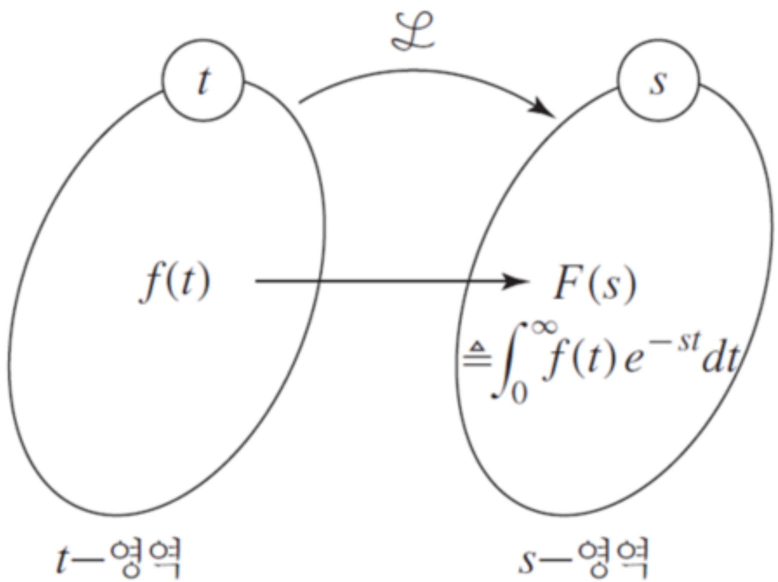
6장 Laplace 변환

- 선형상미분방정식
- 연립 선형상미분방정식
- 회로이론, 제어공학, 신호처리, 물리, 수학 등등
- Fourier 변환, Z 변환

→ 매우 쉽게 풀이

t공간에서는 매우 풀기 어려운 식을 s공간에서는 단순한 연산으로 구할 수 있다

6.1 Laplace 변환의 정의



Laplace 변환의 정의

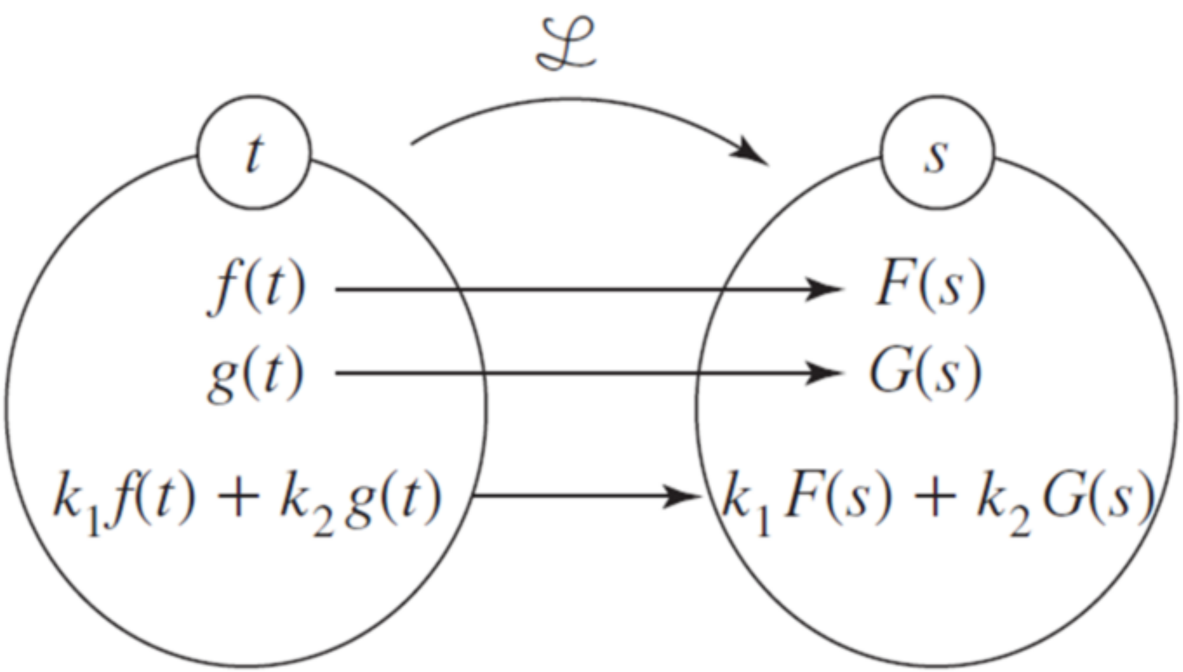
$$\mathcal{L}\{f(t)\} \triangleq F(s) \longleftarrow f(t)\text{의 Laplace 변환}$$

