# 6장  Laplace 변환

# 6.2 미분과 적분의 Laplace 변환

## (1) 도함수의 Laplace 변환

$$\mathscr{L}\{f(t)\} = F(s) \longrightarrow \underline{\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)}$$

$$\because \quad \mathscr{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

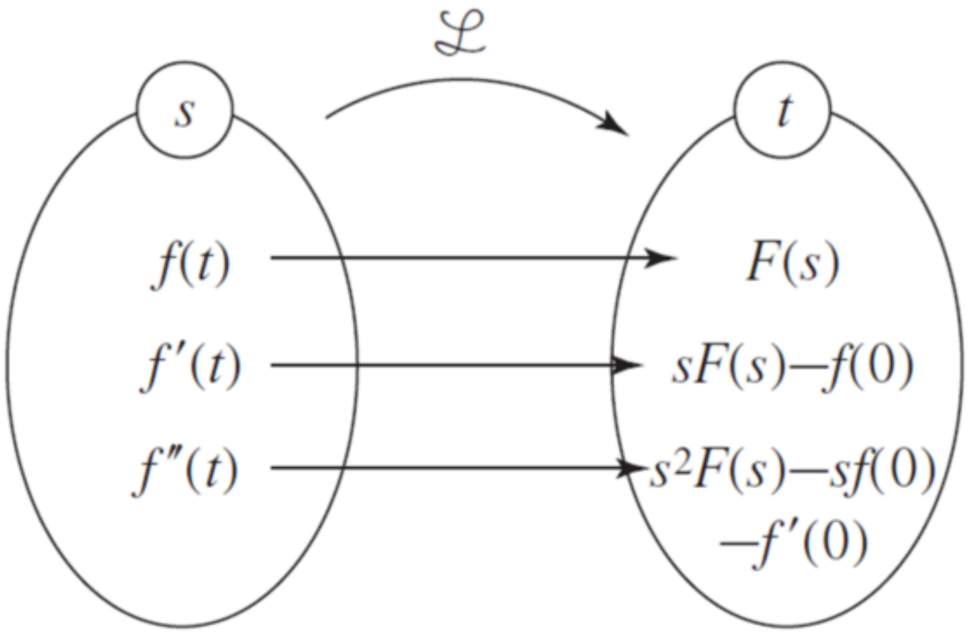
$$= -f(0) + s\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{f(t)\text{의 Laplace 변환}}$$

$$= -f(0) + s\mathscr{L}\{f(t)\} = sF(s) - f(0)$$

$$\mathscr{L}\{f''(t) = \mathscr{L}\{(f'(t))'\}$$
$$= -f'(0) + s\mathscr{L}\{f'(t)\} = -f'(0) + s\{-f(0) + s\mathscr{L}\{f(t)\}\}$$
$$= \underline{s^2F(s) - sf(0) - f'(0)}$$

$n$차 도함수에 대한 Laplace 변환

$$\mathscr{L}\{f^{(n)}(t)\} = \underline{s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \cdots -f^{(n-1)}(0)}$$




도함수의 Laplace 변환

시간영역에서 $f(t)$미분하는 것은 $s$-영역에서 $F(s)$에 $s$를 곱하여 초기값을 빼주는 것과 같다.

$\rightarrow$ 초기치 문제에 Laplace 변환 활용 가능

<예제>

$f(t) = 4\sin^2 t$ 의 Laplace 변환

$$f'(t) = 8\sin t\,\cos t = 4(2\sin t\cos t) = 4\sin 2t$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) = sF(s)$$

$$\frac{8}{s^2+4} \qquad \therefore F(s) = \frac{8}{s(s^2+4)}$$

$f(t)=4\sin^2 t$, $f(0)=0$

$f'(t) = 8\sin t\cos t = 4(2\sin\cos) = 4\sin 2t$

$\mathcal{L}(f'(t)) = \frac{2}{s^2+4}\times 4 = \frac{8}{s^2+4} = sF(s) - f(0)$

$F(s) = \frac{8}{s(s^2+4)}$

## (2) 적분의 Laplace 변환

$$\mathscr{L}\{f(t)\} = F(s) \longrightarrow \mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathscr{L}\{f(t)\} = \frac{1}{s}F(s)$$

