Pricing default baskets

Nicholas Dunbar, *Risk*'s technical editor, introduces the first in a new series of technical papers written by quants at Deutsche Bank.

"Default correlation has been one of the hottest topics in credit derivatives over the past year. So it is a pleasure to welcome Deutsche Bank – a big player in this market – to inaugurate our second Masterclass series with an article on the pricing of basket default swaps. The main problem in valuing such instruments lies in the modelling of default dependencies. Here, Wolfgang Schmidt and Ian Ward investigate aspects of pricing and hedging using an approach based on copulas, with particular focus on the impact of defaults on the spread of remaining credits." ■

The credit derivatives business has grown dramatically in recent years. Credit default swaps are the main plain-vanilla credit derivative product, and also serve as a building block for credit-linked notes and other synthetic credit investments. A credit default swap offers protection against default of a certain underlying entity over a specified time horizon. A premium (spread) s is paid on a regular basis and on a certain notional amount N as an insurance fee against the losses from default of a risky position of notional N, eg, a bond. The payment of the premium s stops at maturity or at default of the underlying credit, whichever comes first. At the time of default before maturity of the trade, the protection buyer receives the payment $N(1 - R)$, where R is the recovery rate of the underlying credit risky instrument.

More sophisticated credit derivative products are linked to several underlying credits. They include synthetic collateralised debt obligations, default swaps on certain tranches of losses from a portfolio or basket default swaps. What these products have in common is that they offer access to tailor-made profiles of credit risk that is appealing to both credit investors and hedgers seeking to redistribute their credit risk or to release regulatory capital.

In this article, we will illustrate the modelling difficulties involved with multi-credit derivative products. We will use a basket default swap as an example. A basket default swap is like an insurance contract that offers protection against the event of the kth default on a basket of n ($n \geq k$) underlying names. It is similar to a plain default swap but the credit event to insure against is the event of the kth default. Again, a premium (spread) s is paid as an insurance fee until maturity or the event of the kth default in return for a compensation for the loss. We denote by s^{kth} the fair spread in a kth-to-default swap, ie, the spread making the value of this swap today equal to zero.

Most popular are first-to-default swaps, ie, $k = 1$. As we will see below, a first-to-default swap offers highly attractive spreads to a credit investor (protection seller).

If the n underlying credits in the basket default swap are independent, the fair spread s^{first} is expected to be close to the sum of the fair default swap spreads s_i over all underlyings $i = 1, \ldots, n$. If the underlying credits are in some sense 'totally' dependent the first default will be the one with the worst spread, therefore $s^{first} = \max(s_1, \ldots, s_n)$. We will provide some intuitive explanation for these two facts later.

So how can we describe dependencies between the underlying credits in our model? Traditionally, dependencies are measured by correlation, which can be problematic because it only quantifies the linear dependence (see Embrechts, McNeil & Straumann, 1999). Also, it is not quite clear which correlation, ie, the correlation between which variables, should be modelled.

Once we have chosen a model for the dependencies between defaults, the most important problem – besides pricing the derivative – is the impact of the dependencies on the hedging strategies. The main focus of this article is to show that if there are dependencies between default times then at the time of default of one credit the spreads of the remaining face a certain spread change (spread widening).

Modelling dependence via copulas

To our knowledge, the concept of copulas applied to the problem of dependent defaults first appeared in Li (2000). Here, we give just an outline of this approach.

We denote by $\tau_1, \ldots, \tau_n$ the random default times for credits $i = 1, \ldots, n$. Write $(B(t))_{t \geq 0}$ for the curve of risk-free discount factors (zero bond prices) and $(P_i(t))_{t \geq 0}$ for the curve of cumulative (risk-neutral) default probabilities for credit i:

$$P_i(t) = \mathbf{P}(\tau_i < t)$$

Let $s_i(T_m)$ denote the fair default swap spread on credit i and maturity T_m as quoted in the market. Then, assuming a deterministic recovery rate R_i for credit i we have by definition[1]:

