Introduction to creation and annihilation operators

Forest Yang

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Note: when only one subscript k is used for an operator it specifies location i and a spin σ.

1 Some relations

The operator $c^\dagger_{i\sigma}$ creates a spin σ particle at position i and the operator $c_{i\sigma}$ annihilates a spin σ particle at position i.
$(c^\dagger_k)^2 = (c_k)^2 = 0$ follows from the anticommutator relations

$$\{c^\dagger_k, c^\dagger_{k'}\}_+ = 0 \qquad \{c_k, c_{k'}\}_+ = 0 \qquad \{c^\dagger_k, c_{k'}\}_+ = \delta_{kk'}$$

The operator $\hat{n}_k$ is defined as $c^\dagger_k c_k$ and can be thought of as counting the number of particles in state k. This gives the following commutator relations

$$[\hat{n}_k, c^\dagger_k] = c^\dagger_k \quad \text{"add one then count vs. count then add one"}$$
$$[\hat{n}_k, c_k] = -c_k \quad \text{"subtract one then count vs. count then subtract one"}$$

2 Representing operators

For a one particle operator $\hat{O}$, we have

$$\hat{O} = \sum_{\ell,\sigma,\ell',\sigma'} \langle \ell,\sigma|\hat{O}|\ell',\sigma'\rangle\, c^\dagger_{\ell,\sigma} c_{\ell',\sigma'}$$

because, (?) for some arbitrary $\ell_1, \sigma_1, \ell_2, \sigma_2$, we have

$$\begin{aligned}
\langle \ell_1,\sigma_1| \sum \langle \ell,\sigma|\hat{O}|\ell',\sigma'\rangle\, c^\dagger_{\ell,\sigma} c_{\ell',\sigma'} |\ell_2,\sigma_2\rangle &= \sum \langle \ell,\sigma|\hat{O}|\ell',\sigma'\rangle \langle \ell_1,\sigma_1|\, c^\dagger_{\ell,\sigma} c_{\ell',\sigma'}\, |\ell_2,\sigma_2\rangle \\
&= \sum \langle \ell,\sigma|\hat{O}|\ell',\sigma'\rangle \langle c_{\ell,\sigma}\ell_1,\sigma_1|\, |c_{\ell',\sigma'}\ell_2,\sigma_2\rangle \\
&= \sum \langle \ell,\sigma|\hat{O}|\ell',\sigma'\rangle \langle 0|0\rangle\, \delta_{\ell_1,\ell}\delta_{\sigma_1,\sigma}\delta_{\ell_2,\ell'}\delta_{\sigma_2,\sigma'} \\
&= \langle \ell_1,\sigma_1|\hat{O}|\ell_2,\sigma_2\rangle
\end{aligned}$$

3 Example of a Two Particle Operator

$$\begin{aligned}
\langle \ell_1\sigma_1\ell_2\sigma_2|V(|\mathbf{r}_1-\mathbf{r}_2|)|\ell_3\sigma_3\ell_4\sigma_4\rangle &= \frac{1}{V^2}\langle\sigma_1|\sigma_4\rangle\langle\sigma_2|\sigma_3\rangle \int d^3\mathbf{r}_2 \int d^3\mathbf{r}_1 e^{-i(\mathbf{k}_1-\mathbf{k}_4)\cdot\mathbf{r}_2} V(|\mathbf{r}_1-\mathbf{r}_2|) e^{-i(\mathbf{k}_2-\mathbf{k}_3)\cdot\mathbf{r}_1} \\
&= \frac{1}{V^2}\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3}|\det\mathbf{J}| \int d^3\mathbf{r} \int d^3\mathbf{R}\, V(r) e^{-i(\mathbf{k}_1-\mathbf{k}_4)\cdot(\mathbf{R}+\frac{\mathbf{r}}{2})} e^{-i(\mathbf{k}_2-\mathbf{k}_3)\cdot(\mathbf{R}-\frac{\mathbf{r}}{2})} \\
&= \frac{1}{V^2}\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3} \int d^3\mathbf{r} \int d^3\mathbf{R}\, V(r) e^{-i(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\cdot\mathbf{R}} e^{-i\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4)\cdot(\mathbf{r}} \\
&\stackrel{?}{=} \frac{1}{V}\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3} \int d^3\mathbf{r}\, \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4) V(r) e^{-i(\mathbf{k}_1-\mathbf{k}_4)} \\
&= \frac{1}{V}\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4) V_{FT}(\mathbf{k}_1-\mathbf{k}_4)
\end{aligned}$$

If $V(r) = \frac{e^2}{r}$ then $V_{FT}(\mathbf{k}_1 - \mathbf{k}_4) = \frac{4\pi e^2}{|\mathbf{k}_1 - \mathbf{k}_4|}\frac{1}{V}$. Note that $\mathbf{r}_1 = \mathbf{R} + \frac{\mathbf{r}}{2}$ and $\mathbf{r}_2 = \mathbf{R} - \frac{\mathbf{r}}{2}$. Also,

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 \end{bmatrix}$$

which has a determinant of -1.