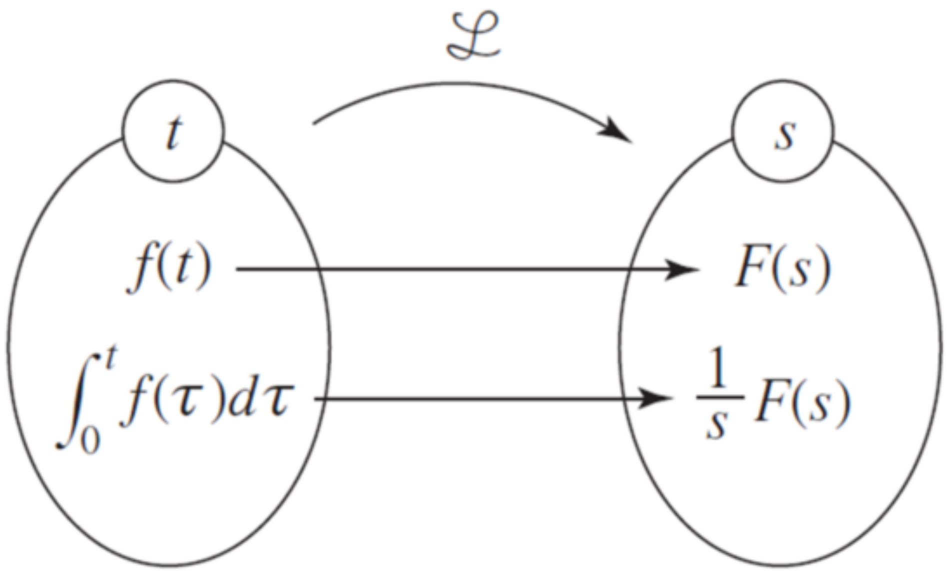
$$\because \quad g(t) \triangleq \int_0^t f(\tau)\,d\tau$$

$$g'(t) = \frac{d}{dt}\left\{\int_0^t f(\tau)\,d\tau\right\} = f(t)$$

$$\underbrace{\mathscr{L}\{g'(t)\}}_{F(s)} = s\mathscr{L}\{g(t)\} - \underbrace{g(0)}_{0} \longrightarrow \therefore \mathscr{L}\{g(t)\} = \mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

$t$영역에서 $f(t)$ 적분하는 것은 $s$-영역에서 에 을 $F(s)$는 $\frac{1}{s}$에 대응한다.

$$\mathscr{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau \longleftarrow \text{Laplace 역변환 정의}$$




적분의 Laplace 변환

<예제> $F(s) = \frac{2}{s(s^2+1)}$의 Laplace 역변환

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\underbrace{\frac{2}{(s^2+1)}}_{\mathcal{L}(2\sin t)}\right\}$$

$$= \int_0^t 2\sin\tau d\tau = -2\cos\tau\Big|_0^t = -2\cos t + 2$$

# Laplace 변환의 미분과 적분

## (1) Laplace 변환의 미분

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$\frac{dF(s)}{ds} = F'(s) = \frac{d}{ds}\left\{\int_0^\infty f(t)\,e^{-st}\,dt\right\} = \int_0^\infty \frac{\partial}{\partial s}\{f(t)\,e^{-st}\}\,dt$$

