GATE CSE NOTES

by

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Q. $\left.\begin{array}{l} |V| = 5 \\ |E| = 7. \end{array}\right\} G.$

For $\bar{G}$, $|V| = 5$

$$|E| = e(K_5) - 7 = \frac{5 \cdot 4}{2} - 7 = 3$$

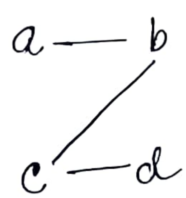
Q Degree seq. of G = $(5, 5, 4, 4, 3, 3, 2)$ | $\Delta = 6$

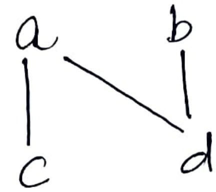
$\therefore$ of $\bar{G}$ = $(1, 1, 2, 2, 3, 3, 4)$

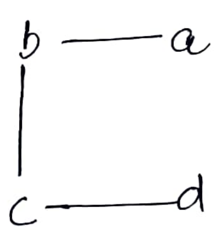
- Complement of disconnected graph is connected, as $G \cup \bar{G} = K_n$

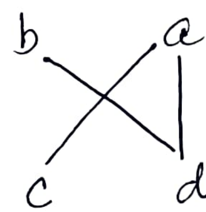
- <u>Self-complementary graph</u>

Complementary graph $\bar{G}$ that is isomorphic to G.

eg. G = 

| $\bar{G}$ = 

G = 

| $\bar{G}$ = 

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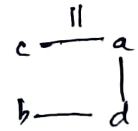


• ϕ_n, K_n are not self-complementary except when $n = 1$.

• <u>Self complementary G.</u>

$G \cup \bar{G} = K_n$ | $|V|_G = |V|_{\bar{G}}$

$|E|_G = |E|_{\bar{G}}$.

$$\therefore \quad |E|_G + |E|_{\bar{G}} = e(K_n) = \frac{n(n-1)}{2}$$

$$\Rightarrow \boxed{|E|_G = |E|_{\bar{G}} = \frac{n(n-1)}{4}}$$

$$\Rightarrow \frac{n(n-1)}{4} = e \Rightarrow n(n-1) = 4e \Rightarrow \begin{matrix} n = 4x \\ \text{or } n-1 = 4x \end{matrix}$$

So, $\boxed{n = 4x \text{ or } 4x+1}$ where $x \in I^+$

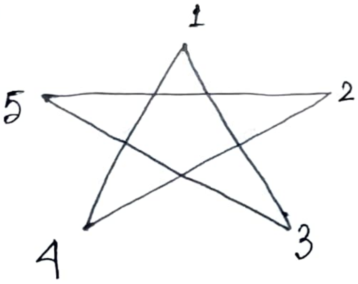
Q $n(G) = 6$

If self-complementary graph exists,

$$e(G) = \frac{n(n-1)}{4} = \frac{6 \cdot 5}{4} \notin I^+$$

Not possible.

eg



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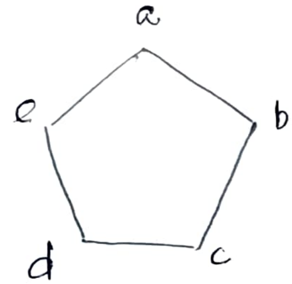


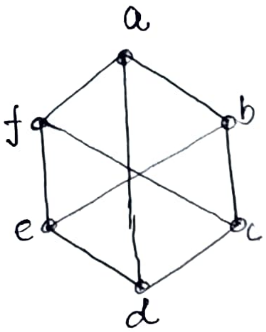
$\bar{G}$

$G, \bar{G}$ self-complementary

$|V|_G = 5$

$|E|_G = \frac{5 \cdot 4}{4} = 5$

eg



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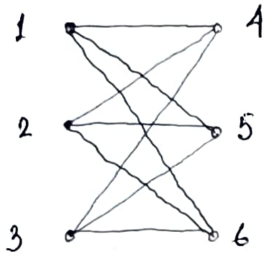


$K_{3,3}$

Isomorphic.

