CHAPTER 13

BARRIER OPTIONS

1. **Check that the solution for the down-and-out call option, $V_{D/O}$, satisfies Black–Scholes, where**

$$V_{D/O}(S,t) = C(S,t) - \left(\frac{S}{S_d}\right)^{1-\frac{2r}{\sigma^2}} C(S_d^2/S,t),$$

and $C(S,t)$ is the value of a vanilla call option with the same maturity and payoff as the barrier option.

Hint: Show that $S^{1-\frac{2r}{\sigma^2}} V(X^2/S,t)$ satisfies Black–Scholes for any X, when $V(S,t)$ satisfies Black–Scholes.

Let $V(S,t)$ satisy Black–Scholes,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Consider

$$W(S,t) = S^\alpha V(X^2/S,t).$$

We set $\xi = X^2/S$, then $V = V(\xi,t)$ and

$$\begin{aligned}
\frac{\partial W}{\partial t} &= S^\alpha \frac{\partial V}{\partial t}, \\
\frac{\partial W}{\partial S} &= \alpha S^{\alpha-1} V - X^2 S^{\alpha-2}\frac{\partial V}{\partial \xi}, \\
\frac{\partial^2 W}{\partial S^2} &= \alpha(\alpha-1)S^{\alpha-2}V + \left(-\alpha X^2 S^{\alpha-3} - (\alpha-2)X^2 S^{\alpha-3}\right)\frac{\partial V}{\partial \xi} \\
&\qquad + X^4 S^{\alpha-4}\frac{\partial^2 V}{\partial \xi^2}.
\end{aligned}$$

Then

$$\frac{\partial W}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + rS\frac{\partial W}{\partial S} - rW = S^\alpha \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2$$

$$
\begin{aligned}
&\left(\alpha(\alpha-1)S^{\alpha-2}V + \left(-\alpha X^2 S^{\alpha-3} - (\alpha-2)X^2 S^{\alpha-3}\right)\frac{\partial V}{\partial \xi}\right.\\
&\qquad \left. + X^4 S^{\alpha-4}\frac{\partial^2 V}{\partial \xi^2}\right) + rS\left(\alpha S^{\alpha-1}V - X^2 S^{\alpha-2}\frac{\partial V}{\partial \xi}\right) - rS^{\alpha}V\\
&= S^{\alpha}\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2\xi^2\frac{\partial^2 V}{\partial \xi^2} + r\xi\frac{\partial V}{\partial \xi} - rV\right)\\
&\qquad + S^{\alpha}\left(\tfrac{1}{2}\sigma^2\alpha(\alpha-1)V + (1-\alpha)\sigma^2\xi\frac{\partial V}{\partial \xi} + r\alpha V - 2r\xi\frac{\partial V}{\partial \xi}\right)\\
&= S^{\alpha}\left(\tfrac{1}{2}\sigma^2(\alpha-1)+r\right)\left(\alpha V - 2\xi\frac{\partial V}{\partial \xi}\right),
\end{aligned}
$$

since $V(\xi,t)$ satisfies Black–Scholes. If

$$\alpha = 1 - \frac{2r}{\sigma^2},$$

then the right hand side equals zero and

$$\frac{\partial W}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 W}{\partial S^2} + rS\frac{\partial W}{\partial S} - rW = 0.$$

Since $C(S,t)$ satisfies Black–Scholes, the above working shows that

$$S^{1-\frac{2r}{\sigma^2}}C(X^2/S,t)$$

satisfies Black–Scholes. Therefore

$$V_{D/O}(S,t) = C(S,t) - \left(\frac{S}{S_d}\right)^{1-\frac{2r}{\sigma^2}} C(S_d^2/S,t)$$

also satisfies Black–Scholes.

2. Why do we need the condition $S_d < E$ to be able to value a down-and-out call by adding together known solutions of Black–Scholes equation (as in Question 1)? How would we value the option in the case that $S_d > E$?

We must synthesise the payoff of the barrier option. If $S_d < E$ then when $S > S_d$, we have $S_d^2/S < S_d < E$ and so the payoff from the synthesising portfolio is

$$C(S,T) - \left(\frac{S}{S_d}\right)^{1-\frac{2r}{\sigma^2}} C(S_d^2/S,T) = \max(S-E,0)$$

as required. On the other hand, if $S_d > E$, then the second term in this expression will have some non-zero payoff and we will not be able to synthesise the call payoff using these options at the same time as replicating the barrier condition.

If $S_d > E$, then we must solve the problem by reducing it to the diffusion equation. We will then have to solve

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

for $\tau > 0$ and $X_0 < x < \infty$, where $X_0 = \log(S_d/E)$. We then use the method of images to extend the problem to the entire x interval and use the fundamental solution of the diffusion equation to find the solution.