$$= \int_0^{\infty} f(t)e^{-st}\,dt$$

$\longrightarrow$ Laplace 변환은 시간영역의 함수 $f(t)$를 s영역의 함수 $F(s)$에 대응하는 변환

Laplace 변환의 선형성

$$\mathscr{L}\{f(t)\} = F(s) \qquad \mathscr{L}\{g(t)\} = G(s) \xrightarrow{\text{선형성}} \begin{aligned} &\mathscr{L}\{k_1 f(t) + k_2 g(t)\} \\ &= k_1 \mathscr{L}\{f(t)\} + k_2 \mathscr{L}\{g(t)\} \\ &= k_1 F(s) + k_2 G(s) \end{aligned}$$



Ex1) LT 구하시오.

$f(t) = 2 \quad , \quad t \geq 0$

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt = \int_0^\infty 2\, e^{-st} dt$$

$$= -\frac{2}{s} e^{-st} \Big|_0^\infty = \frac{2}{s} \quad , \quad s > 0$$

Ex) LT

$f(t) = a \quad , \quad t \geq 0$

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt = \int_0^\infty a\, e^{-st} dt$$

$$= -\frac{a}{s} e^{-at} \Big|_0^\infty = -\frac{a}{s}\left(e^{-\infty} - e^{0}\right)$$

$$= \frac{a}{s}$$

Ex 2) $g(t) = e^{-t} \quad t \geq 0$

$$G(s) = \int_0^\infty g(t) e^{-st} dt = \int_0^\infty e^{-t} e^{-st} dt$$

$$= \int_0^\infty e^{-(s+1)t} dt = -\frac{1}{s+1} e^{-(s+1)t} \Big|_0^\infty$$

$$= \frac{1}{s+1}, \quad s+1 > 0$$

만약 $g(t) = e^{at}$, a ; 상수

$$\rightarrow \frac{1}{s-a}, \quad s-a > 0$$

Ex) 부분적분

$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

└> 양변적분

$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$

t에 대한 LT.

$F(s) = \int_0^\infty t e^{-st} dt$ (부분적분)

$= t(-\frac{1}{s}e^{-st})\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}dt$

$= \frac{1}{s}\int_0^\infty e^{-st}dt$

$= \frac{1}{s}(-\frac{1}{s}e^{-st})\Big|_0^\infty$

$= \frac{1}{s^2}, \quad s > 0$

Ex) t^2 에 대한 LT.

$$F(s) = \int_0^\infty t^2 e^{-st} dt \quad (\text{부분 적분})$$

$$= t^2 \left(-\frac{1}{s} e^{-st}\right)\Big|_0^\infty - \int_0^\infty 2t \left(-\frac{1}{s} e^{-st}\right) dt$$

