FORMULA GUIDE

TRIGONOMETRICAL FORMULA

1 $\quad \text{Sin}\,\theta = \dfrac{1}{\text{Cosec}\,\theta}, \quad \text{Cosec}\,\theta = \dfrac{1}{\text{Sin}\,\theta}, \quad \text{Cos}\,\theta = \dfrac{1}{\text{Sec}\,\theta}, \quad \text{Sec}\,\theta = \dfrac{1}{\text{Cos}\,\theta}$

$\quad \text{Tan}\,\theta = \dfrac{1}{\text{Cot}\,\theta}, \quad \text{Cot}\,\theta = \dfrac{1}{\text{Tan}\,\theta}, \quad \text{Tan}\,\theta = \dfrac{\text{Sin}\,\theta}{\text{Cos}\,\theta}, \quad \text{Cot}\,\theta = \dfrac{\text{Cos}\,\theta}{\text{Sin}\,\theta}$

2. $\quad \sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \text{cosec}^2\,\theta = 1 + \cot^2\theta$

3. (A) <u>Sexagesimal System.</u>

(i) 1 Right Angle = 90 degree (90^o) (ii) 1^o = 60 minutes ($60'$) (iii) 1 minute = $60''$ **seconds ($60''$)**

(B) **<u>Relation between Radian and Degree.</u>**

π radian = 2 right angle = 180^o

Therefore $1^c = \left(\dfrac{180}{\pi}\right)^o$ and $1^o = \left(\dfrac{\pi}{180}\right)^c$

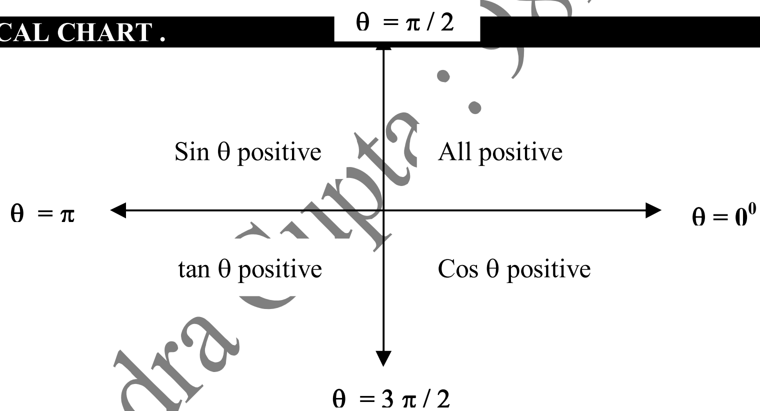
Note : - $1^c = 57^o17'\,44.8''$

4. Relation between radius, length of arc and central angle of circle : -

Suppose the circular measure of a central angle θ subtended by an arc of length l in a circle of radius r , then

$$\boxed{\theta = \frac{l}{r}}$$

5. TRIGONOMETRICAL CHART .



θ	- θ	90 – θ	90 + θ	180 – θ	180 + θ	270 – θ	270 + θ	360 – θ	360 + θ
Sin θ	- Sin θ	Cos θ	Cos θ	Sin θ	- Sin θ	- Cos θ	- Cos θ	- Sin θ	Sin θ
Cos θ	Cos θ	Sin θ	- Sin θ	- Cos θ	- Cos θ	- Sin θ	Sin θ	Cos θ	Cos θ
Tan θ	-Tan θ	Cot θ	- Cot θ	- Tan θ	Tan θ	Cot θ	- Cot θ	- Tan θ	Tan θ

Note:- **1.** Complementary angle (90^0, 270^0, or multiple) = Ratio just opposite. (Sin to Cos , Cos to Sin , Tan to Cot)
2. Supplementary angle (180^0, 360^0, or multiple) = Ratio as it is. (Sin to Sin , Cos to Cos, Tan to Tan)
3. The above chart can be learn by " **ALL STUDENT TAKE COFFEE".**
4. The above formula can be written for positive value of n as,
Sin (360 n + θ) = Sin θ , Cos (360 n + θ) = Cos θ ,Tan (360 n + θ) = Tan θ
Sin (360 n - θ) = –Sin θ , Cos (360 n - θ) = Cos θ, Tan (360 n - θ) = –Tan θ

6. Trigonometry value chart:-

Ratio Angle (θ)	0	π / 6 (30^o)	π / 4 (45^o)	π / 3 (60^o)	π / 2 (90^o)	2 π / 3 (120^o)	7π / 6 (135^o)	5 π / 6 (150^o)	π (180^o)
Sin θ	0	1 / 2	1 / √ 2	√ 3 / 2	1	√ 3 / 2	1 / √ 2	1 / 2	0

Cos θ	1	√3 / 2	1 / √2	1 / 2	0	-1 / 2	-1 / √2	− √3 / 2	-1
Tan θ	0	1 / √3	1	√3	∞	- √3	-1	- 1 / √3	0
Cosec θ	∞	2	√2	2 / √3	1	2 / √3	√2	2	∞
Sec θ	1	2 / √3	√2	2	∞	-2	−√2	-2 / √3	-1
Cot θ	∞	√3	1	1 / √3	0	-1 / √3	-1	−√3	∞

7. $\sin 18^o = \dfrac{\sqrt{5}-1}{4}$, $\cos 18^o = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$, $\cos 36^o = \dfrac{\sqrt{5}+1}{4}$, $\sin 36^o = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

Note: Since angles of 54^o and 72^o are complementary of angle 36^o and 18^o respectively,
$\sin 72^o = \cos 18^o$, $\cos 72^o = \sin 18^o$, $\sin 54^o = \cos 36^o$, $\cos 54^o = \sin 36^o$

8. (i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$ (ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$ (iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ (vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$ (viii) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

9. (i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

10. (i) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$ (ii) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$ (iv) $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

11. (i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ (ii) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ (iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

12. (i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(iv) $\sin 3A = 3\sin A - 4\sin^3 A$

(v) $\cos 3A = 4\cos^3 A - 3\cos A$

(vi) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

13. Half angle formula of T-Ratio :

(i) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(ii) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 1 - 2\sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1 = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(ii) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

14. $1 - \cos 2A = 2\sin^2 A$, $1 + \cos 2A = 2\cos^2 A$, $1 - \cos A = 2\sin^2(A/2)$, $1 + \cos A = 2\cos^2(A/2)$

15. Trigonometrically Equation :

(i) $\sin x = 0 \Rightarrow x = n\pi$ (ii) $\cos x = 0 \Rightarrow x = \frac{(2n+1)}{2}\pi$

(iii) $\tan x = 0 \Rightarrow x = n\pi$ (iv) $\sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n \alpha$

(v) $\cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha$ (vi) $\tan x = \tan\alpha \Rightarrow x = n\pi + \alpha$

(vii) $\sin^2 x = \sin^2\alpha,\ \cos^2 x = \cos^2\alpha,\ \tan^2 x = \tan^2\alpha \Rightarrow x = n\alpha \pm \alpha$

16. If A + B + C = π then,

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ (ii) $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$

(iii) $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$ (iv) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(v) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$

17. Sin Formula : -

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$, Where a, b and c are the side of triangle ABC.

18. Cosine Formula : -

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$ (iii) $\cos C = \frac{b^2 + a^2 - c^2}{2ab}$

(iv) $c^2 = a^2 + b^2 - 2ab\cos C$, (v) $a^2 = b^2 + c^2 - 2bc\cos A$ (vi) $b^2 = a^2 + c^2 - 2ac\cos B$

19. Tangential formula (Napier's analogy)

(i) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)$ (ii) $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\left(\frac{B}{2}\right)$ (iii) $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$

20. To Express the trigonometric ratios of half angles of a triangle in terms of its sides : -

In ΔABC and $2s = a + b + c$ then, Where s is half perimeter of trianlge ABC.

1. $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{b.c}}$ 2. $\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-a)(s-c)}{a.c}}$ 3. $\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-b)(s-a)}{a.b}}$

4. $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{b.c}}$ 5 $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{a.c}}$ 6 $\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{b.a}}$

7. $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ 8. $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$ 9. $\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}}$

21. To express the Sines of the angles of a triangle of the sides : .

$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$, $\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$, $\sin C = \frac{2}{ba}\sqrt{s(s-a)(s-b)(s-c)}$

22. Area of triangle : -

In triangle ABC then,

(i) $\Delta = \frac{1}{2} ab\sin C$ (ii) $\Delta = \frac{1}{2} ac\sin B$ (iii) $\Delta = \frac{1}{2} bc\sin A$

(iv) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ [Hero's Formula]

23. Radius of Circum- Circle : -

Let ABC be a triangle. O be the centre of the circle circum bring triangle ABC. Then OA = OB = OC will be radius of this circle often denoted by as R.

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

24. Relation between area of a triangle and its radius of circum- circle : -

$$R = \frac{abc}{4\Delta}$$

INVERSE TRIGONOMETRIC FUNCTIONS

1. (i) $\sin^{-1}(\sin\theta) = \theta$, $\sin(\sin^{-1}x) = x$ (ii) $\cos^{-1}(\cos\theta) = \theta$, $\cos(\cos^{-1}x) = x$

(iii) $\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1}x) = x$ (iv) $\sec^{-1}(\sec\theta) = \theta$, $\sec(\sec^{-1}x) = x$

(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$ (vi) $\cot^{-1}(\cot\theta) = \theta$, $\cot(\cot^{-1}x) = x$

2. $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$, $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$, $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $\sec^{-1}x = \cos^{-1}\frac{1}{x}$,

$\tan^{-1}x = \cot^{-1}\frac{1}{x}$, $\cot^{-1}x = \tan^{-1}\frac{1}{x}$,

3. (i) $\sin^{-1}(-x) = -\sin^{-1}x$ (ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$ (iii) $\tan^{-1}(-x) = -\tan^{-1}x$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$ (v) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$ (vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

4. (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ (iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

5. (i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ (ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-y^2}.\sqrt{1-x^2})$ (iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{1-y^2}.\sqrt{1-x^2})$

6. (i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ (ii) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy > 1$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ (iv) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(v) $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\left(\frac{xy+1}{x-y}\right)$ (vi) $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\left(\frac{xy-1}{x+y}\right)$

7. (*i*) $2\sin^{-1} = \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1-x^2}\right)$ (ii) $2\cos^{-1} = \cos^{-1}\left(2x^2-1\right) = \cos^{-1}\left(1-2x^2\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(iii) $2\tan^{-1} = \tan^{-1}\left(\frac{2x}{1+x^2}\right)$ (iv) $3\sin^{-1} = \sin^{-1}\left(3x - 4x^3\right)$

(*v*) $3\cos^{-1} = \cos^{-1}\left(4x^3 - 3x\right)$ (vi) $3\tan^{-1} = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

EXPONENTIAL AND LOGARITHMIC SERIES

1. (*i*) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{---------}\infty$ (*ii*) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \text{---------}\infty$

(iii) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} ++ \frac{1}{4!} + \text{---------}\infty$ (iv) $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \text{---------}\infty$

(v) $a^x = 1 + \frac{x\log_e a}{1!} + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)3}{3!} + \text{---------}\infty$ (vi) The value of e lies between 2 and 3 and e = 2.71828

(vii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{---------}\infty$ (viii) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{---------}\infty$

