5장 상미분방정식의 급수해. 특수함수

- Legendre 방정식
- Bessel 방정식

르장드르 함수는 프랑스의 수학자 아드리앵마리 르장드르(Adrien-Marie Legendre; 1752~1833)

물리학, 공학 등

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+n(n+1)y=0$$

거듭제곱 급수 $x - x_0$의 거듭제곱

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

x ; 변수 , $a_0, a_1, \cdots$: 상수 , 급수의 계수

$x_0 = 0$ 이면 $\displaystyle\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

* Maclaurin 급수

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Ex) 거듭제곱급수 해 $y' - y = 0$ 푸시오.

대입

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) - (a_0 + a_1x + a_2x^2 + \cdots) = 0$$

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$$

$$a_1 - a_0 = 0, \quad 2a_2 - a_1 = 0, \quad 3a_3 - a_2 = 0$$

$$a_1 = a_0 \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!} \cdots$$

일반해 $y = a_0 + a_0x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$

일반해 $y = a_0 + a_0x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= a_0 e^x$$

5.2 Legendre 방정식 → 양자역학, 전자기학

Legendre 미분방정식

대입 $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$$y = \sum_{m=0}^{\infty} a_m x^m, \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1},$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m$$

$$+ \sum_{m=0}^{\infty} n(n+1) a_m x^m = 0$$

차수 맞추기 위해 $m \rightarrow m+2$ 대입

$$\sum_{m=0}^{\infty} x^m \Big(\underbrace{(m+2)(m+1) a_{m+2} - m(m-1) a_m - 2m a_m + n(n+1) a_m}_{\hookrightarrow 0} \Big) = 0$$

$$(m+2)(m+1)\,a_{m+2} = \underline{m(m-1)a_m + 2m\,a_m - n(n+1)a_m}$$

$$m^2 - m + 2m - n(n+1)$$

인수 분해 $\left(= m^2 + m - n(n+1) \right.$

$$(m+2)(m+1)\,a_{m+2} = (m-n)(m+n+1)\,a_m$$

$$a_{m+2} = \frac{(m-n)(m+n+1)}{(m+2)(m+1)}\,a_m$$

$$= -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}\,a_m$$

$m = s$

(식4) $$\boxed{a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s}$$

점화식 $s \geq 0$

$\boxed{s=0 \text{ 일 때}}$

$$a_2 = -\frac{n(n+1)}{2!}a_0$$

$$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3} \times \left(-\frac{n(n+1)}{2!}\right)a_0 = \frac{(n-2)\cdot n(n+1)(n+3)}{4!}a_0$$

$$\underbrace{a_0\left(1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots\right)}_{y_1(x)}$$

$\boxed{s = 1 \text{ 일 때}}$

$$a_3 = -\frac{(n-1)(n+2)}{3\cdot 2}\cdot a_1$$

$$a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}\cdot\left(-\frac{(n-1)(n+2)}{3\cdot 2}\right)a_1$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

$$a_1\underbrace{\left(x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 \cdots\right)}_{y_2(x)}$$

일반해 $y(x) = a_0 y_1(x) + a_1 y_2(x)$

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \boxed{?}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \boxed{?}$$

