GATE CSE NOTES

by

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* Generating Functions.

eg. $S_1 = \{2, 5, 7\}$ $\quad$ Sum = 12

$S_2 = \{10, 5, 1\}$ $\quad$ #ways = ?

$$(x^2 + x^5 + x^7)(x^{10} + x^5 + x)$$

Coeff. of $x^{12} = 2$.

eg. Distribute k identical objects to n persons.

→ Each person can get $0, 1, \dots$ or k objects.

$S_1 = \{0, 1, \dots, k\}$ $\quad S_2 = \{0, 1, \dots, k\} \dots S_n = \{0, 1, \dots, k\}$.

#ways to get sum k taking one elem from each set.

Answer = coeff. of x^k in $(1 + x + x^2 + \dots + x^k)^n$

(Also solvable by stars & bars method.

5 objects 3 persons $\quad {}^7C_2$)

$$\binom{n+k-1}{k}$$

$* \mid * \ * \mid * \ *$

Choose 2 positions for bars among the $5 + (3-1)$ places.

* Let $(a_0, a_1, a_2, \dots, a_n)$ be symbolic rep^n of a sequence of events or let it simply be a sequence of numbers, then

(discrete numeric f^n).

the function

$f(x) = a_0 \mu_0(x) + a_1 \mu_1(x) + \dots + a_n \mu_n(x)$ is called

<u>ordinary generating f^n</u> of the sequence

$(a_0, a_1, \dots, a_n)$ where $(\mu_0(x), \dots, \mu_n(x))$ is a

sequence of fun^n of x used as indicators.

* Indicator fun^n that provides unique gen. f^n for ~~some~~ any sequence should be used. If for 2 sequences we get same gen. f^n, we can't use that indicator f^n.

eg/ OGF for $\{1, 1, 1, \dots\}$ $\quad 0 < |x| < 1$

$$1 + x + x^2 + x^3 + \dots = \frac{1 \cdot \cancel{x}}{1 - x}$$

/ OGF for $\{1, 1, 3, 1, 1, \dots\}$

$$(1 + x + x^2 + x^3 + \dots) + 2x^2$$

$$= \frac{1}{1-x} + 2x^2 = \frac{1 + 2x^2 - 2x^3}{1 - x}.$$

/ OGF for $\{1, -1, 1, -1, \dots\}$

$$(1 - x + x^2 - x^3 + x^4 - \dots) = \frac{1}{1 + x}$$ common ratio of GP $= -x$

/ OGF for $\{c_0, c_1, c_2, c_3, \dots, c_n\}$ $\quad c_k = \binom{n}{k}$

$$(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) = \sum_{r=0} \binom{n}{r} x^r = (1 + x)^n$$

// OGF for $\{1, 2, 3, 4, \dots\}$

$$1 + 2x + 3x^2 + 4x^3 + \dots = f(x) \quad \text{--- (i)}$$

*

$$x f(x) = x + 2x^2 + 3x^3 + 4x^4 + \dots \quad \text{--- (ii)}$$

(i) − (ii) ⇒

$$f(x) - x f(x) = 1 + x + x^2 + x^3 + \dots$$

$$\Rightarrow f(x)(1-x) = \frac{1}{1-x}$$

$$f(x) = \frac{1}{(1-x)^2}$$

$$\boxed{\begin{array}{c} a_r = r \\ \{0, 1, 2, \dots\} \\ 0 + x + 2x^2 + 3x^3 + \dots \quad \sum r x^r \\ \downarrow \\ \dfrac{x}{(1-x)^2} \end{array}}$$

// OGF for $\{0, 1, 2, \dots\}$.

