HW 9 Jack Veith

1. let $x_n = (-1)^n \left(\frac{n+1}{n}\right)$ for $n \in \mathbb{N}$. Calculate $\limsup_{n \to \infty} x_n$.

for all $n \in \mathbb{N}$, let $T_n = \{x_k : k \geq n\}$.

assume n is even, Then

$$x_n = (-1)^n \left(\frac{n+1}{n}\right) = \frac{n}{n} + \frac{1}{n} = 1 + 1/n.$$

for $k \geq n$, we have that $k \geq n > 0 \to 1/k \leq 1/n$. Thus,

$$|x_k| = \left|(-1)^k \frac{k+1}{k}\right| = \frac{k+1}{k} = 1 + \frac{1}{k} \leq 1 + \frac{1}{n}$$

Therefore $x_k \leq 1 + 1/n \quad \forall k \geq n \to 1 + 1/n$ is an upr bd of T_n.
As $1 + 1/n = x_n \in T_n$, we can say that $\sup(T_n) = 1 + 1/n$ when n is even.

Assume n is odd. Then

$$x_n = (-1)^n \left(\frac{n+1}{n}\right) = -(1 + 1/n) \quad \text{consider } n \text{ is odd} \to n+1 \text{ is even.}$$

$$x_{n+1} = (-1)^{n+1} \left(\frac{(n+1)+1}{n+1}\right) = (1 + 1/n+1). \longrightarrow \quad x_n < 0 < x_{n+1}$$

for $k \geq n+1$, $\to k \geq n+1 > 0 \to 1/k \leq 1/n+1$ and

$$|x_k| = \left|(-1)^k \frac{k+1}{k}\right| = 1 + 1/k \leq 1 + 1/n+1$$

Thus $x_k \leq 1 + 1/n+1 = x_{n+1}$.

Therefore $x_k \leq x_{n+1}$ holds $\forall k \geq n$, implies that x_{n+1} is an upr bd of of T_n. by $x_{n+1} \in T_n$, it follows that

$$\sup(T_n) = 1 + 1/n+1 \text{ when } n \text{ is odd. Then,}$$

$$(*) \quad \sup(T_n) = \begin{cases} 1 + 1/n & n \text{ is even} \\ 1 + 1/n+1 & n \text{ is odd} \end{cases}$$

cont'd. $\longrightarrow$

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1. contd.) by $(*)$, $\forall n \in \mathbb{N}$, we have

$$|\sup(T_n) - 1| = \frac{1}{n} \text{ or } \frac{1}{n+1}.$$

Thus, by $1/n+1 < 1/n$, $|\sup(T_n)-1| \leq 1/n$ holds $\forall n \in \mathbb{N}$.

by Th25, $\limsup_{n\to\infty}(\sup(T_n)) = 1$. by definition,

$$\limsup_{n\to\infty} X_n = 1. \quad \text{QED.}$$

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2. Let $\{X_n\}$ be a sequence s.t. $\limsup_{n\to\infty}(|X_n|^{1/n}) < 1$. Prove that $\lim_{n\to\infty} X_n = 0$.

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3. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim b_n = 0$, suppse that $|a_n - a_q| \le b_q$ holds whenever $n, q \in \mathbb{N}$ and $n \ge q$. prove $\{a_n\}$ is a cauchy sequence.

By the definition of a limit, $\forall \varepsilon > 0$, $\exists K = K(\varepsilon) \in \mathbb{N}$ s.t.

$$|b_q - 0| < \varepsilon \quad \forall q \ge K.$$

For $n, m \ge K$, there are 2 cases:

(i) $n \ge m$ (ii) $m > n$

(i) for $n \ge m \ge k$, we have that

$$|a_n - a_m| \le b_m < \varepsilon \rightarrow |a_n - a_m| < \varepsilon$$

for $n \ge m$ by assumption that $|a_n - a_q| \le b_q$ for $n, q \in \mathbb{N}$ and $n \ge q$.

(ii) for $m > n \ge K$, we have that

$$|a_m - a_n| \le b_n < \varepsilon \rightarrow |a_n - a_m| < \varepsilon$$

for $m > n$ by same assumption as above.

Then by (i) and (ii), $\exists K = K(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall n, m \ge k.$$

by the definition of a Cauchy sequence, $\{a_n\}$ is Cauchy.

QED.

