Pricing First-to-Default Swaps with Copula

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1. Introduction

- CDS CHINA BDS
  - Although CDS became the most controversial financial tool in the Financial Crisis in 2007, it still plays an important role in risk management
  - On Nov. 5th, 2010，China introduced in CRM(Credit Risk Mitigation) officially
  - Loans are the main assets and source of credit risk for commercial banks in China
  - BDS (a basket of Credit Default Swaps) will be more suitable for Chinese commercial banks to manage credit risk

Related Literature

- Studies on CDS:
- Duffie, firstly gave an introduction to CDS;
- Hull et al. studied the problem of pricing CDS with counterparty’s default risk and estimated the spread of BDS;
- Longstaff et al. calculated the theoretical price of CDS using the reduced form model under the framework of random intensity.

- Studies on BDS：
- after Li (1999, 2000) using the Gaussian Copula function to describe the correlation of defaults, Hull and White (2001) proposed a research model of default risk caused by the multi-asset correlation
- Zhou (2006) used the approximate analytical of Monte Carlo simulation in BDS pricing
- Schonbucher and Schubert (2001) studied the dynamic changes of default intensity

Our contribution

- Considering Chinese commercial banks would like to ensure strict triggering mechanism, the study of First-to-Default Swaps, which means whenever an asset (loan) in the pool defaults the contract terminates, will have more practical significance
- I believe this paper can make a contribution to the financial innovation in China and Chinese commercial banks’ credit risk management.

2.Methodology

- CDS pricing model
- BDS pricing model
- Pricing First-to-Default Swaps

2.1 CDS pricing model

- **Structural approach**: assuming a company which has a simple capital structure will default when it insolvents.
- **Reduced form approach**: assuming a company's default events are unpredictable random events and belong to a Poisson process, at the same time using the characteristic parameters of the Poisson process - default intensity to characterize the likelihood of a default event to get a fair spread based on no arbitrage theory.

2.1 CDS pricing model

- Hybrid approach: assuming a company will suddenly default when its value jumps down. Changes in the value of the assets include two random parts: a continuous diffusion part describing the normal fluctuations and a non-continuous jump part describing the sudden market value changes causing by new information.

2.2 BDS pricing model

- key :determine the default correlation among the basket of credit assets

- the conditional independence of default approach
- the infection model
- the Copula Method

- when calculating a company's value related to a set of system factors coupled with enterprise-specific factors based on the structural model, we need Econometrics regression for the factors;
- the reduced form model which tries to explain defaults infection and agglomeration, still has problems in low default correlation problem and using market price.
- Overall, a Copula reduced form model by entering a single company marginal default probability to get the joint correlation structure is more in line with the academic and practical needs.

F_i

2.3 Pricing First-to-Default Swaps

- Define some variables as follows:

n: the number of the assets in a BDS;

F_i : face value of the credit assets (bonds);

$t_1 < t_2 < \ldots < t_m$ = T : spread payment date and T is the swaps' maturity date;

$\Delta(t_{j-1}, t_j)$:time interval between payments;

τ_k : BDS protect purchaser's premium until the kth default;

$$\tau_1 < \ldots < \tau_k < \ldots < \tau_n$$

S_k : swaps' annul spread which is expressed in basis points (bp)

$B(0, t)$: risk-free interest rate of discount;

R_i :the recovery rate of bonds in the basket, so the compensation ratio is $1 - R_i$

2.3 Pricing First-to-Default Swaps

- Under complete market and risk-neutral conditions and according to no arbitrage theory, the present value of the BDS's buyers' payment (spread) should be equal to the present value of compensation payment, so we can calculate the present value of payment (PL) as:

$$PL = S_k F \sum_{j=1}^{n} (\Delta t_{j-1}, \Delta t_j) B(0, t_j) E[l_{\{\tau^k > t_j\}}] \qquad (1)$$

- $F = \sum_{i=1}^{n} F_i$ means the total face value of the portfolio

  $l_{\{\tau^k > t_j\}}$ means the indicator function of a credit event using risk neutral measure