$$0 + x + 2x^2 + 3x^3 + \dots = f(x).$$

$$f(x) = x\,(1 + 2x + 3x^2 + \dots) = x \cdot \frac{1}{(1-x)^2}$$

// OGF for $\{0^2, 1^2, 2^2, 3^2, \dots\}$

$$x + 2x^2 + 3x^3 + 4x^4 + \dots = \frac{x}{(1-x)^2}$$

* differentiating,

$$1 + 4x + 9x^2 + 16x^3 + \dots = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

$$= \frac{1+x}{(1-x)^3}$$

multiplying x,

$$x + 4x^2 + 9x^3 + \dots = \frac{x(1+x)}{(1-x)^3}$$

eg OGF for $\{0^3, 1^3, 2^3, \dots\}$

$$P(x) = x + 2^3 x^2 + 3^3 x^3 + 4^3 x^4 + \dots$$

Previously

$$x + 2^2 x^2 + 3^2 x^3 + \dots = \frac{x(1+x)}{(1-x)^3}$$

$\frac{d}{dx}$

$$1 + 2^3 x + 3^3 x^2 + \dots = \frac{(1-x)^3(1+2x) + x(1+x)\cdot 3(1-x)^2}{(1-x)^6}$$

$$= \frac{(1-x)(1+2x) + 3x(1+x)}{(1-x)^4}$$

$\times x$

$$x + 2^3 x^2 + 3^3 x^3 + \dots = x\,\frac{1 + 2x - x - 2x^2 + 3x + 3x^2}{(1-x)^4}$$

$$= \frac{x + 4x^2 + x^3}{(1-x)^4}$$

eg OGF for $a_r = Ka^r$

$$f(x) = \sum_{r=0}^{n} a_r x^r = \sum_{r=0}^{n} K a^r x^r \quad \Big| \; n \to \infty$$

$$= K + K ax + K a^2 x^2 + K a^3 x^3 + \dots$$

$$= K\left[1 + ax + (ax)^2 + (ax)^3 + \dots\right]$$

$$= K \cdot \frac{1}{1-ax} \quad ; \quad 0 < |ax| < 1$$

eg $a_r = r$.

$$f(x) = \sum r x^r = x + 2x^2 + 3x^3 + \dots$$

$$= \frac{x}{(1-x)^2}$$

eg $a_r = ba^r$ $\qquad \sum a_r x^r$

$$f(x) = b + bax + ba^2x^2 + ba^3x^3 + \cdots$$

$$= b\left\{\frac{1}{1-ax}\right\}$$

eg $a_r = r b a^r$ $\qquad b\sum_{r=0} r\,(ax)^r = b \cdot \frac{ax}{(1-ax)^2}$

$$f(x) = 0 + bax + 2ba^2x^2 + 3ba^3x^3 + \cdots$$

✓ $$= b\left(\underline{ax + 2(ax)^2 + 3(ax)^3 + \cdots}\right)$$

$$\Downarrow \quad ax = z$$

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$$g(z) = z + 2z^2 + 3z^3 + \cdots$$

$$z\,g(z) = z^2 + 2z^3 + 3z^4 + \cdots$$

$$(1-z)\,g(z) = z + z^2 + z^3 + \cdots = \frac{z}{(1-z)}$$

$$g(z) = \frac{z}{(1-z)^2}$$

$$f(x) = b \cdot \frac{ax}{(1-ax)^2}$$

$$\boxed{a_r = 1/r! \quad \sum_{r=0} \frac{x^r}{r!} = e^x}$$

eg $a_r = 1/r!$ $\qquad f(x) = \sum \frac{x^r}{r!}$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \boxed{e^x}$$

eg $(1, -2, 4, -8, 16, \ldots)$

$$f(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \cdots$$

$$= 1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots$$

$$= \frac{1}{1+2x}$$

eg $\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots\right)$

$$f(x) = 1 + \left(-\frac{1}{2}\right)x + \frac{x^2}{3} + \left(-\frac{x^3}{4}\right) + \frac{x^4}{5} + \dots$$

$$xf(x) = x + \left(-\frac{x^2}{2}\right) + \left(\frac{x^3}{3}\right) + \left(-\frac{x^4}{4}\right) + \left(\frac{x^5}{5}\right) + \dots$$

