GATE CSE NOTES

by

Joyoshish Saha

DM

Set, Relation, Functions.

- $A-(B\cup C) = (A-B)\cap(A-C)$ | $(A\cup B)\cap C = A\cup(B\cap C)$ iff $A\subseteq C$
  $A-(B\cap C) = (A-B)\cup(A-C)$ | $(A\cap B)\cup C = A\cap(B\cup C)$ iff $A\supseteq C$

- $\lambda\in A \iff A^+ = A^*$ | $(A^*)^+ = (A^+)^* = A^*$
  $(A^*B^*)^* = (A\cup B)^* = (A^*\cup B^*)^*$
  $(A^*)^* = A^*A^* = A^*$ | $A^*A^+ = A^+A^* = A^*$
- $A\oplus B = A\Delta B = (A\cup B)-(A\cap B)$
  $= (A-B)\cup(B-A)$
  $= (A'\cap B)\cup(A\cap B')$

- $|P(S)| = 2^{|S|}$, $|P(P(S))| = 2^{2^{|S|}}$.
- Reflexive: R^{-1}, $R\cup S$, $R\cap S$, superset [not subset]
- Irreflexive: R', $R\cup S$, $R\cap S$ · Symmetric: R^{-1}, $R\cup S$, $R\cap S$, R_1-R_2 [not superset, subset] need not be
- Antisymmetric $R\cap S$, $R-S$, subset [not superset, $R\cup S$, R']
- Transitive R^{-1}, $R\cap S$, (not $R\cup S$, R')
- Equivalence rel^n $R\cap S$ [not $R\cup S$]
- Asymmetric subset, intersection, difference (not superset, union, complement)
- Irreflexive and antisym $\rightarrow$ Asymmetric.

If R is antisym, $R\cap R^{-1}\subseteq\Delta_A$ (diagonal rel^n)

Diagonal elements can be present in antisym, not in assymmetric rel^n.

Reflexive cannot be asymmetric.

Every asymmetric rel^n is irreflexive, not vice versa. $\{(1,2),(2,1)\}$.

Every asym. rel^n is antisym., not vice versa.

R on A is transitive iff $R^n\subseteq R$.

A transitive rel^n is asymmetric iff it is irreflexive.

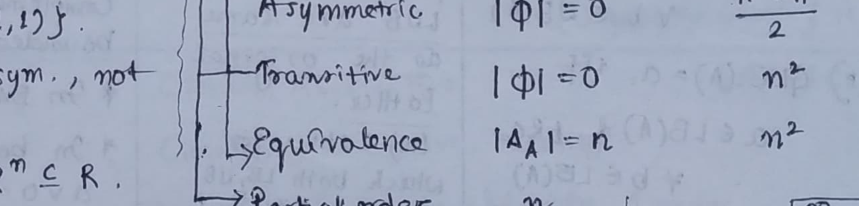
#	Smallest	Largest
Reflexive	$\vert\Delta_A\vert = n$	$\vert A\times A\vert = n^2$
Irreflexive	$\vert\phi\vert = 0$	$\vert A\times A\vert - \vert\Delta_A\vert = n^2-n$
Symmetric	$\vert\phi\vert = 0$	$\vert A\times A\vert = n^2$
Antisym.	$\vert\phi\vert = 0$	~~$\vert A\times A\vert = n^2$~~ $\frac{n}{2}(n+1)$
Asymmetric	$\vert\phi\vert = 0$	$\frac{n^2-n}{2}$
Transitive	$\vert\phi\vert = 0$	n^2
Equivalence	$\vert\Delta_A\vert = n$	n^2
Partial order	n	

* There's a path of length n from a to b iff $(a,b)\in R^n$. | $R^{n+1} = R^n\circ R$ | $R^* = \bigcup_{n=1}^{\infty} R^n$ | $S\circ R$: $R(a,b)$, $S(b,c)$ → (a,c)

Closure of relations: Reflexive: $R\cup\Delta_A$, Symmetric: $R\cup R^{-1}$,
Transitive: Connectivity rel^n $R^* = \{(a,b) \mid \exists$ a path (in digraph) of length at least one in $R\}$

