Math 04B HW10 Jack Veith p1

1. Prove that $\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^n}$ is a convergent series and find its sum.

Pf

$\forall n \in \mathbb{N}$, $\frac{3^{n/2}}{2^n} = \frac{(\sqrt{3})^n}{2^n} = \left(\frac{\sqrt{3}}{2}\right)^n$

Since $\left|\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2} = \sqrt{\frac{3}{4}} < \sqrt{1} = 1$, by the Geometric Series Test $\sum_{n=1}^{\infty} (1)\left(\frac{\sqrt{3}}{2}\right)^n$ is a convergent series, and its sum is

$$\frac{1}{1-(\sqrt{3}/2)}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2-\sqrt{3}} = \frac{(2+\sqrt{3})\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2\sqrt{3}+3}{4-3}$$

$$= 2\sqrt{3}+3, \quad \text{i.e., } \sum_{n=1}^{\infty} (1)\left(\frac{\sqrt{3}}{2}\right)^n = 2\sqrt{3}+3.$$

QED.

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2. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^3-(n-1)^3}$ is a divergent series.

Let $a_n = \frac{(-1)^n n^2}{(n+1)^3-(n-1)^3}$ $\forall n = 2k$ $\forall k \in \mathbb{N}$.

Then $(-1)^n = (-1)^{2k} = ((-1)(-1))^k = (1)^k = 1 \Rightarrow (-1)^n n^2 = n^2$.

Consider the denominator of the quotient a_n:

$$(n+1)^3 - (n-1)^3 = n^3+3n^2+3n+1-(n^3-3n^2+3n-1)$$

$$\longrightarrow \quad = 6n^2+2$$

$$\rightarrow \quad a_n = n^2/6n^2+2$$

$$a_n = \frac{n^2}{6n^2+2} \cdot \frac{n^2}{n^2}$$

$$a_n = \frac{1}{6+2/n^2}$$

$$\lim_{n\to\infty} a_n = \frac{\lim 1}{\lim 6 + \lim(2/n^2)}$$

$$= \frac{1}{6+0} = \frac{1}{6} \neq 0.$$

$\lim_{n\to\infty} a_n$ exists and is nonzero. From this we can conclude via Thm 38 that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^3-(n-1)^3}$$ is a divergent series.

QED

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3. Prove that $\sum_{n=1}^{\infty} \frac{1}{25n^2+5n-6}$ is a convergent series and find its sum.

pf

Let $a_n = \frac{1}{25n^2+5n-6}$ $\forall n \in \mathbb{N}$.

$a_n = 1/25n^2+5n-6$

$\longrightarrow a_n = 1/(5n+3)(5n-2)$

$\longrightarrow = \frac{1}{(5n+3)(5n-2)} = \frac{A}{5n+3} + \frac{B}{5n-2} \longrightarrow A(5n-2)+B(5n+3) = 1$

by letting $n = 2/5$, $A(5(2/5)-2)+B(5(2/5)+3) = 1$

$\longrightarrow A(0)+B(5) = 1$

$\longrightarrow B = 1/5$

by letting $n = -3/5$, $A(5(-3/5)-2)+\frac{1}{5}(5(-3/5)+3 = 1$

$\longrightarrow A(-5)+1/5(0) = 1$

$\longrightarrow A = -1/5$

which implies that $a_n = \frac{1}{(5n-2)(5)} - \frac{1}{(5n+3)(5)}$ $\forall n \in \mathbb{N}$.

$S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n =$

$(1/15 - 1/40)+(1/40-1/65)+\ldots(1/(5(n-1)-2) - 1/(5n+3))$

$= \frac{1}{15} - \frac{1}{5n+3} = \frac{1}{15} - \frac{1/n}{5+3/n}$

Then by Thm 27 $\{S_n\}$ is a convergent sequence,

$\lim_{n\to\infty} S_n = \frac{1}{15} - \frac{\lim 1/n}{5+\lim 1/n} = \frac{1}{15} - \frac{0}{5+0} = \frac{1}{15}$.

by dfn, $\sum_{n=1}^{\infty} a_n$ is a convergent sequence and its sum is $1/15$.

QED

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p4

4. suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series and $\sum_{n=1}^{\infty} y_n$ is divergent series, prove that $\sum_{n=1}^{\infty}(x_n+y_n)$ is divergent.

suppose that $\sum_{n=1}^{\infty}(x_n+y_n)$ is a convergent series. Then let $z_n = x_n+y_n$ $\forall n \in \mathbb{N}$. Thus $\sum_{n=1}^{\infty} z_n$ is a convergent series. Consider that $\sum_{n=1}^{\infty} x_n$ is also a convergent series. Then,

$$\sum_{n=1}^{\infty}(z_n-x_n)$$ is assured to be convergent by

Th27. However,
$$z_n = x_n+y_n$$
$$\rightarrow z_n - x_n = y_n$$
$$\sum_{n=1}^{\infty}(z_n-x_n) = \sum_{n=1}^{\infty}(y_n)$$

Thus arises a contradiction as y_n is already assumed to be a divergent series. Thus $\sum_{n=1}^{\infty}(x_n+y_n)$ must be a divergent series.

QED

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p5

5. prove that $\sum_{n=1}^{\infty} \frac{n+2}{n^3+2n^2+5}$ is a convergent series.

consider that, $\forall n \in \mathbb{N}$, $n^3+2n^2+5 \geq n^3+2n^2$

$$\rightarrow \frac{1}{n^3+2n^2+5} \leq \frac{1}{n^3+2n^2}$$

$$\rightarrow \frac{n+2}{n^3+2n^2+5} \leq \frac{n+2}{n^3+2n^2}$$

RHS; $\frac{n+2}{n^3+2n^2} = \frac{n+2}{n+2} \frac{1}{n^2} = \frac{1}{n^2}$

$$\lim_{n\to\infty}\left(\frac{1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 \cdot 0 = 0.$$

Thus RHS is a convergent sequence and each term of RHS is greater than LHS at n:

$$\frac{n+2}{n^3+2n^2+5} \leq \frac{1}{n^2} \quad \forall n \in \mathbb{N}.$$

Then by the Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n+2}{n^3+2n^2+5}$$ is a convergent series.

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6. is $\sum_{n=1}^{\infty}\left(\sqrt{n^5+2n}-\sqrt{n^5+1}\right)$ convergent or divergent?

let $a=n^5+2n$, $b=n^5+1$. $\forall n\in\mathbb{N}$, $2n>1 \rightarrow a>b$

$$0\le\sqrt{a}-\sqrt{b}=\sqrt{a}-\sqrt{b}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)=\frac{a-b}{\sqrt{a}+\sqrt{b}}$$

$$\sqrt{n^5+2n}-\sqrt{n^5+1}\ge n^2 \longrightarrow$$

$$\frac{1}{\sqrt{n^5+2n}-\sqrt{n^5+1}}\le\frac{1}{n^2}$$

$\rightarrow$ by Comparison Test,

$\sum_{n=1}^{\infty}\left(\sqrt{n^5+2n}-\sqrt{n^5+1}\right)$ is convergent?

n^5+2n-n^5+1

$$\frac{2n+1}{\sqrt{n^5+2n}+\sqrt{n^5+1}}$$

$$\sqrt{a}-\sqrt{b}\quad\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$$

$a-b$

n^5+2n