2. **Common Logarithms** – Logarithm of a number to the base 10 is called **common logarithm**.
3. **Change of Naperian Logarithms to common Logarithms : -**

(i) $\log_{10} N = \frac{1}{\log_e 10} \times \log_e N$ $\left[\because \frac{1}{\log_e 10} = 0.4324\right]$

=> $\log_e 10 = 0.4324 \times \log_e N$

(ii) $\log_a b \times \log_b a = 1$ (ii) $\log_a a = 1$ (iv) $\log_a b = \log_a e \times \log_e b$

HYPERBOLIC FUNCTION

1. $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$, $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$, $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$,

   $\operatorname{cosech} x = \dfrac{2}{e^{x} - e^{-x}}$, $\operatorname{sech} x = \dfrac{2}{e^{x} + e^{-x}}$, $\coth x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$
2. $\cosh^2 x - \sinh^2 x = 1$, $\operatorname{sech}^2 x = 1 - \tanh^2 x$ and $\operatorname{cosech}^2 x = -1 + \coth^2 x$
3. $\sinh(2x) = 2 \sinh x \, . \cosh x$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$
5. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
6. $\sinh(3x) = 3\sinh x + 4\sinh^3 x$, $\cosh(3x) = 4\cosh^3 x - 3\cosh x$

DIFFERENTIAL CALCULUS

1	$\frac{d}{dx}\left(x^{n}\right) = n\,x^{n-1}$	13	$\frac{d}{dx}\left(\operatorname{Sin}^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$
2	$\frac{d}{dx}\left(e^{ax}\right) = a\,e^{ax}$	14	$\frac{d}{dx}\left(\operatorname{Cos}^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}$
3	$\frac{d}{dx}(x) = 1$	15	$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$
4	$\frac{d}{dx}\left[a\,f(x)\right] = a\frac{d}{dx}f(x)$	16	$\frac{d}{dx}\left(\operatorname{Cot}^{-1}x\right) = \frac{-1}{1+x^2}$
5	$\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$	17	$\frac{d}{dx}\left(\operatorname{Sec}^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}$
6	$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x}\log_a e$	18	$\frac{d}{dx}\left(\operatorname{Cosec}^{-1}x\right) = \frac{-1}{x\sqrt{x^2-1}}$
7	$\frac{d}{dx}\left(\operatorname{Sin} ax\right) = a \operatorname{Cos} ax$	19	$\frac{d}{dx}\left(f_1 . f_2\right) = f_1\frac{d}{dx}f_2 + f_2\frac{d}{dx}f_1$
8	$\frac{d}{dx}\left(\operatorname{Cos} ax\right) = -a \operatorname{Sin} ax$	20	$\frac{d}{dx}\left(\frac{f_1}{f_2}\right) = \frac{f_2\frac{d}{dx}f_1 - f_1\frac{d}{dx}f_2}{\left(f_2\right)^2}$
9	$\frac{d}{dx}\left(\tan ax\right) = a \operatorname{Sec}^2 ax$	21	$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$
10	$\frac{d}{dx}\left(\operatorname{Cot} ax\right) = -a \operatorname{Cosec}^2 ax$	22	$\frac{d}{dx}[c] = 0$
11	$\frac{d}{dx}\left(\operatorname{Sec} ax\right) = a \operatorname{Sec} ax \tan ax$		
12	$\frac{d}{dx}\left(\operatorname{Cosec} ax\right) = -a \operatorname{Cosec} ax \operatorname{Cot} ax$		

INTEGRAL CALCULUS

1	$\int x^{n}\,dx = \frac{x^{n+1}}{n+1} + c \,;\, n \neq -1$	10	$\int \sec ax \tan ax\,dx = -\frac{\sec ax}{a} + c$
2	$\int e^{x}\,dx = e^{x} + c$	11	$\int \operatorname{cossec} ax \cot ax\,dx = -\frac{\cot ax}{a} + c$

3	$\int \frac{1}{x}\,dx = \log_e x + c$	12	$\int a\,f(x)\,dx = a\int f(x)\,dx + c$
4	$\int a^x\,dx = \frac{a^x}{\log_e a} + c$	13	$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$
5	$\int 1\,dx = x + c$	14	$\int \frac{-1}{1+x^2}\,dx = \cot^{-1}x + c$
6	$\int \sin ax\,dx = -\frac{\cos ax}{a} + c$	15	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$
7	$\int \cos ax\,dx = \frac{\sin ax}{a} + c$	16	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + c$
8	$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + c$	17	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$
9	$\int \operatorname{cosec}^2 ax\,dx = -\frac{\cot ax}{a} + c$	18	$\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \operatorname{cosec}^{-1}x + c$

SOME IMPORTATN INTEGRAL FORMULAS

1	$\int \tan x\,dx = \log\sec x = -\log\cos x + c$
2	$\int \cot x\,dx = \log\sin x = -\log \operatorname{cosex} x + c$
3	$\int \sec x\,dx = \log(\sec x + \tan x) = \log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) + c$
4	$\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x + \cot x) = \log\tan\left(\frac{x}{2}\right) + c$
5	$\int \frac{1}{x^2+a^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
6	$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$
7	$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$
8	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \frac{1}{a}\sin^{-1}\left(\frac{x}{a}\right) + c$
9	$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left(x+\sqrt{x^2+a^2}\right) + c$
10.	$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right) + c$
11	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right] + c$
12.	$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log\left(x+\sqrt{x^2+a^2}\right)\right] + c$

13.	$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \frac{1}{2}\left[x\sqrt{x^2 - a^2} - a^2 \log\left(x + \sqrt{x^2 - a^2} \right) \right] + c$
14	$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx \right] = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left[bx - \tan^{-1}\left(\frac{b}{a} \right) \right]$
15	$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left[a \cos bx + b \sin bx \right] = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left[bx - \tan^{-1}\left(\frac{b}{a} \right) \right]$

DEFINITE INTEGRAL

1	$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
2	$\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$
3	$\int_a^b f(x)\, dx = \int_a^b f(b + a - x)\, dx$
4	$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx\ ;\ c \in [a, b]$
5	$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$
6	$\int_a^a f(x)\, dx = 0$
7	$\int_{-a}^a f(x)\, dx = \begin{cases} \int_0^a f(x)\, dx & : \text{When } f(-x) = f(x) \\ 0 & : \text{When } f(-x) = -f(x) \end{cases}$
8	$\int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx & : \text{When } f(2a - x) = f(x) \\ 0 & : \text{When } f(2a - x) = -f(x) \end{cases}$
9	$\int_0^a f(x)\, dx = 2\int_0^{a/2} f(x)\, dx$

INTEGRATION BY PARTS

$$\int f_1 f_2\, dx = f_1 \int f_2\, dx - \int\left[\frac{d}{dx} f_1 \int f_2\, dx \right] dx$$

Note :- The order of integration of the function is depend on the nature of function. For our convenience we use the following concept of ILATE

I - INVERSE FUNCTION
L - LOGRITHMIC FUNCTION
A - ALGEBRAIC FUNCTION

T - TRIGONOMETRIC FUNCTION
E - EXPONENTIAL FUNCTION

SHORT TRICKS ON INTEGRATION

1. **LEIBNITZ GENERAL RULE OF INTEGRATION BY PARTS**:

To integrate the product of two functions, one of which is a power of *x*, we apply the generalized rule of integration by parts.

$$\int u\,.\,v\,dx = u.\,v_1 - u'.\,v_2 + u''.\,v_3 - u'''.\,v_4 + \text{-----------}$$

Where $u' = \frac{d}{dx}u,\ u'' = \frac{d}{dx}u',\ \ u''' = \frac{d}{dx}u''$, and $v_1 = \int v\,dx,\ \ v_2 = \int v_1\,dx,\ \ v_3 = \int v_2\,dx,$

NOTE : - 1. **This formula is applicable only when one of the function is x^m and other function is sine, cosine or exponential.**

2. **We taking always $u = x^m$.**

2. **WALLI'S FORMULA**

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) \text{----------} \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad (\text{If n is an even positive integer})$$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) \text{----------} \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} \quad (\text{If n is an odd positive integer})$$

Example : $\int_0^{\pi/2} \cos^4 x\,dx = \left(\frac{4-1}{4}\right)\left(\frac{4-3}{4-2}\right)\left(\frac{\pi}{2}\right) = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$

$$\int_0^{\pi/2} \cos^7 x\,dx = \left(\frac{7-1}{7}\right)\left(\frac{7-3}{7-2}\right)\left(\frac{7-5}{7-4}\right) = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{16}{35}$$

3. **GAMMA INTEGRAL**:-

$$\int_0^{\pi/2} \sin^m x\ \cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$$

Where $\Gamma(n+1) = n\,\Gamma n,\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1. MACLAURIN'S THEOREM: -

Statement : - If f(x) is a function of x, which can be expanded in ascending power of x and each term of the expansion can be differentiated any number of times then,

$$\boxed{f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \text{--------} + \frac{x^n}{n!} f^n(0) + \text{---}}$$

2. **TAYLOR'S SERIES: -**

Statement : - If f (x + h), [where x is an independent of h] be a function of the variable h such that it can be expanded in ascending power of h and its this expansion be differentiable any numbers of times then,

$$\boxed{f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \text{--------} + \frac{h^n}{n!} f^n(x) + \text{----}} \quad \text{-------------- (1)}$$

Corollary : -

- **1. Replace h by x, and x by h, then the expansion in x term of h is,**

$$f(x+h) = f(h) + \frac{x}{1!} f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \text{--------} + \frac{x^n}{n!} f^n(h) + \text{---} \quad \text{-------------- (2)}$$

- **2. Putting x = a, in equation (1), then the expansion in h term of x is,**

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \text{--------} + \frac{h^n}{n!} f^n(a) + \text{----} \quad \text{-------------- (3)}$$

- **3. Expansion in ascending power of (x – a) is,**

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \text{--------} + \frac{(x-a)^n}{n!} f^n(a) + \text{----} \quad \text{------- (4)}$$

- **4. Put h = 0, in equation (2) we get Maclaurin's series is,**

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \text{--------} + \frac{x^n}{n!} f^n(0) + \text{----}$$

OR

$$y = y_0 + \frac{x}{1!} (y')_0 + \frac{x^2}{2!} (y'')_0 + \frac{x^3}{3!} (y''')_0 + \text{--------} + \frac{x^n}{n!} (y^n)_0$$

3. **PARTIAL DIFFERENTIATION:**

1. $$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \Rightarrow \left(\frac{\partial z}{\partial x}\right)_{(a,b)} = \left(\frac{\partial f}{\partial x}\right)_{(a,b)} = \lim_{\delta x \to 0} \frac{f(a+\delta a, b) - f(a,b)}{\delta x}$$

2. $$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \Rightarrow \left(\frac{\partial z}{\partial y}\right)_{(a,b)} = \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = \lim_{\delta y \to 0} \frac{f(a, b+\delta y) - f(a,b)}{\delta y}$$

NOTE : - $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}$

3. **HOMOGENEOUS FUNCTION : -**

A function in which every term is of the same degree is known as a homogenous function **of that degree.**

Let $f(x, y) = a_o x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{n-3} y^3 + \text{-----------------} + a_n y^n$ -------------- (1)

be a function of x and y and each term of the function have the same degree n. Therefore every homogeneous function can be written in the form,

$$f(x,y) = x^n\left[a_0\left(\frac{y}{x}\right)^0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + a_3\left(\frac{y}{x}\right)^3 + ------ + a_n\left(\frac{y}{x}\right)^n\right]$$