4 Deriving angular momentum operators

It was shown that $\hat{L}_z = -ih\frac{\partial}{\partial\phi}$. To get expressions for the other components of angular momentum first note

$$\begin{aligned} \frac{\partial}{\partial\phi} &= -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} \\ \frac{\partial}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z} \\ &= r\cos\theta\cos\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z} \end{aligned}$$

Therefore,

$$\cot\theta\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta} = r\cos\theta\frac{\partial}{\partial y} - r\sin\theta\sin\phi\frac{\partial}{\partial z} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} = \frac{\hat{L}_x}{i\hbar} \quad \Longrightarrow \quad \hat{L}_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi})$$

$$-\cot\theta\sin\phi\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\theta} = r\cos\theta\frac{\partial}{\partial x} - r\sin\theta\cos\phi\frac{\partial}{\partial z} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} = \frac{\hat{L}_y}{-i\hbar} \quad \Longrightarrow \quad \hat{L}_y = -i\hbar(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi})$$

5 Homework 12, Exercise 1

a) Constructing the spin operators:

$$\hat{S}_z = \frac{1}{2}\sum_i \hat{n}_{i\uparrow} - \hat{n}_{i\downarrow} \qquad \hat{S}_+ = \sum_i c^\dagger_{i\uparrow}c_{i\downarrow} \qquad \hat{S}_- = \sum_i c^\dagger_{i\downarrow}c_{i\uparrow}$$

Then we have

$$\begin{aligned} [\hat{S}_+, \hat{S}_-] &= \sum_{j,k} c^\dagger_{j\uparrow}c_{j\downarrow}c^\dagger_{k\downarrow}c_{k\uparrow} - c^\dagger_{k\downarrow}c_{k\uparrow}c^\dagger_{j\uparrow}c_{j\downarrow} \\ &= \sum_{j,k}[c^\dagger_{j\uparrow}c_{j\downarrow}, c^\dagger_{k\downarrow}c_{k\uparrow}] \\ &= \sum_{j,k} c^\dagger_{j\uparrow}[c_{j\downarrow}, c^\dagger_{k\downarrow}c_{k\uparrow}] + [c^\dagger_{j\uparrow}, c^\dagger_{k\downarrow}c_{k\uparrow}]c_{k\downarrow} \end{aligned}$$

Evaluating the remaining commutators:

$$\begin{aligned} [c^\dagger_{k\downarrow}c_{k\uparrow}, c_{j\downarrow}] &= c^\dagger_{k\downarrow}\{c_{k\uparrow}, c_{j\downarrow}\}_+ - \{c^\dagger_{k\downarrow}, c_{j\downarrow}\}_+ c_{k\uparrow} &&= -\delta_{jk}c_{k\uparrow} \\ [c^\dagger_{k\downarrow}c_{k\uparrow}, c^\dagger_{j\uparrow}] &= c^\dagger_{k\downarrow}\{c_{k\uparrow}, c^\dagger_{j\uparrow}\}_+ - \{c^\dagger_{k\downarrow}, c^\dagger_{j\uparrow}\}_+ c_{k\uparrow} &&= \delta_{jk}c^\dagger_{k\downarrow} \end{aligned}$$

Therefore the last expression for $[\hat{S}_+, \hat{S}_-]$ simplifies to

$$\sum_{j,k} c^\dagger_{j\uparrow}c_{k\uparrow}\delta_{jk} - \delta_{jk}c^\dagger_{k\downarrow}c_{k\downarrow} = \sum_i c^\dagger_{i\uparrow}c_{i\uparrow} - c^\dagger_{i\downarrow}c_{i\downarrow} = 2\hat{S}_z$$

Second identity: first, show that if $k \neq k'$ then $[\hat{n}_k, c_{k'}] = [\hat{n}_k, c^\dagger_{k'}] = 0$:

$$\begin{aligned}
[\hat{n}_k, c_{k'}] &= c^\dagger_k c_k c_{k'} - c_{k'} c^\dagger_k c_k \\
&= c^\dagger_k c_k c_{k'} - c_{k'} c^\dagger_k c_k + \{c^\dagger_k, c_{k'}\} c_k \\
&= c^\dagger_k c_k c_{k'} + c^\dagger_k c_{k'} c_k \\
&= c^\dagger_k \{c_k, c_{k'}\} = 0
\end{aligned}$$