(a) Let $\{x_n\}$ be a sequence and $0<\lambda<1$ suppose that $|x_n - x_{n-1}| \le \lambda |x_{n-1} - x_{n-2}|$ holds $\forall n \ge 3$. prove that $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence.

with PMI we will prove that $|x_{n+1} - x_n| \le \lambda^{n-1} |x_2 - x_1|$ holds $\forall n \in \mathbb{N}$. for $n=1$

$$|x_2 - x_1| \le \lambda^{1-1} |x_2 - x_1|$$
$$|x_2 - x_1| = |x_2 - x_1| \quad ✓$$

prove that $P(k)$ "the above statement for $n=k$" implies $P(k+1)$. assume $P(k)$ holds.

$$P(k) \quad |x_{k+1} - x_k| \le \lambda^{k-1} |x_2 - x_1|$$
$$\lambda |x_{k+1} - x_k| \le \lambda^{k} |x_2 - x_1|$$
$$|x_{k+2} - x_{k+1}| \le \lambda |x_{k+1} - x_k| \le \lambda^{k} |x_2 - x_1| \quad \text{by assumption}$$
$$|x_{k+2} - x_{k+1}| \le \lambda^{k} |x_2 - x_1|$$

The previous inequality is of the form $|x_{n+1} - x_n| \le \lambda^{n-1} |x_2 - x_1|$ for $n = k+1$. Thus $P(k) \Rightarrow P(k+1)$ and $|x_{n+1} - x_n| \le \lambda^{n-1} |x_2 - x_1|$ holds $\forall n \in \mathbb{N}$.

Let $n, m \in \mathbb{N}$ and $n > m$.

$$|x_n - x_m| = |x_n + x_{n-1} - x_{n-1} + x_{n-2} - x_{n-2} + \dots + x_{m+1} - x_{m+1} - x_m|$$
$$|x_n - x_m| = |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots + (x_{m+1} - x_m)|$$
$$|x_n - x_m| \le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \dots + |x_{m+1} - x_m| \quad (\text{by } \Delta\text{-ineq})$$
$$|x_n - x_m| \le \lambda^{n-2} |x_2 - x_1| + \lambda^{n-3} |x_2 - x_1| + \dots + \lambda^{m-1} |x_2 - x_1|$$
$$= \lambda^{m-1} (1 + \lambda + \dots + \lambda^{n-m-1}) |x_2 - x_1|$$

The geometric sum of $(1 + \lambda + \dots + \lambda^{n-m-1}) = 1/\lambda(1-\lambda)$. $\rightarrow$

$$|x_n - x_m| \le \frac{\lambda^{m-1} |x_2 - x_1|}{\lambda(1-\lambda)} = \lambda^{m-1}$$

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5. let $\{x_n\}$ be defined by $x_1 = 2$, $x_2 = 7$ and
$x_n = \frac{x_{n-1}}{3} + \frac{2x_{n-2}}{3}$ $\forall n \geq 3$. prove $\{x_n\}$ converges.

$\forall n \geq 3$,

$$|x_n - x_{n-1}| = \left|\left(\frac{x_{n-1}}{3} + \frac{x_{n-2}}{3}\right) - x_{n-1}\right|$$

$$= \left|\frac{2x_{n-2}}{3} - \frac{2x_{n-1}}{3}\right|$$

$$= \left|-\frac{2}{3}\right| |x_{n-1} - x_{n-2}|$$

$$|x_n - x_{n-1}| = \frac{2}{3}|x_{n-1} - x_{n-2}|$$

for $0 < 2/3 < \lambda < 1$,

$$|x_n - x_{n-1}| < \lambda |x_{n-1} - x_{n-2}| \text{ holds } \forall n \geq 3.$$

Then by the definition of a contractive sequence given in 4., $\{x_n\}$ is a contractive sequence and is thus a convergent sequence as well.

Q.E.D.

