Black Scholes Model - Exercises

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$\begin{aligned} d_1 &= \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(S/E) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\ N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-\phi^{\,2}/2)d\phi \end{aligned}$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$

the expiry date, $r \geq 0$ the interest rate, D is the dividend yield and σ is the volatility of S.

1. The Black–Scholes formula for a European call option $C(S,t)$ is given by

   $$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

   By differentiating with respect to S and σ show that the delta and vega are given by

   $$\Delta = \exp(-D(T-t))N(d_1), \quad \text{and} \quad \upsilon = \sqrt{\frac{T-t}{2\pi}}S\exp(-D(T-t))\exp(-{d_1}^2/2).$$

   You may find the following relationship useful:

   $$Se^{(-D(T-t))}\exp\left(-\frac{{d_1}^2}{2}\right) = Ee^{(-r(T-t))}\exp\left(-\frac{{d_2}^2}{2}\right)$$

   (It is quite messy to prove).

2. Given that S is defined by the SDE

   $$dS = a(S,t)\,dt + b(S,t)\,dW \tag{2.1}$$

   where a and b are given functions of S and t, show **using** Itô's lemma that any function $V(S,t)$ satisfies the SDE

   $$dV = \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial S} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}\right)dt + b\frac{\partial V}{\partial S}dW$$

   where we have assumed that all partial derivatives exist.
   Hence derive the partial differential equation

   $$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right) \tag{2.2}$$

for the fair price of an option based on a security S which satisfies (2.1) with r the risk-free interest rate.
Show (by substitution) that $V(S,t) = e^{-\alpha t} S^2$ is a solution of (2.2) provided

$$b^2 = (\alpha - r) S^2$$

and α is a constant.

3. The Black–Scholes formula for a European call option $C(S,t)$ is

$$C(S,t) = S \exp(-D(T-t)) N(d_1) - E \exp(-r(T-t)) N(d_2)$$

From this expression, find the Black–Scholes value of the call option in the following limits:

   a. (time tends to expiry) $t \to T^{-}$, $\sigma > 0$ (*this depends on* S/E);
   b. (volatility tends to zero) $\sigma \to 0^{+}$, $t < T$; (*this depends on* $S \exp(-D(T-t))/E \exp(-r(T-t))$)

4. Suppose S evolves according to the stochastic differential equation

$$dS = \mu S dt + S^{\alpha} dX$$

where μ and α are positive constants. Derive the corresponding Black–Scholes partial differential equation (PDE) for the option based upon this asset S (you are not required to solve any equation). Write this PDE in terms of the greeks.

5. The value of an option $V(S,t)$ satisfies the Black–Scholes equation. Write the option value in the form

$$V(S,t) = e^{(-r(T-t))} q(S,t).$$

Show that the function $q(S,t)$ satisfies the equation

$$\frac{\partial q}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 q}{\partial S^2} + (r-D) S \frac{\partial q}{\partial S} = 0.$$

Recall this is the backward Kolmogorov equation, used for calculating the expected value of stochastic quantities.