* $\frac{dS}{S} = \mu dt + \sigma dX$

$\Rightarrow$ no jumps or discontinuities

* $\sigma, r \in \mathbb{R}$ i.e. constant

Frictionless markets:

No limits to trading.

No transaction costs

No taxes

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dX$$

from Itô v. 4

$$\Delta = \frac{V^+ - V^-}{S^+ - S^-}$$

Subst in 1a

$$d\Pi = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dX$$

$$- \Delta \left(\mu S\, dt + \sigma S\, dX \right)$$

risk

$$\sigma S \frac{\partial V}{\partial S} - \Delta \sigma S = 0 \implies \boxed{\Delta = \frac{\partial V}{\partial S}}$$

$$\Pi = V - \Delta S \implies V = \Pi + \Delta S$$

riskless asset

part of a risky asset

Risky asset - stock

$T = 1$
$t = 0$

riskless asset - cash | bonds

$B_0 = 1$

$$\Pi = \phi S + \psi$$

$$dB_t = r B_t \, dt$$

$$\int_t^T \frac{dB_s}{B_s} = r \int_t^T ds$$

$$\log \frac{B_T}{B_t} = r(T-t)$$

$$\log \frac{B_T}{B_t} = r(T-t)$$

$T = 1$
$t = 0$

$$\frac{B_1}{B_0} = e^r \quad \therefore \quad \boxed{B_1 = e^r}$$

Call	Put
① Final (Payoff) Condition	① Final (Payoff) Condition
$C(S,T) = \max(S_T - E, 0)$	$P(S,T) = \max(E - S_T, 0)$
② B-Cs	② B-Cs
a) $S \to 0 \qquad C \to 0$	a) $S \to \infty \qquad P \to 0$
b) $S \to \infty \qquad C \sim S$	b) $S = 0$ (Put-Call parity)
	$C - P = S - E e^{-r(T-t)}$
	$C \to 0, \quad S \to 0$
	$P = E e^{-r(T-t)}$

B.S.M Model

① Assumptions

② Derived the PDE

③ Famous Nobel prize winning formula.

③ Unwind all the transf^n's done in ①

For ③ i.e solving the PDE

① Use transf^n's & subst to reduce the B.S.E to a 1D heat eq^n

② Solve using similarity reduction

$$\xi = \log s \quad ; \quad \frac{d\xi}{ds} = \frac{1}{s} \quad ; \quad \frac{d^2\xi}{ds^2} = -\frac{1}{s^2}$$

$$\frac{\partial}{\partial s} = \frac{d\xi}{ds}\frac{\partial}{\partial \xi} = \frac{1}{s}\frac{\partial}{\partial \xi}$$

$$\frac{\partial^2}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial \xi}\right) = \frac{1}{s}\underbrace{\frac{\partial}{\partial s}\left(\frac{\partial}{\partial \xi}\right)}_{\substack{\frac{d\xi}{ds}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \xi} \\ \frac{1}{s}\frac{\partial^2}{\partial \xi^2}}} - \frac{1}{s^2}\frac{\partial}{\partial \xi}$$

$$\frac{1}{s^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right)$$

$$\text{new} \begin{cases} x = \xi + (r - \tfrac{1}{2}\sigma^2)\tau & \text{old} \\ \tau = \tau \end{cases}$$

Use chain Rule II

$$\frac{\partial}{\partial \tau} = \frac{\partial \tau}{\partial \tau}\frac{\partial}{\partial \tau} + \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial \tau}{\partial \xi}\frac{\partial}{\partial \tau} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x}$$

$$C = S e^{-D\tau} N(d_1) - E e^{-r\tau} N(d_2)$$

ATMF : $E = S e^{(r-D)\tau}$

$$C = S e^{-D\tau} N(d_1) - S e^{(r-D)\tau - r\tau} N(d_2)$$

$$C = S e^{-D\tau} \left(N(d_1) - N(d_2) \right)$$

where

$$d_{1,2} = \frac{\log\left(\frac{\cancel{S}}{E}\right) + \left(r - D \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = \pm \frac{1}{2}\sigma\sqrt{\tau}$$

$\cancel{S} e^{(r-D)\tau}$

close to expiry

$$C \simeq S e^{-D(T-t)} \times 0.4 \sigma \sqrt{T-t}$$

Kendall & Stuart

$$N(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(x - \frac{x^3}{5} + O(x^5)\right)$$

