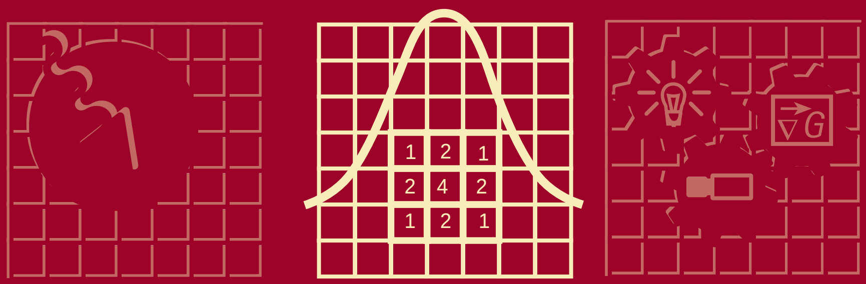

HANDBOOK OF COMPUTER VISION AND APPLICATIONS

Volume 2
Signal Processing and Pattern Recognition

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22 Fuzzy Image Processing

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22.1 Introduction

Our world is *fuzzy*, and so are images, projections of the real world onto the image sensor. Fuzziness quantifies vagueness and ambiguity, as opposed to crisp memberships. The types of uncertainty in images are manifold, ranging over the entire chain of processing levels, from pixel-based grayness ambiguity over fuzziness in geometrical description up to uncertain knowledge in the highest processing level.

The human visual system has been perfectly adapted to handle uncertain information in both data and knowledge. It would be hard to define quantitatively how an object, such as a car, has to look in terms of geometrical primitives with exact shapes, dimensions, and colors. Instead, we are using a descriptive language to define features that eventually are subject to a wide range of variations. The interrelation of a few such "fuzzy" properties sufficiently characterizes the object of interest. Fuzzy image processing is an attempt to translate this ability of human reasoning into computer vision problems as it provides an intuitive tool for inference from imperfect data.

Where is the transition between a gray-value slope and an edge? What is the border of a blurred object? Which gray values exactly belong to the class of "bright" or "dark" pixels? These questions show, that image features almost naturally have to be considered fuzzy. Usually these problems are just overruled by assigning thresholds—heuristic or computed—to the features in order to classify them. Fuzzy logic allows one to quantify appropriately and handle imperfect data. It also allows combining them for a final decision, even if we only know heuristic rules, and no analytic relations.

Fuzzy image processing is special in terms of its relation to other computer vision techniques. It is not a solution for a special task, but rather describes a new class of image processing techniques. It provides a new methodology, augmenting classical logic, a component of any computer vision tool. A new type of image understanding and treatment has to be developed. Fuzzy image processing can be a single image processing routine, or complement parts of a complex image processing chain.

During the past few decades, fuzzy logic has gained increasing importance in control theory, as well as in computer vision. At the same time, it has been continuously attacked for two main reasons: It has been considered to lack a sound mathematical foundation and to be nothing but just a clever disguise for probability theory. It was probably its name that contributed to the low reputation of fuzzy logic. Meanwhile, fuzzy logic definitely has matured and can be considered to be a mathematically sound extension of multivalued logic. Fuzzy logical reasoning and probability theory are closely related without doubt.

They are, however, *not* the same but *complementary*, as we will show in Section 22.1.3.

This chapter gives a concise overview of the basic principles and potentials of state of the art fuzzy image processing, which can be applied to a variety of computer vision tasks.

22.1.1 A brief history

In 1965, Zadeh introduced the idea of *fuzzy sets*, which are the extension of classical crisp sets [1]. The idea is indeed simple and natural: The membership of elements of a set is a matter of grade rather than just zero or one. Therefore, membership grade and membership functions play the key role in all systems that apply the idea of fuzziness. Prewitt was the first researcher to detect the potentials of fuzzy set theory for representation of digital images [2]:

> " ... a pictorial object is a fuzzy set which is specified by some membership function defined on all picture points. From this point of view, each image point participates in many memberships. Some of this uncertainty is due to degradation, but some of it is inherent. The role of object extraction in machine processing, like the role of figure/ground discrimination in visual perception, is uncertainty-reducing and organizational. In fuzzy set terminology, making figure/ground distinctions is equivalent to transforming from membership functions to characteristic functions."

In 1969, Ruspini introduced the fuzzy partitioning in clustering [3]. In 1973, the first fuzzy clustering algorithm called fuzzy c means was introduced by Bezdek [4]. It was the first fuzzy approach to pattern recognition. Rosenfeld extended the digital topology and image geometry to fuzzy sets at the end of the 70s and beginning of the 80s [5, 6, 7, 8, 9, 10, 11]. It was probably the most important step toward the development of a mathematical framework of fuzzy image processing because image geometry and digital topology play a pivotal role in image segmentation and representation, respectively. One of the pioneers of fuzzy image processing is S. K. Pal. Together with coworkers, he developed a variety of new fuzzy algorithms for image segmentation and enhancement [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. In the past decades, many other researchers have also investigated the potentials of fuzzy set theory for developing new image processing techniques. The width and depth of these investigations allow us to speak of a *new methodology* in computer vision: fuzzy image processing. But many questions should be answered: What actually is fuzzy image processing? Why should we use it? Which advantages and disadvantages have fuzzy algorithms for image processing? In following sections of this chapter, we will try to answer these questions.

22.1.2 Basics of fuzzy set theory

The two basic components of fuzzy systems are *fuzzy sets* and *operations on fuzzy sets*. *Fuzzy logic* defines rules, based on combinations of fuzzy sets by these operations. This section is based on the basic works of Zadeh [1, 24, 25, 26, 27].

Crisp sets. Given a universe of discourse $X = \{x\}$, a crisp (conventional) set A is defined by enumerating all elements $x \in X$

$$A = \{x_1, x_2, \dots, x_n\} \tag{22.1}$$

that belong to A. The membership can be expressed by a function f_A, mapping X on a binary value:

$$f_A: X \longrightarrow \{0,1\}, \quad f_A = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases} \tag{22.2}$$

Thus, an arbitrary x either belongs to A, or it does not, partial membership is not allowed.

For two sets A and B, combinations can be defined by the following operations:

$$\begin{aligned} A \cup B &= \{x \,|\, x \in A \text{ or } x \in B\} \\ A \cap B &= \{x \,|\, x \in A \text{ and } x \in B\} \\ \bar{A} &= \{x \,|\, x \notin A, \; x \in X\} \end{aligned} \tag{22.3}$$

Additionally, the following rules have to be satisfied:

$$A \cap \bar{A} = \emptyset, \quad \text{and} \quad A \cup \bar{A} = X \tag{22.4}$$

Fuzzy sets. Fuzzy sets are a generalization of classical sets. A fuzzy set A is characterized by a *membership function* $\mu_A(x)$, which assigns each element $x \in X$ a real-valued number ranging from zero to unity:

$$A = \left\{(x, \mu_A(x)) \,|\, x \in X\right\} \tag{22.5}$$

where $\mu_A(x): \; X \to [0,1]$. The *membership function* $\mu_A(x)$ indicates to which extend the element x has the attribute A, as opposed to the binary membership value of the mapping function f_A for crisp sets Eq. (22.2).

The choice of the shape of membership functions is somewhat arbitrary. It has to be adapted to the features of interest and to the final goal of the fuzzy technique. The most popular membership functions are given by piecewise-linear functions, second-order polynomials, or trigonometric functions.

Figure 22.1 illustrates an example of possible membership functions. Here, the distribution of an optical flow vector (Chapter 13),

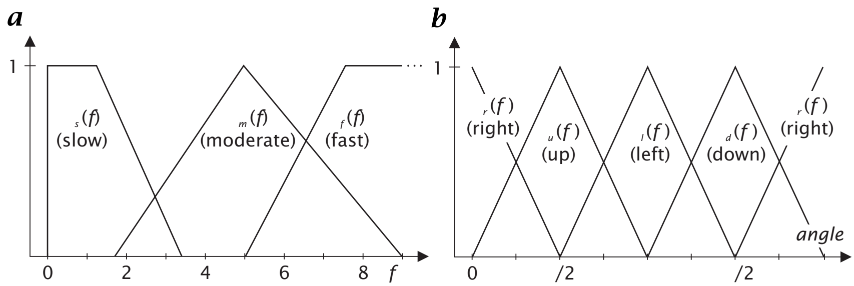

Figure 22.1: *Possible membership functions for* ***a*** *the magnitude and* ***b*** *the direction of an optical flow vector* $\boldsymbol{f}$*.*

is characterized by fuzzy magnitude, $f = \|\boldsymbol{f}\|$, and fuzzy orientation angle, given by two independent sets of membership functions.

It is important to note that the membership functions do not necessarily have to add up to unity:

$$\mu_A(x) + \mu_B(x) + \ldots \neq 1 \tag{22.6}$$

as opposed to relative probabilities in stochastic processes.

A common notation for fuzzy sets, which is perfectly suited for fuzzy image processing, has been introduced by Zadeh [26]. Let X be a finite set $X = \{x_1, \ldots, x_n\}$. A fuzzy set A can be represented as follows:

$$A = \frac{\mu_A(x_1)}{x_1} + \ldots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i} \tag{22.7}$$

For infinite X we replace the sum in Eq. (22.7) by the following integral:

$$A = \int_X \frac{\mu_A(x)}{x} dx \tag{22.8}$$

The individual elements $\mu_A(x_i)/x_i$ represent fuzzy sets, which consist of one single element and are called *fuzzy singletons*. In Section 22.3.1 we will see how this definition is used in order to find a convenient fuzzy image definition.

Operations on fuzzy sets. In order to manipulate fuzzy sets, we need to have operations that enable us to combine them. As fuzzy sets are defined by membership functions, the classical set theoretic operations have to be replaced by function theoretic operations. Given two fuzzy

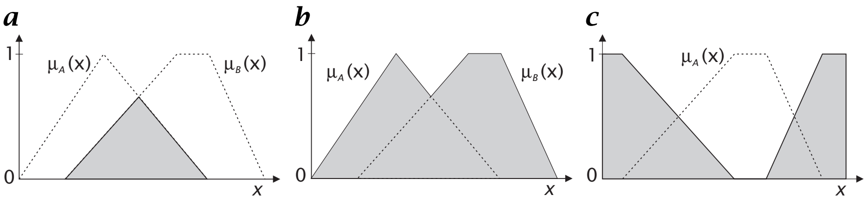

Figure 22.2: *Operations on fuzzy sets. The boundary of the shaded curves represent the **a** intersection $\mu_{A\cap B}$ of the fuzzy sets μ_A and μ_B, **b** the union $\mu_{A\cup B}$ of the fuzzy sets μ_A and μ_B, and **c** complement $\mu_{\bar{A}}$ of the fuzzy set μ_A.*

sets A and B, we define the following pointwise operations ($\forall x \in X$):

$$\begin{array}{lll} \text{equality} & A = B & \Leftrightarrow \ \mu_A(x) = \mu_B(x) \\ \text{containment} & A \subset B & \Leftrightarrow \ \mu_A(x) \le \mu_B(x) \\ \text{complement} & \bar{A}, & \mu_{\bar{A}}(x) = 1 - \mu_A(x) \\ \text{intersection} & A \cap B, & \mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \\ \text{union} & A \cup B, & \mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \end{array} \tag{22.9}$$

It can be easily verified that the conditions of Eq. (22.4) are no longer satisfied

$$\begin{array}{rcl} A \cap \bar{A} & = & \min\{\mu_A(x), 1 - \mu_A(x)\} \neq \emptyset \\ A \cup \bar{A} & = & \max\{\mu_A(x), 1 - \mu_A(x)\} \neq X \end{array} \tag{22.10}$$

for $\mu(x) \neq 1$, due to the partial membership of fuzzy sets.

