BERMUDAN SWAPTIONS IN GAUSSIAN HJM ONE-FACTOR MODEL: ANALYTICAL AND NUMERICAL APPROACHES

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ABSTRACT. A popular way to value (Bermudan) swaption in a Hull-White or extended Vasicek model is to use a tree or PDE approach . A more direct approach through iterated numerical integration is developed. A brute force numerical integration would lead to a complexity exponential in the number of exercise dates in the base of the number of points (p^N). By carefully choosing the integration points and their order, the complexity can be reduced to $4pN$ versus the quadratic complexity $(pN)^2$ in the tree. A semi-explicit formula leading to a faster converging implementation is also proposed.

1. Introduction

Bermudan swaptions are among the most popular exotic interest rate derivatives. This popularity is certainly linked to the steady flow of Bermudan callable bonds issuance. Prepayable fixed rate mortgages are another major product that present a risk profile similar to a Bermudan swaption.

Bermudan swaptions are American-style or compounded options on interest rate swaps. A set of pre-specified dates is agreed at the beginning of the contract. At each exercise date the option owner can either enter into a swap or keep its right up to the next exercise date. The difficulty in dealing with such derivatives stems from the compounded options. At each date an exercise decision must be taken. For that decision to be taken optimaly one need to compute the price of the remaining option, which is it-self is a Bermudan option. In a forward approach (typically Monte Carlo) an exercise strategy is required. In a backward approach (typically tree or PDE), the value is first estimated at the last exercise date for a discretized set of world state and brought to the present, adding the exercise date one by one. Each intermediary Bermudan swaption is constructed recursively on a discrete set of points, usually in a recombining way.

In the martingale approach to option pricing, the prices are obtained through expectations. One way to numerically compute the expectation, if the distribution of the underlying random variable is known, is to perform a numerical integration. Consequently one way to price Bermudan swaptions is to perform a series of numerical integrations representing embedded integrals. The complexity of this computation is exponential in the number of dates; for p points at each date and N dates, one has approximatively p^N computations to do.

In a (trinomial) tree approach, where for one expiry date to the next one uses p steps between each of the N exercise date, the final point number is around $2pN+1$ and the number of computed values is $Np(Np+1)$ with 3 branches for each [9]. A comparison for a usual number of points and dates is clearly in favor of the later. Possible numbers would be $p = 100$ and $N = 10$, giving 10^{20} computations for the multi-integral and 10^6 for the tree.

For swaptions the number of iterated integrals can be reduced by one using an explicit formula [5] for the last optionality which is of European type. Even if the explicit formula is usually faster than the numerical computation it involved the solution of a one-dimensional non-linear equation. Fortunately it is possible to solve the equation only once and to use the result for the different

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points of the numerical integral. A way to achieve this for Bermudan swaptions with only two expiry dates was presented in [6].

But even by reducing the number of expiry dates by one, the brute numerical integration is not efficient. In this note we describe a way to reduce the number of computations in the last integration to $2p(N-1)$ and the total to $pN(N-1)$. Using the previous example with $p=100$ and $N=10$, the number of values is around 310^3. The reduction is possible thanks to a carefull choice of equidistant points in the integration and a separability condition on the volatility structure.

The use of equidistance points can be viewed as similar to the tree approach. But for a (theoretical) binomial tree with 50% probability on each branche, and 100 points, the extreme points have a probability of $(1/2)^{100} \sim 10^{-30}$. The tree approach consequently spends a lot of time on almost useless (tiny probability) computations. While in numerical integration it is possible to *cut* the discretisation at your choosing. In our implementation we chose extreme points such that the left-over is very small at each step. Moreover the numerical integration is cut outside a certain range to avoid similar extremely low probability computations. This may be related to the PDE approach where the total number of points (not increased at each step) is selected. With the cut, the number of values in our implementation is $4(N-1)p$. As a supplementary improvement a second order numerical integration scheme is used, improving significantly the convergence and stability.

Related approach in the literature, even if this one was developed independently, can be found in Gandhi and Hunt [3]. They also propose a numerical integration approach to Bermudan swaption in Hull-White model and recombining properties based on equally spaced points. Their approach is based on short rate while a more direct approach on the bond prices is used here. The difference between the two approaches is similar to the difference existing for European options between the Jamshidian approach and the more direct approach described in [5].

In some sense our approach can also be linked to what Rebonato [14] call *long jump* technique. Computations are done only at price sensitive dates (no intermediary points) and between those date the diffusion is done analytically.

Also like for the 2-Bermudan swaption [6], it is possible to write explicitly the part of the value corresponding to the exercise into a swap at the first date. In practice, for a lot of options, this first option contains most of the value. By computing in an explicit formula the majority of the value we achieve a better convergence of the results. The speed is not improved by this semi-explicit formulation as to estimate the part on which the explicit method apply one need to compute the numerical value for all the points.

The results presented here are valid for Heath-Jarrow-Morton models satisfying the separability condition (H2). The models used in practice that satisfy this condition are the Hull-White and the Ho-Lee models, with the former being the more frequent.