$\boxed{f(x,y) = x^n . F\left(\frac{y}{x}\right)}$, Similarly, $\boxed{f(x,y) = y^n . F\left(\frac{x}{y}\right)}$

4. t- TEST FOR HOMOGENEOUS FUNCTION:

Replace x and y by tx and ty respectively in equation (1), we get

$$f(tx,ty) = t^n[a_o\, x^n + a_1 x^{n-1} y + a_2\, x^{n-2} y^2 + a_3\, x^{n-3} y^3 + --- + a_n\, y^n]$$

$$\boxed{f(tx,ty) = t^n . f(x,y)}$$

Similarly, $\boxed{f(tx,ty,tz) = t^n . f(x,y,z)}$

5. EULER'S THEOREM : -

<u>**Statement :**</u> **-** **If $u = f(x, y)$ is a homogeneous function of x, y of degree n, then**

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

COROLLARY : -

(i). $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y \partial x} = (n-1)\dfrac{\partial u}{\partial x}$

(ii). $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}$

(iii). $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$

(iv). $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u) - 1\right]$

(v). If $u = f(x, y, z)$ is homgeneous function of x, y, z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$$

6. COMPOSITE FUNCTION AND TOTAL DIFFERENTIATION

DEFINITION OF COMPOSITE FUNCTION :-

(i). If $u = f(x, y)$ where $x = \phi_1(t)$ and $y = \phi_2(t)$, then u is called a Composite function of t and we can find $\frac{du}{dt}$

(ii). If $z = f(x, y)$ where $x = \phi_1(t_1, t_2)$ and $y = \phi_2(t_1, t_2)$, then z is called a Composite function of t_1 and t_2 and we can find $\dfrac{\partial z}{\partial t_1}$ *and* $\dfrac{\partial z}{\partial t_2}$

DIFFERENTIATION OF COMPOSITE FUNCTION :-

CASE 1 : -

If u is composite function of t, defined by the relations u = f (x, y) where x = ϕ_1(t) and y = ϕ_2(t), then

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}}$$

CASE 2 :-

If u is composite function of x, defined by the relations u = f (x, y) = constant where x = x and y = ϕ (x), then

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x}.\frac{dx}{dx} + \frac{\partial f}{\partial y}.\frac{dy}{dx} \Rightarrow 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

CASE 3 :-

If z = f (x, y) where x = ϕ_1(u, v) and y = ϕ_2(u, v), then z is called a Composite function of u and v, then

$$\boxed{\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u} \text{ and } \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}}$$

CASE 4 :-

If z = f (x, y), then $\boxed{dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy \Rightarrow dz = p\,dx + q\,dy}$

CASE 5 :-

Let f (x, y) = c be any implicit function then, $\boxed{\dfrac{d^2 y}{dx^2} = -\dfrac{q^2 r - 2pqs + p^2 t}{q^3}}$

7. EVALUATION OF ERROR :-

Since $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

$\Rightarrow \frac{\delta y}{\delta x} = \frac{dy}{dx}$ (Approximately)

$\Rightarrow \delta y = \left(\frac{dy}{dx}\right) . \delta x$ **(Approximately)**

Then,

(i). **δx is known as** Absolute error **in x.**

(ii) $\frac{\delta x}{x}$ **is known as** Relative Error **in x.**

(iii) $\frac{\delta x}{x} \times 100$ **is known as** Percentage Error **in x.**

NOTE :- In case z = f (x, y) is a function of two variable then we have $\delta z = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$ **(Approximately). This formula gives us small error δz in z due to error δx and δy in x and y respectively.**

8. MAXIMA AND MINIMA FOR TWO VARIABLE FUNCTION:

Suppose u = f (x, y) be any two variable function.

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$
2. Taking $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$. Find the value of x and y. Suppose we have x = a and y = b. Then we discuss the maxima and minima at the point (a, b). This point is called the **stationary point**.
3. Find $r = \left(\frac{\partial^2 u}{\partial x^2}\right)_{(a, b)}$, $s = \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{(a, b)}$ and $t = \left(\frac{\partial^2 u}{\partial y^2}\right)_{(a, b)}$
4. Calculate $r t - s^2$
5. (i) If $r t - s^2 > 0$ and $r > 0$ then f (x, y) has a minimum value at (a, b).
   (ii) If $r t - s^2 > 0$ and $r < 0$ then f (x, y) has a maximum value at (a, b).
   (iii) If $r t - s^2 < 0$ then f (x, y) neither maximum nor minimum at (a, b) and the point (a, b) is called **saddle point**.
   (iv) If $r t - s^2 = 0$ then the case is doubtful and further investigation is required.

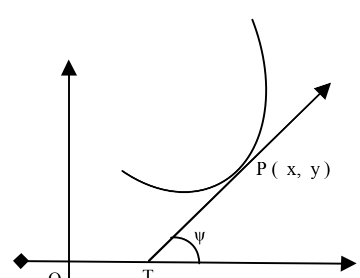
Note :- **The point (a, b) is called a stationary point if $f_x (a, b) = 0$ and $f_y (a, b) = 0$. The value of f (a, b) is called a stationary value. Thus every extreme value is a stationary but converse may not be true.**

9. TANGENT, NORMAL, SUB TANGENT AND SUBNORMAL

DEFINITION OF TANGENT :-

Let y = f (x) be any curve and P (x, y) be any point on the curve. Suppose Q (x + δx, y + δy) be any other neighboring point on the curve y = f (x). If Q $\rightarrow$ P i.e. the chord PQ $\rightarrow$ PT. Then this straight line PT is called the tangent **to the curve y = f (x) at the point P (x, y) and P (x, y) is called the** contact point**.**

EQUATION OF TANGENT AT THE POINT P (X, Y) :-



1. The equation of tangent at the point P (x, y) is : $Y - y = \left(\frac{dy}{dx}\right)_{(x, y)} (X - x)$
2. The equation of tangent at the point P (a, b) is : $Y - b = \left(\frac{dy}{dx}\right)_{(a, b)} (X - a)$

Where $\tan \psi = PT = \frac{dy}{dx}$ = Slope of tangent at the point P (x, y) or P (a, b).

3. Suppose the equation of the curve is given in the parametric form. i.e. x = f (t) and y = ϕ (t), then
$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\varphi' (t)}{f'(t)}$

Then equation of the tangent at any point on the curve is given by $Y - \phi(t) = \dfrac{\varphi'(t)}{f'(t)}(X - f(t))$

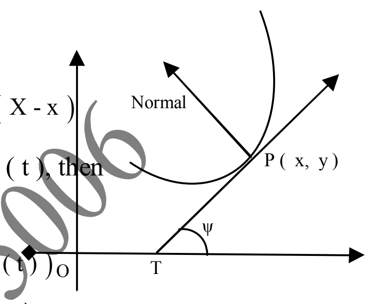
COROLLARY :-

1. If the tangent parallel to the X axes , then $\psi = 0$. Therefore $\tan 0 = 0 \Rightarrow \dfrac{dy}{dx} = 0$
2. If the tangent perpendicular to the X axes then $\psi = 90^0$ Therefore $\tan 90^0 = \infty \Rightarrow \dfrac{dx}{dy} = 0$

DEFINITION OF NORMAL :-

The normal to a curve at any point P is a straight line which passes through the point of contact P and Perpendicular to the tangent to the curve at the point.

EQUATION OF NORMAL AT THE POINT P (X, Y) :-

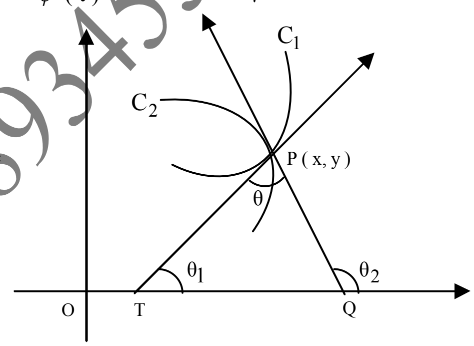


1. The equation of normal at the point P (x, y) of the curve y = f (x) is :- $Y - y = -\left(\dfrac{dx}{dy}\right)_{(x,\,y)}(X - x)$
2. Suppose the equation of the curve is given in the parametric form. i.e. x = f (t) and y = ϕ (t), then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\varphi'(t)}{f'(t)}$$

Then equation of the tangent at any point on the curve is given by $Y - \phi(t) = -\dfrac{f'(t)}{\phi'(t)}(X - f(t))$

ANGLE OF INTERSECTION OF TWO CURVE :-



The angle of intersection of two curves is the angle between the tangents to the two curves at their point of intersection.

Suppose C_1 and C_2 be two curve intersect at P (x, y) and $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$ be the slope of tangent at the point P (x, y) of two curve respectively and θ be the angle of intersection between the two curve C_1 and C_2 then

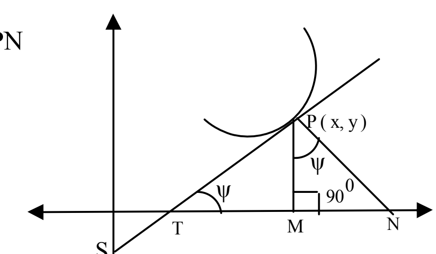
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

NOTE :- 1. The other angle of intersection is $\pi - \theta$.

2. If the curve C_1 and C_2 touch each other then $m_1 = m_2$
3. If the curve C_1 and C_2 cut orthogonally then $m_1.m_2 = -1$

10. SUB TANGENT AND SUBNORMAL

CARTESIAN SUB TANGENT AND SUBNORMAL:-



[In triangle PTM]

Let y = f (x) be the equation of the curve. Let P (x, y)be any point on it. Let PT and PN be the tangent and normal at P, T and N being the points where these cut the axis of x.
Draw PM perpendicular from P on the axis of x.
Then length TM is called the Subtangent and length MN is called subnormal.
Let ψ be the angle which the tangent at P marks with the axis of x. Then

$\tan\psi = \dfrac{dy}{dx}$ and $\angle PTM = \psi$, $\angle MPN = \psi$

1. **The length TM is called Subtangent, then** $TM = y \cot\psi = \dfrac{y}{\frac{dy}{dx}}$
2. **The length MN is called the subnormal, then** $MN = y\tan\psi = y\,.\dfrac{dy}{dx}$

[In triangle PMN]

3. **The length of the Tangent is TP, then** $PT = y\,.\,\text{Cosec}\,\psi = y\,.\dfrac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$
[In triangle PTM]

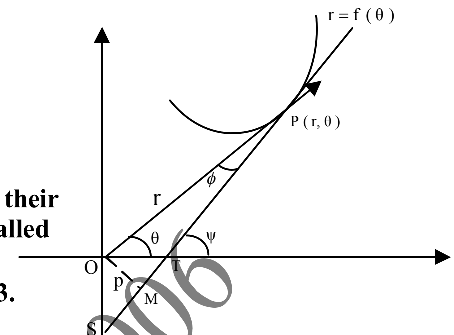
4. **The length of normal is PN, then** $PN = y\,.\sec\psi = y\,.\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Corollary :- 1. **It can easily be shown from the above result that in any curve,** $\dfrac{\text{Subtangent}}{\text{Subnormal}} = \dfrac{(\text{Length of the normal})^2}{(\text{Length of the tangent})^2}$

2. **The length of the intercept OT that the tangent cuts off from the axis of x is** $OT = x - \dfrac{y}{dy/dx}$

3. The length of the intercept OS that the tangent cuts off from the axis of y is $OS = y - x\dfrac{dy}{dx}$

11. ANGLE BETWEEN RADIUS VECTOR AND TANGENT :-



Clearly $\psi = \theta + \phi$ **and** $\tan\phi = r\dfrac{d\theta}{dr}$

12. THE PEDAL EQUATION FOR TANGENT :-

If p denotes the length of the perpendicular from pole on the tangent and r, θ, φ have their Usual meaning, then we have following three important relations these relation are called Pedal equation for tangent.