The results of the complement, intersection, and union operations on fuzzy sets is illustrated in Fig. 22.2. The operations defined in Eq. (22.9) can be easily extended for more than two fuzzy sets and combinations of different operations.

Linguistic variables. An important feature of fuzzy systems is the concept of *linguistic variables*, introduced by Zadeh [26]. In order to reduce the complexity of precise definitions, they make use of words or sentences in a natural or artificial language, to describe a vague property.

A linguistic variable can be defined by a discrete set of membership functions $\{\mu_{A_1}, \ldots, \mu_{A_N}\}$ over the set $\{x\} = U \subset X$. The membership functions quantify the variable x by assigning a partial membership of x with regard to the terms A_i. An example of a linguistic variable could be the property "velocity," composed of the terms "slow," "moderate," and "fast." The individual terms are numerically characterized by the membership functions μ_s, μ_m, and μ_f. A possible realization is shown in Fig. 22.1a.

Linguistic hedges. Given a linguistic variable x represented by the set of membership functions $\{\mu_{A_i}\}$, we can change the meaning of a linguistic variable by modifying the shape (i. e., the numerical representation) of the membership functions. The most important linguistic hedges are *intensity modification*, μ^i, *concentration*, μ^c, and *dilation*, μ^d:

$$\begin{aligned}\mu^i(x) &= \begin{cases} 2\mu^2(x) & \text{if } 0 \le \mu(x) \le 0.5 \\ 1 - 2[1-\mu(x)]^2 & \text{otherwise} \end{cases} \\ \mu^c(x) &= \mu^2(x) \\ \mu^d(x) &= \sqrt{\mu(x)} \end{aligned} \tag{22.11}$$

An application example using dilation and concentration modification is shown in Fig. 22.19.

Fuzzy logic. The concept of linguistic variables allows us to define combinatorial relations between properties in terms of a language.

Fuzzy logic—an extension of classical *Boolean logic*—is based on linguistic variables, a fact which has assigned fuzzy logic the attribute of *computing with words* [28].

Boolean logic uses *Boolean operators*, such as AND ($\wedge$), OR ($\vee$), NOT ($\neg$), and combinations of them. They are defined for binary values of the input variables and result in a binary output variable. If we want to extend the binary logic to a combinatorial logic of linguistic variables, we need to redefine the elementary logical operators. In fuzzy logic, the Boolean operators are replaced by the operations on the corresponding membership functions, as defined in Eq. (22.9).

Let $\{\mu_{A_i}(x_1)\}$ and $\{\mu_{B_i}(x_2)\}$ be two linguistic variables of two sets of input variables $\{x_1\}$ and $\{x_2\}$. The set of output variables $\{x_3\}$ is characterized by the linguistic variable $\{\mu_{C_i}(x_3)\}$. We define the following basic combinatorial rules:

if $(A_j \wedge B_k)$ **then** C_l:

$$\mu'_{C_l}(x_3) = \left(\min\left\{\mu_{A_j}(x_1), \mu_{B_k}(x_2)\right\}\right)\mu_{C_l}(x_3) \tag{22.12}$$

if $(A_j \vee B_k)$ **then** C_l:

$$\mu'_{C_l}(x_3) = \left(\max\left\{\mu_{A_j}(x_1), \mu_{B_k}(x_2)\right\}\right)\mu_{C_l}(x_3) \tag{22.13}$$

if $(\neg A_j)$ **then** C_l:

$$\mu'_{C_l}(x_3) = \left(1 - \mu_{A_j}(x_1)\right)\mu_{C_l}(x_3) \tag{22.14}$$

Thus, the output membership function $\mu_{C_i}(x_3)$ is modified (weighted) according to the combination of A_i and B_j at a certain pair (x_1, x_2). These rules can easily be extended to more than two input variables. A fuzzy inference system consists of a number of if-then rules, one for any membership function μ_{C_i} of the output linguistic variable $\{\mu_{C_i}\}$.

Given the set of modified output membership functions $\{\mu'_{C_i}(x_3)\}$, we can derive a single output membership function $\mu_C(x_3)$ by *accumulating* all μ'_{C_i}. This can be done by combining the μ'_{C_i} by a logical OR, that is, the maximum operator:

$$\mu_C(x_3) = \max_i \left\{\mu'_{C_i}(x_3)\right\} \tag{22.15}$$

Defuzzification. The resulting output membership function $\mu_C(x_3)$ can be assigned a numerical value $x \in \{x\}$ by *defuzzification*, reversing the process of fuzzification. There are a variety of approaches to get a single number from a membership function reported in the literature. The most common techniques are computing the *center of area* (center of mass) or the *mean of maxima* of the corresponding membership function. Applications examples are shown in Section 22.4.3.

The step of defuzzification can be omitted if the final result of the fuzzy inference system is given by a membership function, rather than a crisp number.

22.1.3 Fuzzy logic versus probability theory

It has been a long-standing misconception that fuzzy logic is nothing but another representation of probability theory. We do not want to contribute to this dispute, but rather try to outline the basic difference.

Probability describes the uncertainty in the occurrence of an event. It allows predicting the event by knowledge about its relative frequency within a large number of experiments. After the experiment has been carried out, the event either has occurred or not. There is no uncertainty left. Even if the probability is very small, it might happen that the unlikely event occurs. To treat stochastic uncertainty, such as random processes (e. g., noise), probability theory is a powerful tool, which has conquered an important area in computer vision (Chapter 26).

There are, however, other uncertainties, that can not be described by random processes. As opposed to probability, fuzzy logic represents the *imperfection* in the informational content of the event. Even after the measurement, it might not be clear, if the event has happened, or not.

For illustration of this difference, consider an image to contain a single edge, which appears at a certain rate. Given the probability distribution, we can predict the likelihood of the edge to appear after a certain number of frames. It might happen, however, that it appears in every image or does not show up at all. Additionally, the edge may be corrupted by noise. A noisy edge can appropriately be detected with probabilistic approaches, computing the likelihood of the noisy measurement to belong to the class of edges. But how do we define the edge? How do we classify an image that shows a gray-value slope? A noisy slope stays a slope even if all noise is removed. If the slope is extended over the entire image we usually do not call it an edge. But if the slope is "high" enough and only extends over a "narrow" region, we tend to call it an edge. Immediately the question arises: How large is "high" and what do we mean with "narrow?"

In order to quantify the shape of an edge, we need to have a model. Then, the probabilistic approach allows us to extract the model parameters, which represent edges in various shapes. But how can we treat this problem, without having an appropriate model? Many real world applications are too complex to model all facets necessary to describe them quantitatively. Fuzzy logic does not need models. It can handle vague information, imperfect knowledge and combine it by heuristic rules—in a well-defined mathematical framework. This is the strength of fuzzy logic!

22.2 Why fuzzy image processing?

In computer vision, we have different theories, methodologies, and techniques that we use to solve different practical problems (e. g., digital geometry, mathematical morphology, statistical approaches, probability theory, etc.). Because of great diversity and complexity of problems in image processing, we always require new approaches. There are some reasons to use fuzzy techniques as a new approach. We briefly describe two of them [29].

22.2.1 Framework for knowledge representation/processing

The most important reason why one should investigate the potentials of fuzzy techniques for image processing is that fuzzy logic provides us with a powerful mathematical framework for representation and processing of expert knowledge. Here, the concept of linguistic variables and the fuzzy if-then rules play a key role. Making a human-like processing possible, fuzzy inference engines can be developed using expert knowledge. The rule-based techniques, for example, have the general form:

If condition A_1, **and** condition A_2, **and ...**, **then** action B

In real applications, however, the conditions are often partially satisfied (e.g., the question of homogeneity in a neighborhood can not always be

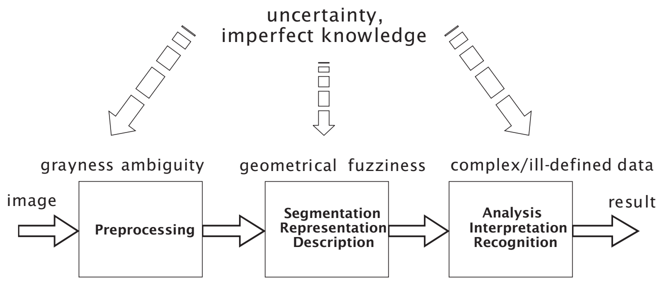

Figure 22.3: *Imperfect knowledge in image processing (similar to [29]).*

answered with a crisp yes or no). Fuzzy if-then rules allow us to perform actions also partially.

22.2.2 Management of vagueness and ambiguity

Where is the boundary of a region? Is the region homogeneous? Which gray level can serve as a threshold? Should we apply noise filtering, edge enhancement, or smoothing technique? What is a road or a tree in a scene analysis situation? These and many other similar questions arise during image processing—from low-level through high-level processing—and are due to vagueness and ambiguity. There are many reasons why our knowledge in such situations is imperfect. Imprecise results, complex class definitions, different types of noise, concurring evidences, and finally, the inherent fuzziness of many categories are just some sources of uncertainty or imperfect knowledge.

Distinguishing between low-level, intermediate-level, and high-level image processing, the imperfect knowledge is due to grayness ambiguity, geometrical fuzziness, and imprecision/complexity (Fig. 22.3). Fuzzy techniques offer a suitable framework for management of these problems.

22.3 Fuzzy image understanding

To use the fuzzy logic in image processing applications, we have to develop a new image understanding. A new image definition should be established, images and their components (pixels, histograms, segments, etc.) should be fuzzified (transformation in membership plane), and the fundamental topological relationships between image parts should be extended to fuzzy sets (fuzzy digital topology).

