Quiz 7

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p1

2. dfn the sequence $\{X_n\}$ by $X_1 = 1$ and $X_{n+1} = (3/4) + \sqrt{X_n}$ $\forall n \in \mathbb{N}$. prove $\{X_n\}$ converges

First use PMI to get that $0 < X_n \leq X_{n+1} \leq 100$ $\forall n \in \mathbb{N}$.

when $n=1$, $X_n = X_1 = 1$

$$X_{n+1} = X_2 = (3/4) + \sqrt{X_1}$$
$$X_2 = 7/4$$

Since $0 < 1 \leq 7/4 \leq 100$, $0 < X_n \leq X_{n+1} \leq 100$ holds.

suppose that for a $k \in \mathbb{N}$,

$$0 \leq X_k \leq X_{k+1} \leq 100. \text{ Then}$$
$$\sqrt{0} < \sqrt{X_k} \leq \sqrt{X_{k+1}} \leq \sqrt{100}$$
$$(3/4) + 0 < (3/4) + \sqrt{X_k} \leq (3/4) + \sqrt{X_{k+1}} \leq (3/4) + 10$$

by the definition of X_{n+1},

$$3/4 < X_{k+1} \leq X_{k+2} \leq (43/4) \leq 100$$

I've have proved that $0 < X_k \leq X_{k+1} \leq 100 \rightarrow 0 < X_{k+1} \leq X_{k+2} \leq 100$ for all $k \in \mathbb{N}$. Thus by PMI $0 < X_n \leq X_{n+1} \leq 100$ is true $\forall n \in \mathbb{N}$.

Thus, $\{X_n\}$ is bounded and monotone. Thus, by the Monotone Convergence Theorem, $\{X_n\}$ is a convergent sequence.

Q.E.D.