n becomes very large (i.e. infinite) this sum tends to the limiting value

$$\frac{a}{1 - r}.$$

Calculate

$$1 + \cos\theta + \cos 2\theta + \ldots\ldots + \cos n\theta$$

$$\sum_{m=0}^{n} \cos m\theta$$

Let $z = \exp(i\theta)$, then

$$\begin{aligned} \cos \theta &= \text{Re}\, z \\ \text{Re} \exp(i\theta)^n &= \text{Re}\,(z^n) \end{aligned}$$

Therefore the geometric series

$$S_n = \text{Re}\left(1 + z + z^2 + \ldots.. + z^n\right)$$

has a value $a = 1$ and common ratio z.

$$S_n = 1 + z + \text{———} + z^n$$

$$zS_n = z + \text{———} + z^{n+1}$$

$$(1-z)S_n = 1 - z^{n+1}$$

$$S_n = \frac{1 - z^{n+1}}{1 - z}$$

$$
\left.
\begin{aligned}
S &= \mathrm{Re}\left(\frac{z^{n+1}-1}{z-1}\right) \quad z \neq 1 \\
&= \mathrm{Re}\left(\frac{\exp\left(i\theta\left(n+1\right)\right)-1}{\exp\left(i\theta\right)-1}\right)
\end{aligned}
\right\} \text{replaced } z \text{ by } e^{i\theta}
$$

$$
\begin{aligned}
S &= \mathrm{Re}\left(\frac{\exp\left(i\theta\left(n+1\right)/2\right)\left(\exp\left(i\theta\left(n+1\right)/2\right)-\exp\left(-i\theta\left(n+1\right)/2\right)\right)}{\exp\left(i\theta/2\right)\left(\exp\left(i\theta/2\right)-\exp\left(-i\theta/2\right)\right)}\right) \\
&= \mathrm{Re}\left(\frac{\exp\left(in\theta/2\right)\left(\sin\left(n+1\right)\theta/2\right)}{\sin\theta/2}\right)
\end{aligned}
$$

$e^{+(\ )} \cdot e^{-(\ )}$

e^{0}

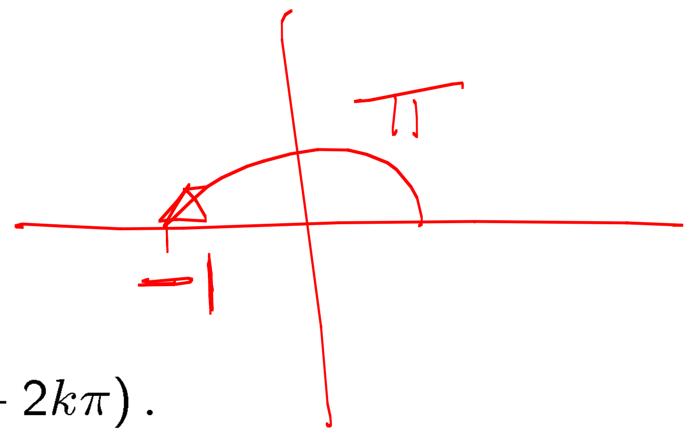
and hence

$$\boxed{S = \frac{\cos n\theta/2\left(\sin\left(n+1\right)\theta/2\right)}{\sin\theta/2}.}$$

Example 4

Find the square roots of -1 , i.e. solve $z^2 = -1$. The complex number -1

has a modulus of one and argument π, so



$$-1 = \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi) \, .$$

Hence,

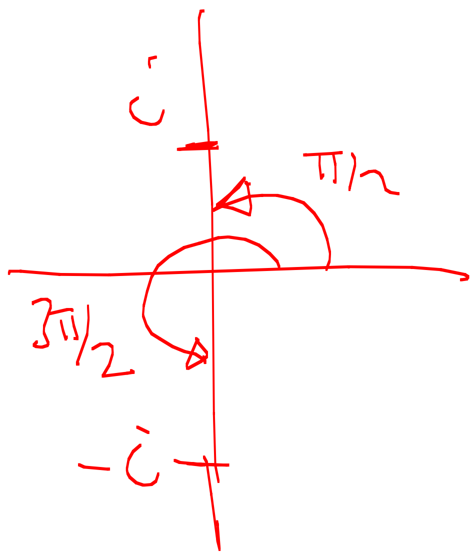
$$\begin{aligned} (-1)^{1/2} &= (\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi))^{1/2} \\ &= \underbrace{\cos\left(\frac{\pi + 2k\pi}{2}\right) + i \sin\left(\frac{\pi + 2k\pi}{2}\right)} \end{aligned}$$

DMT

for $k = 0, 1$:

$$(-1)^{1/2} = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = 0 + i$$

$$(-1)^{1/2} = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) = 0 - i$$



Therefore the square roots of -1 are $z_0 = i$ and $z_1 = -i$.

