Numerical Analysis Review

Chapter 4

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This chapter contains following algorithms.

- Jacobi iterative method
- Gauss-sidel iterative method
- Successive over-relaxation iterative method
- Symmetric over-relaxation iterative method

Iterative Algorithms for Linear System

Consider following linear system

$$Ax = b$$

with full-rank $A \in \mathbb{R}^{n \times n}$. One potential way to solve the problem is by iterative method. i.e., we generate a sequence of iterates $\{x_k\}$ such that $\lim_{k \to \infty} x_k = x^*$ and $Ax^* = b$.

Before considering more details w.r.t. the iterative algorithm for solving linear systems, we first consider a more general problem to solve a non-linear equation in $\mathbb{R}$.

$$f(x) - x = 0.$$

Recall that a well-known method to solve above problem is fixed-point algorithm, which iteratively generates

$$x_{k+1} = f(x_k)$$

until convergence.

i.e., x^* is a fixed-point of f. The iterative method for linear systems follows exactly the principle by creating such f with fixed-point x^*. i.e., $f(x^*) = x^*$. More specifically, we only use a linear function here given by

$$f(x) = Mx + g$$

and the fixed-point satisfies

$$Mx^* + g = x^* \Leftrightarrow Ax^* = b.$$

By simple re-arrangement we have

$$(M - I)x^* = -g \Leftrightarrow Ax^* = b.$$

Hence we can choose arbitrary M and g as long as the equivalence is preserved. But one more thing to consider is convergence. i.e., we must ensure that the sequence generated by $x_{k+1} = Mx_k + g$ converges. An intuitive idea is that M must be contractive. i.e., the linear transformation will only be shrink diameter of space it takes effect on.

Now without loss of generality, we assume that solution to linear system $Ax = b$ lies in the unit ball $\mathbb{B} = \{x: \|x\|_2^2 \leq 1\}$ and diameter of $\mathbb{B}$ is exactly 2. If we consider AB by transforming unit ball with A, then $\operatorname{diam}(B) = 2\|M\|_2$. Hence we need $\|M\|_2 \leq 1$ to ensure the non-expansiveness of f and $\|A\|_2 < 1$ for contraction. By above principle, following iterative mappings have been proposed to make $\|M\|_2$ as small as possible. For brevity we denote $D = \operatorname{diag}(A)$, L, U to be lower/upper-triangular part of A respectively.

- Jacobi

  $M = \operatorname{diag}(A)^{-1}(L + U)$

  $g = D^{-1}b$

- Gauss-Seidel

  $M = (D - L)^{-1}U$

  $g = (D - L)^{-1}b$

- SOR

  $M = (D - \omega L)^{-1}[(1 - \omega)D + \omega U], \omega \in (0, 2)$

  $g = \omega(D - \omega L)^{-1}b$

- SSOR

  $M_1 = (D - \omega L)^{-1}[(1 - \omega)D + \omega U], \omega \in (0, 2)$

  $M_2 = (D - \omega U)^{-1}[(1 - \omega)D + \omega L]$

  $g_1 = \omega(D - \omega L)^{-1}b$

  $g_2 = \omega(D - \omega U)^{-1}b$