# Module 1 Examination Solutions January 2014

## Instructions

All questions must be attempted. Books and lecture notes may be referred to. Help from others is not permitted.

You may assume throughout this examination that $dX_t$ is an increment in a standard Brownian motion $X_t$ and up to Mean Square Convergence

$$\mathbb{E}\left[dX^2\right] = dt$$

**Detailed working** must be presented to obtain maximum credit. If a question asks for a particular method to be used, any other technique employed will result in the loss of marks.

**Submitted work should be neat and easy to read by the tutor. Where a spreadsheet has been used, please submit this together with the relevant graphical results.**

1. Using the expansion of $\sin(kx)$ when $x$ is small, show that

$$\lim_{x\longrightarrow 0}\left(\frac{\alpha\sin(\beta x)-\beta\sin(\alpha x)}{x^2\sin(\alpha x)}\right)\longrightarrow\frac{\beta}{6}\left(\alpha^2-\beta^2\right)$$

**Note: You are not permitted to use L'Hospital's Rule at any stage. You may use any expansions without proof**

$$\begin{aligned}\frac{\alpha\sin(\beta x)-\beta\sin(\alpha x)}{x^2\sin(\alpha x)} &= \frac{\alpha\left(\beta x-\frac{\beta^3x^3}{3!}+...\right)-\beta\left(\alpha x-\frac{\alpha^3x^3}{3!}+...\right)}{x^2\left(\alpha x-\frac{\alpha^3x^3}{3!}+...\right)}\\ &= \frac{\beta\frac{\alpha^3x^3}{3!}-\alpha\frac{\beta^3x^3}{3!}}{x^2\left(\alpha x-\frac{\alpha^3x^3}{3!}\right)}=\frac{1}{6}\frac{\beta\alpha^3-\alpha\beta^3}{\left(\alpha-\frac{\alpha^3x^2}{3!}\right)}\end{aligned}$$

Now take limits

$$\begin{aligned}\frac{1}{6}\lim_{x\longrightarrow 0}\frac{\beta\alpha^3-\alpha\beta^3}{\left(\alpha-\frac{\alpha^3x^2}{3!}\right)} &= \frac{1}{6\alpha}\left(\beta\alpha^3-\alpha\beta^3\right)\\ &= \frac{\beta}{6}\left(\alpha^2-\beta^2\right)\end{aligned}$$

2. Consider the function $z(x,y)=(x+y)\ln\left(\frac{x}{y}\right)$, where $x$ and $y$ are independent variables.

   a. Show (by substitution) that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z.$$

Easier to write as $z(x,y)=(x+y)(\ln x-\ln y)$

$$\begin{aligned}\frac{\partial z}{\partial x} &= (\ln x-\ln y)+(1+y/x)\\ \frac{\partial z}{\partial y} &= (\ln x-\ln y)-(x/y+1)\end{aligned}$$

Now

$$\begin{aligned}x\frac{\partial z}{\partial x} &= x(\ln x-\ln y)+(x+y)\\ y\frac{\partial z}{\partial y} &= y(\ln x-\ln y)-(x+y)\end{aligned}$$

Adding the two expressions

$$\begin{aligned}x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} &= x(\ln x-\ln y)+y(\ln x-\ln y)\\ &= (x+y)(\ln x-\ln y)=z\end{aligned}$$

b. By differentiating the expression in **a.**, find a relationship between $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

$\dfrac{\partial}{\partial x}$ (**i**) :

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial z}{\partial x}\Rightarrow\frac{\partial^2 z}{\partial y\partial yx}=-\frac{x}{y}\frac{\partial^2 z}{\partial x^2}$$