$$= \log(1+x)$$

$$f(x) = \frac{\log(1+x)}{x}.$$

$$\boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \log(1+x)}$$

eg for Fibonacci Numbers

$(1, 1, 2, 3, 5, 8, \dots)$

** $xf(x) = x\{1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots\}$

$$= x + x^2 + 2x^3 + 3x^4 + 5x^4 + 8x^6 + \dots$$

$$x^2 f(x) = x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + \dots$$

$$xf(x) + x^2 f(x) = x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

$$= f(x) - 1$$

$$\Rightarrow f(x) = \frac{1}{1 - x - x^2}$$

eg $a_r = 2r + 3$ for $r = 0, 1, 2, \dots$

$$f(x) = \sum_{r=0}^{\infty} (2r+3)x^r$$

$$= \sum_{r=0}^{\infty} 2rx^r + 3\sum_{r=0}^{\infty} x^r$$

$$= \frac{2x}{(1-x)^2} + \frac{3}{1-x} = \frac{3-x}{(1-x)^2}$$

Numeric function from Generating funⁿ

eg $f(x) = \dfrac{1}{(1-ax)^K}$

$= \underset{b}{(1} - \underset{a}{ax})^{-K}$

$= (1 + (-ax))^{-K}$

$(a+b)^n = \sum_{r=0}^{\infty} \binom{n}{r} a^r b^{n-r}$

$|b| > |a|$

$f(x) = \sum_{r=0}^{n} a_r x^r$

$= \sum_{r=0}^{\infty} \binom{-K}{r} (-ax)^r = \sum_{r=0}^{\infty} \underbrace{\binom{-K}{r} (-a)^r}_{a_r} x^r.$

Sequence =

$\{ {}^{K-1}C_0, a\,{}^{K}C_1, a^2\,{}^{K+1}C_2, \ldots \}$

$= \sum_{r=0}^{\infty} (-1)^r\, {}^{K+r-1}C_r\, (-a)^r x^r$

$= \sum_{r=0}^{\infty} \underbrace{a^r\, {}^{K+r-1}C_r}\, x^r$

eg $f(z) = (a-bz)^{-1} = a^{-1}(1 - b/a\, z)^{-1}$

$= a^{-1} \sum_{r=0}^{\infty} \binom{-1}{r} \cancel{\tfrac{b}{a}} (-b/a\, z)^r$

$= \frac{1}{a} \sum \left(\frac{b}{a}\right)^r \binom{-1}{r} (-1)^r z^r$

$= \sum_{r=0}^{\infty} \underbrace{\binom{-1}{r} \frac{b^r}{a^{r+1}} (-1)^r}\, z^r$

$\therefore a_r = b^r / a^{r+1}$

eg $f(x) = (1-ax)^{-1}$

$$= \sum_{r=0}^{\infty} \binom{-1}{r} (-ax)^r = \sum_{r=0}^{\infty} \binom{-1}{r} (-a)^r x^r$$

$$a_r = \binom{-1}{r} (-1)^r \cdot a^r$$

$$a_r = a^r \qquad Seq = \{1, a, a^2, a^3, a^4, \dots\}$$

eg $f(x) = (1-x)^{-2}$

$$= \sum_{r=0}^{\infty} \binom{-2}{r} x^r (-1)^r$$

$$a_r = \binom{-2}{r} (-1)^r = \binom{2+r-1}{r} = \binom{r+1}{r}$$

$$= r+1.$$

eg $f(x) = \dfrac{3-5x}{1-2x-3x^2} = \dfrac{5x-3}{3x^2+2x-1}$

$$= \frac{5x-3}{(x+1)(3x-1)} = \frac{2}{x+1} - \frac{1}{3x-1}$$

*

$$= 2(x+1)^{-1} - (3x-1)^{-1}$$

$$= 2(x+1)^{-1} + (1-3x)^{-1}$$

$$= 2\sum_{r=0}^{\infty} \binom{-1}{r} x^r + \sum_{r=0}^{\infty} \binom{-1}{r} (-3x)^r$$

$$= \sum_{r=0}^{\infty} \underbrace{\left\{ 2\binom{-1}{r} + \binom{-1}{r}(-3)^r \right\}} x^r$$

$$a_r = 2(-1)^r + 3^r$$

eg $f(x) = \dfrac{2+3x-6x^2}{1-2x} = \dfrac{6x^2-3x-2}{2x-1}.$

$$= \frac{(2x-1)\cdot 3x - 2}{2x-1} = \cancel{\frac{3x}{2x-1}}\; 3x - \frac{2}{2x-1}$$

