Thesis for the Degree of Master of Science

Accelerating BTZ

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Abstract

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It is known that acceleration cuts out spacetime and the blackhole theories in 2+1 AdS spacetime. In this thesis, I reviewed these theoretical contents and wrote an analysis of 2+1 dimensional 'BTZ geometry'. To analyze these geometries, I reviewed stationary, accelerating point particles; one-parameter extensions of the BTZ family resembling an accelerating black hole. This theory is analyzed several years ago but I analyze more details about these geometries using mathematica and what results come out.

The main contents of the first half reviewed this thesis[1].

Contents

1 Introduction

3D gravity is a kind of toy model for real gravity theory. However, this theory has several nice properties. There is no local degree of freedom. We will look into this in Chapter 2. Because of this property, gravitational waves do not exist in 3dimensional gravity. But curiously, that doesn't mean gravitational forces or black holes can't exist. Both are possible and the theories involved are already well-studied. And the name of that blackhole solution is BTZ blackhole, named after Maximo Banados, Claudio Teitelboim, Jorge Zanelli[2].

If you study about the accelerating observer in relativity, you will naturally study about Rindler coordinates. A constantly accelerating particle moves hyperbolically. However, this is not the end, but you will not be able to interact with a certain area of space-time. One sees this phenomenon and expresses that part of space-time is cut out. The fundamental reason this happens is that the speed of light is finite. At this time, the clipped space-time is also expressed as the rindler horizon. At this horizon, the determinant of the metric tensor will be zero. We will not use the Rindler coordinate system directly in this paper. However, if I talk about accelerating time and space, it will naturally be mentioned several times.

The four dimensional C-metric has been historically considered as the prototypical model of an accelerating black hole.

$$ds^2 = \frac{1}{A^2(x+y)^2}(-Fdt^2 + \frac{dy^2}{F} + \frac{dx^2}{Q} + Qd\phi^2) \tag{1.1}$$

where $F = -1 + y^2 - 2MAy^3$ and $Q = 1 - x^2 - 2MAx^3$. This solution contains two constant parameters M and A. It reduces to a form of Minkowski space when $M = 0$, but has no convenient limit as $A \to 0$. Although the x, y coordinates are useful when performing calculations and in analysing the global structure the spacetime, their physical interpretation can be made more clear if we introduce closely related coordinates r, θ

which will play a role of spherical coordinates around the black holes.

$$ds^2 = \frac{1}{(1 + Ar\cos\theta)^2}\left[-Qdt^2 + \frac{dr^2}{Q} + \frac{r^2 d\theta^2}{P} + Pr^2 \sin^2\theta d\phi^2 \right] \tag{1.2}$$

where $P = 1 + 2Am\cos\theta, \quad Q = (1 - A^2r^2)(1 - \frac{2M}{r})$. We plan to do similar work in 3D.

A similar solution exists in 3D. In this paper, we will discuss the story of conical deficit accelerating in three dimensions with a c metric like solution. We plan to set the background space to AdS and calculate mass etc. using AdS/CFT correspondence.

Basically, when we calculate mass for 4D gravity, we derive the stress tensor from the action and integrate it with the time-time element. It starts at r = 0 and gradually progresses to infinity, adding up all the curvatures. However, in AdS, the universe has an end, so it cannot go to infinity and stops at the boundary. Therefore, the physics on the boundary plays an important role, even in the case of acceleration. Therefore, in order to study accelerating black holes in three dimensions, it is necessary to perform related calculations.

In chapter 2, I will briefly review 3D gravity. In chapter 3, I will simpliy review the point particle in 3d gravity and BTZ blackhole. In chapter 4, I will analyze the c metric like solution in earnest. Through this process, it is confirmed that point particles accelerating in 3D can experience various classes of geometry. And I am going to properly classify only those that are physically possible among those geometries. In chapter 5, I will do various calculations for various classes. Through this, I think I will be able to have a higher understanding of the three-dimensional gravity theory. Finally, in the appendix I will share detailed techniques and mathematica codes for doing these calculations.

2 3d Gravity

In this section, we will review the 3d gravity. Pure 3dimensional Einstein gravity is defined by 3d Einstein-Hilbert action:

$$S_{EH} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R - 2\Lambda) \tag{2.1}$$

where G is the gravitational constant, with dimensions of inverse mass in $c = 1$ units, and g is a determinant of the metric tensor. The equation of motion of this action is the Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.2}$$

where $G_{\mu\nu}$ is an Einstein tensor and Λ is a cosmological constant. we can derive the identity with little calculation

$$R^{\alpha\beta}_{\mu\nu} = E^{\alpha\beta\gamma}E_{\mu\nu\rho}G^{\rho}_{\gamma} \tag{2.3}$$

where $R^{\alpha\beta}_{\mu\nu}$ is Riemann curvature tensor, G^{ρ}_{γ} is Einstein tensor and $E^{\alpha\beta\gamma}$ is Levi-Chivita tensor. This identity links curvature and Einstein tensors. so, where $G^{\rho}_{\gamma} = 0$ be empty regions and flat. Although interior ones are not

$$G^{\rho}_{\gamma} = 8\pi G T^{\rho}_{\gamma} \tag{2.4}$$

where T^{ρ}_{γ} is stress tensor. The last thing to consider is the Weyl tensor($C_{\mu\nu\alpha\beta}$) which is the traceless part of the Riemann tensor.

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - (g_{\alpha[\mu}R_{\nu]\beta} + g_{\alpha[\mu}R_{\nu]\beta}) + \frac{1}{2}Rg_{\alpha[\mu}g_{\nu]\beta} \tag{2.5}$$

Since the number of independent components of the Riemann tensor and Ricci tensor are $\frac{d^2}{12}(d^2 - 1)$ and $\frac{d}{2}(d + 1)$ in d-dimension, respectively. In 3 dimnesion, two numbers are same Which means that the Weyl tensor has no independent components and be zero.

$$R_{\mu\nu\alpha\beta} = (g_{\alpha[\mu}R_{\nu]\beta} + g_{\alpha[\mu}R_{\nu]\beta}) - \frac{1}{2}Rg_{\alpha[\mu}g_{\nu]\beta} \tag{2.6}$$

With a little calculation,

$$R_{\mu\nu\alpha\beta} = \Lambda(g_{\alpha\mu}g_{\nu\beta} - g_{\alpha\nu}g_{\mu\beta}) \tag{2.7}$$

By above result, we can know that there is no local degrees of freedom in 3d gravity and spacetime outside sources is locally flat. One may feels that we can describes various analytic solutions in terms of Minkowski spacetime patches. But those patches match with twisted conditions[3]. And here comes the conical deficit.

3 Point particle in 3d gravity

Let's do a brief review of BTZ blackhohle first. To begin this chapter, it might be a good idea to start with this questions : What does the gravitational field of a point particle with mass in 3d look like?. For convenience, let's assume a static, non-rotating point particle. Then, this particle would be located along a single timelike geodesic. The metric would be 3d Schwarzschild metric.

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\phi^2, \tag{3.1}$$

where $0 \leq r \leq \infty, \quad 0 \leq \phi \leq 2\pi$. To make $R_{\mu\nu} = 0$, $A(r)$ and $B(r)$ should be constant. So, Choose $A(r) = A^2$ and $B(r) = B^2$. Then the metric would be

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + r^2 d\phi^2. \tag{3.2}$$

If we redefine a new time coordinate $\tau = At$, then we get

$$ds^2 = -d\tau^2 + B^2 dr^2 + r^2 d\phi^2. \tag{3.3}$$

Now let's finish the coordinate transformation : $\tilde{r} = Br, \tilde{\phi} = \phi/B$. We can get the 3-Minkowski metric in polar coordinates

$$ds^2 = -d\tau^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2. \tag{3.4}$$

As a result of this coordinate transformation, there arise a limitation on polar coordinate : $0 \leq \tilde{r} \leq \infty, \quad 0 \leq \tilde{\phi} \leq 2\pi/B$.