And showing that $[\hat{n}_k, c^\dagger_{k'}] = 0$ is similar.
Now, evaluating the commutator in the problem for $\hat{S}_+$:

$$\begin{aligned}
[\hat{S}_z, \hat{S}_+] &= \frac{1}{2}\Big[\sum_{i,j} (c^\dagger_{i\uparrow} c_{i\uparrow} - c^\dagger_{i\downarrow} c_{i\downarrow}) c^\dagger_{j\uparrow} c_{j\downarrow} - c^\dagger_{j\uparrow} c_{j\downarrow} (c^\dagger_{i\uparrow} c_{i\uparrow} - c^\dagger_{i\downarrow} c_{i\downarrow})\Big] \\
&= \frac{1}{2} \sum_{i,j} [\hat{n}_{i\uparrow}, c^\dagger_{j\uparrow} c_{j\downarrow}] - [\hat{n}_{i\downarrow}, c^\dagger_{j\uparrow} c_{j\downarrow}] \\
&= \frac{1}{2} \sum_{i} [\hat{n}_{i\uparrow}, c^\dagger_{i\uparrow} c_{i\downarrow}] - [\hat{n}_{i\downarrow}, c^\dagger_{i\uparrow} c_{i\downarrow}] \\
&= \frac{1}{2} \sum_{i} c^\dagger_{i\uparrow} [\hat{n}_{i\uparrow} c_{i\downarrow}] + [\hat{n}_{i\uparrow}, c^\dagger_{i\uparrow}] c_{i\downarrow} - (c^\dagger_{i\uparrow} [\hat{n}_{i\downarrow}, c_{i\downarrow}] + [\hat{n}_{i\downarrow}, c^\dagger_{i\uparrow}] c_{i\downarrow}) \\
&= \frac{1}{2} \sum_{i} 0 + c^\dagger_{i\uparrow} c_{i\downarrow} - (c^\dagger_{i\uparrow}(-c_{i\downarrow}) + 0) \\
&= \sum_{i} c^\dagger_{i\uparrow} c_{i\downarrow} = \hat{S}_+
\end{aligned}$$

The procedure for $\hat{S}_-$ is similar.

b)

$$\begin{aligned}
[\hat{J}_+, \hat{J}_-] &= \sum_{i,j=1}^{2} (-1)^{i+j} [c^\dagger_{i\uparrow} c^\dagger_{i\downarrow}, c_{j\downarrow} c_{j\uparrow}] \\
&= \sum_{i,j=1}^{2} (-1)^{i+j} (c^\dagger_{i\uparrow} [c^\dagger_{i\downarrow}, c_{j\downarrow} c_{j\uparrow}] + [c^\dagger_{i\uparrow}, c_{j\downarrow} c_{j\uparrow}] c^\dagger_{i\downarrow}) \\
&= \sum_{i,j=1}^{2} (-1)^{i+j} (c^\dagger_{i\uparrow} c_{j\uparrow} \delta_{ij} - c_{j\downarrow} c^\dagger_{i\downarrow} \delta_{ij}) \\
&= \sum_{i=1}^{2} (-1)^{2i} (c^\dagger_{i\uparrow} c_{i\uparrow} - c_{i\downarrow} c_{i\downarrow} + \{c^\dagger_{i\downarrow}, c_{i\downarrow}\} - 1) \\
&= \sum_{i=1}^{2} c^\dagger_{i\uparrow} c_{i\uparrow} + c^\dagger_{i\downarrow} c_{i\downarrow} - 1 = 2\hat{J}_z
\end{aligned}$$

Second identity:

$$
\begin{aligned}
[\hat{J}_z, \hat{J}_+] &= \frac{1}{2}\sum_{i,j=1}^{2}(-1)^{j+1}[c^\dagger_{i\uparrow}c_{i\uparrow} + c^\dagger_{i\downarrow}c_{i\downarrow}, c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}] \\
&= \frac{1}{2}\sum_{i,j=1}^{2}(-1)^{j+1}([\hat{n}_{i\uparrow}, c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}] + [\hat{n}_{i\downarrow}, c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}]) \\
&= \frac{1}{2}\sum_{i,j=1}^{2}(-1)^{j+1}(c^\dagger_{j\uparrow}[\hat{n}_{i\uparrow}, c^\dagger_{i\downarrow}] + [\hat{n}_{i\uparrow}, c^\dagger_{j\uparrow}]c^\dagger_{j\downarrow} + c^\dagger_{j\uparrow}[\hat{n}_{i\downarrow}, c^\dagger_{i\downarrow}] + [\hat{n}_{i\downarrow}, c^\dagger_{j\uparrow}]c^\dagger_{j\downarrow}) \\
&= \frac{1}{2}\sum_{i,j=1}^{2}(-1)^{j+1}(0 + c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}\delta_{ij} + c^\dagger_{j\uparrow}c_{j\downarrow} + 0) \\
&= \sum_{i=1}^{2}(-1)^{i+1}c^\dagger_{i\uparrow}c^\dagger_{i\downarrow} = \hat{J}_+
\end{aligned}
$$

