

CQF Exercises, Solutions – Binomial Model

(1) Binomial tree for share price is

$$\begin{array}{c} 84 \\ 80 \\ 76 \end{array}$$

Binomial tree for option price V is

$$\begin{array}{c} 5 \text{ (= max(84 - 79, 0))} \\ V \\ 0 \text{ (= max(76 - 79, 0))} \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{c} 5 - 84\Delta \\ V - 80\Delta \\ -76\Delta \end{array}$$

For risk-free portfolio choose Δ such that $5 - 84\Delta = -76\Delta \Rightarrow \Delta = \frac{5}{8}$. So in absence of arbitrage, $V - 80\Delta = -76\Delta$, and $V = 2.5$.

(2) Binomial tree for share price is

$$\begin{array}{c} 98 \\ 92 \\ 86 \end{array}$$

Binomial tree for option price V is

$$\begin{array}{c} 8 \text{ (= max(98 - 90, 0))} \\ V \\ 0 \text{ (= max(86 - 90, 0))} \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{c} 8 - 98\Delta \\ V - 92\Delta \\ -86\Delta \end{array}$$

For risk-free portfolio choose Δ such that $8 - 98\Delta = -86\Delta \Rightarrow \Delta = \frac{2}{3}$. So in absence of arbitrage, since portfolio is riskless, it must earn risk-free rate $r = 2\%$ and $V - 92\Delta = \exp(-0.02)(-86\Delta)$, then $V = \frac{2}{3}(92 - 86 \exp(-0.02)) = 5.14$.

(3) Binomial tree for share price is

$$\begin{array}{c} 17 \\ 15 \\ 13 \end{array}$$

Binomial tree for option price V is

$$\begin{array}{c} 130 \text{ } (= \max(17^2 - 159, 0)) \\ V \\ 10 \text{ } (= \max(13^2 - 159, 0)) \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{c} 130 - 17\Delta \\ V - 15\Delta \\ 10 - 13\Delta \end{array}$$

For risk-free portfolio choose Δ such that $130 - 17\Delta = 10 - 13\Delta \Rightarrow \Delta = 30$. So in absence of arbitrage, $V - 15\Delta = 10 - 13\Delta$, and $V = 70$.

(4) Binomial tree for share price is

$$\begin{array}{c} 92 \\ 75 \\ 59 \end{array}$$

Time is 3 months - i.e. $\frac{1}{4}$ year, interest rate $r = 0$, so risk neutral probability that share price increases satisfies

$$92p + 59(1 - p) = 75$$

Re-arranging gives

$$p = \frac{75 - 59}{33} = 0.485$$

So probability of a fall would be given by $1 - p = 0.515$.

(5) Using data in question 4) now. We know from above risk-neutral probability is $p = 0.485$. The binomial tree for option price V is

$$\begin{array}{c} 7 \text{ } (= \max(92 - 85, 0)) = V^+ \\ V \\ 0 \text{ } (= \max(59 - 85, 0)) = V^- \end{array}$$

Then value of option $V = pV^+ + (1 - p)V^- \Rightarrow V = 0.485(V^+) = 3.395$

(6) Binomial tree for share price is

$$\begin{array}{c} 92 \\ 75 \\ 59 \end{array}$$

Time is 3 months - i.e. $\frac{1}{4}$ year, interest rate $r = 4\%$, so risk neutral probability that share price increases satisfies

$$92p + 59(1 - p) = 75 \exp\left(\frac{0.04}{4}\right)$$

Re-arranging gives

$$p = \frac{75e^{0.01} - 59}{33} = 0.5077$$

So probability of a fall would be given by $1 - p = 0.4923$.

(7)

$$pu + (1 - p)v = e^{\mu \delta t} \quad (a)$$

$$pu^2 + (1 - p)v^2 = e^{(2\mu + \sigma^2) \delta t} \quad (b)$$

$u.(a) + v.(a)$ gives

$$(u + v)e^{\mu \delta t} = pu^2 + uv - puv + pvu + v^2 - pv^2.$$

Rearrange to get

$$(u + v)e^{\mu \delta t} = pu^2 + (1 - p)v^2 + uv$$

and we know from (b) that $pu^2 + (1 - p)v^2 = e^{(2\mu + \sigma^2) \delta t}$ and $uv = 1$.
Hence we have

$$\begin{aligned} (u + v)e^{\mu \delta t} &= e^{(2\mu + \sigma^2) \delta t} + 1 \Rightarrow \\ (u + v) &= e^{-\mu \delta t} + e^{(\mu + \sigma^2) \delta t}. \end{aligned}$$

Now recall that the quadratic equation $ax^2 + bx + c = 0$ with roots α and β has

$$\alpha + \beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}.$$

We have

$$\begin{aligned} (u + v) &= e^{-\mu \delta t} + e^{(\mu + \sigma^2) \delta t} \equiv -\frac{b}{a} \\ uv &= 1 \equiv \frac{c}{a} \end{aligned}$$

hence u and v satisfy

$$(x - u)(x - v) = 0$$

to give the quadratic

$$x^2 - (u + v)x + uv = 0 \Rightarrow$$

$$x = \frac{(u + v) \pm \sqrt{(u + v)^2 - 4uv}}{2}$$

so with $u > 1$

$$u = \frac{1}{2} \left(e^{-\mu \delta t} + e^{(\mu + \sigma^2) \delta t} \right) + \frac{1}{2} \sqrt{\left(e^{-\mu \delta t} + e^{(\mu + \sigma^2) \delta t} \right)^2 - 4}$$

(8)

share price		Option price	
	103		$\max[103 - 100, 0] = 3$
100		V	
	98		$\max[98 - 100, 0] = 0$

Δ hedged portfolio: $V - \Delta S \Rightarrow$

$$\begin{array}{r} 3 - 103\Delta \\ V - 100 \\ -98\Delta \end{array}$$

$$\text{risk-free} \Rightarrow 3 - 103\Delta = -98\Delta \Rightarrow \Delta = 0.6$$

$$V - 100 = -98\Delta \rightarrow V = 1.2$$

$$\Pi = 3 - 103\Delta = \boxed{-58.8}$$

$$103p + 98(1 - p) = 100 \rightarrow p = 0.4 \quad \therefore 1 - p = 0.6$$

If interest rates are non zero, and there is a discount factor of 0.99, how does this affect the results? Fill in the blanks in the following diagram.

As in the first part $\Delta = 0.6$

$$\text{Earnings at risk-free rate, } \therefore V - 100 = 0.99(-98\Delta) \rightarrow V = 1.788$$

$$\Pi = V - 100\Delta = -58.212$$

$$S = [pS^+ + (1 - p)S^-] 0.99 \Rightarrow 100 \div 0.99 = 103p + (1 - p)98 \Rightarrow 0.602$$

$$\therefore 1 - p = 0.398$$