$\dfrac{\partial}{\partial y}$ (**i**) :

$$x\frac{\partial^2 z}{\partial y\partial yx}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=\frac{\partial z}{\partial y}\Rightarrow\frac{\partial^2 z}{\partial y\partial yx}=-\frac{y}{x}\frac{\partial^2 z}{\partial y^2}$$

but since the mixed partial derivatives are equal, we have $\dfrac{x}{y}\dfrac{\partial^2 z}{\partial x^2}=\dfrac{y}{x}\dfrac{\partial^2 z}{\partial y^2}$ and

$$x^2\frac{\partial^2 z}{\partial x^2}=y^2\frac{\partial^2 z}{\partial y^2}$$

3. Consider the partial differential equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}+bu,$$

for the function $u\left(x,t\right)$; where $a$ and $b$ are constants. By using a substitution of the form

$$u\left(x,t\right)=e^{\alpha x+\beta t}v\left(x,t\right),$$

and suitable choice of constants $\alpha$ and $\beta$, show that the PDE can be reduced to the heat equation

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}.$$

From the suggested substitution we have

$$\begin{aligned}\frac{\partial u}{\partial t}&=\left(\beta v+\frac{\partial v}{\partial t}\right)e^{\alpha x+\beta t};\ \frac{\partial u}{\partial x}=\left(\alpha v+\frac{\partial v}{\partial x}\right)e^{\alpha x+\beta t};\\ \frac{\partial^2 u}{\partial x^2}&=\left(\alpha^2 v+2\alpha\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)e^{\alpha x+\beta t}\end{aligned}$$

Subst into the PDE above gives

$$\beta v+\frac{\partial v}{\partial t}=\left(\alpha^2 v+2\alpha\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)+a\left(\alpha v+\frac{\partial v}{\partial x}\right)+bv$$

Rearrange

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+\left(2\alpha+a\right)\frac{\partial v}{\partial x}+\left(\alpha^2+a\alpha+b-\beta\right).$$

Choose $\alpha$ such that

$$2\alpha+a=0,$$

to eliminate the $v$ term. We must therefore choose

$$\alpha=-a/2;\ \beta=b-a^2/4.$$

4. The integral $I_n$ is defined, for positive integers $n$, as

$$I_n=\int_0^\infty\left(1+x^2\right)^{-n}dx.$$

Deduce that

$$I_n=2n\left(I_n-I_{n+1}\right).$$

Hence or otherwise show that

$$I_4 = \int_0^\infty \left(1 + x^2\right)^{-4} dx = \frac{5\pi}{32}$$

To get the recurence relation write the integral as

$$I_n = \int_0^\infty 1. \left(1 + x^2\right)^{-n} dx$$

$$\begin{array}{cc} v = \left(1 + x^2\right)^{-n} & u' = 1 \\ v' = -2nx \left(1 + x^2\right)^{-n-1} & u = x \end{array}$$

$$\left.\frac{x}{(1 + x^2)^n}\right|_0^\infty + 2n \int_0^\infty \frac{x^2}{(1 + x^2)^n} dx = 0 + 2n \left(\int_0^\infty \frac{x^2 + 1 - 1}{(1 + x^2)^{n+1}} dx\right)$$

$$2n \left(\int_0^\infty \frac{x^2 + 1}{(1 + x^2)^{n+1}} dx - \int_0^\infty \frac{1}{(1 + x^2)^{n+1}}\right) = 2n \left(I_n - I_{n+1}\right)$$

Rearrange this by writing as

$$I_{n+1} = \frac{2n - 1}{2n} I_n$$

To obtain $I_4$ put $n = 3$ in the above

$$I_4 = \frac{5}{6} I_3 = \frac{5}{6} . \frac{3}{4} I_2 = \frac{5}{6} . \frac{3}{4} . \frac{1}{2} I_1$$

where

$$I_1 = \int_0^\infty \frac{dx}{1 + x^2} = \arctan x|_0^\infty = \frac{\pi}{2}$$

Hence $I_4 = \frac{5}{32}\pi$

5. A spot rate $r$, evolves according to the popular form

$$dr = u(r)\, dt + \nu r^\beta dX_t, \qquad (*)$$

where $\nu$ and $\beta$ are constants.
Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation.
Show that this implies that the drift structure of $(*)$ is given by

$$u(r) = \nu^2 \beta r^{2\beta - 1} + \frac{1}{2} \nu^2 r^{2\beta} \frac{d}{dr} (\log p_\infty).$$