$$= \int_0^\infty -tf(t)\,e^{-st}\,dt$$

$-tf(t)$의 Laplace 변환

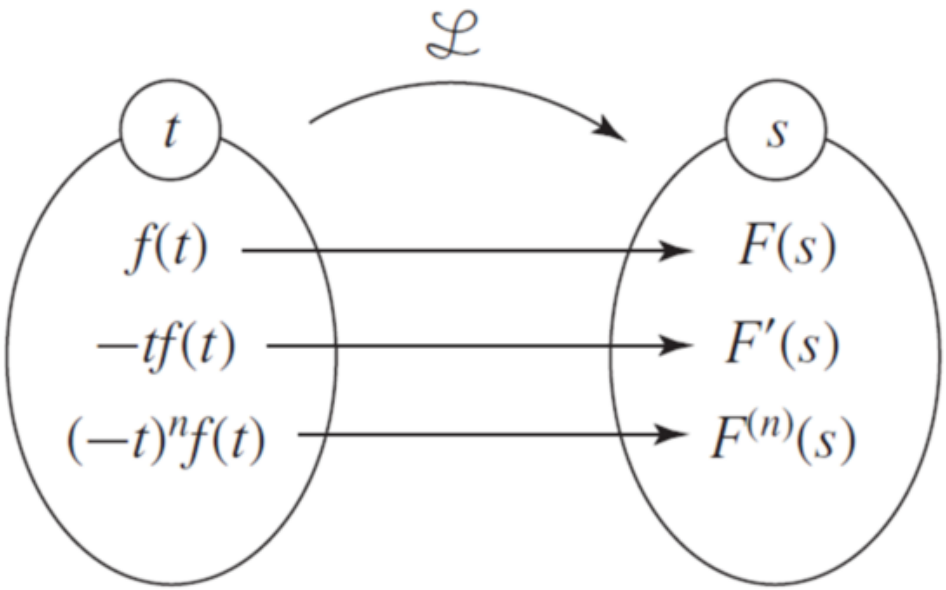
$$\therefore\ F'(s) = \mathscr{L}\{-tf(t)\}$$

$$\frac{d^2 F(s)}{ds^2} = F''(s) = \frac{d^2}{ds^2}\left\{\int_0^\infty f(t)e^{-st}\,dt\right\} = \int_0^\infty \frac{\partial^2}{\partial s^2}\left\{\int_0^\infty f(t)e^{-st}\right\}dt$$

$$= \int_0^\infty (-t)^2 f(t)e^{-st}\,dt = \mathscr{L}\left\{t^2 f(t)\right\}$$

$$\therefore\ F''(s) = \mathscr{L}\{(-t)^2 f(t)\}$$

$$F^{(n)}(s) = \mathscr{L}\{(-t)^n f(t)\}$$




라플라스 변환의 미분

<예제>

$f(t) = te^{-3t}$의 Laplace 변환

$$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$$

$$\mathcal{L}\{te^{-3t}\} = -\mathcal{L}\{\underbrace{-te^{-3t}}_{\{-tf(t)\}}\} = -\frac{d}{ds}\left(\frac{1}{s+3}\right) = f'(s)$$

$$= \frac{1}{(s+3)^2}$$

## (2) Laplace 변환의 적분

$$\mathcal{L}\{f(t)\} = F(s) \triangleq \int_0^{\infty} f(t)\, e^{-st}\, dt$$

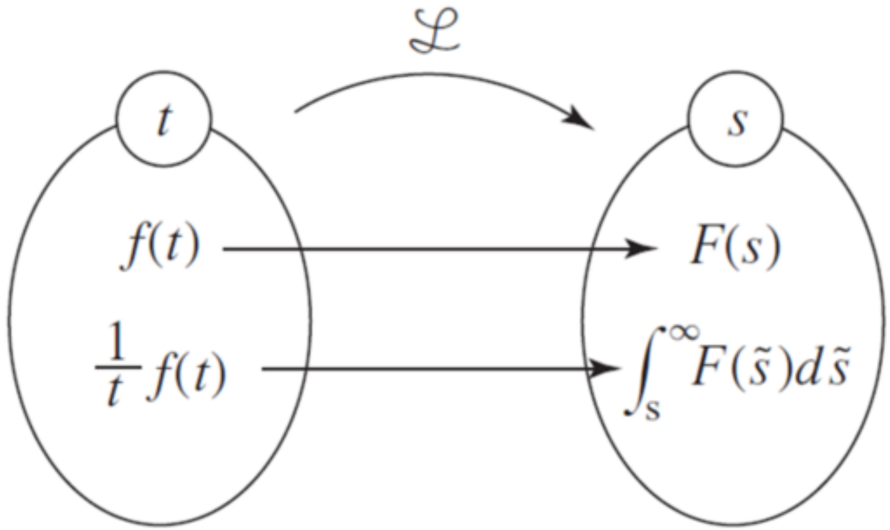
$$\int_s^{\infty} F(\tilde{s})\, d\tilde{s} = \int_s^{\infty} \left\{ \int_s^{\infty} f(t)\, e^{-\tilde{s}t}\, dt \right\} d\tilde{s}$$