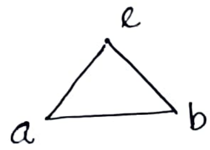
1	2	3	4	5	6
a	c	e	b	d	f

$$\underline{G_1 - G_2 = G_1 \cap \bar{G}_2}$$

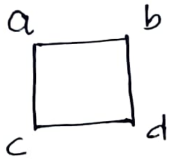
$$V(G_1 - G_2) = V(G_1)$$

$$E(G_1 - G_2) = E(G_1) - E(G_2)$$

eg.

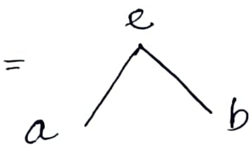


G_1

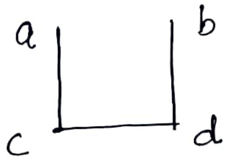


G_2

$G_1 - G_2 =$



$G_2 - G_1 =$



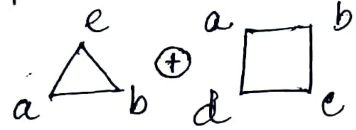
$$\underline{G_1 \oplus G_2 = (G_1 \cup G_2) - (G_1 \cap G_2)}$$

$$= (G_1 - G_2) \cup (G_2 - G_1)$$

$$V = V(G_1) \cup V(G_2)$$

$$E = E(G_1) \oplus E(G_2).$$

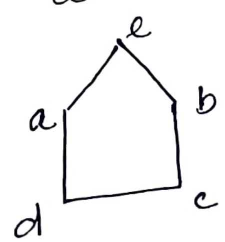
eg.



=

ab, be, ea

ab, bc, cd, da



- <u>Isomorphism</u>

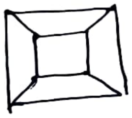
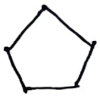
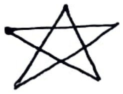
Bijective f^n f exists $f: V_{G_1} \to V_{G_2}$ that reserves adjacency. For $a, b \in V_{G_1}$ where $(a,b) \in E_{G_1}$, $f(a), f(b) \in V_{G_2}$ so that $(f(a), f(b)) \in E_{G_2}$.

→ Checking isomorphism.

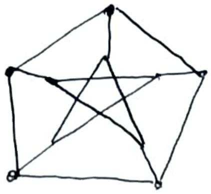
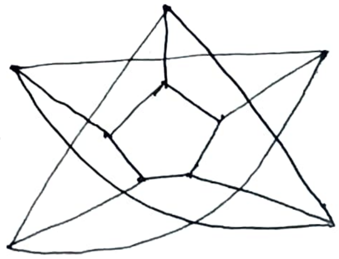
⇒ Check $n!$ bijections & their adjacency. – time consuming!

→ <u>Invariants of isomorphism.</u>

1. $n(G_1) = n(G_2)$, $e(G_2) = e(G_1)$
2. deg. seq.s same for G_1, G_2
3. # cycles of any length same for both
4. $u \in V_{G_1}$, $v \in V_{G_2}$, then all neighbouring properties. vertices of u, v should have same

eg  ≡ ,  ≡ ,  ≡ 

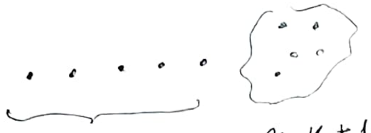
Q_3 pentagram

$K_{3,3}$ ≡ , Petersen Graph  ≡ 

- In any con/discon. graph, if there's exactly 2 odd degree vertices (x, y), then $x \rightsquigarrow y$ exists.