The forward F.P equation for $dr = u(r, t)\, dt + w(r, t)\, dX$ is

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial r^2} \left(w^2 (r, t)\, p (r, t)\right) - \frac{\partial}{\partial r} \left(u (r, t)\, p (r, t)\right)$$

for the probability density $p(r, t)$. The steady state equation for our model becomes

$$\frac{1}{2} \nu^2 \frac{d^2}{dr^2} \left(r^{2\beta} p_\infty (r)\right) - \frac{d}{dr} \left(u (r)\, p_\infty (r)\right) = 0$$

This can be simply integrated once to give

$$\begin{aligned} \frac{1}{2} \nu^2 \frac{d}{dr} \left(r^{2\beta} p_\infty (r)\right) - \left(u (r)\, p_\infty (r)\right) &= \text{const} \\ \frac{1}{2} \nu^2 \left(r^{2\beta} \frac{dp_\infty}{dr}\right) + \nu^2 \beta r^{2\beta - 1} p_\infty (r) - \left(u (r)\, p_\infty (r)\right) &= \text{const} \end{aligned}$$

The constant of integration is zero because as $r$ becomes large

$$\left.\begin{array}{c} p_{\infty}(r) \\ \frac{dp_{\infty}}{dr} \end{array}\right\} \longrightarrow 0$$

$$\begin{aligned} u(r)\, p_{\infty}(r) &= \frac{1}{2}\nu^2 r^{2\beta} \frac{dp_{\infty}}{dr} + \nu^2 \beta r^{2\beta-1} p_{\infty}(r) \\ u(r) &= \frac{1}{2}\nu^2 r^{2\beta} \frac{1}{p_{\infty}(r)} \frac{dp_{\infty}}{dr} + \nu^2 \beta r^{2\beta-1} \end{aligned}$$

We can write $\dfrac{1}{p_{\infty}} \dfrac{dp_{\infty}}{dr}$ as $\dfrac{d}{dr}(\log p_{\infty})$

$$u(r) = \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr}(\log p_{\infty}) + \nu^2 \beta r^{2\beta-1}.$$

6. Consider the following Stochastic Differential Equation for the volatility $\sigma$,

$$d\sigma = a(\sigma, t)dt + b(\sigma, t)dX_t.$$

The drift and diffusion will be abbreviated to $a$ and $b$ respectively. The Forward Kolmogorov Equation, for the transition pdf $p = p(\sigma, t; \sigma', t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial \sigma'^2}(b^2 p) - \frac{\partial}{\partial \sigma'}(ap),$$

where the primed variables refer to future states. Derive the steady state solution given by

$$p_\infty(\sigma') = \frac{C}{b^2}\exp\left(\int \tfrac{2a}{b^2}d\sigma'\right),$$

where $C$ is a constant. Any conditions used should be stated.
At steady state $\frac{\partial p}{\partial t} = 0$. So the FKE is

$$\frac{1}{2}\frac{d^2}{d\sigma^2}(b^2 p) - \frac{d}{d\sigma}(ap) = 0$$

Integrating once we have

$$\frac{1}{2}\frac{d}{d\sigma}(b^2 p) = ap + C$$

Now using the condition $\sigma \longrightarrow \infty; p \longrightarrow 0; \frac{\partial p}{\partial \sigma} \rightarrow 0$, we note that $C = 0$, so

$$\frac{1}{2}\frac{d}{d\sigma}(b^2 p) = ap$$

Now writing the above equation in the following way

$$\frac{d(b^2 p)}{b^2 p} = \frac{2a}{b^2}d\sigma$$

Integrating once again

$$\ln(b^2 p) - b^2 \ln B = \int \frac{2a}{b^2}d\sigma$$

Taking exponentials

$$p(\sigma') = \frac{A}{b^2}e^{\int \frac{2a}{b^2}d\sigma}$$

7. The Ornstein-Uhlenbeck process is given by

$$dU_t = -\gamma U_t dt + \sigma dX_t,\ U_0 = u,$$

where $\gamma$, $\sigma$ are constants. Solve this equation for $U_t$ and hence write down $\mathbb{E}[U_t]$ in its simplest form.
This can solved by an integrating factor or by treating as an Itô integral

$$U_t = u \exp(-\gamma t) + \sigma \left( X_t - \gamma \int_0^t \exp(\gamma(s-t)) X_s \ ds \right)$$