The hyper surface $\tau = constant$ describes a cone with a vertex at $\tilde{r} = 0$ and deficit angle $2\pi(1 - 1/B)$.

In Gott and Alpert[4],the metric(3.4) is embed as a cone $z(\tilde{r}) = (B^2 - 1)^{1/2}\tilde{r}$ in the Minkowski spacetime:

$$ds^2 = -d\tau^2 + dz^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2. \tag{3.5}$$

Moreover, It is well known that the deficit angle is related to the mass of the point particle[5].

$$\mathbf{def}\tilde{\phi} = 2\pi(1 - \sqrt{1 - \frac{\kappa}{\pi}M}) \approx \kappa M + O((\kappa M)^2). \tag{3.6}$$

where $\kappa = 2\pi G$.

3.1 Static BTZ

In polar coordinates, the form of the non-rotating BTZ metric is

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\theta^2, \tag{3.7}$$

where $f = \frac{r^2}{l^2} - M, \quad M$ is the mass parameter.
For $M = -1$, we can obtain the pure AdS metric in polar coordinates,

$$ds^2 = -(\frac{r^2}{l^2} + 1)dt^2 + \frac{dr^2}{(\frac{r^2}{l^2} + 1)} + r^2 d\theta^2. \tag{3.8}$$

So we can interpret it as $M = -1$ metric is the vacuum solution. Detailed energy and counterpart calculations related to this are introduced in this book[5].

3.2 BTZ

The Einstein equation in 3d with negative cosmological constant has blackhole solution. Even, It is characterized by mass, charge and angular momentum which are defined by flux integral at infinity or the end of the AdS world. It is similar to 4d blackhole. The pure AdS spacetime appears as a negative mass state which is separated by a mass gap from the continuous blackhole spectrum.

In this section we will review the brief of BTZ blackhole. For simplicity, I will ignore the coupling to the Maxwell field.

The action is

$$S = \frac{1}{2\pi} \int \sqrt{-g} \Big[R + 2l^{-2} \Big] dx^3 + S_{surf}, \tag{3.9}$$

where S_{surf} is the surface term and l is related to the cosmological constant by $\Lambda = -l^{-2}$.

We can solve (3.9). The equations of motion are solved by the blackhole field

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 (Adt + d\phi)^2 \tag{3.10}$$

where the squared lapse N^2 and the angular shift A are given by

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$$

$$A(r) = -\frac{J}{2r^2}$$

with $-\infty < t < \infty, \quad 0 < r < \infty and 0 \leq \phi < 2\pi$.

The two constants M and J which appers in (3.10) are conserved charges associated with asymptotic time and rotational invariance, respectively. These charges could be calculated by the flux integrals through a large circle at spacelike infinity, the end of the AdS world.

Let;'s see the detail of N. The lapse function vanishes where $r = r_{\pm}$

$$r_{\pm} = l\left[\frac{M}{2}\left(1 \pm \sqrt{1 - (\frac{J}{Ml})^2}\right)\right]^{1/2}. \tag{3.11}$$

Surely, r_+ is the event horizon of the blackhole. Furthermore, In order to exist the blackhole, above equation must has this limitation.

$$M > 0, \quad |J| \leq Ml. \tag{3.12}$$

Then, the extreme case $|J| = Ml$ makes both zeroes coincide.

To go to vacuum state, what is to be regarded as empty space, is obtained by $M \to 0$. Since the equation (3.12), r_+ goes to 0, either. It means that the size of the event horizon goes to zero. The blackhole disappears and the corresponding line element is

$$ds^2 = -(\frac{r^2}{l^2})dt^2 + \frac{dr^2}{(\frac{r^2}{l^2})} + r^2 d\theta^2. \tag{3.13}$$

As M decreases and encounters the negative region, the conical singularity is naked. Like the curvature singularity of a negative mass black hole in 4d. Thus, It have to be excluded from the physical spectrum. But There is an important exceptional case. When M decreases and becomes -1, the singularity disappears. There is no horizon and no singularity to hide : pure AdS spacetime.

4 C-metric-like solutions

In this section, we will analyze the C-metric-like solutions in three dimensions. We start with an ansatz similar to the typical form of the four-dimensional C-metric(1.1).

$$ds^2 = \frac{1}{\Omega^2}(P(y)d\tau^2 - \frac{1}{P(y)}dy^2 - \frac{1}{Q(x)}dx^2) \tag{4.1}$$

where $\Omega = A(x-y)$ is the conformal factor[1]. While removing ansatzs, there are three major solutions. So, Let's call them Class I,II and III.

Class	$Q(x)$	$P(y)$	**Maximal range of** x
I	1 - x^2	$\frac{1}{A^2l^2} + (y^2-1)$	$\lvert x\rvert < 1$
II	x^2 - 1	$\frac{1}{A^2l^2} + (1-y^2)$	$\lvert x\rvert > 1$
III	$1+x^2$	$\frac{1}{A^2l^2} - (1+y^2)$	$\mathbb{R}$

Table. The metric functions in canonical gauge, together with the largest available range of x.

There mathematical common thing of these Classes is that, in Maximal range of x, $Q(x)$ be positive. Which does not affect the sign of the metric. So, metric could be maintained Lorentzian. In particular, in Class II, the domain of x is divided into two places. The place where the sign of x is negative based on 0 would be called $left$, and the place where it is positive would be called $right$.

Each class can also be classified according to the signes of $P(y)$. This classification has physical meaning which sub-divide the classes further by their causal or horizon structure. Since The region that makes $P(y)$ positive is delimited by the signs of A in each Class. In Class I, $P(y)$ is distinguished based on the value of A^2l^2 whether it is equal to 1 or greater/less than 1. So These values would be called *saturated*, *rapid* and *slow*, respectively. In Class II, To make $P(y)$ positive, y always has limited region. But, Class

II would be sub-divided by the choice to take x as mentioned above. In Class III, To make $P(y)$ positive, we have to choice $A^2 l^2 < 1$.

To change the metric to a more familiar form, one can transform the metric to polar style coordinates; identifying $x = \cos\theta$ gives us a polar style coordinates that has an origin if $y \to -\infty$. One can even confirm that $A \to 0$ limit makes polar style coordinate a pure AdS polar coordinate.

$$ds^2 = f(r)d\tau^2 - \frac{dr^2}{f(r)} - r^2 \frac{d\phi^2}{K^2} \tag{4.2}$$

where $f(r) = 1 + r^2/l^2$ and K encodes the range of the x coordinate.

So, From now, Let's call above polar style coordinate to polar coordinate.

5 An Accelerating Particle

A point particle in 3d gravity is represented by a conical deficit. This means we have to have an origin in order to cut out the deficit. The metric shows that the length of an angular arc, represented by x, becomes zero if $y \to -\infty$. Then Table shows that only Class I solutions are able to satisfy this requirement while keeping $P \geq 0$.

To construct the accelerating conical defect, we first glue two copies of the ClassI geometry along $x = \pm 1$, so that we can identify an angular coordinate via $x = \pm\cos\theta$. We then introduce a conical deficit by choosing an $x_+ \in (-1, 1)$ and restricting the range of x to $\pm x \in (x_+, 1]$. The $x = \pm 1$ axis of the newly formed spacetime is regular, but the origin at $y \to -\infty$ has an angular deficit of $2[\pi - \arccos(\pm x_+)]$, with the $x = x_+$ axis.