$$0 = s_i(T_m) \sum_{k=1}^{m} \Delta_k B(T_k)\left(1 - P_i(T_k)\right) - \left(1 - R_i\right) \int_0^{T_m} B(u) P_i(du) \qquad (1)$$

where Δ_k is the day-count fraction for the period k. The first term in the equation above is the present value of the payments of the spread $s_i(T_m)$, which is paid at each T_k provided there has been no default – so the payment is valued using the risk-free discount factor $B(T_k)$ multiplied[2] with the survival probability $(1 - P_i(T_k))$. The integral describes the present value of the payment of $(1 - R_i)$ at the time of default. For a default 'at' time u, we have to discount with $B(u)$ and multiply with the probability $P_i(du)$ that default happens 'around' u.

From these equations, one can extract the market implied (risk-neutral) default curve $P_i(t)$ for each credit i from quoted market spreads $s_i(T_m)$.

Now we come to the problem of modelling the dependence between defaults. The default curve $P_i(t)$ gives us the market implied (risk-neutral) distribution of the random default time τ_i. The cashflows in a basket default swap are functions of the whole random vector $(\tau_1, \ldots, \tau_n)$. To evaluate a basket default swap today following the principles of no-arbitrage pricing, all we need is today's (risk-neutral) joint distribution of the τ_i's:

$$\mathbf{P}(\tau_1 < t_1, \ldots, \tau_n < t_n)$$

So we have to link the given marginal distributions P_i to a joint distribution and this is exactly what a so-called copula is supposed to achieve (see Frees & Valdez, 1998, and Nelson, 1999).

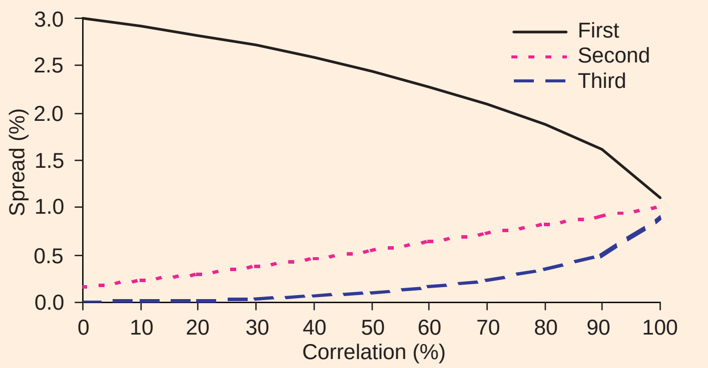
Remember that the transformed default time $U_i = P_i(\tau_i)$ admits a uniform distribution on the interval $[0, 1]$. Therefore:

$$C(u_1, \ldots, u_n) := \mathbf{P}(U_1 < u_1, \ldots, U_n < u_n) \qquad (2)$$

defines a joint distribution with uniform marginals. The function

[1] *Here, we ignore the fact that at default one usually has to pay the accrued premium*
[2] *We assume here that risk-free rates and defaults are independent*

1. Fair kth-to-default spread v. correlation



A. Fair kth-to-default swap spread for maturities from one to five years

Maturity	s^{first} (%)	s^{2nd} (%)	s^{3rd} (%)
1y	2.63	0.34	0.04
2y	2.56	0.42	0.06
3y	2.51	0.47	0.08
4y	2.47	0.51	0.09
5y	2.44	0.55	0.10

$C(u_1, \ldots, u_n)$ is called a copula and the joint distribution of $(\tau_1, \ldots, \tau_n)$ can be written as:

$$\mathbf{P}(\tau_1 < t_1, \ldots, \tau_n < t_n) = C\left(P_1(t_1), \ldots, P_n(t_n)\right) \qquad (3)$$

So we see that the copula links the marginal distributions into a joint distribution, thereby separating the dependence structure C from the marginal distributions.

To build a model for dependent defaults, we start with some dependent uniform random variables $(U_1, \ldots, U_n)$ admitting the copula C and then we define our default times by[3]:

$$\tau_i = P_i^{(-1)}(U_i)$$

Observe that a famous theorem (see Nelson, 1999) on copulas states that any joint distribution can be reduced to a copula and the marginal distributions. Thus, any model for dependent defaults has an equivalent copula representation, although it may be difficult to write down the copula explicitly.