$$= \int_0^{\infty} f(t) \left\{ \int_s^{\infty} e^{-\tilde{s}t}\, d\tilde{s} \right\} dt = \int_0^{\infty} f(t) \left[ -\frac{1}{t} e^{-\tilde{s}t} \right]_{\tilde{s}=s}^{\tilde{s}=\infty} dt$$

$$= \int_0^{\infty} f(t) \frac{1}{t} e^{-st}\, dt = \mathcal{L}\left\{ \frac{1}{t} f(t) \right\}$$

$\frac{1}{t} f(t)$의 Laplace 변환

$t$영역에서 $f(t)$을 $\frac{1}{t}$하는 것은 $s$영역에서 를 적분 $F(s)$ 것에 대응된다.




라플라스 변환의 적분

<예제>

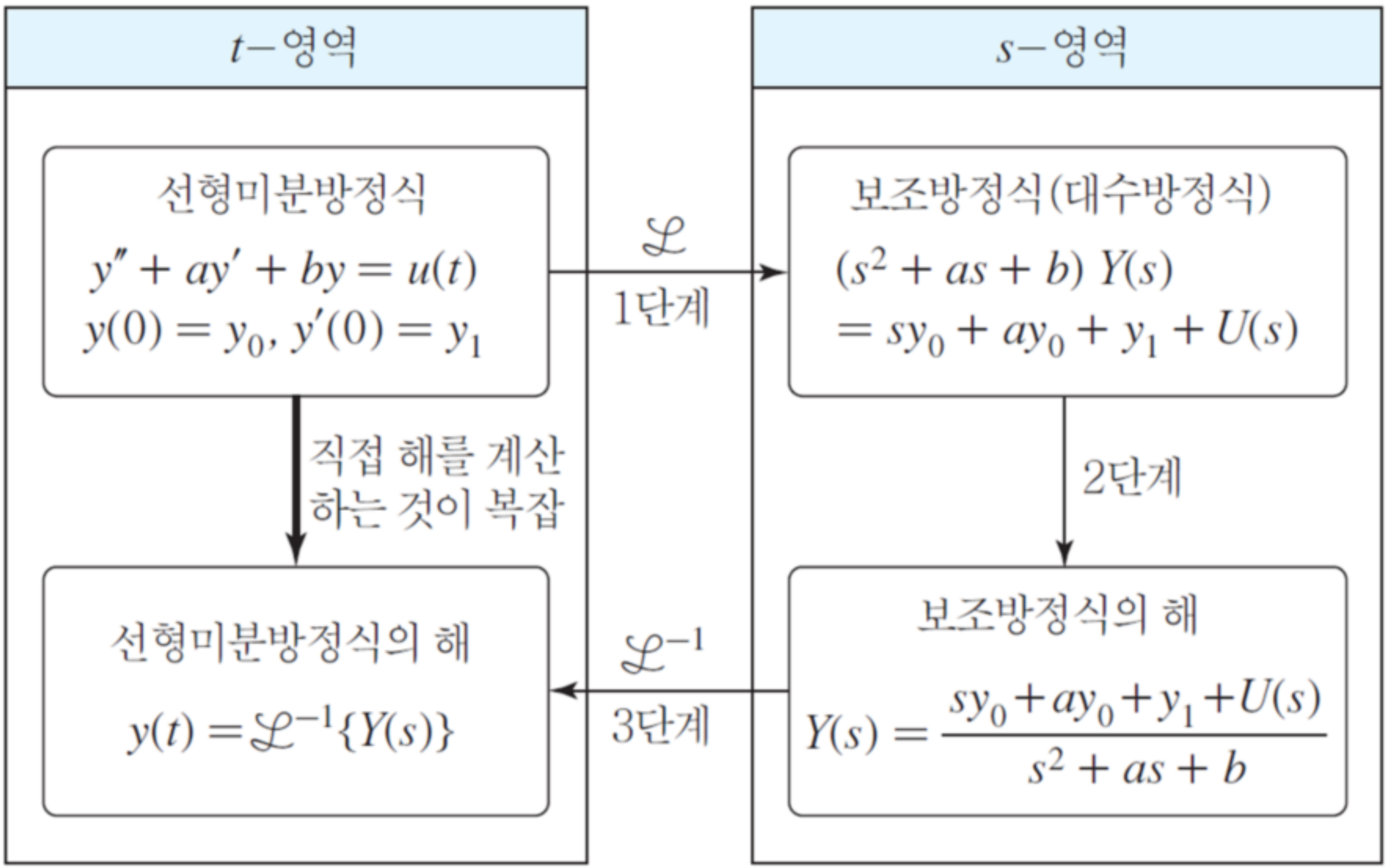
$f(t) = \frac{1}{t}(e^{bt} - e^{at})$의 Laplace 변환