5.1 Pure AdS$_3$

First, define global AdS$_3$ on a hyperboloid in $\mathbb{R}^{2,2}$. The embedding is

$$X_0 = l\sqrt{1+\frac{R^2}{l^2}}\sin\left(\frac{T}{l}\right), \quad X_1 = R\sin\Theta,$$

$$X_2 = R\cos\Theta, \quad X_3 = l\sqrt{1+\frac{R^2}{l^2}}\cos\left(\frac{T}{l}\right).$$

The hyperboloid is

$$\sum_{i=0}^{3}\eta_{ii}X_i^2 = l^2$$

and the induced metric is

$$ds^2 = (1+\frac{R^2}{l^2})dT^2 - \frac{dR^2}{(1+\frac{R^2}{l^2})} - R^2 d\Theta^2 \tag{5.1}$$

η has the signature (+ - - +) and the ranges of coordinate variables are $R \in (0, \infty), \Theta \in (-\pi, \pi)$ and $T \in [-\pi l/2, 3\pi l/2]$.

The global coordinates are defined from the embedding coordinates

$$T_G = \arctan\left(\frac{X_0}{X_3}\right), \quad R_G = \sqrt{X_1^2 + X_2^2},$$

$$X_G = X_1, \quad Y_G = X_2.$$

We can attain the Poincare disk to compactify the spatial two-section

$$\hat{X} = \frac{X_G}{l + \sqrt{l^2 + X_G^2 + Y_G^2}}, \quad \hat{Y} = \frac{Y_G}{l + \sqrt{l^2 + X_G^2 + Y_G^2}} \tag{5.2}$$

Through this transformation, the entire global spacesime could be represented in the form of a cylinder$(T_G, \hat{X}, \hat{Y})$ and the ranges of coordinate variables are $|T_G| < \pi/2$ and $\hat{X}^2 + \hat{Y}^2 < 1$. We will use this definition.

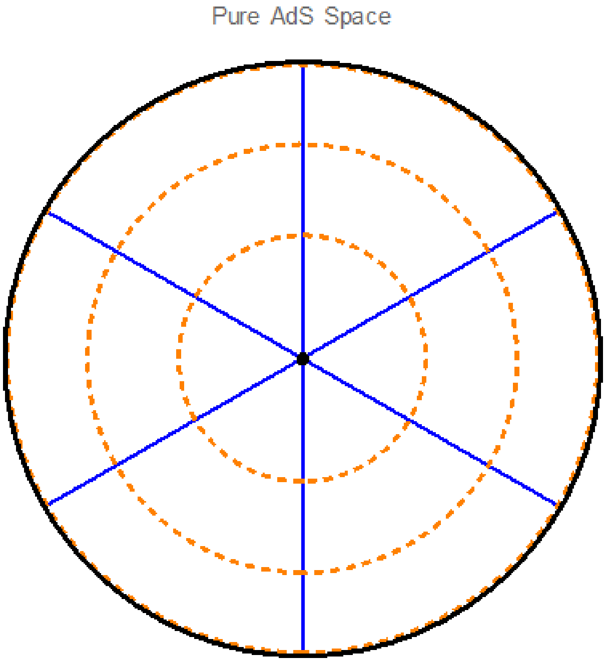


Figure 1: The pure AdS Spacetime, mapped onto the Poincare disk. Several lines of constant Θ are shown in blue, with lines of constant R in dashed orange.

5.2 Non Accelerating, Non Rotating BTZ

The embedding coordinates are

$$X_0 = B(R)\sinh\left(\frac{R_h T}{l^2}\right), \quad X_1 = A(R)\sinh\left(\frac{R_h \Theta}{l}\right),$$

$$X_2 = B(R)\cosh\left(\frac{R_h T}{l^2}\right), \quad X_3 = A(R)\cosh\left(\frac{R_h \Theta}{l}\right).$$

where

$$A(R) = \frac{Rl}{R_h}, \quad B(R) = l\sqrt{(\frac{R}{R_h}) - 1}, \quad R_h = ml.$$

The metric of non accelerating, non rotating BTZ is

$$ds^2 = (-m^2 + \frac{R^2}{l^2})dT^2 - \frac{dR^2}{(-m^2 + \frac{R^2}{l^2})} - R^2 d\Theta^2. \tag{5.3}$$

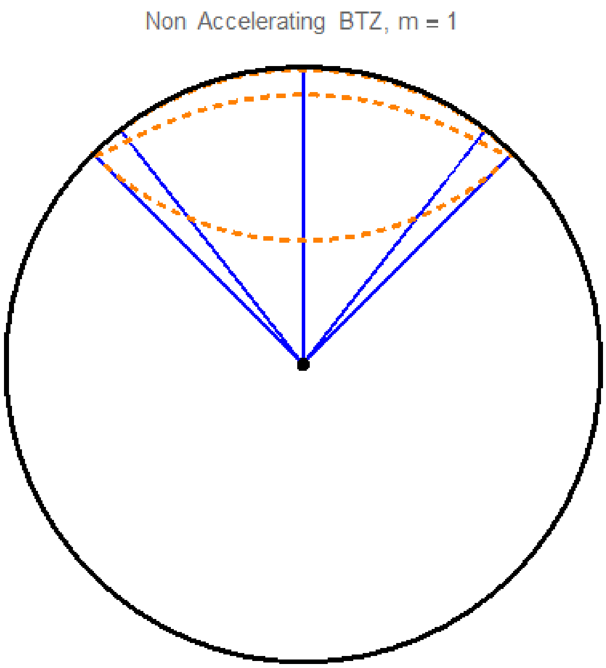


Figure 2: Non Accelerating, Non Rotating BTZ, mapped onto the Poincare disk. Several lines of constant Θ are shown in blue, with lines of constant R in dashed orange.

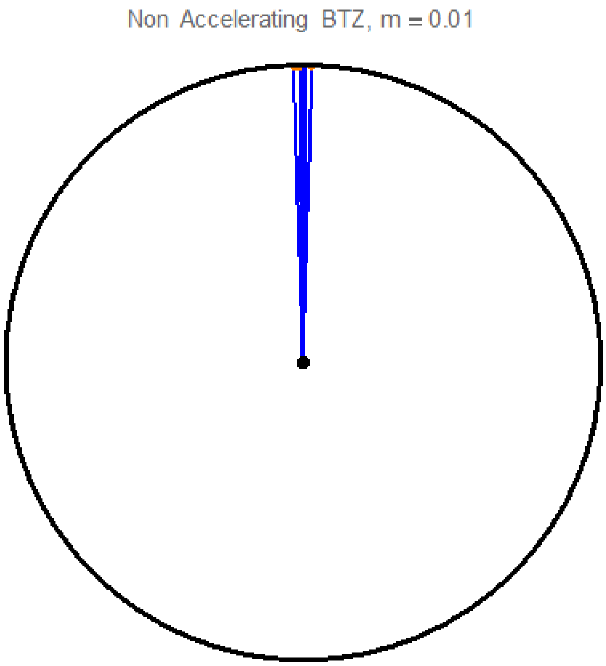


Figure 3: Non Accelerating, Non Rotating BTZ, mapped onto the Poincare disk. when mass is small. We can see the angle deficit is small either. In fact, if you look closely, there is a part that feels strange. It is a deficit angle, but part of the picture is not empty, but only part of it is filled. The root cause of this is that the mass parameter M is a negative number.

5.3 Class I

To change the metric to a more familiar form, we can change the metric to polar coordinates: $r = -\frac{1}{Ay}$ and $t = \frac{\alpha\tau}{A}$, and $x = \cos(\phi/K)$. The parameter $K \equiv \pi/\arccos(x_+)$ now encodes the range of the x coordinate, fixing the range of ϕ to be $(-\pi, \pi)$. Then the metric be

$$ds^2 = \frac{1}{[1 + Ar\cos(\phi/K)]^2}\left[f(r)\frac{dt^2}{\alpha^2} - \frac{dr^2}{f(r)} - r^2\frac{d\phi^2}{K^2}\right] \tag{5.4}$$

where $f(r) = 1 + (1 - A^2l^2)r^2/l^2$. Note that we leave α. It will do important roles.

In this form, it can be easily confirmed that when $A \to 0$, pure AdS metric is recovered. Since a Killing horizon forms at $f(r) = 0$, Only rapidly accelerating class has a Killing horizon which we refer to as an *acceleration horizon*.