One of the most elementary copulas is the normal copula, which appears as follows. Assume $(Y_1, \ldots, Y_n)$ follows an n-dimensional standard normal distribution with correlation matrix (ρ_{ij}). Then $U_i = N(Y_i)$ is uniform on $[0, 1]$ and their joint distribution is[4]:

$$C(u_1, \ldots, u_n) = N\left(N^{(-1)}(u_1), \ldots, N^{(-1)}(u_n), (\rho_{ij})\right)$$

So for a normal copula our default times would be modelled as $\tau_i = P_i^{(-1)}(N(Y_i))$.

There are various different copulas generating all kinds of dependencies. As shown in Frey, McNeil & Nyfeler (2001), the choice of the copula entails a significant amount of model risk. The advantage of the normal copula, however, is that it is related to the latent variable approach to model defaults following the famous Merton firm-value approach (1974). Assume the default event of entity i up to time T is driven by a single random variable A_i (ability to pay variable, eg, the asset value) being below a certain trigger level $c_i(T)$[5]:

$$\tau_i < T \Leftrightarrow A_i < c_i(T)$$

If A_i admits a standard normal distribution[6], then to be consistent with our given default curve, we set $c_i(T) = N^{(-1)}(P_i(T))$. Now, if we calculate pairwise joint default probabilities in that approach, we get:

$$\mathbf{P}(\tau_i < T, \tau_j < T) = \mathbf{P}\left(A_i < c_i(T), A_j < c_j(T)\right)$$
$$= N\left(N^{(-1)}(P_i(T)), N^{(-1)}(P_j(T)), \rho_{ij}^A\right)$$

To make these probabilities coincide with those from the normal copula approach, we see that the asset correlation ρ_{ij}^A above and the correlation ρ_{ij} in the normal copula have to be the same.[7] This makes this approach particularly appealing in practice since those correlations can in principle be estimated from data (if available) or one can use correlations as provided, eg, by KMV (*www.kmv.com*).

Pricing basket default swaps

To price a basket default swap, we need the distribution of the time τ^{kth} of the kth default. In particular, $\tau^{first} = \min(\tau_1, \ldots, \tau_n)$ and its distribution is just:

$$\mathbf{P}\left(\tau^{first} < t\right) = 1 - \mathbf{P}(\tau_1 \geq t, \ldots, \tau_n \geq t) \qquad (4)$$

where the right-hand side can be calculated from the copula and the marginal distributions. A corresponding but more involved formula can be written down for the distribution of τ^{kth}.

The fair spread s^{kth} for maturity T_m is then defined by the relation:

$$0 = s^{kth} \sum_{\iota=1}^{m} \Delta_\iota B(T_\iota) \mathbf{P}\left(\tau^{kth} > T_\iota\right) - \sum_{i=1}^{n} (1 - R_i) \int_0^{T_m} B(u) \mathbf{P}\left(\tau^{kth} \in du, \tau^{kth} = \tau_i\right)$$

The intuition behind this equation is similar to our discussion of equation (1). The first part is the present value of the spread payments, which stop at time τ^{kth} of the kth default and are therefore valued with the respective probabilities. The second term is the value of the payment at the time of the kth default. Since the recovery may be different for the underlying names, we have to sum over all underlying names $i = 1, \ldots, n$. For all times $0 < u \leq T_m$ the respective payment $(1 - R_i)$ has to be discounted with $B(u)$ and weighted with the probabilities that the kth default happens around u and that the kth name defaulting is just i.[8]

As a simple example, consider a basket of $n = 3$ credits with fair spreads $s_1 = 1.10\%$, $s_2 = 1.00\%$ and $s_3 = 0.90\%$, respectively, for all maturities and assuming a recovery rate of $R_i = 20\%$ throughout.[9] We model the default dependence via a normal copula with flat (asset) correlations $\rho_{ij} = 50\%$. Table A shows the fair kth-to-default swap spread for maturities from one to five years.