$\mathcal{L}\{e^{bt} - e^{at}\} = \dfrac{1}{s-b} - \dfrac{1}{s-a}$ 이므로

$$= \mathcal{L}\left\{\frac{1}{t}(e^{bt} - e^{at})\right\} = \int_s^{\infty}\left(\frac{1}{\tilde{s}-b} - \frac{1}{\tilde{s}-a}\right)d\tilde{s} = \ln(\tilde{s}-b) - \ln(\tilde{s}-a)\Big|_s^{\infty}$$

$$= \ln\left(\frac{\tilde{s}-b}{\tilde{s}-a}\right)\Bigg|_s^{\infty} = \ln 1 - \ln\left(\frac{s-b}{s-a}\right) = \ln\left(\frac{s-a}{s-b}\right)$$

# 라플라스 변환을 이용한 선형미분방정식의 해법



Ex] $y' - 3y = 0 \quad , \quad y(0) = 5.7$

$$\frac{dy(t)}{dt} - 3y(t) = 0$$

$\frac{d}{dt}y(t) - 3y(t) = 0$

LT $\rightarrow$ $\mathcal{L}\left\{\frac{dy(t)}{dt} - 3y(t)\right\} = 0$

$\mathcal{L}\{y'(t) - 3y(t)\} = 0$

$sY(s) - y(0) - 3Y(s) = 0$

$SY(s) - y(0) - 3Y(s) = 0$

$sY(s) - 3Y(s) - 5.7 = 0$

$(s-3)Y(s) - 5.7 = 0$

$(s-3)Y(s) = 5.7$

$(s-3)Y(s) = 5.7$

$Y(s) = \frac{5.7}{s-3}$

$Y(s) = \frac{5.7}{s-3}$

$y(t) = \mathcal{L}^{-1}\left(\frac{5.7}{s-3}\right)$

$= 5.7 \times e^{3t}$

$$\boxed{y(t) = 5.7\, e^{3t}}$$

Ex ] $3\frac{dy(t)}{dt} + 5y(t) = 7$ , $y(0) = 6$

$$3\mathcal{L}\{\frac{dy(t)}{dt}\} + 5\mathcal{L}\{y(t)\} = \mathcal{L}\{7\}$$

$3y'(t)+5y(t)=7$

$$3\{sY(s) - y(0)\} + 5Y(s) = \frac{7}{s}$$

$3\mathcal{L}(y'(t))+5(y(t)) = \mathcal{L}(7)$

$= 3[sY(s)-y(0)]+5Y(s) = \frac{7}{s}$

3y(0)=18아님?