Then taking expectations

$$\begin{aligned}
\mathbb{E}[U_t] &= \mathbb{E}[u \exp(-\gamma t)] + \mathbb{E}\left[\sigma \left( X_t - \gamma \int_0^t \exp(\gamma(s-t)) X_s \ ds \right)\right] \\
&= u \exp(-\gamma t) + \sigma \mathbb{E}[X_t] - \gamma \mathbb{E}\left[\int_0^t \exp(\gamma(s-t)) X_s \ ds\right] \\
&= u \exp(-\gamma t) - \gamma \int_0^t \exp(\gamma(s-t)) \mathbb{E}[X_s] \, ds \\
&= u \exp(-\gamma t)
\end{aligned}$$

8. Show that the following process is a martingale

$$e^{aX_t - \frac{1}{2}a^2 t}$$

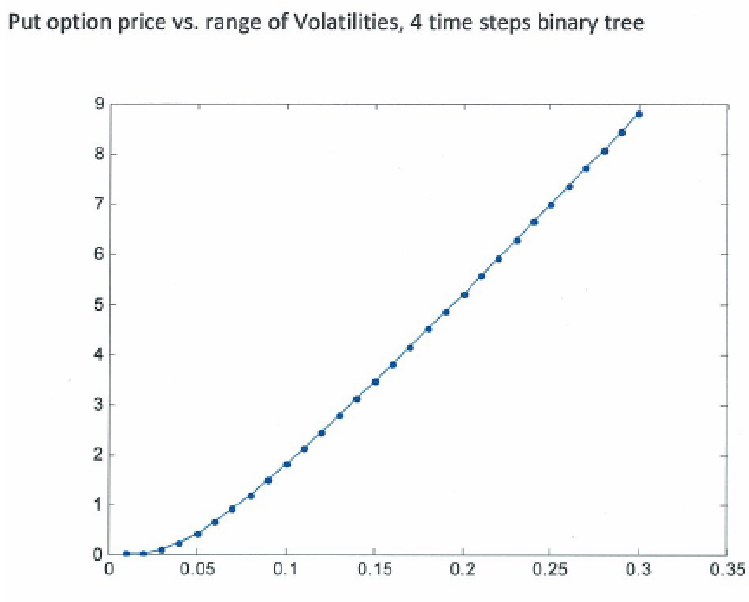
where $a$ is a constant.
Easiest way to show that this is a driftless process so look at Itô

$$\begin{aligned}
df &= \left(\frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial X^2}\right) dt + \frac{\partial f}{\partial X} dX \\
&= \left(\left(-\frac{1}{2}a^2\right) e^{aX_t - \frac{1}{2}a^2 t} + \frac{1}{2}\left(a^2 e^{aX_t - \frac{1}{2}a^2 t}\right)\right) dt + a e^{aX_t - \frac{1}{2}a^2 t} dX \\
&= a e^{aX_t - \frac{1}{2}a^2 t} dX
\end{aligned}$$

Hence it is a martingale.

9. Implement the multi-step **Binomial Method** computationally to price a European put with the following parameters: strike $K = 100$ and maturity $T = 1$. Asset price level $S_0 = 100$ and interest rate $r = 0.05$. The preferred solution method is writing a function in VBA but excel spreadsheets with Binomial trees will be accepted if the plots are correct. For simplicity, use the compound rate as a substitute to simple (discrete) rate.

   (a) For the constant number of time steps in the tree **NTS = 4**, calculate the value of the option for a range of volatilities and plot the result.
   With increasing vol, option price increases. The relationship is near flat for low levels of vol $\sigma < 0.05$ and can be linearly approximated when $\sigma > 0.2$. Such a spline is linked by a curve, convexity of which depends on moneyness. $S/K$



(b) Then, fix the volatility at $\sigma = 0.2$ and plot the value of the option as a function of the number of time steps in the tree, **NTS** $=1, 2, \ldots, 50$. You will need a different tree for each NTS value. Increasing the amount of time steps will lead to decaying oscillatory behaviour that converges around the theoretical (Black-Scholes) option price