Numerical Analysis Review

Chapter 3

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This chapter contains following algorithms.

- Householder transformation
- Givens transformation
- QR decomposition

Ordinary Least Squares

Consider fitting a linear regression model with data $X \in \mathbb{R}^{n \times p}$ and $y \in \mathbb{R}^n$. Without loss of generality we assume that $n \geq p$ and that $\mathrm{rank}(X) = p$. Then OLS is to solve following optimization problem

$$\min_{\beta} \quad \|X\beta - y\|_2^2$$

and we change notation to write

$$\min_{x} \quad \|Ax - b\|_2^2 \quad .$$

Recall that for $Q \in \mathcal{O}_{\mathcal{X}_n}$, we have $\|Q(Ax - b)\|_2^2 = \|Ax - b\|_2^2$ and if we do QR decomposition to obtain $A = QR = Q\begin{pmatrix} R_1 \\ 0 \end{pmatrix}$, where $R \in \mathbb{R}^{n \times p}$ and $R_1 \in \mathbb{R}^{p \times p}$. We equivalently get

$$\|Ax - b\|_2^2 = \|QRx - b\|_2^2 = \|Q^\top QRx - Q^\top b\|_2^2 = \left\| \begin{pmatrix} R_1 x_1 \\ 0 \end{pmatrix} - \begin{pmatrix} Q^\top b_{[1]} = b_1 \\ Q^\top b_{[2]} = b_2 \end{pmatrix} \right\|_2^2 .$$

Hence we can equilvalently solve the linear system $R_1 x = b_1$ to get x_1 and return to $Q\begin{pmatrix} R_1 x_1 \\ 0 \end{pmatrix}$ after solution.