The polar coordinate is a good system to interpret the accelerating conical deficit sits at $r = 0$. Irrespective of acceleration, this is achieved by having $K > 1$ in the metric. The angular deficit, which one may be inclined to refer to as the particle mass[3], is

$$m = \frac{1}{4}(1 - \frac{1}{K}) \tag{5.5}$$

5.3.1 Is A the acceleration?

To check that the point particle is accelerating, one should analyze the four acceleration along the worldline traced by the origin. To define four acceleration, the first step is to define four velocity. The origin($r = 0$) of our coordinate has normalised four velocity

$$\mathbf{v} = \lim_{r\to 0}\frac{\alpha\Omega}{\sqrt{f}}\partial_t$$

Then we can define the associated four acceleration by a covariant derivative

$$\mathbf{a} = \nabla_{\mathbf{v}}\mathbf{v}$$

which has magnitude

$$|\mathbf{a}| = \sqrt{\mathbf{a}^{\mu}\mathbf{a}_{\mu}} = A \tag{5.6}$$

We can see that the acceleration parameter A gives the locally experienced acceleration of the particle.

5.3.2 Slowly accelerating subclass

In this section, we want to calculate how spacetime changes and whether the mass parameters are calculated properly.

At first, In order to see the change in spacetime, it is necessary to compare and match the global AdS metric((5.1)) with the polar coordinates((5.4)) we created previously. The transformation between the slow Class I and Rindler wedge is given by

$$(1 + \frac{R^2}{l^2}) = \frac{f(r)}{\alpha^2[1 + Ar\cos(\phi/K)]^2}, \quad R\sin\Theta = \frac{r\sin(\phi/K)}{1 + Ar\cos(\phi/K)} \tag{5.7}$$

where $\alpha = \sqrt{1 - A^2l^2}$.

This mapping allows for an intuitive picture of the spacetime to be constructed. By compactifying onto the Poincare disk, we obtain figure 4 and figure 4-1 for the slowly accelerating spacetime and conical deficit, respectively, which are plotted at some fixed time. Since there is no deficit angle in figure 4, We can understand figure 4 as the limit $m \to 0$ of figure 4-1. In order to draw figure which mapped onto Poincare disk, it is important to set standards on how to define a cylinder. The basic method is (5.2). All we have to do is set the global time and space coordinate appropriately.

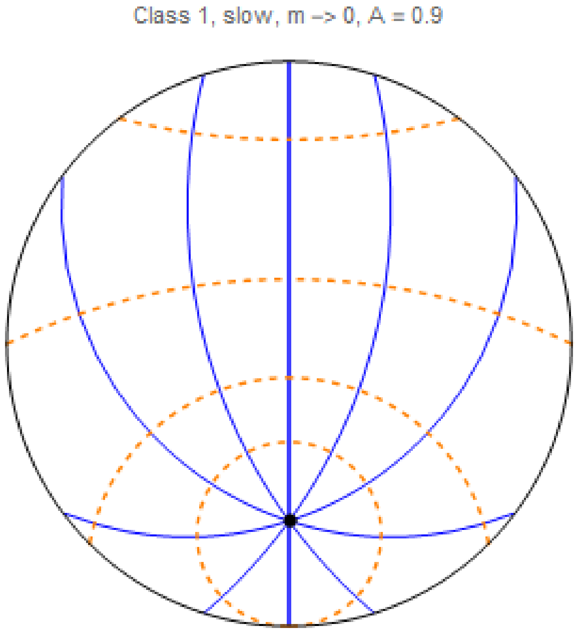


Figure 4: The Class I accelerating spacetime with $Al = 0.9$, mapped onto Poincare disk. In this figure, Several lines of constant Θ are shown in blue, with lines of constant R in dashed orange. Naturally.

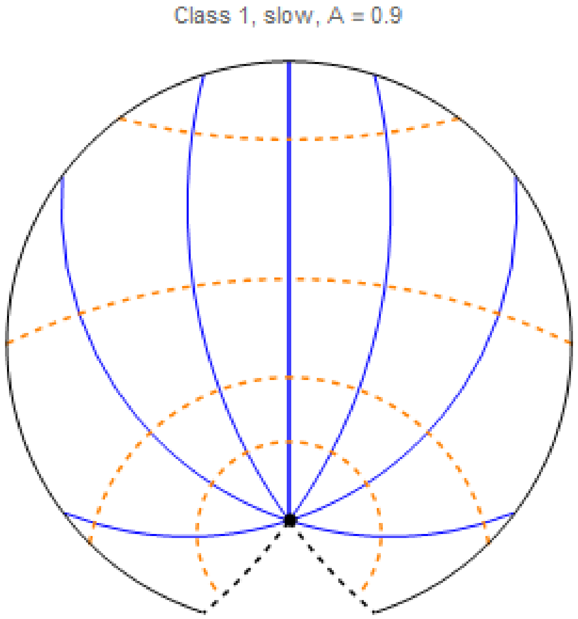


Figure 5: The Class I accelerating BTZ with $Al = 0.9$, mapped onto Poincare disk, which accelerates southwards. In this figure, spacetime wedge is removed from the global spacetime. long-dashed black lines are where edges exist.

The C-metric-like geometry(4.1) is

$$ds^2 = \frac{1}{\Omega^2}\left[Pd\tau^2 - \frac{dy^2}{P} - \frac{dx^2}{Q}\right] \tag{5.8}$$

And the embedding coordinates are

$$X_0 = \frac{\sqrt{P}}{y_h\Omega}\sin(y_h\tau), \quad X_1 = \frac{\sqrt{Q}}{\Omega},$$

$$X_2 = \frac{Al}{\Omega}(y_h x + \frac{y}{y_h}), \quad X_3 = \frac{\sqrt{P}}{y_h\Omega}\cos(y_h\tau).$$

where $P = y^2 + y_h^2, \quad Q = 1 - x^2, \quad \Omega = A(x-y), \quad y_h = \sqrt{\frac{1}{A^2l^2} - 1}.$

At this subclass, the global time coordinate would be made of X_0 and X_3. Similarly, the global space coordinates would be X_1 and X_2.

5.3.3 Holographic mass

In this section, we will calculate the holographic mass of the slowly accelerating conical deficit. The quasi local gravitational energy could be calculated by varying the action with respect to the boundary metric[6]. To accurately compute the variational problem for asymptotically AdS spacetime, we can add an counterterms that regulates the UV divergence that occurs near the boundary[7].

To do the above, one expands the metric in the Fefferman-Graham frame

$$ds^2 = \frac{l^2}{z^2}(-dz^2 + g_{(0)} + z^2 g_{(2)}). \tag{5.9}$$

where $g_{(0)}$ and $g_{(2)}$ are covariant two tensors. In 3d, this series terminates at z^2[8]. And the holographic stress tensor is determined as below[8, 9, 7].

$$T = -\frac{l}{8\pi}(g_{(2)} - g_{(0)} Tr[g_{(0)}^{-1} g_{(2)}]) \tag{5.10}$$

And transform the metric to Fefferman-Graham gauge near the boundary, using the transformation[10].

$$x = \xi + \sum_{m=1}^{\infty} G_m(\xi)(\frac{z}{l})^m, \quad y = \xi + \sum_{m=1}^{\infty} F_m(\xi)(\frac{z}{l})^m \tag{5.11}$$

By repeating the calculation several times, both F_m and G_m can be obtained sequentially. The detailed method is written in Appendix B. For convenience, We 'strategically' define a dimensionless function and F_1[11]

$$\Gamma(\xi) = 1 - A^2 l^2 (1 - \xi^2), \quad F_1(\xi) = \frac{\Gamma^{3/2}}{A l \omega(\xi)} \tag{5.12}$$

where $\omega(\xi)$is the conformal factor. It was said that it was defined 'strategically', but in fact, through several iterations of calculation and differential equations, we can see why it is appropriate to define F_1 in this way. Anyway, through this operation, the metric can be written in a different form.