Figure 1 shows how the fair kth-to-default swap spread varies if we change our correlation in the normal copula from 0% to 100%. The figure as well as the numbers in table A show a somewhat surprising result: the sum over all kth-to-default swap spreads is greater than the sum of the individual default swap spreads:

$$\sum_{k=1}^{n} s^{kth} > \sum_{i=1}^{n} s_i$$

Since all kth-to-default swaps on one side and all plain default swaps on the other side insure exactly the same risks (namely, all defaults), one

[3] $P_i^{(-1)}$ *is the inverse of the distribution function* P_i, *which was assumed to be strictly increasing*

[4] N *denotes the standard normal distribution function and* $N^{(-1)}$ *its inverse. The* n-*dimensional normal distribution function with correlation* (ρ_{ij}) *is* $N(\ldots, (\rho_{ij}))$

[5] *Observe that in the Merton approach default of risky debt is only triggered at its maturity* T *when the asset value at this time turns out to be below the face value of debt*

[6] *This is not a critical assumption, since, for example, a lognormal state variable can be easily transformed into a normal one, transforming the trigger level in the same way*

[7] *However, since the asset value approach above can only model defaults up to a single time horizon, the calibration between the two models can be done only for one fixed horizon* T

[8] *We assume that there are no joint defaults at exactly the same time*

[9] *In practice, it is common to consider baskets comprising names with quite similar credit quality. If one name in the baskets admits a substantial higher spread compared with the others, this name would dominate the basket and the resulting pick-up for the risk of the first default would not be as interesting*

would expect that the spreads add up to the same. However, this is not true since there is a windfall effect in the first-to-default swap: at the time of first default we stop paying the huge spread s^{first} on one side but on the plain-vanilla side we stop paying just the spread s_i of the first defaulting credit i. Of course, the sum of the present values of the plain default swap spread payments and the sum of the present values of all basket default swap spread payments are equal.

Also figure 1 shows a fact already mentioned earlier. Namely, in the extreme case of $\rho_{ij} = 1$ for all i, j the first to default spread is the worst of all underlyings. The reason is obvious: in the case of perfect correlation all state variables Y_i in the normal copula are identical and since the distribution function $P(t)$ of the name with the worst spread dominates all others[10] this credit defaults first.

The other extreme case is that all default times are independent, which means $\rho_{ij} = 0$ for all i, j in the normal copula model. In this case, the fair spread on the first default is very close to (but usually not exactly) the sum of the individual spreads. How can we explain this? An intuitive no-arbitrage argument is the following. Let us assume that the term structure of credit spreads s_i is flat for each name i in our basket. Our first strategy is to buy protection on the first default. Here we have to pay the fair first-to-default spread s^{first} until the time of first default or maturity. The other strategy is to buy protection on all individual names $i = 1, \ldots, n$, via plain default swaps. This requires us to pay all spreads s_i up to the time of default of credit i or maturity. In particular, we have to pay $\Sigma_{i=1}^{n} s_i$ until the time of first default. We also agree that, if there is a default (before maturity), we then unwind our plain default swaps on all the remaining names. Clearly, both strategies protect exactly the same default risks. Since there is independence and the term structure of credit spreads is flat, there is no impact from the realised default on the fair spreads of the remaining names and there will be no costs for unwinding. This proves that the fair first-to-default spread must be the sum of the individual ones.

This brings us to our main topic, namely, how do dependencies between defaults affect hedging strategies? In particular, how do defaults affect spreads in the model?

