CHAPTER 29

MONTE CARLO SIMULATION AND RELATED METHODS

1. **Simulate the risk-neutral random walk for an asset using a spreadsheet package, or otherwise. Use this data to calculate the value of a European call option.**

We can write the risk-neutral stochastic differential equation for S in the form

$$d(\log S) = \left(r - \tfrac{1}{2}\sigma^2\right) dt + \sigma \, dX.$$

This can be integrated exactly to give

$$S(t) = S(0) \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma \int_0^t dX\right).$$

Or, over a time step δt,

$$S(t + \delta t) = S(t) + \delta S = S(t) \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right) \delta t + \sigma \sqrt{\delta t}\, \phi\right).$$

We can then write

$$\text{option value } = e^{-r(T-t)} E\left[\text{payoff}(S)\right],$$

since the fair value of an option in the Black–Scholes world is the present value of the expected payoff at expiry under a risk-neutral random walk for the underlying.

Using this method, we can estimate the value of an option as follows:

(a) Simulate the risk-neutral random walk, starting at today's value of the asset S_0, over the required time horizon. This time period starts today and continues until expiry of the option. This gives one realization of the underlying price path.
(b) For this realization calculate the option payoff.
(c) Perform many more such realizations over the time horizon.
(d) Calculate the average payoff over all realizations.
(e) Take the present value of this average, this is the option value.

The above algorithm is illustrated in the spreadsheet below. The stock begins at time $t = 0$ with a value of 100, and has a volatility of 20%. The spreadsheet simultaneously calculates the values of a call and a put option. They both have an expiry of one year and a strike of 105. The interest rate is 5%.

	A	B	C	D	E	F	G	H	I
1	**Asset**	100		**Time**	**Sim1**	**Sim 2**	**Sim 3**	**Sim 4**	**Sim 5**
2	**Drift**	5%		0	100.00	100.00	100.00	100.00	100.00
3	**Volatility**	20%		0.01	100.30	102.34	97.50	98.38	96.14
4	**Timestep**	0.01		0.02	103.52	103.38	99.31	98.55	90.84
5	**Int.rate**	5%		0.03	106.01	101.98	101.44	96.12	92.94
6				0.04	101.85	106.67	103.02	94.69	93.17
7				0.05	104.75	107.17	105.05	93.25	91.23
8				0.06	106.57	110.24	109.55	94.50	89.97
9				0.07	101.15	107.73	110.77	93.83	81.08
10				[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible].21
11				0.09	101.06	111.06	107.49	93.86	93.23
12				0.1	103.35	108.60	109.52	92.84	93.82
94				0.92	109.27	87.76	118.48	69.38	116.43
95				0.93	110.22	87.59	118.13	72.19	118.66
96				0.94	105.56	88.18	117.37	73.40	117.15
97				0.95	106.53	86.81	118.27	70.86	115.82
98				0.96	103.14	85.72	116.85	69.36	115.81
99				0.97	102.20	85.85	112.89	70.24	119.18
100				0.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]75
101				0.99	101.91	87.63	[illegible]	[illegible]	[illegible]11
102				1	105.99	87.60	111.45	70.67	116.72
103									
104	**Strike**	105	**CALL**	**Payoff**	0.99	0.00	6.45	0.00	11.72
105			**Mean**	8.43					
106			**PV**	8.02					
107			**PUT**	**Payoff**	0.00	17.40	0.00	34.33	0.00
108			**Mean**	8.31					
109			**PV**	7.9					
110									
111									

= D3+$B$4

= E3*EXP(($B$5-0.5*$B$3*$B$3)*$B$4+$B$3*SQRT($B$4)*NORMSINV(RAND()))

= MAX($B$104-F102,0)

= MAX(G102-$B$104,0)

= AVERAGE(E104:IV104)

= D150* EXP(-$B$5*$D$102

Figure 29.1 Spreadsheet showing a Monte Carlo simulation to value a call and a put option.

In this spreadsheet we see a small selection of a large number of Monte Carlo simulations of the random walk for S, using a drift rate of 5%. The time step was chosen to be 0.01. For each realization the final stock price is shown in row 102 (rows 13–93 have been hidden). The option payoffs are shown in rows 104 and 107. The mean of all these payoffs, over all the simulations, is shown in row 105 and 108. In rows 106 and 109 we see the present values of the means, these are the option values.

2. **Modify the spreadsheet in the above to output the option's delta.**

   There are two possible approaches (at least) here.

   One is to run two sets of simulations to value the contract when the underlying is slightly higher than the current value, and another when the asset is slightly lower, and then interpolate to find the option value and use a difference to estimate the slope. Now this will only work if you use exactly the same set of random numbers for both of these calculations. Otherwise randomness will dominate your estimation of the delta.

The other method relies on the underlying being simple lognormal. Write the code to run simulations starting from the current asset value. And then when you get to expiration multiply the asset by 1.01, say, in one cell, and by 0.99, say, in another cell. This gives you two asset values at expiration for the one simulation. Of course, this is just like the asset starting at either 1.01 or 0.99 times its original value because of how the lognormal random walk scales with the original stock value. From the simulations using 1.01 and the simulations using 0.99 you can estimate the option's value at two *different* starting asset values and then calculate the delta by a simple difference method. This is a faster way of achieving the same idea as in the first approach above.

3. **Modify the above spreadsheet to value various exotic options such as barriers, Asians and lookbacks.**

   This is straightforward. As long as you can write down a formula for the payoff in terms of the asset path, even if it is a very complicated formula, then you can price the option by Monte Carlo simulation. The difficulty comes when there is no formula or function of the path, such as in the American option.

4. **Why is it difficult to use Monte Carlo simulations to value American options?**

   The problem is to do with the time direction in which we are solving. We have seen how it is natural in the partial differential equation framework to work backwards from expiry to the present. If we do this numerically then we find the value of a contract at every mesh point between now and expiry. This means that along the way we can ensure that there is no arbitrage, and in particular ensure that the early-exercise constraint is satisfied.

   When we use the Monte Carlo method in it basic form for valuing a European option we only ever find the option's value at the one point, the current asset level and the current time. We have no information about the option value at any other asset level or time. So if our contract is American we have no way of knowing whether or not we violated the early-exercise constraint somewhere in the future.

   In principle, we could find the option value at each point in asset-time space using Monte Carlo. For every asset value and time that we require knowledge of the option value we start a new simulation. But when we have early exercise we have to do this at a large number of points in asset-time space, keeping track of whether the constraint is violated. If we find a value for the option that is below the payoff then we mark this point in asset-time space as one where we must exercise the option. And then for every other path that goes through this point we must exercise at this point, if not before. Such a procedure is possible, but the time taken grows exponentially with the number of points at which we value the option.