1. $p = r\sin\phi$

2. $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

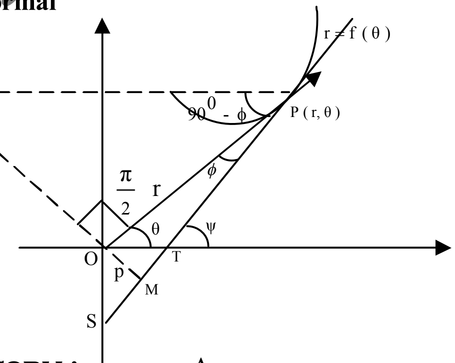
3. $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$, Where $u = \dfrac{1}{r}$

13. POLAR FORM OF SUBTANGENT AND SUBNORMAL :-

Let P (r, θ) be any point on the curve r = f (θ). Through pole O, draw a line GOT perpendicular to the radius vector Op. This meets the tangent and normal at P in T and G respectively. Then we define

Length OM = polar Subtangent **Length OG = polar Subnormal**

Length PM = Length of polar tangent **Length PG = Length of polar normal**



1. **The length OM is called polar Subtangent, then polar Subtangent** $= r^2\dfrac{d\theta}{dr}$

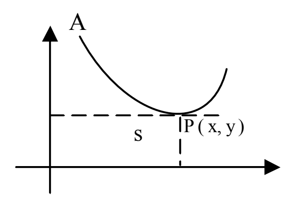
2. **The length OG is called polar subnormal, then polar subnormal** $= \dfrac{dr}{d\theta}$

3. **The length PM is called polar tangent, then polar tangent** $= \sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}$

4. **The length PG is called polar normal, then polar normal** $= \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

14. THE LENGTH OF ARC OF THE CURVE IN CARTESIAN AND POLAR FORM :-

Let A be any fixed point on the curve y = f (x) and P (x, y) be any point on the curve and The length of arc AP is s, then



1. $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

2. $\dfrac{ds}{dy} = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}$

3. $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

4. $\dfrac{ds}{dr} = \sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2}$

5. $\cos\psi = \dfrac{dx}{ds}$ $\sin\psi = \dfrac{dy}{ds}$

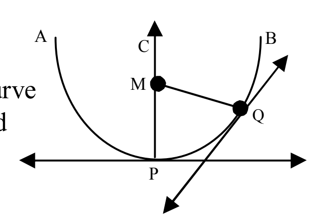
6. $\cos\phi = \dfrac{dr}{ds}$ $\sin\phi = r\dfrac{d\theta}{ds}$

15. CURVATURE

CONCEPT OF CURVATURE:-

The measure of the sharpness of the bending of a curve at a point is called curvature of the curve at that point. In other words "greater the curvature lesser the radius" and vice –versa.

DEFINITION OF CURVATURE:-



Let P and Q be two neighboring point on a curve AB. Let M be the point of intersection of normal At P and Q to the curve. If M → C as Q → P, then the C is called the **centre of curvature** of the curve at the at the point P and is denoted by ρ. The reciprocal of the distance CP i.e. 1/CP i.e. **1/ρ** is called the **curvature** of the curve at P.

FORMULA FOR RADIUS OF CURVATURE:-

1. INTRINSIC FORM:-

The relation between length of arc s and ψ is called the intrinsic form or intrinsic equation Radius of Curvature $\rho = \frac{ds}{d\psi}$

2. CARTESIAN FORM:-

Let $y = f(x)$ be any Cartesian curve and ψ be the angle between the tangent of the curve and the X- axis. Then

$$\text{Radius of Curvature } \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

3. IN PARAMETRIC FORM:-

Let the equation of the curve in the parametric form be $x = f_1(t)$ and $y = f_2(t)$. Where t is parameter.

$$\text{Radius of Curvature } \rho = \frac{\left[(x')^2+(y')^2\right]^{3/2}}{x'y''-y'x''}, \text{ Where } x' = dx/dt,\ x'' = d^2x/dt^2 \text{ and } y' = dy/dt,\ y'' = d^2y/dt^2$$

4. PEDAL FORM:-

The relation between p and r is called the pedal form or pedal equation, then Radius of Curvature $\rho = r\frac{dr}{dp}$

5. IN POLAR FORM :-

The relation between r and θ is called the polar form. Radius of Curvature

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

CENTRE OF CURVATURE AND CHORD OF CURVATURE :-

1. CENTRE OF CURVATURE :-

Let C (α, β) be the centre of curvature corresponding to the point P (x, y) on the curve such that

$$\alpha = x - \frac{y_1\left[1+y_1^2\right]}{y_2} \text{ and } \beta = y + \frac{\left[1+y_1^2\right]}{y_2}$$

2. CIRCLE OF CURVATURE :-

The equation of circle of curvature is, $(x-\alpha)^2+(y-\beta)^2 = \rho^2$

3. CHORD OF CURVATURE IN POLAR FORM :-

1. The Chord of Curvature through the origin i.e. pole is $C_r = 2\rho\cos\phi$
2. The Chord of Curvature perpendicular to radius vector is $C_\theta = 2\rho\sin\phi$

4. CHORD OF CURVATURE PARALLEL TO COORDINATE AXES :-

1. Chord of Curvature parallel to X- axes is $C_x = 2\rho\sin\psi$
2. Chord of Curvature parallel to X- axes is $C_y = 2\rho\cos\psi$

UNIT -2
CALCULUS – II

1. DEFINITE INTEGRAL AS THE LIMIT OF A SUM:-

If $f(x)$ be a single valued continuous function defined in the close interval [a, b] and interval [a, b] is divided into n parts equal Subinterval of width h so that nh = b – a, then we define.

$$\boxed{\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\text{-----}+f(a+\overline{n-1}h)\right]}$$

2. SUMMATION OF SERIES : -

Let f (x) be a continuous function defined in the closed interval [a, b] then by definite integral as the limit of sum,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+f(a+3h)+\text{-----}+f(a+\overline{n-1}h)\right] \quad \text{----------}(1)$$

Where n h = (b – a)

Equation (1) can be written as following

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h \sum_{r=0}^{r=n-1} f[a + rh] \qquad \text{------------ (2)}$$

If we take a = 0 and b = 1. Then n h = 1 $\Rightarrow n = \frac{1}{h}$

If h $\to$ 0 then n $\to \infty$. Then (2) becomes,

$$\boxed{\int_0^1 f(x)\,dx = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{r=n-1} f\left(\frac{r}{n}\right)}$$

Comparing both side, then we get,

$\frac{1}{n} \to dx$, $\frac{r}{n} \to x$ and $\sum \to \int$

$$\boxed{\text{Lower limit} = \lim_{n \to \infty} \left(\frac{r}{n}\right)_{r=0} \quad \text{and} \quad \text{Upper Limit} = \lim_{n \to \infty} \left(\frac{r}{n}\right)_{r=n-1}}$$

3. BETA FUNCTION (FIRST EULERIAN INTEGRAL) : -

If **m > 0** and **n > 0**. Then Beta function of **m** and **n** is denoted by **B (m, n**) and it is defined as

$$\boxed{\beta(m, n) = \int_0^1 x^{m-1} (1 - x)^{n-1}\, dx}$$

It is noted that m, n are positive numbers. For n > 0 and m > 0 the integral is convergent.

PROPERTIES OF BETA FUNCTION : -

1. Beta function is **symmetric** in m and n i.e. B (m, n) = B (n, m)
2. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$
3. $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$
4. $B(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

4. GAMMA FUNCTION (SECOND EULERIAN INTEGRAL) : -

If n > 0, then Gamma function of n is denoted by Γn and it is defined by the improper integral,

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx}$$

Which is convergent for x > 0 and n is a positive number, integral or fractional.

Note : - $\Gamma 0 = \infty$ and $\Gamma -n = \infty$, when n > 0

PROPERTIES OF GAMMA FUNCTION :-

1. Show that : $\Gamma n+1 = n\, \Gamma n$, n > 0
2. Show that : $\Gamma n+1 = n!$, where n = 1, 2, 3, ---
3. Show that : $\Gamma 1 = 1$ **[IMP]**
4. Show that : $\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1}\, dx$

5. RELATION BETWEEN BETA AND GAMMA FUNCTION : -

If m > 0 and n > 0 , then prove that

$$\boxed{\beta(m\,,\,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$

6. SOME MORE RESULT : -

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\qquad \int_{-\infty}^{\infty} e^{-a^2 x^2}\,dx = \frac{\sqrt{\pi}}{a}$ $\qquad \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$

7. LEGENDRE'S DUPLICATION FORMULA: -

If m is positive real numbers then prove that

$$\boxed{\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m}$$

8. DOUBLE INTEGRAL:

Let a single valued and bounded function $f(x, y)$ of two variables x, y be defined in a closed region R of XY–plane.

Divide the region R into n sub-region by drawing lines parallel to co-ordinate axes. Number the rectangles which lie entirely inside the region R from 1 to n. Let area of n sub-intervals are δA_1, δA_2, δA_3.....δA_n. Let (x_r, y_r) be any point lie in r^{th} rectangle whose area is δA_r.

Consider the sum,

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots + f(x_n, y_n)\delta A_n$$

$$= \sum_{r=1}^{r=n} f(x_r, y_r)\,\delta A_r \qquad \ldots(1)$$

Let the number of these sub-regions increase indefinitely. Such that the largest linear dimension (*i.e.* diagonal) of δAr, approachers zero. The limit of the sum (1), if it exist, irrespective of the mode of sub-division, is called the **double integral** of $f(x, y)$ over the region R and is denoted by

$$\iint_R f(x,y)\,dA \text{ i.e, } \iint_R f(x,y)\,dx.dy$$

$$\boxed{\iint_R f(x,y)\,dx\cdot dy = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{r=n} f(x_r, y_r)\,\delta A_r}$$

9. EVALUATION OF DOUBLE INTEGRAL:

Double integrals over a Region R may be evaluated by two successive integration. In other words, the methods of evaluating the double integrals depend upon the nature of the curves bounding the Region R. Suppose the Region R be bounded by the curves $x = x_1, x = x_2$ and $y = y_1, y = y_2$. *i.e.,*

$$\boxed{\iint_R f(x,y)\,dx\cdot dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\cdot dy}$$

The order of integration is depend on nature of limit in the Region R.

