8. 급수 : 수열의 합

멱급수

1. 거듭제곱 급수

2. 다항식을 일반화한 형태의 급수

→ 형태 : $a_0 + a_1x + a_2x^2 \cdots$

3. 등비급수 : $a + ar + ar^2 \cdots \quad (a \neq 0)$

(1) $\sum_{n=0}^{\infty} ar^n \Rightarrow \frac{a}{1-r}$로 수렴, 수렴조건은 $-1 < r < 1$

(2) $a=1$이고 $r=x$인 등비급수 : $1 + x + x^2 \cdots$

→ $\frac{1}{1-x} \ (-1 < x < 1) = \sum_{n=0}^{\infty} x^n$

4. 미분방정식 $xy'' + y = 0$ (이차미분방정식)

$y = a_0 + a_1x + a_2x^2 \cdots$

$y' = a_1 + 2a_2x + 3a_3x^2 \cdots$

$y'' = 2a_2 + (2\cdot3)a_3x + (3\cdot4)a_4x^2 \cdots$

→ $x(2a_2 + 6a_3x + 12a_4x^2 \cdots) + (a_0 + a_1x + a_2x^2 \cdots)$

$= a_0 + (2a_2 + a_1)x + (6a_3 + a_2)x^2 + (12a_4 + a_3)x^3 \cdots = 0$

$a_0 = 0,\ a_2 = -\frac{1}{2}a_1,\ a_3 = \frac{1}{3\cdot2\cdot2}a_1,\ a_4 = -\frac{1}{4\cdot3\cdot3\cdot2}a_1$

$\therefore a_n = (-1)^{n+1}\frac{1}{n!(n-1)!}a_1,\quad y = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{a_1}{n!(n-1)!}x^n$

이항급수

1. 이항 계수를 계수로 하는 멱급수 ≡ 맥클로린 급수 $(a+b)^p$: 항이 두개인 급수

→ $f(x) = (1+x)^p \quad f(0) = 1$

$f'(x) = p(1+x)^{p-1} \quad f'(0) = p$

⋮

$f^{(n)}(0) = p(p-1)(p-2)\cdots(p-(n-1))$

→ $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

$= 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}x^n$

$= (1+x)^p$

※ $f(x) = a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2 + a_3(x-\alpha)^3 \cdots$

$x = \alpha,\ f(\alpha) = a_0$

$f'(x) = a_1 + 2a_2(x-\alpha) + 3a_3(x-\alpha)^2 \cdots$

$x = \alpha,\ f'(\alpha) = a_1$

$f''(x) = 2a_2 + (2\cdot3)a_3(x-\alpha) + \cdots$

$x = \alpha,\ f''(\alpha) = 2a_2$

Taylor, Maclaurin 급수

sin, cos, $\ln x$, e^x, e^{-x}
taylor 기말

$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \frac{f'''(\alpha)}{3!}(x-\alpha)^3 \cdots$ 일 때,

Taylor : $\sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$

Maclaurin : Taylor에서 $\alpha = 0$일 때. = 이항급수.

$\overset{0}{\underset{0}{\sin x}}\quad \overset{1}{\underset{1}{\cos x}}\quad \overset{0}{\underset{2!}{-\sin x}}\quad \overset{-1}{\underset{3!}{-\cos x}}\quad \overset{0}{\underset{4!}{\sin x}}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \cdots$

$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\sin\frac{\pi}{180} = \frac{\pi}{180} - \frac{\left(\frac{\pi}{180}\right)^3}{3!} + \frac{1}{5!}\left(\frac{\pi}{180}\right)^5 + \frac{1}{7!}\left(\frac{\pi}{180}\right)^7$

$f(x) = \ln x \quad f(2) = \ln 2$

$f'(x) = x^{-1} \quad f'(2) = 2^{-1} = \frac{1}{2}$

$f''(x) = -x^{-2} \quad f''(2) = -2^{-2} = -\frac{1}{4}$

$f'''(x) = 2x^{-3} \quad f'''(2) = 2\times2^{-3} = 2^{-2} = \frac{1}{4} \qquad 6\times\frac{1}{16}$

$f''''(x) = -6x^{-4} \quad f''''(2) = -6\times2^{-4} = -\frac{3}{8}$

$f(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4\times2!}(x-2)^2 + \frac{1}{4\times3!}(x-2)^3 - \frac{1}{8\times4!}(x-2)^4 + \frac{1}{8\times5!}(x-2)^5 \cdots$

$f(x) = e^x \quad f(0) = f'(0) = f''(0) - \cdots f^{(n)}(0) = 1$

$f'(x) = e^x \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f''(x) = e^x \quad = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \cdots$

$\approx 2.708 \cdots$

$f(x)=e^{-x}$ $f(0)=1$ $e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\cdots$

$f'(x)=-e^{-x}$ $f'(0)=-1$ $=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$

$f''(x)=e^{-x}$ $f''(0)=1$