Again the calculation for $\hat{J}_-$ is similar.
$[\hat{J}_z, \hat{S}_z] = 0$ because once expanded the commutator yields a sum of commutators of the form $[\hat{n}_k, \hat{n}_{k'}]$. If $k = k'$ clearly the commutator is 0, otherwise it is 0 due to the earlier result that $[\hat{n}_k, c_{k'}] = [\hat{n}_k, c^\dagger_{k'}] = 0$ if $k \neq k'$.

$$
\begin{aligned}
[\hat{J}_z, \hat{S}_+] &= \frac{1}{2}\sum_{i,j}[n_{i\uparrow}, c^\dagger_{j\uparrow}c_{j\downarrow}] + [\hat{n}_{i\downarrow}, c^\dagger_{j\uparrow}c_{j\downarrow}] \\
&= \frac{1}{2}\sum_{ij} c^\dagger_{j\uparrow}[n_{i\uparrow}, c_{j\downarrow}] + [\hat{n}_{i\uparrow}c^\dagger_{j\uparrow} + c^\dagger_{j\uparrow}[\hat{n}_{i\downarrow}, c_{j\downarrow}] + [\hat{n}_{i\downarrow}, c^\dagger_{j\uparrow}]c_{j\downarrow} \\
&= \frac{1}{2}\sum_{ij} 0 + \delta_{ij}c^\dagger_{j\uparrow}c_{j\downarrow} - \delta_{ij}c^\dagger_{j\uparrow}c_{j\downarrow} + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{S}_z, \hat{J}_+] &= \frac{1}{2}\sum_{i,j}(-1)^{j+1}([\hat{n}_{i\uparrow}, c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}] - [\hat{n}_{i\downarrow}, c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}]) \\
&= \frac{1}{2}\sum_{i,j}(-1)^{j+1}(c^\dagger_{j\uparrow}[\hat{n}_{i\uparrow}, c^\dagger_{j\downarrow}] + [\hat{n}_{i\uparrow}, c^\dagger_{j\uparrow}]c^\dagger_{j\downarrow} - c^\dagger_{j\uparrow}[\hat{n}_{i\downarrow}, c^\dagger_{j\downarrow}] - [\hat{n}_{i\downarrow}, c^\dagger_{j\uparrow}]c^\dagger_{j\downarrow}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{J}_+, \hat{S}_+] &= \sum_{i,j}(-1)^{i+1}[c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}, c^\dagger_{j\uparrow}c_{j\downarrow}] \\
&= \sum_{i,j}(-1)^{i+1}(c^\dagger_{j\uparrow}[c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}, c_{j\downarrow}] + [c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}, c^\dagger_{j\uparrow}]c_{j\downarrow}) \\
&= \sum_{i,j}(-1)^{i+1}\left[c^\dagger_{j\uparrow}(c^\dagger_{i\uparrow}\{c^\dagger_{i\downarrow}, c_{j\downarrow}\} - \{c^\dagger_{i\uparrow}, c_{j\downarrow}\}c^\dagger_{i\downarrow}) + (c^\dagger_{i\uparrow}\{c^\dagger_{i\downarrow}, c^\dagger_{j\uparrow}\} - \{c^\dagger_{i\uparrow}, c^\dagger_{j\uparrow}\}c^\dagger_{i\downarrow})c_{j\downarrow}\right] \\
&= \sum_{i}(-1)^{i+1}(c^\dagger_{i\uparrow})^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{J}_+, \hat{S}_-] &= \sum_{i,j}(-1)^{i+1}[c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}, c^\dagger_{j\downarrow}c_{j\uparrow}] \\
&= \sum_{i,j}(-1)^{i+1}(c^\dagger_{j\downarrow}[c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}, c_{j\uparrow}] + [c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}, c^\dagger_{j\downarrow}]c_{j\uparrow}) \\
&= \sum_{i,j}(-1)^{i+1}\left[c^\dagger_{j\downarrow}(c^\dagger_{i\uparrow}\{c^\dagger_{i\downarrow}, c_{j\uparrow}\} - \{c^\dagger_{i\uparrow}, c_{j\uparrow}\}c^\dagger_{j\downarrow}) + (c^\dagger_{i\uparrow}\{c^\dagger_{i\downarrow}, c^\dagger_{j\downarrow}\} - \{c^\dagger_{i\uparrow}, c^\dagger_{j\downarrow}\}c^\dagger_{i\downarrow})c_{j\uparrow}\right] \\
&= \sum_{i}(-1)^{i}(c^\dagger_{i\downarrow})^2 = 0
\end{aligned}
$$