This scripts verifies the (in)stability of the dynamical system formed by the Polyak stepsize with Mathematica.

In[∘]:=
```
Clear["`*"];
```

Define functions and related assumptions.

In[∘]:=
$$f[x\_, y\_] := \frac{\kappa}{2} x^2 + \frac{1}{2} y^2;$$
$$g[x\_, y\_] := \begin{pmatrix} \kappa x \\ y \end{pmatrix};$$
$$\text{alphapoly}[x\_, y\_] := \frac{f[x, y]}{\text{Norm}[g[x, y]]^2};$$
$$\text{asp} = x \in \mathbb{R} \wedge y > 0 \wedge 0 \le \gamma \le 2 \wedge \kappa > \frac{4-\gamma}{\gamma} \wedge \kappa \ge 2;$$
```
ResourceFunction["JacobianMatrix"];
```

Obtain the expression of the next state

In[∘]:=
$$\begin{pmatrix} x \\ y \end{pmatrix} - \alpha\, g[x, y]$$

Out[∘]=
$$\{\{x - x \alpha \kappa\}, \{y - y \alpha\}\}$$

In[∘]:=
$$\text{Simplify}\left[\frac{\begin{pmatrix} x \\ y \end{pmatrix} - \alpha\, g[x, y]}{\text{Norm}\left[\begin{pmatrix} x \\ y \end{pmatrix} - \alpha\, g[x, y]\right]}, \text{Assumptions} \to \text{asp} \wedge \alpha > 0\right]$$
```
(* Next normalized point*)
```

Out[∘]=
$$\left\{\left\{\frac{x - x \alpha \kappa}{\sqrt{(y - y \alpha)^2 + (x - x \alpha \kappa)^2}}\right\}, \left\{\frac{y - y \alpha}{\sqrt{(y - y \alpha)^2 + (x - x \alpha \kappa)^2}}\right\}\right\}$$

In[∘]:=
$$\text{eplus} = \gamma\, \text{Simplify}[\text{alphapoly}[x - x \alpha \kappa, y - y \alpha], \text{Assumptions} \to \text{asp} \wedge \alpha > 0]$$
```
(*Next Polyak stepsize*)
```

Out[∘]=
$$\frac{\gamma \left(y^2 (-1+\alpha)^2 + x^2 \kappa (-1+\alpha \kappa)^2\right)}{2 \left(y^2 (-1+\alpha)^2 + x^2 \kappa^2 (-1+\alpha \kappa)^2\right)}$$

Assemble the next state.

In[∘]:=
$$s = \begin{pmatrix} \frac{x - x \alpha \kappa}{\sqrt{(y - y \alpha)^2 + (x - x \alpha \kappa)^2}} \\ \frac{y - y \alpha}{\sqrt{(y - y \alpha)^2 + (x - x \alpha \kappa)^2}} \\ \text{eplus} \end{pmatrix};$$

Solve for the initial conditions.

In[ ]:= $\texttt{Solve}\left[\left\{x^2 == \dfrac{4-\gamma(\kappa+1)}{\gamma\kappa(\kappa+1)-4\kappa^2}y^2,\ x^2+y^2==1\right\},\ \{x,y\},\ \texttt{Reals}\right]\ \texttt{// Simplify}$

Out[ ]=

$$\left\{\left\{x\to -\sqrt{\frac{4-\gamma(1+\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition},\ y\to -\sqrt{\frac{\kappa(\gamma-4\kappa+\gamma\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition}\right\},\right.$$

$$\left\{x\to -\sqrt{\frac{4-\gamma(1+\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition},\ y\to \sqrt{\frac{\kappa(\gamma-4\kappa+\gamma\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition}\right\},$$

$$\left\{x\to \sqrt{\frac{4-\gamma(1+\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition},\ y\to -\sqrt{\frac{\kappa(\gamma-4\kappa+\gamma\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition}\right\},$$

$$\left.\left\{x\to \sqrt{\frac{4-\gamma(1+\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition},\ y\to \sqrt{\frac{\kappa(\gamma-4\kappa+\gamma\kappa)}{(-4+\gamma)(-1+\kappa^2)}}\ \text{if condition}\right\}\right\}$$

Evaluate the Jacobian matrix.