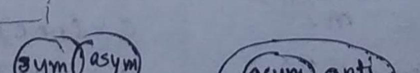
Counting #rel^{n}s. (total $= 2^{mn}$ or 2^{n^2})

1. #ref $=$ #irref $= 2^{n^2-n}$
2. #sym $= 2^{n(n+1)/2}$
3. #antisym $= 2^n\times 3^{(n^2-n)/2}$
4. #asym $= 3^{(n^2-n)/2}$
5. #transitive $= 2, 13, 171, 3994, \ldots$ (from $n=1$)
6. #eqv. rel^{n}s $=$ Bell(n). $= \sum_{k=1}^{n} S(n,k)$
7. Either ref or irr $= 2\cdot 2^{n^2-n}$
8. neither ref nor irref $= 2^{n^2}-(2\cdot 2^{n^2-n})$
9. Both sym & ref (compatibility rel^n) $= 2^{(n^2-n)/2}$
10. Both sym & irr. $= 2^{(n^2-n)/2}$
11. Both sym, antisym $= 2^n$ $\subseteq\Delta_A$
12. Ref & Antisym $= 3^{(n^2-n)/2}$
13. Irref & antisym $= 3^{(n^2-n)/2}$
14. Sym & asym $= 1$.



Bell triangle:
n = 1: 1
2: 1 2
3: 2 3 5
4: 5 7 10 15
⋮

#partitions possible for a set into nonempty subsets.



15. # total order $= n!$
16. # partial order $= 1, 3, 19, 219, \ldots$ (from $n=1$)

* Partial order relⁿ (PO): Ref, Antisym, Trans.   * POSET [A; PO]

* Total order: Partial order such that aRb or bRa $\forall a,b \in A$ [Connexity]

Connex order ↳ or, Binary relⁿ which is antisym, transitive & a connex relⁿ.

* TOSET ([A; TO]) Toset's hasse diagram is a chain, [$\#v = n$, $\#e = n-1$].

* [Maximal (a) / Minimal (b) / Maximum (1) (c) / Minimum (0) (d) / LUB ($\vee$, Join, Supremum) (g) / GLB ($\wedge$, Meet, Infimum) (h) / LB (e) / UB (f) / Join Semi Lattice (i) / Meet Semi Lattice (j) / Lattice (k) / Sublattice (l), Bounded Lattice (m), Complement (n), Complemented lattice (o), Distributive lattice (p), Boolean algebra (q).

$(S,\leq)$ be the poset or (S,R)

(a) $\nexists b \in S$, aRb or $a \leq b$ — a maximal

(b) $\nexists b \in S$, bRa or $b \leq a$ — a minimal

(c) a maximal & bRa $\forall b \in S$

(d) a minimal & aRb $\forall b \in S$.

(e) LB of $A \subseteq S$, $a \in LB(A)$, iff aRb $\forall b \in A$

(f) UB of $A \subseteq S$, $a \in UB(A)$, iff bRa $\forall b \in A$.

(g) $LUB(A) = a$, iff $a \in UB(A)$ & aRb $\forall b \in UB(A)$

(h) $GLB(A) = a$, iff $a \in LB(A)$ & bRa $\forall b \in LB(A)$

(i) Poset for which every pair of elems has LUB.

(j) Poset for which every pair of elems has GLB.

(k) Poset, every pair has both GLB & LUB.

(l) Subset of a lattice such that the subset also is a lattice and for any pair GLB, LUB are same as the original lattice.

(m) Lattice for which both LB, UB exists.

(n) (of an elem) for a $\exists$ a' s.t. $a \vee a' = 1$, $a \wedge a' = 0$

(o) Each elem has a complement.

(p) That follows distributive propʸ.
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(q) Lattice that is distributive & Complemented (thus bounded).

* Every TOSET is a lattice.

* LUB: $\subseteq$ — union of sets; $\leq$ — maximum elem; $\div$ — LCM

* GLB: $\subseteq$ — intersection; $\leq$ — least elem; $\div$ — GCD.

