Chapter 5

Elementary Stochastic Calculus

In all of these $X(t)$ is Brownian motion.

1. By considering $\mathbf{X^2(t)}$, show that

$$\int_0^t X(\tau)\,dX(\tau) = \tfrac{1}{2}X^2(t) - \tfrac{1}{2}t.$$

We use Itô's Lemma for a function $F(X(t))$:

$$dF = \frac{dF}{dX}dX + \tfrac{1}{2}\frac{d^2F}{dX^2}dt.$$

Note that

$$\int_0^t dF = [F]_0^t,$$

and let $F = X^2(t)$. Then

$$\int_0^t dF = X^2(t) - X^2(0) = X^2(t),$$

as $X(0) = 0$. Applying Itô's Lemma to F gives us that

$$dF = 2XdX + dt,$$

and therefore

$$\int_0^t (2X(\tau)dX(\tau) + d\tau) = X^2(t).$$

Rearranging, we find that

$$\int_0^t X(\tau)dX(\tau) = \tfrac{1}{2}X^2 - \tfrac{1}{2}t.$$

2. Show that

$$\int_0^t \tau dX(\tau) = tX(t) - \int_0^t X(\tau)\,d\tau.$$

We use Itô's Lemma for a function $F(X(t), t)$:

$$dF = \frac{dF}{dX}dX + \left(\frac{dF}{dt} + \frac{1}{2}\frac{d^2F}{dX^2}\right)dt.$$

Let $F = tX(t)$. Then

$$\int_0^t dF = tX(t).$$

Applying Itô's Lemma to F gives us that

$$dF = t\,dX + X\,dt,$$

and therefore

$$\int_0^t (\tau\,dX(\tau) + X(\tau)\,d\tau) = tX(t).$$

Rearranging, we find that

$$\int_0^t \tau\,dX(\tau) = tX(t) - \int_0^t X(\tau)\,d\tau.$$

3. Show that

$$\int_0^t X^2(\tau)\,dX(\tau) = \tfrac{1}{3}X^3(t) - \int_0^t X(\tau)\,d\tau.$$

Let $F = X^3(t)$. Then

$$\int_0^t dF = X^3(t) - X^3(0) = X^3(t),$$

as $X(0) = 0$. Applying Itô's Lemma to F gives us that

$$dF = 3X^2\,dX + 3X\,dt,$$

and therefore

$$\int_0^t (3X^2(\tau)\,dX(\tau) + 3X(\tau)\,d\tau) = X^3(t).$$

Rearranging, we find that

$$\int_0^t X^2(\tau)\,dX(\tau) = \tfrac{1}{3}X^3 - \int_0^t X(\tau)\,d\tau.$$

4. Consider a function $f(t)$ which is continuous and bounded on $[0, t]$. Prove integration by parts, i.e.

$$\int_0^t f(\tau)\,dX(\tau) = f(t)X(t) - \int_0^t X(\tau)\,df(\tau).$$

Let $F = f(t)X(t)$. Then

$$\int_0^t dF = f(t)X(t) - f(0)X(0) = f(t)X(t).$$

Applying Itô's Lemma to F gives us that

$$dF = f dX + X df,$$

and therefore

$$\int_0^t (f(\tau)\, dX(\tau) + X(\tau)\, df(\tau)) = f(t)X(t).$$

Rearranging, we find that

$$\int_0^t f(\tau)\, dX(\tau) = f(t)X(t) - \int_0^t X(\tau)\, df(\tau).$$

5. **Find $u(W,t)$ and $v(W,t)$ where**

$$dW(t) = u\; dt + v\; dX(t)$$

and

(a) $W(t) = X^2(t)$,
(b) $W(t) = 1 + t + e^{X(t)}$,
(c) $W(t) = f(t)X(t)$,

where f is a bounded, continuous function.