- # components $= k$

$$\boxed{n-k \le e \le \frac{(n-k)(n-k+1)}{2}}$$



$K-1$ $\qquad$ $n-K+1$

min $\quad n-K+1-1$

max $\quad {}^{n-K+1}C_2 = \frac{(n-K)(n-K+1)}{2}.$

Cor. For a disconnected graph with n vertices,

$k=2$ $\quad$ max # edges $= \frac{(n-2)(n-1)}{2}$

- Vertex connectivity $\le \delta(G)$.
  α

To remove the vertex with min. degree we can remove the adjacent vertices to the vertex.

- Same way,

Edge connectivity $\le \delta(G)$.
λ

- $\alpha \le \lambda$ as for each vertex, we must remove at least one edge.

$\alpha \le \delta$, $\lambda \le \delta$, $\delta \le \frac{2e}{n} \le \Delta$, $\alpha \le \delta$

$$\Rightarrow \boxed{\alpha \le \lambda \le \delta \le \frac{2e}{n} \le \Delta}$$

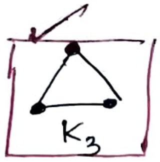
- $\alpha = 1$, G is separable (1-connected)

- Eulerian graph

  - ϕ_n - not a Eulerian
  - K-regular graph.
    K even $\Rightarrow$ Eulerian
  - K_n - n odd
    $\Rightarrow$ Eulerian

  - C_n - Eulerian
  - W_n - not Eulerian

2-regular

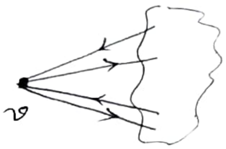


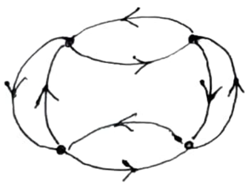
3-regular
K_4

- Unicursal graph
  Containing Eulerian path. $\Rightarrow$ only 2 vertices of odd degree
  These 2 must be starting & endings of path.

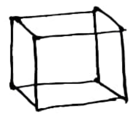
- Traversable / Traceable graph
  If G is either unicursal or Eulerian.

- Directed Euler Graph
  If directed Eulerian cycle exists.
  $\forall v \in V$, $indeg_v = outdeg_v$

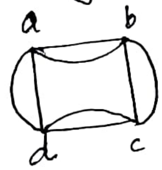




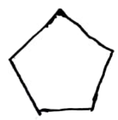
- Hamiltonian graph



Q_n : can't say (may or may not be Hamiltonian)



a b c d a



\# hamiltonian cycles in K_n =

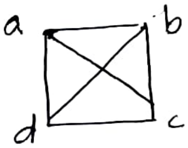
$$\frac{n!}{n \cdot 2} = \frac{(n-1)!}{2}$$

$n \cdot 2$ → mirror img

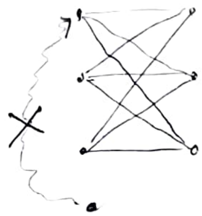
circular arrangement

1 4 2 3	4 3 2 1	2 1 4 3	3 2 4 1

- ϕ_n - not hamiltonian
- K-regular - can't say
- K_n ($n \geq 3$) - Hamiltonian
- C_n - Hamiltonian | W_n - Hamiltonian



a b c d a

$K_{m,n}$ ⇝ 

If $m = n$, $m, n \geq 2$
Hamiltonian

When $m \neq n$, not possible to have Hamiltonian cycle.

- If G is Hamiltonian, no pendant vertex.

- <u>Checking for Hamiltonian graph</u>

1. <u>Dirac's theorem</u> : If G is connected & $\forall v$, $\deg \geq n/2$ & $n \geq 3$ $\Rightarrow$ G is Hamiltonian.
   $\not\Leftarrow$

$\Downarrow$ Extension

2. <u>Ore's theorem</u> : If G is connected, & $\forall u, v$, $\deg_u + \deg_v \geq n$ $(n \geq 2)$
   $\Rightarrow$ G is Hamiltonian.
   $\not\Leftarrow$

- <u>Planar Graph</u> : Embedding on plane s.t. no edges intersect.