$$f(x) = 3x + 2(1-2x)^{-1}$$

$$= 3x + 2\sum_{r=0}^{\infty} \binom{-1}{r} (-2x)^r$$

$$= 3x + 2\sum_{r=0}^{\infty} 2^r x^r$$

$t_r = 2^{r+1}$

$= 3 + 2^{r+1}$, when $x = 1$

Seq $= (2, 7, 8, 16, \dots)$

eg $f(x) = \dfrac{x^4}{1-2x} = x^4 (1-2x)^{-1}$

$$= x^4 \sum_{r=0}^{\infty} \binom{-1}{r} (-2x)^r = x^4 \sum_{r=0}^{\infty} 2^r x^r$$

$$= \sum_{r=0}^{\infty} 2^r x^{4+r}.$$

eg $f(x) = \dfrac{1-x^{n+1}}{1-x}$

$$= (1-x^{n+1})(1-x)^{-1} = (1-x^{n+1}) \sum_{r=0}^{\infty} (-1)^r (-x)^r$$

$$= (1-x^{n+1}) \sum_{r=0}^{\infty} x^r$$

$$= \sum_{r=0}^{\infty} x^r - \sum_{r=0}^{\infty} x^{n+r+1}.$$

$1 \cdot \dfrac{1-x^{n+1}}{1-x}$

GP $\Rightarrow$ $x^0, x^1, \dots, x^n$

Seq $= (1, 1, 1, \dots)$ $\qquad a_r = 1$

※ Application of Gen f^ns to counting problems

eg # sol^ns of $e_1 + e_2 + e_3 = 17$; $2 \le e_1 \le 5$

$3 \le e_2 \le 6$

$4 \le e_3 \le 7$

$(2,3,4,5)$ $(3,4,5,6)$ $(4,5,6,7)$

$\_ + \_ + \_ = 17$

$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)$

$(x^4 + x^5 + x^6 + x^7)$

coeff of x^{17} ?

$$= x^2(1 + x + x^2 + x^3)^3 \cdot x^3 \cdot x^4 .$$

$$= x^9 (1 + x + x^2 + x^3)^3$$

$$= x^9 \left(\frac{1 - x^4}{1 - x}\right)^3$$

$$= x^9 (1 - x^4)^3 \; (1 - x)^{-3}$$

$$= x^9 \sum_{r=0}^{3} \binom{3}{r} (-x^4)^r \cdot \sum_{r=0}^{\infty} \binom{-3}{r} (-x)^r$$

$$= x^9 \sum_{r=0}^{3} \binom{3}{r} (-1)^r x^{4r} \cdot \sum_{r=0}^{\infty} {}^{r+2}C_r \, x^r$$

$$= \underbrace{\sum_{r=0}^{3} \binom{3}{r} (-1)^r x^{4r+9}}_{} \; \underbrace{\sum_{r=0}^{\infty} {}^{r+2}C_r \, (x^r)}_{}$$

$r = 0$	x^9	x^8
$r = 1$	x^{13}	x^4
$r = 2$	x^{17}	x^0

$\# = 1 \times {}^{10}C_2 + 3(-1) \times {}^{6}C_2 + 3$

$= 45 - 45 + 3 = 3.$ (Ans)

eg Solⁿs of $x_1 + x_2 + x_3 + x_4 + x_5 = 15.$

$1 \le x_1 \le 5,\ 1 \le x_2 \le 5,\ x_3 \ge 2,\ x_4 \ge 2,\ x_5 \ge 2$

$(x + x^2 + x^3 + x^4 + x^5)^2 (x^2 + x^3 + x^4 + \cdots)^3$

$= x^2 \left(\frac{1 - x^5}{1 - x}\right)^2 x^6 \left(\frac{1}{1 - x}\right)^3$

$= x^8 (1 - x^5)^2 (1 - x)^{-5}$

$= x^8 (1 + x^{10} - 2x^5) \sum_{r=0}^{\infty} \binom{-5}{r} (-x)^r$

$= (x^8 + x^{18} - 2x^{13}) \sum_{r=0}^{\infty} {}^{r+4}C_4\, x^r \quad \Big|\ x^{15}$