Hedging the risks

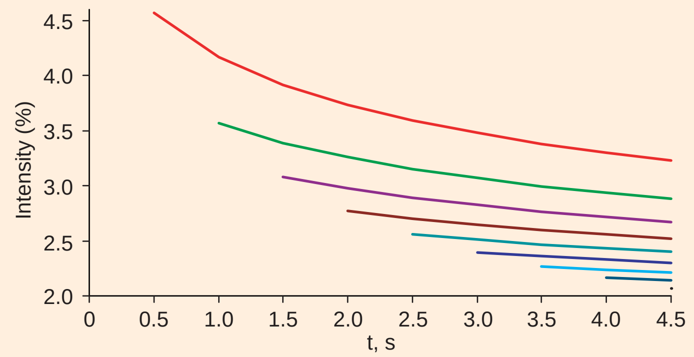
When we have to hedge a position in a basket default swap, we face two main types of risk. The first is spread risk, which is the risk arising from the fact that the spreads s_i for the underlying names, which we have used to back out the default curve $P_i(t)$, will not be stationary. This is the risk arising from the changing credit quality of the individual names. The second is event risk. In the case of an event occurring in which a payment on the basket default swap is triggered, we have to make an event payment, for which we need an appropriate offsetting hedge position.

Although difficult in practice, to hedge these two sources of risk simultaneously we need in principle at least two default-sensitive instruments per credit to hedge with. However, in our discussion of hedging below, we will focus on the event risk.[11]

Assume we have bought protection on the first to default. As a hedge, we will sell protection via plain-vanilla default swaps on each of the underlying names. To hedge the spread risk, we calculate the sensitivities of the basket with regard to changes in the individual default curves $P_i(t)$. A spread hedge position is then created by offsetting these risks by appropriate positions in plain-vanilla default swaps.

The event risk is trickier. Assume that our dynamic hedge at time t in the underlying names $i = 1, \ldots, n$ consists of sold protection for percentage notional amounts $N_1(t), \ldots, N_n(t)$ and fair spreads $s_1(t), \ldots, s_n(t)$. If there is a default at time t and assuming that credit i is the first to default, we receive from the first-to-default swap the event premium $(1 - R_i)$ whereas in our hedge position we have to pay $(1 - R_i)N_i(t)$ in the plain default swap on credit i. The remaining positions in credits $j \neq i$ now have to be unwound in the market at the prevailing market conditions. But, because the underlying names are dependent, the spreads in these underlyings will face a spread widening[12] as a consequence of the default of credit i.

2. Intensity of credit 3 after first default (credit 1)



B. Implied spread widenings

$\tau_{min} = t$	$\Delta_1^2(t)$	$\Delta_1^3(t)$	$\Delta_2^1(t)$	$\Delta_2^3(t)$	$\Delta_3^1(t)$	$\Delta_3^2(t)$
0.5	2.26%	2.08%	2.47%	2.12%	2.52%	2.35%
1.0	1.78%	1.63%	1.96%	1.67%	2.00%	1.86%
1.5	1.51%	1.38%	1.67%	1.41%	1.71%	1.58%
2.0	1.32%	1.21%	1.47%	1.24%	1.50%	1.39%
2.5	1.18%	1.08%	1.32%	1.11%	1.35%	1.25%
3.0	1.07%	0.98%	1.20%	1.01%	1.23%	1.14%
3.5	0.98%	0.90%	1.10%	0.92%	1.13%	1.04%
4.0	0.91%	0.82%	1.02%	0.85%	1.05%	0.96%
4.5	0.84%	0.76%	0.95%	0.79%	0.98%	0.90%

Spread widenings implied from the copula

Denote by $\Delta_i^j(t)$ the spread widening on credit j in case of credit i defaulting first and at time t. It is interesting to see how the implied spread widening is reflected in our copula model approach. If there is nothing known about defaults until time t, the default curve for credit j as implied from our approach will be basically the forward default curve:

$$1 - \frac{1 - P_i(s)}{1 - P_i(t)}, \qquad s \geq t$$

However, at the time of the first default $\tau_{min} = \min(\tau_1, \ldots, \tau_n)$, to get the default curves for the surviving names we have to look at the conditional distribution of τ_j given $\tau_{min} = t$. Since the default times are dependent via the copula C, this distribution will be different from the forward default curve, thereby implying some spread widening. Observe that at time t, assuming that there has been no default until t, the conditional default curves for each credit will also be different from the forward curve. Assuming a 'positive' dependence, the no-default conditional default curve will, in general, be below the forward curve.