Example 5

Find the fifth roots of -1 , i.e. solve $z^5 = -1$. The complex number -1 has a modulus of one and argument π, so

$$\begin{aligned}(-1)^{1/5} &= \left(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\right)^{1/5} \\ &= \cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\end{aligned}$$

for $k = 0, 1, 2, 3, 4$:

$$z_0 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$$

$$z_1 = \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)$$

$$z_2 = \cos(\pi) + i\sin(\pi)$$

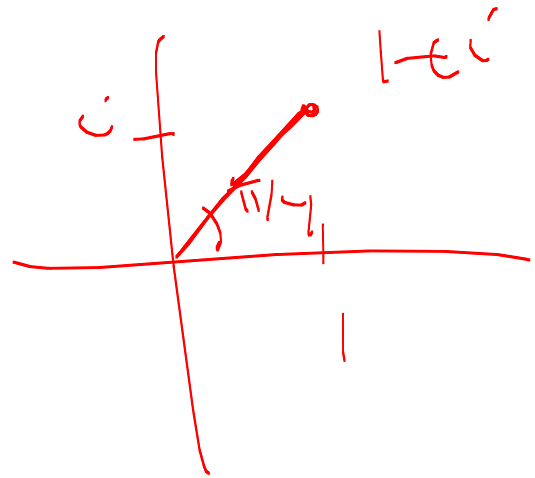
$$z_3 = \cos\left(\frac{7\pi}{5}\right) + i \sin\left(\frac{7\pi}{5}\right)$$

$$z_4 = \cos\left(\frac{9\pi}{5}\right) + i \sin\left(\frac{9\pi}{5}\right)$$

Example 6

Find all $z \in \mathbb{C}$ such that $z^3 = 1 + i$. So we wish to find the cube roots of $(1+i)$. The argument of this complex number is $\theta = \arctan 1 = \pi/4$. The modulus of $(1+i)$ is $r = \sqrt{2}$. We can express $(1+i)$ compactly in $r\exp(i\theta)$ as

$$1 + i = \sqrt{2}\exp\left(i\frac{\pi}{4}\right)$$



So

$$(1+i)^{1/3} = 2^{1/6}\exp\left(i\frac{\pi(8k+1)}{12}\right)$$

for $k = 0, 1, 2$.

$$z_0 = 2^{1/6} \exp\left(i \frac{\pi}{12}\right)$$

$$z_1 = 2^{1/6} \exp\left(i \frac{9\pi}{12}\right)$$

$$z_2 = 2^{1/6} \exp\left(i \frac{17\pi}{12}\right)$$

Function of a Complex Variable

Functions

Polynomial Functions: A polynomial function of z has the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + O\left(z^3\right) = \sum_{n=0}^{\infty} a_n z^n$$

and is on degree n. The domain is the set $\mathbb{C}$ of all complex numbers. So for example a 3rd degree polynomial is $2 - z + a_2 z^2 + 3z^3$.

Rational Functions: A rational function has the form

$$R(z) = \frac{P_1(z)}{P_2(z)}$$

where $P_1,\ P_2$ are polynomials. The domain is the set $\mathbb{C}-$zeroes of $P_2(z)$. For example

$$\begin{aligned} f(z) &= \frac{2z+3}{z^2-3z+2} \\ &= \frac{2z+3}{(z-1)(z-2)} \end{aligned}$$

$z \neq 1$
$z = +2$

Singularities

$f(z)$ is $\boxed{\text{irregular}}$ at $z = 1, 2$

and domain is $\mathbb{C} - \{1, 2\}$.

Exponential Function: $\exp(z) = e^z = e^{x+iy} = e^x e^{iy}$. $= e^x(\cos y + i \sin y)$

$\text{Re}\, e^z : u(x, y) = e^x \cos y$

$\text{Im}\, e^z : v(x, y) = e^x \sin y$

$|\exp z| = e^x$ and y is the argument.

$\lim_{\delta x \to 0}$

$\lim_{\delta z \to 0}$

Power Series:

$$
\begin{aligned}
\exp(\pm z) &= 1 \pm z + \frac{1}{2!}z^2 \pm \frac{1}{3!}z^3 + \ldots.. = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \\
\sinh z &= z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots.. = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\
\cosh z &= 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \ldots.. = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \\
\sin z &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \ldots.. = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\
\cos z &= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \ldots.. = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\end{aligned}
$$