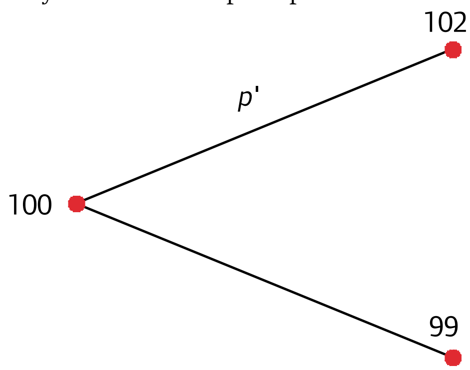
Lyceum

Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

Another Look at Risk-neutral probabilities

We can hardly stress enough the importance of the concept of risk neutrality, in the form of risk-neutral pricing and risk-neutral probabilities. It is by no means the end of the subject of pricing and hedging, indeed in some respects it clouds one's view of more realistic possibilities, but a thorough understanding is nevertheless a *sine qua non* for quantitative finance.

We have seen the idea of a binomial tree, how it is built up and how it is used to price options. Let's look at this in a slightly different way. I am going to give you the tree of asset prices and I want you to calculate option prices. Ok?



Suppose we are one day from expiry, the stock stands at 100, tomorrow the stock could be at either 102 or 99. A call option, with strike 100, expires tomorrow. Interest rates are zero. What is the price of the call?

We've seen how to set up a delta-hedged portfolio that is risk free, and thereby derived an algorithm for pricing the option. Recall that the option price does not depend on the real probability of the stock price rising or falling, but on a quantity called the risk-neutral probability, which we shall denote by p'.

There is a shortcut to finding the p'. Ask the question, what p' makes the expected stock price tomorrow the same as the stock price today? The answer is given by the solution of

$$102 \times p' + 99 \times (1 - p') = 100\,,$$

so $p' = 1/3$. This is the required risk-neutral probability. This probability is then used in the option price calculation. Simply calculate the expected option price at expiry using p' and, with zero interest rates, this is the option price today:

$$V = 2 \times \frac{1}{3} + 0 \times \frac{2}{3} = \frac{2}{3}.$$

Before discussing this idea and building up an entire tree, let's put interest rates back in. The idea is very similar.

Suppose that the interest rate applying at the present time is r, to find p' we must solve the problem: What p' makes the expected stock price tomorrow, a time dt away, the same as the forward stock price? In other words,

$$102 \times p' + 99 \times (1 - p') = 100 \times (1 + r\,dt).$$

So

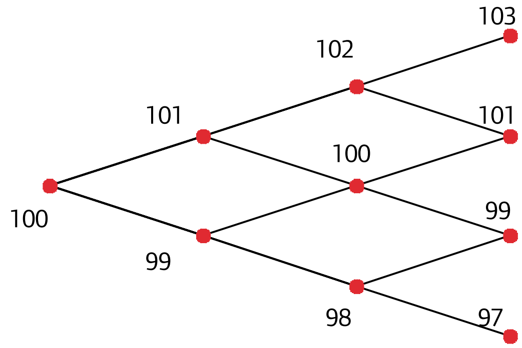
$$p' = \frac{100 \times (1 + r\,dt) - 99}{3}.$$

The final step is to use this probability to calculate the expected option price at expiry and then discount that amount at the risk-free rate:

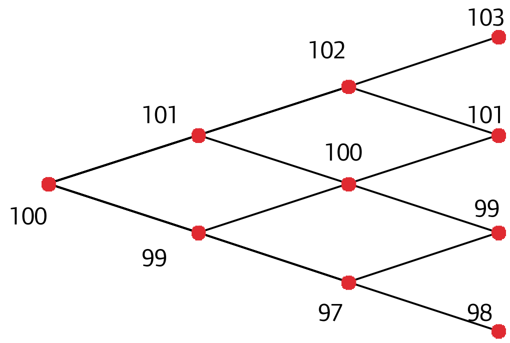
$$V = \frac{1}{1 + r\,dt}\left(2 \times \frac{100 \times (1 + r\,dt) - 99}{3} + 0 \times \frac{3 - 100 \times (1 + r\,dt) - 99}{3}\right).$$

Below is a very simple binomial tree. What are the probabilities associated with each branch?

In the absence of interest rates, it's obvious that all of the probabilities are 0.5.



What if I make the simple modification as below?



All that I've done is to swap around the two numbers at the bottom. But this has messed up the probabilities. There is no probability (i.e. a number from zero to one) such that we can go from 97 to either 98 or 99. In financial terms, this tree permits arbitrage.

When it comes to setting up the tree for pricing derivatives there are two things we must ensure. First that there are no arbitrage opportunities, and second that the branching captures the volatility that we want to model. There are many ways to set up the tree to satisfy these requirements, and you should re-read the earlier Lyceum where this is discussed in detail.