$x^8 \cdot \underline{\underline{x^7}} \rightsquigarrow \cancel{{}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2} = 120}$

${}^{7+4}C_4 = {}^{11}C_4$

x^{18} ✗

$x^{13} \cdot \underline{\underline{x^2}} \rightsquigarrow \cancel{{}^{5}C_3 = \frac{5 \times 4 \times 3}{3 \times 2} = 10}$ ${}^{6}C_2$

$\# = {}^{11}C_4 - 2\,{}^{6}C_2 = 300$ (Ans) eg # ways to distribute 8 identical cookies among 3 distinct children if each child receives at least 2 cookies & no more than 4 cookies.

Combinatorially,
$8 - 3\times 2 = 2$ cookies among 3 children

$|*|$.

$|*|*$ $\binom{2+2}{2} = 6$.

cookie

$\rightarrow$ $a + b + c = 8$

$2 \leq a, b, c \leq 4$.

$(x^2 + x^3 + x^4)^3$ coeff of x^8

$= x^6 (x + 1 + x^2)^3$

$= x^6 \left(\frac{1 - x^3}{1 - x}\right)^3 = x^6 (1 - x^3)^3 (1 - x)^{-3}$

$= x^6 \cdot (1 - 3x^3 + 3x^6 - x^9) \sum_{r=0}^{\infty} \binom{-3}{r} (-x)^r$

$= (x^6 - 3x^9 + 3x^{12} - x^{15}) \sum_{r=0}^{\infty} {}^{r+2}C_2 \, x^r$

only contributing for x^8

$x^6 \cdot x^2$

$\# = {}^4C_2 = 6$.

eg How many ways we can choose a committee of 9 members from 3 political parties such that no party has absolute majority in committee?

$\rightarrow$ $p_1 + p_2 + p_3 = 9$

$p_1, p_2, p_3 \leq 4$

$\Downarrow$ if $p_1 = 0$, $p_2, p_3 > 4$
so,

$p_1, p_2, p_3 \geq 1$

$(x+x^2+x^3+x^4)^3$ coeff of x^9

$= x^3(1+x+x^2+x^3)^3$

$= x^3 \cdot \frac{1-x^4}{1-x} = x^3(1-x^4)^3(1-x)^{-3}$

$= (x^3-x^7)\sum_{r=0}^{\infty}(-1)^r(-x)^r = (x^3-x^7)\sum_{r=0}^{\infty} x^r$

$x^3 \cdot x^6 = x^9 \rightsquigarrow 1$

$x^7 \cdot x^2 = x^9$

$= x^3(1-3x^4+3x^8-x^{12})\sum_{r=0}^{\infty}\binom{-3}{r}(-x)^r$

$= (x^3-3x^7+3x^{11}-x^{15})\sum_{r=0}^{\infty} {}^{r+2}C_2\, x^r$

$x^3 \cdot x^6 = x^9 \rightsquigarrow {}^8C_2 = \frac{8\times 7}{2} = 28$

$x^7 \cdot x^2 = 9 \rightsquigarrow {}^4C_2 = \frac{4\times 3}{2} = 6$

$28 - 3\cdot 6 = 10$ (Ans)

eg #Ways of selecting r objects from n kinds of objects with unlimited repetitions.

→ e_i represents # times ith objects getting selected $(0 \le e_i \le r)$.

So, $e_1+e_2+e_3+\dots+e_n = r$

$(x^0+x^1+x^2+\dots+x^r)^n$

If $r \to \infty$, $(x^0+x^1+x^2+\dots)^n$

$= \left(\frac{1}{1-x}\right)^n$

Also, combinatorially, $n-1$ bars r stars

$$= \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r$$

eg Selecting $\bar{r}$ objects from n distinct (kinds of) objects if we must selects at least 1 of each kind.