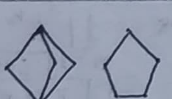
* Finite lattice is always bounded.

* A complemented lattice is always bounded.

* If a complemented lattice is distributive then each elem has unique complement.
↳ In BA each elem has unique complement.

* In BA, if $a \vee b = a \vee c$, and $a \wedge b = a \wedge c$, then $b = c$.

* In bounded lattice, $a \vee 1 = 1$, $a \wedge 1 = a$, $a \vee 0 = a$, $a \wedge 0 = 0$.

*  If in some lattice, any of these is a sublattice, then it can't be distributive.

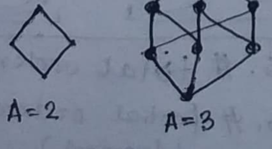
* A lattice with 4 or fewer elems is distributive.

* Every TOSET, is a distributive lattice.

* Every sublattice of a distributive lattice is also a dis. lattice.

* TOSET with 2 elems is BA. TOSET with ≥ 3 elems can't be complemented lattice (so no BA).

* Every lattice to be a BA, its hasse dgm has to be isomorphic to that of $[P(A), \subseteq]$.



A=2   A=3

$\#v = 2^n$
$\#e = n\,2^{n-1}$
for any BA.

* $[D_n, \div]$ is a BA iff n is a squaree free number (unique primes in prime factorizⁿ).
↳ If BA, then compl. $(a) = n/x$

* Examples of POSET: $(R, \subseteq)$, $(N, \leq)$, $(D_n, \div)$.

* Examples of Distributive Lattice: Every BA, Every TOSET, $(N, \leq)$, $(D_n, /)$ — set of +ve divisors

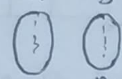
* Examples of BA: $(P(A), \subseteq)$, $(D_n, /)$

* A TOSET can't be a complemented lattice if $|S| \geq 3$.

* Propⁿ of lattice (opⁿs $\vee, \wedge$) (DMAb)
closure, commutative, associative, idempotent, absorption ($a \vee (a \wedge b) = a$), consistency ($a, b \leq a \vee b$, $a, b \geq a \wedge b$), distributive inequality —
$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
$a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$

* Well ordered set: $(S, \leq)$ if it is a poset s.t. $\leq$ is a total order & every non empty subset of S has a least element.

* **Function** $\forall x \in A,\ \exists (x,y) \in R,\ y \in B.$ | Domain
$(x,y) \in R \wedge (x,z) \in R \Rightarrow y = z$ | Codomain ~ Range

A B

m n

$\#f^n s = n^m$

> Partial f^n : Subset of elems in A has mapping to B ↘ $\# = (n+1)^m$

> One to one (Injective) : $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ | $\# = {}^nP_m$
Choosing m elems to map: $n(n-1)(n-2)\cdots(n-m+1)$ ↲ ↓ $0,\ n<m$

Fun^n: Subset of $A \times B$ such that each elem of A is related to only one elem of B.

> Many to one $\#\ n^m - {}^nP_m$

> Onto Codomain = range
(Surjective) $\# = n!\left\{ {m \atop n} \right\}$
$= n!,\ n = m$

> Into Codomain ⊃ Range
$\#\ n^m - n!\left\{ {m \atop n} \right\}$

> Bijection One to one, onto. $|A| = |B| = n$
$\# = n!$

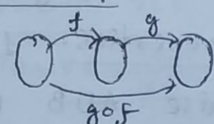
$|A| > |B|\ \&\ |B| = 2$ then # onto $f^n s$ $A \to B$ $= 2^{|A|} - 2$

• $\left\{ {m \atop n} \right\} = S(m,n)$ Stirling's number of 2nd kind
= #ways to partition set of m elements into n subsets
$S(m,n) = S(m-1,n-1) + n\,S(m-1,n). = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m$
Building table: ⇓
$\#$ onto $f^n s = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m$

m \ n	0	1	2	3	4
0	1				
1	0	1			
2	0	1	1		
3	0	1	3	1	
4	0	1	7	6	1
⋮					

> Inverse exists only for bijection

> Composition of f^n : $f\circ g(x) = f(g(x))$



$g\circ f() = g(f())$

> $g\circ f \neq f\circ g$ > $g\circ(f\circ h) = (g\circ f)\circ h$

> If f & g i) one to one → i) $g\circ f$ one to one
ii) onto → ii) $g\circ f$ onto
iii) bijection → iii) $g\circ f$ bijection.

