CHAPTER 3

THE BINOMIAL MODEL

1. **Solve the three equations for u, v and p using the alternative condition $p = \frac{1}{2}$ instead of the condition that the tree returns to where it started, i.e. $uv = 1$.**

   We must solve

$$p = \tfrac{1}{2},$$
$$pu + (1 - p)v = e^{\mu\delta t},$$
$$pu^2 + (1 - p)v^2 = e^{(2\mu+\sigma^2)\delta t}.$$

   Substituting for p, we find

$$u + v = 2e^{\mu\delta t},$$
$$u^2 + v^2 = 2e^{(2\mu+\sigma^2)\delta t}.$$

   We substitute for v from the former equation into the latter, to find

$$2u^2 - 4ue^{\mu\delta t} + 2e^{2\mu\delta t}\left(2 - e^{\sigma^2\delta t}\right) = 0.$$

   We can solve this quadratic in u to find (since $u > 1$) that

$$u = e^{\mu\delta t}\left(1 + \sqrt{e^{\sigma^2\delta t} - 1}\right).$$

2. **Starting from the approximations for u and v, check that in the limit $\delta t \to 0$ we recover the Black–Scholes equation.**

   We approximate

$$u \sim 1 + \sigma\sqrt{\delta t},$$
$$v \sim 1 - \sigma\sqrt{\delta t}.$$

   We expand V^+ and V^- using Taylor's theorem to find

$$V^+ = V(uS, t + \delta t) \sim V + \sigma\sqrt{\delta t}S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2\delta t S^2\frac{\partial^2 V}{\partial S^2} + \delta t\frac{\partial V}{\partial t},$$
$$V^- = V(vS, t + \delta t) \sim V - \sigma\sqrt{\delta t}S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2\delta t S^2\frac{\partial^2 V}{\partial S^2} + \delta t\frac{\partial V}{\partial t}.$$

Then

$$V = \frac{V^+ - V^-}{u - v} + \frac{uV^- - vV^+}{(1 + r\delta t)(u - v)}$$
$$= \frac{2\sigma\sqrt{\delta t}\,S}{2\sigma\sqrt{\delta t}}\frac{\partial V}{\partial S} + \frac{(1 + \sigma\sqrt{\delta t})V^- - (1 - \sigma\sqrt{\delta t})V^+}{(1 + r\delta t)2\sigma\sqrt{\delta t}}.$$

Rearranging, we find

$$(1 + r\delta t)2\sigma\sqrt{\delta t}\,V = 2\sigma\sqrt{\delta t}\,S(1 + r\delta t)\frac{\partial V}{\partial S} + (V^- - V^+)$$
$$+ \sigma\sqrt{\delta t}(V^- + V^+),$$

and so

$$(1 + r\delta t)2\sigma\sqrt{\delta t}\,V = 2\sigma\sqrt{\delta t}\,S(1 + r\delta t)\frac{\partial V}{\partial S} - \left(2\sigma\sqrt{\delta t}\,S\frac{\partial V}{\partial S}\right)$$
$$+ 2\sigma\sqrt{\delta t}\left(V + \tfrac{1}{2}\sigma^2\delta t S^2\frac{\partial^2 V}{\partial S^2} + \delta t\frac{\partial V}{\partial t}\right),$$

which simplifies to

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

3. **A share price is currently \$80. At the end of three months, it will be either \$84 or \$76. Ignoring interest rates, calculate the value of a three-month European call option with exercise price \$79. You must use both the methods of setting up the delta-hedged portfolio, and the risk-neutral probability method.**

The binomial tree for the share price is

$$\begin{array}{ccc} & & 84 \\ & 80 & \\ & & 76 \end{array}$$

The binomial tree for the option price is

$$\begin{array}{ccc} & & 5 \quad (= \max(84 - 70, 0)) \\ C & & \\ & & 0 \quad (= \max(76 - 79, 0)) \end{array}$$

If we set up a Black–Scholes hedged portfolio, $C - \Delta S$, then the binomial tree for its value is

$$\begin{array}{ccc} & & 5 - 84\Delta \\ C - 80\Delta & & \\ & & -76\Delta \end{array}$$

For a risk-free portfolio, we choose Δ such that

$$5 - 84\Delta = -76\Delta,$$

i.e. $\Delta = 5/8$. Then, in absence of arbitrage,

$$C - 80\Delta = -76\Delta,$$

and $C = 2\frac{1}{2}$.