$\rightarrow$ $$e_1 + e_2 + \dots + e_n = \bar{r} \qquad 1 \le e_i \le r$$

$$(x + x^2 + x^3 + \dots)^n$$

$$= \left(\frac{x}{1-x}\right)^n = x^n (1-x)^{-n}$$

$$= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r = x^n \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r$$

$$= \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^{n+r} \qquad r+n = \bar{r}, \quad r = \bar{r} - n$$

$$= \sum_{\bar{r}=n}^{\infty} {}^{n+(\bar{r}-n)-1}C_{\bar{r}-n} \, x^{\bar{r}}$$

$$= \sum_{\bar{r}=n}^{\infty} {}^{\bar{r}-1}C_{\bar{r}-n} \, x^{\bar{r}} \qquad \# = {}^{\bar{r}-1}C_{\bar{r}-n}$$

$\hookrightarrow$ Combinatorially,

select $r-n$ objects from n kinds of object with unlimited repetition

$$\binom{n+r-n-1}{r-n} = \binom{r-1}{r-n}.$$

Contd. $\rightarrow$

Contd. ←

Exp^l GF | Integer Partition

eg How many ways we can choose 3 letters when the letters are to be chosen from unlimited supply of a's & b's ?

→ $e_1 + e_2 = 3$

$$(x^0 + x^1 + x^2 + x^3)^2 = \left(\frac{1-x^4}{1-x}\right)^2 = (1 + x^8 - 2x^4) \sum_{r=0}^{\infty} \binom{-2}{r} (-x)^r$$

$$= (1 + x^8 - 2x^4) \sum_{r=0}^{\infty} (1+r) x^r$$

coeff of x^3 = 4.

- OGF - solving selection
- EGF - solving arrangement

※ Exponential Generating Functions

eg # different words of 3 letters when the letters are to be chosen from an unlimited supply of a's & b's.

word ~ arrangement

aaa - 1

aab - $\frac{3!}{2!1!} = 3$

abb - $\frac{3!}{2!1!} = 3$

bbb - 1

8 ways

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)^2$$

coeff x^3 $= \left(\frac{x^0 x^3}{0!\,3!} + \frac{x^1 x^2}{1!\,2!} + \frac{x^2 x^1}{2!\,1!} + \frac{x^3 x^0}{3!\,0!}\right)$

$$= \underbrace{\left(\frac{1}{0!\,3!} + \frac{1}{1!\,2!} + \frac{1}{2!\,1!} + \frac{1}{3!\,0!}\right)}_{\left(A \times 3!\right)} x^3$$

Exp. GF

$$A_0 x^0 + A_1 x^1 + A_2 x^2 + \dots$$

$$= \sum_{r=0} A_r x^r = \sum_{r=0} (r!\,A_r) \frac{x^r}{r!} = \sum_{r=0} a_r \frac{x^r}{r!}$$

$$\boxed{f(x) = \sum_{r=0}^{n} a_r \frac{x^r}{r!}}$$

While finding coeff. we find a_r.

$$f(x) = a_0 \frac{x^0}{0!} + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots + a_n \frac{x^n}{n!}$$

eg Find EGF for a_r, where a_r is the # of arrangements without repetition of n objects.

$$\rightarrow \quad a_r = {}^nP_r = \frac{n!}{(n-r)!}$$

$$e_1 + e_2 + \dots + e_n = r \qquad e_i \in \{0,1\}$$

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!}\right)^n = (1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

$$= \sum_{r=0}^{n} \underbrace{\frac{n!}{(n-r)!}}_{\Downarrow\ {}^nP_r} \frac{x^r}{r!}$$

eg $a_r \sim$ # arrangements of r objects from 4 different types of objects with each type of object appearing at least 2 & no more than 5 times. $a_1 + a_2 + a_3 + a_4 = r$

$$\left(e^2/2! + e^3/3! + e^4/4! + e^5/5!\right)^4 \sim \text{coeff of } e^r$$

Ans

eg $a_r \sim$ # ways to place r distinct people into 3 rooms with at least one person in each room.

$$l_1 + l_2 + l_3 = r \qquad l_i \geq 1$$

a | b | cd
b | a | cd — different

$$\left(\frac{e}{1!} + \frac{e^2}{2!} + \frac{e^3}{3!} + \frac{e^4}{4!} + \cdots\right)^3 \rightarrow \text{coeff. of } e^r$$

$\Rightarrow$ If we want even # people in each room,

$$\left(\frac{e^2}{2!} + \frac{e^4}{4!} + \frac{e^6}{6!} + \cdots\right)^3$$

* Only a few EGF's coeff. can be easily evaluated.

eg $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{nx} = 1 + nx + \frac{n^2x^2}{2!} + \frac{n^3x^3}{3!} + \cdots$

$= \sum_{r=0}^{n} \frac{n^r x^r}{r!}$

eg Even powers.