$\Rightarrow$ <u>Euler's formula</u> $\boxed{v + f - e = k + 1}$ | f # faces, k # components.

$\Rightarrow$ # Open face = 1 for connected graph.
closed faces = # faces − 1

eg Planar G, $|V| = 10$, every face bounded by 3 edges, # faces = ?

$\rightarrow$ $v + f - e = 2$
$\Rightarrow 10 + f - \frac{3f}{2} = 2 \Rightarrow f = 16$

$3f = 2e$
$\Rightarrow e = 3f/2$

$\boxed{3f \leq 2e}$

$\boxed{e \leq 3n - 6}$

- Complete bipartite graph. $K_{m,n}$

$v + f - e = 2$

$\Rightarrow (m+n) + f - mn = 2 \Rightarrow f = mn - (m+n) + 2$

- <u>Kuratowski's theorem</u> (Test for planarity)

G is planar $\Leftrightarrow$ G does not contain a subgraph that is a subdivision of K_5, or $K_{3,3}$.

(K_5 - smallest complete graph i.e. not planar

$K_{3,3}$ - smallest " bipartite " i.e. not planar).

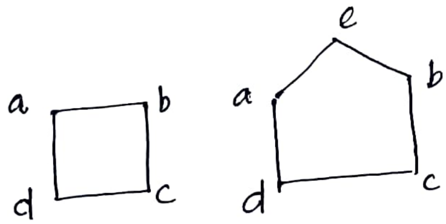
$\rightsquigarrow$ <u>Subdivision / Expansion</u>

$\downarrow$ Opposite

<u>Smoothing</u>

- <u>Homeomorphism</u> $G_1 =_h G_2$

If we can arrive at G_2 by subdividing G_1's some edges.



- For connected simple planar graph, with no K_3, $\boxed{e \leq 2n - 4}$

$4f \leq 2e$

$n + f - e = 2 \Rightarrow f = e - n + 2 \Rightarrow e - n + 2 \leq \frac{2e}{4}$

$\Rightarrow e \leq 2n - 4$

$\delta \le 5.$

- If we have a connected simple planar graph,

$$\delta \le \frac{2e}{n} = \frac{2(3n-6)}{n} \Rightarrow \delta \le 6 - \frac{6}{n} \qquad \boxed{\delta \le 5} \quad n = 6$$

- For complete simple planar graph, with no K_3, $\boxed{\delta \le 3}$

$$\delta \le \frac{2e}{n} = \frac{2(2n-4)}{n} \Rightarrow \delta \le 4 - \frac{8}{n}$$

- Tree graph.

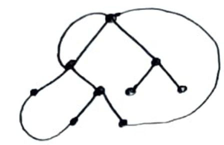
Bichromatic & bipartite. # $n-1$ edges # 1-connected

Acyclic connected # $\forall u,v$, $\exists$ exactly one $u \rightsquigarrow v$

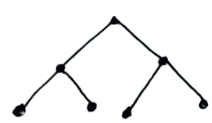
$\Rightarrow$ Fundamental cycle

Cycle obtained by adding exactly one edge.

fundamental cycles $= {}^{n}C_2$



$\Rightarrow$ k-ary tree.



K^0
K^1
K^2
$\vdots$
K^h

$\boxed{l \le K^h}$

l #leaves

n # nodes

h #height $\qquad i$ #internal nodes

$$\boxed{n \le \frac{K^{h+1} - 1}{K-1}}$$

$$\boxed{i \le \frac{K^h - 1}{K-1}}$$

$\boxed{n \ge h+1}$

chain $\boxed{i \ge h}$

$$\boxed{h+1 \le n \le \frac{K^{h+1} - 1}{K-1} \quad \Big| \quad h \le i \le \frac{K^h - 1}{K-1}}$$

$$l \le K^h \Rightarrow \boxed{h \ge \log_K l} \Rightarrow h_{min} = \lceil \log_K l \rceil$$

