Itô's Lemma and Stochastic Differential Equations

Throughout this problem sheet, you may assume that X_t is a Brownian Motion (Weiner Process) and dX_t is its increment. $X_0 = 0$.

1. The change in a share price $S(t)$ satisfies

$$dS = A(S,t)\,dX_t + B(S,t)\,dt,$$

for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dX_t.$$

Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.

2. Show that $F(X_t) = \arcsin(2aX_t + \sin F_0)$ is a solution of the stochastic differential equation

$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dX_t,$$

where F_0 and a is a constant.

3. Show that

$$\int_0^t X_t\left(1 - e^{-X_t^2}\right)dX_t = \overline{F}(X_t) + \int_0^t G(X_\tau)\,d\tau$$

where the functions $\overline{F}$ and G should be determined.

4. Consider a two factor model in which the stock price dynamics S_t, follows Geometric Brownian Motion and the stock variance v_t is itself stochastic and follows a square root process

$$dS_t = \mu S_t dt + \sqrt{v_t}S_t dX_1(t),$$

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t}dX_2(t).$$

The two processes have a correlation coefficient ρ, i.e.

$$dX_1(t)dX_2(t) = \rho dt$$

The parameters $\mu,\ \lambda,\ \bar{v}$ and η are all constant. Let $F = F(t, S_t, v_t)$. Using Itô, consider the SDE for dF and integrate over $[0,t]$ to obtain an expression for $F(t, S_t, v_t)$.

5. Consider the stochastic differential equation

$$dG\left(t\right)=a\left(G,t\right)dt+b\left(G,t\right)dX_t.$$

Find $a\left(G,t\right)$ and $b\left(G,t\right)$ where

(a) $G\left(t\right)=X_t^2$

(b) $G\left(t\right)=1+t+e^{X_t}$

(c) $G\left(t\right)=f_tX_t$, where f_t is a bounded and continuous function.

6. Show that

$$G=\exp\left(t+a\exp\left(X\left(t\right)\right)\right)$$

is a solution of the stochastic differential equation

$$dG\left(t\right)=G\left(1+\tfrac{1}{2}\left(\log G-t\right)+\tfrac{1}{2}\left(\log G-t\right)^2\right)dt+G\left(\log G-t\right)dX$$

7. Use Itô's lemma to show that

$$d\left(\cos X_t\right)=\alpha\left(\cos X_t\right)dt+\beta\left(\sin X_t\right)dX_t$$

&

$$d\left(\sin X_t\right)=\alpha\left(\sin X_t\right)dt-\beta\left(\cos X_t\right)dX_t$$

and determine the constants α & β.