# 5. The Feynman-Kač Formula

During our investigation of the martingale methods, we derived the fundamental asset pricing equation (8).

Since in the Black-Scholes problem we have assumed that the interest rate is constant, we can rewrite this formula as:

EMM

FAPF

$$\chi(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T] \tag{13}$$

Discounted CF = Payoff

It turns out that this type of expectation has a PDE representation, thanks to the **Feynman-Kač formula**.

## Key Fact (The Feynman-Kač Formula)

*Assume that* $V(t,s)$ *solves the boundary value problem*

Discounting

$$\frac{\partial V}{\partial t}(t,s) + \mu(t,s)\frac{\partial V}{\partial s}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 V}{\partial s^2}(t,s) - rV(t,s) = 0$$

state

$$V(T,s) = G(s) \qquad (14)$$

TC

*and that the process* $S(t)$ *follows the dynamics*

state

$$dS_t = \mu(t,S_t)dt + \sigma(t,S_t)dX(t)$$

*where* $X(t)$ *is a Brownian motion. Then, the function* $V$ *has the representation*

$$V(t,S_t) = e^{-r(T-t)}\mathbb{E}\left[G(S_T)|\mathcal{F}_t\right] \qquad (15)$$

Payoff = TC

# Application

In the Black-Scholes model, the option value under the risk-neutral measure can be expressed as the expectation:

FAPF

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T]$$

where $S_t$ follows the dynamics:

$$dS_t = rS_t dt + \sigma S_t dX^{\mathbb{Q}}(t) \tag{16}$$

in which $X^{\mathbb{Q}}(t)$ is a Brownian motion under $\mathbb{Q}$.

By the **Feynman-Kač formula**, the value $V(t, S_t)$ of the option solves the boundary value problem

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$
$$V(T, s) = G(s)$$

which is the **Black-Scholes PDE**.

# A Few Remarks...

1. **the Feynman-Kač formula works both ways:** we can represent a PDE of the form (14) as an expectation, and we can represent an expectation of the form (15). However, since historically Feynman-Kač was established to represent PDEs as expectation, we generally quote the formula as the representation of a PDE;
2. **the Feynman-Kač formula is independent from the measure:** Feynman-Kač works for any measure and does not imply any change of measure. Indeed, in the previous slide, we have used Feynman-Kač in the measure of the expectation, i.e the risk-neutral measure. **Tip:** make sure that you are using the "correct" dynamics for the process $S(t)$ (i.e. the dynamics under the same measure as the expectation), otherwise the $\frac{\partial V}{\partial s}$ coefficient in the PDE will be wrong!

# A Few Remarks...

3. since we have not gone through the $\Delta$-hedging argument of the PDE method, we do not know that $\frac{\partial V}{\partial s}$ represents the number of stocks to be held to hedge/replicate an option.

Since interest rates are deterministic, the futures price is equal to the forward price and we can write:

$$f_t = F(t; T) = S_t e^{r(T-t)} \tag{17}$$

and in particular

$$f_0 = S_0 e^{rT}$$

Recall that

$$dS_t = \mu S_t dt + \sigma S_t dX, \quad S(0) = S_0$$

Applying Itô to the relationship (17), we can now express the dynamics of $f_t$ as

$$df_t = (\mu - r) f_t dt + \sigma f_t dX, \quad f(0) = S_0 e^{rT}$$

and thus we can see now that

$$\begin{aligned} \mu_f &= \mu - r \\ \sigma_f &= \sigma \end{aligned}$$

namely, the volatility of the futures is equal to the volatility of the spot and the drift of the futures is the discounted drift of the spot.

As a result we can also see clearly that the dynamics (SDE) for $f_t$ is of the same form as the dynamics (SDE) for $\frac{S_t}{B_t} = S_t^*$. In fact, we even have

$$f(t) = \frac{S(t)}{B(t)} e^{rT} = S^*(t) e^{rT}$$

where $T$ is fixed by the contract. This relationship reveals that the futures price is already a (naturally) discounted process. The immediate conclusion from this observation is that we will not need to consider the discounted futures price process. We can just go ahead with the futures price process as it stands.

We will now proceed as in the Black-Scholes model, first defining a self-financing futures strategy with equation

$$V_t(\phi) \quad = \quad V_0(\phi) + \int_0^t \phi_u^f df_u + \int_0^t \phi_u^B dB_u, \quad \forall t \in [0, T]$$

**Please note** that now, the trading strategy involves the futures and the risk-free asset. It does not involve the underlying asset $S$.

The martingale measure $\mathbb{Q}$ is still unique, with the process $\theta$ used in the change of measure defined as

$$\theta = \frac{\mu_f}{\sigma}$$

As expected, under the martingale measure $\mathbb{Q}$, the dynamics of $f_t$ is given by

$$df_t = \sigma f_t dX_t^{\mathbb{Q}}, \quad S_0 > 0$$

In addition, the no-arbitrage pricing equation (7) is still valid, and as a consequence, the value of a derivative is given by:

$$\chi_t = e^{-r(T-t)}\mathbb{E}\left[G(S_T)|\mathcal{F}_t\right]$$

Solving this equation leads us to Black's formula for a European call on a futures:

$$\chi(t, f_t) = e^{-r(T-t)}\left[f_t N(d_1) - EN(d_2)\right]$$

with $d_1$ and $d_2$ given by

$$d_1 = \frac{\ln\left(\frac{f_t}{E}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{f_t}{E}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$