> If $g\circ f$ is one to one, then f is one to one but g may not be.

> If $g\circ f$ is onto, then g is onto but f may not be.

> $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.

> $f: A \to B$, X, Y be non-empty subsets of A, then
1. If $X \subseteq Y$, $f(X) \subseteq f(Y)$, 2. $f(X \cup Y) = f(X) \cup f(Y)$, 3. $f(X \cap Y) \subseteq f(X) \cap f(Y)$
(equality when f is one-one)

Group Theory (DM$_1$)

CAIdIC

* **Semigroup** : Closure, Associative

* **Monoid** : Closure, Associative, Identity

* **Group** : Closure, Associative, Identity, Inverse

* **Abelian group** : Closure, Associative, Identity, Inverse, Commutative.

* **Binary operation** (*)
– $a*b$ is closed.
– $a*b$ is unique.

* **Algebraic structure** (wrt some bin. operation).
eg. $(P(A), \cup)$
$(\{1,-1\}, \times)$

* **Identity** : $a*e = e*a = a$
(Unit)

* **Inverse** : $a*a^{-1} = a^{-1}*a = e$

* **Group properties** : i) e is unique. ii) $a*e = e*a$ iii) a^{-1} unique.
iv) $(a*b)^{-1} = b^{-1}*a^{-1}$ v) $(a^{-1})^{-1} = a$ vi) $ab = ac \Leftrightarrow b = c$
vii) $ba = ca \Leftrightarrow b = c$ viii) $ax = b \Rightarrow x = a^{-1}b$ is unique, $xa = b \Rightarrow x = ba^{-1}$ is unique. ix) Cayley's table of a finite group has no repetition, in any row or column.

* **Subgroup** : $(H,*)$ is a subgroup of $(G,*)$ iff i) $H \subseteq G$,
ii) $(H,*)$ is also a group. or $a*b^{-1} \in H$ $\forall a,b \in H$.

* **Cyclic group** : $(G,*)$ such that $\exists g \in G$, $\forall a \in G$, $a = g^n$, $n \in \mathbb{Z}$
(multiple g's may exist) g ~ generator

• e can't be the generator unless only elem is e.
• if g is generator, g^{-1} too.
• A cyclic group is always abelian. (contrative also)

* **Power of an element**
$a^0 = e,\ a^1 = a,\ a^2 = a*a, \ldots$
$a^{-1} = $ inverse(a)
$a^{-2} = $ inverse(a^2)
$(a^m)^n = a^{mn}$
$(a^m.a^n) = a^{m+n}$

* **Order of group** : $(G,*)$ #elems in G. Can be ∞.

* **Order of elem** : $o(a)$ = smallest $n \in \mathbb{Z}$ s.t. $a^n = e$

* Abelian group properties. 1. $a*b = b*a$, 2. $\forall a \in G$, $a^{-1} = a$, 3. $(ab)^2 = a^2b^2$
4. Every cyclic group is abelian.

$a*b \approx ab$

* Properties of order of elem 1. order of an elem $o(a)$ $\leq$ order of group $o(G)$
2. $O(a)$ divides $o(G)$ for finite group. 3. for finite group $o(a)$ exists & is finite.
4. $o(G) = \text{prime}(p) \Rightarrow o(a) = 1$ or p, $o(e) = 1$; $o(a) = p \;\forall a \neq e$, 5. $o(ab) = n \Leftrightarrow (ba)^n = e$
$\Leftrightarrow (ab)^n = e$ 6. $o(ab) = o(ba)$ for all groups.

