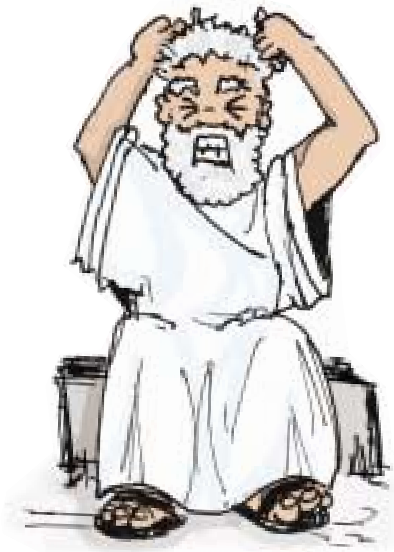
Lyceum

Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

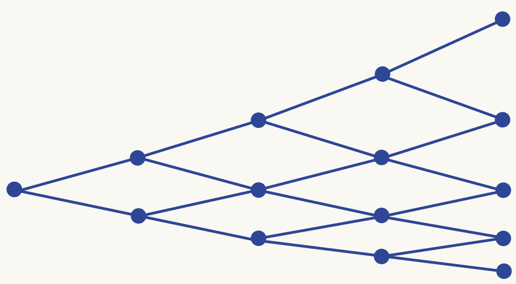
The Binomial Model III



The binomial tree

The binomial model allows the stock to move up or down a prescribed amount over the next time step. If the stock starts out with value S then it will take either the value uS or vS after the next time step. We can extend the random walk to the next time step. After two timesteps the asset will be at either u^2S, if there were two up moves, uvS, if an up was followed by a down or vice versa, or v^2S, if there were two consecutive down moves. Imagine extending this random walk out all the way until expiry. The result is the binomial tree.

THE BINOMIAL TREE



Observe how the tree bends due to the geometric nature of the asset growth.

Valuing back down the tree

Recall how in the last Lyceum we derived the simple algorithm for working back down t ree. This algorithm was

$$(1 + r\,dt)V = p'V^+ + (1 - p')V^-,$$

where

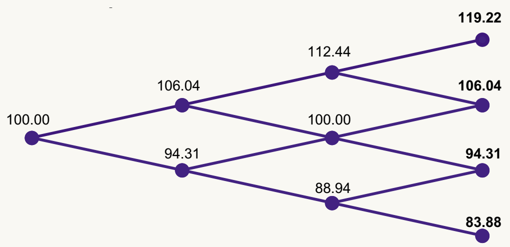
$$p' = \frac{1}{2} + \frac{r\sqrt{dt}}{2\sigma}.$$

In this r is the risk-free interest rate, dt is the time step, V is the option value at one node, V^+ and V^- are the options values at the next time step and σ is the asset volatility.

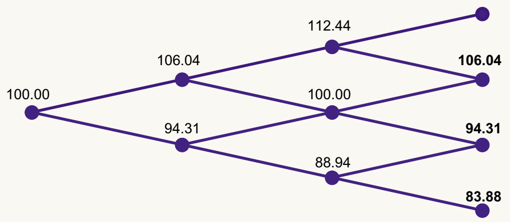
We know all of the parameter values and we also know V^+ and V^- at expiry, time T, because we know the option value as a function of the asset then, this is the payoff function.

If we know the value of the option at expiry we can find the option value at the time $T - dt$ for all values of S on the tree. But knowing these values means that we can find the option values one step further back in time. This idea is shown in the following two diagrams, one is the tree for the asset price and the other is the tree for the option vlues. We build up the former from left to right, with time increasing. The latter is built from right to left going backwards in time.

TREE OF ASSET PRICES



TREE OF OPTION PRICES



The continuous-time limit

Let's examine the pricing equation as the time step gets smaller and smaller. We'll end up with a partial differential equation, the famous Black-Scholes equation.

First of all, we have

$$u = 1 + \sigma\sqrt{dt}$$

and

$$v = 1 - \sigma\sqrt{dt}.$$

Next we write

$$V = V(S,t), \quad V^+ = V(uS, t + dt)$$

and $V^- = V(vS, t + dt)$

Expanding these expressions in Taylor series for small dt and substituting into the pricing algorithm we find that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

This is the well-known Black-Scholes partial differential equation.