2. Model, hypothesis and preliminary results

The HJM framework describes the behavior of $P(t,u)$, the price in t of the zero-coupon bond paying 1 in u $(0 \leq t, u \leq T)$. When the discount curve $P(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t,u)$ such that

$$P(t,u) = \exp\left(-\int_t^u f(t,s)ds\right). \tag{1}$$

The idea of [4] was to exploit this property by modeling f with a stochastic differential equation

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)dW_t$$

for some suitable (potentially stochastic) μ and σ and deducing the behavior of P from there. To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of [10].

The probability space is $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W_t)_{0\leq t\leq T}$. To simplify the writing in the rest of the paper, the notation

$$\nu(t,u) = \int_t^u \sigma(t,s)ds$$

is used.

Let $N_t = \exp(\int_0^t r_s ds)$ be the cash-account numeraire with $(r_s)_{0\leq s\leq T}$ the short rate given by $r_t = f(t,t)$. The equations of the model in the numeraire measure associated to N_t are

$$df(t,u) = \sigma(t,u)\nu(t,u)dt + \sigma(t,u)dW_t$$

or

$$dP^N(t,u) = -P^N(t,u)\nu(t,u)dW_t$$

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$.

The following *separability hypothesis* will be used:

H: The function σ satisfies $\sigma(t,u) = g(t)h(u)$ for some positive functions g and h.

Note that this condition is essentially equivalent to the condition (H2) of [5] but written on σ instead of on ν. The condition on ν was $\nu(s,t_2) - \nu(s,t_1) = f(t_1,t_2)g(s)$.

Separability conditions have been widely used in interest rate modeling. The results cover Markov process in [2], explicit swaption formulas in Gaussian HJM model in [5] and [8], efficient approximation for LMM in [12] and [1] and hybrids in [13].

We recall the generic pricing theorem [10, Theorem 7.33-7.34].

Theorem 1. *Let V_T be some $\mathcal{F}_T$-measurable random variable. If V_T is attainable, then the time-t value of the derivative is given by $V_t^N = V_0^N + \int_0^t \phi_s dP_s^N$ where ϕ_t is the strategy and*

$$V_t = N_t \,\mathrm{E}^{\mathbb{N}}\left[V_T N_T^{-1} \middle| \mathcal{F}_t\right].$$

We now state two technical lemmas that were presented in [6].

Lemma 1. *Let $0 \leq t \leq u \leq v$. In a HJM one factor model, the price of the zero coupon bond can be written has,*

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp\left(-\frac{1}{2}\int_t^u \left(\nu^2(s,v) - \nu^2(s,u)\right) ds + \int_t^u \left(\nu(s,v) - \nu(s,u)\right) dW_s \right).$$

Lemma 2. *In the HJM one factor model, we have*

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_s ds \right) = P(u,v) \exp\left(\int_u^v \nu(s,v)dW_s - \frac{1}{2}\int_u^v \nu^2(s,v)ds \right).$$

We give the pricing formula for swaptions for a future time ([6, Theorem 2]. The notations used to describe the swaption are the following. Let θ be the expiry date and the swap is represented by its cash-flow equivalent $(t_i, c_i)_{i=0,\ldots,n}$. The date t_0 is the swap start date and t_i $(i = 1,\ldots,n)$ are the fix coupon dates. The amounts c_0 is -1[1], $c_i > 0$ $(i = 1,\ldots,n-1)$ are the coupons and $c_n > 0$ is the final coupon plus 1 for the notional.

Theorem 2. *Suppose we work in the HJM one-factor model with a volatility term of the form (H2). Let $\theta \leq t_0 < \cdots < t_n$, $c_0 < 0$ and $c_i \geq 0$ $(1 \leq i \leq n)$. The price of an European receiver swaption, with expiry θ on a swap with cash-flows c_i and cash-flow dates t_i is given at time t by the $\mathcal{F}_t$-measurable random variable*

$$\sum_{i=0}^n c_i P(t,t_i) N(\kappa + \alpha_i)$$

[1]It is $-K$ for a bond option of strike K.

where κ is the $\mathcal{F}_t$-measurable random variable defined as the (unique) solution of

$$\sum_{i=0}^{n} c_i P(t,t_i) \exp\left(-\frac{1}{2}{\alpha_i}^2 - \alpha_i \kappa\right) = 0 \tag{2}$$

and

$${\alpha_i}^2 = \int_t^\theta \left(\nu(s,t_i) - \nu(s,\theta)\right)^2 ds.$$

The price of the payer swaption is

$$-\sum_{i=0}^{n} c_i P(t,t_i) N(-\kappa - \alpha_i)$$

The following result describes the change of probability for conditional expectation (find a reference!).