$$g_{(0)} = \frac{\omega^2}{A^2}\left[d\tau^2 - A^2l^2\frac{d\xi^2}{(1-\xi^2)\Gamma^2}\right] \tag{5.13}$$

This result is similar to the same calculation in 4 dimensions[11]. To add a bit of comment, There is only one more term in 4d. But, the amount of computation increases enormously.

$$\begin{aligned} g_{(2)} = &\frac{1}{2A^2l^2}\left[1 - A^2l^2 + (1-\xi^2\Gamma^2(\frac{\omega'}{\omega})^2)\right]d\tau^2 \\ &+ \left[\frac{1-A^2l^2}{2(1-\xi^2)\Gamma^2} + (1-3A^2l^2(1-\xi^2))\frac{\xi}{(1-\xi^2)\Gamma}(\frac{\omega'}{\omega})^2 + \frac{3}{2}(\frac{\omega'}{\omega})^2 - \frac{\omega''}{\omega}\right]d\xi^2 \end{aligned} \tag{5.14}$$

The mass with respect to ∂_τ is

$$M = 2A\int_{x_+}^{1}\sqrt{-g_{(0)}}T^\tau_\tau d\xi. \tag{5.15}$$

The first 2 is the effect of the contributions from both patches. The reason why both patches have effects is that only when two patches are combined,complete spacetime could be created. If only one patch is used, we can only describe half of the total spacetime.

We can fix the conformal factor to be $\omega = 1$. The integral then takes the form

$$M = -\frac{1}{8\pi}\sqrt{1-A^2l^2}\int_{x_+}^{1}\frac{d\xi}{(1-\xi^2)\Gamma} = -\frac{1}{8\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{\cot(\pi/K)}{\sqrt{1-A^2l^2}}\right)\right] \tag{5.16}$$

Replacing ξ with sin makes this calculation simple. Maybe one wants to compare and analyze this result and (5.5)[1]. Take the limit of vanishing acceleration parameter A, then one can recover exactly the Casimir energy of pure (global) AdS_3 with a conical deficit[7, 12, 13].

$$M_{AdS_3} = -\frac{1}{8K} \tag{5.17}$$

5.3.4 Rapidly accerating subclass

Since defining a mass in the rapidly accelerating case is difficult, we have to determine the appropriate Killing vector[1]. First, Consider the portion of AdS_3 described by the metric

$$ds^2 = (-1+\frac{R^2}{l^2})dT^2 - \frac{dR^2}{(-1+\frac{R^2}{l^2})} - R^2 d\Theta^2 \tag{5.18}$$

where $R > l$ and $\Theta \in \mathbb{R}$. This geometry has the Killing horizon at $R = l$ which is created by a ∂_T. We will call it as *Rindlerwedge*. As R increases and gets closer to l, T becomes the timelike coordinates of a typical Poincare patch and gives the zero-mass state for AdS_3[7].A good normalization of T for such rapidly accelerating particles is given by choosing α such that the time coordinates of the solution coincide with the time coordinates of the Rindler wedge(5.18).

The transformation between the rapid Class I and Rindler wedge is given by

$$(-1+\frac{R^2}{l^2}) = \frac{f(r)}{\alpha^2[1+Ar\cos(\phi/K)]^2}, \quad R\sinh\Theta = \frac{r\sin(\phi/K)}{1+Ar\cos(\phi/K)} \tag{5.19}$$

where $\alpha = \sqrt{A^2l^2-1}$.

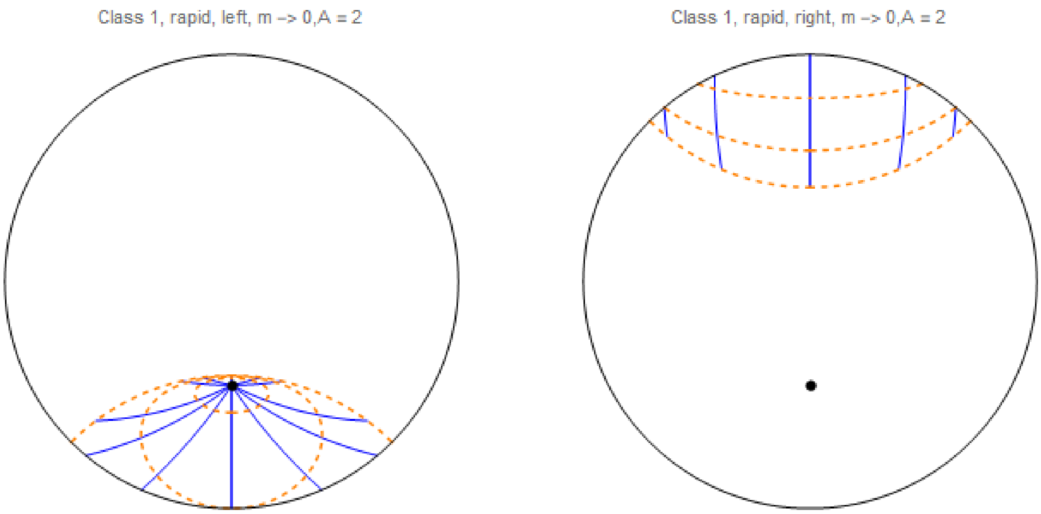
The embedding coordinates are

$$X_0 = \frac{\sqrt{P}}{y_h\Omega}\sinh(y_h\tau), \quad X_1 = \frac{\sqrt{Q}}{\Omega},$$

$$X_2 = \frac{\sqrt{P}}{y_h\Omega}\cosh(y_h\tau), \quad X_3 = \frac{Al}{\Omega}(y_h x - \frac{y}{y_h}).$$

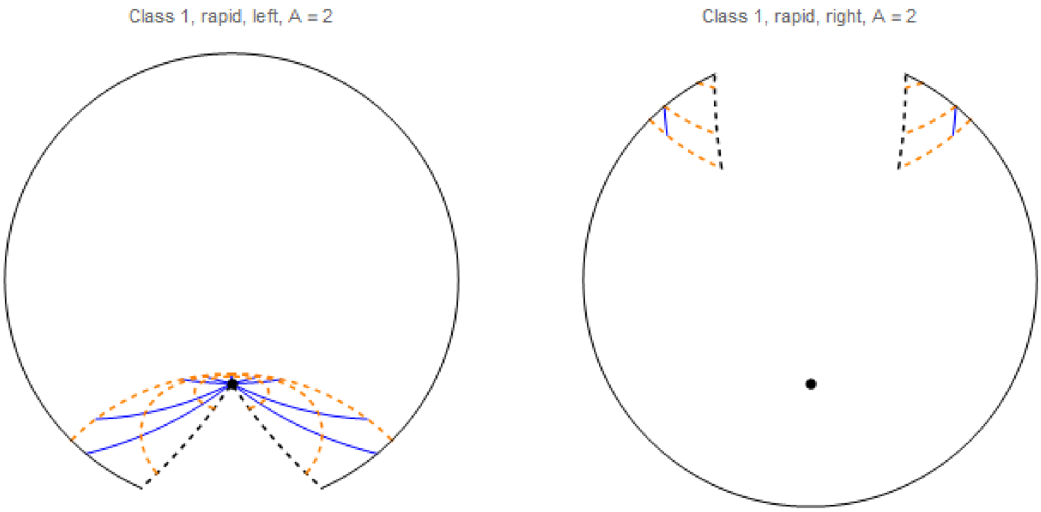
where $P = y^2 - y_h^2, \quad Q = 1 - x^2, \quad \Omega = A(x-y), \quad y_h = \sqrt{1-\frac{1}{A^2l^2}}$.