In the absence of interest rates the symmetry in this problem means that the risk-neutral probability of the stock rising is 0.5.

4. **A share price is currently \$92. At the end of one year, it will be either \$86 or \$98. Calculate the value of a one year European call option with exercise price \$90 using a single-step binomial tree. The risk-free interest rate is 2% p.a. with continuous compounding. You must use both the methods of setting up the delta-hedged portfolio, and the risk-neutral probability method.**

The binomial tree for the share price is

$$\begin{array}{ccc} & & 98 \\ & 92 & \\ & & 86 \end{array}$$

The binomial tree for the option price is

$$\begin{array}{cc} & 8 \quad (= \max(98 - 90, 0)) \\ C & \\ & 0 \quad (= \max(86 - 90, 0)) \end{array}$$

If we set up a Black–Scholes hedged portfolio, $C - \Delta S$, then the binomial tree for its value is

$$\begin{array}{cc} & 8 - 98\Delta \\ C - 92\Delta & \\ & -86\Delta \end{array}$$

For a risk-free portfolio, we choose Δ such that

$$8 - 98\Delta = -86\Delta,$$

i.e. $\Delta = 2/3$. Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$C - 92\Delta = e^{-0.02}(-86\Delta),$$

then

$$C = \tfrac{2}{3}\left(92 - 86e^{-0.02}\right) = 5.14.$$

The risk-neutral probability, p', of the stock rising comes from equating the present value of the expected risk-neutral asset with the current asset:

$$p'\,98\,e^{-0.02} + (1 - p')\,86\,e^{-0.02} = 92,$$

from which

$$p' = 0.655.$$

To get the option value just calculated expected risk-neutral payoff:

$$8p' + 0 \times (1 - p') = 5.239.$$

Now discount this to get the option value to be

$$5.239\,e^{-0.02} = 5.14$$

again.

5. **A share price is currently \$45. At the end of each of the next two months, it will change by going up \$2 or going down \$2. Calculate the value of a two-month European call option with exercise price \$44. The risk-free interest rate is 6% p.a. with continuous compounding. You must use both the methods of setting up the delta-hedged portfolio, and the risk-neutral probability method.**

The binomial tree for the share price is

$$\begin{array}{ccc} & & 49 \\ & 47 & \\ 45 & & 45 \\ & 43 & \\ & & 41 \end{array}$$

The binomial tree for the option price is

$$\begin{array}{ccll} & & 5 & (= \max(49 - 44, 0)) \\ & C_1 & & \\ C & & 1 & (= \max(45 - 44, 0)) \\ & C_2 & & \\ & & 0 & (= \max(41 - 44, 0)) \end{array}$$

We must first find the values of C_1 and C_2 before we can solve for C.

If we set up a Black–Scholes hedged portfolio, $C_1 - \Delta_1 S$, for C_1, then the binomial tree for its value is

$$\begin{array}{ccc} & & 5 - 49\Delta_1 \\ C_1 - 47\Delta_1 & & \\ & & 1 - 45\Delta_1 \end{array}$$

For a risk-free portfolio, we choose Δ_1 such that

$$5 - 49\Delta_1 = 1 - 45\Delta_1,$$

i.e. $\Delta_1 = 1$. Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$C_1 - 47\Delta_1 = e^{-0.06/12}(5 - 49\Delta_1),$$

then

$$C_1 = 47 - e^{-0.06/12}(44) = 3.22.$$

If we set up a Black–Scholes hedged portfolio, $C_2 - \Delta_2 S$, for C_2, then the binomial tree for its value is

$$\begin{array}{ccc} & & 1 - 45\Delta_2 \\ C_2 - 43\Delta_2 & & \\ & & -41\Delta_2 \end{array}$$

For a risk-free portfolio, we choose Δ_2 such that

$$1 - 45\Delta_2 = -41\Delta_2,$$

i.e. $\Delta_2 = 1/4$. Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$C_2 - 43\Delta_2 = e^{-0.06/12}(-41\Delta_2),$$

then

$$C_2 = 43/4 - e^{-0.06/12}(41/4) = 0.55.$$

We can now set up a Black–Scholes hedged portfolio, $C - \Delta S$, for C. The binomial tree for its value is

$$\begin{array}{ccc} & & C_1 - 47\Delta \\ C - 45\Delta & & \\ & & C_2 - 43\Delta \end{array}$$