* Cyclic group properties. 1. Group having order = prime#, it's cyclic. generators = $G - \{e\}$.
2. $(G,*)$ is cyclic $\Leftrightarrow \exists a$, s.t. $o(a) = o(G)$. 3. #generators of a cyclic group of order n = $\phi(n)$ Euler's totient f^n. $\phi(n) = n-1$ when n prime. If n not prime, use prime factorization to determine $\phi(n)$. $n = p_1^{m_1} p_2^{m_2} p_3^{m_3} \ldots$, $\phi(n) = (p_1^{m_1} - p_1^{m_1 - 1})(p_2^{m_2} - p_2^{m_2-1}) \ldots$ 4. If g_i, g_j are generators of cyclic group, $g_i = g_j^x$, 5. $o(g_i)$ is relatively prime (gcd = 1) to $o(G)$. | $\phi(n) \to$ # integers (i) from 1 to n s.t. $\gcd(i,n) = 1$

* Subgroup properties. 1. A, B be subgroups of G. Then, $A \cap B$ is also a subgroup, while $A \cup B$ need not be. 2. Lagrange's th^m: $O(H)$ divides $O(G)$ when H is a subgroup of G. 3. If $|G| = \text{prime}$, $o(G) = p$ [no proper subgroup],
4. H, K be subgroups of G then $HK = \{h*k \mid h \in H, k \in K\}$ is also subgroup.
5. $O(HK) = \dfrac{O(H) \cdot O(K)}{O(H \cap K)}$ • $O(H \cap K) \leq O(H), O(K)$, $O(HK) \mid O(H), O(K)$,
$O(H \cap K) \mid O(H), O(K)$ (where H, K are subgroups of G). 6. $a * b^{-1} \in H \;\forall a, b \in H$.

* $(M, *, e)$ be a monoid. If element x in M is invertible, then there is a unique inverse element. $xx' = x'x = e \wedge xx'' = x''x = e \Rightarrow x' = x''$.

* Ring: Algebraic structure consisting of a nonempty set R & 2 binary operations $(+, \cdot)$ such that i) $R(+)$ is abelian group, ii) $R(\cdot)$ is a semigroup, iii) $R(+)$ & $R(\cdot)$ satisfy : $x \cdot (y+z) = x \cdot y + x \cdot z$, $(x+y) \cdot z = x \cdot z + y \cdot z$.
> If R is a ring with at least 2 elements (different), then additive & multiplicative identities are not equal.

* Field: $R(\cdot, +)$ such that i) $(R, +)$ is abelian group, ii) $(R - \{0\}, \cdot)$ is an abelian group. iii) $\cdot$ is right & left distributive over $+$.

• For a finite cyclic group, $o(G) = o(g)$ ~ $g = \text{gen}^r$.

For any infinite cyclic group, there are only 2 generators.

• (nth roots of unity, $\times$) is a finite cyclic group. Generator $= e^{2\pi/n}$

• Every finite group of order <6 must be abelian.
(non abelian has order ≥ 6).

Graph Theory

* $\delta_G \le \frac{2e}{v} \le \Delta_G$
* $\sum d(v) = 2e$

* Order # vertices
  Size # edges

#graphs with n vertices possible = $2^{{}^nC_2}$ ↓

* Ramsey's Theorem (3,3) (3,4)

* Cayley's Theorem: #labeled trees = n^{n-2} $(n \ge 2)$

* Tree → n ver. ↓ n−1 edges ↳ Cor.: (↓)

* Total # trees possible with n nodes = $2^n - n$.

* # unlabeled binary trees = C_n | # labeled binary trees = $n! \cdot C_n$

  # BSTs with n keys = C_n

  # structurally diff binary trees = C_n

  $n = 3$ $C_n = 5$

* Cayley's formula: # trees on n labeled vertices = n^{n-2}
  ↳ isomorphic not counted | $n = 3. \Rightarrow \# = 3.$
  ↳ Also, # spanning trees in K_n on n labeled nodes.

