Math 0413 Midterm

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1. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be 2 injective fxns. defn $h: \mathbb{R} \to \mathbb{R}$ by $h(x) = g(f(3x)-2) + f(1)$ $\forall x \in \mathbb{R}$. Prove h is injective

$\forall x_1, x_2 \in \mathbb{R}$, suppose $h(x_1) = h(x_2)$. Then

$h(x_1) = g(f(3x_1)-2) + f(1) = h(x_2) = g(f(3x_2)-2) + f(1)$

$\rightarrow g(f(3x_1)-2) = g(f(3x_2)-2)$ subtract $f(1) \in \mathbb{R}$, $f(1) = f(1)$

$\rightarrow f(3x_1) - 2 = f(3x_2) - 2$ by injectivity of g

$\rightarrow f(3x_1) = f(3x_2)$ add 2

$\rightarrow 3x_1 = 3x_2$ by injectivity of f

$\rightarrow x_1 = x_2$ divide by 3

Then, $\forall x_1, x_2 \in \mathbb{R}$, $h(x_1) = h(x_2)$ implies $x_1 = x_2$. Thus, h is an injective function. QED

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p2

2. Use mathematical induction to prove that
$1+5+9+...+(4n-3) = n(2n-1) \quad \forall n \in \mathbb{N}$.

let $P(n)$ be the statement that the above equality holds for some $n \in \mathbb{N}$.

$P(1)$: $1 = 1(2(1)-1) = 1(1) = 1$ ✓ $P(1)$ holds

we will prove that, for any $k \in \mathbb{N}$, $P(k) \Rightarrow P(k+1)$. Assume $P(k)$ is true.

$$P(k) = 1+5+...+(4k-3) = k(2k-1)$$
$$1+5+...+(4k-3)+(4(k+1)-3) = k(2k-1)+(4(k+1)-3)$$
$$= 2k^2-k+4k+4-3$$
$$= 2k^2+3k+1$$
$$= (2k+1)(k+1)$$
$$= (k+1)(2k+1)$$
$$= (k+1)(2(k+1)-1)$$

which is of the form $n(2n-1)$ for $n=k+1$. Thus, $P(k)$ implies $P(k+1)$. Thus, by PMI, $P(n)$ is true $\forall n \in \mathbb{N}$. □

QED

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3. Prove that $2|5x+7| + 5|2x+3| \geq 1$ holds $\forall x \in \mathbb{R}$.

Let x be an arbitrary element of $\mathbb{R}$.

$$
\begin{aligned}
& 2|5x+7| + 5|2x+3| \\
= \ & |2||5x+7| + |-5||2x+3| && |2| = 2,\ |-5| = 5. \\
= \ & |10x+14| + |-10x-15| \\
\geq \ & |10x+14-10x-15| && |a| + |b| \geq |a+b| \quad \Delta\text{-ineq} \\
\geq \ & |-1| \\
\geq \ & 1.
\end{aligned}
$$

Thus it has been shown that for any $x \in \mathbb{R}$, $2|5x+7| + 5|2x+3| \geq 1$ holds.

QED.

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4. Suppose that $a, t \in \mathbb{R}$ and $t > 0$. Prove $\exists z \in \mathbb{R} \setminus \mathbb{Q}$ such that $|z-a| < t/2$.

Because $t > 0$ and $2 > 1 > 0$, we have that $t/2 \in \mathbb{R}$ and $t/2 > 0$.

$$\rightarrow 0 < t/2$$

$$\rightarrow a < t/2 + a \quad \text{add } a$$

by $a, (t/2 + a) \in \mathbb{R}$ and $a < t/2 + a$, and the density of the irrationals, $\exists z \in \mathbb{R} \setminus \mathbb{Q}$ s.t

$$a < z < t/2 + a$$

$$\rightarrow 0 < z - a < t/2$$

because $z - a > 0$, we have that

$$z - a = |z - a|$$

$$\rightarrow 0 < |z-a| < t/2$$

$$\rightarrow |z-a| < t/2.$$

Thus it has been shown that there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ that satisfies $|z-a| < t/2$.

QED.

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5. Let $S=\left\{\frac{4n+5}{5n+6} : n \in \mathbb{N}\right\}$. Prove that $\inf(S) \in \mathbb{R}$ and $\inf(S) \geq 4/5$.

By definition, $S \subset \mathbb{R}$ and is nonempty.

$\forall x \in S$, $\exists n \in \mathbb{N}$ such that $x = (4n+5)/(5n+6)$. by $n \in \mathbb{N}$, $n \geq 1 > 0$. We also have that $4n > 0$ by $4 > 0$, and that $25 > 24 \rightarrow 25/5 > 24/5 \rightarrow 5 > 24/5$, as well as $5 > 0 \rightarrow 5n > 0$, and $6 > 0 \rightarrow 5n+6 > 0$. Thus $1/(5n+6) \in \mathbb{R}$ and $1/(5n+6) > 0$. Then,

$$5 > 24/5$$

$$4n+5 > 4n + 24/5$$

$$\frac{1}{5n+6}(4n+5) > \frac{1}{5n+6}(4n+24/5)$$

$$\frac{4n+5}{5n+6} > \frac{4\cancel{(n+6/5)}}{5\cancel{(n+6/5)}}$$

$$x > 4/5 \quad \forall x \in S.$$

Then $4/5$ is a lower bound of S; S is bounded below.

by Axiom (C), $\inf(S)$ exists in $\mathbb{R}$. As $4/5$ is a lower bound of S, it follows from the definition of $\inf(S)$ that

$$\inf(S) \geq 4/5 \qquad \text{QED}$$