Generic Notes

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1 Introduction to the propagator

Consider the classical mechanics problem of two blocks of mass m in a well coupled by springs to the walls and a spring between them (three springs in total). Denoting x_1 and x_2 as the displacement of each block from the equilibrium position, (note there is a unique equilibrium configuration of the masses, since we can set up two equations with two unknowns and solve them) Hooke's law implies that

$$\begin{aligned} \ddot{x}_1 &= -2\frac{k}{m}x_1 + \frac{k}{m}x_2 \\ \ddot{x}_2 &= \frac{k}{m}x_1 - 2\frac{k}{m}x_2 \end{aligned}$$

Written in matrix form, this is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{matrix} \Omega_{11} = -\frac{2k}{m} & \Omega_{12} = \frac{k}{m} \\ \Omega_{21} = \frac{k}{m} & \Omega_{22} = -\frac{2k}{m} \end{matrix}$$

From which it is apparent that the matrix Ω is hermitian. This can be represented in a more abstract form as

$$|\ddot{x}(t)\rangle = \Omega\,|x(t)\rangle$$

In this representation, the equivalent of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $|1\rangle$, representing displacement of the first mass from its equilibrium position and the equivalent of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $|2\rangle$, representing displacement of the second mass from its equilibrium position. Therefore

$$|x(t)\rangle = x_1\,|1\rangle + x_2\,|2\rangle$$

The abstract vector equation can be put into a form that more closely relates to the original matrix equation by projecting onto $|1\rangle$ and $|2\rangle$, and using the completeness relation $|1\rangle\langle 1| + |2\rangle\langle 2| = I$:

$$\begin{aligned} \ddot{x}_1 = \langle 1|\ddot{x}(t)\rangle &= \langle 1|\,\Omega(|1\rangle\langle 1| + |2\rangle\langle 2|)\,|x(t)\rangle &= \langle 1|\Omega|1\rangle\langle 1|x(t)\rangle + \langle 1|\Omega|2\rangle\langle 2|x(t)\rangle &= \Omega_{11}x_1 + \Omega_{12}x_2 \\ \ddot{x}_2 = \langle 2|\ddot{x}(t)\rangle &= \langle 2|\,\Omega(|1\rangle\langle 1| + |2\rangle\langle 2|)\,|x(t)\rangle &= \langle 2\Omega|1\rangle\langle 1|x(t)\rangle + \langle 2|\Omega|2\rangle\langle 2|x(t)\rangle &= \Omega_{21}x_1 + \Omega_{22}x_2 \end{aligned}$$

To be continued.

2 Green's Function, Equation of Motion

The Schrödinger picture considers the wavefunction time-dependent and the operator time-independent. On the other hand, the Heisenberg picture considers the wavefunction time-independent and the operator time-dependent. These viewpoints are mathematically equivalent, because the Heisenberg time-dependent operator $\hat{o}_{\mathcal{H}}$ is defined in terms of the Schrödinger time independent operator $\hat{o}_{\mathcal{S}}$ as

$$\hat{o}_{\mathcal{H}} = e^{i\hat{H}t}\hat{o}_{\mathcal{S}}e^{-i\hat{H}t}$$

so that the evolution of the expectation of an operator remains the same, as seen in

$$\langle\Psi(t)|\hat{o}_{\mathcal{S}}|\Psi(t)\rangle = \langle e^{-i\hat{H}t}\psi_0|\hat{o}_{\mathcal{S}}|e^{-i\hat{H}t}\psi_0\rangle = \langle\psi_0|e^{i\hat{H}t}\hat{o}_{\mathcal{S}}e^{-i\hat{H}t}|\psi_0\rangle = \langle\psi_0|\hat{o}_{\mathcal{H}}|\psi_0\rangle$$

Heisenberg's equation can be derived using the chain rule as follows:

$$\frac{\partial \hat{o}_{\mathcal{H}}}{\partial t} = (i\hat{H})e^{i\hat{H}t}\hat{o}_{\mathcal{S}}e^{-i\hat{H}t} + e^{i\hat{H}t}\hat{o}_{\mathcal{S}}e^{-i\hat{H}t}(-i\hat{H}) = i[\hat{H}, \hat{o}_{\mathcal{H}}]$$