HW 9

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pG

7. find the limit of $\{X_n\}$ from problem 5.

by definition $X_n = X_{n-1}/3 + 2X_{n-2}/3 \quad \forall n \geq 3$.
Let $P(n)$ be the statement that $X_{n+1} = (-2/3)X_n + 25/3$.
prove $P(2)$ is true:

$$X_3 = X_2/3 + 2X_1/3$$
$$= 7/3 + 4/3 = 11/3 \quad \text{by dfn of } \{X_n\}$$

$$P(2) \quad X_3 = (-2/3)X_2 + 25/3$$
$$= 7(-2/3) + 25/3 = -14/3 + 25 = 11/3 \checkmark$$

suppose $P(k)$ is true for $k \in \mathbb{N}$. Prove $P(k) \rightarrow P(k+1)$:

$$P(k): \quad X_{k+1} = \frac{-2X_k}{3} + 25/3$$
$$X_{k+1} - (2/3)X_{k+1} + (2/3)X_k = 25/3 - (2/3)X_{k+1}$$
$$(1/3)X_{k+1} + (2/3)X_k = (-2/3)X_{k+1} + 25/3$$

by dfn of X_n, LHS $= X_{k+2}$. Then
$$X_{k+2} = (-2/3)X_{k+1} + 25/3$$

Thus, $P(k) \rightarrow P(k+1)$ $\forall k \in \mathbb{N}$. By PMI, $X_{n+1} = (-2/3)X_n + 25/3$ (*) holds $\forall n \in \mathbb{N}$.

by $\{X_n\}$ converges, $\lim_{n\to\infty} X_n$ exists. Let $X = \lim X_n$. Then

$$X = (-2/3)X + 25/3 \qquad (*)$$
$$5X/3 = 25/3$$
$$X = 5.$$

Thus the limit of the sequence $\{X_n\}_{n=1}^{\infty}$ is 5.

Q.E.D.

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p4

4. Let $\{X_n\}$ be a sequence and $0<\lambda<1$ suppose that $|X_n - X_{n-1}| \leq \lambda |X_{n-1} - X_{n-2}|$ holds. $\forall n \geq 3$. prove that $\{X_n\}$ is a Cauchy sequence.

via PMI we will prove that $|X_{n+1} - X_n| \leq \lambda^{n-1}|X_2 - X_1|$ holds $\forall n \in \mathbb{N}$. for $n=1$:

$$|X_2 - X_1| \leq \lambda^{1-1}|X_2 - X_1|$$
$$|X_2 - X_1| = |X_2 - X_1| \quad \checkmark$$

prove that $P(k)$ = "the above statement for $n=k$" implies $P(k+1)$. assume $P(k)$ holds.

$$P(k) = |X_{k+1} - X_k| \leq \lambda^{k-1}|X_2 - X_1|$$
$$\lambda|X_{k+1} - X_k| \leq \lambda^k |X_2 - X_1|$$
$$|X_{k+2} - X_{k+1}| \leq \lambda |X_{k+1} - X_k| \leq \lambda^k |X_2 - X_1| \quad \text{by assumption}$$
$$|X_{k+2} - X_{k+1}| \leq |\lambda^k|X_2 - X_1|$$

The previous inequality is of the form $|X_{n+1} - X_n| \leq \lambda^{n-1}|X_2 - X_1|$ for $n=k+1$. Thus $P(k) \to P(k+1)$ and $|X_{n+1} - X_n| \leq \lambda^{n-1}|X_2 - X_1|$ holds $\forall n \in \mathbb{N}$.

for $n,m \in \mathbb{N}$ and $n>m$,

$$|X_n - X_m| = |X_n + X_{n-1} - X_{n-1} + X_{n-2} - X_{n-2} + \ldots + X_{m+1} - X_{m+1} - X_m|$$
$$|X_n - X_m| = |(X_n - X_{n-1}) + (X_{n-1} - X_{n-2}) + \ldots + (X_{m+1} - X_m)|$$
$$|X_n - X_m| \leq |X_n - X_{n-1}| + |X_{n-1} - X_{n-2}| + \ldots + |X_{m+1} - X_m| \quad (\text{by } \Delta\text{-ineq})$$
$$|X_n - X_m| \leq \lambda^{n-2}|X_2 - X_1| + \lambda^{n-3}|X_2 - X_1| + \ldots + \lambda^{m-1}|X_2 - X_1|$$
$$= \lambda^{m-1}(1 + \lambda + \ldots + \lambda^{n-m-1})|X_2 - X_1|$$

The geometric sum of $(1+\lambda + \ldots + \lambda^{n-m-1}) = 1/\lambda(1-\lambda)$. $\rightarrow$

$$|X_n - X_m| \leq \frac{\lambda^{m-1}|X_2 - X_1|}{\lambda(1-\lambda)} = \lambda^{m-1}$$