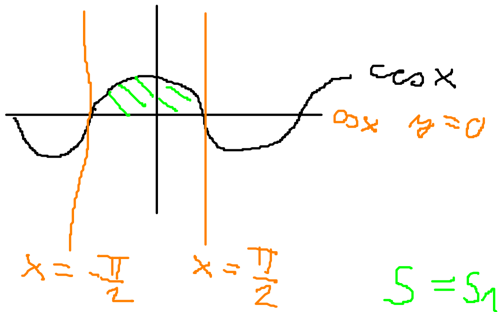
Vypočítajte obsah oblasti ohraničenej:
$f(x) = \cos x$
$x = -\frac{\pi}{2}$
$x = \frac{\pi}{2}$
os x $y=0$



$$S=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos x - 0)\,dx$$

Oblasť je symetrická, dá sa to vypočítať jednoduchšie

$$S = S_1+S_2 = 2S_2$$

$$S = 2\int_{0}^{\frac{\pi}{2}}(\cos x - 0)\,dx = 2\left[\sin x\right]_0^{\frac{\pi}{2}} = 2\left(\sin\frac{\pi}{2} - \sin 0\right) =$$

$$= 2(1-0) = 2\ j^2$$

Vypočítajte objem telesa, ktoré vznikne rotáciou tejto oblasti okolo osi x.

$$V=\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2 x - 0^2)\,dx$$

Keďže je to symetrická oblasť, vieme to vypočítať jednoduchšie

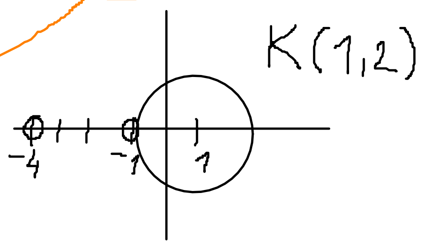
$$V = 2\pi\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx = \quad j^3$$

Vypočítajte Taylorov rozvoj funkcie $f(z)=\frac{z+2}{z^2+5z+4}$ so stredom v bode $a = 1$ a určte jeho konvergenciu.

$(z+1)(z+4)$

f(z) je analytická na $\mathbb{C}\setminus\{-1,-4\}$

hľadáme $f(z)=\sum_{n=0}^{\infty} c_n(z-1)$



$$f(z)=\frac{z+2}{(z+1)(z+4)} = \frac{A}{z+1}+\frac{B}{z+4} = \frac{A(z+4)+B(z+1)}{(z+1)(z+4)} = \frac{Az+4A+Bz+B}{(z+1)(z+4)} =$$

$z:\ 1 = A + B \quad |\cdot(-1) \longrightarrow B = 1 - A$

$z^0:\ 2 = 4A + B \quad ||.$

$1 = 3A \quad ||.-|.$

$A = \frac{1}{3}$

$B = 1 - \frac{1}{3}$

$B = \frac{2}{3}$

$$= \frac{\frac{1}{3}}{z+1} + \frac{\frac{2}{3}}{z+4} = \frac{\frac{1}{3}}{z-1+1+1} + \frac{\frac{2}{3}}{z-1+4+1} =$$

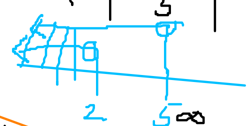
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \forall z,\ |z|<1$$

$$= \frac{\frac{1}{3}}{\left(\frac{2}{2}+\frac{z-1}{2}\right)2} + \frac{\frac{2}{3}}{\left(\frac{5}{5}+\frac{z-1}{5}\right)5} = \frac{1}{6}\cdot\frac{1}{1+\frac{z-1}{2}} + \frac{2}{15}\cdot\frac{1}{1+\frac{z-1}{5}}$$

$$= \frac{1}{6}\cdot\frac{1}{1-\left(-\frac{z-1}{2}\right)} + \frac{2}{15}\cdot\frac{1}{1-\left(-\frac{z-1}{5}\right)}$$

$$\left[\left\{\left|-\frac{z-1}{2}\right|<1 \wedge \left|-\frac{z-1}{5}\right|<1\right\} \Leftrightarrow \left\{\left|\frac{z-1}{2}\right|<1 \wedge \left|\frac{z-1}{5}\right|<1\right\}\right.$$

$$\left.\{|z-1|<2 \wedge |z-1|<5\} \Leftrightarrow |z-1|<2\right]$$



$$= \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{-z-1}{2}\right)^n + \frac{2}{15}\sum_{n=0}^{\infty}\left(\frac{-z-1}{5}\right)^n = \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z-1)^n + \frac{2}{15}\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n}(z-1)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3\cdot 2^{n+1}} + \frac{2(-1)^n}{3\cdot 5^{n+1}}\right)(z-1)^n$$