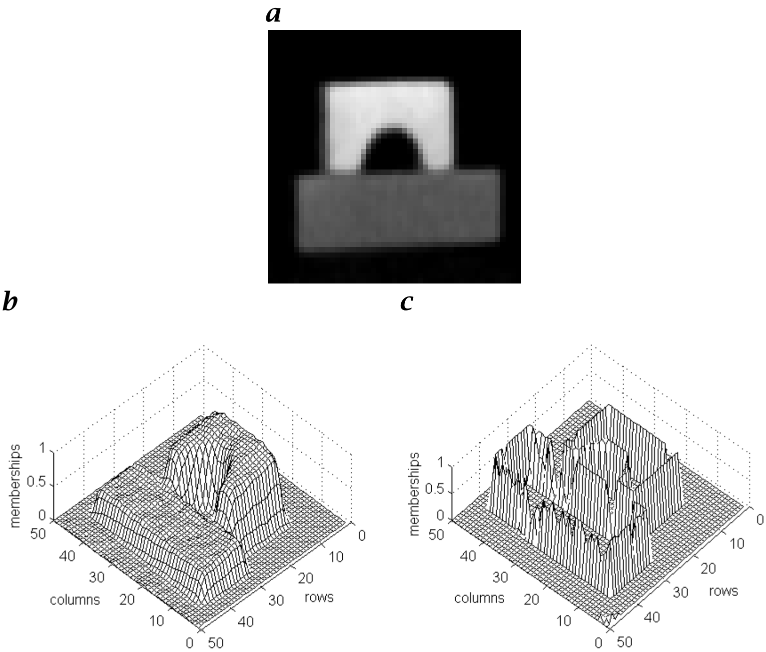

Figure 22.4: *Images as an array of fuzzy singletons.* ***a*** *test image as a fuzzy set regarding* ***b*** *brightness (bright pixels have higher memberships), and* ***c*** *edginess (edge pixels have higher memberships).*

22.3.1 A new image definition: Images as fuzzy sets

An image G of size $M \times N$ with L gray levels can be defined as an array of fuzzy singletons (fuzzy sets with only one supporting point) indicating the membership value μ_{mn} of each image point x_{mn} regarding a predefined image property (e.g., brightness, homogeneity, noisiness, edginess, etc.) [13, 15, 29]:

$$G = \bigcup_{m=1}^{M} \bigcup_{n=1}^{N} \frac{\mu_{mn}}{x_{mn}} \tag{22.16}$$

The definition of the membership values depends on the specific requirements of particular application and on the corresponding expert knowledge. Figure 22.4 shows an example where brightness and edginess are used to define the membership grade of each pixel.

22.3.2 Image fuzzification: From images to memberships

Fuzzy image processing is a kind of nonlinear image processing. The difference to other well-known methodologies is that fuzzy techniques

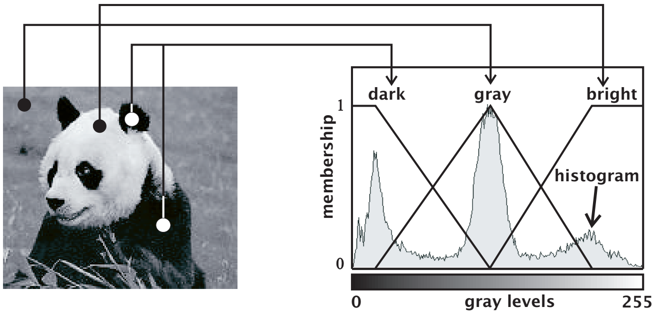

__Figure 22.5:__ Histogram-based gray-level fuzzification. The location of membership functions is determined depending on specific points of image histogram (adapted from [31]).

operate on membership values. The image fuzzification (generation of suitable membership values) is, therefore, the first processing step. Generally, three various types of image fuzzification can be distinguished: histogram-based gray-level fuzzification, local neighborhood fuzzification, and feature fuzzification [29].

As in other application areas of fuzzy set theory, the fuzzification step should be sometimes optimized. The number, form, and location of each membership function could/should be adapted to achieve better results. For instance, genetic algorithms are performed to optimize fuzzy rule-based systems [30].

Histogram-based gray-level fuzzification [29]. To develop any point operation (global histogram-based techniques), each gray level should be assigned with one or more membership values regarding to the corresponding requirements.

Example 22.1: Image brightness

The *brightness* of an image can be regarded as a fuzzy set containing the subsets *dark*, *gray*, and *bright* intensity levels (of course, one may define more subsets such as very dark, slightly bright, etc.). Depending on the normalized image histogram, the location of the membership functions can be determined (Fig. 22.5). It should be noted that for histogram-based gray-level fuzzification some knowledge about image and its histogram is required (e.g., minimum and maximum of gray-level frequencies). The detection accuracy of these histogram points, however, should not be very high as we are using the concept of fuzziness (we do not require precise data).

Local neighborhood fuzzification [29]. Intermediate techniques (e.g., segmentation, noise filtering etc.) operate on a predefined neighborhood of pixels. To use fuzzy approaches to such operations, the fuzzi-

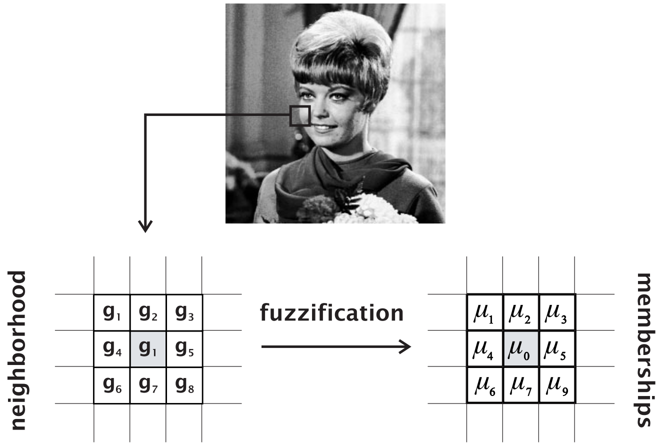

Figure 22.6: *On local neighborhood fuzzification [31].*

fication step should also be done within the selected neighborhood (Fig. 22.6). The local neighborhood fuzzification can be carried out depending on the task to be done. Of course, local neighborhood fuzzification requires more computing time compared with histogram-based approach. In many situations, we also need more thoroughness in designing membership functions to execute the local fuzzification because noise and outliers may falsify membership values.

Example 22.2: Edginess

Within 3×3-neighborhood U we are interested in the degree of membership of the center point to the fuzzy set *edge pixel*. Here, the edginess μ_e is a matter of grade. If the 9 pixels in U are assigned the numbers $0, \ldots 8$ and G_0 denotes the center pixel, a possible membership function can be the following [29]:

$$\mu_e = 1 - \left[1 + \frac{1}{\Delta} \sum_{i=0}^{8} \| G_0 - G_i \| \right]^{-1} \tag{22.17}$$

with $\Delta = \max_U (G_i)$.

Example 22.3: Homogeneity

Within 3×3-neighborhood U, the homogeneity is regarded as a fuzzy set. The membership function μ^h can be defined as:

$$\mu_h = 1 - \frac{G^{max,l} - G^{min,l}}{G^{max,g} - G^{min,g}} \tag{22.18}$$

where $G^{min,l}$, $G^{max,l}$, $G^{min,g}$, and $G^{max,g}$ are the local and global minimum and maximum gray levels, respectively.

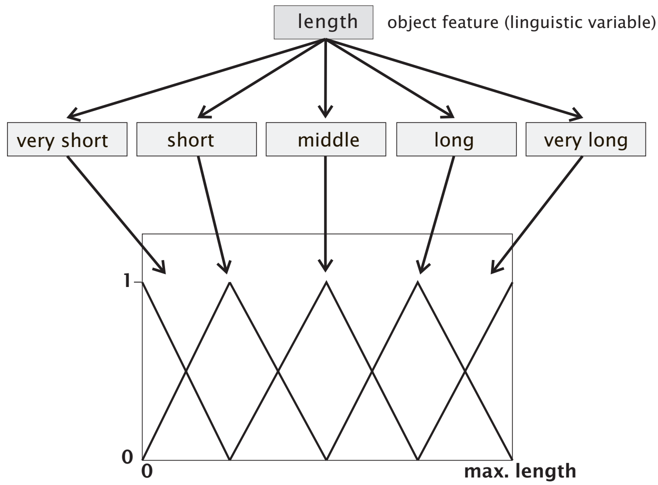

Figure 22.7: *Feature fuzzification using the concept of linguistic variables [29].*

Feature fuzzification [29]. For high-level tasks, image features should usually be extracted (e.g., length of objects, homogeneity of regions, entropy, mean value, etc.). These features will be used to analyze the results, recognize the objects, and interpret the scenes. Applying fuzzy techniques to this tasks, we need to fuzzify the extracted features. It is necessary not only because fuzzy techniques operate only on membership values but also because the extracted features are often incomplete and/or imprecise.

Example 22.4: Object length

> If the length of an object was calculated in a previous processing step, the fuzzy subsets *very short*, *short*, *middle-long*, *long* and *very long* can be introduced as terms of the linguistic variable *length* in order to identify certain types of objects (Fig. 22.7).

22.3.3 Fuzzy topology: Noncrisp definitions of topological relationships

Image segmentation is a fundamental step in all image processing systems. However, the image regions can not always be defined crisply. It is sometimes more appropriate to consider the different image parts, regions, or objects as fuzzy subsets of the image. The topological relationships and properties, such as *connectedness* and *surroundedness*, can be extended to fuzzy sets. In image analysis and description, the digital topology plays an important role. The topological relationships between parts of an image are conventionally defined for (crisp) subsets of image. These subsets are usually extracted using different types

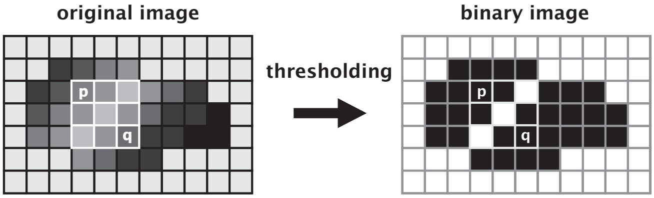

Figure 22.8: On crisp and fuzzy connectedness. The pixels p and q are fuzzy connected in original image, and not connected in the binary image.

of segmentation techniques (e. g., thresholding). Segmentation procedures, however, are often a strong commitment accompanied by loss of information. In many applications, it would be more appropriate to make soft decisions by considering the image parts as fuzzy subsets. In these cases, we need the extension of (binary) digital topology to fuzzy sets. The most important topological relationships are connectedness, surroundedness and adjacency. In the following, we consider an image g with a predefined neighborhood $U \subset g$ (e.g., 4- or 8-neighborhood).

