Certificate in Quantitative Finance
Final Project

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Abstract

This final project consists of two sub-projects: Basket Credit Default Swap (Section 1) and Interest Rate Derivatives (Section 2). Section 1 addresses the pricing of Basket Credit Default Swaps (a.k.a k-th-default Basket Crdit Default Swap). We will see that correlation plays an important role in the pricing of these products (these products are generally traded by the correlation desk); we will therefore deep dive in the estimation of the correlation matrix and other input parameter before covering the sampling by copulae (Gaussian copulae and Student's t copulae). Section 2 addresses the pricing of various IBOR[1] fixings linked derivatives (Bonds, Caps/Floors and swaptions) using the Heath–Jarrow–Morton framework. This framework relies on the estimation of historical forward rates volatility in order to simulate the evolution of the forward rates in the future. We will describe the principal component analysis (PCA) which will help us to identify the main factors which influance the volatility forward curve before covering the Monte Carlo simulation which projects the forward curve, compute the IBOR rates and price interest derivatives instruments.

Keywords: Basket Default Swaps, Monte Carlo method, Gaussian copulae, Student t copulae

[1] Interbank Offered Rate (IBOR) are daily reference rates based on the averaged interest rates at which banks offer to lend unsecured funds to other banks in interbank markets. LIBOR is the most famous reference but there are other (EURIBOR, SIBOR, etc.)

Contents

Todo

The report should cover the following items:

• A full mathematical description of the models employed as well as numerical methods. Include discussion and tests for accuracy. Inlcude discussion and tests for accuracy and convergence

• Results presented using a plenty of tables and figures, together with sensible interpretation

• Instructions on how to use the software if that's not obvious. The code must be thoroughly tested amd well documented

• Pros and cons of a particular models and its implementation, together with possible improvements

• Demonstrate "the specials" of you implementation, such as research, re-coding of numerical methods, using the "industrial-strength" libraries of C++, Python or NAG

1 Basket Credit Default Swap

1.1 Introduction

A basket default swap is like an insurance contract that offers protection against the event of the k^{th} default on a basket of n ($n \geq k$) reference names. It is similar to a plain vanilla credit default swap but the credit event to insure against is the event of the k^{th} default. The spread (premium) is paid as an insurance fee until maturity or the event of the k^{th} default in return for a compensation for the loss. If the k^{th} default occurs, protection payment We denote by $s^{k^{th}}$ the fair spread in a k^{th}-to-default swap, i.e, the spread making the value of this swap today equal to zero (Default leg = Premium leg). The most common type of basket default swaps is the first-to-default swap (FTDS), i.e $k = 1$ where the seller compensates the buyer for the loss on the reference name which defaults first.

We will study baskets with 5 reference names of 5 years maturity in order to keep the calculation fast; however the code that has been implemented can handle n references names.The appropriate default correlation and other input parameters are estimated from historical credit curves (probabilities of survival and hazard rates; historical credit spreads are used as input to bootstrap historical credit curves).The fair spread of the k^{th}-to-default swap is calculated as an expectation over the joint distribution of default times and will be computed by sampling from copulae (using pseudo-random and quasi-random numbers). We will explore 2 types of elliptical (Gaussian and Student t copulas) and compare the results we obtain. We use a LIBOR curve as discount curve for the historical credit curve bootstrapping (used to estimate input parameters), the hazard rates bootstrapping and the basket spread calculation.

1.2 Model and hypothesis

1.2.1 Poisson process

Definition 1. Default indicator function

To model the arrival of a credit event we need to model an unknow random point in time $\tau \in \mathbb{R}_+$. We define the default indicator function (indicator function of the default event) to model default risk

$$I_{Default}(t) = 1_{\{\tau > T\}} = \begin{cases} 1 & \text{if } \tau < T \\ 0 & \text{if } \tau \geq T \end{cases}$$

Definition 2. Default survival function

We define also the survival indicator function $I_{Survival}(t) = 1 - I_{Default}(t)$

$$I_{Survival}(t) = 1_{\{\tau > T\}} = \begin{cases} 1 & \text{if } \tau > T \\ 0 & \text{if } \tau \leq T \end{cases}$$

Definition 3. Poisson process

We use a Poisson process characterized by its parameter λ(intensity of the process) to describe the default time of a company. The default time can be viewed as the first jump of a Poisson process.

Based on the nature of the intensity function, the Poisson process can be classified as time homogeneous Poisson process or time inhomogeneous Poisson process.

- Poisson processes have no memory
- The inter-arrival times of Poisson processes ($\tau_{n+1} - \tau_n$) are exponentially distributed
- Two or more jumps at excately the same time have probability zero

Definition 4. Homogeneous Poisson process
A homogeneous Poisson process is defined as a process with stationary independant increments and initial value of zero. The times between two consecutive jumps are independently and identically distributed as an exponential random variable with mean $\frac{1}{\lambda}$ where λ is constant in time. If we define the default time τ as the first jump, the probability of an event can be expressed as:

$$P\{\tau \in [t, t+dt] | \tau \geq t\} = \lambda dt$$

Therefore, we can express the survival probability $P(t,T)$ between t and T as:

$$P(t,T) = \exp(-\lambda(T-t))$$

Definition 5. Inhomogeneous Poisson process
A inhomogeneous Poisson process is defined as a process with a non-negative deterministic and time varying intensity function $\lambda(t)$. In this case, we can derive the survival probability as:

$$P(t,T) = \exp\left(-\int_t^T \lambda(s)\, ds\right)$$

Remark. In this case the intensity (default hazard rate) depends on the time horizon T: we obtain a term structure of hazard rates when matching market prices.

1.2.2 Credit Default Swap pricing

There are two main families of model to price Credit Default Swap: the structural models and the reduced form models. Structural models are based on the complete knowledge of a detailed information set (e.g. Merton model 1974, Black and Cox 1976). In contrast, the reduced form approach (a.k.a. the "standard model" or JP Morgan approach in the literature) is based on the information set available in the market. In this section, we will describe the reduced form approach.

Definition 6. Mathematical setup
Let us suppose that there are N perdiods, indexed by $n = 1, \ldots, N$. Each perdiod is of length Δt,expressed in years. Therefore, time intervals are:

$$\{(0, \Delta t), (\Delta t, 2\Delta t), \ldots, ((N-1)\Delta t, N\Delta t)\}$$

The end of period can be expressed as:

$$T_n = n\Delta t$$

Definition 7. Premium leg
We denote the N-period CDS spread as S_N,stated as an annualized percentage of the nominal value of the contract. We set the nominal value to 1. We assume that defaults occur only at the end of the period, so the premiums will be paid until the end of the period. Since the premium payments are made as long as the reference security survives, the expected present value of the premium leg is:

$$PL_N = S_N \sum_{n=1}^{N} D(0,T_n)Q(T_n)(\Delta t_n)$$

where Δn is the year fraction corresponding to $T_{n-1} - T_n$ and $Q(T_n)$ is the survival probability up to time T_n. This accounts for the expected present value of payments made from the buyer to the seller.

Definition 8. Default leg

The expected loss payment in period n is based on the probability of default in period n, conditional on no default in a prior period. This probability is given by the probability of surviving until period $n-1$ and then defaulting in period n. Therefore, the expected present value of loss payments is:

$$DL_N = (1-R)\sum_{n=1}^{N} D(0,T_n)(P(T_{n-1}) - P(T_n))$$

Definition 9. Fair spread

The fair pricing of the $N-$period CDS, i.e. the fair quote of the spread S_N, must be such that the expected present value of payments made by buyer and seller are equal, i.e. $PL_N = DL_N$. Thus we obtain

$$S_N = \frac{(1-R)\sum_{n=1}^{N} D(0,T_n)(P(T_{n-1}) - P(T_n))}{\sum_{n=1}^{N} D(0,T_n)P(T_n)(\Delta t_n)}$$

1.2.3 Bootstrapping survival probailities and hazard rates

We have described previously how to calculate the spread S_n from the survival probabilities $P(T_n)$ with $n \in [1,N]$ and the discounting curve. The discounting curve is quoted in the market (bond markets, money market, etc.) but the survival probabilities are not quoted in the market. The participants in the Credit Default Swap market (Over-the-counter market) are quoting CDS spread instead (i.e. S_n are known but $P(T_n)$ are not known). This is a typical example of inverse problem that we encounter often in quantitative finance. As in other similar cases, we will calculate the survival probabilities $P(T_n)$ with $n \in [1,N]$ recursively. i.e. We will start by find $P(T_1)$ which will help us to find $P(T_2)$ which will help us to find $P(T_3)$, etc.

Definition 10. First step for survival probabilities

For $n = 1$, we obtain the following equation after a couple of algebraic manipulation:

$$\begin{aligned}
PL_1 &= DL_1 \\
S_1 D(0,T_1)P(T_1)\Delta t_1 &= (1-R)D(0,T_1)(P(T_0) - (P(T_1)) \\
&\dots \\
D(0,T_1)P(T_1)(S_1\Delta t_1 + L) &= LD(0,T_1)P(T_0) \\
P(T_1) &= \frac{L}{L + \Delta t_1 S_1}
\end{aligned}$$

with R the recovery rate and L the loss given defaut defined as $L = 1 - R$

Definition 11. Subsequent steps ($n = N$) for survival probabilities

The general expression is given by this formula

$$P(T_N) = \frac{\sum_{n=1}^{N-1} D(0,T_n)\left[(1-R)P(T_{n-1}) - ((1-R) + \Delta t_n S_N)P(T_n)\right]}{D(0,T_N)((1-R) + \Delta t_N S_N)} + \frac{P(T_{N-1})(1-R)}{((1-R) + \Delta t_N S_N)}$$

Remark. We can clearly see that$P(T_N)$ depends on $P(T_{N-1})$. This explains why we need to apply a bootsrapping process

Definition. Hazard rates (non-cumulative)

We can calculate the non-cumulative hazard rate λ_n from the two survival probabilities $P(T_n)$ and $P(T_{n-1})$. We have seen earlier the relationship between the survival probability and the hazard rate in the case of an inhomogeneous Poisson process. This is the same relationship is if we assume the hazard rate function to be piecewise constant:

$$P(0,T_n) = \exp(-\int_0^{T_n} \lambda_s \, ds) = \exp(-\sum_{i=1}^{n} \lambda_i \Delta t_i)$$

From the previous equation, we can deduce that

$$\begin{aligned} P(0,T_n) &= \exp(-\sum_{i=1}^{n-1} \lambda_i \Delta t_i - \lambda_n \Delta t_n) \\ &= \exp(-\sum_{i=1}^{n-1} \lambda_i \Delta t_i) \times \exp(-\lambda_n \Delta t_n) \\ &= P(0,T_{n-1}) \times \exp(-\lambda_n \Delta t_n) \end{aligned}$$

This automatically give the expression that expresses the hazard rate as the "log" difference of two survival probabilities:

$$\lambda_n = -\frac{1}{\Delta t} \log(\frac{P(0,T_n)}{P(0,T_{n-1})}) \tag{1.1}$$

1.2.4 Marginal default time// Converting uniform variables to exact default times

The calculation of the basket spread that is part of the sampling from copula procedure[2] requires default times. The procedure will first generate a vector of correlated uniform variable $(u_1, \ldots u_n)$ where n is the number of reference names in the basket CDS. It will then need to convert this vector of correlated uniform variables $(u_1, \ldots u_n)$ into a vector of default times. This section explains how the exact default time can be calculated using the hazard rates (non-cumulative) term strcuture of a given reference name.