* # different labeled binary trees of n nodes (where $k_1, k_2, \dots, k_m$ are the repetitions of $a_1, a_2, \dots, a_m$ values of the tree, $m \le n$). =

$$\frac{n!}{k_1!\, k_2! \cdots k_m!} \cdot C_n.$$

(Unique or non-unique values at nodes)

$\boxed{\text{* every tree is bipartite}}$

for K_n, # STs = n^{n-2}

* Any 2 vertices in a tree - unique path b/w them.

* # spanning trees: Kirchoff's Matrix tree Theorem

$$L_{ij} = \begin{cases} d(i) & i = j \\ -1 & i \ne j \text{ ; } i, j \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Delete one row + one col. → get L^*

sp. trees = $\det(L^*)$

* Planar graphs (4-colorable): $v + f - e = 2$ $(= k+1)$ | $3f \le 2e$ & $e \le 3v - 6$ for $v \ge 3$ | $kf = 2e$
  • If $e > 3v - 6$ for $v \ge 3$, then G is nonplanar.

* Kuratowski's theorem: Every non-planar G contains either K_5, or $K_{3,3}$ or a subdivision of K_5 or $K_{3,3}$.

* (↑) Forest F of order n + k conn. comp.s, $e = n - k$.

* For every connected graph, $e \ge v - 1$ (↑)

Connected simple graphs with n labelled vertices $= n\,2^{\binom{n}{2}}$

* Walk — neither edges, ver's repeat → Path $\xrightarrow{\text{Closed}}$ Cycle
  - no edges repeat → Trail $\xrightarrow{\text{Close}}$ Circuit

(Path ⇓ Trail, Cycle ⇓ Circuit)

(If vertices don't repeat, automatically edges can't repeat.)

* Connectivity : Th] If for every 2 non-adj vertices (u, v), degree $d_u + d_v \geq |V| - 1$ then G is connected.

Th] If in G, $\delta(G) \geq \frac{|V|-1}{2}$, then G is connected.

Th] If G is disconnected, G' is connected.

* Euler trail : Includes all the edges of G. (Circuit if closed)
  ↳ Th] A connected graph or multigraph G is Eulerian iff each vertex is of even degree. (In case of a trail except start & end vertex.) ✓
  ↳ Fleur's algo (finding Euler trail) At each step we move across an edge whose deletion does not result in >1 components, unless we have no choices (visiting an edge ⇒ delete from G for further consideration). In the end, no edges are left.

K_{2n+1}, C_n, K (even) regular ✓
(DM 4a) Q_n, W_n ✗

* Hamiltonian path : Includes all vertices in G. (If G is hamiltonian, no pendant vertex.)
(No known tests for Hamiltonian).

If a graph is hamiltonian it has a hamiltonian cycle.

↳ Dirac's Th] If G is a simple graph with $(n \geq 3)$ & if $d(v) \geq n/2$ then G is Hamiltonian. ← (Last pg)

↳ # hamiltonian cycles in K_n $= \frac{(n-1)!}{2}$ (circular arrangement of beads)

K_n, C_n, W_n ✓ $K_{n,n}$ ✓
Q_n ✗ | Q_n, K regular ?
$K_{m,n}$ ✗

* Bipartite graph. V divided into A, B. for each $e \in E$, connects A & B's vertex.
  ↳ $\sum_A d_v = \sum_B d_v$
  ↳ A graph is bipartite iff it not have any odd cycle.
  ↳ A non-null graph is bipartite iff it is bichromatic or 2-colorable.
  ↳ Any acyclic graph is bipartite.

* Eccentricity (of vertex)

$\varepsilon(u) = \max_v \{d(u, v)\}$

$d(u, v)$ being shortest distance

Radius

$r(G) = \min_v \varepsilon(v)$

Diameter

$d(G) = \max_v \varepsilon(v)$

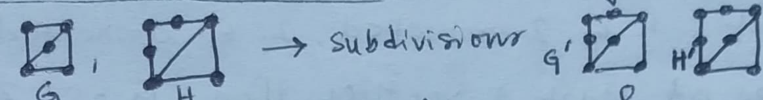
largest of the shortest paths b/w any pair of vertices.