Consider the simplest example of $n = 2$ names linked by a normal copula:

$$\tau_i = P_i^{(-1)}\left(N\left(Y_i\right)\right), \qquad i = 1, 2$$

with standard normal random variables Y_1, Y_2 with correlation ρ. The default probability for credit 2 given that credit 1 defaulted at time t is:

$$\mathbf{P}\left(\tau_2 < s \middle| \tau_1 = t\right) = \mathbf{P}\left(Y_2 < N^{(-1)}\left(P_2(s)\right) \middle| Y_1 = N^{(-1)}\left(P_1(t)\right)\right)$$

Now the distribution of Y_2 given $Y_1 = x$ is again normal but with mean ρx

[10] *Of course, we have to assume that the recoveries are the same across all credits*
[11] *The copula model makes no explicit assumptions about the dynamics of the spreads, which is why in the copula framework spread risk is basically a model risk*
[12] *Assuming a kind of 'positive' dependence*

and variance $1 - \rho^2$:

$$Y_2 | Y_1 = x \sim N\left(\rho x, 1 - \rho^2\right)$$

Consequently:

$$\mathbf{P}\left(\tau_2 < s | \tau_1 = t\right) = N\left(\frac{N^{(-1)}\left(P_2(s)\right) - \rho N^{(-1)}\left(P_1(t)\right)}{\sqrt{1-\rho^2}}\right)$$

and we obtain[13] for $s > t$:

$$\mathbf{P}\left(\tau_2 < s | \tau_{min} = \tau_1 = t\right) = 1 - \mathbf{P}\left(\tau_2 \geq s | \tau_2 > t, \tau_1 = t\right)$$
$$= 1 - \frac{N\left(\frac{-N^{(-1)}\left(P_2(s)\right) + \rho N^{(-1)}\left(P_1(t)\right)}{\sqrt{1-\rho^2}}\right)}{N\left(\frac{-N^{(-1)}\left(P_2(t)\right) + \rho N^{(-1)}\left(P_1(t)\right)}{\sqrt{1-\rho^2}}\right)}$$

Now let us investigate the general case. Let the joint distribution of the default times $(\tau_1, \ldots, \tau_n)$ be given by (3), ie, $\tau_i = P_i^{(-1)}(U_i)$ with uniforms $(U_1, \ldots, U_n)$ and copula (2). Denote by $\hat{C}$ the copula corresponding to the uniforms $(1 - U_1, \ldots, 1 - U_n)$:

$$\hat{C}\left(x_1, \ldots, x_n\right) = \mathbf{P}\left(1 - U_1 < x_1, \ldots, 1 - U_n < x_n\right)$$

Using $\hat{C}$, the joint survival probability can be written as:

$$\mathbf{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n\right) = \hat{C}\left(1 - P_1(t_1), \ldots, 1 - P_n(t_n)\right)$$

Now the default curve for credit j at the time of first default $\tau_{min} = \tau_i = t$ is given by[14]:

$$\mathbf{P}\left(\tau_j < s | \tau_{min} = \tau_i = t\right) = 1 - \frac{\frac{\partial}{\partial x_i}\hat{C}\left(x_1, \ldots, x_n\right)\Big|_{x_j = 1 - P_j(s),\, x_k = 1 - P_k(t),\, k \neq j}}{\frac{\partial}{\partial x_i}\hat{C}\left(x_1, \ldots, x_n\right)\Big|_{x_j = 1 - P_j(t),\, j = 1, \ldots, n}}, s > t \quad (5)$$

To illustrate the impact of the default of credit i at time t on credit j we translate the default curve for credit j into the corresponding default intensity (hazard rate):

$$\lambda_j^{\tau_{min} = \tau_i = t}(s), s > t$$

which is defined by the relationship:

$$P\left(\tau_j < s | \tau_{min} = \tau_i = t\right) = 1 - \exp\left(-\int_t^s \lambda_j^{\tau_{min} = \tau_i = t}(u)\, du\right)$$

Intuitively, the hazard rate is the continuously compounded spread for the case of no recovery.

