Itô's Lemma and Stochastic Differential Equations

Throughout this problem sheet, you may assume that X_t is a Brownian Motion (Weiner Process) and dX_t is its increment. $X_0 = 0$.

1. The change in a share price $S(t)$ satisfies

$$dS = A(S,t)\,dX_t + B(S,t)\,dt,$$

for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dX_t.$$

Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.
A function $g(S)$ will satisfy the shorter SDE

$$dg = \left(B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2}\right)dt + A\frac{dg}{dS}dX.$$

For $g(S)$ to have a zero drift but non-zero diffusion, we require the condition

$$B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2} = 0.$$

i.e.

$$\frac{dg}{dS} + \frac{1}{2}\frac{A^2}{B}\frac{d^2 g}{dS^2} = 0$$

We can find a solution to this problem if $\frac{A^2}{B}$ is independent of time.

2. Show that $F(X_t) = \arcsin(2aX_t + \sin F_0)$ is a solution of the stochastic differential equation

$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dX_t,$$

where F_0 and a is a constant.
$F = \arcsin(2aX(t) + \sin F_0)$ implies $\sin F = 2aX(t) + \sin F_0$ hence

$$\frac{dF}{dX} = \frac{2a}{\sqrt{1-(2aX+\sin F_0)^2}} = 2a\left\{1-(2aX+\sin F_0)^2\right\}^{-1/2}$$

$$\frac{d^2F}{dX^2} = \frac{(2a)^2(2aX(t)+\sin F_0)}{\left\{1-(2aX+\sin F_0)^2\right\}^{3/2}}$$

So Itô gives

$$dF = \frac{2a}{\sqrt{1-(2aX+\sin F_0)^2}}dX + \frac{1}{2}\frac{(2a)^2(2aX(t)+\sin F_0)}{\left\{1-(2aX+\sin F_0)^2\right\}^{3/2}}dt$$

We know $\cos^2 F + \sin^2 F = 1 \Longrightarrow \cos F = \sqrt{1-\sin^2 F} = \sqrt{1-(2aX+\sin F_0)^2}$. Some more trig.

$$\sec F = \frac{1}{\cos F} = \frac{1}{\sqrt{1-(2aX+\sin F_0)^2}}$$

and

$$(\tan F)\left(\sec^2 F\right) = \frac{\sin F}{\cos F}\frac{1}{\cos^2 F} = \frac{\sin F}{\cos^3 F} = \frac{2aX+\sin F_0}{\left\{1-(2aX+\sin F_0)^2\right\}^{3/2}}$$

which gives

$$dF = 2a^2 (\tan F) \left(\sec^2 F\right) dt + 2a (\sec F) dX.$$

3. Show that

$$\int_0^t X_t \left(1 - e^{-X_t^2}\right) dX_t = \overline{F}(X_t) + \int_0^t G(X_\tau) d\tau$$

where the functions $\overline{F}$ and G should be determined.
We can do this two ways. Perform Itô's lemma and then integrate or use the stochastic integral version as given in the first question. Comparing

$$\int_0^t X(\tau) \left(1 - e^{-X^2(\tau)}\right) dX(\tau) = \overline{F}(X(t)) + \int_0^t G(X(t)) d\tau$$

with

$$\int_0^t \frac{\partial F}{\partial X} dX(\tau) = F(X(t), t) - F(X(0), 0) + \int_0^t -\left(\frac{\partial F}{\partial \tau} + \tfrac{1}{2}\frac{\partial^2 F}{\partial X^2}\right) d\tau$$

suggests that

$$\frac{\partial F}{\partial X} = X(\tau) \left(1 - e^{-X^2(\tau)}\right)$$

so integrating over $[0, t]$ gives $\overline{F}(X(t), t)$, which we will do by substitution, i.e. put $u = X^2$ which gives

$$F(X(t), t) - F(X(0), 0) = \tfrac{1}{2}X^2(t) + \tfrac{1}{2}e^{-X^2(t)} - \tfrac{1}{2}.$$

Also knowing $\frac{\partial F}{\partial X}$ allows us to easily obtain $\frac{\partial^2 F}{\partial X^2} = 2X^2(t) e^{-X^2(t)} - e^{-X^2(t)} + 1$. Hence

$$G(X(t)) = -\tfrac{1}{2}\frac{\partial^2 F}{\partial X^2} = -\tfrac{1}{2}\left(1 - e^{-X^2(t)}\right) - X^2(t) e^{-X^2(t)}$$

and we have shown

$$\int_0^t X(\tau) \left(1 - e^{-X^2(\tau)}\right) dX(\tau) = \overline{F}(X(t)) + \int_0^t G(X(t)) d\tau$$

where

$$\begin{aligned}
\overline{F}(X(t), t) &= \tfrac{1}{2}X^2(t) + \tfrac{1}{2}e^{-X^2(t)} - \tfrac{1}{2} \\
G(X(t)) &= -\tfrac{1}{2}\left(1 - e^{-X^2(t)}\right) - X^2(t) e^{-X^2(t)}.
\end{aligned}$$

4. Begin by writing a 3D Taylor expansion for $F(t, S_t, v_t)$

$$\begin{aligned}
& V(t + dt, S_t + dS, r_t + dr) - V(t, S_t, v_t) \\
= \;& \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial v^2} dv^2 + \frac{\partial^2 V}{\partial v \partial S} dv dS
\end{aligned}$$