Case 1: If R is the region bounded by the curves $x_1 = f_1(y)$, $x_2 = f_2(y)$ and two straight lines $y = b_1$, $y = b_2$. Then,

$$\boxed{\iint_R f(x,y)\,dx\cdot dy = \int_{b_1}^{b_2}\left[\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\right]dy}$$

In this integration we first integrate the function. $f(x, y)$ w.r.t x keeping y constant from $f_1(y)$ to $f_2(y)$ and the resuting function of y from b_1 to b_2.

Case 2: If R is the region bounded by the curves $y_1 = \phi_1(x)$, $y_2 = \phi_2(x)$ and two straight lines $x = a_1$, $x = a_2$. Then,

$$\boxed{\iint_R f(x,y)\,dx\cdot dy = \int_{a_1}^{a_2}\left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right]dx}$$

In this integration we first integrate the function $f(x, y)$ w.r.t y keeping x constant from $\phi_1(x)$ to $\phi_2(x)$ and the resulting function of x from a_1 to a_2.

Case 3: If Region R is bounded by the straight lines $x_1 = a_1$, $x_2 = a_2$ and $y_1 = b_1$, $y_2 = b_2$. *i.e.* both the limit are constant. Then

$$\boxed{\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y)\,dx\cdot dy = \int_{a_1}^{a_2}\left[\int_{b_1}^{b_2} f(x,y)\,dy\right]dx = \int_{b_1}^{b_2}\left[\int_{a_1}^{a_2} f(x,y)\,dx\right]dy}$$

In this integration the order of integration is immaterial, provided the limits of integration are changed accordingly.

10. APPLICATION OF DOUBLE INTEGRAL

(1) Area by Double Integration

Suppose the Area of the region lying between the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$, $x = a$ and $x = b$ is obtained by putting $f(x, y) = 1$ in the definition of double integral and it is given by,

$$\boxed{\iint_R dA = \int_a^b\int_{f_1(x)}^{f_2(x)} dx\cdot dy} \Rightarrow \boxed{\text{Area } = \int_a^b\int_{f_1(x)}^{f_2(x)} dx\cdot dy}$$

(2) Volume Under a Surface

Let R be a region in the XY – plane. Then the volume inside the cylinder, enclosed by the surface $z = f(x, y) > 0$ and XY – plane is

$$\boxed{V = \iint_R f(x,y)\,dx\cdot dy}.$$

The base of the cylinder is boundary of the region R in XY – plane and generators are parallel to the z – axes.
If the region R may be considered as enclosed by the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$, $x = a$ and $x = b$. We can write the volume as

$$\boxed{V = \int_a^b\left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx}$$

11. CHANGE OF VARIABLES IN DOUBLE INTEGRAL (CARTESIAN TO POLAR FORM)

The evaluation of some double integrals becomes simpler by change of the variables. In the case of change from Cartesian to polar for two dimensional problems, the element of area $\delta x\ \delta y$ is replaced by $r\ \delta r\ \delta\theta$
With the relation,

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$r^2 = x^2 + y^2 \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

$$And\ \ \theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$Simmilarly,\ \ \frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Using the Jacobians concept,

$$\iint_R f(x,y)\,dx.dy = \iint_R f(r\cos\theta, r\sin\theta)\,|J|\,dr.d\theta$$

$$\text{Where } |J| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \frac{\partial x}{\partial r}.\frac{\partial y}{\partial\theta} - \frac{\partial x}{\partial\theta}.\frac{\partial y}{\partial r} = r$$

$$\therefore \boxed{\iint_R f(x,y)\,dx.dy = \iint_R f(r\cos\theta, r\sin\theta)\,r.dr.d\theta}$$

12. TRIPLE INTEGRAL:

Let $f(x, y, z)$ be a function of the independent variable x, y, z defined at every point in a closed and bounded region V of the three dimensional space. Divide the region V into n elementary volumes.
$\delta V_1, \delta V_2, \ldots.\delta V_n$. Let (x_r, y_r, z_r) be any point inside the rth sub-division Then the limit of the sum,

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r \qquad \text{-------------} (1)$$

as $n \to \infty$ and dimensions of each sub-division tends to zero, is called **triple integral** of $f(x, y, z)$ over the region V and it is denoted by $\iiint_V f(x,y,z)\,dV$. Thus,

In other words, if the region V is bounded by the surfaces $x = x_1, x = x_2, y = y_1, y = y_2, z = z_1, z = z_2$. Then,

$$\boxed{\iiint_V f(x,y,z)\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx \cdot dy\,dz}$$

13. EVALUATION OF TRIPLE INTEGRATION

1. **If V is described as $a_1 \le x \le a_2$, $b_1 \le y \le b_2$ and $c_1 \le z \le c_2$ *i.e* all the limits are constant. Then the order of integration is immaterial provided the limits are changed accordingly. Thus,**

$$\iiint_V f(x,y,z)\,dx \cdot dy\,dz = \int_{x=a_1}^{x=a_2}\left[\int_{y=b_1}^{y=b_2}\left(\int_{z=c_1}^{z=c_2} f(x,y,z)\,dz\right)dy\right]dx = \int_{y=b_2}^{y=b_1}\left[\int_{z=c_1}^{z=c_2}\left(\int_{x=a_1}^{x=a_2} f(x,y,z)\,dx\right)dz\right]dy$$
$$= \int_{z=c_1}^{z=c_2}\left[\int_{x=a_1}^{x=a_2}\left(\int_{y=b_1}^{y=b_2} f(x,y,z)\,dx\right)\right]dz.$$

2. **If V is described as** $a_1 \le x \le a_2$ and $f_1(x) \le y \le f_2(x)$, $\phi_1(x,y) \le z \le \phi_2(x,y)$. **then,**

$$\iiint_V f(x,y,z)\,dx \cdot dy \cdot dz = \int_{a_1}^{a_2}\left[\int_{f_1(x)}^{f_2(x)}\left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\right)dy\right]dx.$$

3. **If V is described as** $b_1 \le y \le b_2, f_1(y) \le x \le f_2(y)$ and $\phi_1(x,y) \le z \le \phi_2(x,y)$. **then,**

$$\iiint_V f(x,y,z)\,dx \cdot dy \cdot dz = \int_{b_1}^{b_2}\left[\int_{f_1(y)}^{f_2(y)}\left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\right)dx\right]dy.$$

4. **If V is described as** $a \le z \le b$ and $f_1(z) \le x \le f_2(z)$ and $\phi_1(x, z) \le y \le \phi_2(x, z)$. Then

$$\iiint_V f(x,y,z)\ dx\,dy\,dz = \int_{z=a}^{z=b}\left[\int_{x=f_1(z)}^{x=f_2(z)}\left(\int_{y=\phi_1(x,z)}^{y=\phi_2(x,z)} f(x,y,z)\,dy\right)dx\right]dz$$

14. VOLUME OF SOLIDS AS A TRIPLE INTEGRAL : -

1. The volume of a three dimensional region is given by .
   If the region is bounded by

$$x = f_1(y,z),\ x = f_2(y,z),\quad y = \phi_1(z),\quad y = \phi_2(z),\quad z = a,\ z = b$$

2. The order of integration may be changed with a suitable change in the limits of integration

(i) In Cylindrical coordinate (r, ϕ, z). Then $V \iiint_r r\,dr\,d\phi\,dz$

(ii) In Spherical coordinate (r, θ, ϕ).Then $V \iiint_r r^2 \sin\theta\,d\theta\,d\phi\,dr$, Where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$
$dV = dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\phi$

15. LENGTH OF THE CURVE OR RECTIFICATION

DEFINITION OF RECTIFICATION : -

The method of obtaining the length of an arc of a curve between two given points on it is said to be the Rectification.

FORMULA FOR LENGTH OF A CURVE

For Cartesian Curve : -

1. The length of the arc of a curve $y = f(x)$, between $x = a$ and $x = b$ is ,

$$\boxed{\text{arc } AB = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx,\ (b > a)}$$

2. The length of the arc of a curve $x = f(y)$, between $y = a$ and $y = b$ is

$$\boxed{\text{arc } AB = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dx,\ (b > a)}$$

For Parametric Curve : -
If the equation of the curve be the given in the form $x = f_1(t)$ and $y = f_2(t)$. Then the length of arc between t_1 and t_2 is

$$\boxed{S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

For Polar Curve : -
1. If the polar equation of curve is $r = f(\theta)$, Then the length of arc of the polar curve between $\theta = \alpha$ and $\theta = \beta$ is

$$\boxed{S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

2. If the polar equation of curve is $\theta = f(r)$.Then the length of arc of the polar curve between $r = r_1$ and $r = r_2$ is

$$\boxed{S = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr}$$

For Pedal equation : -

If the curve is $p = f(r)$ between the point $r = a$ and $r = b$ Then the length of curve is $\boxed{S = \int_a^b \frac{r}{\sqrt{r^2 - p^2}}\, dr}$

UNIT – 3
DIFFERENTIAL EQUATION

CHAPTER – 01
SOLUTION OF DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

Method 1:- **Separation of variables**

Suppose the differential equation is

$$\frac{dy}{dx} = \frac{f_1(x)}{f_2(y)}$$

$\Rightarrow \quad f_2(y)\, dy = f_1(x)\, dx$

Integrate both side, we get

$F_2(y) = F_1(x) + c$ Where c is integration constant.

METHOD 2:- **HOMOGENEOUS DIFFERENTIAL EQUATION (H.D.E)**

A differential equation,

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} \qquad \text{-------------- } (1)$$

Where $f_1(x, y)$, $f_2(x, y)$ are homogeneous functions of the same degree in x and y is called **H. D. E**.
Changing a HDE. to reducible to separable form. For this
put $y = vx$ where **v** is function of ***x***

$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$

From equation. (1) and (2) we get the separable and solve by previous method.

METHOD 3:- **NON – HOMOGENEOUS DIFFERENTIAL EQUATION**

The differential Equation of the form,

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

As known as differential equation reducible to homogeneous form.

Working rule : -
1. Putting $x = u + h$, $y = v + k$ where h & k are constant.
2. Then we get $dy/dx = dv/du$

3. Find h & k such that ah + bk + c = 0 & Ah + Bk + C = 0.
4. Then it is Reduce in H. D. E. put v = t. u. where t is function of u.
5. Then we get dv/du = t + u dt/du
6. Reduce in separable and solve by previous method.

METHOD 4:- LINEAR DIFFERENTIAL EQUATION (LEIBNITEZ'S LINEAR EQUATION)

First type When y is dependent and x is an independent then a differential of the form,

$$\frac{dy}{dx} + Py = Q \quad \text{------------------ (1)}$$

Where P & Q is function of x only and $I.F = e^{\int P\,dx}$

Then Solution of differential equation is,

$$y.\ I.F = c + \int I.F.\ Q\,dx$$

Second Type. When **x** is dependent and **y** is an independent variable, then the differential equation in the form,

$$\frac{dx}{dy} + P\,x = Q \quad \text{---------------------- (2)}$$

Where P & Q is function of y only and $I.F = e^{\int P\,dy}$

Then Solution of differential equation is,

$$x.\ I.F = c + \int I.F.\ Q\,dy$$

METHOD 5:- EQUATION REDUCIBLE TO LINEAR FORM (BERNOULLI'S EQUATION)

An equation of the form

$$\frac{dy}{dx} + P.\,y = Q.\,y^n \quad \text{-------------------- (1)}$$

Where P & Q be the function of x only or constant is known as **Bernoulli's equation**.