For a Poisson process, the probability distribution of the waiting time until the next event (i.e. default in our case) is an exponential distribution. We can therefore use the Poisson CDF $(F(\tau, \lambda))$to convert τ to a probability $u \in [0,1]$:

$$u = F(\tau; \lambda_\tau) = \begin{cases} 1 - \exp(-\lambda_\tau \tau) & \text{when} \tau \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

By imversing the above equation, we obtain:

$$\tau = F^{-1}(u; \lambda_\tau) = -\frac{\log(1-u)}{\lambda_\tau} \text{for u} \in [0,1]$$

We have assumed hazard rate to piecewise constant when we bootstrapped the credit curve; therefore we don't have the exact value of λ_τ (hazard rate $\lambda$for a given value of τ). We will therefore need to apply the following procedure to estimate the exact default time τ for a given uniform variable u:

1. Find the year of default (i.e. find n so that the default occurs between $t_{n-1}$and t_n
2. Estimate the exact default time δt within the year of default

[2] Sampling from copula will be described in detail later in this document

Definition 12. Year of default

We iterate through the term structure of hazard rates by summing up λ_i i until the inequality begins to hold.

$$\tau = \inf\left\{t > 0 : \log(1-u) \geq -\sum_{i=1}^{t} \lambda_i \Delta t_i\right\}$$

Where inf is the matematical operator that represent the infinimum of a set

Remark. If the inequality holds after adding λ_n,then it means that $t_{n-1} < \tau < t_n$

Definition 13. Exact default time

We define the defaut time τso that $\tau = t_{n-1} + \delta t$ where τ_n is the year of default and is the "year fraction" within the year of default. From the previous formula, we can calculate δt as:

$$\delta t = -\frac{1}{\lambda_n} \log(\frac{1-u}{P(0, t_{n-1})})$$

1.2.5 Join distribution

The most important problem in the pricing of basket credit default swaps is the modelling of the joint default times. A marginal distribution of a random variable X is defined by its CDF:

$$F(x) = \Pr(X \leq x)$$

The joint distribution function of two random variable $X_1$and $X_2$is:

$$F(x_1, x_2) = \Pr(X_1 \leq x_1, X_2 \leq x_2)$$

The joint distribution of n random variables $X_1, \ldots, X_n$ is:

$$F(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

We can model the default risk of our basket CDS if the joint default distribution function is known:

$$F(t_1, \ldots, t_n) = \Pr(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n)$$

where $\tau_i \leq t_i$means a default event.

A multivariate joint distribution is a multidimensional analytical construction $(F(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n)$that is hard to work with especially in high dimension. The better approach to isolate dependence structure among multiple variables is a copula. Copula approach separates the joint distribution into two parts:

- marginal distributions for each variable
- and their dependance structure

The joint distribution function of n uniform random variables $U_1, \ldots, U_n$is the Copula function:

$$C(u_1, \ldots, u_n; \Sigma_\rho)$$

where the dependance structure is defined by either a correlation matrix Σ_ρ:

$$\Sigma_\rho = \begin{pmatrix} 1 & \sigma_{i,j} & \sigma_{i,j} \\ \sigma_{i,j} & \ddots & \sigma_{i,j} \\ \sigma_{i,j} & \sigma_{i,j} & 1 \end{pmatrix}$$

Remark. Equicorrelation between all reference names is sometimes assumed, in this case we have a single parameter $\forall i \forall j, \sigma_{i,j} = \rho$ that defines the dependance structure.

Definition 14. Copula as a joint distribution

In order to obtain the expression for the copula function in terms of u_i, we transform a random variable by its own CDF. For the random variables $X_1, \ldots, X_n$ with a flexible choice of marginal dsitribution $F_1(x_1), \ldots, F_n(x_n)$

$$
\begin{aligned}
C(u_1, u_2, \ldots, u_n) &\equiv \Pr(U_1 \leq u_1, \ldots, U_n \leq u_n) \\
&= \Pr(F_1(X_1), \ldots, F_n(X_n)) \\
&= \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) \\
&\equiv F(x_1, \ldots, x_n)
\end{aligned}
$$

Definition 15. Sklar theorem

Sklar proved that for every multivariate CDF F with marginals F_i, there exist some copula fonction C such that:

$$F(x_1, x_2, \ldots, x_{n)} \equiv C(u_1, u_2, \ldots, u_n)$$

and if the joint distribution is continuous then the copula function C is unique.

Remark. Sklar theorem shows that we can obtain and rely on the copula function to model dependence structure

Definition 16. Multivariate Gaussian copula

The multivariate Gaussian Copula function can be expressed as

$$C(u_1, u_2, \ldots, u_n) = \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n); \Sigma)$$

where $\Phi_n$is the CDF for the multivariate standard normal distribution. There is no closed-form solution for this copula function

Definition 17. Multivariate Student's t copula

The multivariate Student's t Copula function can be expressed as

$$C(u_1, u_2, \ldots, u_n) = T_\nu(T_\nu^{-1}(u_1), \ldots, T_\nu^{-1}(u_n); \Sigma)$$

Student's t copula density function is

$$c(U; \nu, \Sigma) = \frac{1}{\sqrt{|\Sigma|}} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})} \left(\frac{\Gamma(\frac{v}{2})}{\Gamma(\frac{\nu+1}{2})} \right)^n \frac{\left(1 + \frac{T_\nu^{-1}(U') \sum^{-1} T_\nu^{-1}(U)}{\nu}\right)^{-\frac{\nu+n}{2}}}{\prod_{i=1}^n \left(1 + \frac{T_\nu^{-1}(u_i^2)}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

where $U = (u_1, \ldots, u_n)$

1.2.6 Sampling from copula

Sampling from copula is a method to structure Monte-Carlo simulation. For each simulation, do the following:

- Generate a vector $(z_1, \ldots, z_n)$ of n uncorrelated random normal variables (n is the number of reference name is the basket CDS)
- Impose correlation on the vector $(z_1, \ldots, z_n)$ using a correlation matrix and obtain the vector $(x_1, \ldots, x_n)$
- Use the Normal / Student's t CDF to transform the vector $(x_1, \ldots, x_n)$ to a vector $(u_1, \ldots, u_n)$ of variables uniformly distributed over $[0, 1]$

- Convert the vector $(u_1, \ldots, u_n)$ to a vector of default times $(\tau_1, \ldots, \tau_n)$ using the hazard rate term structure of each reference name (i.e. use the hazard rates term structure of the reference name n to convert u_n into τ_n)

The benefit of the copula method is separation of:

- dependance structure and
- marginal distribution for the default time τ_i

1.2.7 Correlation matrix for Gaussian copulae

Gaussian copulae require linear correlation. Linear correlation is always estimated from normal variables. The following steps explain how to estimate the correlation matrix that can be used in sampling from Gaussian copulae:

1. Transfrom data $\mathbf{X}$ into uniform $\mathbf{U} = \hat{\mathbf{F}}(\mathbf{X})$, where $\mathbf{U}$ is properly scaled $\in [0, 1]$and $\hat{\mathbf{F}}(\mathbf{X})$ is the empirical cumulative density function (CDF)
2. Apply inverse Normal CDF $\mathbf{Z} = \mathbf{\Phi^{-1}}(\mathbf{U})$
3. Calculate a linear correlation (Pearson) matrix $\hat{\Sigma} = \mathrm{corr}(\mathbf{Z})$

Remark. Correlating returns by a linear measure (Pearson) assumes they are Normally distributed

1.2.8 Correlation matrix for Student's t copulae

Student's t copulae require linear correlation (this is true for all elliptical copulas). The following steps explain how to estimate the correlation matrix that can be used in sampling from Student's t copulae:

1. Transfrom data matrix$\mathbf{X}$ into uniform using$\mathbf{U} = \hat{\mathbf{F}}(\mathbf{X})$, where $\hat{\mathbf{F}}(\mathbf{X})$ is the empirical cumulative density function (CDF)
2. Calculate correlation matrix $\hat{\sum}_K$ by applying Kendall's tau or another rank measure to U
3. Calculate a linear correlation matrix $\hat{\Sigma}$by applying the formula $\rho = \sin(\frac{\pi}{2}\rho_K)$on each of the element $\rho_K$of the matrix$\hat{\sum}_K$

1.2.9 Maximum likelihood estimation for Student T copula

The maximum likelihood estimation (MLE) for Student's t copula done by summing up contributions of daily/weekly observations to the log-likelihood. We maximize the following quantity with regards to ν:

$$\arg\max_\nu \left\{ \sum_{t=1}^{T} \log c(U_t^{Hist}; \nu, \Sigma) \right\}$$

where the function $c(U; \nu, \Sigma)$ is the Student's t copula density function that we have seen earlier. The MLE is a two-step procedure that is repeated for each value of ν from 1 to 25:

1. Calculate the log of Student's t copula density function using the input of $T_\nu^{-1}(U')$, where U' is a row vector of $1 \times n$, a set of observations for n reference names
2. Add to the sum of log-likelihood for the previous days. Continue until T = NumberObservations

1.2.10 Sampling from Gaussian copulae

The detailed procedure for sampling from Gaussian copulae is as follows:

- Compute decomposition of correlation matrix $\hat{\sum}_{Gaussian} = AA'$ using Cholesky (if matrix is positive definite)
- For each simulation, do the following:
  - Generate a vector $Z = (z_1, \ldots, z_n)$ of n uncorrelated random normal variables (n is the number of reference name is the basket CDS)
  - Impose correlation on the vector $Z = (z_1, \ldots, z_n)$ using the correlation matrix A and obtain the vector $X = AZ = (x_1, \ldots, x_n)$
  - Use the Normal CDF to transform the vector $X = (x_1, \ldots, x_n)$ to a vector $U = (u_1, \ldots, u_n)$ of variables uniformly distributed over $[0, 1]$
  - Convert the vector $U = (u_1, \ldots, u_n)$ to a vector of default times $\tau = (\tau_1, \ldots, \tau_n)$ using the hazard rate term structure of each reference name (i.e. use the hazard rates term structure of the reference name n to convert u_n into τ_n)
  - Calculate the Default leg
  - Calculate the Premium leg
- Obtain the expectation of the default leg by averaging the default leg values calculated for all simulations
- Obtain the expectation of the premium leg by averaging the premium leg values calculated for all simulations
- Calculate the expectation of the fair spread as

$$\mathbb{E}^{\mathbb{Q}}\left[s^k\right] = \frac{< DL >}{< PL >}$$

1.2.11 Sampling from Student t copulae

The detailed procedure for sampling from Student's t copulae is as follows:

