Value-at-Risk and ES Exercises

CQF

1. **Portfolio Risk warm up** – Consider a position of £5 million in a single asset X with daily volatility of 1%. What are the annualised and 10-day standard deviations? Using the Normal factor calculate 99%/10day VaR in money terms.

2. Now, consider a portfolio of two assets X and Y, £100,000 investment each. The daily volatilities of both assets are 1% and correlation between their returns is $\rho_{XY} = 0.3$. Calculate 99%/5day Analytical VaR for this portfolio.

3. **Fully Analytical VaR for Efficient Markets** – Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile* in order to derive the formula for VaR calculation,
$$\Pr\left(x \leq \mathrm{VaR}(X)\right) = 1 - c.$$

4. What about Expected Shortfall? The universal definition of ES in terms of expectations algebra is given as follows:
$$\begin{aligned} \mathrm{ES}_c(X) &= \mathbb{E}[\, X \mid X \leq \mathrm{VaR}_c(X)] \\ \mathrm{ES}_c(X) &= \frac{1}{1-c}\int_0^{1-c} \mathrm{VaR}_u(X)du \end{aligned}$$
The actual ES calculation formula will vary depending on the distribution of P&L X, a random variable. Derive ES calculation formula for the case of Normal Distribution using the result $\mathrm{VaR}(X) = \mu + \Phi^{-1}(1-c) \times \sigma$.

5. Let's figure out a few numbers for these efficient, 'elliptical' markets – the term means asset (market index) returns are Normally distributed or close. Consider the left tail.
   - What percentage of returns are outside 2σ from the mean?
   - What is an average tail loss? To compute mean of the tail, divide the first moment by the total probability mass of the tail (computed by CDF).

   Derive formula solutions and provide numerical answers for Standard Normal.
   Normal Distribution PDF $N(\mu, \sigma^2)$ is
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

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6. **The Tail Story...** Improvement to risk measures made by choosing a distribution more adequate to characterise the tail data. $\text{GPD}_{\xi,\beta}$ is a proven approximation for the losses $X = v$ being above the threshold u_0 (for the right tail). How much above? That is regulated by another variable y.

$$\Pr\left[(\text{Loss} - u_0) \le y \,|\, \text{Loss} > u_0\right] = \text{GPD}_{\xi,\beta}(y)$$

Extreme Value Theory shows that tails of the wide range of distribution share common properties. EVT Analytical VaR is known – it has been derived as inversion of $F(\text{VaR}) = c$, where $c = 99\%$ confidence, N_u is the number of exceedances among N observations, and $u = u_0$ is chosen threshold for loss, expressed as DV or percentage.

$$\text{VaR}_c = u + \frac{\beta}{\xi}\left[\left(\frac{N}{N_u}(1-c)\right)^{-\xi} - 1\right].$$

The useful expectation result below holds for exceedances of ANY random variable and is known as **mean excess function**,

$$e(u) = \mathbb{E}[X - u|X > u] = \frac{\beta + \xi(u - u_0)}{1 - \xi}.$$

Question: derive the formula for Expected Shortfall under EVT from its definition,

$$\text{ES}_c = \mathbb{E}[X|X \ge \text{VaR}_c].$$

7. **Back to ABC.** Assume three bonds A,B and C from, each has a face value of £1,000 payable at maturity and the independent probability of default 0.5%, when the loss is the face value in full.

   (a) For the portfolio equally invested in bonds A, B and C, the 99% VaR is £1,000. Explain this result.

   (b) Calculate the Expected Shortfall for the bond A only. Assume 1% tail and partial loss possible.

   (c) Calculate the Expected Shortfall of a portfolio *equally invested* in bonds A, B and C. Assume 1% tail and continuous distribution.

   (d) Compare results (b) and (c) to conclude whether ES is *sub-additive.*

8. **The practice** – What are two main numerical methods that support VaR Backtesting and Stress-testing in terms of generating rather than merely sampling asset returns? What are their drawbacks?

9. **A useful risk modelling technique...** Covariance matrix can be decomposed as $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$ by Cholesky method (presented in Credit Risk module). The result is a lower triangular matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Let $X_1(t)$ and $X_2(t)$ are two uncorrelated Wiener processes (orthogonal). How would you use the Cholesky result in order to construct two correlated processes?