In rapidly accelerating subclass, y is constrained unlike slowly accelerating subclass. The line of the boundary $y \to x$ is allowed on the arc through the bulk. Even, If the mass of the particle is sufficiently heavy, it would cut out all of the conformal boundary. In that case, the horizon structure is similar to that of de Sitter space.



(a) $y < -y_h$ (b) $y > y_h$

Figure 6: The Class I rapidly accelerating spacetime with $Al = 2$, mapped onto Poincare disk. In these figured, Only plotted in the region where P and Q are positive. We can see that the covering regions of each subclasses are different. Several lines of constant Θ are shown in blue, with lines of constant R in dashed orange.



(a) $y < -y_h$ (b) $y > y_h$

Figure 7: Class I rapidly accelerating BTZ. Since BTZ has mass, we can see the angle deficit. In this figure, spacetime wedge is removed from the global spacetime. long-dashed black lines are where edges exist. To understand why the removed orientations of the two pictures are different, just go back to creating the patch.

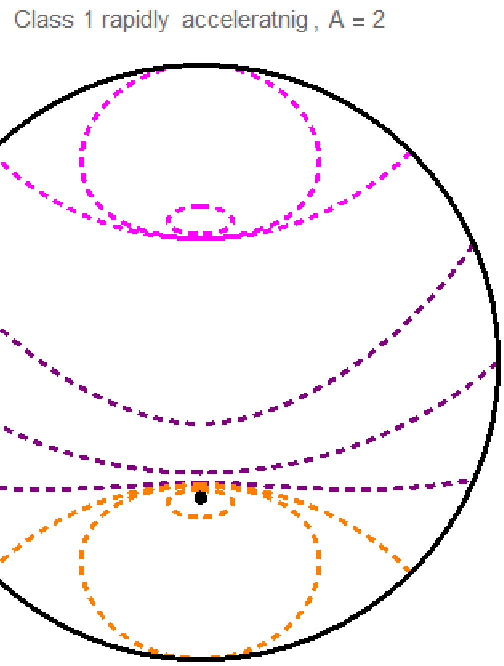


Figure 8: this figure represents only constant y ($=$ constant R) lines. Lines of constant R in dashed magenta ($y < -y_h$), purple ($-y_h < y < y_h$) and orange ($y > y_h$). This figure shows how y coordinate covers whole spacetime.

5.4 Class II

Since the results of the subclass of left($x < 1$) and right($x > 1$) are similar, so I will write based on right sub class. The major difference between two sub classes is the process of creating patches. At right sub class, choose $x_+ > 1$ and define patch with $x \in [1, x_+)$. Glue two copies of this patch, mirroring along both $x = 1$ and $x = x_+$. Most of the remaining procedures for calculation are the same as for class I. The transformation rule between the Class II and Rindler wedge, and the calculation method of holographic mass. But to cast the metric in more intuitive coordinate, x be $\cosh(\phi/K)$. Everything else is the same. Then, the metric becomes

$$ds^2 = \frac{1}{[1 + Ar\cos(\phi/K)]^2}[f(r)\frac{dt^2}{\alpha^2} - \frac{dr^2}{f(r)} - r^2\frac{d\phi^2}{K^2}] \tag{5.20}$$

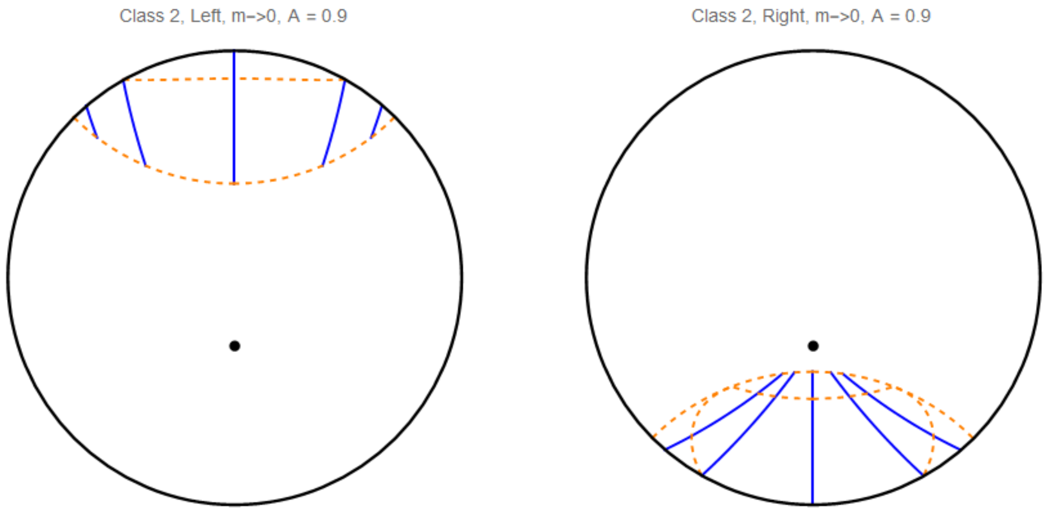
where $f(r) = -1 + (1 + A^2 l^2) r^2/l^2$.

The embedding coordinates are

$$X_0 = \frac{P}{y_h \Omega}\sinh(y_h \tau), \quad X_1 = \frac{Q}{\Omega},$$

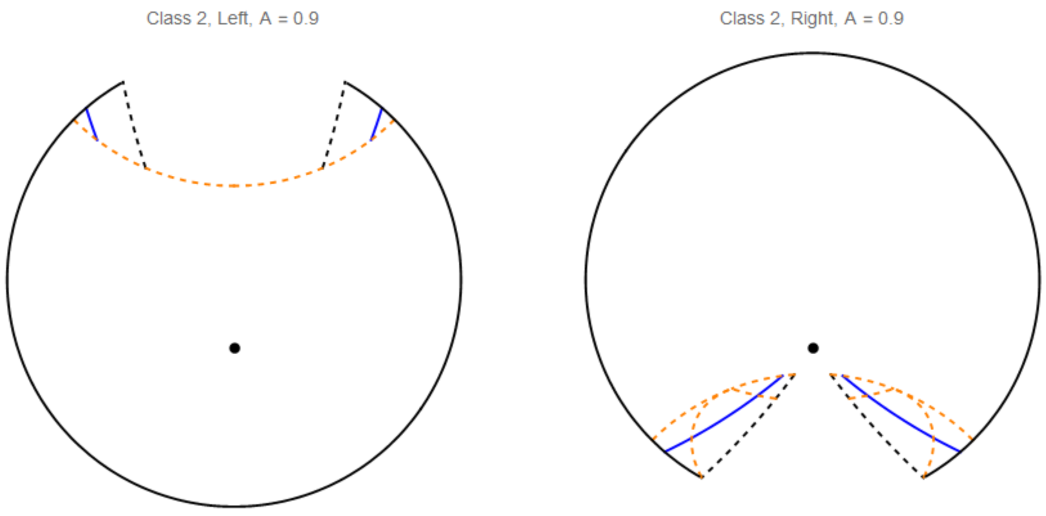
$$X_2 = \frac{P}{y_h \Omega}\cosh(y_h \tau), \quad X_3 = \frac{Al}{\Omega}(y_h x - \frac{y}{y_h}).$$

where $P = -y^2 + y_h^2, \quad Q = x^2 - 1, \quad \Omega = A(x - y), \quad y_h = \sqrt{1 + \frac{1}{A^2 l^2}}.$