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	$y = P_n(x)$

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m}$$

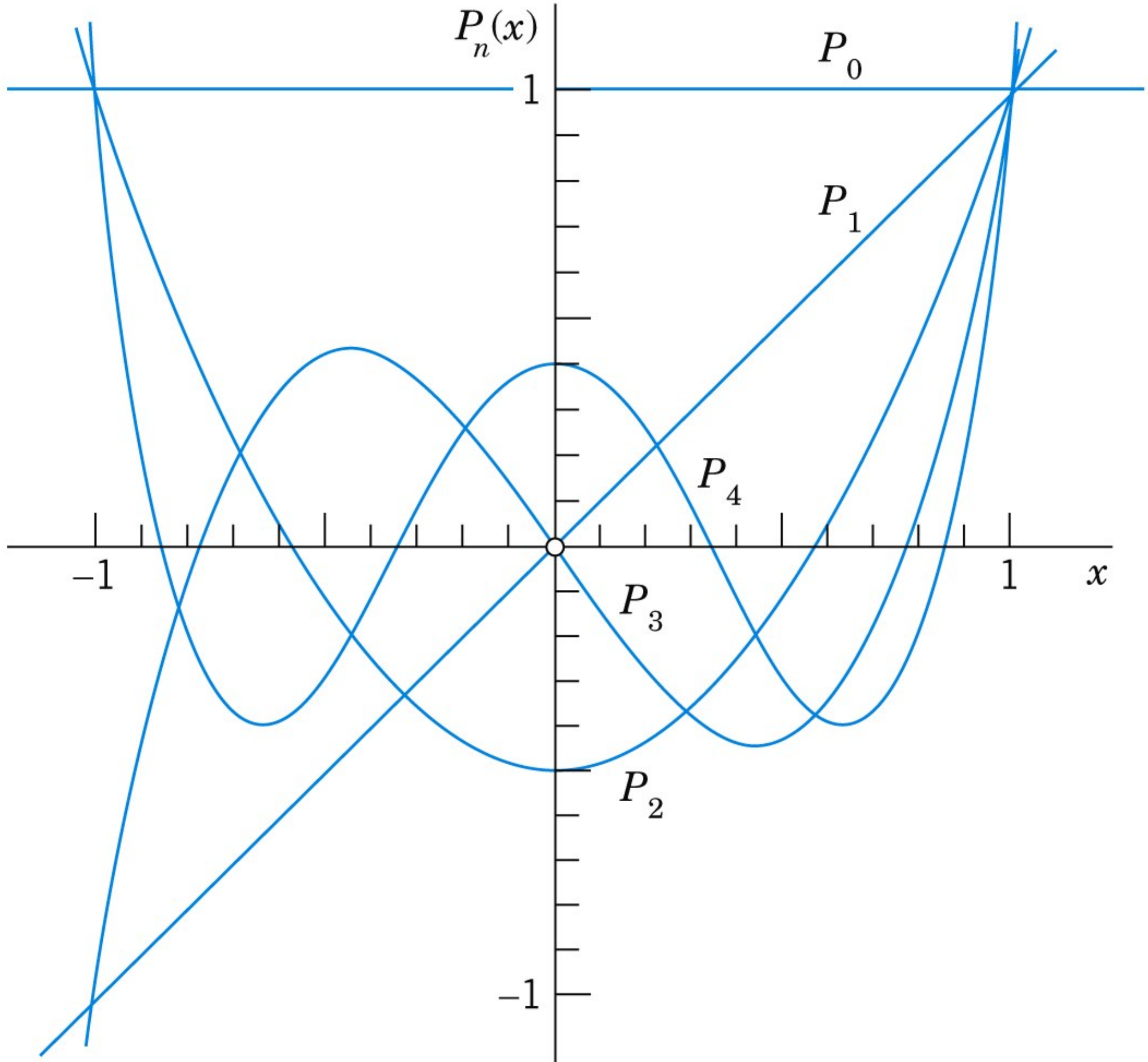
$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n\, 1!\,(n-1)!\,(n-2)!} x^{n-2} + - \cdots$$

$P_0(x) = 1, \quad P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$-1 \le x \le 1$



$$P_0(x) = 1, \quad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}\left(3x^2-1\right)$
3	$\frac{1}{2}\left(5x^3-3x\right)$
4	$\frac{1}{8}\left(35x^4-30x^2+3\right)$
5	$\frac{1}{8}\left(63x^5-70x^3+15x\right)$
6	$\frac{1}{16}\left(231x^6-315x^4+105x^2-5\right)$
7	$\frac{1}{16}\left(429x^7-693x^5+315x^3-35x\right)$
8	$\frac{1}{128}\left(6435x^8-12012x^6+6930x^4-1260x^2+35\right)$
9	$\frac{1}{128}\left(12155x^9-25740x^7+18018x^5-4620x^3+315x\right)$
10	$\frac{1}{256}\left(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63\right)$

레포트

$$(1-x^2)y'' - 2xy' = 0$$

정답

$$y = c_1 + c_2 \ln\left(\frac{1+x}{1-x}\right)$$

$-1 \le x \le$
1

5.4 Bessel 방정식

베셀 방정식	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$	$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$
		$y = c_1 J_n(x) + c_2 Y_n(x)$