In[ ]:= $\texttt{JacobianMatrix}\left[\left[\left\{\dfrac{x-x\alpha\kappa}{\sqrt{(y-y\alpha)^2+(x-x\alpha\kappa)^2}},\ \dfrac{y-y\alpha}{\sqrt{(y-y\alpha)^2+(x-x\alpha\kappa)^2}},\ \texttt{eplus}\right\},\ \{x,y,\alpha\}\right]\ \texttt{// FullSimplify // MatrixForm}$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{y^2(-1+\alpha)^2(-1+\alpha\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & \frac{x\,y(-1+\alpha)^2(-1+\alpha\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & \frac{x\,y^2(-1+\alpha)(-1+\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} \\ \frac{x\,y(-1+\alpha)(-1+\alpha\kappa)^2}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & -\frac{x^2(-1+\alpha)(-1+\alpha\kappa)^2}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & -\frac{x^2y(-1+\kappa)(-1+\alpha\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} \\ -\frac{x\,y^2(-1+\alpha)^2\gamma(-1+\kappa)\kappa(-1+\alpha\kappa)^2}{(y^2(-1+\alpha)^2+x^2\kappa^2(-1+\alpha\kappa)^2)^2} & \frac{x^2y(-1+\alpha)^2\gamma(-1+\kappa)\kappa(-1+\alpha\kappa)^2}{(y^2(-1+\alpha)^2+x^2\kappa^2(-1+\alpha\kappa)^2)^2} & \frac{x^2y^2(-1+\alpha)\gamma(-1+\kappa)^2\kappa(-1+\alpha\kappa)}{(y^2(-1+\alpha)^2+x^2\kappa^2(-1+\alpha\kappa)^2)^2} \end{pmatrix}$$

Verify that the Polyak stepsize is unchanged after one iteration.

In[ ]:= $\eta = \texttt{FullSimplify}\left[\gamma\,\texttt{alphapoly}[x,y]\ /.\ x\to -\sqrt{-\dfrac{y^2(-4+\gamma+\gamma\kappa)}{\kappa(\gamma-4\kappa+\gamma\kappa)}},\ \texttt{Assumptions}\to\texttt{asp}\right]$

$\texttt{FullSimplify}\left[\texttt{eplus}\ /.\ \left\{x\to -\sqrt{-\dfrac{y^2(-4+\gamma+\gamma\kappa)}{\kappa(\gamma-4\kappa+\gamma\kappa)}},\ \alpha\to\eta\right\},\ \texttt{Assumptions}\to\texttt{asp}\right]$

Out[ ]=

$$\frac{2}{1+\kappa}$$

Out[ ]=

eplus

Validate the spectrum of product Jacobians

In[ ]:= $J = \begin{pmatrix} -\frac{y^2(-1+\alpha)^2(-1+\alpha\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & \frac{x\,y(-1+\alpha)^2(-1+\alpha\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & \frac{x\,y^2(-1+\alpha)(-1+\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} \\ \frac{x\,y(-1+\alpha)(-1+\alpha\kappa)^2}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & -\frac{x^2(-1+\alpha)(-1+\alpha\kappa)^2}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} & -\frac{x^2y(-1+\kappa)(-1+\alpha\kappa)}{((y-y\alpha)^2+(x-x\alpha\kappa)^2)^{3/2}} \\ -\frac{x\,y^2(-1+\alpha)^2\gamma(-1+\kappa)\kappa(-1+\alpha\kappa)^2}{(y^2(-1+\alpha)^2+x^2\kappa^2(-1+\alpha\kappa)^2)^2} & \frac{x^2y(-1+\alpha)^2\gamma(-1+\kappa)\kappa(-1+\alpha\kappa)^2}{(y^2(-1+\alpha)^2+x^2\kappa^2(-1+\alpha\kappa)^2)^2} & \frac{x^2y^2(-1+\alpha)\gamma(-1+\kappa)^2\kappa(-1+\alpha\kappa)}{(y^2(-1+\alpha)^2+x^2\kappa^2(-1+\alpha\kappa)^2)^2} \end{pmatrix};$

In[ ]:=
```
J1 = FullSimplify[J /. {α → η, x → Sqrt[(4 - γ (1 + κ))/((-4 + γ) (-1 + κ^2))], y → Sqrt[(κ (γ - 4 κ + γ κ))/((-4 + γ) (-1 + κ^2))]},
    Assumptions → asp];
J2 = FullSimplify[J /. {α → η, x → -Sqrt[(4 - γ (1 + κ))/((-4 + γ) (-1 + κ^2))], y → Sqrt[(κ (γ - 4 κ + γ κ))/((-4 + γ) (-1 + κ^2))]},
    Assumptions → asp];
```

$$J1 = \text{FullSimplify}\left[J\ /.\ \left\{\alpha \to \eta,\ x \to \sqrt{\frac{4-\gamma(1+\kappa)}{(-4+\gamma)\left(-1+\kappa^2\right)}},\ y \to \sqrt{\frac{\kappa(\gamma-4\kappa+\gamma\kappa)}{(-4+\gamma)\left(-1+\kappa^2\right)}}\right\},\ \text{Assumptions} \to \text{asp}\right];$$

$$J2 = \text{FullSimplify}\left[J\ /.\ \left\{\alpha \to \eta,\ x \to -\sqrt{\frac{4-\gamma(1+\kappa)}{(-4+\gamma)\left(-1+\kappa^2\right)}},\ y \to \sqrt{\frac{\kappa(\gamma-4\kappa+\gamma\kappa)}{(-4+\gamma)\left(-1+\kappa^2\right)}}\right\},\ \text{Assumptions} \to \text{asp}\right];$$

