Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

Differential Equations

You've seen the Black-Scholes equation, either in its Greek form

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$$

or in its differential form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Whichever form you've seen or prefer, this is just an example of a differential equation. To be precise it's an example of a partial differential equation (PDE). 'Partial' means that the dependent variable V (so called because it 'depends on' other variables) is a function of more than one independent variable, here S, the asset price and t, the time. There are many, many different types of differential equation, but the ones you are most likely to encounter in the financial world are usually similar in form to the Black-Scholes equation.

These equations are called 'differential equations' because they involve differences between quantities. That's what the curly d's are all about, the ∂s. A term such as

$$\frac{\partial V}{\partial t}$$

means the difference in V values divided by difference in t values. But that's not how we think of it. Instead think of it as the slope of the function V in the t direction. But what does 'slope' mean in this context?

The option value is a function of two variables, asset price S and time t. If it helps, think of V as being the height of a hill with the two variables being distances in a northerly and westerly directions. We're going to be looking at the slope of this mountain in each of the two directions, these will be sensitivities of the option price to changes in the asset and in time. These slopes or gradients are what you experience in your car when you see a sign such as '1-in-10 gradient.' That is precisely the same as a slope of 0.1.

Similarly

$$\frac{\partial V}{\partial S}$$

means the slope in the S direction. But what about

$$\frac{\partial^2 V}{\partial S^2}?$$

This can also be written as

$$\frac{\partial\left(\frac{\partial V}{\partial S}\right)}{\partial S}.$$

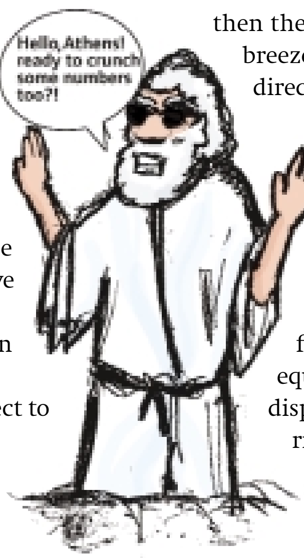
This can be interpreted as the slope in the S direction of...the slope in the S direction! The slope of the slope.

The Black-Scholes equation is a relationship between the height of the hill and the slopes, and slopes of slopes, in various directions. Readers of *The Hitchhikers Guide to the Galaxy* will recall the character who designed the fjords of Norway. Well, imagine another character who designed a Welsh hillside to satisfy the Black-Scholes equation.

The Black-Scholes PDE is a parabolic equation, meaning that it has a second derivative with respect to one variable, S, and a first derivative with respect to the other, t. Equations of this form are also known as heat or diffusion equations.

The equation, in its simplest form, goes back to almost the beginning of the 19th century. Diffusion equations have been successfully used to model

- diffusion of one material within another, smoke particles in air
- flow of heat from one part of an object to another
- chemical reactions, such as the Belousov-Zhabotinsky reaction which exhibits fascinating wave structure
- electrical activity in the membranes of living organisms, the Hodgkin–Huxley model
- dispersion of populations, individuals move both randomly and to avoid overcrowding
- pursuit and evasion in predator-prey systems
- pattern formation in animal coats, the formation of zebra stripes
- dispersion of pollutants in a running stream



In most of these cases the resulting equations are more complicated than the Black-Scholes equation.

The Black-Scholes equation can be accurately interpreted as a reaction-convection-diffusion equation. The basic diffusion equation is a balance of a first-order t derivative and a second-order S derivative. If these were the only terms in the Black-Scholes equation it would still exhibit the smoothing-out effect, that any discontinuities in the payoff would be instantly diffused away. The only difference between these terms and the terms as they appear in the basic heat or diffusion equation, is that the diffusion coefficient is a function of one of the variables S. Thus we really have diffusion in a non-homogeneous medium. The greater the amount of diffusion, or here the volatility, the faster the diffusion.

The first-order S-derivative term can be thought of as a convection term. If this equation represented some physical system, such as the diffusion of smoke particles in the atmosphere, then the convective term would be due to a breeze, blowing the smoke in a preferred direction.

The final term, $-rV$, is a reaction or absorption term. Think of this as the passive-smoking effect, atmospheric smoke is absorbed by the lungs.

Putting these terms together and we get a reaction-convection-diffusion equation. An almost identical equation would be arrived at for the dispersion of pollutant along a flowing river with absorption by the sand. In this, the dispersion is the diffusion, the flow is the convection, and the absorption is the reaction.