베셀 방정식	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$	$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$
		$y = c_1 J_n(x) + c_2 Y_n(x)$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

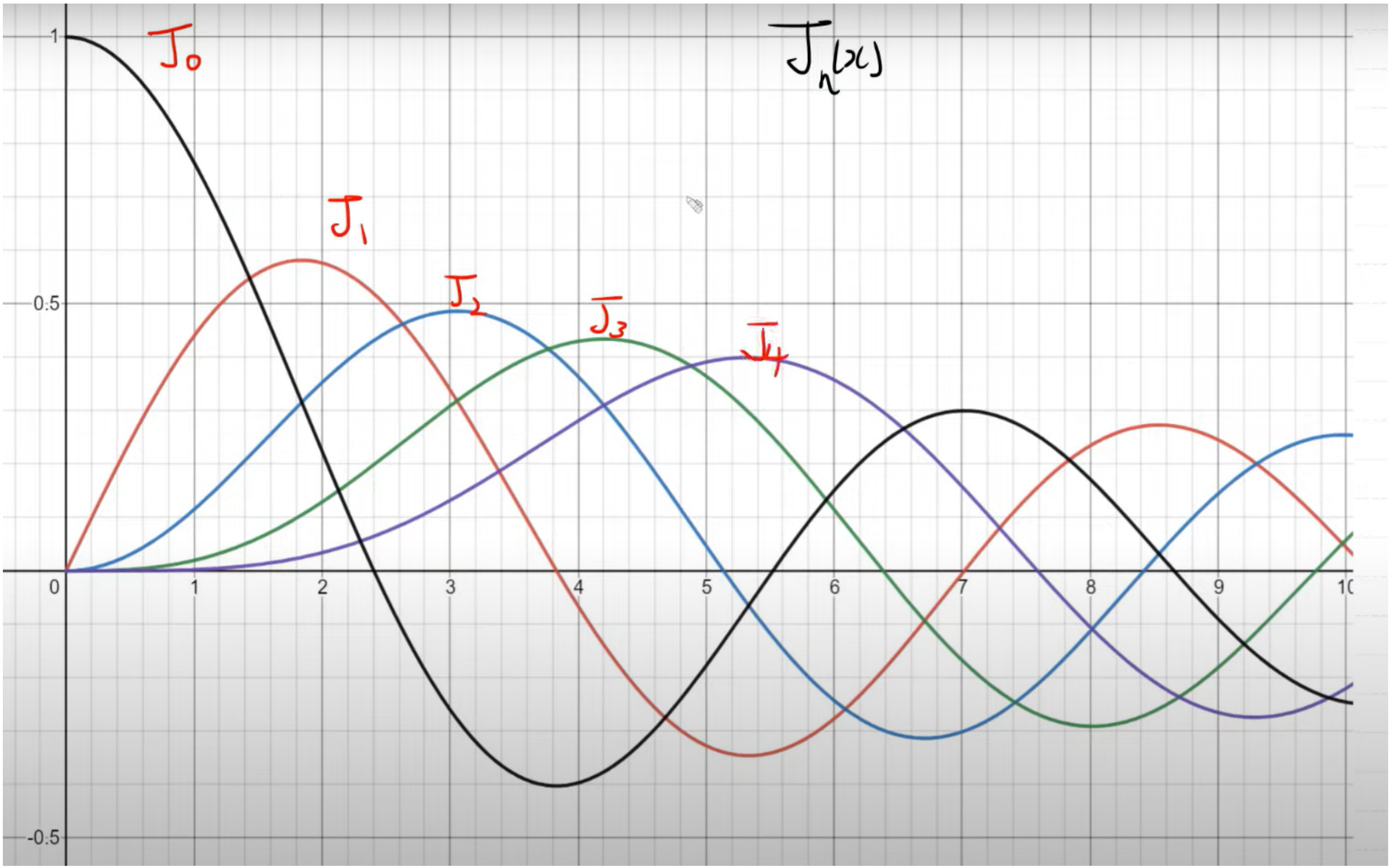
$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