Fuzzy connectedness [5]. Let p and $q \in U(\subset g)$ and let μ be a membership function modeling G or some regions of it. Further, let δ_{pq} be paths from p to q containing the points r. The degree of *connectedness* of p and q in U with respect to μ can be defined as follows (Fig. 22.8):

$$\text{connectedness}_{\mu}(p,q) \equiv \max_{\delta_{pq}} \left[\min_{r \in \delta_{pq}} \mu(r) \right] \tag{22.19}$$

Thus, if we are considering the image segments as fuzzy subsets of the image, the points p and q are connected regarding to the membership function μ if the following condition holds:

$$\text{connectedness}_{\mu}(p,q) \geq \min\left[\mu(p), \mu(q)\right] \tag{22.20}$$

Fuzzy surroundedness [5, 11, 32]. Let μ_A, μ_B and μ_C be the membership functions of fuzzy subsets A, B and C of image G. The fuzzy subset C separates A from B if for all points p and r in $U \subset G$ and all paths δ from p to q, there exists a point $r \in \delta$ such that the following condition holds:

$$\mu(C)(r) \geq \min\left[\mu_A(p), \mu_B(q)\right] \tag{22.21}$$

In other words, B surrounds A if it separates A from an unbounded region on which $\mu_A = 0$. Depending on particular application, appropriate membership functions can be found to measure the surroundedness. Two possible definitions are given in Example 22.5, where $\mu_{B \odot A}$

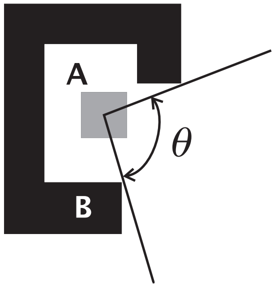

Figure 22.9: *Example for calculation of fuzzy surroundedness.*

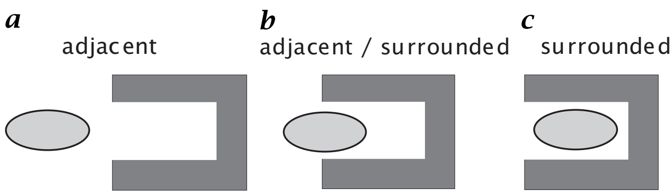

Figure 22.10: *Relationship between adjacency and surroundedness.*

defines the membership function of the linguistic variable 'B surrounds A' (Fig. 22.9) [29, 32].

Example 22.5: Surroundedness

$$\mu_{B\odot A}(\theta) = \begin{cases} \dfrac{\pi-\theta}{\pi} & 0 \le \theta < \pi \\ 0 & \text{otherwise} \end{cases}, \quad \mu_{B\odot A}(\theta) = \begin{cases} \cos^2\left(\dfrac{\theta}{2}\right) & 0 \le \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

Fuzzy adjacency [5, 11, 17]. The adjacency of two disjoint (crisp) sets is defined by the length of their common border. Following, a brief description of generalization of this definition to fuzzy sets.

Let μ_1 and μ_2 be piecewise-constant fuzzy sets of G. The image G can be partitioned in a finite number of bounded regions G_i, meeting pairwise along arcs, on each of which $\mu_1(i)$ and $\mu_2(j)$ are constant. If μ_1 and μ_2 are disjoint then in each region G_i either $\mu_1 = 0$ or $\mu_2 = 0$. Let $A(i,j,k)$ be the k-th arc along which G_i and G_j meet. Then the adjacency of μ_1 and μ_2 can be defined as follows:

$$\text{adjacency}(\mu_1,\mu_2) = \sum_{i,j,k\ i\neq j} \mu_1(i)\mu_2(j)\|A(i,j,k)\| \tag{22.22}$$

where $\|A(i,j,k)\|$ indicates the length of the k-th arc. This definition may not fully agree with our intuition in some situations. For instance, consider the following cases:

1. $\mu_1 = 0.1$, $\mu_2 = 0.15 \longrightarrow$ adjacency $= 0.015$
2. $\mu_1 = 0.7$, $\mu_1 = 0.75 \longrightarrow$ adjacency $= 0.525$

The difference of membership values is the same in both cases, namely 0.05. Intuitively, one may expect that the adjacency should also be the same in both cases. Therefore, it may be useful to use other definitions of fuzzy adjacency:

$$\text{adjacency}(\mu_1, \mu_2) = \sum_{i,j,k} \frac{\|A(i,j,k)\|}{1 + \|\mu_1(i)\mu_2(j)\|} \tag{22.23}$$

Now, to introduce a definition for degree of adjacency for fuzzy image subsets, let us consider two segments S_1 and S_2 of an image G. Further, let $B(S_1)$ be the set of border pixels of S_1, and p an arbitrary member of B. The degree of adjacency, can be defined with respect to the definition of adjacency in Eq. (22.22) as follows:

$$\text{degree of adjacency}(\mu_1, \mu_2) = \sum_{p \in B(S_1)} \frac{1}{1 + d(p)} \tag{22.24}$$

where $d(p)$ is the shortest distance of pixel p from the border of segment S_2. Considering the adjacency definition in Eq. (22.23), the degree of adjacency can also be defined as follows:

$$\text{degree of adjacency}(\mu_1, \mu_2) = \sum_{p \in B(S_1)} \frac{1}{1 + \|\mu_1(i)\mu_2(j)\|} \frac{1}{1 + d(p)} \tag{22.25}$$

where $p \in S_1$ and $q \in S_2$ are border pixels, and $d(p)$ is the shortest distance between p and q. Here, it should be noted that there exists a close relationship between adjacency and surroundedness (Fig. 22.10a,b). Depending on particular requirements, one may consider one or both of them to describe spatial relationships.

22.4 Fuzzy image processing systems

Fuzzy image processing consists (as all other fuzzy approaches) of three stages: fuzzification, suitable operations on membership values, and, if necessary, defuzzification (Fig. 22.11). The main difference to other methodologies in image processing is that input data (histograms, gray levels, features, ...) will be processed in the so-called membership plane where one can use the great diversity of fuzzy logic, fuzzy set theory and fuzzy measure theory to modify/aggregate the membership values, classify data, or make decisions using fuzzy inference. The new membership values are retransformed in the gray-level plane

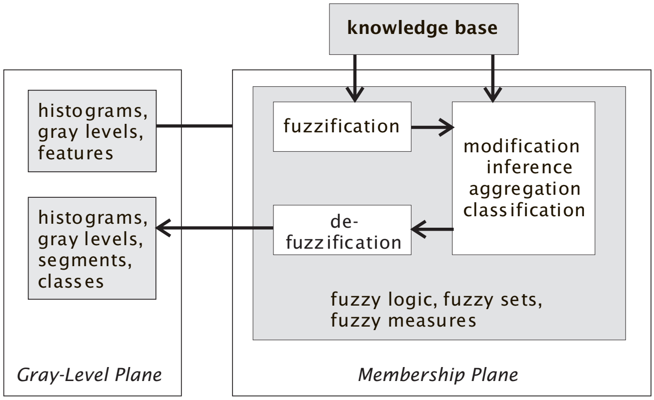

Figure 22.11: *General structure of fuzzy image processing systems [29].*

Table 22.1: *On relationships between imperfect knowledge and the type of image fuzzification [29].*

Problem	Fuzzification	Level	Examples
Brightness ambiguity/ vagueness	histogram	low	thresholding
Geometrical fuzziness	local	intermediate	edge detection, filtering
Complex/ill-defined data	feature	high	recognition, analysis

to generate new histograms, modified gray levels, image segments, or classes of objects. In the following, we briefly describe each processing stage.

22.4.1 Fuzzification (coding of image information)

Fuzzification is in a sense a type of input data coding. It means that membership values are assigned to each input (Section 22.3.2). Fuzzification does mean that we assign the image (its gray levels, features, segments, ...) with one or more membership values with respect to the properties of interest (e.g., brightness, edginess, homogeneity). Depending on the problem we have (ambiguity, fuzziness, complexity), the suitable fuzzification method, should be selected. Examples of properties and the corresponding type of fuzzification are given in Table 22.1.

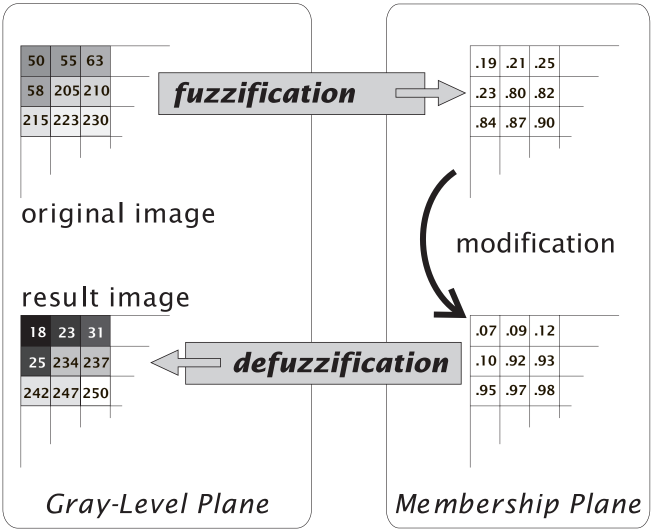

Figure 22.12: *Example for modification-based fuzzy image processing [29].*

22.4.2 Operations in membership plane

The generated membership values are modified by a suitable fuzzy approach. This can be a modification, aggregation, classification, or processing by some kind of if-then rules.

Aggregation. Many fuzzy techniques aggregate the membership values to produce new memberships. Examples are fuzzy hybrid connectives, and fuzzy integrals, to mention only some of them. The result of aggregation is a global value that considers different criteria, such as features and hypothesis, to deliver a certainty factor for a specific decision (e. g., pixel classification).

Modification. Another class of fuzzy techniques modify the membership values in some ways. The principal steps are illustrated in Fig. 22.12. Examples of such modifications are linguistic hedges, and distance-based modification in prototype-based fuzzy clustering. The result of the modification is a new membership value for each fuzzified feature (e. g., gray level, segment, object).

Classification. Fuzzy classification techniques can be used to classify input data. They can be numerical approaches (e. g., fuzzy clustering algorithms, fuzzy integrals, etc.) or syntactic approaches (e. g., fuzzy grammars, fuzzy if-then rules, etc.). Regarding to the membership values, classification can be a kind of modification (e. g., distance-based adaptation of memberships in prototype-based clustering) or aggregation (e. g., evidence combination by fuzzy integrals).

Inference. Fuzzy if-then rules can be used to make soft decisions using expert knowledge. Indeed, fuzzy inference can also be regarded as a kind of membership aggregation because they use different fuzzy connectives to fuse the partial truth in premise and conclusion of if-then rules.

22.4.3 Defuzzification (decoding of the results)

In many applications we need a crisp value as output. Fuzzy algorithms, however, always deliver fuzzy answers (a membership function or a membership value). In order to reverse the process of fuzzification, we use defuzzification to produce a crisp answer from a fuzzy output feature. Depending on the selected fuzzy approach, there are different ways to defuzzify the results. The well-known defuzzification methods such as *center of area* and *mean of maximum* are used mainly in inference engines. One can also use the inverse membership function if point operations are applied. Figure 22.12 illustrates the three stages of fuzzy image processing for a modification-based approach.

22.5 Theoretical components of fuzzy image processing

Fuzzy image processing is knowledge-based and nonlinear. It is based on fuzzy logic and uses its logical, set-theoretical, relational and epistemic aspects. The most important theoretical frameworks that can be used to construct the foundations of fuzzy image processing are: fuzzy geometry, measures of fuzziness/image information, rule-based approaches, fuzzy clustering algorithms, fuzzy mathematical morphology, fuzzy measure theory, and fuzzy grammars. Any of these topics can be used either to develop new techniques, or to extend the existing algorithms [29]. In the following, we give a brief description of each field. Here, the soft computing techniques (e. g., neural fuzzy, fuzzy genetic) are not mentioned due to space limitations.

Combined approaches, such as neural fuzzy and fuzzy genetic techniques are not considered here because of space limitations. In Sections 22.5.1–22.5.7 we will briefly introduce each of these topics.