For the example considered above, ie, $n = 3$ credits with fair spreads $s_1 = 1.10\%$, $s_2 = 1.00\%$ and $s_3 = 0.90\%$ and recoveries $R_i = 20\%$, we obtain figure 2 for the default intensity for credit $j = 3$ if the first default is $\tau_{min} = \tau_1 = t$ and assuming a correlation of 30%. For five-year maturity default swaps, table B shows how this translates into a spread widening $\Delta_i^j(t)$ for credit j at the time t of the first default, which is assumed to be credit i.

Given a flat correlation structure, we see that the size of the spread widenings depends on the quality of the credit first defaulting. If the first name defaulting is less risky, then its impact is higher, eg, $\Delta_3^2 > \Delta_1^2$. Also the implied spread widening admits a pronounced term structure. The earlier the first default, the higher the impact on the remaining spreads.

Conclusion

We discussed the problem of modelling dependent defaults, which is particularly important for pricing credit derivatives on baskets of underlying names. Using an approach based on copulas, we investigated the impact of dependencies on pricing and hedging basket default swaps. We quantified the default implied spread widenings in the case of a normal copula. The normal copula is particularly appealing since the correlations needed are basically asset correlations.

There are various other approaches to the problem of dependent defaults based on other types of copulas, structural models, etc. When deciding on a modelling approach one has to be careful concerning the appropriateness of the hedging implications. ■

Wolfgang Schmidt is director and Ian Ward is vice-president in the global markets, research and analytics department at Deutsche Bank in Frankfurt and London, respectively. **They would like to thank two anonymous referees for valuable comments**

Comments on this article may be posted on the technical discussion forum on the Risk website at http://www.risk.net

[13] *Observe that in the extreme case of* $P_1(t) >> P_2(s)$ *even if* $\rho > 0$ *this conditional default probability is not always higher than the forward default probability. However, in practice this is just an academic situation*

[14] *Of course, we have to impose some technical smoothness conditions on* $\hat{C}$

Appendix: outline of the proof of formula (5)

The proof of formula (5) requires only elementary probability calculus. To simplify the notation, consider the particular case of $j = n$ and $i = 1$ in equation (5). First, it is easy to see that for $s > t$:

$$\mathbf{P}\left(\tau_n > s | \tau_{min} = \tau_1 = t\right) = \frac{\mathbf{P}\left(\tau_n > s, \tau_k > t, k \neq 1 | \tau_1 = t\right)}{\mathbf{P}\left(\tau_k > t, k \neq 1 | \tau_1 = t\right)}$$

If $f(x_1, \ldots, x_n)$ denotes the joint density of the random vector $(\tau_1, \ldots, \tau_n)$ and $f_1(x)$ is the marginal density of τ_1, then the conditional density for $(\tau_k, k \neq 1) | \tau_1 = t$ is known to be:

$$\frac{f\left(t, x_2, \ldots, x_n\right)}{f_1(t)}$$

This implies:

$$\mathbf{P}\left(\tau_n > s | \tau_{min} = \tau_1 = t\right) = \frac{\int_t^\infty \ldots \int_t^\infty \int_s^\infty f\left(t, x_2, \ldots, x_n\right) dx_2 \ldots dx_n}{\int_t^\infty \ldots \int_t^\infty f\left(t, x_2, \ldots, x_n\right) dx_2 \ldots dx_n}$$

The integrals above are related to the joint survival function $F(x_1, \ldots, x_n)$, which is defined as:

$$F\left(t_1, \ldots, t_n\right) = \mathbf{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n\right) = \hat{C}\left(1 - P_1(t_1), \ldots, 1 - P_n(t_n)\right)$$

by the relation:

$$\int_t^\infty \ldots \int_t^\infty \int_s^\infty f\left(t, x_2, \ldots, x_n\right) dx_2 \ldots dx_n = (-1)\frac{\partial}{\partial x_1} F\left(t, t, \ldots, t, s\right)$$

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