For a risk-free portfolio, we choose Δ such that

$$C_1 - 47\Delta = C_2 - 43\Delta,$$

i.e. $\Delta = (C_1 - C_2)/4$. Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$C - 45\Delta = e^{-0.06/12}(C_1 - 47\Delta),$$

then

$$C = 45(C_1 - C_2)/4 - e^{-0.06/12}(C_1 - 47(C_1 - C_2)/4) = 2.03.$$

The risk-neutral probabilities of rising are for the first part of the tree 0.556, then 0.559 when the asset is at 47, or 0.554 when at 43. Now take expectations, working back down the tree. Alternatively, the risk-neutral probability of ending up at 49 is $0.556 \times 0.559 = 0.311$. The probability of ending up at 41 is similarly 0.198. So the probability of being at 45 is $1 - 0.311 - 0.198 = 0.491$. The option value is therefore

$$(0.311 \times 5 + 0.491 \times 1 + 0.198 \times 0)\, e^{0.01} = 2.03.$$

6. **A share price is currently \$63. At the end of each three-month period, it will change by going up \$3 or going down \$3. Calculate the value of a six-month American put option with exercise price \$61. The risk-free interest rate is 4% p.a. with continuous compounding. You must use both the methods of setting up the delta-hedged portfolio, and the risk-neutral probability method.**

The binomial tree for the share price is

$$\begin{array}{ccc} & & 69 \\ & 66 & \\ 63 & & 63 \\ & 60 & \\ & & 57 \end{array}$$

The binomial tree for the option price is

$$\begin{array}{ccl} & & 0 \quad (= \max(61-69,0)) \\ & P_1 & \\ P & & 0 \quad (= \max(61-63,0)) \\ & P_2 & \\ & & 4 \quad (= \max(61-57,0)) \end{array}$$

We must first find the values of P_1 and P_2 before we can solve for P.

If we set up a Black–Scholes hedged portfolio, $P_1 - \Delta_1 S$, for P_1, then the binomial tree for its value is

$$\begin{array}{ccc} & & -69\Delta_1 \\ P_1 - 66\Delta_1 & & \\ & & -63\Delta_1 \end{array}$$

For a risk-free portfolio, we choose Δ_1 such that

$$-69\Delta_1 = -63\Delta_1,$$

i.e. $\Delta_1 = 0$. Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$P_1 - 66\Delta_1 = e^{-0.04/4}(-69\Delta_1),$$

i.e.

$$P_1 = 0.$$

If we were to exercise early at this point, we would receive $\max(61 - 66, 0) = 0$, so there is no benefit in exercising early.

If we set up a Black–Scholes hedged portfolio, $P_2 - \Delta_2 S$, for P_2, then the binomial tree for its value is

$$\begin{array}{ccc} & & -63\Delta_2 \\ P_2 - 60\Delta_2 & & \\ & & 4 - 57\Delta_2 \end{array}$$

For a risk-free portfolio, we choose Δ_2 such that

$$-63\Delta_2 = 4 - 57\Delta_2,$$

i.e. $\Delta_2 = -2/3$. Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$P_2 - 60\Delta_2 = e^{-0.04/4}(-63\Delta_2),$$

then

$$P_2 = -40 + e^{-0.04/4}(42) = 1.58.$$

If we were to exercise early at this point, we would receive $\max(61 - 60, 0) = 1 < 1.58$, so there is no benefit in exercising early.

We can now set up a Black–Scholes hedged portfolio, $P - \Delta S$, for P. The binomial tree for its value is

$$\begin{array}{ccc} & & P_1 - 66\Delta \\ & P - 63\Delta & \\ & & P_2 - 60\Delta \end{array}$$

For a risk-free portfolio, we choose Δ such that

$$P_1 - 66\Delta = P_2 - 60\Delta,$$

i.e. $\Delta = -P_2/6$. Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$P - 63\Delta = e^{-0.04/4}(P_1 - 66\Delta),$$

then

$$P = 63\Delta + e^{-0.04/4}(-66\Delta) = 0.62.$$

If we were to exercise early at this point, we would receive $\max(61 - 63, 0) = 0 < 0.62$, so there is no benefit in exercising early and the value of the put option is 0.62.