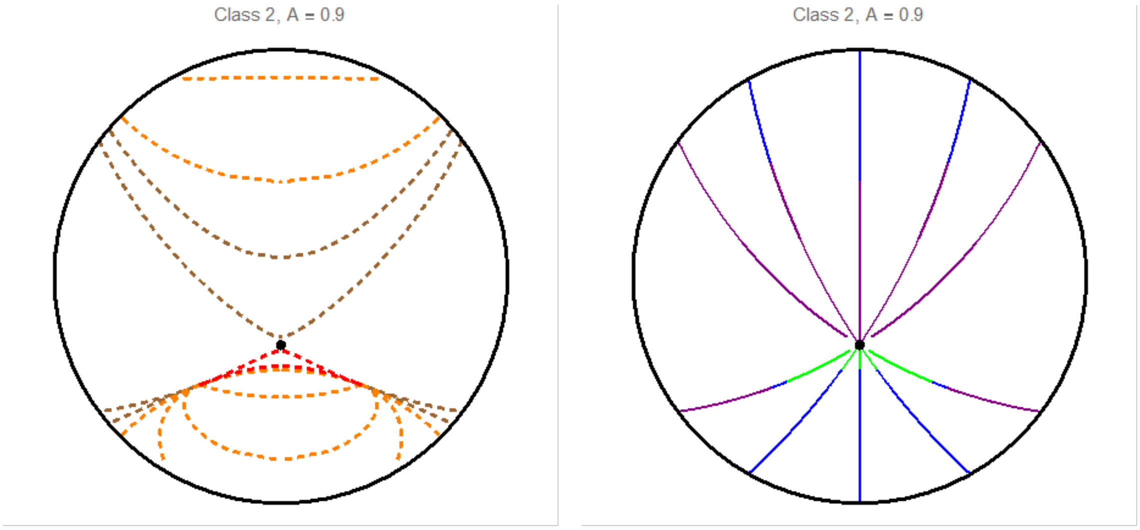
(a) Left : $x < -1$ (b) Right : $x > 1$

Figure 9: The Class II accelerating spacetime with $Al = 0.9$, mapped onto Poincare disk. In this figure, Only plotted in the region where P and Q are positive. Several lines of constant Θ are shown in blue, with lines of constant R in dashed orange.



(a) Left : $x < -1$ (b) Right : $x > 1$

Figure 10: Class II rapidly accelerating BTZ. Since BTZ has mass, we can see the angle deficit. In this figure, spacetime wedge is removed from the global spacetime. long-dashed black lines are where edges exist.



(a) constant y (b) constant x

Figure 11: The first figure represents only constant $y(=$ constant $R)$ lines. Lines of constant R in dashed brown $(y < -y_h)$, orange $(-y_h < y < y_h)$ and red $(y > y_h)$. This shows how y coordinate covers whole spacetime. Similarly, the Second figure represents only constant $x(=$ constant $\Theta)$ lines. Lines of constant Θ in dashed green $(y < -y_h)$, blue $(-y_h < y < y_h)$ and purple $(y > y_h)$. This shows how x coordinate covers whole spacetime.

5.4.1 Holographic mass

The overall calculation technique is the same as in Class I, so only the results are presented. This result is same as the result of the left sub class[1].

$$M = 2mA\int_1^{x_+}\sqrt{-g_{(0)}}T^{\tau}_{\tau}d\xi = \frac{m}{8\pi\alpha}\sqrt{1+A^2l^2}\tanh^{-1}[\sqrt{1+A^2l^2}\tanh(m\pi)] \tag{5.21}$$

5.5 Class III

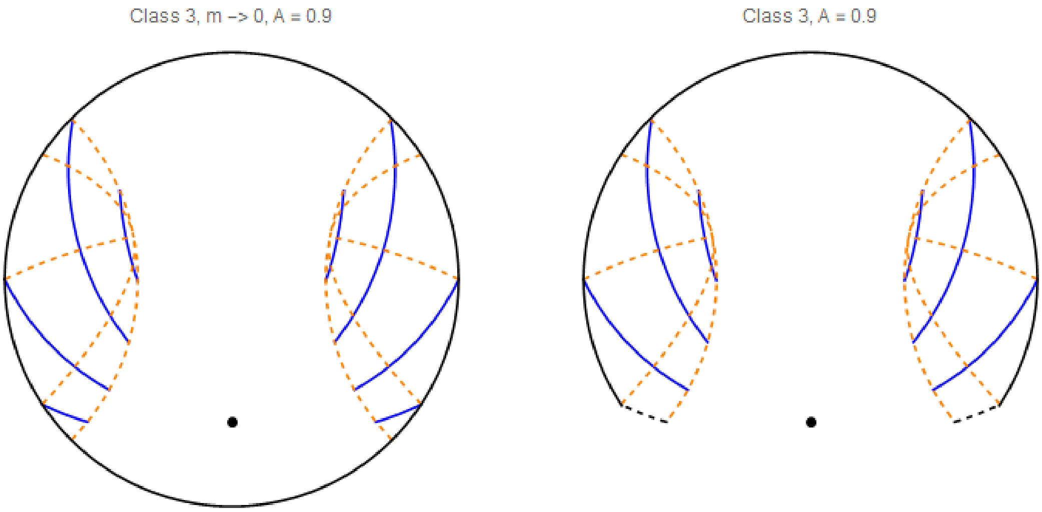
By the definition of Class III, P is always negative when $A^2l^2 \geq 1$. So, there exists only slowly accelerating sub class.

The embedding coordinates are

$$X_0 = \frac{\sqrt{P}}{y_h\Omega}\sinh(y_h\tau), \quad X_1 = \frac{Al}{\Omega}(y_h x + \frac{y}{y_h}),$$

$$X_2 = \frac{\sqrt{P}}{y_h\Omega}\cosh(y_h\tau), \quad X_3 = \frac{\sqrt{Q}}{\Omega}.$$

where $P = -y^2 + y_h^2, \quad Q = 1 + x^2, \quad \Omega = A(x-y), \quad y_h = \sqrt{\frac{1}{A^2l^2} - 1}.$



(a) Class III geometry

(b) Class III BTZ

Figure 12: (a). The Class III slowly accelerating spacetime with $Al = 0.9$, mapped onto Poincare disk. In this figure, Only plotted in the region where P is positive. Several lines of constant Θ are shown in blue, with lines of constant R in dashed orange. (b). The Class III accelerating BTZ with Al = 0.9, mapped onto Poincare disk, which accelerates southwards. In this figure, spacetime wedge is removed from the global spacetime. long-dashed black lines are where edges exist.

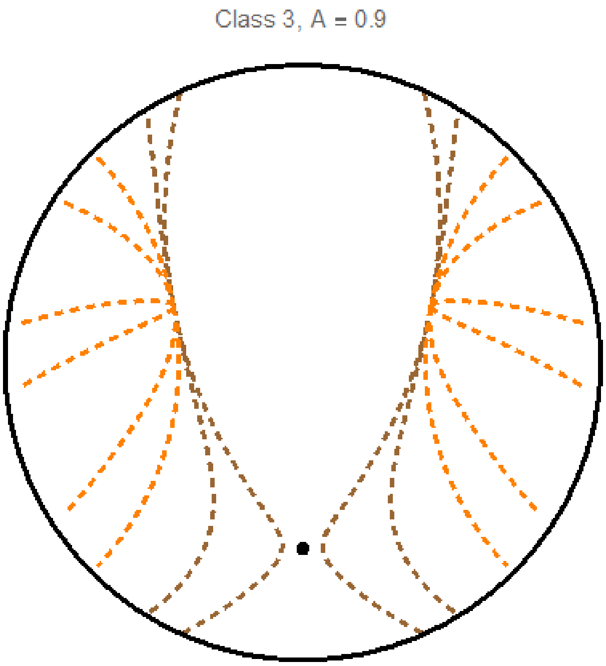


Figure 13: In this figure, only constant $y(=\text{constant } R)$ lines are drawn. Lines of constant R in dashed brown ($|y| > y_h$), orange ($-y_h < y < y_h$). In this figure, we can see that y coordinate in real number domain can't cover whole spacetime. So We have to treat Class III specially.

6 Conclusions

In this paper, starting with a simple theorem on three-dimensional gravity, various calculations and analyzes are made on accelerating black holes and conical deficits. Based on a solution similar to the C metric in 4-dimensional gravity, various possible geometries were examined, and related diagrams were drawn to enhance physical understanding. When we talk about the accelerating coordinate system, we usually start with a story about the Rindler coordinate, but naturally, that content seems to have melted. And when calculating the holographic mass, a negative number appears.