We use Itô's Lemma for a function $W(X(t), t)$:

$$dW = \frac{dW}{dX} dX + \left(\frac{dW}{dt} + \tfrac{1}{2} \frac{d^2W}{dX^2} \right) dt.$$

(a)

$$dW = 2X\, dX + dt = 2\sqrt{W}\, dX + dt.$$

Therefore

$$u(W,t) = 2\sqrt{W} \text{ and } v(W,t) = 1.$$

(b)

$$dW = e^{X(t)}\, dX + \left(1 + e^{X(t)}\right) dt.$$

Rearranging the formula for $W(t)$, we find that

$$e^{X(t)} = W(t) - 1 - t,$$

and so

$$dW = (W(t) - 1 - t)\, dX + (W(t) - t)\, dt.$$

Therefore

$$u(W, t) = W(t) - 1 - t \text{ and } v(W, t) = W(t) - t.$$

(c)

$$dW = f dX + X\frac{df}{dt}dt.$$

Therefore

$$u(W, t) = f(t) \text{ and } v(W, t) = \frac{W(t)}{f(t)}\frac{df}{dt}.$$

6. If S follows a lognormal random walk, Use Itô's lemma to find the differential equations satisfied by

(a) $f(S) = AS + B$,
(b) $g(S) = S^n$,
(c) $h(S, t) = S^n e^{mt}$,

where A, B and n are constants.

Itô's lemma for a function $f(S)$ is

$$df = \sigma S\frac{df}{dS}dX + \left(\mu S\frac{df}{dS} + \tfrac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}\right) dt.$$

Then

(a) $df = A\sigma S dX + A\mu S dt = A dS.$
(b) $dg = n\sigma S^n dX + nS^n \left(\mu + \tfrac{1}{2}(n-1)\sigma^2\right) dt.$
(c) $dh = n\sigma S^n e^{mt} dX + S^n e^{mt} \left(m + n\mu + \tfrac{1}{2}n(n-1)\sigma^2\right) dt.$

7. If $dS = \mu S\, dt + \sigma S\, dX$, use Itô's lemma to find the stochastic differential equation satisfied by $f(S) = \log(S)$.

Itô's lemma for a function $f(S)$ is

$$df = \sigma S\frac{df}{dS}dX + \left(\mu S\frac{df}{dS} + \tfrac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}\right) dt.$$

Now

$$\frac{df}{dS} = 1/S \text{ and } \frac{d^2 f}{dS^2} = -1/S^2,$$

so

$$df = \sigma dX + \left(\mu - \tfrac{1}{2}\sigma^2\right) dt.$$

Note that this stochastic differential equation for $\log(S)$ has constant coefficients. For this reason, S is described as satisfying a lognormal random walk.

8. **The change in a share price satisfies**

$$dS = A(S,t)dX + B(S,t)dt,$$

for some functions A, B, what is the stochastic differential equation satisfied by $f(S,t)$? Can A, B be chosen so that a function $g(S)$ has a zero drift, but non-zero variance?

We could use Itô's Lemma directly to answer this and the following question, but as a teaching aid, will derive the results informally from first principles.

We apply Taylor's theorem to find the change in f over a small time step, $f(S+\delta S, t+\delta t)$:

$$f(S+\delta S, t+\delta t) = f(S,t) + \frac{\partial f}{\partial S}\delta S + \frac{\partial f}{\partial t}\delta t + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\delta S^2 + \frac{\partial^2 f}{\partial S \partial t}\delta S \delta t + \tfrac{1}{2}\frac{\partial^2 f}{\partial t^2}\delta t^2 + \dots$$

Substitute for $\delta S = \sigma S \delta X + \mu S \delta t$ to get

$$\delta f = (A\delta X + B\delta t)\frac{\partial f}{\partial S} + \delta t\frac{\partial f}{\partial t} + \tfrac{1}{2}(A^2\delta X^2 + B^2\delta t^2 + 2AB\delta X\delta t)\frac{\partial^2 f}{\partial S^2}\delta S^2 + (A\delta X\delta t + B\delta t^2)\frac{\partial^2 f}{\partial S\partial t} + \tfrac{1}{2}\delta t^2\frac{\partial^2 f}{\partial t^2} + \dots$$