$$= \frac{2}{s} \underline{\int_0^\infty t e^{-st} dt} \quad (\text{부분 적분})$$

$$= \frac{2}{s} \cdot \frac{1}{s^2}$$

$$= \frac{2}{s^3} \quad s > 0$$

Ex) t^n 에 대한 LT

$$F(s) = \frac{n!}{s^{n+1}} \quad s > 0$$

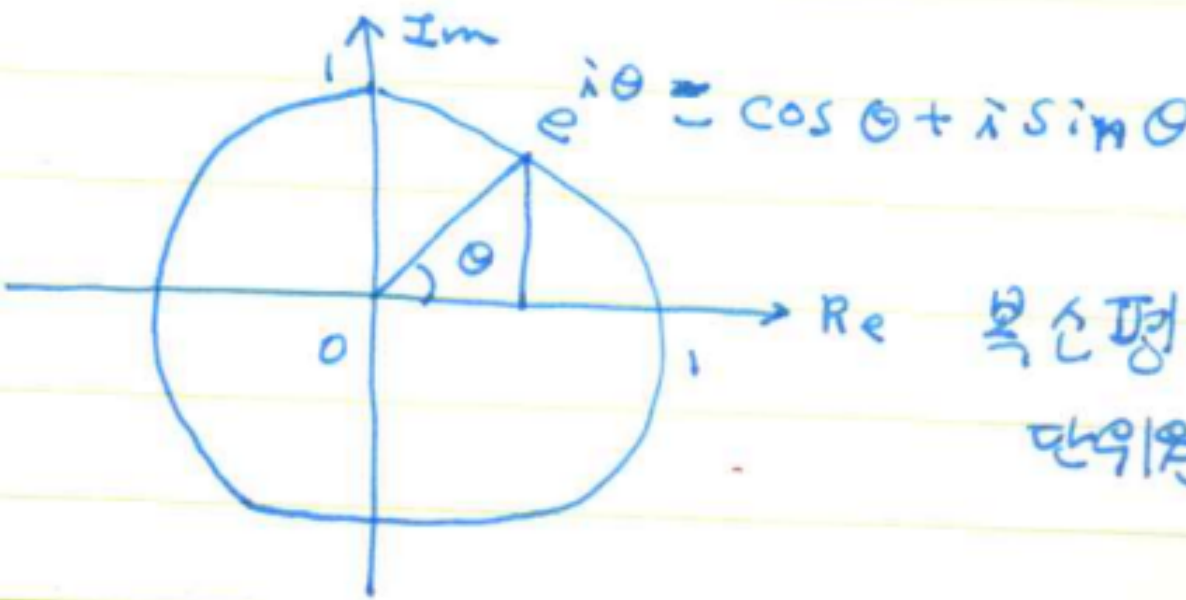
오일러 공식

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{+ix} - e^{-ix}\right)$$



복소평면에서 단위원.

Date

Ex) $\cos \omega t = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$

$$F(s) = \int_0^{\infty} \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right] e^{-st} dt$$

$$= \frac{1}{2} \int_0^{\infty} e^{i\omega t} e^{-st} dt + \frac{1}{2} \int_0^{\infty} e^{-i\omega t} e^{-st} dt$$

$$= \frac{1}{2} \frac{1}{s - i\omega} + \frac{1}{2} \frac{1}{s + i\omega}$$

$$= \frac{s}{s^2 + \omega^2}$$

Ex) $\sin \omega t = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}$

$$F(s) = \int_0^{\infty} \left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right] e^{-st}\, dt$$

$$= \frac{1}{2i} \left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega} \right)$$

$$= \frac{1}{2i} \; \frac{(s + i\omega) - (s - i\omega)}{(s - i\omega)(s + i\omega)}$$

$$= \frac{\omega}{s^2 + \omega^2}$$

<예제> $f(t) = \sinh(at)$

$$\sinh(at) = \frac{1}{2}e^{at} - \frac{1}{2}e^{-at}$$

선형성에 의해 $\mathscr{L}\{\sinh(at)\} = \mathscr{L}\{\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}\}$

$$= \frac{1}{2}\mathscr{L}\{e^{at}\} - \frac{1}{2}\mathscr{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2}$$