  τ^k means when the underlying assets' kth default happens in the portfolio

2.3 Pricing First-to-Default Swaps

- During the contract period, if the k^{th} asset in the portfolio defaults, the contract's seller will pay the buyer for asset k $F_i(1-R_i)$,the present value of this may-happen payment DL (default leg) can be calculated as
- $$DL=\sum_{i=1}^{n}F_i(1-R_i)E[B(0,\tau^k)l_{\{\tau_i=\tau^k\}}] \qquad (2)$$
- According to no arbitrage theory we can get the fair spread pricing formula: PL =DL

Then we can get the fair spread of BDS:

$$S_k=\frac{\sum_{i=1}^{n}F_i(1-R_i)E[B(0,\tau^k)l_{\{\tau_i=\tau^k\}}]}{\sum_{j=1}^{m}(\Delta t_{j-1},t_j)B(0,t_j)E[l_{\{\tau^k>t_j\}}]} \qquad (3)$$

3. Process

- BDS Pricing Based On Copula Function
- Pricing First-to-Default Swaps in detail
- Numerical Example

3.1 BDS Pricing Based On Copula Function

- 1.establish each single CDS' default probability function $F(i)$
- 2.establish the joint default probability function $G(t_1,t_2,\ldots,t_n)$ of the credit portfolio
- 3.establish multivariate random variable U_k according to the structure of $G(t_1,t_2,\ldots,t_n)$
- 4.calculate the default time , $t_k = F^{-1}(U_k)$ of k^th single CDS;
- 5.rank the default time in ascending order, namely the bonds' default times are $\tau_1,\tau_2,\ldots,\tau_n$
- 6.repeat steps 2) – step 5) we can get the distribution function $H_k(g)$ of τ_k
- 7.use equation (3) to calculate the j^th-to-default Swaps' price.

- We can get each single credit asset's rating and recovery rate R_i and different credit assets' default intensity from the model CreditMetrics.

- we use Hull's (2005) method: $\lambda_i = \dfrac{r_i - r_f}{1 - R_i}$ (r_i means yield rate of credit asset i, r_f means risk-free yield rate, R_i means recovery rate of asset i) to get each credit asset's default intensity
- If the default intensity (hazard function) has a horizon term structure, the default distribution function can be written as follows: $F_i(t) = 1 - \exp(-\lambda_i t)$
- we use the credit asset correlation matrix to get the ρ of Normal Copula. And we use Cholesky matrix factorization to get Normal Copula's random numbers with the ρ.

3.2 Pricing First-to-Default Swaps

- 1.from CreditManger we can get the default intensity of the credit assets, and assuming that the default intensity has a horizon term structure;
- 2.from CreditManager we can get the correlation structure of credit assets
- 3.we extract a series of random number $\{\mu_i'\}_{i=1}^{n}$ from [0,1]-mean distribution under Normal Copula function
- 4.establish each bond’s marginal distribution of default time according to the default intensity from step 1)
- 5.calculate a simulated default time using the random number generated from step 3), as $t = -\ln(1-u)$ in Normal Copula, we can get the simulated default time $t_1, t_2, \ldots, t_n$ of all the credit assets.

3.2 Pricing First-to-Default Swaps（cont）

- 6.rank the default time got in step 5) in ascending order to get a sequence $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_n$
- 7. Repeat step 2) to step 6) 10000 times and get the default time sequence's 10000 simulated values $\{\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_n\}_{i=1}^{10000}$
- 8.calculate the average time $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n$ of each credit asset's default time, compare them to the deadline of the contract to calculate equation (1)
- 9. denote: $\vec{t}_1 = t_i'$, then 10,000 simulated values is divided into n groups, First-to-Default simulated time sequences can be written as $D_1, D_2, \ldots D_n$ .We take the first sequences D_1 to calculate the equation as follow: $DL_i = F_i R_i E[B(0, t_i')]$ So equation (2) can be obtained as: $DL = DL_1 + DL_2 \cdots + DL_n$
- 10. according to equation 3), we can get the default time under a Normal Copula structure and get the fair spread of a First-to-Default Swaps

3.3 Numerical Example

bonds	rating	F_i	R_i	r_i	h_i
1	CCC	100 000	0. 5380	0. 1505	0. 266799
2	CCC	100 000	0. 5113	0. 1505	0. 250231
3	BBB	100 000	0. 3852	0. 0410	0. 008146
4	BBB	100 000	0. 3274	0. 0410	0. 007443
5	BB	100 000 F_i	0. 1709	0. 0555	0. 023567
6	B	100 000	0. 5380	0. 0605	0. 053808
7	B	100 000	0. 51113	0. 0605	0. 050792
8	A	100 000	0. 3852	0. 0372	0. 001953
9	B	100 000	0. 3274	0. 0605	0. 036647
10	B	100 000	0. 1709	0. 0605	0. 029626