From equation (1) we get,

$$y^{-n}\frac{dy}{dx} + P.\,y^{1-n} = Q \quad \text{-------------------- (2)}$$

Put $y^{1-n} = V$, Then

$$(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx} \quad \text{----------- (3)}$$

Then from equation (2) and (3) we get

$$\frac{1}{1-n}\frac{dv}{dx} + P.\,V = Q$$

$$\Rightarrow \frac{dv}{dx} + PV(1-n) = Q(1-n)$$

This is L. D. E of Ist order, then solve by previous method.

CHAPTER – 02

ORDINARY DIFFERENTIAL EQUATION OF FIRST ORDER AND HIGHER DEGREE

1. **EQUATION SOLVABLE FOR "p**

An equation of the first order and nth degree is,

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \text{-------------------} + A_n = 0 \quad \text{----------------- (1)}$$

Where A_1, A_2, A_3 -------------- A_n are constants or function of f (x, y) and p = dy/dx

If Equation (1) is solvable for p, its L.H.S can be resolved into n rational factors of the degree. Then equation (1) can be written as

$$(p - f_1)(p - f_2)(p - f_3) \text{ ---------- } (p - f_n) = 0 \quad \text{--------------------- (2)}$$

Where f_1, f_2, f_3, ------- f_n are function of x and y.

Equating each factor of the L.H.S to zero. We get

$$p - f_1 = 0,\ p - f_2 = 0 \text{ ---------- } p - f_n = 0 \quad \text{------------------------------------ (3)}$$

Let the solutions of these n factors are

$$F_1(x, y, c_1) = 0,\ F_2(x, y, c_2) = 0,\ F_3(x, y, c_3) = 0 \text{ ------------ } F_n(x, y, c_n) = 0 \quad \text{--------------------- (4)}$$

Where c_1, c_2, c_3 ---------- c_n arbitrary constant of integration.
Since the given equation is of the first order so its general solution we have only one arbitrary constant.
Therefore taking $c_1 = c_2 = c_3 = \text{----------} = c_n = c$
Hence the general solution of (1) is,
$$F_1(x, y, c).F_2(x, y, c).F_3(x, y, c) \text{------------} F_n(x, y, c) = 0$$
This is required solution of given differential equation.

2. EQUATION SOLVABLE FOR "Y".

A differential equation of the form,
$$y = f(x, p) \qquad \text{---------------------------------} (1)$$
is called differential equation solvable for y. In other words if it is possible to collect y in terms of x and p. Then we say equation (1) is solvable for y.
Equation (1) differentiating w.r.t x, we get,
$$\frac{dy}{dx} = p = \phi\left[x, p, \frac{dp}{dx}\right] \qquad \text{-------------------} (2)$$
$$\Rightarrow \quad F(x, p, c) = 0 \qquad \text{----------------} (3)$$
Eliminating p from (1) and (3), we get required solution of equation (1).

3. EQUATION SOLVABLE FOR "X".

A differential equation of the form,
$$x = f(y, p) \qquad \text{---------------------------------} (1)$$
is called differential equation solvable for x. In other words if it is possible to collect x in terms of y and p. Then we say equation (1) is solvable for x.
Equation (1) differentiating w.r.t x, we get,
$$\frac{dx}{dy} = \frac{1}{p} = \phi\left[y, p, \frac{dp}{dy}\right] \qquad \text{-------------------} (2)$$
$$\Rightarrow \quad F(y, p, c) = 0 \qquad \text{--------------------} (3)$$
Eliminating p from (1) and (3), we get required solution of equation (1).

4. CLAIRAUT'S EQUATION

The differential equation of the form,
$$y = px + f(p) \qquad \text{----------------------------} (1)$$
is called **Clairaut's equation** after the name of " **Alexis Claude Clairaut**" in which p is written for dy/dx.
The solution of **Clairaut's equation** is,
$$y = cx + f(c) \qquad [\text{ Here p replace by c }]$$

Note : -
1. Some times the given differential equation can be transformed into Clairaut's form by a change of variable or suitable transformation formula.
2. The special case of Clairaut's form is **Lagrange's equation**
$$y = x f(p) + F(p)$$
is associated with the name of "**Joseph Louis Lagrange's**". This is **generalization form** of Clairaut's equation.

CHAPTER – 03

LINEAR DIFFERENTIAL EQUATION OF HIGHER ORDER WITH CONSTANT COEFFICIENT'S

DEFINITION : -

The Standard form of linear differential equation of nth order is,
$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \text{-----} + a_{n-1}\frac{dy}{dx} + a_n y = Q \qquad \text{-------------------------} (1)$$

Where a_1, a_2 ----- a_{n-1}, a_n are constant and Q is the function of x (only) is called the **linear differential equation of higher order with constant coefficient.** This L. D. E. can be written in the following form,

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \text{-----} + a_n) y = Q$ ------------------------- (2)

$\Rightarrow \quad f(D)\, y = Q$ ---------- (3)

Where $D \equiv \frac{d}{dx}$, $D^2 \equiv \frac{d^2}{dx^2}$, --------$D^n \equiv \frac{d^n}{dx^n}$

And $f (D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \text{-----} + a_n$.

The Complete solution of equation (1) is,

$y = C.F + P.I$ ------------------------- (4)

Where C. F = Complementry function (when $Q = 0$) and P. I = Particular integral (When $Q \neq 0$)

Rules to C.F : When Q = 0

Then from (3) we get,

$f (D) y = 0$ -------------------------- (5)

The Auxillary Equation (i.e) is,

$f (m) = 0$ [Where D replace by m and y neglect]

$\Rightarrow \quad m^n + a_1 m^{n-1} + a_2 m^{n-2} + \text{-----} + a_n = 0$

This is polynomial equation in m of degree n.

Case 1 :- **When the roots of A. E are distinct**

Say $m = m_1, m_2, m_3, m_4$, etc.

Then solution is,

$y = C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$

Case 2 :- **When the roots of A. E. are repeated.**

Say $m = m_1, m_1, m_1$ etc.

Then solution is

$y = (c_1 + x c_2 + x^2 c_3) e^{m_1 x}$

Case 3 :- **When the roots of A. E. are distinct complex number. (i.e in an imaginary)**

Say $m = \alpha \pm \beta i$

Then solution is,

$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ **or** $y = c_1 e^{\alpha x} \sin (\beta x + c_2)$ **or** $y = c_1 e^{\alpha x} \cos (\beta x + c_2)$

Case 4 :- **When the roots of A. E. are in radicals (Surds) form**

Say $m = \alpha \pm \sqrt{\beta}$

Then solution is,

$y = e^{\alpha x} (c_1 \cosh \beta x + c_2 \sinh \beta x)$ **or** $y = c_1 e^{\alpha x} \sinh (\beta x + c_2)$ **or** $y = c_1 e^{\alpha x} \cosh (\beta x + c_2)$

Case 5 :- **When the roots of A.E. are in distinct and repeated in real numbers.**

Say $m = m_1, m_2, m_3$ and m_4, m_4 etc

Then solution is

$y = C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + (c_4 + x c_5) e^{m_4 x}$.

Case 6 : - **When the roots of A.E. are repeated in complex form.**

Say $m = \alpha \pm \beta\, i$ and $m = \alpha \pm \beta\, i$

Then solution is,

$y = e^{\alpha x} \left[(c_1 + x c_2) \cos \beta x + (c_3 + x c_4) \sin \beta x \right]$

Rules to P.I : When Q ≠ 0

The formula for P.I. is

$P.I. = \frac{1}{f (D)} Q$, Where Q is function of x only and $D \equiv \frac{d}{dx}$

$D\, Q = \frac{d}{dx} Q$ $\qquad \frac{1}{D} Q = \int Q\, dx$

1\.

2\.

3\. $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, When $f(a) \neq 0$

4\. $\frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(D+a)} 1$, When $f(a) = 0$

5\. $\frac{1}{f(D^2)} \text{Sin } ax = \frac{1}{f(-a^2)} \text{Sin } ax$, When $f(-a^2) \neq 0$

6\. $\frac{1}{f(D^2)} \text{Cos } ax = \frac{1}{f(-a^2)} \text{Cos } ax$, When $f(-a^2) \neq 0$

Note : - $-a^2 \neq (-a)^2$

7\. $\frac{1}{D^2 + a^2} \text{Sin } ax = \frac{-x}{2a} \text{Cos } ax$, When $f(-a^2) = 0$

8\. $\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$, When $f(-a^2) = 0$

9\. $\frac{1}{f(D)} x^m = \left[f(D) \right]^{-1} x^m$ The expansion of $[f(D)]^{-1}$ is depend on power of x.

NOTE :-

1. $(1 + D)^{-1} = 1 - D + D^2 - D^3 + D^4$ ---------
2. $(1 - D)^{-1} = 1 + D + D^2 + D^3 + D^4 +$ ---------
3. $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + 5D^4$ ------
4. $(1 + D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + 5D^4$ ------
5. $[1 + D]^{-n} = 1 - nD + \frac{n(n+1)}{2!} D^2 - \frac{n(n+1)(n+2)}{3!} D^3 + --$
6. $[1 - D]^{-n} = 1 + nD + \frac{n(n+1)}{2!} D^2 + \frac{n(n+1)(n+2)}{3!} D^3 + ---$

10\. $\frac{1}{f(D)} e^{ax}.V = e^{ax} \frac{1}{f(D+a)} V$, Where V is function of x.

11\. $\frac{1}{f(D)} x.V = x \frac{1}{f(D)} V + \frac{d}{dD}\left[\frac{1}{f(D)}\right] V$, Where V is function of x but not e^{ax}.

NOTE :- This formula applicable when power of x is one only. And this formula is not applicable for those conditions in which the denominator zero when V operated.

12\. $\frac{1}{f(D)} x e^{ax}.V = e^{ax} \frac{1}{f(D+a)} (x.V)$

SPECIAL CONDITION :-

13\. $\frac{1}{f(D)} x^m \text{Sin } ax = \text{I. P. } \frac{1}{f(D)} x^m [\text{Cos } ax + i \sin ax] = \text{I. P. } \frac{1}{f(D)} x^m e^{aix}$ Where $m \geq 1$

14\. $\frac{1}{f(D)} x^m \text{Cos } ax = \text{R. P. } \frac{1}{f(D)} x^m [\text{Cos } ax + i \sin ax] = \text{R. P. } \frac{1}{f(D)} x^m e^{aix}$ Where $m \geq 1$

15\. $\frac{1}{D - a} Q = e^{ax} \int e^{-ax} Q \, dx$

16\. $\frac{1}{D + a} Q = e^{-ax} \int e^{ax} Q \, dx$

CHAPTER – 04

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION { CAUCHY'S LINEAR EQUATION } [H.L.D.E]

DEFINITION : -

A linear differential of the type,

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \text{-----} + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x)$$

Where a_1, a_2 ----- a_n are constant and Q is a function of *x* is called a **Homogeneous linear differential equation (H. L. D. E.) or Cauchy's linear equation.** It is also called **Differential equation reducible to linear equation with constant coefficient.**

METHOD OF SOLVING THE H.L.D.E.

In this differential equation first we reduce in L. D. E. with constant coefficient for this

we put $x = e^z$

$\Rightarrow z = \log x$

$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$ and $x \frac{d}{dx} \equiv D$, $x^2 \frac{d^2}{dx^2} \equiv D(D-1)$, $x^3 \frac{d^3}{dx^3} \equiv D(D-1)(D-2)$ -----.

