Lyceum

Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

No Arbitrage and Risk Neutrality in Horse Racing I

Previously in the Lyceum…

A few Lyceums ago we saw how the absence of arbitrage opportunities led to the idea of risk-neutral pricing. The value of an option can be interpreted as the present value of the expected payoff, with the expectation being with respect to the risk-neutral asset price path. In this context risk-neutral just means that the asset price increases with a growth rate that is the same as the risk-free interest rate. In other words, what we really believe that the asset price is going to do in the future (in terms of its growth rate) is irrelevant. We don't even need to know the growth rate of an asset to price its options, only its volatility.

Something related happens in the world of sports betting.

Setting the odds in a sporting game

When it's a horse race, football or baseball game the odds are set not to reflect the real probabilities of a horse or a team winning but to reflect the betting that has occurred. Depending on how the betting goes, the odds will be set so that the House/Bookie cannot lose. For example, in a soccer match between England and Germany the Germans are more likely to win, but the patriotic English will bet more heavily on England (presumably). The odds given by the bookies will reflect this betting and make it look like England is more likely to win.

Of course, in Germany the situation is reversed. The best bet might be on Germany, but placed in England, and one on England placed in Germany!

In practice bookies in one country would lay off their bets on bookies in other countries so all bookies have roughly the same odds. Otherwise there would be straightforward arbitrage opportunities.

The odds will be set so that the House/Bookie cannot lose

In practice it's unlikely for there to be a sure-fire bet (unless the bookie has made a mistake, the race is fixed, or you can find two or more bookies that aren't directly or indirectly laying off their bets on each other).

But you can win, on average. By exploiting the difference between the real probability of a horse winning and the odds you can get. (There are differences between real odds and what you get paid in all casino games, but it's only in Blackjack that this can be exploited.)

The mathematics

Suppose that there are N horses in a race, with an amount W_i bet on the ith horse. The odds set by the bookie are $q_i : 1$. This means that if you bet 1 on horse i you will lose the 1 if the horse loses, but will take home $q_i + 1$ if the horse wins, your original 1 plus a further q_i. How does the bookie set the odds to ensure he never loses?

The total takings before the race is

$$\sum_{i=1}^{N} W_i.$$

If horse j wins the bookie has to pay out

$$(q_j + 1)W_j.$$

All that the bookie has to do is to ensure that

$$\sum_{i=1}^{N} W_i \geq (q_j + 1)W_j,$$

Nothing too complicated.

But see how the odds have been chosen to reflect the betting. Nowhere was there any mention of the likelihood of horse j winning!

Arbitrage

Suppose the bookie made an error when setting the odds. How could you determine whether there was an arbitrage opportunity? (Don't forget that only positive bets are allowed, there's no going short here!)

Let's introduce some more notation. The w_i are the bets that you place. (We can forget about the wagers made by everyone else, the Ws.) Let's assume that your total wager is 1, so that

$$\sum_{i=1}^{N} w_i = 1. \qquad (1)$$

The amount you win is

$$(q_j + 1).w_j - 1 \qquad (2)$$

if horse j is the winner.

Can you find w_i for all i such that they add up to one, are all positive and that expression (2) is positive for all j? If you can there is an arbitrage opportunity.