$$(3s+5)Y(s) - 6 = \frac{7}{s}$$

$$Y(s) = \frac{6s+7}{s}\cdot\frac{1}{3s+5} = \frac{(6s+7)}{s(3s+5)}$$

$$= \frac{2s+\frac{7}{3}}{s(s+\frac{5}{3})} = \frac{a}{s} + \frac{b}{s+\frac{5}{3}}$$

$$= \frac{a(s+\frac{5}{3}) + bs}{s(s+\frac{5}{3})} = \frac{as + \frac{5}{3}a + bs}{s(s+\frac{5}{3})}$$

$a+b = 2$ , $\frac{5}{3}a = \frac{7}{3}$

$\hookrightarrow a = \frac{7}{5}$

$b = \frac{3}{5}$

$$Y(s) = \frac{7/5}{s} + \frac{3/5}{s+5/3}$$

$\mathcal{L}^{-1}[(\frac{7}{5}\times\frac{1}{s}) + (\frac{3}{5}\times\frac{1}{s+\frac{5}{3}})]$

$=(\frac{7}{5}\times 1)+(\frac{3}{5}\times e^{-\frac{5}{3}t})$

$$\boxed{\therefore\ y(t) = \frac{7}{5} + \frac{3}{5}e^{-\frac{5}{3}t}}$$

<예제> $y'' + 3y' + 2y = e^{-3t},\ \ y(0) = 1,\ \ y'(0) = 0$

양변에 Laplace 변환을 취하면

$$\{s^2Y(s) - sy(0) - y'(0)\} + 3\{sY(s) - y(0)\} + 2Y(s) = \frac{1}{s+3}$$

$$(s^2 + 3s + 2)\ Y(s) = s + 3 + \frac{1}{s+3} = \frac{s^2+6s+10}{s+3}$$

$$\therefore\ Y(s) = \frac{s^2+6s+10}{(s+1)(s+2)(s+3)}$$

부분분수 전개

$$\frac{s^2+6s+10}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{\frac{5}{2}}{s+1} + \frac{-2}{s+2} + \frac{\frac{1}{2}}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+6s+10}{(s+1)(s+2)(s+3)}\right\}$$

$$= \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{5}{2}e^{-t} - 2e^{-2t} + \frac{1}{2}e^{-3t}$$

Ex) $y'' - y = t \quad , \; y(0) = 1 \; , \; y'(0) = 1$

$$y(t) = e^{t} + \sinh t - t$$

# 6.7 선형연립미분방정식

$$
\begin{array}{ccc}
\boxed{\text{선형연립미분방정식}} & \xrightarrow{\mathscr{L}} & \boxed{\text{연립대수방정식}} \\
\downarrow \text{Not Easy} & & \downarrow \text{Easy} \\
\boxed{t\text{-영역의 해}} & \xleftarrow{\mathscr{L}^{-1}} & \boxed{s\text{-영역의 해}}
\end{array}
$$

<예제>

$$\begin{cases} y_1' = -y_1 + y_2 \\ y_2' = -y_1 - y_2 \end{cases} \quad y_1(0) = 1, \quad y_2(0) = 0 \longrightarrow \text{연립미분방정식}$$

변환

$\mathscr{L}\{y_1(t)\} \triangleq Y_1(s)$, $\mathscr{L}\{y_2(t)\} \triangleq Y_2(s)$로 가정하고 Laplace 변환을 취하면

$$\begin{cases} sY_1(s) - y_1(0) = -Y_1(s) + Y_2(s) \\ sY_2(s) - y_2(0) = -Y_1(s) - Y_2(s) \end{cases}$$

$$Y_1 = -\frac{Y_2}{(s+1)^2} + \frac{s+1}{(s+1)^2}$$

$(S+1)Y_1 - Y_2 = 1$

$Y_1 = \frac{Y_2+1}{S+1}$ $\quad Y_2 = -\frac{Y_1}{S+1}$

$Y_1 + (S+1)Y_2 = 0$

$\frac{Y_2}{S+1} = \frac{-\frac{Y_1}{S+1}}{S+1}$

$$\begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \text{연립대수방정식}$$

$(S+1)^2 Y_1 = -Y_1 + S+1$

$$\therefore \quad Y_1(s) = \frac{s+1}{(s+1)^2+1} \xrightarrow{\mathcal{L}^{-1}} y_1(t) = e^{-t}\cos t$$