$x^n \frac{d^n}{dx^n} \equiv D(D-1)(D-2)$ ----- $(D - n + 1)$ where $D \equiv \frac{d}{dz}$.

CHAPTER – 05

SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATION

DEFINITION : -

If t be the independent variable and *x*, y be two dependent variable in the given differential. Hence the two differential equation Will be of the form.

$$f_1(D)\,x + f_2(D)\,y = f(t) \qquad \text{----- (1)}$$

and

$$g_1(D)\,x + g_2(D)\,y = g(t) \qquad \text{----- (2)}$$

Where $D = d/dt$, $f_1(D)$, $g_1(D)$ etc. being differential operators with constant coefficient.
We solve to this equation by simultaneous form and find the value of x & y.

UNIT – 4
MATRIX

CHAPTER – 01
RANK AND NULLITY OF MATRIX

1. RANK OF A MATRIX A :-

The rank of a matrix **A** is said to be **r** or **ρ (A),** if they satisfy following two properties.

1. It has at least one non – zero minor of order **r**.
2. Every minor of **A** of order higher than **r** is zero.

In other words the **Rank of matrix A = Number of non – zero rows in upper triangular matrix.**

NOTE : - 1. None zero rows are that row which does not contain all the elements zero.
2. The rank of zero matrix (Null matrix) is zero.
3. The rank of unit matrix I_n of order **n** is **n**.
4. The rank of a non-singular matrix $[\,|A| \neq 0\,]$ **A** of order **n** is **n**.
5. $\rho(A') = \rho(A)$

2. NULLITY OF A SQUARE MATRIX:-

If **A** be any square matrix of order **n**. Then nullity of **A** is denoted by **N (A)** and it is defined by $N(A) = n - \rho(A)$

NOTE :- 1. The rank of non–singular square matrix of order **n**. Then the nullity is **zero**.

3. ECHELON FORM OF A MATRIX: -

A matrix is called in **Echelon form** if

1. if each non-zero row the leading entry is 1 or any number.
2. in each column that contains the leading entry of a row then all other entries are zero.

For example. The matrix
$$\begin{bmatrix} 1 & 0 & -2 & 7 \\ 0 & 2 & -5 & 5 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
is in Echelon form.

NOTE : - 1. If a matrix is in Echelon form the rank of matrix is equal to the number of non-zero rows in it. i.e. in the above example there are three non – zero rows in the matrix. Then rank of matrix is 3.
2. **To find the Echelon form any matrix applies only row operation.**
3. Every matrix is row – equivalent to a row reduced Echelon matrix equivalent to a given **matrix is unique.**
4. The row reduced Echelon matrix.
5. All Echelon matrices have one property in common i.e. they have the same number of non – zero rows.

4. NORMAL FORM (CANONICAL FORM) OF MATRIX: -

If $A = [a_{ij}]$ be any matrix of order $\mathbf{m \times n}$ and rank **r**, then the following form

(i) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (ii) $[I_r \;\; 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (i v) $[I_r]$

are called **normal or canonical form** of matrix **A.** Where $\mathbf{I_r}$ is a unit matrix of order **r** and 0 is zero matrix of any order.

NOTE : - Using only row and column operation.

5. ELEMENTARY OPERATION OR ELEMENTARY TRANSFORMATIONS OF A MATRIX:-

An elementary transformation (or an E – Transformation) is an operation of any of the following three types.

1. The interchange of any two row or two columns.

2. Multiplication of (each element) a row or column by a non – zero number k.
3. The addition of k times the element of a row (or column) to the corresponding elements of another row (or column).

NOTE : - An **E – operation** is called row operation or columns according as it applies to row or columns.

6. METHOD OF FINDING THE RANK OF MATRIX A : -

There are three method to find the rank of matrix A. These are

1. by minor method
2. by Echelon form or row operation method.
3. by normal form or canonical form.

NOTE : - To find the rank of matrix for **best accuracy using only normal form**.

CHAPTER – 02

SOLUTION OF SIMULTANEOUS EQUATION BY ELEMENTARY OPERATION & CONSISTENCY OF EQUATION.

1. HOMGENEOUS SIMUALTANEOUS LINEAR EQUATIONS:

Suppose the simultaneous homogeneous linear equations in three variable is,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \mathbf{A\,X = B} \qquad \text{- - - - - - - - - - - - - - - - - - - (1)}$$

There are many methods to solve the simultaneous equation.

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the coefficient matirx and $C = \left[A \,\vdots\, B \right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \vdots & b_2 \\ a_{31} & a_{32} & a_{33} & \vdots & b_3 \end{bmatrix}$ is Called the **Agumented matrix**.

of the given system of equations.

The matrix equation **A X = B** need not always have a solution. It may have **no solution** or **an unique solution** or **an infinite number of solution**

CONSISTENCE OF LINEAR EQUATIONS: -

There are two types of consistence.

.1. **CONSISTENT SOLUTION : -**

If the system of equation having **one or more solution**, then the system is called **consistent..**

(i) **Unique Solution : -**

If the system of equation has only one solution then we say that the system has unique solution. In this case

$\rho(C) = \rho(A) = n$ (No. of unknown variabes)

(ii) **Infinite number of many Solution : -**

If the system of equation has more then one solution then we say that the system has infinite many solutions. In this case

$\rho(C) = \rho(A) = r < n$ (No. of unknown variabes)

NOTE : - If the system is consistent, then find the value of unknown variable.

2. **INCONSISTENT SOLUTION : -**

If the system of equation having **no solution** is called an **inconsistent system of equations** In this case

$\rho(C) \neq \rho(A)$

HOMOGENEOUS LINEAR EQUATION A X = 0 : -

The standard form of homogeneous linear equation is,
$a_{11}x + a_{12}y + a_{13}z = 0$
$a_{21}x + a_{22}y + a_{23}z = 0$
$a_{31}x + a_{32}y + a_{33}z = 0$

$\Rightarrow$ **A X = 0** - - - - - - - - - - - - - - - - - - - **(1)**

This system is always **Consistent** because x = 0 , y = 0 and z = 0 always satisfy the system. There are two types of solution of Homogeneous linear equation.

(i) **TRIVIAL SOLUTION OR NULL SOLUTION FOR A X = 0 : -**
The solution in which each **unknown has the value zero** is called the **Null solution or the Trivial solution**. Thus a homogeneous system is always consistent.

(ii) **NON- TRIVIAL SOLUTION OR NON- ZERO SOLUTION FOR A X = 0 : -**
If the system of equation has **infinite many solution,** then we say that the system has **non-trivial or non-zero solution**.

Note : - To Solve the simultaneous equation for unknown variable. **We apply only row operation (i.e Echelon form) in Augmented matrix. We never use Column operation.**

CHAPTER – 03
EIGEN VALUE AND EIGEN VECTOR

1. EIGEN VALUES OF A MATRIX. : -

Let A be **square matrix** of order n (i.e. **n × n**) we can form the matrix **A - λ I**, where **λ** is a scalar and **I** is the unit matrix of order n. Then **λ** is called **Eigen values** of **A** if

$$|A - \lambda I| = 0$$

The Eigen values are also called **Characteristic roots** or **latent roots or Proper values.**

NOTE : - 1. The sum of the Eigen values of a matrix A is equal to trace of A.
2. The trace of a square matrix is the sum of its diagonal elements.

2. CHARACTERISTIC MATRIX: -

Suppose **A** be any square matrix. Then **A - λ I** called the characteristic matrix where **λ** is scalar and **I** is the unit matrix.

$$\text{Suppose } A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Called Characteristic matrix.

3. CHARACTERISTIC POLYNOMIAL: -

The determinate **|A – λ I|,** when expanded will give a polynomial in **λ** is called the **characteristic polynomial** of matrix **A**.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = \lambda^3 - 7\lambda^2 + 11\lambda - 5$$

, Called Characteristics polynomial of matrix **A**.

4. CHARACTERISTIC EQUATION: -

The equation **|A – λ I| = 0** is called the **Characteristic equation** of matrix **A**.
i.e. $|A - \lambda I| = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$ characteristic equation of matrix A.

5. CHARACTERISTIC ROOTS OR EIGEN VALUE: -

The roots of characteristic equation |A – λ I| = 0 are called **characteristic roots or Eigen values** of matrix A.

Ex :- $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \Rightarrow (\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$
$\Rightarrow \lambda = 1, 1, 5$

Characteristic roots or **Eigen value** are 1, 1, 5.

6. PROPERTIES OF EIGEN VALUES :-

1. Any square matrix A and its transpose A' have the same Eigen values.
2. The sum of the elements on the principal diagonal of a matrix is called the trace of the matrix.
3. The sum of the Eigen values of a matrix is equal to the trace of the matrix.
4. The product of the Eigen values of matrix A is equal to the determinant of A. i.e. **Product of Eigen values of A = |A |**
5. If $\lambda_1, \lambda_2, \lambda_3,$ ——— λ_n are the Eigen value of A, then the Eigen value of
   (i) K A are $K\lambda_1, K\lambda_2, K\lambda_3,$ ——— $K\lambda_n$
   (ii) A^m are λ_1^m, λ_2^m --- λ_n^m
   (iii) A^{-1} are $1/\lambda_1, 1/\lambda_2$ --- $1/\lambda_m$

7. CHARACTERISTIC VECTORS OR EIGEN VECTOR:-

Let A be n × n square matrix and λ be Eigen value of matrix A. Then there exist **non-zero matrix** X of order n × 1 such that

$$A X = \lambda X$$

is called an Eigen vector of A corresponding to the Eigen value λ.

NOTE : - The Eigen are also called **Latent vector** or **Characteristics vector** or **spectral vector** or **proper vector**.

8. PROPERTIES OF EIGEN VECTORS:-

1. The Eigen vector X of a matrix A is **not unique numerically** and always **non-zero**.
2. If λ_1, λ_2 ----- λ_n distinct Eigen values of an n × n matrix then corresponding Eigen vector X_1, X_2 ----- X_n form a linearly independent set.
3. If two or more Eigen values are equal it may or may not be possible to get linearly independent Eigen vectors corresponding to equal Eigen values.
4. Two Eigen vectors X_1 and X_2 are called **orthogonal vectors** if $X_1 . X'_2 = 0$ or $X'_1 . X_2 = 0$
5. Eigen vectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

9. SPECTRUM: -

The set of Eigen values of A is known as Spectrum of A.

Example :- If $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, and their eigen value are -2, 4. Then Spectrum of A will be E = { -2, 4 }

10. EIGEN SPACE: -

Let A be any square matrix of order n × n and λ_1, λ_2 ----- λ_n are Eigen value and suppose X_1, X_2 ----- X_m are Eigen vector corresponding their Eigen value then the set $E_\lambda = \{X_1, X_2 \text{ ----- } X_m\}$ is called Eigen space.

CHAPTER – 03
CAYLEY HAMILTON THEOREM

Statement: - Every square matrix satisfies its own characteristic equation.

Ex.:- Suppose $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ be any square matrix. The Characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

Replace λ by A, we get $A^3 - A^2 - 4A + 4I = 0$ [Where 0 is null matrix and I is an unit matrix of order is an order of A]
Putting the value of A^3, A^2 & A in above equation if we get R.H.S. = L.H.S. then we say the Cayley Hamilton theorem is satisfies.