Theorem 3. *Let X be a random variable, $\mathcal{G}$ be a sub-σ-algebra and ξ be the Radon-Nikodym derivative $\frac{dQ}{dP}$, then*

$$\mathrm{E}^P\left[\xi|\,\mathcal{G}\right]\mathrm{E}^Q\left[X|\,\mathcal{G}\right] = \mathrm{E}^P\left[X\xi|\,\mathcal{G}\right].$$

3. Main result

The notations we use to describe the swaption are the following. The N expiry dates are $0 < \theta_1 < \theta_2 < \cdots \theta_N$ and to simplify some notations we set $\theta_0 = 0$. For each expiry i $(1 \leq i \leq N)$ the swap that can be enter into by exercising the option in θ_i has n_i fixed coupons. The swap is represented by its cash-flow equivalent $(t_{i,j}, c_{i,j})_{j=0,\ldots,n_i}$. The date $t_{i,0}$ is the swap start date and $t_{i,j}$ $(j = 1,\ldots,n_i)$ are the fix coupon dates. The amounts $c_{i,0}$ are -1[2], $c_{i,j} > 0$ $(j = 1,\ldots,n_i - 1)$ are the coupons and $c_{i,n_i} > 0$ is the final coupon plus 1 for the notional.

For the study of the swaptions we will use a change of probability. We define $\nu^\#(s) = \nu(s,\theta_{i+1})$ for $s \in [\theta_i,\theta_{i+1})$ and the Dolean exponential of its stochastic integral

$$L_t = \mathcal{E}\left(\int_0^t \nu^\#(s)dW_s\right) = \exp\left(\int_0^t \nu^\#(s)dW_s - \frac{1}{2}\int_0^t {\nu^\#}^2(s)ds\right).$$

The numeraire associated to the new measure is the rolled bond with successive maturities at θ_i, the expiry dates.

Theorem 4. *Suppose we work in a HJM one-factor model with a volatility structure of the form (H2). Consider a N-Bermudan receiver swaption with expiry dates $\theta_1 < \theta_2 < \cdots < \theta_N$ on swaps represented by $(t_{i,j}, c_{i,j})_{i=1,\ldots,N;j=0,\ldots,n_i}$. Let $\alpha_{i,j,k}$ $(i = 1,\ldots,N; j = 0,\ldots,n_i, k = 1,\ldots,i)$ be the positive number defined by*

$${\alpha_{i,j,k}}^2 = \int_{\theta_{k-1}}^{\theta_k} \left(\nu(s,t_{i,j}) - \nu(s,\theta_k)\right)^2 ds.$$

Let $V_{\theta_k}^{N-k}$ be the value of the $(N-k)$-Bermudan swaption at time θ_k $(k = 0,\ldots,N-1)$. Let $\tilde{V}_{\theta_k}^{N-k} = N_{\theta_0} N_{\theta_k}^{-1} V_{\theta_k}^{N-k} L_{\theta_k}^{-1}$. The $\tilde{V}_k^{N-k}$ are $\mathcal{F}_{\theta_k}$-measurable random variables given recursively by

$$\tilde{V}_{\theta_{k-1}}^{N-(k-1)} = \mathrm{E}^\#\left[\max\left(\sum_{j=0}^{n_k} c_{k,j} P(0,t_{k,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{k}\alpha_{k,j,l}^2 - \sum_{l=1}^{k}\alpha_{k,j,l}X_l\right), \tilde{V}_{\theta_k}^{N-k}\right)\,\middle|\,\mathcal{F}_{\theta_{k-1}}\right] \tag{3}$$

[2] It is $-K$ for a bond option of strike K.

and $\tilde{V}^1_{\theta_{N-1}}$ is the European receiver swaption given by

$$\tilde{V}^1_{\theta_{N-1}} = \sum_{j=0}^{n_N} c_{N,j} P(0,t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N-1}\alpha^2_{N,j,l} - \sum_{l=1}^{N-1}\alpha_{N,j,l}X_l\right) N(\kappa + \alpha_{N,j,N}) \tag{4}$$

with κ given by

$$\sum_{j=0}^{n_N} c_{N,j} P(0,t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N}\alpha^2_{N,j,l} - \sum_{l=1}^{N-1}\alpha_{N,j,l}X_l - \alpha_{N,j,N}\kappa\right) = 0 \tag{5}$$

and the X_l are independent $\mathcal{F}_l$-measurable $\mathbb{P}^\#$-standard normal random variable.

The payer swaption can be valued in a similar way by replacing Equation 4 by the equvalent payer European swaption and replacing $c_{k,j}$ by $-c_{k,j}$ in Equation 3.