- Compute decomposition of correlation matrix $\hat{\sum}_{Studentt} = AA'$ using Cholesky (if matrix is positive definite)
- For each simulation, do the following:
  - Generate a vector $Z = (z_1, \ldots, z_n)$ of n uncorrelated random normal variables (n is the number of reference name is the basket CDS)
  - Draw an independent chi-squared random variable $s \sim \chi^2_\nu$
  - Compute the n-dimensional vector Student's t vector $Y = \frac{Z}{\sqrt{\frac{s}{\nu}}}$
  - Impose correlation on the vector $Z = (z_1, \ldots, z_n)$ using the correlation matrix A and obtain the vector $X = AY = (x_1, \ldots, x_n)$
  - Use the Student's t CDF to transform the vector $X = (x_1, \ldots, x_n)$ to a vector $U = (u_1, \ldots, u_n)$ of variables uniformly distributed over $[0, 1]$
  - Convert the vector $U = (u_1, \ldots, u_n)$ to a vector of default times $\tau = (\tau_1, \ldots, \tau_n)$ using the hazard rate term structure of each reference name (i.e. use the hazard rates term structure of the reference name n to convert u_n into τ_n)
  - Calculate the Default leg
  - Calculate the Premium leg

- Obtain the expectation of the default leg by averaging the default leg values calculated for all simulations
- Obtain the expectation of the premium leg by averaging the premium leg values calculated for all simulations
- Calculate the expectation of the fair spread as

$$\mathbb{E}^{\mathbb{Q}}\left[s^k\right] = \frac{< DL >}{< PL >}$$

1.2.12 Spread Calculation

Theory

The par spread of the k-th to default swap is derived by equating the Default leg to the Premium leg under the risk neutral measure:

$$\begin{aligned}
< DL > &= (1-R) \times NP \sum_{i=1}^{m} Z(0,t_i)(F_k(t_i) - F_k(t_{i-1})) \\
< PL > &= s^k \times NP \times \Delta t \sum Z(0,t_i)(1 - F_k(t_i))
\end{aligned}$$

where m is the number of periods
We can therefore calculate the fair spread of the k-th to default basket CDSs^k as:

$$s^k = \frac{< DL >}{< PL >} = \frac{(1-R) \times NP \sum_{i=1}^{m} Z(0,t_i)(F_k(t_i) - F_k(t_{i-1}))}{NP \times \Delta t \sum_{i=1}^{m} Z(0,t_i)(1 - F_k(t_i))}$$

However the joint distribution for k-th to default time accross all reference names $\tau_k \sim F_k(t_1, t_2, \ldots, t_n)$ is unknow.

The alternative formula for the fair spread uses the Loss function as an expectation over the join distribution $L_k = \mathbb{E}\left[F_k(t)\right]$

The fair spread can then be calculated as

$$s^k = \frac{(1-R) \times \sum_{i=1}^{m} Z(0,t_i)(L_i - L_{i-1})}{NP \times \Delta t \sum_{i=1}^{m} Z(0,t_i)(NP - L_i)}$$

The fair spread calculation is carried out multiples and averaged (Monte-Carlo method)

Practical implementation

I have taken the following assumptions for the pricing of k-th to default basket CDS:

- Notional principal NP is equal to 1
- Discrete-time annual payments for the Premium leg
- Continuous-time payment for the Default leg
- Reference entities that have defaulted before k-th default are removed from the basket; each default will reduce the value of the portfolio by $\frac{1}{n} \times NP$ (n being the number the number reference names in the basket)

Credit entity	Ticker	Tier	Doc Clause	Rating	Sector
Bristol-Myers Squibb	BMY	Senior Unsecured	XR	A	Healthcare
Thomson Reuters	TRI	Senior Unsecured	XR	BBB	Consumer Services
Hewlett-Packard	HPQ	Senior Unsecured	XR	BBB	Technology
Int. Business Machines	IBM	Senior Unsecured	XR	AA	Technology
Pfizer	PFE	Senior Unsecured	XR	AA	Healthcare

Table 1: Credit entities used for Basket CDS Pricing

1.3 Main result

1.3.1 Correlation matrix for gaussian copula

One of the most important step in pricing Basket Credit Default Swap is the estimation of the correlation matrix. The correlation matrix is generally obtained through historical sampling. Sereval ways to estimate the Gaussian correlation matrix have been researched:

- Difference of default probabilities / Log difference of default probabilities (daily)
- Hazard rates / Difference of hazard rates / Log difference of hazard rates (daily)
- Stock return (daily)
- Weekly observation vs. daily observations

Methodology

Historical credit spreads have been obtained from Markit for year 2011 until today. We are looking at 5 credit entities which are all corporates head-quartered in the United States. The below table (Table 1)[3] gives more details on the CDS contracts we have worked with. The restructuring clause (Doc Clause) stipulates whether restructuring is considered as a credit event or not. “XR” means “No restructuring”. All CDS have been choosen with the restructuring clause “XR” because these are considered as the most liquid in the United States.

Historical yield curves (discount factors) have been obtained from Bloomberg for year 2011 until today. I have only snapped one set of one curve per week (Fridays) to reduce the amount of market data required. This seems to be a reasonable simplification considering the movements that can be observed in the yield curve over short period of time (week) and considering the sensitivity of credit curves to interest rates.

Historical credit curves (probability of default, non-cumulative hazard rates) are bootstrapped using the historical credit spreads and the historical yield curves. The “standard model” (described earlier) is used bootstrap the credit curve. The CDS payment frequency is assumed to be 1 (annual payments). Linear correlations (Pearson correlation) is then estimated on changes (difference or log difference) in the variable (default probability or hazard rates) or on stock log returns. 5Y maturity point has been choosen as reference point for default probability and hazard rates (i.e. hazard rates between 4Y and 5Y) because the 5Y CDS are generally the most liquid contracts.

The proces to estimate the correlation matrix for Gaussian copula is the one summarized in the “Model and hypothesis” section.

Historical stock prices (closing prices) have been obtained from Yahoo Finance.

Remark. I have tried several option to obtain the empirical CDF $\hat{\mathbf{F}}(\mathbf{X})$: the function density() which computes kernel density estimates combine with a trapezoid integration and the function ecdf() which returns a empirical cumulative distribution step function. The below smoothing test (Figure 1.1) shows that the function ecdf() works best in our case.

[3]Market data as of 30/04/2014

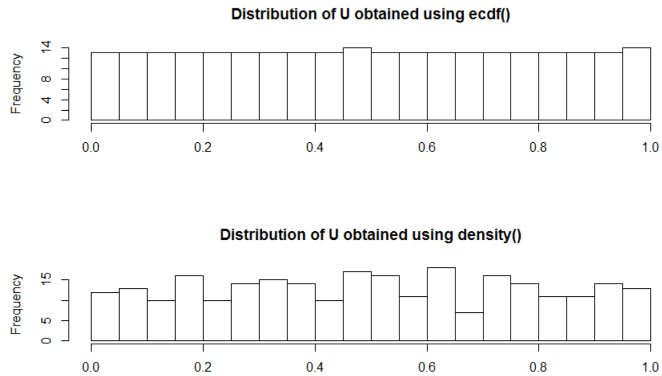

Figure 1.1: Two ways to transform data to uniform pseudo-samples - $\hat{\mathbf{F}}(\mathbf{X})$

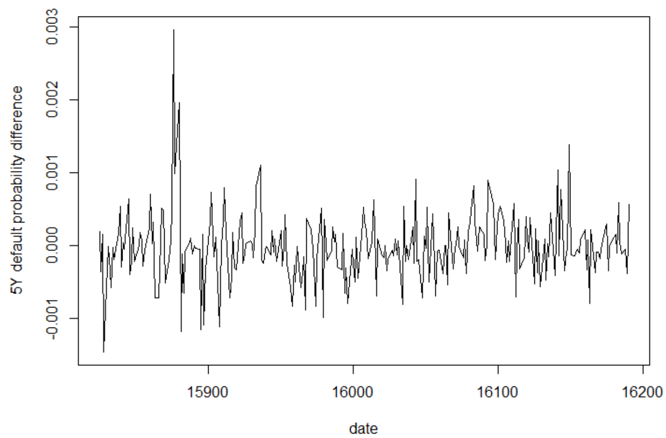

Figure 1.2: Bristol-Myers Squibb - 5Y default probability difference (Date origin in R is 01/01/1970)

Default Probabilities

We have worked with 2 data sets of historical market data:

Rate set A 1 year of daily CDS spreads between 29/04/2013 to 30/04/2014

Rate set B 2 years of daily CDS spreads between 30/04/2012 to 30/04/2014

Daily 5Y default probabilities are obtained through the bootstrapping process. Differences (or log differences) are calculated for each day (except for the first one), finally this data is transformed to a standard normal data using the empirical CDF (Pearson correlation assumes normally distributed data)

Remark. It occurs that due to extreme market movements, some differences (or log differences) are so large (outliers) that their CDF is calculated as 1 when converting data to uniform (i.e. conversion from X to U) using the empirical CDF $\hat{\mathbf{F}}$. These "1" are then converted to Infinity when we apply the inverse standard normal (i.e $\mathbf{Z} = \mathbf{\Phi^{-1}}(\mathbf{U})$). These infinite values cause issues while calculating the Pearson correlation; I have excluded them. Since we are interested in the base correlation (long term correlation), it seems acceptable to exclude a few extreme values while estimating the correlation matrix. The figure (Figure 1.2) shows one of the extreme market movements for Bristol-Myers Squibb on 20/06/2013 (15876 on the diagram).

As we have seen earlier, we have to convert our data to normal before we can calculate the linear correlation. We will get better results if our orginal data follows a distribution that is close to the normal

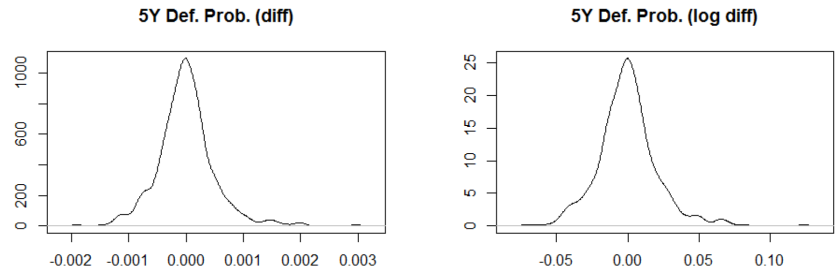

Figure 1.3: Distribution of 5Y Default Probabilities difference vs log difference

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	-0.93	27.81	30.00	48.60
TRI	-0.93	100	2.55	-3.72	-0.43
HPQ	27.81	2.55	100	45.43	35.96
IBM	30.00	-3.72	45.43	100	32.11
PFE	48.60	-0.43	35.96	32.11	100

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	-1.20	27.04	28.80	47.60
TRI	-1.20	100	3.62	-3.41	-0.45
HPQ	27.04	3.62	100	44.38	35.60
IBM	28.80	-3.41	44.38	100	31.97
PFE	47.60	-0.45	35.60	31.97	100

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	3.35	30.38	30.75	47.68
TRI	3.35	100	1.42	-0.32	2.30
HPQ	30.38	1.42	100	42.97	39.66
IBM	30.75	-0.32	42.97	100	38.28
PFE	47.68	2.30	39.66	38.28	100

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	2.78	29.20	29.02	46.78
TRI	2.78	100	-0.01	1.09	1.49
HPQ	29.20	-0.01	100	42.01	38.25
IBM	29.02	1.09	42.01	100	38.67
PFE	46.78	1.49	38.25	38.67	100

Table 2: 5Y Def. Probability difference (left) and log difference (right) for Rate set A (top) and B (bottom)

distribution. The figure (Figure 1.3) plots the kernel density of differences and log differences 5Y default probabilities. We can observe that both distributions are relatively close from the normal distribution.