기본함수에 대한 Laplace 변환

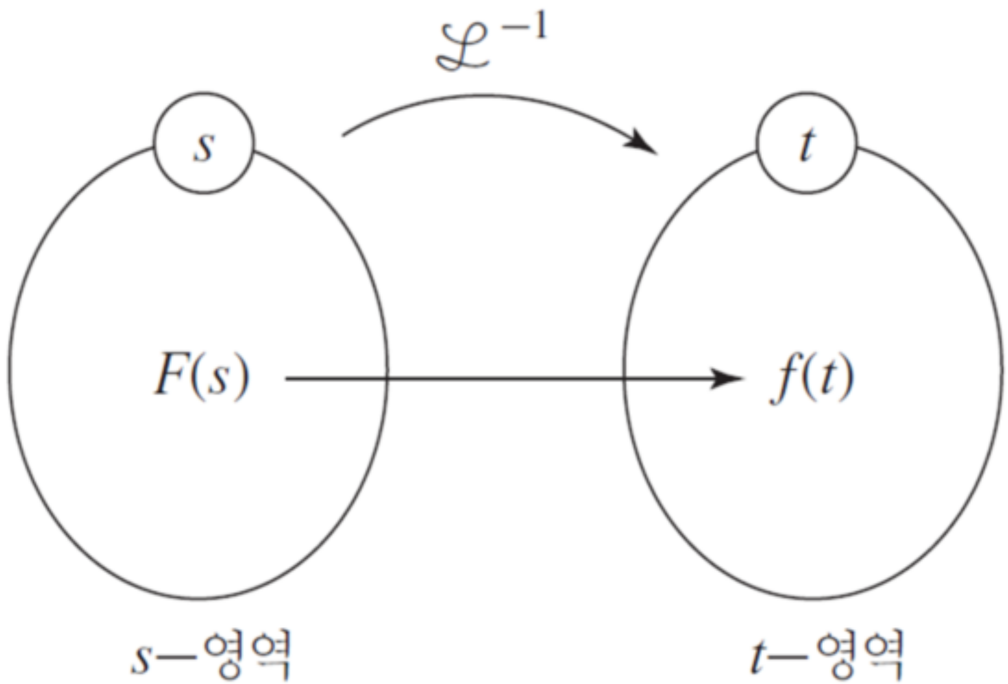
$f(t)$		$\mathscr{L}\{f(t)\}=F(s)$	
1	t	$\frac{1}{s}$	$\frac{1}{s^2}$
t^2	t^3	$\frac{2!}{s^3}$	$\frac{3!}{s^4}$
t^n		$\frac{n!}{s^{n+1}}$	
e^{at}, a는 상수		$\frac{1}{s-a}$	
$\cos\omega t$	$\sin\omega t$	$\frac{S}{s^2+\omega^2}$	$\frac{\omega}{s^2+\omega^2}$
$\cosh\omega t$	$\sinh\omega t$	$\frac{S}{s^2-\omega^2}$	$\frac{\omega}{s^2-\omega^2}$

Laplace 역변환과 부분분수

(1) Laplace 역변환

s-영역에서 어떤 함수 $F(s)$가 주어져 있을 때 t-영역에서 이에 유일하게 $F(s)$에 대응되는 함수 $f(t)$를 찾는 것을 Laplace 역변환으로 정의

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \triangleq \frac{1}{2\pi i}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$



<예제>

$$\mathscr{L}\{e^{-t}\} = \frac{1}{s+1} \quad \longleftrightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$\mathscr{L}\{\cos 3t\} = \frac{s}{s^2+9} \quad \longleftrightarrow \quad \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t$$