Case of $\gamma = 2$

In[ ]:=
```
M2 = FullSimplify[J2.J1 /. γ → 2, Assumptions → asp];
```

In[ ]:=
```
FullSimplify[Eigenvalues[M2], Assumptions → κ > 1]
```

Out[ ]=
```
{0, 0, 1}
```

Case of $\gamma = 1$

In[ ]:=
```
M1 = FullSimplify[J2.J1 /. γ → 1, Assumptions → asp];
e1 = FullSimplify[Eigenvalues[M1], Assumptions → κ > 100];
```

In[ ]:=
```
e1
```

In[ ]:=
```
e1
```

Out[ ]=

$$\left\{\left(2\,\text{Root}\left[\left(\frac{18\kappa}{\sqrt{-1+\kappa^2}} - \frac{126\kappa^2}{\sqrt{-1+\kappa^2}} + \frac{162\kappa^3}{\sqrt{-1+\kappa^2}} + \frac{306\kappa^4}{\sqrt{-1+\kappa^2}} - \frac{306\kappa^5}{\sqrt{-1+\kappa^2}} - \frac{162\kappa^6}{\sqrt{-1+\kappa^2}} + \frac{126\kappa^7}{\sqrt{-1+\kappa^2}} - \frac{18\kappa^8}{\sqrt{-1+\kappa^2}}\right)\#1^2 + \#1^3\ \&,\ 1\right]\right) \Big/ \left(9\,(-1+\kappa)^4\,\kappa\,(1+\kappa)\,\sqrt{-1+\kappa^2}\right),\right.$$
$$\left(2\,\text{Root}\left[\left(\frac{18\kappa}{\sqrt{-1+\kappa^2}} - \frac{126\kappa^2}{\sqrt{-1+\kappa^2}} + \frac{162\kappa^3}{\sqrt{-1+\kappa^2}} + \frac{306\kappa^4}{\sqrt{-1+\kappa^2}} - \frac{306\kappa^5}{\sqrt{-1+\kappa^2}} - \frac{162\kappa^6}{\sqrt{-1+\kappa^2}} + \frac{126\kappa^7}{\sqrt{-1+\kappa^2}} - \frac{18\kappa^8}{\sqrt{-1+\kappa^2}}\right)\#1^2 + \#1^3\ \&,\ 2\right]\right) \Big/ \left(9\,(-1+\kappa)^4\,\kappa\,(1+\kappa)\,\sqrt{-1+\kappa^2}\right),$$
$$\left.\left(2\,\text{Root}\left[\left(\frac{18\kappa}{\sqrt{-1+\kappa^2}} - \frac{126\kappa^2}{\sqrt{-1+\kappa^2}} + \frac{162\kappa^3}{\sqrt{-1+\kappa^2}} + \frac{306\kappa^4}{\sqrt{-1+\kappa^2}} - \frac{306\kappa^5}{\sqrt{-1+\kappa^2}} - \frac{162\kappa^6}{\sqrt{-1+\kappa^2}} + \frac{126\kappa^7}{\sqrt{-1+\kappa^2}} - \frac{18\kappa^8}{\sqrt{-1+\kappa^2}}\right)\#1^2 + \#1^3\ \&,\ 3\right]\right) \Big/ \left(9\,(-1+\kappa)^4\,\kappa\,(1+\kappa)\,\sqrt{-1+\kappa^2}\right)\right\}$$

The first two roots of the Polynomial are 0. Hence it suffices to consider the last one, with closed form given as follows.

In[ ]:=

$$v1 = -2\,\frac{\frac{18\kappa}{\sqrt{-1+\kappa^2}} - \frac{126\kappa^2}{\sqrt{-1+\kappa^2}} + \frac{162\kappa^3}{\sqrt{-1+\kappa^2}} + \frac{306\kappa^4}{\sqrt{-1+\kappa^2}} - \frac{306\kappa^5}{\sqrt{-1+\kappa^2}} - \frac{162\kappa^6}{\sqrt{-1+\kappa^2}} + \frac{126\kappa^7}{\sqrt{-1+\kappa^2}} - \frac{18\kappa^8}{\sqrt{-1+\kappa^2}}}{9\,(-1+\kappa)^4\,\kappa\,(1+\kappa)\,\sqrt{-1+\kappa^2}};$$

In[ ]:=
```
FullSimplify[v]
```

Out[ ]=

$$\frac{4\ (1 + (-4 + \kappa)\ \kappa)^2}{(-1 + \kappa)^4}$$

Solve for the final range of κ

$$\text{Reduce}\left[\frac{4\ (1 + (-4 + \kappa)\ \kappa)^2}{(-1 + \kappa)^4} > 1,\ \kappa\right]$$

Out[ ]=

$$\kappa < 3 - 2\sqrt{2}\ \ ||\ \ \frac{1}{3} < \kappa < 1\ \ ||\ \ 1 < \kappa < 3\ \ ||\ \ \kappa > 3 + 2\sqrt{2}$$