The correlation matrices estimated using 5Y Default Probability differences and the 5Y Default Probability log differences are very close (See **Rate set A** in Table 2 and **Rate set B** in Table 3). The correlations we obtain are all between -5% and 50%; must of them are positive or close to zero. The 2 hightest correlations we observe are for Bristol Myers Squibb/Pfizer and HP/IBM. This makes senses these companies belong to the same industry sector: Healthcare for Bristol Myers Squibb and Pfizer and Technology for IBM and HP. I have deliberately choosen a fifth corporate that belong to a third industry sector (Thomson Reuters belong to Consumer Services). We can see that Thomson Reuters has very small correlation with the 4 other corporates.

Correlations observed over 1 year (**Rate set A**) and over 2 years (**Rate set B**) are also quite similar. In order to test the correlation stability, I have calculated the rolling correlation over 60 days (Figure 1.3). We can observe that the 60 days correlation is not very stable over time.

Hazard rates

We know that hazard rates are already log difference with regards of probability of default (1.1); however when we plot its kernel density (Figure 1.5) we can see that its data is not normally distributed. However hazard rates differences and log differences are close to a normal distribution. It is therefore suggested to work with differences (or the log differences) of hazard rates. Correlation matrix for hazard rates differences and log differences are again very similar (Table 3). Finally, the 60 days rolling correlation gives a result similar to what we observed with default probabilities; correlation is not very stable over time.

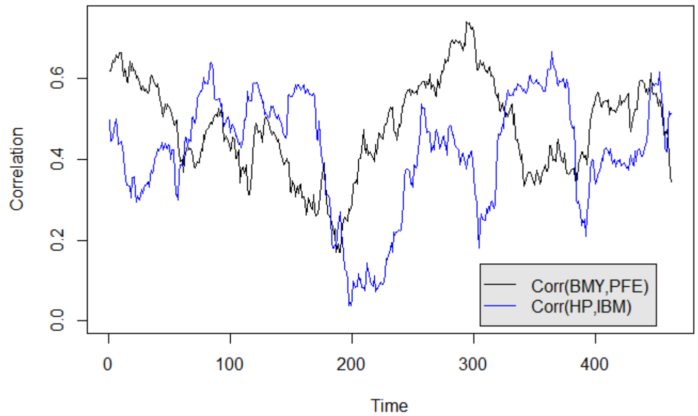

Figure 1.4: 60 days rolling correlation for Corr(BMY,PFE) and Corr(HP,IBM)

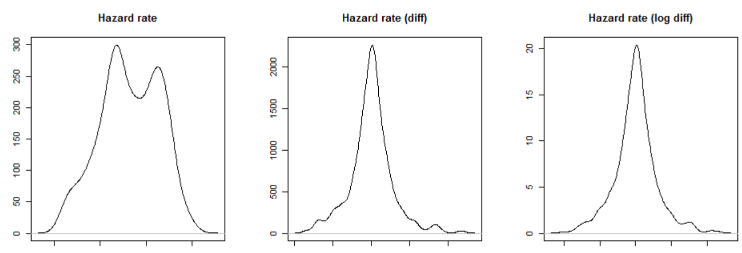

Figure 1.5: Hazard rate, hazard rates differences and log differences

%	BMY	TRI	HPQ	IBM	PFE		%	BMY	TRI	HPQ	IBM	PFE
BMY	100	-3.07	21.08	14.13	25.18		BMY	100	-3.36	21.30	14.02	25.48
TRI	-3.07	100	-3.46	0.04	-7.75		TRI	-3.36	100	-3.92	0.39	-6.18
HPQ	21.08	-3.46	100	18.63	23.84		HPQ	21.30	-3.92	100	19.02	24.69
IBM	14.13	0.04	18.63	100	6.78		IBM	14.02	0.39	19.02	100	8.37
PFE	25.18	-7.75	23.84	6.78	100		PFE	25.48	-6.18	24.69	8.37	100

Table 3: Gaussian correlation matrix for differences (left) and log differences (right)

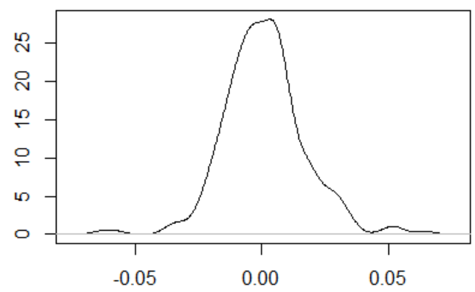

Figure 1.6: Bristol-Myers Squibb - Stock log returns

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	-1.20	27.04	28.80	47.60
TRI	-1.20	100	3.62	-3.41	-0.45
HPQ	27.04	3.62	100	44.38	35.60
IBM	28.80	-3.41	44.38	100	31.97
PFE	47.60	-0.45	35.60	31.97	100

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	2.78	29.20	29.02	46.78
TRI	2.78	100	-0.01	1.09	1.49
HPQ	29.20	-0.01	100	42.01	38.25
IBM	29.02	1.09	42.01	100	38.67
PFE	46.78	1.49	38.25	38.67	100

Table 4: Correlation matrix from stock log return vs. default probability log diff (1 year)

Stock log return

I have also studied how a correlation matrix estimated from stocks log returns could be used as proxy. Without surprise, log returns are almost normally distributed (Figure 1.6). The correlation matrix obtained from stock log returns is pretty close from the correlation matrix obtained from 5Y default probabilities log differences (Table 4). Again, the highest correlations are observed for Bristol Myers Squibb/Pfizer and HP/IBM. At least in this case, we can conclude that stock log returns are a good proxy for correlation matrix estimation. Finally, I tested at the stability of the correlation estimated from stocks log returns by calculating the 60 days rolling correlation (Figure 1.7). Correlations of stock log returns are still pretty unstable but slightly more stable that the correlation of 5Y default probability log difference.

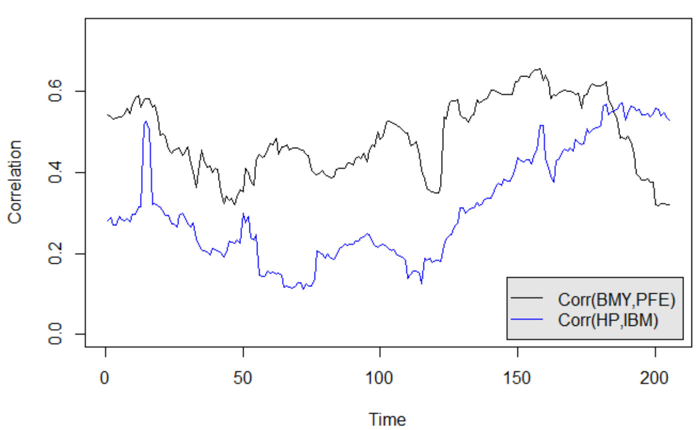

Figure 1.7: 60 days rolling correlation for Corr(BMY,PFE) and Corr(HP,IBM)

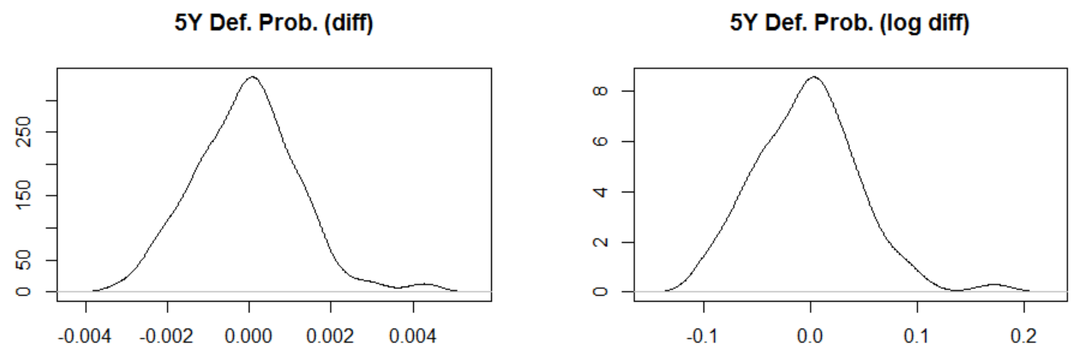

Figure 1.8: diff / log diff

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	21.82	33.05	47.18	57.12
TRI	21.82	100	11.65	13.98	23.92
HPQ	33.05	11.65	100	61.60	47.65
IBM	47.18	13.98	61.60	100	63.49
PFE	57.12	23.92	47.65	63.49	100

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	2.78	29.20	29.02	46.78
TRI	2.78	100	-0.01	1.09	1.49
HPQ	29.20	-0.01	100	42.01	38.25
IBM	29.02	1.09	42.01	100	38.67
PFE	46.78	1.49	38.25	38.67	100

Table 5: Weekly observations log diff / daily observation 2Y log diff

Weekly observations

I have snapped weekly CDS spreads data (Fridays) between 29/04/2011 and 30/04/2014 and bootstrap the credit curve on these dates. The differences and the log differences are also normally distributed (Figure 1.8). The estimated correlation matrix keeps some similarity with the one calculated with daily observations. For instance, the largest correlations are still observed for are observed for Bristol Myers Squibb/Pfizer and HP/IBM. But there are also differences: we start to see correlations between Thomson Reuters and the other reference names whereas these correlation were always very close to zero with daily observations. We can observe that the “weekly” correlations are higher than the “daily” correlations on average. Finally, I tested the stability of these correlations over time by plotting the 60 weeks rolling correlations. “Weekly” correlations appear to be more stable over “time”; they probably reflect better base correlation. We will use compare the pricing results using daily and weekly data 5Y default probability log differences.

1.3.2 Correlation matrix and degree of freedom for Student T copula

Correlation matrix

We used the same input data as for the correlation matrix for Gaussian copulae. The proces to estimate the correlation matrix for Student’s t copula is the one summarized in the “Model and hypothesis” section. The correlations observed for Bristol Myers Squibb/Pfizer and HP/IBM are still among the largest.