Assuming the distribution of τ_i is $F_i(t)$, we can get the survival time's joint distribution based on Copula function as follow:

$$F(t_1,,t_2,\ldots,t_{10}) = C(F_1(t_1),F_2(t_2),\ldots,F_{10}(t_{10}))$$

We take Normal Copula and get:

$$F(t_1,,t_2,\ldots,t_{10}) = \Phi_{10}\left(\phi^{(-1)}(F_1(t_1)),?\phi^{(-1)}(F_2(t_2)),\ldots,\phi^{(-1)}(F_{10}(t_{10}))\right)$$

（Φ_{10} is a 10-dimensional normal cumulative distribution function, we can get its correlation coefficient matrix from CreditMetrics's CreditManager）

{1.0000 0.3367 0.3490 0.3343 0.3865 0.3804 0.3993 0.3797 0.3433 0.1190,
0.3367 1.0000 0.4725 0.5021 0.6686 0.4853 0.4056 0.5040 0.5519 0.3334,
0.3490 0.4725 1.0000 0.3133 0.4621 0.4189 0.2694 0.3555 0.4456 0.2873,
0.3343 0.5021 0.3133 1.0000 0.5945 0.4989 0.4422 0.4899 0.5605 0.2936,
0.3865 0.6686 0.4621 0.5945 1.0000 0.5952 0.6021 0.4752 0.6457 0.4000,
0.3804 0.4853 0.4189 0.4989 0.5952 1.0000 0.5455 0.4457 0.5042 0.3455,
0.3993 0.4056 0.2694 0.4422 0.6021 0.5455 1.0000 0.4693 0.5553 0.3876,
0.3797 0.5040 0.3555 0.4899 0.4752 0.4457 0.4693 1.0000 0.4118 0.3297,
0.3433 0.5519 0.4456 0.5605 0.6457 0.5042 0.5553 0.4118 1.0000 0.4728,
0.1190 0.3334 0.2873 0.2936 0.4000 0.3455 0.3876 0.3297 0.4728 1.0000})

- To simulate the survival time with correlation we introduce another sequence of random variables $Y_1, Y_2, \cdots Y_n$ to make $Y_1 = \Phi^{-1}(F_1(t_1))$, , $\ldots, Y_{10} = \Phi^{-1}(F_{10}(t_{10}))$. We can simulate $Y_1, Y_2, \cdots Y_n$ instead of $\{\tau_i \mid i = 1, 2, \ldots, 10\}$.
- $\{Y_i \mid i = 1, 2, \ldots, 10\}$ is N-dimensional normal distribution and the correlation coefficient matrix is $\sum$
- according to $T_i = F_i^{-1}(N(Y_i))$ we can get $T_1, T_2, \ldots, T_n$. As each credit asset's hazard rate is constant, the density function of survival time T is $he^{(-ht)}$.
- default time sequences is $\tau = \min(T_1, T_2, \ldots, T_n)$
- assuming the interest rate is constant r=0.036, and the contract period is 2 years. If T<2,the present value of the contract is 100000
- according to the above steps, we can get the price of the contract with SAS software programming. It is 57976.031

4. Conclusions

- To simulate default time with Copula should indentify three things: the marginal distribution of default time; Copula function’s form and parameters; under the known Copula function the premise of simulation technology to simulate the random number.
- In this paper, we assuming the default intensity has a horizon term structure to describe the marginal distribution of default time, and give the default intensity parameters of market data; then determine the form and parameters of Copula function, select the Normal Copula function, and use simulation technology - Cholesky matrix decomposition for simulating random numbers, and at last we simulate default time with Copula function.

Conclusions (cont.)

- The contribution of this paper lies in: under the reduced form model we describe the default time correlation of the assets in a BDS using Copula function and take use of SAS to simulate pricing and numerical examples. All of these will make a contribution to the study on CDS and BDS.

Main References

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Thank you for your attention.

Any comments are welcome.