$A_G = PA_HP^T$ P - Permutation matrix

* If diameter of a simple graph is ≥ 3, then $diam(\bar{G}) \leq 3$.
Also, " " a graph $\geq 4 \Rightarrow diam(\bar{G}) \leq 2$.

* <u>Isomorphism</u>: If there's a <u>bijection</u> b/w 2 vertex sets. (Checking: # v's, e's, same deg. seq., same # of circuits of particular length, # comp^{nt}s., diameter, radius, length of longest path, checking isomorphism on $\bar{G}_1$ & $\bar{G}_2$, <u>checking planarity, bipartite or not</u>

eg. G, H → subdivision G', H'



* <u>Homeomorphism</u>: If there's graph isomorphism from some $\boxed{\text{subdivision}}$ of G to some subdivision of G'.
(<u>subdivision of edges</u> - u —— v → u — w — v, reverse op^n is <u>smoothing</u> an edge).

* <u>Matching</u> (max): Set of disjoint edges. (Perfect matching) (Max^m & max^l matching, matching no. $\mu(G)$) • <u>Hall's cond^n/Matching cond^n</u> (in Bipartite):
Bipartite (A, B) Existence of matching saturating A. $|N(S)| \geq |S| \ \forall\ S \in A$. $N(S)$ neighbourhood of S. (vertices of $G \setminus S$ adj. to S's at least one vertex)
• Stable matching (no rogue couples, algorithm terminates at most $n(n-1)+1$ rounds (n men, n women) - repeats until no man is rejected in a round - algo is male optimal, female pessimal

m w
m' w'

* <u>Vertex cover</u> (min) (Q): $Q \subseteq V(G)$, at least one endpoint of each edge in Q. <u>Vertex cover number</u> $\tau(G)$. $\mu \leq \tau$ as one end vertex of each edge $\in$ Matching is in vertex cover. For <u>bipartite</u> G, $\mu = \tau$.

* <u>Edge cover</u> (min) (F): $F \subseteq E(G)$, each v incident to an edge in F, Edge cover number ($\rho(G)$). $\rho \geq \lceil n/2 \rceil$

* <u>Stable set/Independent set</u> (max) (I): $I \subseteq V$, no 2 vertices in I are adjacent. Independence no. $\alpha(G)$.

* <u>Gallai's theorem</u>.
$\alpha + \tau = |V| = \mu + \rho$
α - max I, τ - min Q, μ - max M, ρ - min F

* For bipartite graph with no isolated ver's, $\alpha = \rho$, $\mu = \tau$

* $S \subseteq V$, is vertex cover (independent set iff $\bar{S}$ is a)
$\Rightarrow \alpha + \tau = |V|$
$\Rightarrow$ max I is complement of min Q.

* <u>Tutte's theorem</u>: G has a perfect matching iff for every $S \subseteq V$, $odd(G \setminus S) \leq |S|$
* If G is a k-regular bipartite graph with $k > 0$, G has a perfect matching
* <u>Connectivity</u>: <u>k-connected</u> (there doesn't exist k-1 vertices whose removal disconnects G), <u>Vertex connectivity</u> (κ): Largest k s.t. G is k-connected. <u>$\kappa \leq \delta(G)$</u>, <u>Vertex set/cut</u>: <u>Whose removal disconnects G</u>. Size of minimal vertex cut = κ, <u>Edge set/cutset</u>: Set of edges, <u>edge connectivity</u> (κ'): size of smallest edge set; If G is k-connected, it's k-1 connected; $\kappa(K_{n,n}) = n$.

If G is a simple G, $\kappa \le \kappa' \le \delta$. Th| Every G has at least $|V|-|E|$ connected components. <u>Articulation pt./ Cut vertex</u>, <u>Bridge</u>, <u>Cut edge</u>. Th| If $\exists$ exactly 2 vertices of odd degree (x,y), then $\exists\ x \rightsquigarrow y$. Th| In a simple graph, (k components) $n-k \le e \le \frac{(n-k)(n-k+1)}{2}$. $\underbrace{\kappa \le \kappa' \le \delta}$ $\le \frac{2e}{n} \le \Delta$.
↳ with each v we remove at least one e.