• Counting spanning trees (Also by Kirchoff's matrix tree theorem)

* 1. Cycle disjoint graph (Common vertices, no common edges among cycles)

eg

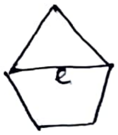
cycle 4 vertices

$$\# \text{STs} = {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_3$$

↓

Choose 2 edges among 3

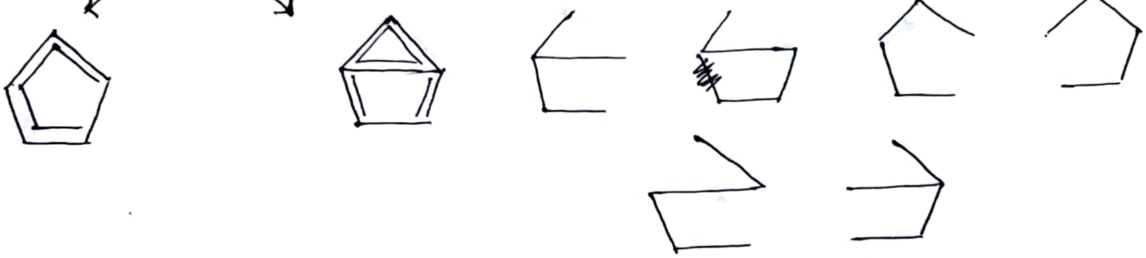
2.



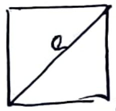
$\#$ STs w/o e $= {}^5C_4 = 5$

$\#$ STs w e $= {}^3C_2 \cdot {}^2C_1 = 6$

$+\ 11$



3.



W/o e $= {}^4C_3 = 4$

w e $= {}^2C_1 \cdot {}^2C_1 = 4.$

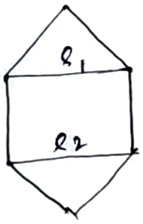
$+\ 8$

	a	b	c	d
a	2	-1	0	-1
b	-1	3	-1	-1
c	0	-1	2	-1
d	-1	-1	-1	3

$3(6-1) + 1(-3-1) - 1(1+2)$

$15 - 4 - 3 = 8$

4.



 w/o e_1, e_2 $= {}^6C_5 = 6$

 w e_1, not e_2 $= {}^2C_1 \cdot {}^4C_3 = 8$

 w e_2, not e_1 $= {}^2C_1 \cdot {}^4C_3 = 8$

 w e_1, e_2 $= {}^2C_1 \cdot {}^2C_1 \cdot {}^2C_1 = 8$

30

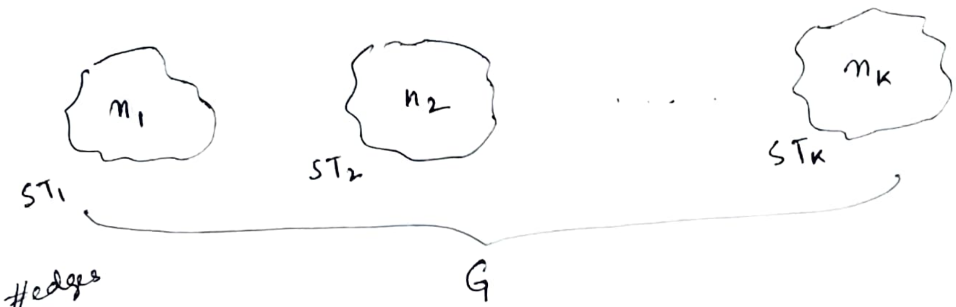
- Rank of Graph $= n - k$

k # components
n # vertices
e # edges

Nullity $= e -$ rank
$= e - n + k$

Rank + Nullity $= e$.

Nullity – Cyclomatic complexity

- 

disconnected graph G

$$\underbrace{\Sigma \text{ size } (ST_i)}_{\text{\#edges}} = \Sigma(n_i - 1) = \Sigma n_i - k = n - k$$

edges in the spanning forest $= n - K =$ rank of G.
(or spanning tree for connected G)
size $= n - 1$.