Discarding terms of $O\left(\delta t^{3/2}\right)$ and smaller,

$$\delta f = A\frac{\partial f}{\partial S}\delta X + \left(B\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t}\right)\delta t + \tfrac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\delta X^2 + O\left(\delta t^{3/2}\right).$$

As $\delta t \to 0$, replace δt by dt, δX by dX and δX^2 by dt to find the stochastic differential equation satisfied by $f(S,t)$:

$$df = A\frac{\partial f}{\partial S}dX + \left(B\frac{\partial f}{\partial S} + \tfrac{1}{2}A^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt.$$

A function $g(S)$ will therefore satisfy the equation

$$dg = A\frac{dg}{dS}dX + \left(B\frac{dg}{dS} + \tfrac{1}{2}A^2\frac{d^2 g}{dS^2}\right)dt.$$

For $g(S)$ to have a zero drift but non-zero variance, we require that

$$B\frac{dg}{dS} + \tfrac{1}{2}A^2\frac{d^2g}{dS^2} = 0.$$

We can find a solution to this problem if A^2/B is independent of time.

9. Two shares follow geometric Brownian motions, i.e.

$$dS_1 = \mu_1 S_1\, dt + \sigma_1 S_1\, dX_1,$$
$$dS_2 = \mu_2 S_2\, dt + \sigma_2 S_2\, dX_2,$$

The share price changes are correlated with correlation coefficient ρ. Find the stochastic differential equation satisfied by a function $f(S_1, S_2)$.

We apply Taylor's theorem to find the change in f over a small time step - $f(S_1 + \delta S_1, S_2 + \delta S_2)$:

$$f(S_1 + \delta S_1, S_2 + \delta S_2) = f(S) + \delta S_1\frac{\partial f}{\partial S_1} + \delta S_2\frac{\partial f}{\partial S_2} + \tfrac{1}{2}\delta S_1^2\frac{\partial^2 f}{\partial {S_1}^2} + \delta S_1\delta S_2\frac{\partial^2 f}{\partial S_1\partial S_2} + \tfrac{1}{2}\delta S_2^2\frac{\partial^2 f}{\partial {S_2}^2} + \dots$$

Substituting for δS_1 and δS_2, and discarding terms of $O\left(\delta t^{3/2}\right)$ and smaller, we find that

$$\delta f = \sigma_1 S_1\frac{\partial f}{\partial S_1}\delta X_1 + \sigma_2 S_2\frac{\partial f}{\partial S_2}\delta X_2 + \mu_1 S_1\frac{\partial f}{\partial S_1}\delta t + \mu_2 S_2\frac{\partial f}{\partial S_2}\delta t + \tfrac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 f}{\partial {S_1}^2}\delta X_1^2 + \tfrac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 f}{\partial {S_2}^2}\delta X_2^2 + \sigma_1\sigma_2 S_1 S_2\frac{\partial^2 f}{\partial S_1\partial S_2}\delta X_1\delta X_2 + O\left(\delta t^{3/2}\right).$$

As $\delta t \to 0$, replace δt by dt, δX_1 by $dX1$, δX_2 by $dX2$, δX_1^2 by dt, δX_2^2 by dt and $\delta X_1\delta X_2$ by ρdt to find the stochastic differential equation satisfied by $f(S_1, S_2)$:

$$df = \sigma_1 S_1\frac{\partial f}{\partial S_1}dX_1 + \sigma_2 S_2\frac{\partial f}{\partial S_2}dX_2 + \left(\mu_1 S_1\frac{\partial f}{\partial S_1} + \mu_2 S_2\frac{\partial f}{\partial S_2} + \tfrac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 f}{\partial {S_1}^2} + \tfrac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 f}{\partial {S_2}^2} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 f}{\partial S_1\partial S_2}\right)dt.$$