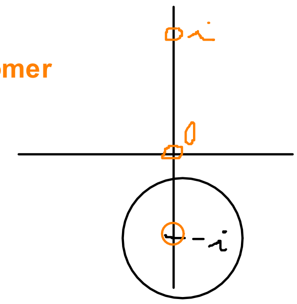
konverguje to na množine $M = \{z\in\mathbb{C}:\ |z-1|<2\} = K(1,2)$

Vypočítajte pomocou CIV alebo CIF

$\int_C \frac{\sin z}{(z^3+z)(z-i)}$, kde C je kladne orientovaná kružnica $|z+i| = \frac{1}{2}$

$|z-(-i)| = \frac{1}{2}$ polomer

stred



$z(z^2+1)(z-i)$

$z(z+i)(z-i)(z-i)$

$z(z+i)(z-i)^2$

f(z) nie je analytická v bodoch $0, -i, +i$

Keďže f(z) nie je analytická v bode z = - i nemôžeme použiť CIV a použijeme CIF

Podmienky pre CIF:

1. krivka C je JPČHU kladne orientovaná
2. $f_1(z)$ je analytická na oblasti

$$\int_C f(z)\,dz = \int_C \frac{\sin z}{z(z+i)(z-i)^2}\,dz = \int_C \frac{\overbrace{\frac{\sin z}{z(z-i)^2}}^{f_1(z)}}{z-(-i)}$$

podmienky pre CIF sú splnené

$$= 2\pi i\, f_1(-i)$$

$$= 2\pi i\, \frac{\sin(-i)}{-i(-i-i)^2} = \frac{2\pi \cancel{i} \sin(-i)}{-\cancel{i}(-2i)^2} =$$

$$= -\frac{2\pi \sin i}{-4i^2} = \frac{-2\pi \sin i}{4} = \frac{-\pi \sin i}{2}$$

$\sin(-i) = -\sin i$

z nepárnosti funkcie sin z

Ďalší príklad na samostatnom slajde

Parametrizácie

$|z - \text{stred}| = \text{polomer}$ $\qquad \varphi(t) = \text{stred} + \text{polomer} \cdot e^{it}$

$|z-2| = 7 \qquad \varphi(t) = 2 + 7 \cdot e^{it}$

$|z+3| = 8$

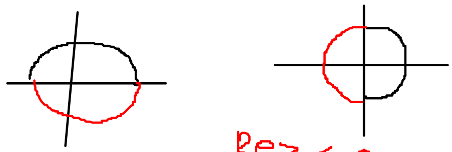
$|z-(-3)| = 8 \qquad \varphi(t) = -3 + 8 \cdot e^{it}$

$|z| = 5 \qquad \varphi(t) = 5e^{it}$

$|z-2| = 1 \qquad \varphi(t) = 2 + e^{it}$

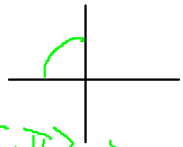
$|z| = 1 \qquad \varphi(t) = e^{it}$

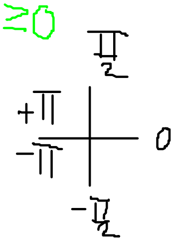
$\operatorname{Im} z \geq 0$ $\qquad \operatorname{Re} z \geq 0$



