Module 1 Examination January 2014

Instructions

All questions must be attempted. Books and lecture notes may be referred to. Help from others is not permitted.

You may assume throughout this examination that dX_t is an increment in a standard Brownian motion X_t and up to Mean Square Convergence

$$\mathbb{E}\left[dX^2\right] = dt$$

Detailed working must be presented to obtain maximum credit. If a question asks for a particular method to be used, any other technique employed will result in the loss of marks.

Submitted work should be neat and easy to read by the tutor. Where a spreadsheet has been used, please submit this together with the relevant graphical results.

1. Using the expansion of $\sin(kx)$ when x is small, show that

$$\lim_{x \longrightarrow 0} \left(\frac{\alpha \sin(\beta x) - \beta \sin(\alpha x)}{x^2 \sin(\alpha x)} \right) \longrightarrow \frac{\beta}{6}\left(\alpha^2 - \beta^2\right)$$

   Note: You are not permitted to use L'Hospital's Rule at any stage. You may use any expansions without proof

2. Consider the function $z(x,y) = (x+y)\ln\left(\frac{x}{y}\right)$, where x and y are independent variables.

   a. Show (by substitution) that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

   b. By differentiating the expression in **a.**, find a relationship between $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$.

3. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu,$$

   for the function $u(x,t)$; where a and b are constants. By using a substitution of the form

$$u(x,t) = e^{\alpha x + \beta t} v(x,t),$$

   and suitable choice of constants α and β, show that the PDE can be reduced to the heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

4. The integral I_n is defined, for positive integers n, as

$$I_n = \int_0^\infty \left(1+x^2\right)^{-n} dx.$$

   Deduce that

$$I_n = 2n\left(I_n - I_{n+1}\right).$$

   Hence or otherwise show that

$$I_4 = \int_0^\infty \left(1+x^2\right)^{-4} dx = \frac{5\pi}{32}$$

5. A spot rate r, evolves according to the popular form

$$dr = u(r)\,dt + \nu r^\beta dX_t, \qquad (*)$$

   where ν and β are constants.
   Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation.
   Show that this implies that the drift structure of $(*)$ is given by

$$u(r) = \nu^2 \beta r^{2\beta - 1} + \frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}\left(\log p_\infty\right).$$

6. Consider the following Stochastic Differential Equation for the volatility σ,

$$d\sigma = a(\sigma,t)dt + b(\sigma,t)dX_t.$$

The drift and diffusion will be abbreviated to a and b respectively. The Forward Kolmogorov Equation, for the transition pdf $p = p(\sigma,t;\sigma',t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial \sigma'^2}(b^2 p) - \frac{\partial}{\partial \sigma'}(ap),$$

where the primed variables refer to future states. Derive the steady state solution given by

$$p_\infty(\sigma') = \frac{C}{b^2}\exp\left(\int \tfrac{2a}{b^2}d\sigma'\right),$$

where C is a constant. Any conditions used should be stated.
Note: You must solve the differential equation and then apply the three conditions.

7. The Ornstein-Uhlenbeck process is given by

$$dU_t = -\gamma U_t dt + \sigma dX_t,\ U_0 = u,$$

where γ, σ are constants. Solve this equation for U_t and hence write down $\mathbb{E}[U_t]$ in its simplest form.

8. Show that the following process is a martingale

$$\exp\left\{aX_t - \frac{1}{2}a^2 t\right\},$$

where a is a constant.

9. Implement the multi-step **Binomial Method** computationally to price a European put with the following parameters: strike $K = 100$ and maturity $T = 1$. Asset price level $S_0 = 100$ and interest rate $r = 0.05$.

   (a) For the constant number of time steps in the tree **NTS = 4**, calculate the value of the option for a range of volatilities and plot the result.

   (b) Then, fix the volatility at $\sigma = 0.2$ and plot the value of the option as a function of the number of time steps in the tree, **NTS** $=1, 2, \ldots, 50$. You will need a different tree for each NTS value.