Chapter 18

Interest Rate Derivatives

1. Write down the problem we must solve in order to value a puttable bond.

The puttable bond satisfies

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0,$$

with

$$V(r, T) = 1,$$

and

$$V(r, t_c^-) = V(r, t_c^+) + K_c,$$

across coupon dates. If the bond can be sold back for an amount $P(t)$ then we have the constraint on the bond's value

$$V(r, t) \geq P(t),$$

together with continuity of $\partial V/\partial r$.

2. Derive a relationship between a floorlet and a call option on a zero-coupon bond.

A floorlet has the following cashflow

$$\tau \max(r_f - r_L, 0).$$

This is received at time t_i but the rate r_L is set at t_{i-1}, and $\tau = t_i - t_{i-1}$. This cashflow is exactly the same as the cashflow

$$\frac{\tau}{1 + \tau r_L}\max(r_f - r_L, 0)$$

received at time t_{i-1}. We can rewrite this cashflow as

$$\max\left(\frac{1 + \tau r_f}{1 + \tau r_L} - 1, 0\right).$$

But

$$\frac{1 + \tau r_f}{1 + \tau r_L}$$

is the price at time t_{i-1} of a bond paying $1+\tau r_f$ at time t_i. We can conclude that a floorlet is equivalent to a call option expiring at time t_{i-1} on a bond maturing at time t_i.

3. **How would a collar be valued practically? What is the explicit solution for a single payment?**

A collar is equivalent to long a cap and short a floor. Market practice gives the explicit value of a caplet as

$$e^{-r^*(t_i-t)}\left(F(t,t_{i-1},t_i)N(d_1')-r_cN(d_2')\right),$$

where $F(t,t_{i-1},t_i)$ is the forward rate today between t_{i-1} and t_i, r^* is the yield to maturity for a maturity of t_i-t,

$$d_1'=\frac{\log(F/r_c)+\frac{1}{2}\sigma^2(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}}\quad\text{and}\quad d_2'=d_1'-\sigma\sqrt{t_i-t_{i-1}}.$$

σ is the volatility of the (three-month) interest rate.

Similarly, the explicit value of a floorlet is

$$e^{-r^*(t_i-t)}\left(-F(t,t_{i-1},t_i)N(-d_1'')+r_fN(-d_2'')\right),$$

where

$$d_1''=\frac{\log(F/r_f)+\frac{1}{2}\sigma^2(t_i-t_{i-1})}{\sigma\sqrt{t_i-t_{i-1}}}\quad\text{and}\quad d_2''=d_1''-\sigma\sqrt{t_i-t_{i-1}}.$$

The explicit solution for a single collar payment is therefore

$$\begin{aligned}
&e^{-r^*(t_i-t)}\left(F(t,t_{i-1},t_i)N(d_1')-r_cN(d_2')\right)\\
&\quad-e^{-r^*(t_i-t)}\left(-F(t,t_{i-1},t_i)N(-d_1'')+r_fN(-d_2'')\right)\\
&=e^{-r^*(t_i-t)}\big(F(t,t_{i-1},t_i)N(d_1')-r_cN(d_2')\\
&\quad+F(t,t_{i-1},t_i)N(-d_1'')-r_fN(-d_2'')\big)\\
&=e^{-r^*(t_i-t)}\big(F(t,t_{i-1},t_i)(N(d_1')\\
&\quad+N(-d_1''))-r_cN(d_2')-r_fN(-d_2'')\big).
\end{aligned}$$

4. **When an index amortizating rate swap has a lockout period for the first year, we must solve**

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV=0,$$

with jump condition

$$V(r,P,t_i^-)=V(r,g(r,i)P,t_i^+)+(r-r_f)P,$$

where

$$g(r, i) = 1 \quad \textbf{if} \quad t_i < 1,$$

and with final condition

$$V(r, P, T) = (r - r_f)P.$$

In this case, reduce the order of the problem using a similarity reduction of the form

$$V(r, P, t) = PH(r, t).$$

We set $V(r, P, t) = PH(r, t)$, then

$$\frac{\partial V}{\partial t} = P\frac{\partial H}{\partial t},$$
$$\frac{\partial V}{\partial r} = P\frac{\partial H}{\partial r},$$

and

$$\frac{\partial^2 V}{\partial r^2} = P\frac{\partial^2 H}{\partial r^2}.$$

Substituting into the partial differential equation for V, we find

$$\frac{\partial H}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 H}{\partial r^2} + (u - \lambda w)\frac{\partial H}{\partial r} - rH = 0.$$

The jump condition becomes

$$H(r, t_i^-) = g(r, i)H(r, t_i^+) + (r - r_f),$$

and the final condition becomes

$$H(r, T) = r - r_f.$$

5. **Find the approximate value of a cashflow for a floorlet on the one month LIBOR, when we use the Vasicek model.**

The yield curve is given, for small maturities, by

$$-\frac{\log Z}{T - t} \sim r + \tfrac{1}{2}(u - \lambda w)(T - t) + \ldots \quad \text{as} \quad t \to T.$$

For the Vasicek model with one month Libor, we find

$$r_L \sim r + \tfrac{1}{2}(\eta - \gamma r)(1/12).$$

A floorlet cashflow therefore has approximate value

$$\max(r_f - r_L, 0) \sim \max\left(r_f - r - \frac{1}{24}(\eta - \gamma r), 0\right).$$