Basic Mathematics Revision

A1. Consider the probability density function $f(x)$ given by

$$f(x) = \begin{cases} Ax^2 \exp\left(-\lambda x^2\right) & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Deduce that

$$A = 4\sqrt{\frac{\lambda^3}{\pi}}.$$

Show that

$$\mathbb{E}[X] = \frac{2}{\sqrt{\pi\lambda}}.$$

By using integration by parts, or otherwise, deduce that for $n = 0, 1, 2, ...$ the even moments of this distribution are given by

$$\mathbb{E}\left[X^{2n}\right] = \frac{1.3...(2n+1)}{(2\lambda)^n}$$

and the odd moments are given by

$$\mathbb{E}\left[X^{2n+1}\right] = \frac{2}{\sqrt{\pi}} \frac{(n+1)!}{\lambda^{(2n+1)/2}}.$$

A2. Given the probability density function $p(x)$

$$p(x) = \begin{cases} \mu \exp(-\mu x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\mu\,(> 0)$ is a constant, we know

$$\mathbb{E}[X^n] = \frac{n!}{\mu^n}, \qquad n = 0,\ 1,\ 2, ...$$

Calculate the *skew* and *kurtosis* for this distribution.

A3. The ordinary differential equation

$$\mu S \frac{du}{dS} + \tfrac{1}{2}\sigma^2 S^2 \frac{d^2u}{dS^2} = -1,$$

for the function $u(S)$ is to be **solved** with boundary conditions

$$\begin{aligned} u(S_0) &= 0 \\ u(S_1) &= 0. \end{aligned}$$

μ and σ are constants. Show that the solution is given by

$$u(S) = \frac{1}{\frac{1}{2}\sigma^2 - \mu}\left(\log(S/S_0) - \frac{1 - (S/S_0)^{1-2\mu/\sigma^2}}{1 - (S_1/S_0)^{1-2\mu/\sigma^2}} \log(S_1/S_0)\right)$$

Hint: When solving for the particular integral, assume a solution of the form $C \log S$, where C is a constant.