Proof. Using the generic pricing formula we have for $k = 0, \ldots, N-2$,

$$V^{N-(k-1)}_{\theta_{k-1}} = N_{\theta_{k-1}} \operatorname{E}\left[N^{-1}_{\theta_k} \max\left(\sum_{j=0}^{n_k} c_{k,j} P(\theta_k, t_{k,j}), V^{N-k}_{\theta_k}\right) \middle| \mathcal{F}_{\theta_{k-1}}\right].$$

Using the explicit formula for European swaptions [5],

$$V^1_{\theta_{N-1}} = \sum_{j=0}^{n_N} c_{N,j} P(\theta_{N-1}, t_{N,j}) N(\kappa + \alpha_{N,j,N})$$

where κ is the solution of

$$\sum_{j=0}^{n_N} c_{N,j} P(\theta_{N-1}, t_{N,j}) \exp\left(-\frac{1}{2}\alpha^2_{N,j,N} - \alpha_{N,j,N}\kappa\right) = 0.$$

Using Lemmas 1 and 2 recursively k times together with the definition of L_t, we have that

$$N_{\theta_0} N^{-1}_{\theta_k} P(\theta_k, t_{i,j}) = L_{\theta_k} P(0, t_{i,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{k}\alpha^2_{i,j,l} - \sum_{l=1}^{k}\alpha_{i,j,l}X_l\right).$$

where the X_l are independent $\mathcal{F}_l$-measurable standard normal random variables with respect to the probability $\mathbb{P}^\#$. To obtain the result we used Girsanov Theorem [11, Section 4.2.2, p. 72] with $\nu^\#$. The random variables X_l are the same for all i and j thanks to the property (H2) of the volatility function.

With this result and using the result on conditional expectation Theorem 3, we can rewrite the value of the options

$$\begin{aligned}
\tilde{V}^{N-(k-1)}_{\theta_{k-1}} &= N_{\theta_0} N^{-1}_{\theta_{k-1}} V^{N-(k-1)}_{\theta_{k-1}} L^{-1}_{\theta_{k-1}} \\
&= L^{-1}_{\theta_{k-1}} N_{\theta_0} \operatorname{E}\left[N^{-1}_{\theta_k} \max\left(\sum_{j=0}^{n_k} c_{k,j} P(\theta_k, t_{k,j}), V^{N-k}_{\theta_k}\right) \middle| \mathcal{F}_{\theta_{k-1}}\right] \\
&= \operatorname{E}^\#\left[\max\left(\sum_{j=0}^{n_k} c_{k,j} P(0, t_{k,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{k}\alpha^2_{k,j,l} - \sum_{l=1}^{k}\alpha_{k,j,l}X_l\right), \tilde{V}^{N-k}_{\theta_k}\right) \middle| \mathcal{F}_{\theta_{k-1}}\right]
\end{aligned}$$

Similarly by replacing P in the equation defining κ, we obtain an implicit definition of κ which depend on X_l:

$$\sum_{j=0}^{n_N} c_{N,j} P(0,t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N}\alpha^2_{N,j,l} - \sum_{l=1}^{N-1}\alpha_{N,j,l}X_l - \alpha_{N,j,N}\kappa\right) = 0.$$

And for the European swaption we obtain

$$N_{\theta_0} N_{\theta_{N-1}} V^1_{\theta_{N-1}} L^{-1}_{\theta_{N-1}} = \sum_{j=0}^{n_N} c_{N,j} P(0, t_{N,j}) \exp\left(-\frac{1}{2} \sum_{l=1}^{N-1} \alpha^2_{N,j,l} - \sum_{l=1}^{N-1} \alpha_{N,j,l} X_l \right) N(\kappa + \alpha_{N,j,N}).$$

□

Like for 2-Bermudan swaption we can write explicitly the expected value for the exercise at θ_1. We obtain the following semi-explicit valuation theorem.

Theorem 5 (Semi-explicit formula). *Let μ be defined by*

$$\mu = \min_{y \in \mathbb{R}} \left\{ \sum_{j=0}^{n_1} c_{1,j} P(0, t_{1,j}) \exp\left(-\frac{1}{2} \alpha^2_{1,j,1} - \alpha_{1,j,1} y \right) \leq \tilde{V}^{N-1}_{\theta_1}(y) \right\}$$

with the convention that if the set is empty, $\mu = +\infty$ and if the set a no minimum, $\mu = -\infty$.

The value of the Bermudan swaption of the previous theorem can then be written as

$$\begin{aligned} V_0^N &= \sum_{j=0}^{n_1} c_{1,j} P(0, t_{1,j}) N(\mu + \alpha_{1,j,1}) \\ &\quad + \mathrm{E}\left[1\!\!1(X_1 \geq \mu) \max\left(\sum_{j=0}^{n_1} c_{1,j} P(0, t_{1,j}) \exp\left(-\frac{1}{2} \alpha^2_{1,j,1} - \alpha_{1,j,1} X_1 \right), \tilde{V}^{N-1}_{\theta_1} \right) \right] \end{aligned}$$

with $\tilde{V}^{N-1}_{\theta_1}$ defined in the previous theorem.

Knowing if μ is non trivial ($-\infty < \mu < \infty$) is in general not obvious. Also the uniqueness (or non uniqueness) of the number for which we have an equality is a non-trivial question.