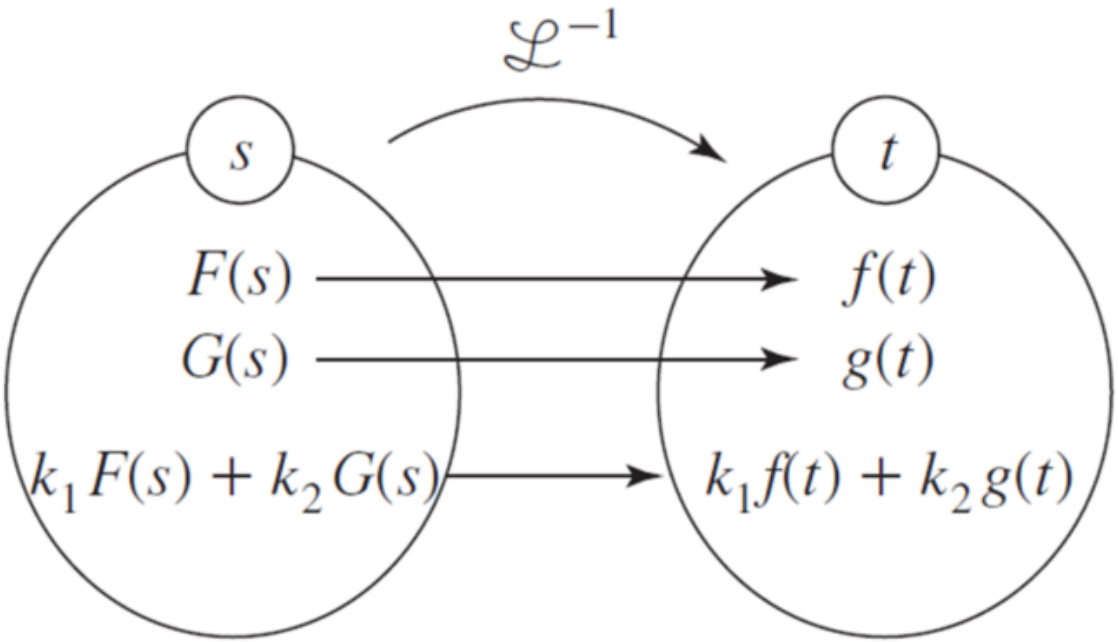
Laplace 역변환의 선형성

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

$$g(t) = \mathscr{L}^{-1}\{G(s)\}$$

$$\xrightarrow{\text{선형성}} \mathscr{L}^{-1}\{k_1F(s) + k_2G(s)\}$$

$$= k_1\mathscr{L}^{-1}\{F(s)\} + k_2\mathscr{L}^{-1}\{G(s)\}$$



Laplace 역변환의 선형성

<예제>

$$F(s)=\frac{3}{s+1}+\frac{4}{s-1}$$

$$\mathscr{L}^{-1}\{F(s)\}=\mathscr{L}^{-1}\left\{\frac{3}{s+1}+\frac{4}{s-1}\right\}=3\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}+4\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$=3e^{-t}+4e^{t}$$

(2) 부분분수 전개

$$F(s)=\frac{s^2+4}{(s+1)(s+2)(s+3)}$$ ← Laplace 역변환을 구하기 어렵다.

$$=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}$$ ← 부분분수 전개
Laplace 역변환을 구하기 쉽다.

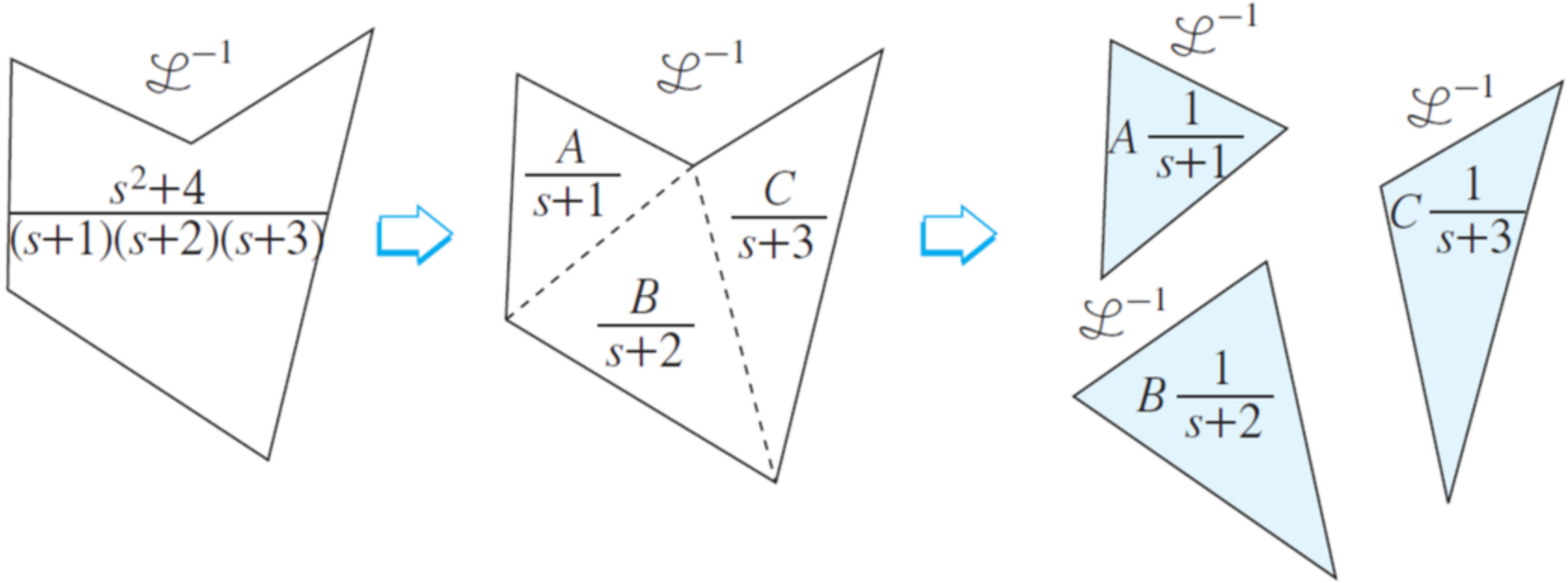
$$=\frac{\frac{5}{2}}{s+1}+\frac{-8}{s+2}+\frac{\frac{13}{2}}{s+3}$$