22.5.1 Fuzzy geometry

Geometrical relationships between the image components play a key role in intermediate image processing. Many geometrical categories such as area, perimeter, and diameter, are already extended to fuzzy sets [5, 6, 7, 8, 9, 10, 11, 16, 17]. The geometrical fuzziness arising

Table 22.2: *Theory of fuzzy geometry [5, 6, 7, 8, 9, 10, 11, 16, 17, 29]*

Aspects of fuzzy geometry	Examples of subjects and features
digital topology	connectedness, surroundedness, adjacency
metric	area, perimeter, diameter, distance between fuzzy sets
derived measures	compactness index of area coverage, elongatedness
convexity	convex/concave fuzzy image subsets
thinning/medial axes	shrinking, expanding, skeletonization
elementary shapes	fuzzy discs, fuzzy rectangles, fuzzy triangles

during segmentation tasks can be handled efficiently if we consider the image or its segments as fuzzy sets. The main application areas of fuzzy geometry are feature extraction (e. g., in image enhancement), image segmentation, and image representation ([12, 16, 17, 29, 29, 33], see also Table 22.2).

Fuzzy topology plays an important role in fuzzy image understanding, as already pointed out earlier in this chapter. In the following, we describe some fuzzy geometrical measures, such as compactness, index of area coverage, and elongatedness. A more detailed description of other aspects of fuzzy geometry can be found in the literature.

Fuzzy compactness [7]. Let $\boldsymbol{G}$ be an image of size MN, containing one object with the membership values $\mu_{m,n}$. The *area* of the object—interpreted as a fuzzy subset of the image—can be calculated as:

$$\mathrm{area}(\mu) = \sum_{m=0}^{M} \sum_{n=0}^{N} \mu_{m,n} \tag{22.26}$$

The *perimeter* of the object can be determined as

$$\mathrm{perimeter}(\mu) = \sum_{m=1}^{M} \sum_{n=1}^{N-1} \|\mu_{m,n} - \mu_{m,n+1}\| + \sum_{m=1}^{M-1} \sum_{n=1}^{N} \|\mu_{m,n} - \mu_{m+1,n}\| \tag{22.27}$$

The *fuzzy compactness*, introduced by Rosenfeld [7] can be defined as

$$\mathrm{compactness}(\mu) = \frac{\mathrm{area}(\mu)}{[\mathrm{perimeneter}(\mu)]^2} \tag{22.28}$$

In the crisp case, the compactness is maximum for a circle. It can be shown that the compactness of fuzzy sets is always more than a corresponding case. Many fuzzy techniques are, therefore, developed for image segmentation, which minimizes the fuzzy compactness.

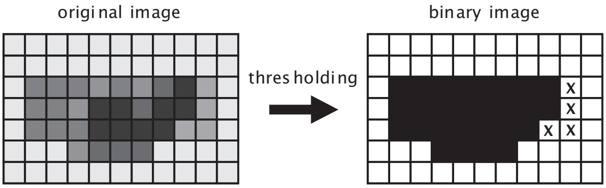

Figure 22.13: Calculation of elongatedness of crisp image subsets is often accompanied with loss of information (pixels marked with "x" are lost during the thresholding task).

Index of area coverage [16, 17]. The *index of area coverage* of a fuzzy image subset μ, introduced by Pal and Ghosh [16], represents the fraction of the maximum image area actually covered by this subset. It is defined as follows:

$$\mathrm{ioac}(\mu) = \frac{\mathrm{area}(\mu)}{\mathrm{length}(\mu)\mathrm{breadth}(\mu)} \tag{22.29}$$

Here, the length and breadth of the fuzzy image subset are calculated as follows:

$$\mathrm{length}(\mu) = \max_m \left\{ \sum_n \mu_{m,n} \right\} \tag{22.30}$$

$$\mathrm{breadth}(\mu) = \max_n \left\{ \sum_m \mu_{m,n} \right\} \tag{22.31}$$

The definition of the index of area coverage is very similar to compactness. For certain cases, it can be shown that there exists a relationship between the two definitions.

Fuzzy elongatedness [7]. As an example for cases that have no simple generalization to fuzzy sets, we briefly explain the *elongatedness* of an object. The elongatedness can serve as a feature to recognize a certain class of objects. Making strong commitments to calculate such geometrical features (e. g., thresholding), it can lead to loss of information and falsification of final results (Fig. 22.13).

Let μ be the characteristic function of a crisp image subset. The elongatedness can be defined as follows:

$$\mathrm{elongatedness}(\mu) = \frac{\mathrm{area}(\mu)}{[\mathrm{thickness}(\mu)]^2} \tag{22.32}$$

Now, letting μ be the membership function of a fuzzy image subset, a closely related definition of fuzzy elongatedness is introduced by

Rosenfeld [5]:

$$\text{fuzzy elongatedness}(\mu) = \max_{\delta>0} \frac{\text{area}(\mu - \mu_{-\delta})}{(2\delta)^2} \tag{22.33}$$

Here, μ_δ denotes the result of a shrinking operation in a given distance δ, where the local "min" operation can be used as a generalization of shrinking.

22.5.2 Measures of fuzziness and image information

Fuzzy sets can be used to represent a variety of image information. A central question dealing with uncertainty is to quantify the "fuzziness" or uncertainty of an image feature, given the corresponding membership function. A goal of fuzzy image processing might be to minimize the uncertainty in the image information.

Index of fuzziness. The intersection of a crisp set with its own complement always equals zero (Eq. (22.4)). This condition no longer holds for two fuzzy sets. The more fuzzy a fuzzy set is, the more it intersects with its own complement. This consideration leads to the definition of the *index of fuzziness* γ. Given a fuzzy set A with the membership function μ_A defined over an image of size $M \times N$, we define the *linear index of fuzziness* γ_l as follows:

$$\gamma_l(G) = \frac{2}{MN} \sum_{m,n} \min(\mu_{mn}, 1 - \mu_{mn}) \tag{22.34}$$

Another possible definition is given by the *quadratic index of fuzziness* γ_q defined by

$$\gamma_q(G) = \frac{1}{\sqrt{MN}} \left[\left(\sum_{m,n} \min(\mu_{mn}, 1 - \mu_{mn}) \right)^2 \right]^{1/2} \tag{22.35}$$

For binary-valued (crisp sets) both indices equal zero. For maximum fuzziness, that is, $\mu_{mn} = 0.5$ they reach the peak value of 1.

Fuzzy entropy. An information theoretic measure quantifying the information content of an image is the *entropy*. The counterpart in fuzzy set theory is given by the *fuzzy entropy*, quantifying the uncertainty of the image content. The *logarithmic fuzzy entropy* H_{log}, is defined by [34]

$$H_{log}(G) = \frac{1}{MN \ln 2} \sum_{m,n} S_n(\mu_{mn}) \tag{22.36}$$

where

$$S_n(\mu_{mn}) = -\mu_{mn}\ln(\mu_{mn}) - (1-\mu_{mn})\ln(1-\mu_{mn}) \tag{22.37}$$

Another possible definition, called the *exponential fuzzy entropy* has been proposed by Pal and Pal [21]:

$$H_{exp}(G) = \frac{1}{MN(\sqrt{e}-1)}\sum_{m,n}\left\{\mu_{mn}e^{(1-\mu_{mn})} + (1-\mu_{mn})e^{\mu_{mn}} - 1\right\} \tag{22.38}$$

The fuzzy entropy also yields a measure of uncertainty ranging from zero to unity.

Fuzzy correlation. An important question in classical classification techniques is the *correlation* of two different image features. Similarly, the *fuzzy correlation* $K(\mu_1, \mu_2)$ quantifies the correlation of two fuzzy features, defined by the membership functions μ_1 and μ_2, respectively. It is defined by [13]

$$K(\mu_1,\mu_2) = 1 - \frac{4}{\Delta_1 + \Delta_2}\sum_{m,n}(\mu_{1,mn} - \mu_{2,mn})^2 \tag{22.39}$$

where

$$\Delta_1 = \sum_{m,n}(2\mu_{1,mn} - 1)^2, \quad \text{and} \quad \Delta_2 = \sum_{m,n}(2\mu_{2,mn} - 1)^2 \tag{22.40}$$

If $\Delta_1 = \Delta_2 = 0$, K is set to unity. Fuzzy correlation is used either to quantify the correlation of two features within the same image or, alternatively, the correlation of the same feature in two different images. Examples of features are brightness, edginess, texturedness, etc.

More detailed information about the theory on common measures of fuzziness can be found in [13, 14, 21, 35, 36, 37, 38]. A variety of practical applications are given by [19, 20, 29, 39, 40, 41, 42].

22.5.3 Rule-based systems

Rule-based systems are among the most powerful applications of fuzzy set theory. They have been of utmost importance in modern developments of fuzzy-controllers. Thinking of fuzzy logic usually implies dealing with some kind of rule-based inference, in terms of incorporating expert knowledge or heuristic relations. Whenever we have to deal with combining uncertain knowledge without having an analytical model, we can use a rule-based fuzzy inference system. Rule-based approaches incorporate these techniques into image processing tasks.

Rule-based systems are composed of the following three major parts: *fuzzification*, *fuzzy inference*, and *defuzzification*.

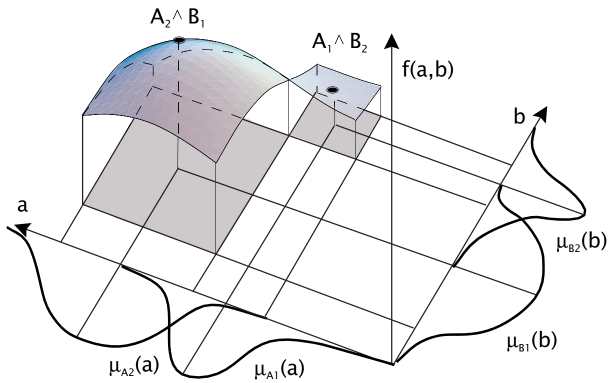

***Figure 22.14:** The rules of a fuzzy-inference system create* fuzzy patches *in the product space $A \times B$. These regions constitute the support of the function $\mu_C(a,b)$.*

We outlined the components fuzzification and defuzzification earlier in this chapter. They are used to create fuzzy sets from input data and to compute a crisp number from the resulting output fuzzy set, respectively.

The main part of rule-based systems is the *inference engine*. It constitutes the brain of the fuzzy technique, containing the knowledge about the relations between the individual input fuzzy sets and the output fuzzy sets. The fuzzy inference system comprises a number of rules, in terms of if-then conditions, which are used to modify the membership functions of the corresponding output condition according to Eqs. (22.12) to (22.14). The individual output membership functions are accumulated to a single output fuzzy set using Eq. (22.15).