In the case where all the underlying swaps have the same payment dates and amounts after their respective expiry, the different parts are decreasing with their difference decreasing up to a common factor. This is the most frequent case in practice. In practice μ exists and there is only one intersection. The formula becomes

$$V_0^N = \sum_{j=0}^{n_1} c_{1,j} P(0, t_{1,j}) N(\mu + \alpha_{1,j,1}) + \mathrm{E}\left[1\!\!1(X_1 \geq \mu) \tilde{V}^{N-1}_{\theta_1} \right].$$

4. Numerical implementation

4.1. **Alpha.** In the case of the extended Vasicek or one-factor Hull and White model, one has $\sigma(s,t) = \eta \exp(-a(t-s))$ and $\nu(s,t) = (1 - \exp(-a(t-s)))\eta/a$ in the constant volatility case. The time-dependent volatility is also covered with $\sigma(s,t) = \eta(s) \exp(-a(t-s))$ and $\nu(s,t) = (1 - \exp(-a(t-s)))\eta(s)/a$. The α used in the theorems are given in the constant volatility case by

$$\alpha^2_{i,j,k} = \frac{\eta^2}{2a^3} \left(\exp(-a\theta) - \exp(-at_{i,j}) \right)^2 \left(\exp(2a\theta_k) - \exp(2a\theta_{k-1}) \right).$$

In the time-dependent case, η is piece-wize constant with $\eta(s) = \eta_i$ for $s_{i-1} \leq s \leq s_i$ and $0 = s_0 < s_1 < \cdots < s_n = +\infty$. The expiry dates are between some of those dates and the relevant dates are denoted $s_p \leq \theta_{k-1} < \theta_k \leq s_q$. To shorten the notation a intermediary notation is used: $r_p = \theta_{k-1} < r_l = s_l < r_q = \theta_k$. With those notations, one has

$$\alpha^2_{i,j,k} = \frac{1}{2a^3} \left(\exp(-a\theta) - \exp(-at_{i,j}) \right)^2 \sum_{l=p}^{q-1} \eta_l^2 \left(\exp(2ar_{l+1}) - \exp(2ar_l) \right).$$

4.2. **Kappa.** From the definition it seems that equation (5) needs to be solved for each draw of $\{X_l\}$. In the next lemma we show that this is not the case.

Lemma 3. *Under the hypothesis of Theorem 4, the solution κ of (5) is given by*

$$\kappa = \frac{1}{\beta_N}\left(\Lambda - \sum_{l=1}^{N-1} \beta_l X_l\right)$$

where Λ is the (unique) solution of

$$\sum_{j=0}^{n_N} c_{N,j} P(0, t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N} \alpha_{N,j,l}^2 - H(t_{N,j})\Lambda\right) = 0, \tag{6}$$

$H(t) = \int_0^t h(s)ds$ and $\beta_l = \sqrt{G(\theta_l) - G(\theta_{l-1})}$ for $G(t) = \int_0^t g^2(s)ds$ (h and g are defined in (H2)).

Proof. By definition

$$\nu(t,u) = \int_t^u \sigma(t,s)ds = \int_t^u h(s)ds\, g(t) = (H(u) - H(t))\, g(t).$$

From there we have that

$$\alpha_{N,j,l}^2 = \int_{\theta_{l-1}}^{\theta_l} (\nu(s, t_{N,j}) - \nu(s,\theta_l))^2 ds = \beta_l^2 (H(t_{N,j}) - H(\theta_l))^2.$$

The last two terms in the equation (5) are

$$-\sum_{l=1}^{N-1} \alpha_{N,j,l} X_l - \alpha_{N,j,N}\kappa = -\left(\sum_{l=1}^{N-1} \beta_l X_l + \beta_N \kappa\right) H(t_{N,j}) + \left(\sum_{l=1}^{N-1} \beta_l X_l H(\theta_l) + \beta_N \kappa H(\theta_N)\right).$$

The second last term in this last expression being independent of j, it can be simplified in the equation (5) and we obtain the result. □

To compute the root in 6, one can use the equivalent equation

$$\sum_{j=0}^{n_N} c_{N,j} P(0, t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N} \alpha_{N,j,l}^2 - (H(t_{N,j}) - H(t_{N,0})\Lambda\right) = 0,$$

in which the left hand side function is monoteneous and non-degenerated in Λ.

In the piece-wize constant extended Vasicek set-up, the β's are given by

$$\beta_k^2 = \frac{1}{2a}\sum_{l=p}^{q-1} \sigma_l^2 \left(\exp(2ar_{l+1}) - \exp(2ar_l)\right).$$

4.3. **Some notation.** To shorten the writing we use the following notations ($1 \leq k \leq N-1$):

$$Y_k = \textstyle\sum_{l=1}^k \beta_l X_l, \qquad Z_l = \exp\left(\beta_l X_l H(\theta_l)\right),$$
$$T_k = \textstyle\sum_{j=0}^{n_k} c_{k,j} P(0, t_{k,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^k \alpha_{k,j,l}^2 - H(t_{k,j}) Y_k\right),$$
$$W_{N-1} = \textstyle\sum_{j=0}^{n_N} c_{N,j} P(0, t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N-1} \alpha_{N,j,l}^2 - H(t_{N,j}) Y_{N-1}\right) N(\kappa(Y_{N-1}) + \alpha_{N,j,N}).$$