This equation may appear without the factor of i attached to the commutator when working with imaginary time. A useful property is that

$$[\hat{a}_{\mathcal{H}}, \hat{b}_{\mathcal{H}}] = [\hat{a}_{\mathcal{S}}, \hat{b}_{\mathcal{S}}]_{\mathcal{H}}$$

Similarly

$$\{\hat{a}_{\mathcal{H}}, \hat{b}_{\mathcal{H}}] = \{\hat{a}_{\mathcal{S}}, \hat{b}_{\mathcal{S}}\}_{\mathcal{H}}$$

The Green's function can be approached using the framework of equation of motion. As a reminder, the Green's function was

$$G(\tau, \tau') = -\frac{1}{z}\operatorname{Tr}\left[e^{-\beta\hat{H}}T_\tau C(\tau)C^\dagger(\tau')\right] = -\langle T_\tau C(\tau)C^\dagger(\tau')\rangle \qquad \langle\hat{o}\rangle := \frac{1}{z}\operatorname{Tr} e^{-\beta\hat{h}}\hat{o}$$

For now, as a notation that may come in handy in the future, let the subscripts ij in $G_{ij}(\tau, \tau')$ denote subscripts of the $C(\tau)$ and $C^\dagger(\tau')$ respectively. Also, for the purpose of taking a derivative, it's helpful to rewrite the function as

$$G_{ij}(\tau, \tau') = -\theta(\tau - \tau')\langle C(\tau)C^\dagger(\tau')\rangle + \theta(\tau' - \tau)\langle C^\dagger(\tau')C(\tau)\rangle$$

Since the derivative of the heaviside function is the delta function. Then (using time-translational invariance and an implicit 0 for the second argument of Green's function)

$$\begin{aligned}\frac{\partial G_{ij}(\tau)}{\partial \tau} &= -\delta(\tau)\langle C_i(\tau)C_j^\dagger(0) + C_j^\dagger(0)C_i(\tau)\rangle - \theta(\tau)\langle[\hat{H}, C_i(\tau)]C_j^\dagger(0)\rangle + \theta(-\tau)\langle C_j^\dagger(0), [\hat{H}, C_i(\tau)]\rangle \\ &= -\delta(\tau)\delta_{ij} - \theta(\tau)\langle\mu C_i(\tau)C_j^\dagger(0)\rangle + \theta(-\tau)\langle C_j^\dagger(0), \mu C_i(\tau)\rangle \\ &= -\delta(\tau)\delta_{ij} + \mu G_{ij}(\tau)\end{aligned}$$

Rearranging this:

$$\implies (\mu - \frac{\partial}{\partial\tau})G_{ij} = \delta(\tau)\delta_{ij}$$

If the right side is interpreted to be "unity," then one might interpret the left side as "$G^{-1}G$". Now, what happens when the Hamiltonian is switched from $-\mu c^\dagger c$ to, say,

$$-\mu(c_\uparrow^\dagger c_\uparrow + c_\downarrow^\dagger c_\downarrow) + U\hat{n}_\uparrow\hat{n}_\downarrow$$

The only thing that changes in the above derivation is $[\hat{H}, C(\tau)]$. Recalling some commutator algebra with creation/annihilation operators:

$$\begin{aligned}[\hat{n}_{i\sigma}, c_{j\sigma'}] &= c_{i\sigma}^\dagger\{c_{i\sigma}, c_{j\sigma'}\} - \{c_{i\sigma}^\dagger, c_{j\sigma}\}c_{i\sigma} \\ &= -\delta_{ij}\delta_{\sigma\sigma'}c_{i\sigma} \\ [\hat{n}_{i\uparrow}\hat{n}_{i\downarrow}, c_{j\sigma}] &= \hat{n}_{i\uparrow}[\hat{n}_{i\downarrow}, c_{j\sigma}] + [\hat{n}_{i\uparrow}, c_{j\sigma}]\hat{n}_{i\downarrow} \\ &= -\delta_{ij}(\delta_{\sigma\downarrow}\hat{n}_{i\uparrow}c_{i\downarrow} + \delta_{\sigma\uparrow}\hat{n}_{i\downarrow}c_{i\uparrow})\end{aligned}$$