$$\mathscr{L}^{-1}\left\{\frac{s^2+4}{(s+1)(s+2)(s+3)}\right\}=\mathscr{L}^{-1}\left\{\frac{\frac{5}{2}}{s+1}+\frac{-8}{s+2}+\frac{\frac{13}{2}}{s+3}\right\}$$

$$=\frac{5}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}-8\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{13}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$=\frac{5}{2}e^{-t}-8e^{-2t}+\frac{13}{2}e^{-3t}$$

부분분수 전개법은 복잡한 형태로 되어있는 $F(s)$를 간단한 함수 형태로 적절히 조각을 냄으로써 Laplace 역변환을 구하는 매우 편리한 방법이다.



부분분수 분해에 의한 Laplace 역변환

	$F(s) = \mathcal{L}\{f(t)\}$		$f(t)$
1	$1/s$		1
2	$1/s^2$		t
3	$1/s^n$	$(n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$
4	$1/\sqrt{s}$		$1/\sqrt{\pi t}$
5	$1/s^{3/2}$		$2\sqrt{t/\pi}$
6	$1/s^a$	$(a > 0)$	$t^{a-1}/\Gamma(a)$
7	$\dfrac{1}{s-a}$		e^{at}
8	$\dfrac{1}{(s-a)^2}$		te^{at}
9	$\dfrac{1}{(s-a)^n}$	$(n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!}t^{n-1}e^{at}$
10	$\dfrac{1}{(s-a)^k}$	$(k > 0)$	$\dfrac{1}{\Gamma(k)}t^{k-1}e^{at}$
11	$\dfrac{1}{(s-a)(s-b)}$	$(a \neq b)$	$\dfrac{1}{a-b}(e^{at} - e^{bt})$
12	$\dfrac{s}{(s-a)(s-b)}$	$(a \neq b)$	$\dfrac{1}{a-b}(ae^{at} - be^{bt})$
13	$\dfrac{1}{s^2+\omega^2}$		$\dfrac{1}{\omega}\sin \omega t$
14	$\dfrac{s}{s^2+\omega^2}$		$\cos \omega t$
15	$\dfrac{1}{s^2-a^2}$		$\dfrac{1}{a}\sinh at$
16	$\dfrac{s}{s^2-a^2}$		$\cosh at$
17	$\dfrac{1}{(s-a)^2+\omega^2}$		$\dfrac{1}{\omega}e^{at}\sinh \omega t$
18	$\dfrac{s-a}{(s-a)^2+\omega^2}$		$e^{at}\cos \omega t$
19	$\dfrac{1}{s(s^2+\omega^2)}$		$\dfrac{1}{\omega^2}(1 - \cos \omega t)$
20	$\dfrac{1}{s^2(s^2+\omega^2)}$		$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$