With those notations, the swap that compose the term of the max in the description of $\tilde{V}_{\theta_{k-1}}$ is

$$S_{\theta_k}^k = T_k Z_k \prod_{l=1}^{k-1} Z_l \quad \text{and} \quad \tilde{V}_{\theta_{N-1}} = W_{N-1} \prod_{l=1}^{N-1} Z_l.$$

Let

$$W_{k-1} = \mathrm{E}^{\#}\left[Z_k \max(T_k, W_k) \middle| \mathcal{F}_{\theta_{k-1}}\right]$$

Value number	Tree $(Np)^2$	Numerical integration $pN(N-1)$	NI with cut $4p(N-1)$
p **x** N			
10 x 100	10^6	$9\ 10^3$	$3.6\ 10^3$
20 x 100	$4\ 10^6$	$3.8\ 10^4$	$7.6\ 10^3$
20 x 50	10^6	$1.9\ 10^4$	$3.8\ 10^3$
Table note			

Table 1. Number of call dates and discretization and the impact of value number in different implementations.

then

$$\tilde{V}_{\theta_{k-1}}^{N-(k-1)} = W_{k-1} \prod_{l=1}^{k-1} Z_l. \tag{7}$$

4.4. **How to compute the integrals.** To compute the nested integrals we sample $\beta_l X_l$ using $2p+1$ equally spaced points $[-p\epsilon, \dots, p\epsilon]$. Then Y_k is sampled with $2pk+1$ equally spaced points $[-pk\epsilon, \dots, pk\epsilon]$. The Z_l are sampled directy from X_l.

Note that $W_{N-1} = W_{N-1}(Y_{N-1})$ and its $2p(N-1)+1$ values can be easily computed from W_{N-1}. We are interested by $V_{\theta_0}^N = \tilde{V}_{\theta_0}^N = W_0$. We only need to compute (recursively) the W_k.

The T_k depend only on Y_k and are sampled with $2pk+1$ points. We suppose that they have been computed from the Y_k.

The W_{k-1} are expected values depending on Z_k, T_k and W_k. To compute the m-th point ($-pk \leq m \leq pk$) of the sample of W_{k-1} we compute the expected value over the $2p+1$ points of Z_k multiplied by the $2p+1$ points that symmetrically surround the m-th point of $\max(T_k, W_k)$.

The implementation uses explicitely the fact that the points of the different $\beta_l X_l$ are equidistant with the same distance at each level. The number of points could potentially be different but this would introduce some complication in the algorithm. Like in the trinomial tree implementation the extreme points of the latter steps have an extremely lo probability and almost no impact on the final price. Moreover on the extreme out-of-the-money options the value is very close to zero and on the extreme in-the-money options the swap price will be used. The Bermuda value on extreme mouvements is almost not used. This suggest that extreme points can be cut. This is done in our implementation where the cut is done at twoce the initial number of points. This can be compared to a PDE implementation where the spatial discretization has a fixed number of points for all steps. With the cut the number of values computed for the swaps and the Bermuda swaptions through the procedure is around $N-1$ steps multiplied by a maximum of $2p$ points. The exact number is $4(N-1)p$ values. Some examples are given in Table 1.

The test are run with several implementations. All of them uses the general algorithm described above. In the first implementation, the integral is computed using simple *rectangles*. For each point, the maximum of T and W is taken and multiplied by the probability of the interval surrounding it. In the second implementation a *trapezoidal* approach is used. Each point is consider as the end point of an interval. The numerical integral is computed using the usual trapeze approach. Around the maximum a special process is used. The intersection point is computed by linear interpolation between the two adjacent points. This point is added in the integration points. The integral is computed on each side using the same trapezes approach. Taking the intersection into account does not make a large difference in term of convergence. Nevertheless it improve the stability significantly. The third implementation uses parabola fit instead of trapezoid. The maximum interestion is also computed with the parabola fit. The last two are similar to the previous two with the analytical part for the swap at first exercise date.

The semi-analytical approach described in Theorem 5 is marginally slower. The reason is that one has to first compute all the points of the numerical integration to estimate the μ. Once this is done the points on one side of μ are used in the numerical integration and the other side are disregarded.