An interesting aspect of rule-based systems is that they can be interpreted as a nonlinear interpolation technique approximating arbitrary functions from partial knowledge about relations between input and output variables. Consider $f(a,b)$ to be a function of the two variables a, and b. In case we do not know the analytical shape of f we need an infinite number of relations between a, b, and $f(a,b)$ in order to approximate f. If we quantify a and b by fuzzy sets A_i and B_i, it is sufficient to know the relations between the finite number of pairs (A_i, B_j). The continuous function f over the entire parameter space $A \times B$ can be interpolated, as illustrated in Fig. 22.14. In control theory, the function $f(a,b)$ is called the *control surface*. It is, however, necessary to carefully choose the shape of the membership functions μ_{A_i} and μ_{B_i}, as they determine the exact shape of the interpolation between the sparse support points, that is, the shape of the control surface.

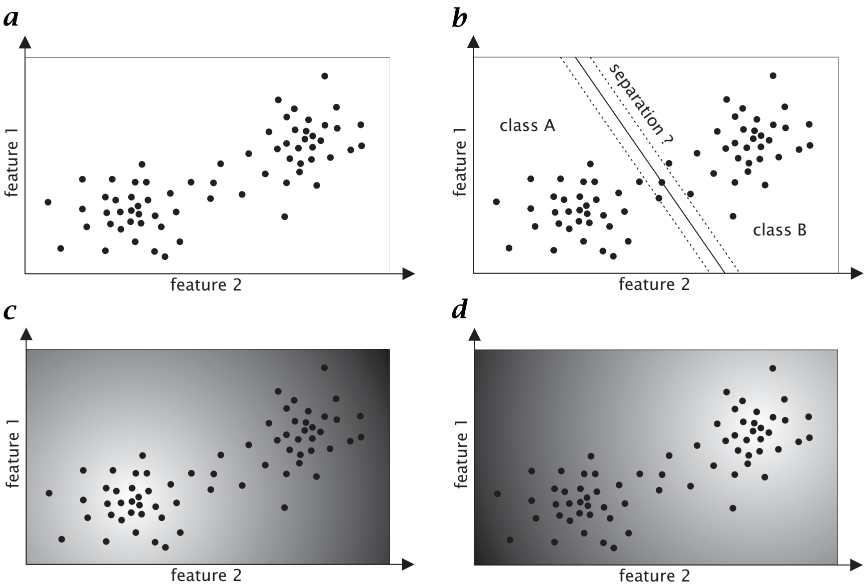

Figure 22.15: Crisp versus fuzzy classification. ***a*** *Set of feature points.* ***b*** *Crisp classification into two sets A and B. Features close to the separation line are subject to misclassification.* ***c*** *Fuzzy membership function μ_A and μ_B used for fuzzy clustering.*

More detailed information about the theory on rule-based systems can be found in [24, 25, 26, 27]. A variety of practical applications are given by [29, 43, 44, 45, 46, 47].

22.5.4 Fuzzy/possibilistic clustering

In many image processing applications, the final step is a classification of objects by their features, which have been detected by image processing tools. Assigning objects to certain classes is not specific to image processing but a very general type of technique, which has led to a variety of approaches searching for *clusters* in an n-dimensional *feature space.*

Figure 22.15a illustrates an example of feature points in a 2-D space. The data seem to belong to two clusters, which have to be separated. The main problem of all clustering techniques is to find an appropriate partitioning of the feature space, which minimizes misclassifications of objects. The problem of a crisp clustering is illustrated in Fig. 22.15b. Due to a long tail of "outliers" it is not possible to unambiguously find a separation line, which avoids misclassifications. The basic idea of *fuzzy clustering* is not to classify the objects, but rather to quantify the partial membership of the same object to more than one class, as illustrated in Fig. 22.15. This accounts for the fact that a small transition in the feature of an object—eventually crossing the separation line—should only lead to a small change in the membership, rather than changing the final classification. The membership functions can be used in subsequent processing steps to combine feature properties until, eventually, a final classification has to be performed.

Within the scope of this handbook we are not able to detail all existing clustering techniques. More detailed information about the theory of fuzzy-clustering and the various algorithms and applications can be found in the following publications [4, 29, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57].

22.5.5 Fuzzy morphology

Fuzzy morphology extends the concept of classical morphology (Chapter 21) to fuzzy sets. In the following we assume the image to be represented by a fuzzy membership function μ. In addition to the membership function at any pixel of the image of size $M \times N$, we need a "fuzzy" *structuring element*, ν. The structuring element can be thought of as the membership function. The shape of the structuring element, that is, the values of the membership function ν_{mn}, determine the spatial area of influence as well as the magnitude of the morphological operation.

Without going into details of the theoretical foundations, we show two possible realizations of the two basic morphological operations *fuzzy dilation* and *fuzzy erosion*, respectively [29].

Example 22.6: Fuzzy erosion

1. [58, 59]:

$$E_\nu(x) = \inf \max \left[\mu(y), (1 - \nu(y - x))\right], \quad x, y \in X \qquad (22.41)$$

2. [60]:

$$E_\nu(x) = \inf \left[\mu(y)\nu(y - x) + 1 - \nu(y - x)\right], \quad x, y \in X \qquad (22.42)$$

Example 22.7: Fuzzy dilation

1. [58, 59]:

$$E_\nu(x) = \sup \min \left[\mu(y), \nu(y - x)\right], \quad x, y \in X \qquad (22.43)$$

2. [60]:

$$E_\nu(x) = \sup \left[\mu(y)\nu(y - x)\right], \quad x, y \in X \qquad (22.44)$$

Other realizations and more detailed information about the theory of morphology can be found in the following publications [29, 61, 62, 63, 64, 65, 66, 67, 68, 69].

22.5.6 Fuzzy measure theory

Fuzzy sets are useful to quantify the inherent vagueness of image data. Brightness, edginess, homogeneity, and many other categories are a matter of degree. The class boundaries in these cases are not crisp. Thus, reasoning should be performed with partial truth and incomplete knowledge. Fuzzy set theory and fuzzy logic offer the suitable framework to apply heuristic knowledge within complex processing tasks.

Uncertainty arises in many other situations as well, even if we have crisp relationships. For instance, the problem of thresholding is not due to the vagueness because we have to extract two classes of pixels belonging to object and background, respectively. Here, the main problem is that the decision itself is uncertain—namely assigning each gray level with membership 1 for object pixels and membership 0 for background pixels. This uncertainty, however, is due to the ambiguity, rather than to vagueness. For this type of problems, one may take into account *fuzzy measures* and *fuzzy integrals*.

Fuzzy measure theory—introduced by Sugeno [70]—can be considered as a generalization of classical measure theory [71]. *Fuzzy integrals* are nonlinear aggregation operators used to combine different sources of uncertain information [29, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82].

Fuzzy measures. Let X be a universe of discourse (a set of features, algorithms, images of different sources, etc.). A fuzzy measure

$$g: 2^X \longrightarrow [0,1] \tag{22.45}$$

over the set X in a measurable space (X, K), satisfies the following conditions (K is the power set of X):

1. Boundedness:

$$g(\emptyset) = 0 \quad \text{and} \quad g(X) = 1 \tag{22.46}$$

2. Monotony:

$$A \in K, \; B \in K, \; A \subset B \;\; \Rightarrow \; g(A) \le g(B) \tag{22.47}$$

3. Lower continuity:

$$\{A_n\} \subset K, \; A_1 \subset A_2 \subset \ldots, \; \bigcup_{n=1}^{\infty} A_n \in K \;\; \Rightarrow \; \lim_{n\to\infty} g(A_n) = g\left(\bigcup_{n=1}^{\infty} A_n\right) \tag{22.48}$$

4. Upper continuity:

$$\{A_n\} \subset K, \; A_1 \supset A_2 \supset \ldots, \; \bigcap_{n=1}^{\infty} A_n \in K, \;\; \Rightarrow \lim_{n\to\infty} g(A_n) = g\left(\bigcap_{n=1}^{\infty} A_n\right) \tag{22.49}$$

A fuzzy measure is a set function and represents the (subjective) estimation of importance of each information source. Sugeno [70] introduced a class of fuzzy measures, called λ-fuzzy measures, also referred to as *Sugeno measures*. A fuzzy measure g_λ is a Sugeno measure in (X, K) if it satisfies the following rule (λ-rule):

1. A, B, and $A \cup B \in K$, $A \cap B = \emptyset$
2. $g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$
3. $\lambda \in (-1/\sup g_\lambda(A), \infty) \cup \{0\}$

In pattern recognition and image processing applications, we generally have to deal with finite numbers of elements. The λ-rule can be formulated as follows:

$$g_\lambda\left(\bigcup_{i=1}^{n} A_i\right) = \begin{cases} \sum_{i=1}^{n} g_\lambda(A_i) & \text{if } \lambda = 0 \\ \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1 + \lambda g_\lambda(A_i)) - 1\right] & \text{if } \lambda \neq 0 \end{cases} \tag{22.50}$$

The Sugeno measure can be completely constructed if the value of λ is known. Assuming the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, we consider the case that the Sugeno measure is not a probability measure ($\lambda \neq 0$):

$$g_\lambda(X) = \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1 + \lambda g_\lambda(\{x_i\})) - 1\right] \tag{22.51}$$

The value of λ can be calculated from the following equation:

$$1 + \lambda g_\lambda(X) = \prod_{i=1}^{n}(1 + \lambda g_\lambda(\{x_i\})) \tag{22.52}$$

For the case that $g_\lambda(X) = 1$ we receive the following polynomial expression:

$$1 + \lambda = \prod_{i=1}^{n}(1 + \lambda \mu(\{x_i\})) \tag{22.53}$$

Example 22.8:

Let $X = \{a, b, c\}$. Suppose that a fuzzy measure g is defined as follows:

$$g(x) = \begin{cases} 0.0 & \text{if } x = \emptyset \\ 0.4 & \text{if } x = \{a\} \\ 0.2 & \text{if } x = \{b\} \\ 0.3 & \text{if } x = \{c\} \\ 1.0 & \text{if } x = \{a, b, c\} = X \end{cases} \tag{22.54}$$

We solve Eq. (22.52) to find the corresponding λ-fuzzy measure:

$$1 + \lambda = (1 + 0.4\lambda)(1 + 0.2\lambda)(1 + 0.3\lambda) \tag{22.55}$$

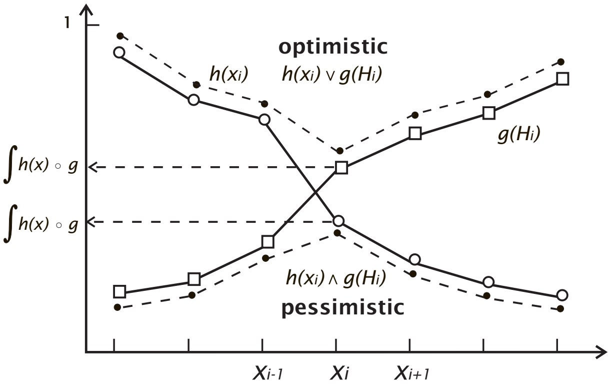

Figure 22.16: *Illustration of the fuzzy integral.*

which has the two solutions $\lambda_1 = 0.372$ and $\lambda_2 = -11.2$, respectively. The only useful solution is given by λ_1, as $\lambda_2 < -1$. The fuzzy λ-fuzzy measure can be completely constructed:

$$\begin{aligned}
g_\lambda(\{\emptyset\}) &= g(\{\emptyset\}) &&= 0.0,\\
g_\lambda(\{a\}) &= g(\{a\}) &&= 0.4,\\
g_\lambda(\{b\}) &= g(\{b\}) &&= 0.2,\\
g_\lambda(\{c\}) &= g(\{c\}) &&= 0.3,\\
g_\lambda(\{a,b\}) &= g(\{a\}) + g(\{b\}) + \lambda g(\{a\})g(\{b\}) &&= 0.63,\\
g_\lambda(\{a,c\}) &= g(\{a\}) + g(\{c\}) + \lambda g(\{a\})g(\{c\}) &&= 0.74,\\
g_\lambda(\{b,c\}) &= g(\{b\}) + g(\{c\}) + \lambda g(\{b\})g(\{c\}) &&= 0.52,\\
g_\lambda(\{a,b,c\}) &= g(\{X\}) &&= 1.0.
\end{aligned}$$