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	2.92	20.59	19.61	31.32
TRI	2.92	100	-0.34	0.61	0.99
HPQ	20.59	-0.34	100	30.56	27.15
IBM	19.61	0.61	30.56	100	25.38
PFE	31.32	0.99	27.15	25.38	100

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	4.59	31.78	30.32	47.24
TRI	4.59	100	-0.54	0.96	1.55
HPQ	31.78	-0.54	100	46.18	41.37
IBM	30.32	0.96	46.18	100	38.82
PFE	47.24	1.55	41.37	38.82	100

Table 6: Daily observations - Kendall Tau (left) and Student T (right) correlation matrix

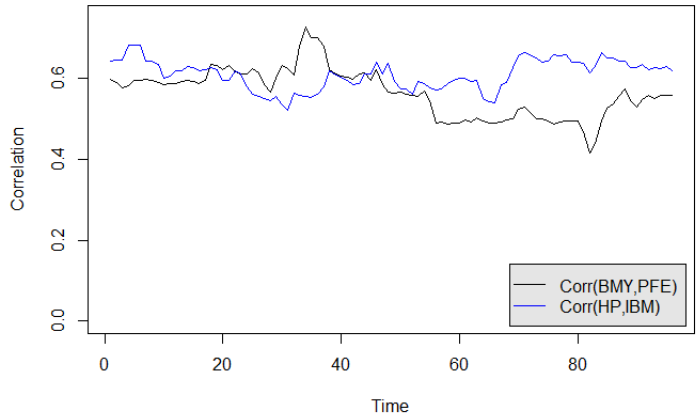

Figure 1.9: 60 days rolling correlation for Corr(BMY,PFE) and Corr(HP,IBM) - weekly data

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	15.77	24.81	34.75	43.43
TRI	15.77	100	7.75	11.78	18.08
HPQ	24.81	7.75	100	44.02	34.52
IBM	34.75	11.78	44.02	100	50.08
PFE	43.43	18.08	34.52	50.08	100

%	BMY	TRI	HPQ	IBM	PFE
BMY	100	24.53	37.99	51.91	63.05
TRI	24.53	100	12.15	18.40	28.02
HPQ	37.99	12.15	100	63.76	51.61
IBM	51.91	18.40	63.76	100	70.80
PFE	63.05	28.02	51.61	70.80	100

Table 7: Weekly observations - Kendall Tau (left) and Student T (right) correlation matrix

Degree of freedom

I have applied the maximum likelihood estimation procedure to calculate the degree of freedom. The procedure is described in the "Model and hypothesis" section. The degree of freedom has been calculed for daily observations and weekly observations; we obtain the below results:

Degree of Freedom	
Daily data	Weekly data
13.78	10.76

1.3.3 Spread calculation

Sampling from Gaussian copulae and Student's t copulae follows the procedures described in the "Model and hypothesis" section. Spread calculated is also detailed in the "Model and hypothesis" section. The following results have been obtained by running 1,000,000 simulations.

I have priced the k^{th} to default basket using the correlation matrix estimated from the weekly observations and using the correlation matrix estimated from the daily observations.

The fair spread of the k^{th} to default basket decreases as k increases. As k increases, the risk of default of the basket (i.e. at least k reference name default within the basket) decreases. It is therefore logical that the protection premium's value (i.e fair spread) decreases. The spread of the first-to-default basket is higher with the Gaussian copulae than with the Student's t copulae. On the other hand, the spread of the $k^{th}$to-default for $k > 1$ are smaller with the Gaussian copulae than with the Student's t copulae. The t-distribution (when $\nu < 25$)has heavier tails than the normal distribution; it means that it is more prone to producing values that fall far from its mean. I believe this explains why the the spread of $k^{th}$to-default for $k > 1$ are higher with the Student's copule than with the Gaussian copulae.

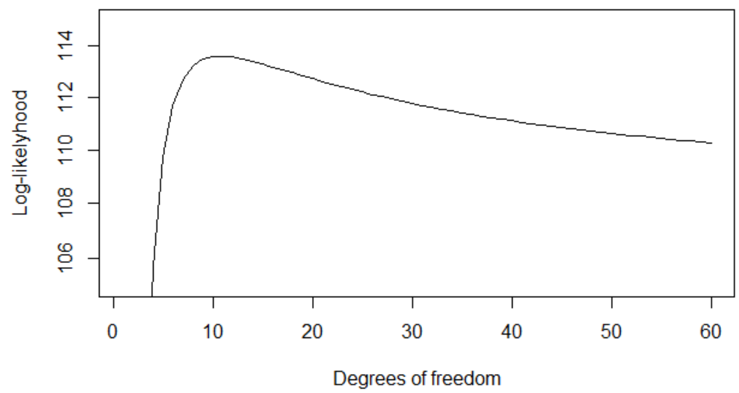

Figure 1.10: Log-Likelihood function for Student's t copula vs degree of freedom parameter (weekly data)

	Daily observations		Weekly observations	
	Gaussian Copula	Student's T Copula	Gaussian Copula	Student's T Copula
$k^{th}$to default	Fair spread (bp)	Fair spread (bp)	Fair spread (bp)	Fair spread (bp)
1	43.6619	41.7913	39.8578	37.8018
2	6.5649	7.5139	8.4065	9.1998
3	1.1302	1.6419	2.2866	2.9420
4	0.1506	0.2943	0.5283	0.8390
5	0.0067	0.0217	0.0616	0.1436

1.3.4 Risk and sensitivity analysis

This section explores the impacts of the model input parameters on the k^{th} to default fair spread. We are moving one parameter at a time in order to clearly identify its impact; also we simplify the overall setup by taking the following assumptions:

- Homogeneous basket of 5 reference names with constant CDS spreads (flat term structure)
- Equicorrelation between all reference names
- Constant interest rate

Impact of credit quality of individual reference names

- Equicorrelation = 30%
- Interest rate = 1%
- Degree of freedom (for Student's t): 10
- 1,000,000 simulations

The decresase of credit quality of the individual reference names (i.e increase of their CDS spread) does not have the impact in all $k^{th}$to default fair spread. The bigger k is, the bigger is the impact of the decrease of credit quality on the spread $s^k$is (Figure 1.11). For instant the increase of the CDS spread from 50 basis points to 500 basis points multiplies the $1^{st}$to default spread by 8 ($\frac{347.4121}{42.6791} \sim 8$) but it multiplies by 45($\frac{74.6040}{1.6561} \sim 45$) the 3^{rd} to default spread. We can observe that first-to-default spread is always lower than the CDS spread of any reference name. Since the reference names are correlated, Why??? less invidual default, therefore less 1st default alone provide explanations ....

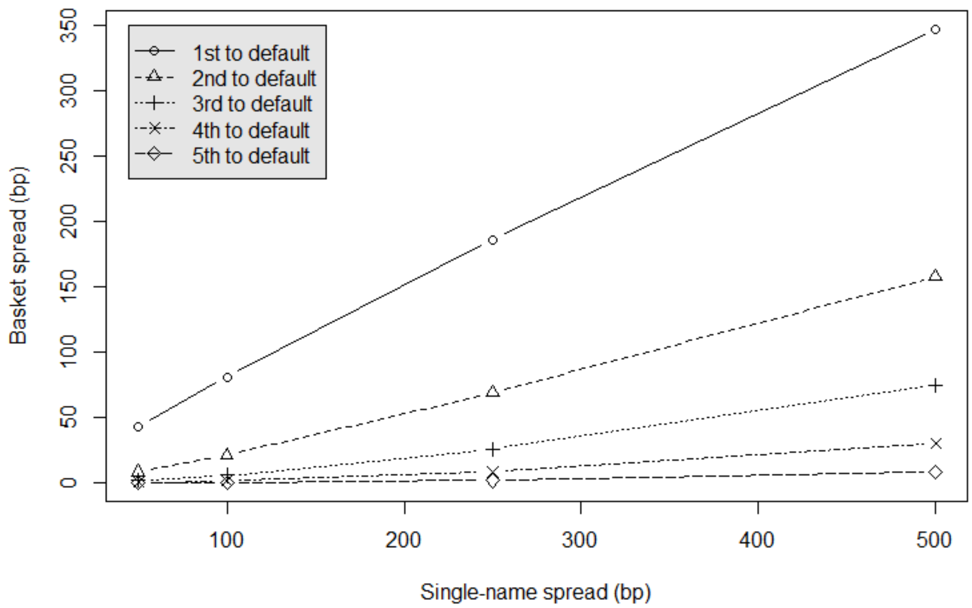

Figure 1.11: Impact of credit quality on kth to default fair spread (Gaussian copula)

shows

	Gaussian Copula			
$n^{th}$to default	50bp	100bp	250bp	500bp
1	42.6791	80.9760	186.3073	347.4121
2	8.1186	21.0202	68.7467	157.8954
3	1.6561	5.6255	25.8877	74.6040
4	0.2917	1.2642	8.1271	29.9927
5	0.0307	0.1759	1.6065	7.8849

Student T Copula			
50bp	100bp	250bp	500bp
39.7199	76.4019	180.1036	340.5270
9.4424	22.3590	69.1917	157.6017
2.5171	6.9085	27.1378	75.1341
0.6065	1.8239	9.0312	30.5448
0.0942	0.3233	1.9560	8.1306

Impact of correlation on the fair spread

- CDS spread = 100 basis points
- Interest rate = 1%
- Degree of freedom (for Student's t): 10
- 1,000,000 simulations

As default correlation increases to very high levels, spreads for different k^{th} to default instrument lapse. Why?

What are your levels of correlations obtained from changes in traded CDS price?

as correlation increases, less default alone and more group / joined defaults. therefore the first to default decrease and the 3/4/5 default increase.

second-to-default increase until a certain inflexion point (around 70%); after that the spread of the second-to-default decreases

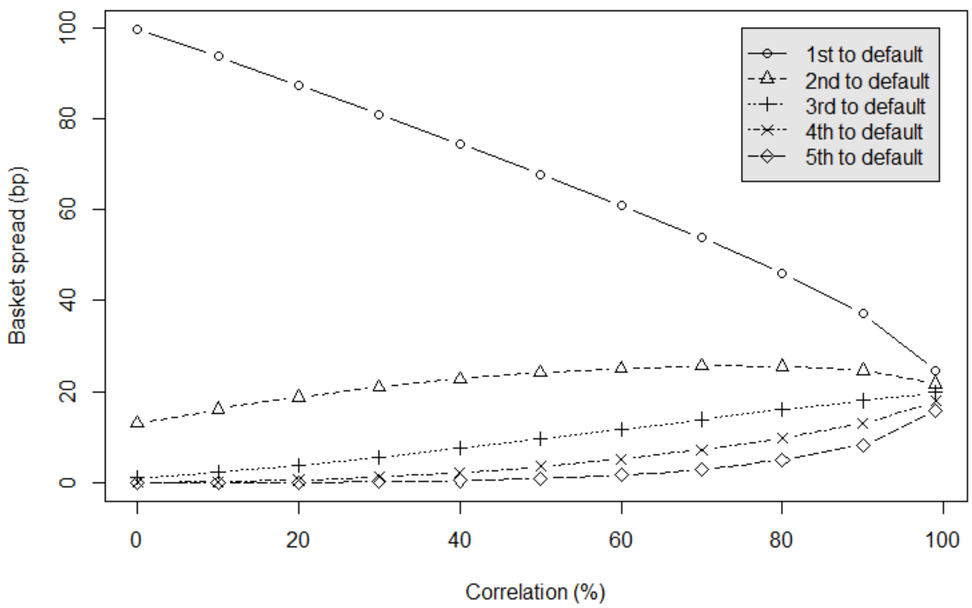

Figure 1.12: Impact of correlation on kth to default fair spread (Gaussian copula)

	Gaussian Copula				Student T Copula			
$n^{th}$to default	0	0.3	0.6	0.99	0	0.3	0.6	0.99
1	99.6695	80.9760	60.9459	24.7272	93.8157	76.3525	58.2076	24.4443
2	13.1045	21.0202	25.1358	21.7586	16.0760	22.2168	25.3058	21.6490
3	1.0561	5.6255	11.7798	19.7831	2.1289	6.9054	12.5474	19.7686
4	0.0434	1.2642	5.1764	17.9658	0.1809	1.8475	5.8387	18.0612
5	0.0002	0.1759	1.7027	15.7702	0.0085	0.3126	2.1023	15.9467

Impact of the recovery rate on the fair spread

- CDS spread = 100 basis points
- Equicorrelation = 30%
- Interest rate = 1%
- Degree of freedom (for Student's t): 10
- 1,000,000 simulations

explore larger recovery rate ????