The risk-neutral probabilities of rising are for the first part of the tree 0.606, then 0.611 when the asset is at 66, or 0.601 when at 60. Now take expectations, working back down the tree. Alternatively, the risk-neutral probability of ending up at 69 is $0.606 \times 0.611 = 0.370$. The probability of ending up at 57 is similarly 0.158. So the probability of being at 63 is $1 - 0.370 - 0.158 = 0.473$. The option value is therefore

$$(0.370 \times 0 + 0.473 \times 0 + 0.158 \times 4)\, e^{0.02} = 0.62.$$

7. A share price is currently \$15. At the end of three months, it will be either \$13 or \$17. Ignoring interest rates, calculate the value of a three-month European option with payoff $\max(S^2 - 159, 0)$, where S is the share price at the end of three months. You must use both the methods of setting up the delta-hedged portfolio, and the risk-neutral probability method.

The binomial tree for the share price is

$$\begin{array}{ccc} & & 17 \\ & 15 & \\ & & 13 \end{array}$$

The binomial tree for the option price is

$$\begin{array}{ccl} & & 130 \quad (= \max(17^2 - 159, 0)) \\ V & & \\ & & 10 \quad (= \max(13^2 - 159, 0)) \end{array}$$

If we set up a Black–Scholes hedged portfolio, $V - \Delta S$, then the binomial tree for its value is

$$\begin{array}{ccc} & & 130 - 17\Delta \\ V - 15\Delta & & \\ & & 10 - 13\Delta \end{array}$$

For a risk-free portfolio, we choose Δ such that

$$130 - 17\Delta = 10 - 13\Delta,$$

i.e. $\Delta = 30$. Then, in absence of arbitrage,

$$V - 15\Delta = 10 - 13\Delta,$$

and $V = 70$.

By symmetry the risk-neutral probability of rising is 0.5. The payoff if the stock rises is 130 or 10 if it falls. Therefore the option value is

$$0.5 \times 130 + 0.5 \times 10 = 70.$$

8. A share price is currently \$180. At the end of one year, it will be either \$203 or \$152. The risk-free interest rate is 3% p.a. with continuous compounding. Consider an American put on this underlying. Find the exercise price for which holding the option for the year is equivalent to exercising immediately. This is the break-even exercise price. What effect would a decrease in the interest rate have on this break-even price?

In order to find a non-zero solution to this problem, we must have $152 < E < 203$. Then the binomial tree for the share price is

$$\begin{array}{ccc} & & 203 \\ 180 & & \\ & & 152 \end{array}$$

The binomial tree for the option price is

$$\begin{array}{ccl} & & 0 \quad (= \max(E - 203, 0)) \\ P & & \\ & & E - 152 \quad (= \max(E - 152, 0)) \end{array}$$

If we set up a Black–Scholes hedged portfolio, $P - \Delta S$, then the binomial tree for its value is

$$\begin{array}{ccc} & & -203\Delta \\ P - 180\Delta & & \\ & & E - 152 - 152\Delta \end{array}$$

For a risk-free portfolio, we choose Δ such that

$$-203\Delta = E - 152 - 152\Delta,$$

i.e.

$$\Delta = \frac{152 - E}{51}.$$

Then, in absence of arbitrage, since the portfolio is risk-free, it must earn the risk-free rate and

$$P - 180\Delta = e^{-0.03}(-203\Delta),$$

then

$$P = \left(180 - 203e^{-0.03}\right)\left(\frac{152 - E}{51}\right).$$

If holding the option for the year is equivalent to exercising immediately, then

$$E - 180 = \left(180 - 203e^{-0.03}\right)\left(\frac{152 - E}{51}\right),$$

and $E = 129.33$.

A decrease in the interest rate would raise this exercise price (the benefit of the increased exercise price is offset by the decrease in interest rate).

9. A share price is currently \$75. At the end of three months, it will be either \$59 or \$92. What is the risk-neutral probability that the share price increases? The risk-free interest rate is 4% p.a. with continuous compounding.

The binomial tree for the share price is

$$\begin{array}{ccc} & & 92 \\ 75 & & \\ & & 59 \end{array}$$

The risk-neutral probability that the share price increases satisfies

$$92p + 59(1-p) = e^{0.04/4}75.$$

Then

$$p = \frac{75e^0.01 - 59}{33} = 0.51.$$