Fuzzy integrals. The fuzzy integral of a function $h: X \longrightarrow [0,1]$ over X, with respect to the fuzzy measure g is defined as follows:

$$\int h(x) \circ g = \sup_{a \in [0,1]} [\alpha \wedge g(F_\alpha)] \tag{22.56}$$

where $F(\alpha) = \{x | h(x) \geq \alpha\}$. Some basic properties of fuzzy integrals are

1. $\int a \circ g = a$, $a \in [0,1]$,
2. $\int h_1 \circ g \leq \int h_2 \circ g$, if $h_1 \leq h_2$,
3. $\int_A h \circ g \leq \int_B h \circ g$, if $A \subset B$.

Let X be a set with a finite numbers of elements $x_1, x_2, \ldots, x_n$. Further let h be a decreasing function of x:

$$h(x_1) \geq h(x_2) \geq \ldots \geq h(x_n) \tag{22.57}$$

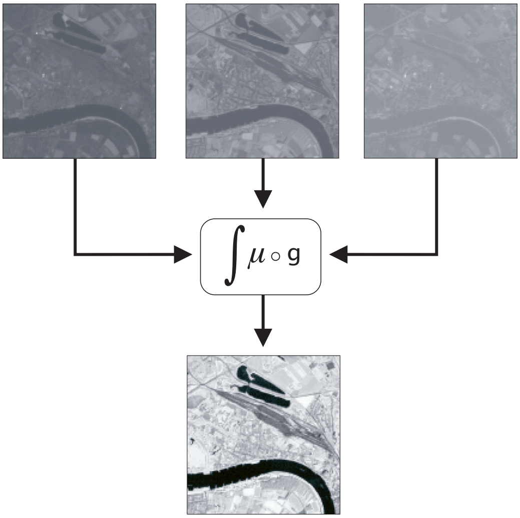

Figure 22.17: *Segmentation by fusion of multispectral images [29].*

The fuzzy integral can be reformulated as follows:

$$\int h(x) \circ g = \bigvee_{i=1}^{n} [h(x_i) \wedge g(H_i)] \tag{22.58}$$

where $H_i = \{x_1, x_2, \ldots, x_i\}$. The operators $\vee$ and $\wedge$ represent the maximum and minimum operator, respectively. This reformulation of fuzzy integral reduces the computational cost from 2^n to n calculations, taking into account that the function h should be sorted in a previous step.

The calculation of the fuzzy integral in Eq. (22.58) can be regarded as a *pessimistic fusion* of *objective evidence* (value of the function h) and *subjective importance* of the information source (fuzzy measure g). One may develop a more optimistic fusion by exchanging the order of maximum and minimum operators (Fig. 22.16).

Applications. Fuzzy integrals as nonlinear aggregation operators can be applied to different problems in image processing and pattern recognition. The main application areas are the fusion of different decisions (differing experts, algorithms, etc.) and fusion of different sensors [72, 73, 74, 75, 76, 77, 78, 80, 81]. For instance, Keller et al. [78] used fuzzy integration for image segmentation. Tizhoosh [29] applied the fuzzy integral to segment images by fusing multispectral images (Fig. 22.17) [29] and for fusion of subjective image quality evaluations in medical applications [82].

One of the problems using fuzzy integral as an aggregation operator relates to constructing the underlying fuzzy measure. The most simple way is to interpret the subjective evaluation of the expert as fuzzy densities. This way, however, is not possible in many applications. Therefore, in the literature some techniques are introduced for construction of fuzzy measures. For instance, the use of a confusion matrix [72, 78], a genetic approach [83], or an approach based on relations equations [84], are some examples for automatic generation of fuzzy measures.

22.5.7 Fuzzy grammars

Language is a powerful tool to describe patterns. The structural information can be qualitatively described without a precise numerical quantification of features. The theory of formal languages has been used for speech recognition before it has been considered to be relevant for pattern recognition. The main reason was that formal languages have been criticized for being not flexible enough for an application in pattern recognition, especially for dealing with disturbances such as noise or unpredictable events.

Fuzzy grammars, introduced by Zadeh and Lee [85], are an extension of classical formal languages that are able to deal with uncertainties and vague information. Fu [86] uses the theory of fuzzy grammars for the first time in image processing. Theoretical and practical aspects of fuzzy languages are detailed by [87, 88, 89, 90]. Practical examples can be found in [23, 91, 92].

22.6 Selected application examples

22.6.1 Image enhancement: contrast adaptation

Image enhancement tries to suppress disturbances, such as noise, blurring, geometrical distortions, and illumination corrections, only to mention some examples. It may be the final goal of the image processing operation to produce an image, with a higher contrast or some other improved property according to a human observer. Whenever these properties cannot be numerically quantified, fuzzy image enhancement techniques can be used. In this section we illustrate the example of *contrast adaptation* by three different algorithms.

In recent years, some researchers have applied the concept of fuzziness to develop new algorithms for contrast enhancement. Here, we briefly describe following fuzzy algorithms:

1. Minimization of image fuzziness
2. Fuzzy histogram hyperbolization
3. Rule-based approach

a b

*Figure 22.18: Example for contrast enhancement based on minimization of fuzziness: **a** original image; and **b** contrast enhanced image.*

Example 22.9: Minimization of image fuzziness [15, 18, 33]

This method uses the intensification operator to reduce the fuzziness of the image that results in an increase of image contrast. The algorithm can be formulated as follows:

1. setting the parameters (F_e, F_d, g_{max}) in Eq. (22.59)
2. fuzzification of the gray levels by the transformation G:

$$\mu_{mn} = G(g_{mn}) = \left[1 + \frac{g_{max} - g_{mn}}{F_d}\right]^{-F_e} \tag{22.59}$$

3. recursive modification of the memberships ($\mu_{mn} \longrightarrow \mu'_{mn}$) by following transformation (intensification operator [24]):

$$\mu'_{mn} = \begin{cases} 2\,[\mu_{mn}]^2 & 0 \le \mu_{mn} \le 0.5 \\ 1 - 2\,[1 - \mu_{mn}]^2 & 0.5 \le \mu_{mn} \le 1 \end{cases} \tag{22.60}$$

4. generation of new gray levels by the inverse transformation G^{-1}:

$$g'_{mn} = G^{-1}(\mu'_{mn}) = g_{max} - F_d\left((\mu'_{mn})^{-1/F_e} - 1\right) \tag{22.61}$$

Figure 22.18 shows an example for this algorithm. The result was achieved after three iterations.

Example 22.10: Fuzzy histogram hyperbolization [29, 42]

Due to the nonlinear human brightness perception, this approach modifies the membership values of original image by a logarithmic function. The algorithm can be formulated as follows (Fig. 22.19):

1. setting the shape of membership function
2. setting the value of fuzzifier β (Fig. 22.19)
3. calculation of membership values
4. modification of the membership values by β

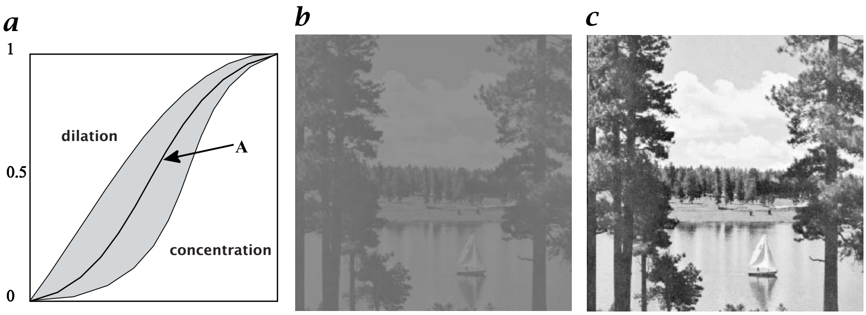

Figure 22.19: ***a*** *Application of dilation (*$\beta = 0.5$*) and concentration (*$\beta = 2$*) operators on a fuzzy set. The meaning of fuzzy sets may be modified applying such operators. To map the linguistic statements of observers in the numerical framework of image processing systems, linguistic hedges are very helpful.* ***b*** *and* ***c*** *are examples for contrast enhancement based on hyperbolization (*$\beta = 0.9$*).*

5. generation of new gray levels by following equation:

$$g'_{mn} = \left(\frac{L-1}{\exp(-1)-1}\right)\left(\exp\left(-\mu^{\beta}(g_{mn})\right)-1\right) \tag{22.62}$$

Example 22.11: Fuzzy rule-based approach [29, 42]

The fuzzy rule-based approach is a powerful and universal method for many tasks in the image processing. A simple rule-based approach to contrast enhancement can be formulated as follows (Fig. 22.20):

1. setting the parameter of inference system (input features, membership functions, ...)
2. fuzzification of the actual pixel (memberships to the dark, gray and bright sets of pixels, see Fig. 19)
3. inference (if dark then darker, if gray then gray, if bright then brighter)
4. defuzzification of the inference result by the use of three singletons

22.6.2 Edge detection

Another important application example of fuzzy techniques is *edge detection*. Edges are among the most important features of low-level image processing. They can be used for a variety of subsequent processing steps, such as object recognition and motion analysis.