In Chapter 4, various calculations were made based on the various classes classified in Chapter 2. Since the thesis is based on a solution inspired by C metric, you may naturally think of A as an acceleration factor. In order to prevent this in advance, I proceeded with the start of the work to check whether the acceleration of A is correct. However, similar calculations were not performed in all classes, but when you do these calculations, the results are natural. And by drawing various figures, I could see that the x and y of the allowed domain region in some sub classes did not cover all areas. So, I was able to find out what the results would be if I unrestricted these restrictions within the range of real numbers and whether it could cover all areas. However, even in that case, some sub classes could not cover all areas. Perhaps in order to solve this phenomenon, it is necessary to expand x and y into the realm of complex numbers. However, if x and y become complex, the domains of r and θ involved must also be complex, which seems to lose its physical meaning. You can approach it as a simple mathematical curiosity, but it doesn't seem like a good physical approach. This case occurred in Class III, and I found out that it should be interpreted as $\mathrm{AdS}_3\mathrm{xAdS}_3$, not one AdS_3[1]. The resulting space is some subset of two copies of the anti de Sitter covering space and This is a 'braneworld solution'. I won't go into the detailed properties of these solutions here, but they're worth admitting as a curiosity.

The preliminary work to calculate the holographic mass was also difficult, but the results in Class I and Class II are quite different. The most notable difference is the sign of the two values. You can immediately see that the values calculated in Class I cannot always be positive. However, it is found that the values calculated in Class II always had positive values. Since my calculations are expected mass and integrated energy density, it always seemed like it should be a positive number, but the actual calculations are quite different. Researching this in depth seems to be one direction of pretty good research.

Finally, it seems that more research is needed on the subclass of the saturated class. Let's take a quick look. In the case of a saturated subclass in Class I, the value of P is greater than or equal to 0. However, since it cannot be 0, it can be expected that it will be divided into subclasses of right and left. For saturated subclasses in Class III, the value of P cannot be positive. Therefore, in this case, there is no possible model, and there is no saturated subclass in Class III. However, the case of the saturated subclass in Class II is quite different from the case in the other two classes, because it does not have much effect due to the characteristics of P in Class II. In Class II, regardless of the value of A, P is always a positive number, so it goes smoothly from slow to rapid, and there is no need to divide and distinguish subclasses. For this reason, you can see that there will not be a very interesting story in the saturated subclass. Therefore, this story is not included in this thesis.

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A Appendix A

Mathematica codes to draw figures of classes at constant time in a nutshell.

```
   (* State of sub class *)
boundary := PolarPlot[
   1,
   { θ, −π, π},
   PlotStyle → Black, Axes → False]
```

$A :=$

$Q[x\_] :=$

$\Omega[x\_, y\_] := A(x - y)$

$y_h :=$

$P[y\_] :=$

To Define X and Y, We have to choose what will be the appropriate spatial coordinate system. For simplicity, I write X_{global} as X. Similarly, XD means $X_{Poincare\ disk}$.

$X[x\_, y\_] :=$

$Y[x\_, y\_] :=$

$XD[x\_, y\_] := X[x, y]/(1 + \sqrt{1 + X[x, y]^2 + Y[x, y]^2})$

$YD[x\_, y\_] := Y[x, y]/(1 + \sqrt{1 + X[x, y]^2 + Y[x, y]^2})$

In order to check whether the results of calculation are correct, the location of the particle is picked separately and compare with the result of the hand calculation.

```
   particlelocation[x_, y_] = {XD[x, y], YD[x, y]} // Limit[#, y → Infinity] &
particle := Graphics[Black, PointSize[Large], Point[particlelocation[x, y]]]
equation_x := Table[
   ParametricPlot[
```

```
    {sign XD[i APV, y], YD[i APV, y]},
    {y, min_y, max_y},
    PlotStyle → Blue, PlotRange → {-1.1, 1.1}, Axes -> False],
    {i, { APV }}, {sign, {-1, 1}}]
Show[equations, boundary, particle, PlotLabel → StringJoin["State of sub class,
A = ", ToString[A]]]
```

APV = Appropriate values to draw a good-looking picture

You can view the calculation results by inserting various values and colors to suit your taste. What is written as equation_x is the main code that draws the blue line, and in this way, I am able to draw lines in a various range by using several equations. Taking Class III as an example, it can be seen that there is an area that cannot be covered in the figure. In order to cover this area, it seems that x and y values should be brought to the complex number or take the imaginary part of whole complex domain, but I did not do this. Since I am not sure whether this is physically appropriate and meaningful.

B Appendix B

Mathematica codes to calculate F_n and G_n in a nutshell.

```
   Maxn :=
Γ[ξ_] := 1 − A^2(1 − ξ^2)
x[ξ_, z_] := ξ + Sum[G_m[ξ]z^m, {m, 1, Maxn }]
dx[ξ_, z_] := (1 +Sum[G'_m[ξ]z^m, {m, 1, Maxn}])dξ + Sum[m G_m[ξ]z^(m−1), {m, 1, Maxn
}] dz
y[ξ_, z_] := ξ + Sum[F_m[ξ]z^m, {m, 1, Maxn }]
dy[ξ_, z_] := (1 +Sum[F'_m[ξ]z^m, {m, 1, Maxn}])dξ + Sum[m F_m[ξ]z^(m−1), {m, 1, Maxn
}] dz
Ω[ξ_, z_] := Simplify[
   A(x[ξ, z] − y[ξ, z])]
OO[ξ,z_] := Simplify[
   Collect[
   Series[1/Ω[ξ,z]^2, {z, 0, APV}]
   , z]] P[ξ_, z_] := Simplify[
   P(x, y)]
PP[ξ_, z_] := Simplify[
   Collect[
   Series[1/P[ξ,z], {z, 0, APV}]
   , z]]
Q[ξ_, z_] := Simplify[
   Q(x, y)]
QQ[ξ_, z_] := Simplify[
   Collect[
   Series[1/Q[ξ,z], {z, 0, APV}]
   , z]]
Collect[
```

```
Simplify[
OO[ξ, z] (P[ξ, z] dτ^2 - PP[ξ, z] dy[ξ, z]^2 - QQ[ξ, z] dx[ξ, z]^2)
], {z, dz, dτ, dξ} ]
```

APV = Appropriate number values to expand depending on the situation

Through this process, the overall outline is drawn, and F and G for low values of m are found. Continue tracking F and G while increasing the values of APV and Maxn. You can obtain as many differential equations as you need by making the coefficients of $dxd\xi$ and the odd power of z equal to zero. Differential equations can be solved using DSolve. Even why F_1 should be defined like this is naturally derived in this process. So I don't write down the definition of F_1 in the code.

You may worry about what to do if the result obtained by calculating when the values of Maxn and APP are small is different from the value obtained by calculating when the two values are large. However, since series calculation is performed, the m value that can be attached to each order of z is fixed. Therefore, iterative work of first calculating F and G for a small m with both values small, then increasing the two values and calculating again is very effective in these calculations. Even the logic for these iterations is very appropriate for calculations in 4 dimensions.

And when you repeat a task, there are times when you need to be sure that the calculation you are doing is correct. In that case, it is good to insert the code below and use it.

```
StringJoin[
  "P = ", ToString[G_m[ξ, z], StandardForm]
]
```

Acknowledgments

I would like to express my gratitude and appreciation to my supervisor, professor Nakwoo Kim. He taught me various courses and advised me to get necessary attitudes toward research. In particular, at the final stage, he helped me fill in the gaps in many ways, such as pointing out flaws in logic or giving suggestions for improvement.

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Fianlly, I would like to thank deeply my family. Thanks to their help, I was able to successfully complete my master's degree. Thank you for trusting me for a long time.