$\{[(S+1)^2+1]\}Y_1 = S+1$

$$Y_2(s) = \frac{-1}{(s+1)^2+1} \xrightarrow{\mathcal{L}^{-1}} y_2(t) = -e^{-t}\sin t$$

$Y_1 = \frac{S+1}{(S+1)^2+1}$

$Y_2 = -\frac{1}{(S+1)^2+1}$

$\mathcal{L}^{-1}\{Y_1(S)\} = e^{t}\cos -$

1. 제1이동정리 : 시간 영역 $f(t) \times e^{at} \rightarrow F(s-a)$로 평행이동

① $f(t) = e^{5t} t^2$

$\mathcal{L}\{e^{5t} t^2\}$

$\mathcal{L}\{t^2\} = \frac{2}{s^3} \longrightarrow s = s-5$

$\therefore \frac{2}{(s-5)^3}$

② $f(t) = e^{-t} \sin 2t$

$\mathcal{L}\{e^{-t} \sin 2t\}$

$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} \longrightarrow s = s+1$

$\therefore \frac{2}{(s+1)^2+4}$

③ $F(s) = \frac{1}{s^2+2s-8} = \frac{1}{(s+1)^2-9}$

$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2-9}\right\} = \underbrace{\frac{1}{3}\sinh(3t) \times e^{-t}}_{f(t)}$

2. 제 2 이동정리 : s 영역 $F(s) \times e^{-as} \rightarrow f(t-a)$로 평행이동 ($\longleftrightarrow$ 제 1 이동정리)

① $g(t) = \sin t \, u(t-2\pi)$

$\mathcal{L}\{\sin t \, u(t-2\pi)\} = \mathcal{L}\{\sin(t-2\pi) u(t-2\pi)\}$

$= e^{-2\pi s} \mathcal{L}\{\sin(t-2\pi)\}$

$= e^{-2\pi s} \times \frac{1}{s^2+1}$ //

② $F(s) = e^{-3s} \times \frac{1}{(s-1)^3}$ [ S 영역에서 $e^{\square} \rightarrow t$영역 평행이동 : 제2이동정리 ]

$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = \frac{1}{2} \times e^{t} \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2} e^{t} t^2$ [ 제1이동 ]

$\mathcal{L}^{-1}\left\{e^{-3s} \times \frac{1}{(s-1)^3}\right\} = \frac{1}{2} e^{t-3} (t-3)^2 u(t-3)$

3. 컨볼루션(합성곱): $\mathcal{L}\{f*g\} = F(s)G(s)$

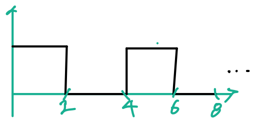
$\underbrace{f(t)*g(t)}_{\text{시간의 함수}} = \int_{-\infty}^{\infty} \underbrace{f(\tau)}_{\text{함수}}\,\underbrace{g(t-\tau)}_{\text{변화체크}}\,d\tau$ : f함수의 반응을 체크

$g(t-\tau)$ : 시간을 뒤집어줌

$9\ 7\ 5 \longrightarrow \boxed{f} \xrightarrow{f(5),\ f(7),\ f(9)}$

4. 주기함수 라플라스: $\mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt\,(1+e^{-sT}+e^{-2sT}\cdots)$

$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$



$T=4$

$\mathcal{L}\{f(t)\} = \int_0^4 f(t)e^{-st}dt \times \frac{1}{1-e^{-4s}}$

$\int_0^4 f(t)e^{-st}dt = \int_0^2 e^{-st} = -\frac{1}{s}e^{-st}\Big|_0^2 = -\frac{1}{s}e^{-2s}+\frac{1}{s} = \frac{1}{s}(1-e^{-2s})$

$\mathcal{L}\{f(t)\} = \frac{1}{s} \times \frac{1-e^{-2s}}{1-e^{-4s}}$

$= \frac{1}{s} \times \frac{1}{1+e^{-2s}}$ //

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가우스 소거법

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$