The concept of fuzziness has been applied to develop new algorithms for edge detection, which are perfectly suited to quantify the presence of edges in an intuitive way. The different algorithms make use of various aspects of fuzzy theory and can be classified into the following three principal approaches:

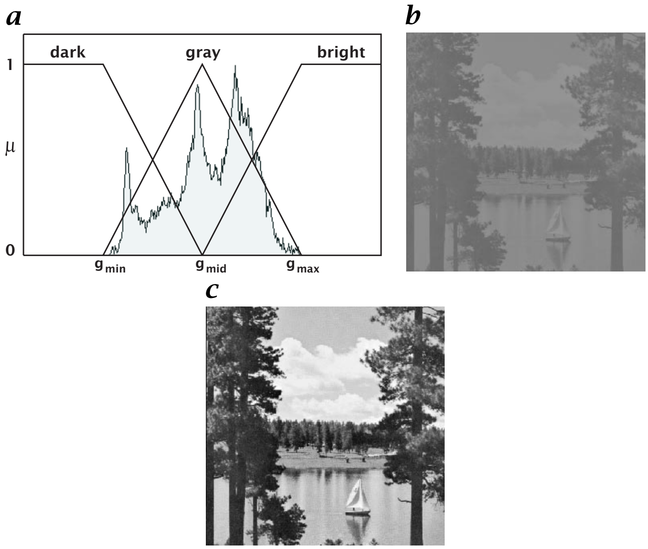

Figure 22.20: ***a*** *Input membership functions for rule-based enhancement based on the characteristic points of image histogram;* ***b*** *and* ***c*** *example for contrast enhancement based on fuzzy if-then rules.*

1. Edge detection by optimal fuzzification [93]
2. Rule-based edge detection [46, 47]
3. Fuzzy-morphological edge detection [29]

Here, we briefly describe the rule-based technique, which is the most intuitive approach using fuzzy logic for edge detection. Other approaches to fuzzy-based edge detection can be found in [43, 44].

Example 22.12: Rule-based edge detection [46, 47]

A typical rule for edge extraction can be defined as follows:

if a pixel belongs to an edge
then it is assigned a dark gray value
else it is assigned a bright gray value

This rule base is special in terms of using the "else" rule. In that way only one explicit logical relation is used and anything else is assigned the complement. It would be harder and more costly to specify all possible cases that can occur.

The input variables are differences between the central point P of a small 3×3 neighborhood U and all neighbors $P_i \in U$. Instead of computing all possible combinations of neighboring points, only eight different clusters of three neighboring points are used [29]. Each of the eight differences is fuzzified according to a membership function μ_i, $i = \{1, \ldots, 8\}$.

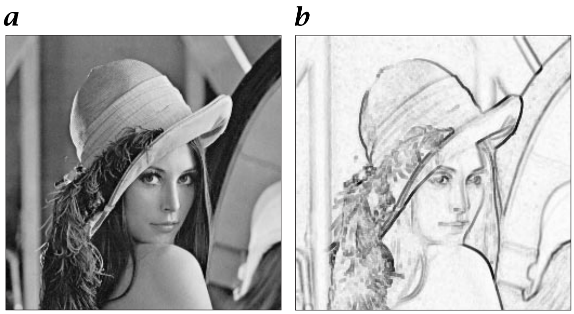

Figure 22.21: *Example for rule-based edge detection:* ***a*** *original image; and* ***b*** *fuzzy edge image.*

The output membership function μ_e corresponding to "edge" is taken as a single increasing wedge. The membership function μ_n of "no edge" is its complement, that is, $\mu_n = 1 - \mu_e$.

The fuzzy inference reduces to the following simple modification of the output membership functions:

$$\mu_e = \max\{\mu_i; i = 1, \ldots, 8\}, \quad \text{and} \quad \mu_n = 1 - \mu_e \tag{22.63}$$

Figure 22.21 illustrates the result of this simple rule-based approach. The final mapping of edges onto gray values of an edge image can be changed by modifying the shape of the individual membership functions. If small differences are given less weight, the noise of the input image will be suppressed. It is also very straightforward to construct directional selective edge detectors by using different rules according to the orientation of the neighboring point clusters.

22.6.3 Image segmentation

The different theoretical components of fuzzy image processing provide us with diverse possibilities for development of new segmentation techniques. The following description gives a brief overview of different fuzzy approaches to image segmentation [29].

Fuzzy rule-based approach If we interpret the image features as linguistic variables, then we can use fuzzy if-then rules to segment the image into different regions. A simple fuzzy segmentation rule may seem as follows: IF the pixel is dark AND its neighborhood is also dark AND homogeneous, THEN it belongs to the background.

Fuzzy clustering algorithms Fuzzy clustering is the oldest fuzzy approach to image segmentation. Algorithms such as fuzzy c-means (FCM, [49]) and possibilistic c-means (PCM, [55]) can be used to build clusters (segments). The class membership of pixels can be interpreted as similarity or compatibility with an ideal object or a certain property.

Measures of fuzziness and image information Measures of fuzziness (e. g., fuzzy entropy) and image information (e. g., fuzzy divergence) can also be used in segmentation and thresholding tasks (see the example that follows).

Fuzzy geometry Fuzzy geometrical measures such as fuzzy compactness [5] and index of area coverage [16] can be used to measure the geometrical fuzziness of different regions of an image. The optimization of these measure (e. g., minimization of fuzzy compactness regarding the cross-over point of membership function) can be applied to make fuzzy and/or crisp pixel classifications.

Fuzzy integrals Fuzzy integrals can be used in different forms:

1. Segmentation by weighting the features (fuzzy measures represent the importance of particular features)
2. Fusion of the results of different segmentation algorithms (optimal use of individual advantages)
3. Segmentation by fusion of different sensors (e. g., multispectral images, fuzzy measures represent the relevance/importance of each sensor)

Example 22.13: Fuzzy thresholding

In many image processing applications, we often have to threshold the gray-level images to generate binary images. In these cases, the image contains a background and one or more objects. The production of binary images serves generally the feature calculation and object recognition. Therefore, image thresholding can be regarded as the simplest form of segmentation, or more general, as a two-class clustering procedure. To separate the object gray levels g_0 from the background gray levels g_B, we have to determine a threshold T. The thresholding can be carried out by the following decision:

$$g = \begin{cases} g_0 = 0 & \text{if } \ 0 \le g_i \le T \\ g_B = 1 & \text{if } \ T \le g_i \le L-1 \end{cases} \tag{22.64}$$

The basic idea is to find a threshold T that minimizes/maximizes the amount of image fuzziness. To answer the question of how fuzzy the image G of size $M \times N$ and L gray levels $g = 0, 1, \ldots, L-1$ is, measures of fuzziness-like fuzzy entropy [34]:

$$H = \frac{1}{MN \ln 2} \sum_{g=0}^{L-1} h(g) \left[-\mu(g) \ln(\mu(g)) - (1-\mu(g)) \ln(1-\mu(g)) \right] \tag{22.65}$$

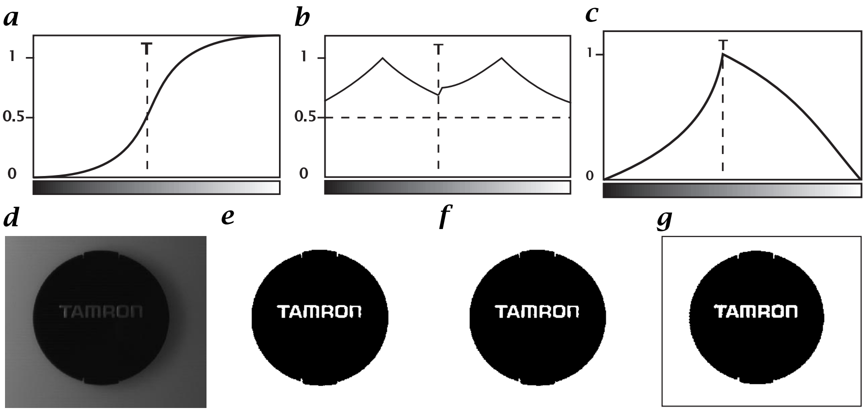

Figure 22.22: *Different membership functions for fuzzy thresholding applied by: **a** Pal and Murthy [20]; **b** Huang and Wang [39]; and **c** [29]; **d** original image; Results of thresholding: **e** Pal and Murthy [20]; **f** Huang and Wang [39]; and **g** Tizhoosh [41].*

or index of fuzziness [38]

$$\gamma = \frac{2}{MN} \sum_{g=0}^{L-1} h(g) \min\left(\mu(g), 1 - \mu(g)\right) \tag{22.66}$$

can be used, where $h(g)$ denotes the histogram value and $\mu(g)$ the membership value of the gray level g, respectively.

The general procedure for fuzzy thresholding can be summarized as follows:

1. Select the type of membership function (Fig. 22.22)
2. Calculate the image histogram
3. Initialize the membership function
4. Move the threshold and calculate in each position the amount of fuzziness using fuzzy entropy or any other measure of fuzziness
5. Find out the position with minimum/maximum fuzziness
6. Threshold the image with the corresponding threshold

The main difference between fuzzy thresholding techniques is that each of them uses different membership function and measures of fuzziness, respectively. Figure 22.22 illustrated three examples of fuzzy membership functions applied to thresholding together with the corresponding results on a test image. For the analytical form of the various membership functions, we would like to refer to the literature [20, 33, 39, 41].

Table 22.3: *The practical and theoretical ripeness of different fuzzy approaches [29]*

Fuzzy approach	Theoretical/practical ripeness
rule-based systems	extensively investigated
fuzzy-clustering	↓ ↓ ↓ ↓ ↓ ↓ ↓
measures of fuzziness	↓ ↓ ↓ ↓ ↓ ↓
fuzzy geometry	↓ ↓ ↓ ↓ ↓
neural fuzzy approaches	↓ ↓ ↓ ↓
fuzzy genetic approaches	↓ ↓ ↓
fuzzy measures/integrals	↓ ↓
fuzzy grammars	↓
fuzzy morphology	more investigations necessary

22.7 Conclusions

Among all publications on fuzzy approaches to image processing, fuzzy clustering and rule-based approaches have the greatest share. Measures of fuzziness and fuzzy geometrical measures are usually used as features within the selected algorithms. Fuzzy measures and fuzzy integrals seem to become more and more an interesting subject of research. The theoretical research on fuzzy mathematical morphology seems still to be more important than practical reports. Only a few applications of fuzzy morphology can be found in the literature. Fuzzy grammars, finally, seem to be still as unpopular as its classical counterpart. Table 22.3 gives an overview of theoretical/practical ripeness of different fuzzy approaches (here, the ripeness, as a fuzzy number, may also be interpreted as degree of popularity measured by number of corresponding publications).

The topics detailed in Sections 22.4.1–22.5.7 can also be used to extend the existing image processing algorithms and improve their performance. Some examples are: *fuzzy Hough transform* [94], fuzzy mean filtering [95], and fuzzy median filtering [96].

Besides numerous publications on new fuzzy techniques, the literature on introduction to fuzzy image processing can be divided into overview papers [13, 14, 77, 97], collections of related papers [49], and textbooks [15, 29, 31, 56, 72].

Fuzzy clustering algorithms and rule-based approaches will certainly play an important role in developing new image processing algorithms. Here, the potentials of fuzzy if-then rule techniques seem to be greater than already estimated. The disadvantage of rule-based approach, however, is its expensive computing in local operations. Hardware developments will be presumably a subject of investigations. Fuzzy integrals will find more and more applications in image data fusion. The theoretical research on fuzzy morphology will be completed with regard to its fundamental questions, and more practical reports will be published in this area. Fuzzy geometry will be further investigated and play an indispensable part of fuzzy image processing.

It is not possible (and also not meaningful) to do everything in image processing with fuzzy techniques. Fuzzy image processing will mainly play a supplementary role in computer vision. Its part will be possibly small in many applications; its role, nevertheless, will be a pivotal and decisive one.

22.8 References

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