$$y = C_1 J_{\frac{1}{3}}(x) + C_2 J_{-\frac{1}{3}}(x)$$

$$x^2y'' + xy' + (x^2 - 9)y = 0$$

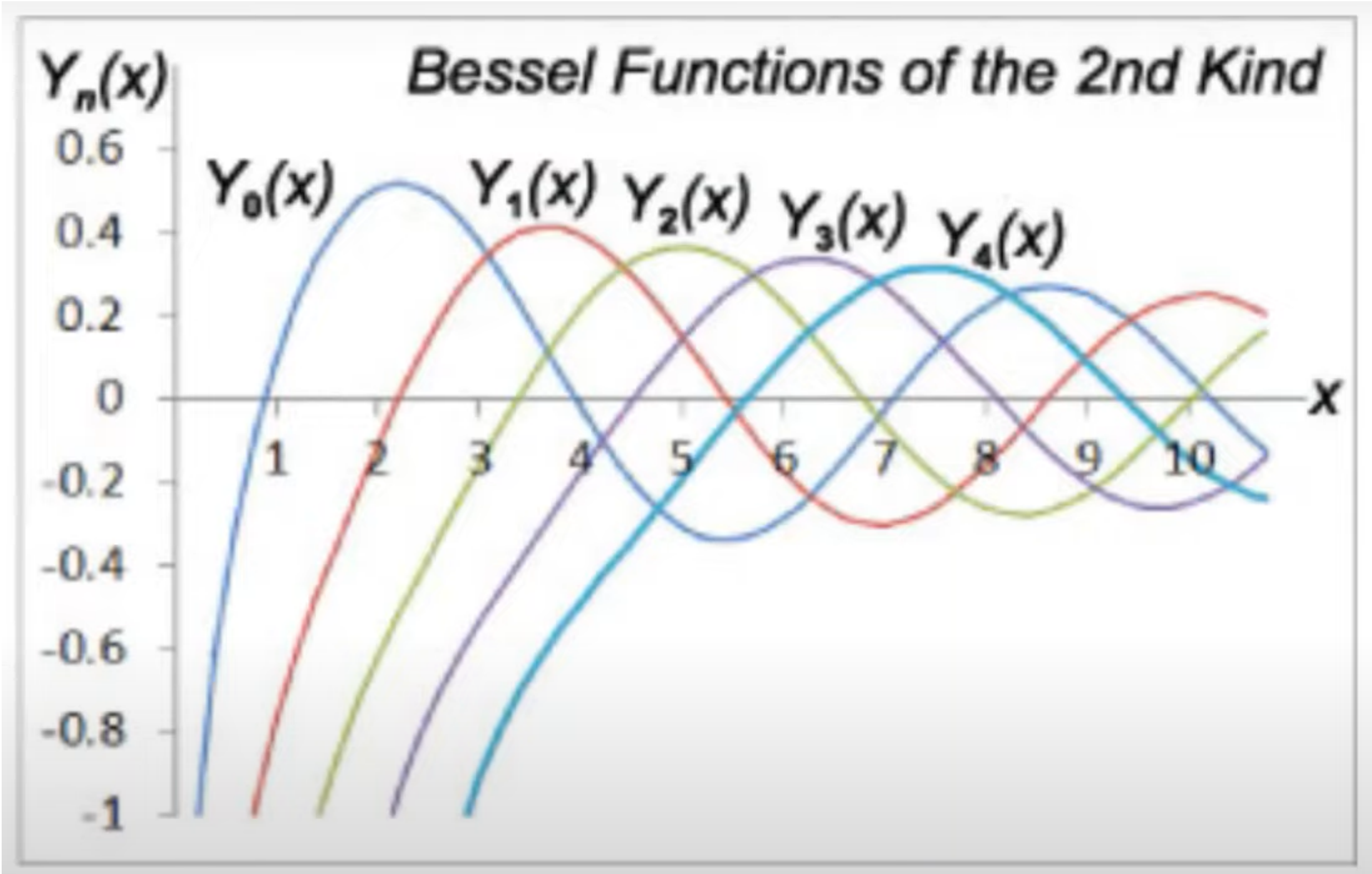
$n = 3$

$$y = C_1 J_3(x) + C_2 Y_3(x)$$



$J_n(x)$
J_0
J_1
J_2
J_3
J_4
1
0.5
-0.5
0
1
2
3
4
5
6
7
8
9
10

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178



Bessel Functions of the 2nd Kind
$Y_n(x)$
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
1 2 3 4 5 6 7 8 9 10
x
$Y_0(x)$
$Y_1(x)$
$Y_2(x)$
$Y_3(x)$
$Y_4(x)$