	Gaussian Copula				Student T Copula			
$n^{th}$to default	0.30	0.40	0.50	0.60	0.30	0.40	0.50	0.60
1	94.4720	80.9760	67.4800	53.9840	89.2530	76.3525	63.5134	50.9287
2	24.5236	21.0202	17.5168	14.0134	26.2039	22.2168	18.5329	14.8813
3	6.5631	5.6255	4.6879	3.7503	8.1251	6.9054	5.7410	4.6235
4	1.4749	1.2642	1.0535	0.8428	2.1734	1.8475	1.5155	1.2343
5	0.2052	0.1759	0.1466	0.1173	0.3709	0.3126	0.2543	0.2155

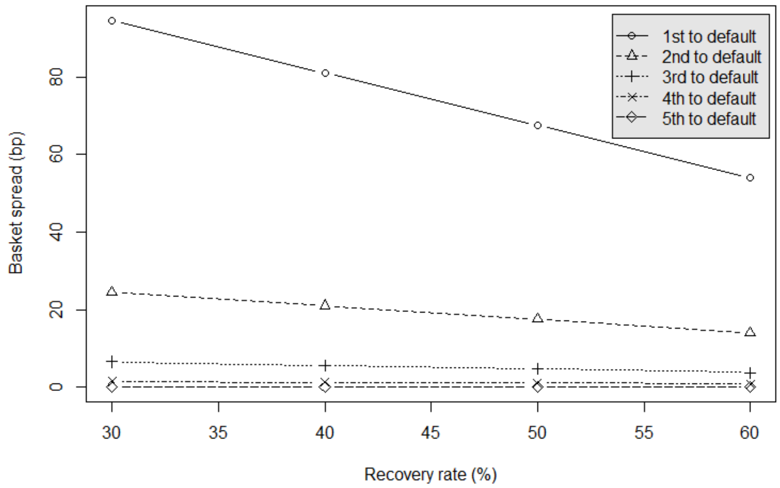

Figure 1.13: Impact of the recover rate

Impact of the discount curve on the fair spread

- CDS spread = 50 basis points
- Equicorrelation = 30%
- Degree of freedom (for Student's t): 10
- 1,000,000 simulations

Comments ???? explore granular

	Gaussian Copula				Student T Copula			
$n^{th}$to default	1%	2%	3%	5%	1%	2%	3%	5%
1	42.6791	42.9262	43.17	43.6827	39.8855	39.8275	40.2818	40.5920
2	8.1186	8.1230	8.1272	8.1356	9.5513	9.4597	9.5547	9.5998
3	1.6561	1.6524	1.6486	1.6410	2.5188	2.5091	2.5337	2.5866
4	0.2917	0.2905	0.2893	0.2868	0.6117	0.5856	0.5911	0.6021
5	0.0307	0.0305	0.0304	0.0300	0.0927	0.0849	0.0864	0.0983

1st and 2nd to default spread increase. 3rd to default spread onwards decrease

Stress testing of correlation matrix

historical sampling of default correlation matrix, and
choice of the stress-testing levels of correlation.

1.4 Convergence and stability

I have run 1,000,000 simulations using pseudo-random numbers (rnorm native function in R) and quasi-random numbers (Sobol, Niederreiter) through the NAG library. With this data, i have a convergence diagram that compares the convergence of these 3 types of numbers. The simulation that uses pseudo-random number is the slowest to converge: we can still observe significant variation in spread after 300,000 simulations. On the other hand, simulations that use Sobol and Niederreiter numbers seem to stabilize after 100,000 simulations. See Peter jackel lecture for more comments.

Add Halton using r?

Comments on accuracy???

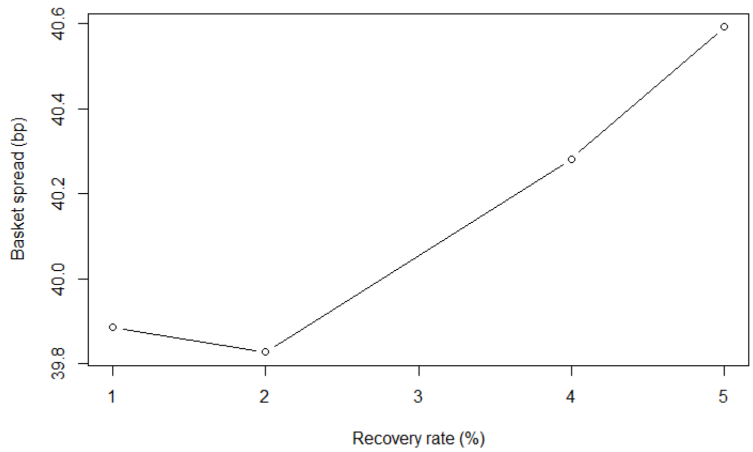

Figure 1.14: Impact of yield curve

include graph in color??? close-up on the end???

Impact of the increase of the number of reference names

?????

1.5 Numerical implementation

1.5.1 Highlights

- Integration with NAG library (professional routine). identify closest positive definite correlation matrix using NAG???
- Pricing code that allows pricing of n reference names basket

1.5.2 Algorithm

1.5.3 Market data

Two types of market data have been required for this implementation:

- Credit default swap spreads
- Discounting curve

Credit default swap spreads

Historical CDS spreads for the following entities have been obtained from Markit:

Ticker	RedCode	Tier	Curreny	DocClause	Sector	Region
BMY	1C1134	SNRFOR	USD	XR	Healthcare	N.Amer
TRI	8GD65J	SNRFOR	USD	XR	Consumer Services	N.Amer
HPQ	46AA59	SNRFOR	USD	XR	Technology	N.Amer
IBM	49EB20	SNRFOR	USD	XR	Technology	N.Amer
PFE	7I8789	SNRFOR	USD	XR	Healthcare	N.Amer

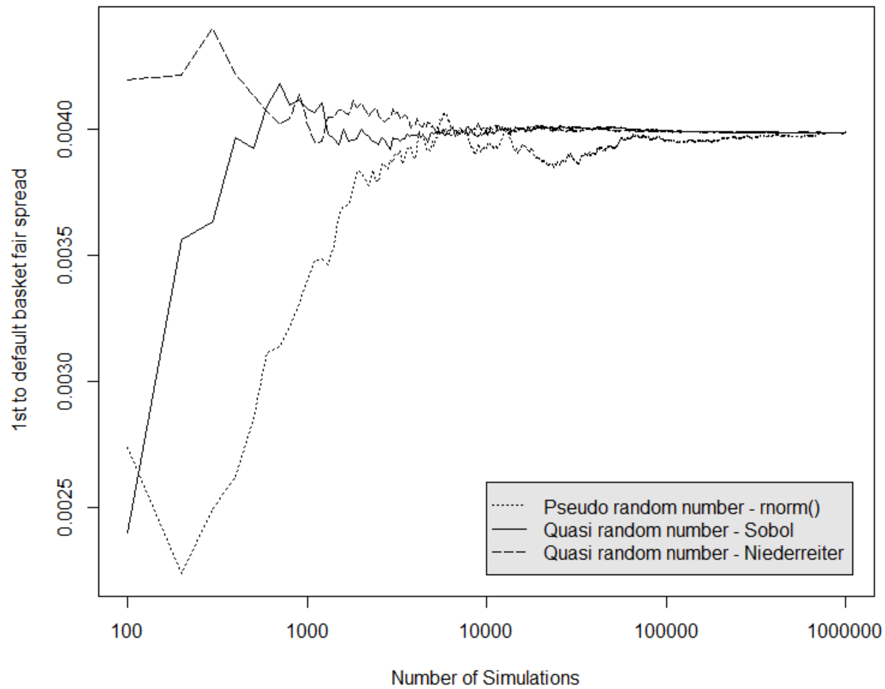

Figure 1.15: Sobol / Niederreiter / Rnorm convergence diagram

Discounting curve

Historical discounting curves have been obtained from Bloomberg using the following Bloomberg Tickers:

Bloomberg Ticker	**Description**
S0023D 1Y BLC2	Discount factor (LIBOR curve) for 1Y
S0023D 2Y BLC2	Discount factor (LIBOR curve) for 2Y
S0023D 3Y BLC2	Discount factor (LIBOR curve) for 3Y
S0023D 4Y BLC2	Discount factor (LIBOR curve) for 4Y
S0023D 5Y BLC2	Discount factor (LIBOR curve) for 5Y

1.5.4 Pros and Cons

- Pros
  - Usage of NAG
  - ?????
- Cons
  - Premium leg - annual payment. standard is quaterly
  - Default leg - should be in discrete time (quaterly payment) instead of continuous time
  - Should be implemented in C++ for better design object oriented design and faster computation (multithreading, etc.)

1.5.5 Instructions and package description

This program has been written using R (version 3.1.0, 64-bit) and Rstudio as IDE. In order to launch this program, please follow this procedure:

1. Copy the whole folder "creditbasket" to your local computer (e.g. C:\temp\creditbasket)
2. Open the file "creditbasket\src\master_program.R" (e.g. C:\temp\creditbasket\src\master_program.R)
3. Update the value of the variable **current_dir** to match your environment (e.g. current_dir = "C:/temp/creditbasket/src")
4. Launch R
5. type: source ("C:/temp/creditbasket/src/master_program.R") (replace the path to master_program.R to match your environment)
6. The program will start.

Folder	**File**	**Description**
.\interestrate-hjm		Top folder
.\interestrate-hjm\src		Folder containing source code (R)
	master_program.R	Main program that calls all the other programs
	class_definition.R	Defines several classes which are used throughout the project
	install_register_packages.R	Installs and registers external packages which are required
	market_data_functions.R	Defines the function that loads the forward curve (BLC) and the OIS spot curve
	market_data_loading.R	Loads historical forward curve (short and long end) and load OIS spot curve
	jacobi_transformation_functions.R	Defines functions required for the Jacobi transformation
	principal_component_analysis.R	Performs principal component analysis
	black76.R	Defines Black76 function, cap/caplets and swaptions pricing functions that use Black76
	nag_library_wrapper.R	Defines a function that allows R to call the NAG Fortran 64 bit library
	monte_carlo_simulation_functions.R	Main pricing functions: Monte Carlo loop and bond pricing, libor calculation, cap/caplet pricing and swaption pricing within one simulation matrix
	monte_carlo_simulation.R	Calls above mentionned functions; plots volatility surfaces and convergence diagrams
.\interestrate-hjm\data		Folder containing market data files
	ukblc05_mdaily_fwdcurve_shortend.csv	BOE BLC forward curve (short end)
	ukblc05_mdaily_fwdcurve_longend.csv	BOE BLC forward curve (long end)
	ukois09_mdaily_spotcurve.csv	BOE OIS spot curve

2 Interest Rate Derivatives

2.1 Introduction

The Heath–Jarrow–Morton (HJM) framework is a framework that model the evolution of the yield curve by representing the evolution of the instantenuous forward rates. This model impose the no-arbitrage restriction: the drift can't be random and depends on the volatility of forward rates. In HJM, volatility of forward rates is estimated from historical interest rate data (historical forward rates in our case). We will show how the principal component analysis is conducted (estimation and analysis of covariance matrices) to identify the major factors which influance the evolution of the yield curve before describing the projection of the forward curve and the pricing of various interest rate derivatives instrument using Monte Carlo simulation (using pseudo-random and quasi-random numbers). We use an OIS[4] spot curve as discounting curve in order to reflect the cost of funding.