3. **Check the value for the down-and-in call option using the explicit solutions for the down-and-out call and the vanilla call option.**

We have

$$C_{BS}(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$
$$C_{D/O}(S,t) = C_{BS}(S,t) - \left(\frac{S}{S_d}\right)^{1-\frac{2r}{\sigma^2}} C_{BS}(S_d^2/S,t).$$

Now, a portfolio of a down-and-out call plus a down-and-in call is equivalent to a vanilla call, as we get the call payoff regardless of whether or not we hit the barrier. Consequently,

$$\begin{aligned} C_{D/I}(S,t) &= C_{BS}(S,t) - C_{D/O}(S,t) \\ &= C_{BS}(S,t) - \left(C_{BS}(S,t) - \left(\frac{S}{S_d}\right)^{1-\frac{2r}{\sigma^2}} C_{BS}(S_d^2/S,t)\right) \\ &= \left(\frac{S}{S_d}\right)^{1-\frac{2r}{\sigma^2}} C_{BS}(S_d^2/S,t). \end{aligned}$$

4. **Formulate the following problem for the accrual barrier option as a Black–Scholes partial differential equation with appropriate final and boundary conditions:**

The option has barriers at levels S_u and S_d, above and below the initial asset price, respectively. If the asset touches either barier before expiry then the option knocks out with an immediate payoff of $\Phi(T-t)$. Otherwise, at expiry the option has a payoff of $\max(S-E,0)$.

We must solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with final data

$$V(S,T) = \max(S-E,0),$$

and boundary conditions

$$V(S_d, t) = V(S_u, t) = \Phi(T - t).$$

5. **Formulate the following barrier option pricing problems as partial differential equations with suitable boundary and final conditions:**
   (a) **The option has barriers at levels S_u and S_d, above and below the initial asset price, respectively. If the asset touches both barriers before expiry, then the option has payoff** $\max(S - E, 0)$**. Otherwise the option does not pay out.**
   (b) **The option has barriers at levels S_u and S_d, above and below the initial asset price, respectively. If the asset price first rises to S_u and then falls to S_d before expiry, then the option pays out** $\$1$ **at expiry.**

   (a) If we hit the upper barrier first, we receive a contract which has call payoff if we hit a lower barrier (i.e. a down-and-in call). If we hit the lower barrier first, we receive a contract which has call payoff if we hit an upper barrier (i.e. an up-and-in call). If we hit neither barrier before expiry, then we will receive nothing. We can write this problem as:
   Solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with final data

$$V(S, T) = 0,$$

and boundary conditions

$$V(S_d, t) = V_1(S, t),$$
$$V(S_u, t) = V_2(S, t),$$

where V_1 is the solution of

$$\frac{\partial V_1}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + rS\frac{\partial V_1}{\partial S} - rV_1 = 0,$$

with final data

$$V_1(S, T) = 0,$$

and boundary conditions

$$V_1(S_u, t) = C_{BS}(S, t),$$
$$V_1(0, t) = 0,$$

and V_2 is the solution of

$$\frac{\partial V_2}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + rS\frac{\partial V_2}{\partial S} - rV_2 = 0,$$

with final data

$$V_2(S, T) = 0,$$

and boundary conditions

$$V_2(S_d, t) = C_{BS}(S, t),$$
$$V_2(S, t) \to 0 \text{ as } S \to \infty.$$

i.e.

$$V_1(S, t) = C_{U/I}(S, t) \text{ and } V_2(S, t) = C_{D/I}(S, t).$$

(b) If we hit the upper barrier, then we receive a contract which has a payoff of 1 at expiry if we hit the lower barrier before expiry. This payoff has present value $e^{-r(T-t)}$ when we hit the lower barrier (at time t). Otherwise, we receive nothing. We can write this problem as:
Solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with final data

$$V(S, T) = 0,$$

and boundary conditions

$$V(S_u, t) = V_1(S, t),$$
$$V(0, t) = 0,$$

where V_1 is the solution of

$$\frac{\partial V_1}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + rS\frac{\partial V_1}{\partial S} - rV_1 = 0,$$

with final data

$$V_1(S, T) = 0,$$

and boundary conditions

$$V_1(S_d, t) = e^{-r(T-t)},$$
$$V_1(S, t) \to 0 \text{ as } S \to \infty.$$

6. Price the following double-knockout option: The option has barriers at levels S_u and S_d, above and below the initial asset price, respectively. The option has payoff \$1, unless the asset touches either barrier before expiry, in which case the option knocks out and has no payoff.

