CHAPTER 12

MULTI-ASSET OPTIONS

1. **N shares follow geometric Brownian motions, i.e.**

$$dS_i = \mu_i S_i \, dt + \sigma_i S_i \, dX_i,$$

for $1 \le i \le N$. The share price changes are correlated with correlation coefficients ρ_{ij}. Find the stochastic differential equation satisfied by a function $f(S_1, S_2, \dots, S_N)$.

Consider a smooth function $f(S_1, S_2, \dots, S_N)$. We apply Taylor's theorem to find the change in f over a small time step:

$$f(S_1 + \delta S_1, S_2 + \delta S_2, \dots, S_N + \delta S_N))$$
$$= f(S) + \sum_{i=1}^{N} \delta S_i \frac{\partial f}{\partial S_i} + \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \delta S_i \delta S_j \frac{\partial^2 f}{\partial S_i \partial S_j} + \dots$$

Substituting for δS_1 and δS_2 in the second term, and discarding terms of $O\left(\delta t^{3/2}\right)$ and smaller, we find that

$$\delta f = \sum_{i=1}^{N} \frac{\partial f}{\partial S_i} \delta S_i + \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j S_i S_j \frac{\partial^2 f}{\partial S_i \partial S_j} \delta X_i \delta X_j + O\left(\delta t^{3/2}\right).$$

As $\delta t \to 0$, replace δS_i by dS_i and $\delta X_i \delta X_j$ by $\rho_{ij} dt$ to find the stochastic differential equation satisfied by $f(S_1, S_2, \dots, S_N)$:

$$df = \sum_{i=1}^{N} \frac{\partial f}{\partial S_i} \delta S_i + \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 f}{\partial S_i \partial S_j} dt.$$

2. **Using tick-data for at least two assets, measure the correlations between the assets using the entirety of the data. Split the data in two halves and perform the same calculations on each of the halves in turn. Are the correlation coefficients for the first half equal to those for the second? If so, do these figures match those for the whole data set?**

Figure 12.1 shows a 60-day correlation time series for two assets. It is clear that the correlation is far from constant.

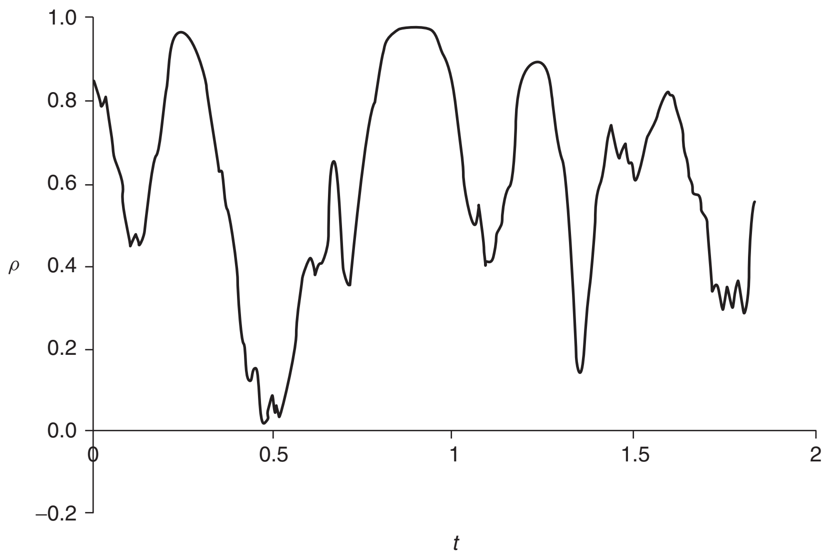

Figure 12.1 A correlation time series.

3. **Check that if we use the pricing formula for a non-path-dependent European option on dividend-paying assets, but for a single asset (i.e. in one dimension), we recover the solution found in Chapter 6:**

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty$$
$$\times\, e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\ \textbf{Payoff}(S')\frac{dS'}{S'}.$$

We have

$$V = e^{-r(T-t)}\,(2\pi(T-t))^{-d/2}\,(\mathrm{Det}\boldsymbol{\Sigma})^{-1/2}(\sigma_1\cdots\sigma_d)^{-1}$$
$$\int_0^\infty \cdots \int_0^\infty \frac{\mathrm{Payoff}(S_1'\cdots S_d')}{S_1'\cdots S_d'} \exp\left(-\frac{1}{2}\boldsymbol{\alpha}^T\ \boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\right) dS_1'\cdots dS_d',$$

where

$$\alpha_i = \tfrac{1}{\sigma_i(T-t)^{1/2}}\left(\log\left(\tfrac{S_i}{S_i'}\right) + \left(r - D_i - \tfrac{\sigma_i^2}{2}\right)(T-t)\right).$$

When $d = 1$,

$$\boldsymbol{\Sigma} = (\sigma) \text{ and}$$
$$\boldsymbol{\alpha} = \left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{S}{S'}\right) + (r - D - \tfrac{1}{2}\sigma^2)(T-t)\right)\right).$$

Then

$$
\begin{aligned}
V(S,t) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}}\frac{1}{\sigma}\int_0^\infty \frac{\text{Payoff}(S')}{S'}e^{-\frac{1}{2}\alpha^2}\,dS' \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty e^{-\left(\log(S/S')+\left(r-D-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} \\
&\qquad \times\ \text{Payoff}(S')\frac{dS'}{S'},
\end{aligned}
$$

where we have included an underlying dividend yield.

