Implementation of Biharmonic operator in polar coordinates using python

ME 223 : Term paper

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Introduction

- Inspiration : While studying airy stress function, we come across a daunting term known as biharmonic. While in cartesian system of coordinates I found it is easy to check if a particular mathematical function satisfies this equation :-

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = 0$$

In case of polar coordinates , it is an ordeal :-

$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0$$

—

- Simplifying it further it takes quite a gruesome form : -

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)\right)\right)+\frac{2}{r^2}\frac{\partial^4 \varphi}{\partial \theta^2 \partial r^2}+\frac{1}{r^4}\frac{\partial^4 \varphi}{\partial \theta^4}-\frac{2}{r^3}\frac{\partial^3 \varphi}{\partial \theta^2 \partial r}+\frac{4}{r^4}\frac{\partial^2 \varphi}{\partial \theta^2}=0$$

$$\frac{r^4\frac{\partial^4}{\partial r^4}\phi(r,\theta)+2r^3\frac{\partial^3}{\partial r^3}\phi(r,\theta)-r^2\frac{\partial^2}{\partial r^2}\phi(r,\theta)+2r^2\frac{\partial^4}{\partial \theta^2 \partial r^2}\phi(r,\theta)+r\frac{\partial}{\partial r}\phi(r,\theta)-2r\frac{\partial^3}{\partial \theta^2 \partial r}\phi(r,\theta)+4\frac{\partial^2}{\partial \theta^2}\phi(r,\theta)+\frac{\partial^4}{\partial \theta^4}\phi(r,\theta)}{r^4}$$

As a person who is quite lazy doing gruesome repetitive mathematical operations, I found it my duty to do something about this and my trials has resulted in this term paper.

What did I do?

- So of late I had found **python** to be a very useful slave to automate stuffs and implement mathematical and scientific operations. I found about SymPy . It has got a whole bunch of functions to implement calculus and trigonometry-based operations by a single line of codes. What was required was the math I had learnt in college and school, to put code together, and reach at the final result.

Back to basics

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\,\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \qquad (a)$$

For the present purpose we need this equation transformed to polar coordinates. The relation between polar and Cartesian coordinates is given by

$$r^2 = x^2 + y^2, \qquad \theta = \arctan\frac{y}{x}$$

from which

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

Courtesy : Theory of Elasticity by Timoshenko and Goodier.

Similarly we can get derivative of phi with respect to y

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x}$$

Road ahead.. With jupyter notebook

- We will 1st define del /del x as a function in terms of r and theta in SymPy and make the same function operate on del phi by del x again to get del^2 (phi) / del x^2 and same in case of y.

Turning idea into code

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So here I am getting directly to the jupyter notebook. 1st step is obviously to import sympy.

[168]:
```
import sympy as sym

r, theta = sym.symbols('r, theta')

delr_by_delx = sym.cos(theta)

delr_by_dely = sym.sin(theta)

deltheta_by_delx = -sym.sin(theta)/r

deltheta_by_dely = sym.cos(theta)/r
```

In the above code we basically have defined r and theta , and following derivatives using the basics of polar coorindatees which we discussed.

$$\frac{\partial r}{\partial x} = \cos\theta \tag{1}$$
$$\frac{\partial r}{\partial y} = \sin\theta \tag{2}$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \tag{3}$$
$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \tag{4}$$
$$\tag{5}$$

[169]:
```
def del_by_delx(f):
    return delr_by_delx*sym.diff(f,r) + deltheta_by_delx*sym.diff(f,theta)






def del_by_dely(f):
    return delr_by_dely*sym.diff(f,r) + deltheta_by_dely*sym.diff(f,theta)
```

```python
def del2_by_delx2(f):
    return del_by_delx(del_by_delx(f))



def del2_by_dely2(f):
    return del_by_dely(del_by_dely(f))
```

I think it is pretty obvious what i am doing above, to define laplacian on f , (so that we can get biharmonic by taking laplacian on laplacian of f) , in terms of polar coorindates, we need

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \tag{6}$$

so

1st we defined this in the 1st two functions : −

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{7}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{8}$$

$$\tag{9}$$

and to get the partial derivatives of order 2, we called the same function again.