$x \to 0$ $\quad N(x) \simeq \frac{1}{2} + \frac{x}{\sqrt{2\pi}}$

$N(x) - N(-x) = \frac{2x}{\sqrt{2\pi}}$ if $x = \frac{1}{2}\sigma\sqrt{\tau}$

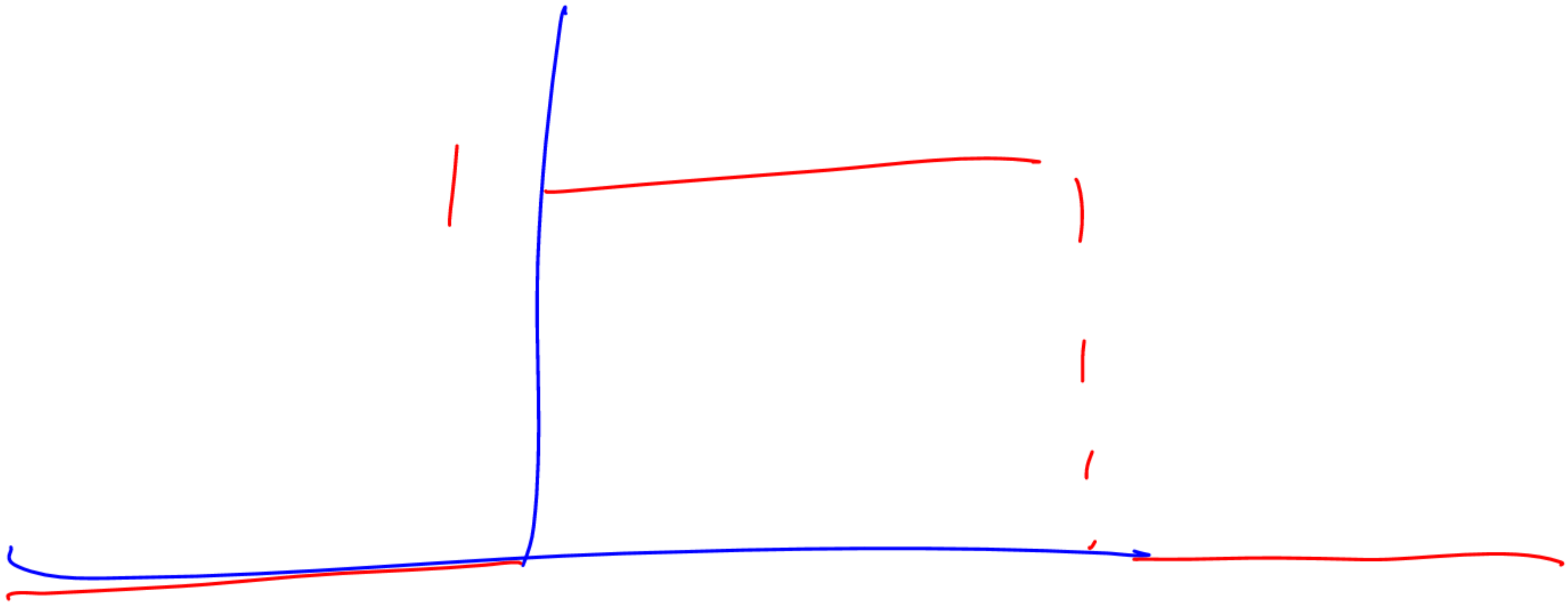
$$N(d_1) - N(d_2) = \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{2}\sigma\sqrt{\tau} = \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}$$

$$\simeq 0.4\,\sigma\sqrt{T-t}$$

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$



$V(S)$

$S \longrightarrow S + \delta S$

$$V(S+\delta S, t) = V(S, t) + \frac{\partial V}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\delta S^2$$

$$+ \cdots$$

$$\frac{\partial V}{\partial S} = \frac{V(S+\delta S, t) - V(S, t)}{\delta S} + O(\delta S)$$

$$= \Delta$$

$$C = S N(d_1) - E e^{-r(T-t)} N(d_2)$$

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) + S \underbrace{\frac{\partial}{\partial S} N(d_1)}_{}$$

$$\frac{\partial}{\partial S} N(d_1) = \frac{\partial N}{\partial d_1} \frac{\partial d_1}{\partial S}$$

$$\frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S} \left[\frac{(\log S - \log E)}{\sigma \sqrt{T-t}} + \frac{(r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \right]$$

$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial \xi}\right)$$

$$\boxed{\frac{\partial \xi}{\partial s}\frac{\partial}{\partial \xi}}\frac{\partial}{\partial \xi}$$

$$\frac{1}{s}$$

$$\boxed{\sigma} \, dX$$

$$\frac{\partial u}{\partial S} = \frac{\partial \xi}{\partial S} \frac{\partial u}{\partial \xi} = \frac{1}{S} \frac{\partial u}{\partial \xi}$$

X R.V

$$\frac{\partial}{\partial \tau}$$

price = I.V + T.V

$$\mathbb{E}\left[h(X) \right] = \int_{-\infty}^{\infty} \tilde{p}(x) \, h(x) \, dx$$