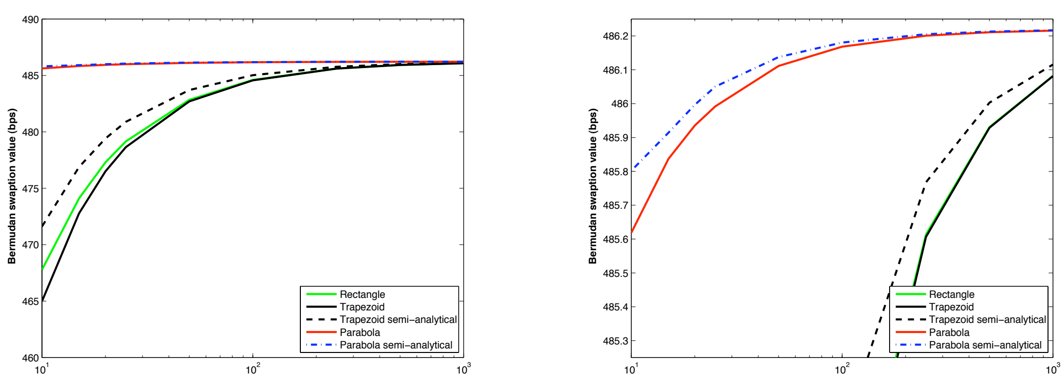
5. Convergence and stability

It was shown in a previous article [8] that numerical integration and semi-analytical approaches are faster and more precise than the classical tree approach for 2-Bermudan swaptions. In particular it was shown that in the tree case the delta figures compute by finite difference are unstable and gamma figures meaningless. The method called rectangle here is related to the method called equi-spaced in the mentioned article on 2-Bermudan swaptions. That method was proved to be significantly faster and more stable than the trees. The trapezoidal and parabola methods proposed here will be showed to the significantly more efficient than the rectangle one. By transitivity of the improvement there is no need to compare the present approach to the trinomial trees.

The term *number of points* (or *step-point*) used in the analysis is the one mention in the introduction. Let N be the number of expiry dates. A number of points equal to p means that the final integration has $2p(N-1)$ points and the total number of value estimated is $pN(N-1)$.The integrals are computed with $2p-1$ points. The total number of points is reduced to $4p(N-1)$ in the cut approach. Remember that in the trinomial tree approach the number of value estimated is of the order of $(Np)^2$.

All the computations are done with a 1y x 9y Bermudan swaption with annual expiry dates. There are nine expiry dates between one and nine years from now. The swaps to be entered into have tenors between nine and one year with a common final maturity ten years from now. The coupon is set at 4% and the swaption notional is 100m. The curve is flat at 4%.

In convergence term the semi-analytical versions propose an improvement. A large part of the integral is computed explicitly. For that part, the discretization is used only to estimate the integral limits in a way similar to the bond futures approach presented in [7]. In that sense the error is reduce by two. This is roughly what is visible in Figure 1. In that figure the price of the bermudan option mentioned above is computed for number of points between 10 and 1,000. The price with very high number of points is computed only for the interest of the graph. In practice 25 points seems to be sufficient. The error is below 0.2 basis points for the best approach.

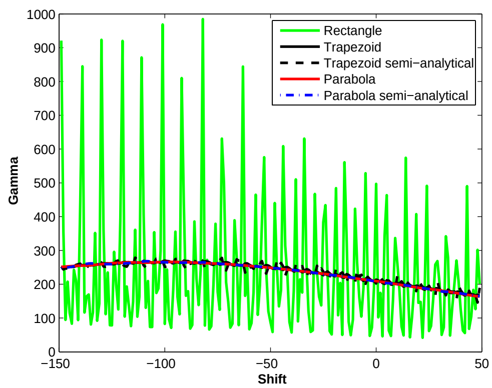


(a) Full picture

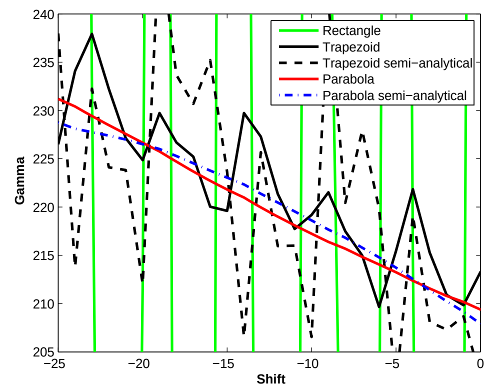
(b) Enlarged picture

Figure 1. Price convergence according to the number of points for numerical integration and semi-analytical methods. Figures in basis points for a 1y x 9y swaption.

The numerical stability of the implementation is analyzed. This is done by computing the gamma through finite difference. It means that the second order derivative is numerically computed as $P_+ + P_- - 2P$ where $P_{0/+/-}$ are the price with the initial curve and the curve after parallel moves by ± 1 bp. This computation is performed not only for one curve but for a set of curves. The initial curve described above is moved parallel by one basis point at a time. This move is done for changes between -150 and +50 basis points. For each curve the gamma to the curve parallel move is computed. Clearly this is a very demanding task and minute numerical instability are revealed in this process.