$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(using expansion of e^x, e^{-x})

odd powers

$\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

eg # r arrangements chosen from unlimited supply of n objects.

$\underbrace{\underline{n}\ \underline{n}\ \cdots\ \underline{n}\ \underline{n}}_{r} = n^r.$

→ Using EGF,

$e_1 + e_2 + \cdots + e_n = r$

$0 \le e_i < \infty$

$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^n = e^{nx} = \sum_{r=0}^{n} n^r \frac{x^r}{r!}$

eg # r-digit quarternery sequences (0,1,2,3) with an even number of 0's & odd # 1's.

$$\rightarrow \quad \overset{0}{e_1} + \overset{1}{e_2} + \overset{2}{e_3} + \overset{3}{e_4} = r \qquad e_1 \in \{0,2,4,\dots\}$$

$$e_2 \in \{1,3,5,\dots\}$$

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(x+1+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^2$$

$$= \frac{1}{2}(e^x+e^{-x})\,\frac{1}{2}(e^x-e^{-x}) \cdot e^{2x}.$$

$$= \frac{1}{4}(e^{2x}-e^{-2x})\,e^{2x} = \frac{1}{4}(e^{4x}-1)$$

$$= \frac{1}{4}\left(\sum_{r=0}^{\infty} 4^r \frac{x^r}{r!} - 1\right)$$

$$\# = 4^{r-1}$$

eg # ways to place 25 distinct people into 3 rooms with at least one person each room.

$$\rightarrow \quad e_1 + e_2 + e_3 = 25 \qquad e_i \geq 1$$

$$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3 = (e^x-1)^3$$

$$= e^{3x} - 3e^{2x} + 3e^x - 1$$

$$= \sum 3^r \frac{x!}{r!} - 3\sum 2^r \frac{x!}{r!} + 3 \cdot \sum \frac{x^r}{r!} - 1$$

$$\# = 3^r - 3 \cdot 2^r + 3 \qquad (\text{Ans}) = 3^{25} - 3 \cdot 2^{25} + 3$$

* <u>Partition of Integers.</u>

Partition of the integer n is a multiset of +ve integers that sum to n.

eg. $p_5 = 7$ as $5 = 5, 41, 32, 311, 221, 2111, 11111$

$\Rightarrow$ A partition is uniquely defined by the # 1's, # 2's & so on, ie, by the repetition numbers of the multiset.

$$(1+x+x^2+x^3+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\cdots(1+x^k+x^{2k}+\dots)$$

if 1 occurs 3 times in p_n

if 3 occurs once in p_n

if k occurs 2 times in p_n

$$= \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} x^{ik}$$

When the product is expanded, we pick one term from each factor in all possible ways, with the further condition that we only pick a finite # of non-ve terms.

Now, the k^{th} factor is in GP, so it sums to $\frac{1}{1-x^k}$

So the generating fn is $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$

- <u>Note</u>: If we're interested in some p_n, we don't need the entire infinite product, or even any complete factor, since no partition of n can use any integer $> n$ & also can't use more than n/k copies of k.

eg Find p_8. $n = 8$ | Each factor have $(\lfloor n/k \rfloor + 1)$ terms, $k \in \mathbb{N}$

$$(1+x+x^2+\dots+x^8)(1+x^2+x^4+x^6+x^8)(1+x^3+x^6)(1+x^4+x^8)$$
$$(1+x^5)(1+x^6)(1+x^7)(1+x^8)$$

coeff of $x^8 = 22$.

$$= \frac{1-x^9}{1-x} \cdot \frac{1-x^{10}}{1-x^2} \cdot \dots$$

- For every n, $p_d(n) - p_o(n)$

$p_d(n)$: # partitions of n into distinct parts

$6 = 6, 51, 42, 321$

$p_o(n)$: # partitions into odd parts

$6 = 5+1, 33, 3111, 111111$.