NOTE : - **We find the inverse of A with the help of Cayley Hamilton theorem.**

UNIT – 5

CHAPTER – 01
ALGEBRA OF LOGIC

PROPOSITION OR STATEMENT:-

Collection of words ad their combination expressing some idea in any language is called a sentence. But in mathematical every collection of words is not a sentence.

A proposition is a declarative sentence which is true or false, but not both i.e. proposition or statement is a declarative sentence which has a definite truth value.

LOGICAL CONNECTIVES:-
Logical connectives are the symbols or words used to combine two statements or sentences to form a compound statement.
The following symbols are known as logical connectives.

No.	Connective words	Name of connectives	Symbol	Orders
1.	Not	Negative	$\sim$ or $\neg$	1.
2.	And	Conjunction	$\wedge$	2.
3.	Or	Disjunction	$\vee$	3.
4.	If --- then	Conditional (Impales)	$\Rightarrow$	4.
5	Iff or if and only if	Bi -conditional	$\Leftrightarrow$	5.

TRUTH TABLES : -

Fundamental Connectors:-

(i) **Negation (~)**

p	~ p
T	F
F	T

(ii) **Conjunction ($\wedge$)**

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(iii) **Disjunction ($\vee$)**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(iv) **Implies or Conditional ($\Rightarrow$)**

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(v) **Bi – conditional ($\Leftrightarrow$)**

p	q	$P \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TAUTOLOGY: - A compound proposition which is always true regardless of the truth values of its components are called a **tautology**.

CONTRADICTION: - A compound proposition which is always false regardless of truth values of its components are called a **Contradiction.**

LOGICAL EQUIVALENCE: -
Two propositions are called logically equivalent if the truth values of both the proposition are always identical.
If two proposition P & Q are logically equivalent then these are written as $P \equiv Q$, if $P \equiv Q$ then $P \Leftrightarrow Q$ will be a tautology.

SOME IMPORTANT LAW: -

1. **Idempotent law: -** (i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$
2. **Commutative law** (i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$
3. **Associative law** (i) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
4. **De – morgan's law** (i) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$ (ii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$
5. **Distributive Law** (i). $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

CHAPTER – 02
BOOLEAN ALGEBRA AND BOOLEAN FUNCTION

1. DEFINITON OF BOOLEAN ALGEBRA

Let B be non empty set which is defined on the binary operation '+' (for addition) and '.' (for multiplication) and one unary operation ' $'$ ' (for compliment), then the algebraic structure **(B, +, . , ′)** is called **Boolean Algebra** if they satisfy the following axiom, where a, b, c are any element of B.

B_{11} **Closure Laws**

If $a.b \in B$ then

(i) $a + b \in B \quad \forall \quad a, b \in B$

(ii) $a . b \in B \quad \forall \quad a, b \in B$

B_{12} **Commutative laws**

If $a.b \in B$ then

(i) $a + b = b + a$ (ii) $a . b = b . a$ for all $a, b \in B$

B_{13} **Associative laws**

(i) $a + (b + c) = (a + b) + c$ (ii) $a . (b . c) = (a . b) . c$ for all $a, b, c \in B$

B_{14} **Identity laws**

(i) $a + 0 = 0 + a = a$ (ii) $a . 1 = 1. a = a$ for all $a \in B$

B_{15} **Distributive laws**

(i) $a . (b + c) = a . b + a . c$ (ii) $a + (b . c) = (a + b) . (a + c)$ for all $a, b, c \in B$

B_{16} **Complement laws**

For each $a \in B$ there exist $a' \in B$ such that

(i) $a + a' = 1$ (ii) $a . a' = 0$

for all $a \in B$

Where 0 and 1 are additive and multiplication identities respectively

2. DUALITY IN BOOLEAN ALGEBRA : -

Let (B , + , . , ′) be a Boolean algebra . If in a statement of Boolean algebra the position of ' + ' and ' . ' are interchanged i. e. (**+** to **.** and **.** to **+**) and also the identity element 0 and 1 are interchanged i. e. (1 to 0 and 0 to 1) Then the new statement is called dual of the given statement. In other words any result which is true for Boolean algebra remains true even ' + 'and ' . ' are interchanged together with the interchange of 0 and 1 .

3. PROPERTIES (THEOREM) OF BOOLEAN ALGEBRA : -

Theorem 1 : -

(i) Additive identity is unique.

(ii) Multiplicative identity is unique.

(iii) For each $\mathbf{a} \in \mathbf{B}$ its complement $\mathbf{a}'$ is an unique.

Theorem 2 : - Idempotent Law

For every element of a Boolean algebra B

(i) $\mathbf{a + a = a}$ (ii) $\mathbf{a . a = a}$ **for all** $\mathbf{a \in B}$

Theorem 3 : - Involution Law

If for each $\mathbf{a \in B}$, then $\mathbf{(a')' = a}$

Theorem 4 : - Bounded ness Law

For every element of a Boolean algebra B

(i) $\mathbf{a + 1 = 1}$ (ii) $\mathbf{a . 0 = 0}$

Theorem 5 : - Absorption Law

For every element of a Boolean algebra B

(i) $\mathbf{a + a . b = a}$ (ii) $\mathbf{a . (a + b) = a}$

Theorem 6 : - In a Boolean algebra B, the identity element are complementary to each other i.e. $0, 1 \in B$, then

(i) $\mathbf{0' = 1}$ (ii) $\mathbf{1' = 0}$

Theorem 7 : - If a, b, c $\mathbf{\in B}$

$\mathbf{b . a = c . a}$ **and** $\mathbf{b . a' = c . a'}$**, then** $\mathbf{b = c}$

Note : - The Cancellation Laws does not hold in Boolean algebra.

Theorem 8 : - Demorgan's Law

If **(B, +, . , ′)** be a Boolean algebra and a, b be any elements of B, then

(i) $\mathbf{(a + b)' = a' . b'}$ (ii) $\mathbf{(a . b)' = a' + b'}$

3. DISJUNCTIVE NORMAL FORM (DNF) OR CANONICAL FORM : -

A Boolean function f (x_1, x_2, x_3, ---- x_n), when expressed as a polynomial in the form of sum of products (SOP) in such a way that each term involved all the n-variables, then it is known as DNF or Canonical form of the given Boolean function.

Example : -

(i) f (x, y) = x.y′ + x′.y is a DNF of two variable x, y.

(ii) f (x, y, z) = x.y.z + x′.y.z + x.y′.z′ is DNF of three variable x, y, z.

4. COMPLETE DISJUNCTIVE NORMAL FORM OR COMPLETE CANONICAL FORM : -

If all the minimal function of the given number of variables constitute the terms of canonical form, then that function is called the complete canonical form of the Boolean function of the given number of variable. In other word the complete DNF of a Boolean function in n-variables **contains 2^n terms**.

Example : -

(i). The Complete DNF of two variable is f (x, y) = x.y + x′.y + x.y′ + x′.y′

(ii). The Complete DNF of three variable is f (x, y, z) = x.y.z + x′.y.z + x.y′.z + x.y.z′ + x′.y′.z + x.y′.z′ + x′.y.z′ + x′.y′.z′

Note : - The value of the complete DNF in n-variables is 1.

5. DUAL CANONICAL FORM OR CONJUNCTIVE NORMAL FORM (CNF) : -

If a Boolean function f (x_1, x_2, x_3, ---- x_n) is expressed in factored form and each factor is the sum of all n-variable (POS). Then that function is called Dual canonical form or CNF.

Example : -

(i) f (x, y) = (x + y) (x′ + y) is a dual canonical form of two variable x and y

(ii) f (x, y, z) = (x + y + z) (x + y′ + z′) is a CNF of three variables x, y, z.

6. COMPLETE DUAL CANONICAL FORM OR COMPLETE CONJUNCTIVE NORMAL FORM : -

A Conjunctive normal form in n-variables is called Complete conjunctive normal form if there are 2^n distinct factors.

Example : -

(i) The Complete CNF of two variables is f (x, y) = (x + y). (x′ + y) (x + y′) (x′ + y′)

(ii) The Complete CNF of three variables is

f (x, y, z) = (x + y + z) (x′ + y + z) (x + y′ + z) (x + y + z′) (x′ + y′ + z) (x + y′ + z′) (x′ + y + z′) (x′ + y′ + z′)

Note : - The value of Complete CNF for n-variables is 0.

CHAPTER – 03
APPLICATION OF BOOLEAN ALGEBRA TO SWITCHING

Switching circuit and its applications :-

Switch is a device which can control the flow of the current in electric circuits. The properties of the Boolean algebra can be used in the logical designing of certain network. The switches can have precisely two mutually exclusive states, namely **on** and **off** i.e. **Closed** and **Open** .

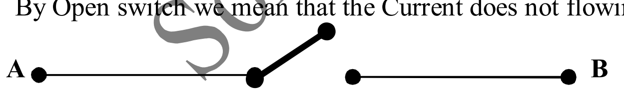
1. **CLOSED (OR ON) SWITCH : -**

By closed switch we mean that the Current is flowing in the electric circuit.

If x represent a switch then in the **Closed** position its is taken to be **1**.
Hence if switch x is **Closed**, then **x = 1** and current flows in this position.

2. **OPEN (OR OFF) SWITCH : -**

By Open switch we mean that the Current does not flowing in the electric circuit.



If x represent a switch then in the **Open** position its is taken to be **0**.
Hence if switch x is **Open**, then **x = 0** and current does not flows in this position.

RELATION BETWEEN CLOSED AND OPEN SWITCHES : -

If x represent a **Closed** switch then x’ will represent an **Open** switch. Conversely if x represent an **Open** switch then x’ will be a **Closed** switch.

Hence If x = 1 then x’ = 0 and if x = 0 then x’ = 1

COMBINATION OF TWO SWITCHES IN A ELECTRIC CIRCUIT : -

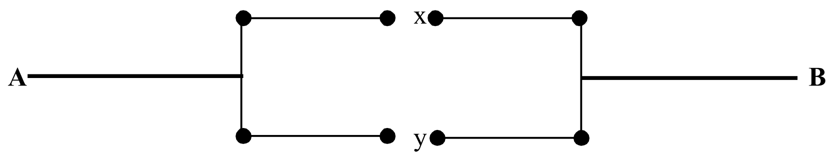
Two switches in an electric circuit can combined in two ways.

1. In Parallel 2. In Series

1. **SWITCHES IN PARALLEL : -**

Let x and y be the variables states of the two switches connected in parallel then the combined state of the two switches is represent by the combined form x + y.

Diagrammatically,



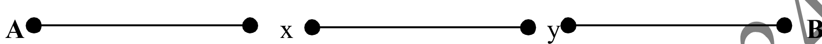
The truth table for above circuit will be as follows.

x	y	x + y
1	1	1
1	0	1
0	1	1
0	0	0

Thus x + y ⇔ Two switches x and y are connected in parallel and the Boolean function f (x, y) = x + y.

2. **SWITCHES IN SERIES : -**

Let x and y be the variable states of the two switches Connected in series then the combined state of the two switches is represented by the compound form (x. y) or simply x.y. Diagrammatically,



The truth table for above circuit will be follows.

x	**Y**	**x.y**
1	1	1
1	0	0
0	1	0
0	0	0