2.2 Model, hypothesis

2.2.1 Pricincipal component analysis

Definition. Covariance matrix
dfg

Definition 18. Matrix decomposition

Please see PCA Note. HJM Lecture Solutions also consider steps and properties of the matrix decomposition by Jacobi Transformation.

PCA is carried out numerically, by nding the orthogonal representation to a covariance matrix of the original factors. This is known under the general terms of 'matrix factorisation' and 'matrix decomposition'. Orthogonal decomposition is specically known as 'eigendecompo- sition' and more widespread spectral decomposition.

Application of PCA to some data covariance matrix means spectral decomposition to

$$\Sigma = V \Lambda V^T$$

where is a matrix with eigenvalues along its diagonal and zeros elsewhere, and V ec(V) is a matrix with eigenvectors in its columns. Eigenvalues must be positive $\lambda_1 > \ldots > \lambda_n > 0$and are often ranked.

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Polynomial tting by cubic spline is a popular choice

The alternatives to the cubic spline $\beta_0 + \beta_1 \tau + \beta_2 \tau^2 + \beta_3 \tau^3$ are

2.2.2 Heath–Jarrow–Morton

Instantenuous forward rate f(t,T), as seen from time t, starts at time T and applies over an instant $\Delta t \to 0$

It is a number from a continusous proces.

$$\begin{aligned} f(t,T) &= -\lim_{\Delta t \to 0} \frac{\log Z(t;T+\Delta t) - \log Z(t;T)}{\Delta t} \\ &= -\frac{\partial}{\partial T} \log\ Z(t;T) \end{aligned}$$

In general, the N-dimensional SDE (multi-factor HJM) is as follows:

[4] Overnight indexed swap

$$df(t,T) = m(t,T)dt + \sum_{i=1}^{N} \nu_i(t,T)dX_i$$

with the risk-neutral drift

$$m(t,T) = \sum_{i=1}^{N} \nu_i(t,T) \int_t^T \nu_i(t,s)\ ds$$

Multi-Factor Musiela Parametrization

$$d\overline{f}(t,\tau) = \left(\sum_{i=1}^{k} \overline{\nu}_i(t,\tau) \int_t^T \overline{\nu}_i(t,s)\ ds + \frac{\partial \overline{f}(t,\tau)}{\partial \tau} \right) dt + \sum_{i=1}^{k} \overline{\nu}_i(t,\tau)dX_i$$

2.2.3 Zero-Coupon Bond pricing

Zero coupon bond (ZCB) price comes from the solution for dZ = Zrtdt, under the risk- neutral measure

$$Z(t;T) = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_s\ ds \right) \right] = \exp\left(-\int_t^T \overline{f}(s,0)\ ds \right)$$

2.2.4 Cap/Floor pricing

We also need to price simple interest rate options, such as caps and oors. A cap can be treated as a sum of caplets, each re-setting (expiring) at Ti and maturing at Ti+1. The rate is paid for the time i = Ti+1 to Ti. The amount of interest accrued is equal to L(Ti; Ti+1)(Ti+1 Ti) = fii

In order to calculate the cash ow for a caplet for the notional capital of one, we use

$$Z(t;T_{i+1}) \times \max\left[L(T_i, T_{i+1}) - K, 0 \right] \times \tau$$

$$L(T_i, T_{i+1}) \equiv \overline{L}(T_i, \tau) = \frac{1}{\tau} \int_0^{\tau} \overline{f}(t,0)\ ds$$

$$\mathbb{E}\left[L(T_i, T_{i+1}) \right] = \frac{1}{\tau} \sum_{T_i}^{T_{i+1}} \overline{f}_t(t,\tau)$$

$$Caplet - Floorlet = (L(0, T_{i-1}, Ti) \times \tau_i \times Z(0, T_i) \times N$$

$$Cap - Floor = N \sum_{i=\alpha+1}^{\beta} (L(0, T_{i-1}, Ti) \times \tau_i \times Z(0, T_i)$$

N

2.2.5 Vanilla Payer Swaption pricing

2.2.6 Black76

The formula is used to convert cap/floor prices to implied volatilities. The discussion below assumes that caplet prices are known However, in practice we strip caplet prices (3M period) from market-traded cap prices (1Y period). Prices are quoted in terms of implied volatility.

Proposing that a caplet priced under LMM (lognormal factors) and Black's caplet price (market) are the same,

$$\mathrm{Cpl}^{LMM}(0, T_i, T_{i+1}, K) = \mathrm{Cpl}^{Bl}(0, T_i, T_{i+1}, K, \zeta_i)$$

$$\begin{aligned}
\mathrm{Cpl}^{LMM}(0, T_i, T_{i+1}, K) &= \mathrm{Cpl}^{Bl}(0, T_i, T_{i+1}, K, \zeta_i) \\
&= Z(0, T_{i+1}) \times \tau_i \times \mathrm{Bl}(K, F_i(0), \zeta_i)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Bl}(K, F_i(0), \zeta_i) &= \mathbb{E}^{\mathbb{Q}}\left[F_i(t) - K\right]^+ \\
&= F_i(0)\Phi(d_1) - K\Phi(d_2)
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\log \frac{F}{K} + \frac{\zeta^2}{2}}{\zeta} \\
d_2 &= \frac{\log \frac{F}{K} - \frac{\zeta^2}{2}}{\zeta}
\end{aligned}$$

in these formulae $\zeta^2 = \zeta_i^2 T_i$
The formula is used to convert swaption prices to implied volatilities.

$$\mathrm{PayerSwaption}(t, T, \tau, K, \sigma_{\alpha,\beta}) = \sum_{i=\alpha+1}^{\beta} \tau_i \times Z(t, T_{i+1}) \times \left[S(t, \alpha, \beta)\Phi(d_1) - K\Phi(d_2)\right]$$

with

$$\begin{aligned}
d_1 &= \frac{\log\left(\frac{S(t,\alpha,\beta)}{K}\right) + \frac{\sigma^2(T_\alpha - t)}{2}}{sigma\sqrt{(T_\alpha - t)}} \\
d_2 &= \frac{\log\left(\frac{S(t,\alpha,\beta)}{K}\right) - \frac{\sigma^2(T_\alpha - t)}{2}}{sigma\sqrt{(T_\alpha - t)}}
\end{aligned}$$

2.3 Main result

Implementation of PCA, curve-tting, and numerical integration will enable you to conduct Monte-Carlo. To achieve price convergence it might be necessary to conduct $>$ 10; 000 simulations.

2.3.1 Principal Component Analysis

The covariance between daily dierences has to be re-annualised by the factor of 252.The Bank of England's data provides rates as percentages, and therefore, we adjust covariance by 1 1002 . The total adjustment is 252 10;000 .

A technical condition ensuring real and positive eigenvalues (they represent variance) are obtained by decomposition is for the matrix to be positive denite or semi-positive denite.

PCA on log-difference in forward rates $\log f(t + \Delta t) - \log f(t)$

- log difference / difference
- 1 year / 2 years / 3 years

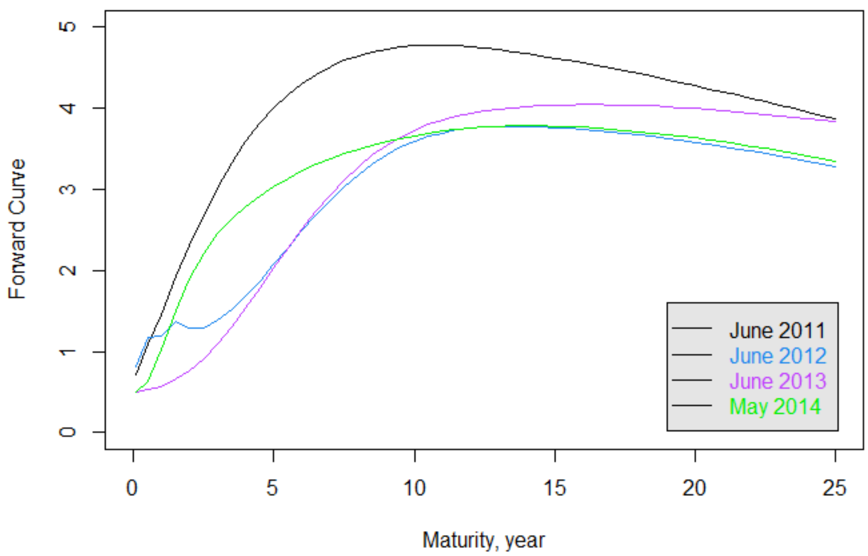

Figure 2.1: GBP Forward rate term structure between June 2011 and May 2014

The principal component analysis (PCA) is a major step in any implementation of the Heath–Jarrow–Morton (HJM) framework. It is the step that estimates the volatility function for the historical forward rates and therefore influance the drift of the forward rates in our projection (the HJM drift depends on volatility as we have seen in equation 2.2.2).

We are working with Bank of England data (Bank Liability Curve constructed from LIBOR-linked instruments) observed beween the 1st June 2010 and the 30th May 2014. We can observe a certain homogeneity between these curves

Principal components:

Principal Component	λ	Time	Weight (%)	Cumul. Weight (%)
PC1	0.0002959422	2	68.18	68.18
PC2	0.00007297595	0.5	16.81	84.99
PC3	0.00003116608	25	7.18	92.18
PC4	0.00001409952	3.5	3.24	95.42
PC4	0.000007983922	1	1.84	97.26

dfgdg

Not a perfect fit

Report on this topic should discuss model risk that comes from calibraiton (tting) of volatility functions and discounting curve (if involved).

2.3.2 Discounting

- Dual curve pricing
- OIS spot curve available until 5Y - so we use it. after 5Y, we must use the OIS spread over LIBOR
- Usage of OIS or SONIA rates to construct discount factors is compatible with the market model that relies on shifting numerarie Q(m) (an account for which rate is xed every night).
- We are using the OIS spot curve that is available on the Bank of England website.