$\operatorname{Im} z \leq 0$ $\qquad \operatorname{Re} z \leq 0$

$\operatorname{Re} z \leq 0 \wedge \operatorname{Im} z \geq 0$





$C: \varphi\langle \frac{\pi}{2}, \pi \rangle \to \mathbb{C}$

Parametrizácia úsečky

$C: \varphi\langle 0,1\rangle \to \mathbb{C}, \quad \varphi(t) = A + t(B-A)$

Dosádzanie

$2e^{it} = \qquad z = 2e^{it} \qquad \bar{z} = 2e^{-it} \qquad |z| = 2$

$= 2(\cos t + i \sin t)$

$\operatorname{Re} z = 2\cos t$

$\operatorname{Im} z = 2\sin t$

$5+i \; + t(2+i) =$

$5 + t \cdot 2 + i(1+t) \qquad \bar{z} = 5 + 2t - i(1+t)$

$\operatorname{Re} z = 5 + 2t$

$\operatorname{Im} z = 1 + t$

$|z| = \sqrt{(5+2t)^2 + (1+t)^2}$

$2 + 3i \qquad |z| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

$2^5 = \underbrace{\overbrace{2 \cdot 2 \cdot 2}^{8} \cdot \overbrace{2 \cdot 2}^{4}}_{5} = 32$

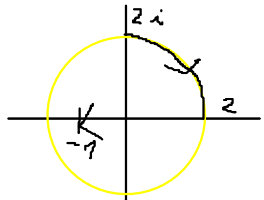
$(a \mp b)^2 = a^2 \mp 2ab + b^2$

$a^2 - b^2 = (a-b)(a+b)$

Príklad na ďalšom slajde

$\int_c z \cdot \operatorname{Im} z \, dz$, kde $c: |z|=2$, $\operatorname{Re} z \geq 0 \wedge \operatorname{Im} z \geq 0$ **od bodu 2i po bod 2**

a druhá časť krivky je daná úsečkou od bodu 2 po bod -1.



$C = C_1 + C_2$

$C_1: \varphi: \langle \overset{\alpha}{0}, \overset{\beta}{\frac{\pi}{2}} \rangle \to \mathbb{C}, \ \varphi(t) = 0 + 2e^{it} = 2(\cos t + i \sin t)$

$\varphi'(t) = i 2 e^{it}$

$C_2: \varphi: \langle \overset{\alpha}{0}, \overset{\beta}{1} \rangle \to \mathbb{C}, \ \varphi(t) = 2 + t(-1-2) = 2 - 3t + i \cdot 0 \ \operatorname{Im} z$

$\varphi'(t) = -3$

$\int_c f(z)\, dz = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\, dt$

$\int_c f(z)\, dz = \int_{c_1} f(z)\, dz + \int_{c_2} f(z)\, dz$

$= -\int_0^{\frac{\pi}{2}} \underbrace{2e^{it}}_{z} \underbrace{2 \sin t}_{\operatorname{Im} z} \underbrace{i 2 e^{it}}_{\varphi'(t)} dt + \int_0^1 \underbrace{(2-3t)}_{z} \cdot 0 \underbrace{(-3)}_{\varphi'(t)} dt =$

$= -8i \int_0^{\frac{\pi}{2}} e^{it} \sin t \, e^{it} dt + 0 = -8i \int_0^{\frac{\pi}{2}} e^{2it} \sin t \, dt = -\cancel{8}^{4} i \int_0^{\frac{\pi}{2}} e^{2it} \frac{e^{it} - e^{-it}}{\cancel{2i}}$

$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$= -4 \int_0^{\frac{\pi}{2}} (e^{3it} - e^{it})\, dt = -4 \left[\frac{e^{3it}}{3i} \right]_0^{\frac{\pi}{2}} + 4 \left[\frac{e^{it}}{i} \right]_0^{\frac{\pi}{2}} =$

$= \frac{-4}{3i}\frac{i}{i} \left(e^{3i\frac{\pi}{2}} - e^0 \right) + \frac{4}{i}\frac{i}{i} \left(e^{i\frac{\pi}{2}} - e^0 \right) = \frac{4}{3} i \left(e^{\frac{3}{2}\pi i} - 1 \right) - 4i \left(e^{i\frac{\pi}{2}} - 1 \right) =$

$e^{\frac{3}{2}\pi i} = \cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi = 0 - i = -i$

$e^{\frac{\pi}{2} i} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i = i$

$= \frac{4}{3} i(-i - 1) - 4i(i - 1) = -\frac{4}{3} \overset{-1}{i^2} - \frac{4}{3} i - 4 \overset{-1}{i^2} + 4i =$

$= 4i - \frac{4}{3}i + \frac{4}{3} + 4 = \frac{+12+4}{3} + i \frac{12-4}{3} =$

$= \frac{16}{3} + i \frac{8}{3}$