Nullity = min # edges to be removed from G to make it a spanning tree or forest.
= # edges to be removed to break all cycles.

- **Branch set** is set of all edges in ST or SF.
  | Branch set | = rank (G).

- **Chord set** Set of edges to be removed to make ST/SF.
  | Chord set | = nullity (G).

Counting graphs

1. # labelled graphs with n vertices $= 2^{{}^nC_2}$

* 2. # simple labelled graphs given n, e $= {}^{({}^nC_2)}C_e$

3. # labelled trees $= n^{n-2}$. (Cayley's Formula)

STs in K_n $= n^{n-2}$

4. # rooted labelled trees $= n \cdot n^{n-2} = n^{n-1}$

5. # labelled subgraphs of K_n =

$$\sum_{i=1}^{n} {}^nC_i \; 2^{{}^iC_2}$$

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eg # graphs with n vertices & at least $\frac{n(n-1)}{4}$ edges.

$\rightarrow$ $e(K_n) = \frac{n(n-1)}{2}$ $\cancel{\frac{n(n-1)}{2} \setminus \frac{n(n-1)}{4}} =$

$${}^{{}^nC_2}C_{\frac{n(n-1)}{4}} + {}^{{}^nC_2}C_{\frac{n(n-1)}{4}+1} + \cdots + {}^{{}^nC_2}C_{{}^nC_2}$$

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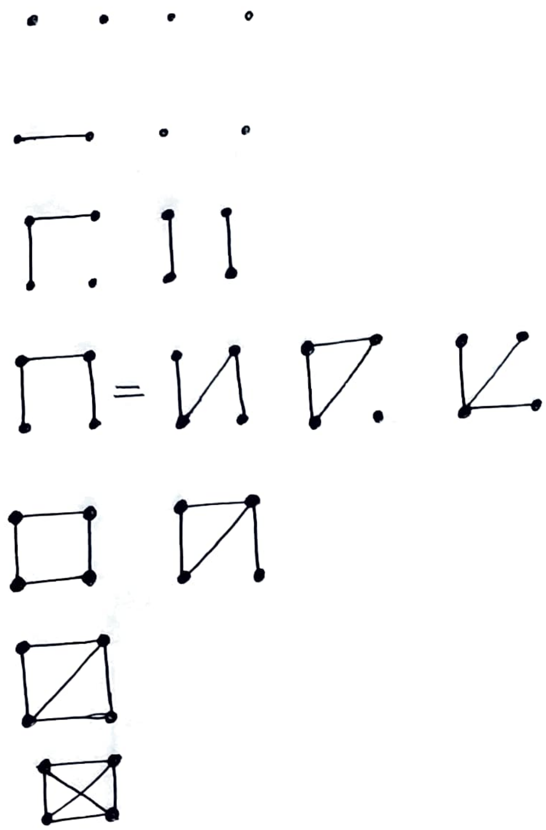
6. # unlabelled ~~graphs~~ binary trees with n vertices $= C_n$ | Catalan no.

$$T(n) = \sum_{i=1}^{n} T(i-1)\,T(n-i) \qquad \frac{1}{n+1}\binom{2n}{n}$$

labelled binary trees with n vertices $= (n! \cdot C_n)$

• Unlabelled graphs with $n = 4$



$e = 0$	1
$e = 1$	1
$e = 2$	2
$e = 3$	3
$e = 4$	2
$e = 5$	1
$e = 6$	1
	+ 11

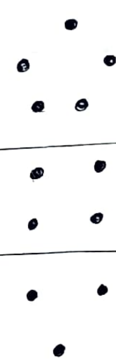
• # subgraphs for a labelled G, with $n = 5$, $e = 3$.

