Lyceum

Aristotle's Lyceum is the institution considered to be the forerunner of the modern university. Opened in 335 BC, the Lyceum was a center of study and research in both science and philosophy.

No Arbitrage and Risk Neutrality in Horse Racing II

Previously in the Lyceum…

Let's continue with the search for an arbitrage opportunity in a horse race.

The w_i are the bets that you place. Let's assume that your total wager is 1, so that

$$\sum_{i=1}^{N} w_i = 1 . \qquad (1)$$

The amount you win is

$$(q_j + 1)w_j - 1 \qquad (2)$$

if horse j is the winner. (The qs are the bookie's odds as explained in the last issue.)

Can you find a w_i for all i such that they add up to one, are all positive and that expression (2) is positive for all j? If you can there is an arbitrage opportunity.

The requirement that (2) is positive can be written as

$$w_j \geq \frac{1}{q_j + 1} . \qquad (3)$$

Can we find positive ws such that (1) and (3) hold? This is very easy to visualize, at least when there are two or three horses. Let's look at the two-horse race.

In the figure the axes represent the amount of the wager on each of the two horses. The red line shows the constraint (1). The wagers must lie on this line. The two dots, one black and the other orange, mark the point

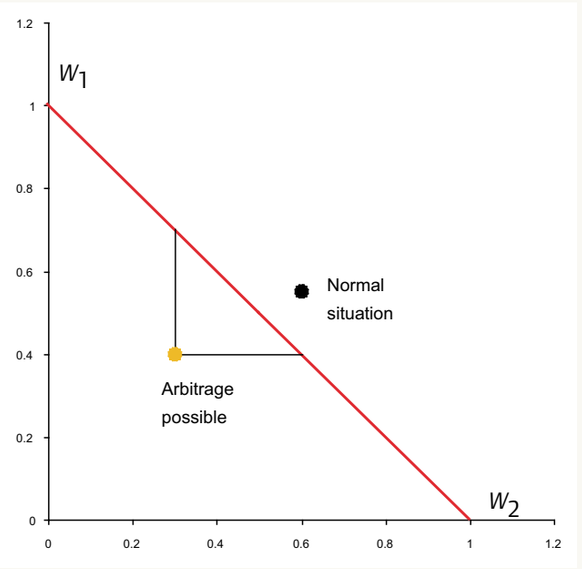
$$\left(\frac{1}{q_1 + 1}, \frac{1}{q_1 + 1} \right) \qquad (4)$$

in each of two situations. The black dot is the typical situation where there is no arbitrage opportunity and the orange dot does have an associated arbitrage opportunity. Let's see the details.

To find an arbitrage opportunity we must find a pair (w_1, w_2) lying on the red line such that each coordinate is greater than a certain quantity, depending on the qs. Plot the point (4) and draw a line vertically up, and another line horizontally to the right, as shown in the figure, emanating from the orange dot.

Does the quadrant defined by these two lines include any of the red line? If not, as would be the case with the black dot, then there is no arbitrage possible. If some of the red line is included then arbitrage is possible.

ARBITRAGE IN A TWO-HORSE RACE



How best to profit from the opportunity?

There's a simple test to see whether we are in a black dot or an orange dot situation. In general, if

$$\sum_{i=1}^{N} \frac{1}{q_i + 1} \geq 1$$

then there is no arbitrage. If the sum is less than one, there is an arbitrage.

You can benefit from the arbitrage by placing wagers wi such that they lie on the part of the red line encompassed by the quadrant. Which part of the red line, though, is up to you. By that I mean that you must make some statement about what you are trying to achieve or optimize in the arbitrage. One possibility is to look at the worst-case scenario and maximize the payback in that case. Alternatively, specify real probabilities for each of the horses winning

In the next Lyceum we'll see more about how to optimally place your bets.