EX)

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0$$

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

$$\nu = \frac{1}{2}$$

$$y = C_1 J_{\frac{1}{2}}(x) + C_2 J_{-\frac{1}{2}}(x)$$

N차 제1종 베셀함수

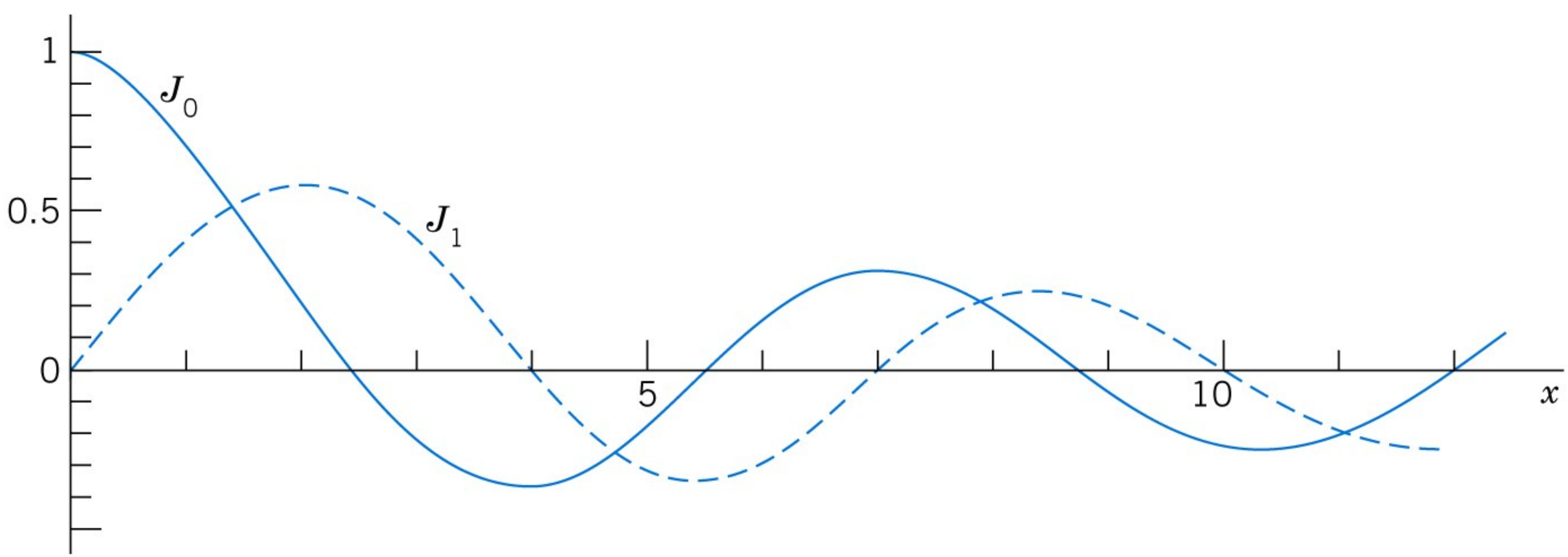
$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (n+m)!} \qquad (n \geq 0)$$

0차 베셀함수

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} = 1 - \frac{x^2}{2^2 (1!)^2} + \frac{x^4}{2^4 (2!)^2} - \frac{x^6}{2^6 (3!)^2} + - \cdots$$

1차 베셀함수

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m! (m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1! 2!} + \frac{x^5}{2^5 2! 3!} - \frac{x^7}{2^7 3! 4!} + - \cdots$$



제2종 베셀함수

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2}+\gamma\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\right]$$

$$Y_n(x) = \frac{2}{\pi}J_n(x)\left(\ln\frac{x}{2}+\gamma\right) + \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m+h_{m+n})}{2^{2m+n}m!(m+n)!}x^{2m}$$
$$-\frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n}m!}x^{2m}$$