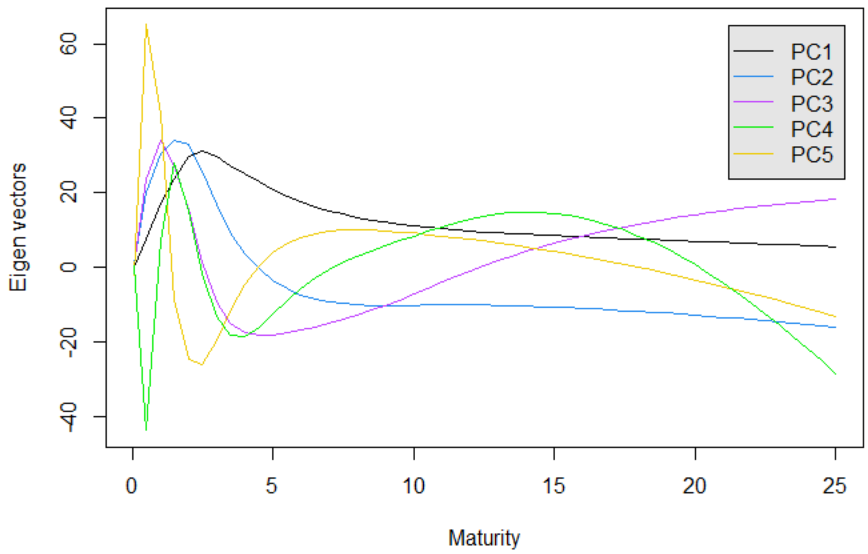

Figure 2.2: Eigen vectors for the 5 major Principal Components

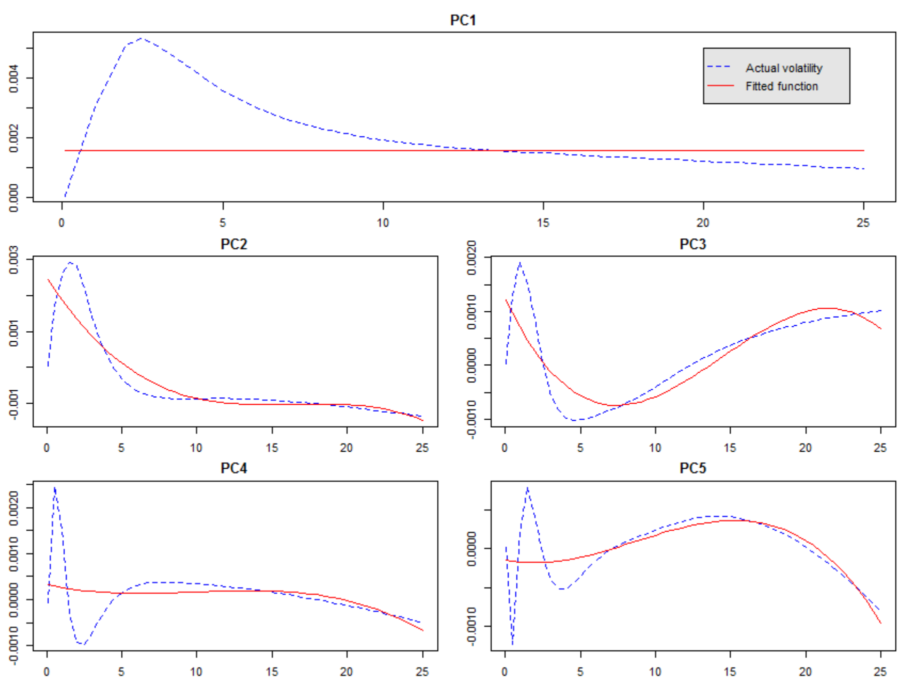

Figure 2.3: Actual volatility vs. Fitted function

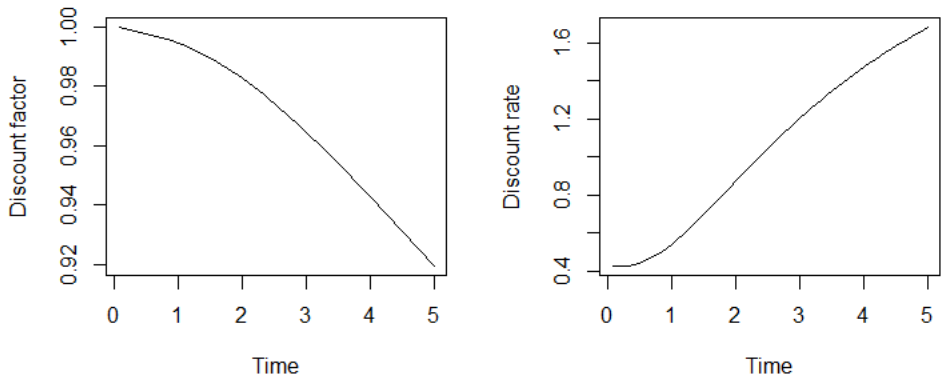

Figure 2.4: GBP OIS Dsicount Curve

2.3.3 Zero Coupond Bond pricing

10,000 simulations.

$Z(t,T)$	T				
t	1	2	3	4	5
0	0.9923	0.9770	0.9555	0.9304	0.9037
1	NA	0.9844	0.9627	0.9375	0.9105

2.3.4 Cap pricing

30,000 simul

- volatility smile

	Price						
Cap(t,T,K)	K (%)						
T	0.1	0.2	0.5	1	2	3	4
3	0.0340	0.0321	0.0267	0.0191	0.0083	0.0022	0.0001

Implied Volatilit			
K (%)			
0.1	0.2	0.5	1
120.29	80.99	42.84	22.72

instert plot

- volatility surface

	Price						
Cap(t,T,K)	K (%)						
T	0.1	0.2	0.5	1	2	3	4
2	0.0076	0.0066	0.0042	0.0014	0.0000	0	0
3	0.0340	0.0321	0.0267	0.0191	0.0083	0.0022	0.0001
4	0.0838	0.0809	0.0727	0.0603	0.0400	0.0244	0.0130
5	0.159	0.1552	0.1442	0.1272	0.0976	0.0727	0.0521

Implied Vola			
K (%)			
0.1	0.2	0.5	1
81.53	42.88	24.17	15.99
120.30	81.07	43.00	23.00
154.42	108.51	64.15	37.43
203.57	133.43	81.17	51.51

instert plot

2.3.5 Swap pricing

explain notation

SwapParSpread(t,T)	T (%)			
t	2	3	4	5
1	0.0087	0.0183	0.0298	0.0424
2	NA	0.0131	0.0255	0.0389
3	NA	NA	0.0161	0.0301

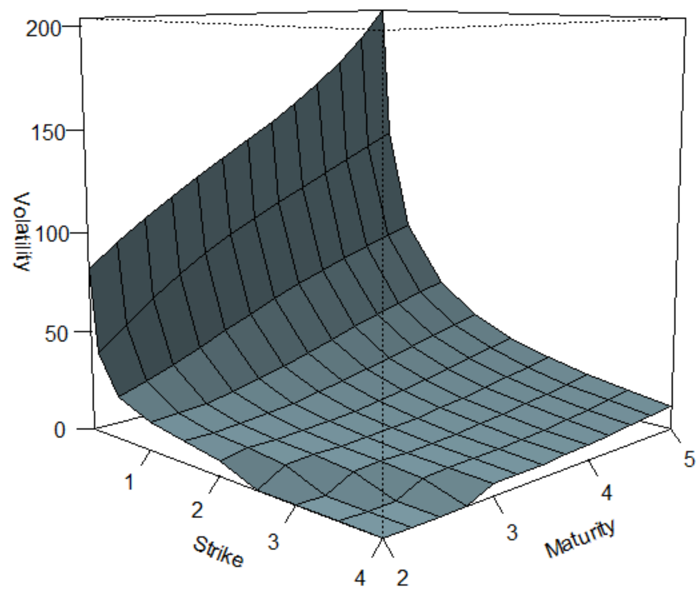

Figure 2.5: Cap volatility surface

2.3.6 European swaptions

- volatility smile

	Price							
Cap(t, T, K)	K (%)							
T	0.1	0.2	0.5	1	2	3	4	5
3								

Implied Volatility							
K (%)							
0.1	0.2	0.5	1	2	3	4	5

instert plot

- volatility surface

	Price							
PayerSwaption(t, T, K)	K (%)							
T	0.1	0.2	0.5	1	2	3	4	5
2								
3								
4								
5								

Implied Volatility							
K (%)							
0.1	0.2	0.5	1	2	3	4	5

instert plot

2.4 Convergence and stability

Comparison between Pseudo-random and Quasi random numbers (Sobol, Niederreiter, Faure).
===> Niederreiter not applicable because it's limited to 318 dimension (in nag lib)

2.5 Numerical implementation

2.5.1 Highlights

Integration with NAG

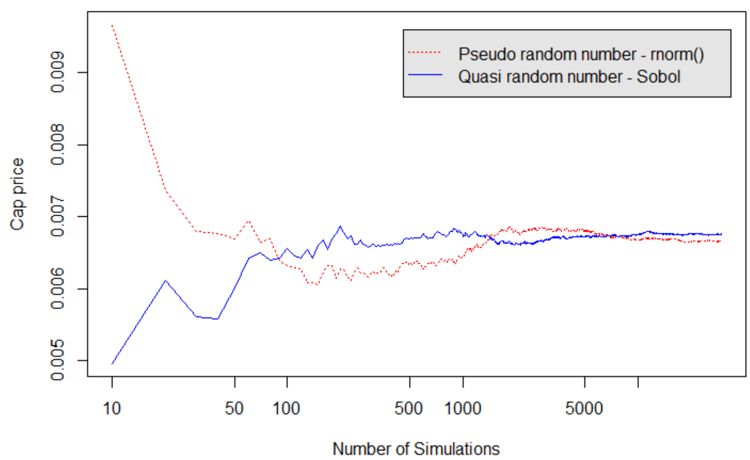

Figure 2.6: Convergence diagram for a cap price

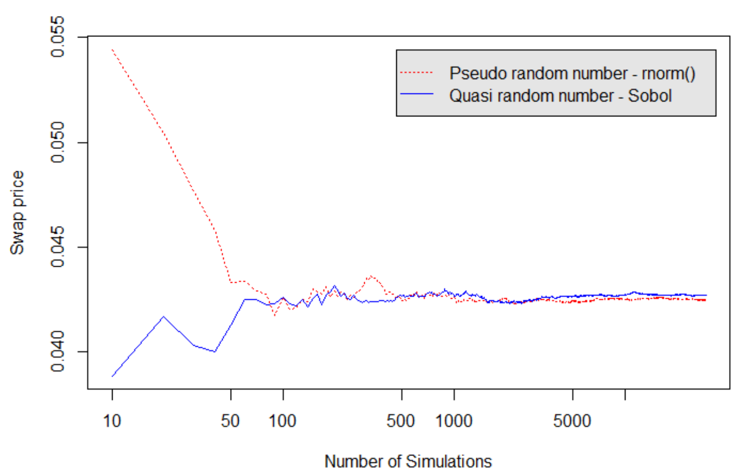

Figure 2.7: Convergence diagram for a swap price (1*5)

Numerical integration
Numerical integration to calculate LIBOR
Multi-threading to improve performance
Newton-Raphson to calculate Implied Volatility

2.5.2 Market data

2.5.3 Algorithm

The algorithm implemented for this project (Interest Rate Derivatives using HJM framework) can be summarized as follows:

1. Load

2.5.4 Pros and Cons

Cons: did not use monthly forward for PCA
Cons: very computing intensive
Pro: numerical integration instead of trapezium rule
Pro: dual-curve - OIS

2.5.5 Instructions and package description

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	ukois09_mdaily_ spotcurve.csv	BOE OIS spot curve

3 Conclusion

La conclusion ...

Appendix

Appendix A

Appendix B

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References

[1] G. Darocczi, M. Puhle and E. Berlinger. Introduction to R for Quantitative Finance. Wiley Finance

[1] D. J. Duffy and A. Germani. *C# for Financial Market*. Wiley Finance

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[3] Duffie [1998]