Since $dX_i^2 \to dt$ in the mean square limit for $i = 1, 2$, we see that

$$dS_t^2 \to v_t S_t^2 dt,$$

$$dv_t^2 \to \eta^2 v dt,$$

Also, since $dX_1 dX2 = \rho dt$, we see that

$$dS_t dv_t \to \rho\eta v_t S_t dt$$

This gives us a *bivariate* version of Itô's Lemma, the SDE for F is given by

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} - \lambda(v_t - \bar{v}) \frac{\partial V}{\partial v_t} \right. \\ &\quad \left. + \frac{1}{2} v_t S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\eta^2 \frac{\partial^2 V}{\partial v_t^2} + \rho\eta v_t S_t \frac{\partial^2 V}{\partial v_t \partial S} \right) dt \\ &\quad + \sqrt{v_t} S_t \frac{\partial V}{\partial S} dX_1 + \eta\sqrt{v_t} \frac{\partial V}{\partial v_t} dX_2 \end{aligned}$$

Integrating over $[0, t]$, we get

$$\begin{aligned} V(t, S_t, v_t) &= v + \int_0^t \left(\frac{\partial V}{\partial \tau} + \mu S_\tau \frac{\partial V}{\partial S} - \lambda(v_\tau - \bar{v}) \frac{\partial V}{\partial v_\tau} \right. \\ &\quad \left. + \frac{1}{2} v_\tau S_\tau^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\eta^2 \frac{\partial^2 V}{\partial v_\tau^2} + \rho\eta v_\tau S_\tau \frac{\partial^2 V}{\partial v_\tau \partial S} \right) d\tau \\ &\quad + \int_0^\tau \sqrt{v_\tau} S_\tau \frac{\partial V}{\partial S} dX_1 + \int_0^\tau \eta\sqrt{v_\tau} \frac{\partial V}{\partial v_\tau} dX_2 \end{aligned}$$

5. We use Itô's lemma on a function $G(X(t), t)$:

$$dG = \frac{\partial G}{\partial X} dX + \left(\frac{\partial G}{\partial t} + \tfrac{1}{2} \frac{\partial^2 G}{\partial X^2} \right) dt.$$

(a) $G(t) = X_t^2$

$$dG = 2X dX + dt = 2\sqrt{G} dX + dt.$$

Therefore

$$a(G, t) = 1 \text{ and } b(G, t) = 2\sqrt{G}$$

(b)

$$dG = \exp(X(t)) dX + \left(1 + \tfrac{1}{2} \exp(X(t))\right) dt.$$

Rearranging the formula for $G(t)$ we have $\exp(X(t)) = G(t) - 1 - t$, and so

$$dG = \underbrace{(G(t) - 1 - t)}_{b(G,t)} dX + \underbrace{\tfrac{1}{2}(1 + G(t) - t)}_{a(G,t)} dt.$$

(c)

$$dG = f(t) dX + X(t) \frac{df}{dt} dt = f(t) dX + \frac{G(t)}{f(t)} \frac{df}{dt} dt$$

therefore

$$a(G, t) = \frac{G(t)}{f(t)} \frac{df}{dt} \text{ and } b(G, t) = f(t)$$

6. Show that

$$G = \exp(t + a \exp(X(t)))$$

is a solution of the stochastic differential equation

$$dG(t) = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right) dt + G(\ln G - t) dX$$

$$\frac{\partial G}{\partial t} = G, \quad \frac{\partial G}{\partial X} = aGe^X, \quad \frac{\partial^2 G}{\partial X^2} = ae^X G + ae^X \frac{\partial G}{\partial X} = ae^X G + a^2 e^{2X} G$$

In Itô, i.e.

$$\begin{aligned} dG &= \left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} \right) dt + \frac{\partial G}{\partial X} dX \\ &= \left(G + \tfrac{1}{2} ae^X G + \tfrac{1}{2} a^2 e^{2X} G\right) dt + ae^X G dX \end{aligned}$$

From $G = \exp(t + a \exp(X(t)))$ we have

$$ae^X + t = \ln G \Longrightarrow ae^X = \ln G - t$$

so we can write the SDE in terms of the process G

$$dG = G\left(1 + \tfrac{1}{2} ae^X + \tfrac{1}{2} a^2 e^{2X}\right) dt + ae^X G dX$$

So

$$dG = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right) dt + G(\ln G - t) dX$$

7.

$$\left.\begin{array}{c} F = \cos\left(X\left(t\right)\right) \\ G = \sin\left(X\left(t\right)\right) \end{array}\right\} \Rightarrow \text{ Itô gives}$$

$$\left.\begin{array}{rl} dF & = \dfrac{\partial F}{\partial X} dX + \frac{1}{2} \dfrac{\partial^2 F}{\partial X^2} dt = -\sin(X) dX - \frac{1}{2} \cos\left(X\right) dt \\ dG & = \dfrac{\partial G}{\partial X} dX + \frac{1}{2} \dfrac{\partial^2 G}{\partial X^2} dt = \cos(X) dX - \frac{1}{2} \sin\left(X\right) dt \end{array}\right\}$$

comparing with earlier expressions gives

$$\alpha = -\tfrac{1}{2}; \quad \beta = -1$$