Subgraph also has 3 edges

5 vertices, 3 edges $\quad {}^5C_5 \times {}^{{}^5C_1}C_3$

4 vertices, 3 edges $\quad {}^5C_4 \times {}^{{}^4C_2}C_3$

3 vertices, 3 edges $\quad {}^5C_3 \times 1$



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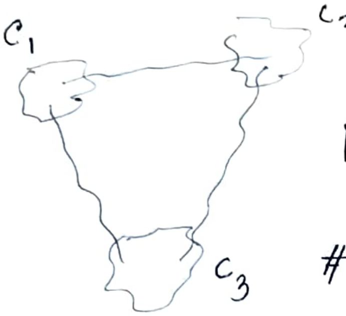
• # simple graphs (labelled) with n vertices, e edges $= {}^{{}^nC_2}C_e$

• # graphs with n edges (simple, unlabeled, no isolated nodes, not necessarily connected)

- Chromatic number $\chi(G)$.



K_χ

$\# \text{edges} \geq {}^{\chi}C_2$

$$\boxed{|E| \geq {}^{\chi}C_2}$$

$\boxed{\chi \leq \Delta}$ except for K_n, odd C_n

K_n: $\chi = n$, $\Delta = n-1$

odd C_n: $\chi = 3$, $\Delta = 2$

$$\boxed{\chi \leq \Delta + 1 \leq n}$$

$\boxed{\chi_{\text{planar } G} \leq 4.}$ 4-color theorem

$\chi_{\text{multigraph}} = \chi_{\text{simple graph dropping multiedges}}$

- Independence / stable set

Ind. no. β_G.

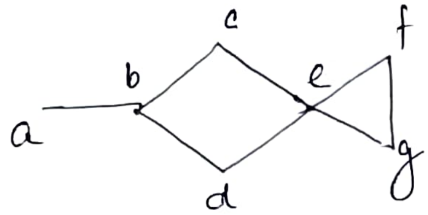
$$\boxed{\beta_G \geq \frac{|V|}{\chi(G)}}$$

$|V|$ pigeons in χ holes. 1 hole with $\geq |V|/\chi$ vertices

1 ind. set with $\geq \frac{|V|}{\chi}$ vertices

- Dominant Set (DS)

Set of vertices from which all vertices are one step away.



e.g. $\{b, e\}$

$\{a, c, d, f\}$

Domination number

Size of smallest DS.

→ If a set is maximal independent set

→ it is DS.

- Domination # $\leq$ Independence #

- Matching : Disjoint edge set.

Covering : Set of edges that covers all vertices. (Edge covering)

Size of smallest cover = Covering # $\geq \lceil n/2 \rceil$

- Perfect matching possible when # vertices is even.

- # perfect matching $(K_{2n}) = \dfrac{(2n)!}{n! \cdot 2^n}$ $\cancel{\dfrac{n! \binom{2n}{n}}{2^n}}$

Proof K_{2n} all vertices adj to each other.

$(2n-1)$ ways to choose 2nd vertex after choosing 1st.

$(2n-3)$ " " 2nd pair

$(2n-5)$ " " 3rd pair

⋮

1 way " nth pair

Finding set of n disjoint pairs

$(2n-1)(2n-3)\cdots 1$

$$\Rightarrow \frac{(2n-1)(2n-3)\cdots 1 \cdot 2n(2n-2)(2n-4)\cdots 2}{2n(2n-2)(2n-4)\cdots 2} = \frac{2n!}{2^n \cdot n!} \quad \blacksquare$$

- If n = odd in K_n, no perfect matching.
- #edges in perfect matching $= \frac{|V|}{2}$

- <u>Thm</u> An edge covering is minimal iff every component is a star graph.

1) order : $(2n)!$

2) n pairs. $(1,2)(3,4) \equiv (3,4)(1,2)$ $\Big\}\ \frac{(2n)!}{n!}$

3) $(1,2) \equiv (2,1)$: $\frac{(2n)!}{n!\,2^n}$

(n pairs, 2 ways