We must solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with final data

$$V(S, T) = 1,$$

and boundary conditions

$$V(S_d, t) = V(S_u, t) = 0.$$

We transform

$$S = e^x, \;\; t = T - \frac{2\tau}{\sigma^2}, \;\; V(S, t) = v(x, \tau),$$

to find

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv,$$

where $k = 2r/\sigma^2$, with initial data

$$v(x, 0) = 1,$$

and boundary conditions

$$v(X_1, \tau) = v(X_2, \tau) = 0,$$

where $X_1 = \log(S_d)$ and $X_2 = \log(S_u)$. We then transform

$$v = e^{\alpha x + \beta \tau} u(x, \tau),$$

where

$$\alpha = -\tfrac{1}{2}(k-1) \text{ and } \beta = -\tfrac{1}{4}(k+1)^2.$$

These transformations give us the following problem for u:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

for $\tau > 0$ and $X_1 < x < X_2$, with initial data

$$u(x, 0) = e^{\frac{1}{2}(k-1)x},$$

and boundary conditions

$$u(X_1, \tau) = u(X_2, \tau) = 0.$$

We now transform the problem onto the interval $-\pi$ to π so that we can solve it by a Fourier Sine series.

Set

$$x = \frac{X_1 + X_2}{2} - \frac{X_1 - X_2}{2\pi} y,$$

and consider

$$w(y, \tau) = u(x, \tau).$$

We must solve

$$\frac{\partial w}{\partial \tau} = K \frac{\partial^2 w}{\partial y^2},$$

where

$$K = \left(\frac{2\pi}{X_2 - X_1} \right)^2,$$

with initial data

$$w(y, 0) = e^{\frac{1}{2}(k-1)\left(\frac{X_1+X_2}{2} - \frac{X_1-X_2}{2\pi} y \right)},$$

and boundary conditions

$$w(-\pi, \tau) = w(\pi, \tau) = 0.$$

This problem has a solution of the form

$$w(y, \tau) = \sum_{n=1}^{\infty} c_n(\tau) \sin(ny).$$

Substituting into the differential equation, we find

$$\sum_{n=1}^{\infty} \frac{dc_n}{d\tau} \sin(ny) = K \sum_{n=1}^{\infty} -n^2 c_n \sin(ny).$$

Since the sin terms are orthogonal,

$$\frac{dc_n}{d\tau} = -Kn^2 c_n,$$

and

$$c_n(\tau) = a_n e^{-Kn^2 \tau},$$

for some a_n. Hence

$$w(y, \tau) = \sum_{n=1}^{\infty} a_n e^{-Kn^2 \tau} \sin(ny).$$

We use the initial data to find the a_n terms, since

$$w(y,0) = \sum_{n=1}^{\infty} a_n \sin(ny),$$

and find

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} w(y,0)\sin(ny)\,dy \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\frac{1}{2}(k-1)\left(\frac{X_1+X_2}{2}-\frac{X_1-X_2}{2\pi}y\right)}\sin(ny)\,dy. \end{aligned}$$

We have now found the solution for $w(y,\tau)$. Converting back into S and t variables, we find

$$u(x,\tau) = \sum_{n=1}^{\infty} a_n e^{-Kn^2\tau}\sin\left(n\left(\frac{2\pi}{X_1-X_2}\left(\frac{X_1+X_2}{2}-x\right)\right)\right),$$

and

$$V(S,t) = e^{\alpha\log S+2\beta(T-t)/\sigma^2}\sum_{n=1}^{\infty} a_n e^{-2Kn^2(T-t)/\sigma^2}$$
$$\sin\left(\frac{2\pi n}{X_1-X_2}\left(\frac{X_1+X_2}{2}-\log S\right)\right).$$

7. **Prove put-call parity for simple barrier options:**

$$C_{D/O} + CD/I - P_{D/O} - P_{D/I} = S - Ee^{-r(T-t)},$$

where $C_{D/O}$ is a European down-and-out call, $C_{D/I}$ is a European down-and-in call, $P_{D/O}$ is a European down-and-out put and $P_{D/I}$ is a European down-and-in put, all with expiry at time T and exercise price E.

A portfolio of a down-and-out plus a down-and-in is equivalent to the vanilla option, since we get the payoff regardless of whether or not we hit the barrier. Consequently,

$$C_{D/O} + C_{D/I} = C_{BS},$$
$$P_{D/O} + P_{D/I} = P_{BS}.$$

Then

$$\begin{aligned} C_{D/O} + C_{D/I} - P_{D/O} - P_{D/I} &= C_{BS} - P_{BS} \\ &= S - Ee^{-r(T-t)}, \end{aligned}$$

from put-call parity for vanilla options.

8. **Why might we prefer to treat a European up-and-out call option as a portfolio of a vanilla European call option and a European up-and-in call option?**

The vanilla European call and the up-and-in call both have a gamma of one sign, whereas the up-and-out call has a gamma which changes sign. As seen in Section 13.7, it is far easier to price options in practice (when we are not completely sure of the volatility) when the gamma is one-signed.