(a) Full picture



(b) Enlarged picture

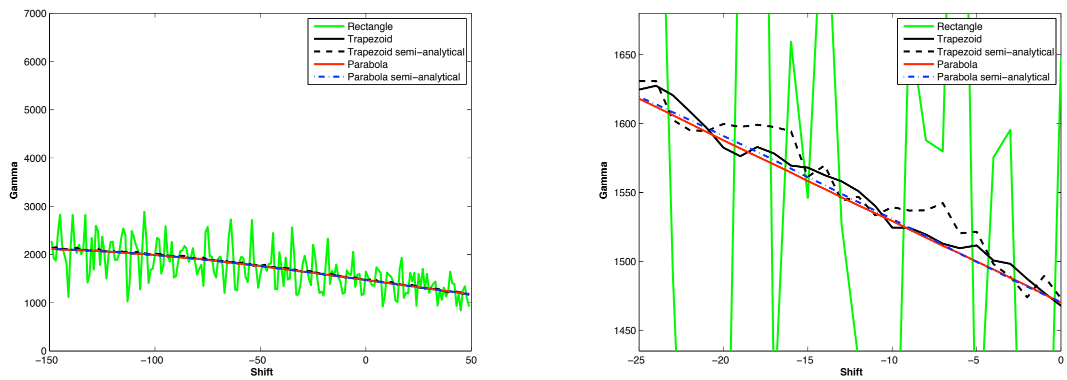
FIGURE 2. Gamma profile for the numerical and semi-analytical methods with 25 step-points.

The profile with 25 points is given in Figure 2. Even if the numerical integration with rectangles is more stable than the trinomial tree, the gamma profile is erratic. The importance of treating the maximum change properly appears clearly. The trapezoidal approach and the semi-analytical trapezoidal one provide similar patterns. Even if the semi-analytical advantage appears in the convergence, the numerical noise is similar. The parabola and semi-analytical parabola approaches provide almost clean figures with only 25 pts.

The numerical errors depend on the exact position of exercise boundaries with respect to the discritezation points. The error will be different for different (even similar) swaptions. Working on a portfolio of swaption has a smoothing effect on the relative error. Figure 3 presents the same profile than the previous figure but for a portfolio of nine swaptions with different strikes and maturities. The smoothing effect appears clearly.

The extra precision brough by the integration methods requires a parallel increase in the algorithm complexity. The relation execution time/stability is presented in Figure 4. The time refers to the time needed to price a swaption and is mesured in seconds. The computation are done in a non-optimized Matlab implementation, nevertheless their relative value are a good extimate of relative efficiency.

The stability measure is more arbitrary. The gamma profile described in the previous figures is computed for each method with different number of points. The error is the standard deviation of the relative error of the method gamma error to the exact gamma. As the error depend on the exact swaption details relative to the integration points, a (small) increase of number of points may lead in some cases to a (small) precision decrease. Note also that even if the semi-analytical approach is better in convergence term, nevertheless is is less stable under our stability measure.

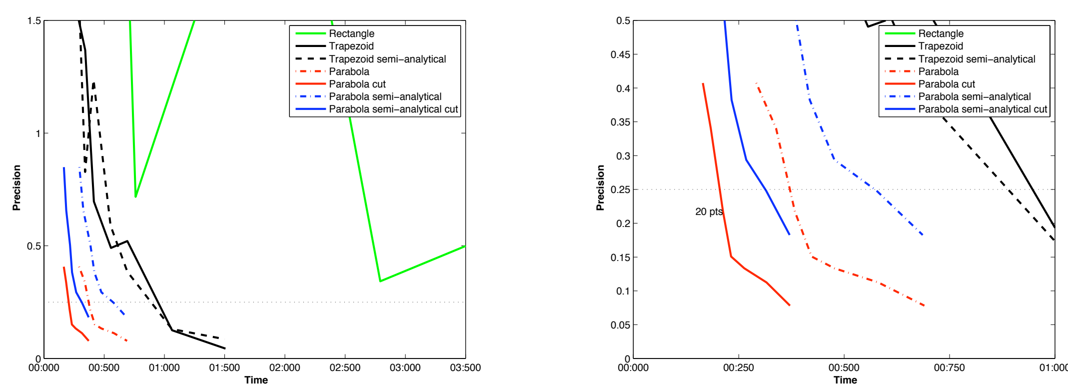


(a) Full picture

(b) Enlarged picture

FIGURE 3. Gamma profile of a portfolio of nine different options for the numerical and semi-analytical methods with 25 step-points.

In our best implementation, 20 points are enough to have a gamma profile standard deviation precision below 0.25% of the exact figures. For the same precision, the required time is around five time larger for the trapezoid approach and more than twenty times larger for the rectangles.



(a) Full picture

(b) Enlarged picture

FIGURE 4. Time/error relation in Bermudan swaptions pricing. Figures for an at-the-money 1yx9y swaption.

6. CONCLUSION

A numerical integration approach to Bermudan swaptions valuation is presented in the Gaussian HJM one factor model. The approach uses the explicit zero-coupon price to compute the required value only at financially meaningfull dates (expiry dates) and not in between. In that sense the method is superior to PDE or tree approaches (no time discretization). The recombining structure of the model is used to reduce significantly the number of computation in the multidimentional integration. This lead to a very efficient and stable approach to Bermudan swaptions valuation.

Different approaches to the numerical integration are proposed. The most efficient uses parabola approximations. A semi-analytical computation can be added to the process to improve the convergence. The algorithm is efficient and robust enough to compute the prices, delta and gamma in a precise and almost instantaneous way.

Disclaimer: The views expressed here are those of the author and not necessarily those of his employer.

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Contents

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