Edge cut $([S,\bar{S}])$. • G <u>separable</u> iff 1-connected ($\kappa = 1$).

DM4a

• <u>k-line connected</u> ($\kappa' = k$) • Th| If G has at least 3 vertices, following are equivalent: 1. G 2-connected, 2. G connected & has no cut vertex.

Th| G has at least 3 vertices, then G is 2-connected iff every 2 vertices u & v are contained in a cycle. • <u>Menger's th</u>: If G has at least k+1 vertices then G is k-connected iff b/w every 2 vertices u, v there are k pairwise internally disjoint paths.

* <u>Coloring</u>: Chromatic no. $\chi(G) = \max\{\chi(C);\ C \text{ is conn. component of } G\}$

$\chi \ge \omega$ (clique no.), $\Delta + 1 \ge \chi \ge \frac{|V|}{\alpha}$ (indep. no.) $[\text{for } K_n,\ \chi = \Delta + 1 = n]$.

DM4a

<u>Brook's th|</u> If G is not K_n or C_{2n+1}, then $\chi \le \Delta$. <u>Edge coloring</u>: $\chi'(G) = \Delta$ or $\Delta + 1$ for simple graph. ($=\Delta$ for bipartite G). • Each color class is an independent set. • If G is 2-colorable it is bipartite. • If G is not regular, $\chi \le \Delta$. • Every k-chromatic graph has at least $\binom{k}{2}$ edges. • $\chi_{planar} \le 4$

• <u>Degree seq.</u> of $G = (a,b,c,d,\dots)$ and Δ = max deg., deg. seq of $\bar{G} = (\Delta-a, \Delta-b, \Delta-c\dots)$

• <u>Complement</u> of disconnected G is connected, (not vice versa). [A graph or its complement must be connected). • <u>Self complementary graphs</u> ($G, \bar{G}$ isomorphic, $|E|_G = |E|_{\bar{G}} = \frac{n(n-1)}{4}$, $n = 4p$ or $4p+1$, $p \in I^+$) ↳ always connected.

• $G_1 - G_2$: $V = V(G_1)$, $E = E(G_1) - E(G_2)$ • $G_1 \oplus G_2$: $V = V(G_1) \cup V(G_2)$, $E = E(G_1) \oplus E(G_2)$ • <u>Fundamental cycle</u>: Cycle obtained by adding one edge • # f. cycles in a tree $= {}^nC_2$ • k-ary tree: $h+1 \le n \le (k^{h+1}-1)/(k-1)$, $h \le \#i \le \frac{k^h - 1}{k-1}$, $\#l \le k^h$

• Perfect matching possible only when $|V|$ = even • #perfect matchings in $K_{2n} = \frac{(2n)!}{2^n \cdot n!}$ • #edges in perfect matching $= \frac{|V|}{2}$ • An edge covering is minimal, iff every component of it is a star graph.

• #perfect matchings in $K_{2n+1} = 0$.

* <u>Branch set</u> * <u>Chord set</u>

↳ #edges in the spanning forest

* Enumeration of graphs: DM4b

rank + nullity = $|E|$

* <u>Rank</u>: $n-k$ * <u>Nullity</u>: $|E|$ − rank/#edges to be removed to break all cycles

* <u>Directed Eulerian</u>: $\forall v \in V$, indegree = outdegree

* <u>Traversable/Tracable</u>: If G is unicursal or Eulerian

* <u>Unicursal graph</u>: Only 2 vertices of odd degree, have Eulerian path.

* <u>Ore's th|</u> $n \ge 3$ $\deg_u + \deg_w \ge n$ for each pair of distinct non-adj vertices, then G is <u>Hamiltonian</u>.

* <u>Clique</u>: Complete subgraph. <u>Clique no $\omega(G)$</u>: Size of max clique. • <u>Girth</u>: Smallest cycle.

* Every G with $\delta \ge 2$, has a cycle of length at least $\delta + 1$.