4. Set-up the following problems mathematically (i.e. what equations do they satisfy and with what boundary and final conditions?) The assets are correlated.

(a) An option that pays the positive difference between two share prices S_1 and S_2 and which expires at time T.
(b) An option that has a call payoff with underlying S_1 and strike price E at time T only if $S_1 > S_2$ at time T.
(c) An option that has a call payoff with underlying S_1 and strike price E_1 at time T if $S_1 > S_2$ at time T and a put payoff with underlying S_2 and strike price E_2 at time T if $S_2 > S_1$ at time T.

(a) We must find $V(S_1, S_2, t)$ where

$$
\frac{\partial V}{\partial t} + \tfrac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i\partial S_j} + \sum_{i=1}^{2} rS_i\frac{\partial V}{\partial S_i} - rV = 0,
$$

with final condition

$$
V(S_1, S_2, T) = |S_1 - S_2|,
$$

and boundary conditions

$$
\begin{aligned}
V(S_1, 0, t) &= S_1, \\
V(0, S_2, t) &= S_2, \\
V(S_1, S_2, t) &\sim S_1 - S_2 \text{ as } S_1 \to \infty, \text{ when } S_1 > S_2, \\
V(S_1, S_2, t) &\sim S_2 - S_1 \text{ as } S_2 \to \infty, \text{ when } S_2 > S_1.
\end{aligned}
$$

(b) We must find $V(S_1, S_2, t)$ where

$$
\frac{\partial V}{\partial t} + \tfrac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i\partial S_j} + \sum_{i=1}^{2} rS_i\frac{\partial V}{\partial S_i} - rV = 0,
$$

with final condition

$$V(S_1, S_2, T) = \mathcal{H}(S_1 - S_2)\max(S_1 - E, 0),$$

and boundary conditions

$$V(S_1, 0, t) = C_{BS}(S_1, t; E, T),$$
$$V(0, S_2, t) = 0,$$
$$V(S_1, S_2, t) \sim S_1 \text{ as } S_1 \to \infty, \text{ when } S_1 > S_2,$$
$$V(S_1, S_2, t) \to 0 \text{ as } S_2 \to \infty, \text{ when } S_2 > S_1.$$

(c) We must find $V(S_1, S_2, t)$ where

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i\partial S_j} + \sum_{i=1}^{2} rS_i\frac{\partial V}{\partial S_i} - rV = 0,$$

with final condition

$$V(S_1, S_2, T) = \mathcal{H}(S_1 - S_2)\max(S_1 - E_1, 0) + \mathcal{H}(S_2 - S_1)\max(E_2 - S_2, 0),$$

and boundary conditions

$$V(S_1, 0, t) = C_{BS}(S_1, t; E_1, T),$$
$$V(0, S_2, t) = \mathcal{H}(S_2)P_{BS}(S_2, t; E_2, T),$$
$$V(S_1, S_2, t) \sim S_1 \text{ as } S_1 \to \infty, \text{ when } S_1 > S_2,$$
$$V(S_1, S_2, t) \to 0 \text{ as } S_2 \to \infty, \text{ when } S_2 > S_1.$$

5. What is the explicit formula for the price of a quanto which has a put payoff on the Nikkei Dow index with strike at E and which is paid in yen. $S_\$$ is the yen-dollar exchange rate and S_N is the level of the Nikkei Dow index. We assume

$$dS_\$ = \mu_\$ S_\$\,dt + \sigma_\$ S_\$\,dX_\$$$

and

$$dS_N = \mu_N S_N\,dt + \sigma_N S_N\,dX_N,$$

with a correlation of ρ.

The value of the quanto, W, satisfies

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma_N^2 S_N^2\frac{\partial^2 W}{\partial S_N^2} + S_N\frac{\partial W}{\partial S_N}\left(r_f - \rho\sigma_\$\sigma_N\right) - r_\$ V = 0,$$

with final data

$$W(S_N, T) = \max(E - S_N, 0).$$

This is the simple one-factor Black–Scholes equation. Pricing the quanto is equivalent to pricing a European put option using a dividend yield of

$$r_\$ - r_f + \rho\sigma_\$\sigma_N.$$

The value of the quanto is therefore

$$W(S_N, t) = P_{BS}(S_N, t; E, T),$$

with a risk-free interest rate of r and a dividend yield of $r_\$ - r_f + \rho\sigma_\$\sigma_N$.