[170]:
```python
def polarLaplacian(f):
    return (del2_by_delx2(f) + del2_by_dely2(f)).simplify()

def polarbiharmonic(f):
    return polarLaplacian(polarLaplacian(f))
```

for laplacian we used the addition of the two functions we defined above, to get this.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \tag{10}$$

and for biharmonic, we called the laplacian on the laplacian of f in terms of polar coordinates, and over!

[171]:
```python
phi = sym.Function('phi')(r,theta)
```

```python
polarbiharmonic(phi)
```

[171]:
$$\frac{r^4\frac{\partial^4}{\partial r^4}\phi(r,\theta)+2r^3\frac{\partial^3}{\partial r^3}\phi(r,\theta)-r^2\frac{\partial^2}{\partial r^2}\phi(r,\theta)+2r^2\frac{\partial^4}{\partial\theta^2\partial r^2}\phi(r,\theta)+r\frac{\partial}{\partial r}\phi(r,\theta)-2r\frac{\partial^3}{\partial\theta^2\partial r}\phi(r,\theta)+4\frac{\partial^2}{\partial\theta^2}\phi(r,\theta)+\frac{\partial^4}{\partial\theta^4}\phi(r,\theta}{r^4}$$

now since we have polar biharmonic, we can easily perform this opertion by calling the function. Let's calculate the biharmonic of some functions.

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Results:

[172]:
```python
print("biharmonic of",sym.cos(theta))
polarbiharmonic(sym.cos(theta))
```

biharmonic of cos(theta)

[172]:
$$-\frac{3\cos(\theta)}{r^4}$$

[173]:
```python
print("biharmonic of",sym.cos(theta)+sym.sin(theta))
polarbiharmonic(sym.cos(theta)+sym.sin(theta))
```

biharmonic of sin(theta) + cos(theta)

[173]:
$$-\frac{3\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)}{r^4}$$

[174]:
```python
print("biharmonic of",pow(sym.cos(theta)+sym.sin(theta),8))
polarbiharmonic(pow(sym.cos(theta)+sym.sin(theta),8))
```

biharmonic of (sin(theta) + cos(theta))**8

[174]:
$$\frac{(\sin(2\theta)+1)^2\left(1152\sqrt{2}(1-\cos(2\theta))^2\cos\left(2\theta+\frac{\pi}{4}\right)-1536(1-\cos(2\theta))^2-1632\sin(2\theta)-1152\sin(4\theta)-5088\cos}{r^4}$$

As you can see, we can practically calculate the biharmonic of any complicated function, even hyperbolic , exponential functions and their combination.

Results:-

Results:

In [179]:
```
print("biharmonic of",sym.cos(theta))
polarbiharmonic(sym.cos(theta))
```

```
biharmonic of cos(theta)
```

Out[179]:

$$-\frac{3\cos(\theta)}{r^4}$$

In [180]:
```
print("biharmonic of",sym.cos(theta)+sym.sin(theta))
polarbiharmonic(sym.cos(theta)+sym.sin(theta))
```

```
biharmonic of sin(theta) + cos(theta)
```

Out[180]:

$$-\frac{3\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)}{r^4}$$

In [181]:
```
print("biharmonic of",pow(sym.cos(theta)+sym.sin(theta),8))
polarbiharmonic(pow(sym.cos(theta)+sym.sin(theta),8))
```

```
biharmonic of (sin(theta) + cos(theta))**8
```

Out[181]:

$$\frac{(\sin(2\theta)+1)^2\left(1152\sqrt{2}(1-\cos(2\theta))^2\cos\left(2\theta+\frac{\pi}{4}\right)-1536(1-\cos(2\theta))^2-1632\sin(2\theta)-1152\sin(4\theta)-5088\cos(2\theta)-288\sqrt{2}\cos\left(6\theta+\frac{\pi}{4}\right)+5184\right)}{r^4}$$

As you can see, we can practically calculate the biharmonic of any complicated function, even hyperbolic , exponential functions and their combination.

Concluding remarks

. No matter how complicated the function, its biharmonic can be caculated in seconds

- We can also calculate the transformation of stress functions into polar coordinates by extending this. All we would need is to implement two lines of code containing matrix multiplication, and we are done.

- I have enclosed my code too, which can be opened using jupyter notebook. [".ipynb" extension]

Thank you.

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. Professor Arup Kumar Nandy

- James N. Goodier and Stephen Timoshenko

- Python Programming language.

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