Therefore, the final equation of motion doesn't change much. Concluding the derivation

$$\begin{aligned}\frac{\partial G_{\sigma\sigma'}}{\partial\tau} &= -\delta(\tau) - \theta(\tau)\langle[\hat{H}, C_\sigma(\tau)]C_{\sigma'}^\dagger(0)\rangle + \theta(-\tau)\langle C_{\sigma'}^\dagger(0), [\hat{H}, C_\sigma(\tau)]\rangle \\ &= -\delta(\tau) - \theta(\tau)\langle(\mu C_\sigma(\tau) - U\hat{N}_{\bar{\sigma}}C_\sigma(\tau))C_{\sigma'}^\dagger(0)\rangle + \theta(\tau)\langle C_{\sigma'}^\dagger(0)(\mu C_\sigma(\tau) - U\hat{N}_{\bar{\sigma}}C_\sigma(\tau))\rangle \\ &= -\delta(\tau) + \mu G_{\sigma\sigma'} + U\,\theta(\tau)\langle\hat{N}_{\bar{\sigma}}C_\sigma(\tau)C_{\sigma'}^\dagger(0)\rangle - U\,\theta(\tau)\langle C_{\sigma'}^\dagger(0)\hat{N}_{\bar{\sigma}}C_\sigma(\tau)\rangle \\ &= -\delta(\tau) + (\mu - U\hat{N}_{\bar{\sigma}})G_{\sigma\sigma'}\end{aligned}$$

$\hat{N}$ is the number operator so it can be moved out of the expectation freely (? prove this later). Therefore the obtained equation of motion is

$$(\mu - U\hat{N}_{\bar{\sigma}} - \frac{\partial}{\partial\tau})G_{\sigma\sigma'} = \delta(\tau)$$

2.1 Action

What if you introduce a disturbance to your system over a period of time? Say the hamiltonian is

$$\hat{H} = -\mu c^\dagger c$$

and the disturbance is modeled by

$$S(U) = e^{-i\int \theta(t)\,\theta(t_0-t)Uc^\dagger(t)c(t)dt} = e^{-i\int_0^{t_0} Uc^\dagger(t)c(t)dt}$$

which changes the Green's function (this one is the real time version) to

$$G(t,t') = -i\langle T_t C(t)C^\dagger(t')S(U)\rangle$$

The presence of the $S(U)$ modifies the value of the normalization constant z (so that $S(U)$ really defines a new expectation) to

$$\begin{aligned}
z(u) &= \mathrm{Tr}\left[e^{-i\int_0^{t_0} Uc^\dagger c dt} e^{\mu\beta c^\dagger c}\right] \\
&= 1 + e^{\mu\beta} e^{-i\int_0^{t_0} U dt} \\
&= 1 + e^{\mu\beta - iUt_0}
\end{aligned}$$

Now let's compute this Green's function for a specific order of $0, t_0, t, t'$. There are essentially 12 orders. The order we will compute Green's function for is $t < 0 < t' < t_0$. The first step is to find an expression for $C(t) = e^{i\hat{H}t}ce^{-i\hat{H}t}$ and one for $C^\dagger(t') = e^{i\hat{H}t'}c^\dagger e^{-i\hat{H}t'}$. Using Heisenberg's equation:

$$\begin{aligned}
\frac{\partial C(t)}{\partial t} &= i[-\mu c^\dagger c, C(t)] \\
&= i\mu C(t) \\
\Rightarrow C(t) &= e^{i\mu t}c \\
\frac{\partial C^\dagger(t')}{\partial t'} &= i[-\mu c^\dagger c, C^\dagger(t')] \\
&= -i\mu C^\dagger(t') \\
\Rightarrow C^\dagger(t') &= e^{-i\mu t}c^\dagger
\end{aligned}$$

Evaluating Green's function supposing $t < 0 < t' < t_0$:

$$\begin{aligned}
G(t,t') &= -i\,\theta(t' > t_{int} > t)\langle C^\dagger(t')e^{-i\int_0^{t'} Uc^\dagger c\,dt}C(t)\rangle + i\,\theta(t_{int} > t' > t)\langle e^{-i\int_{t'}^{t_0} Uc^\dagger c\,dt}C^\dagger(t')C(t)\rangle \\
&= -\frac{i}{z}\,\mathrm{Tr}\left[e^{-\beta\hat{H}}e^{-i\mu t'}c^\dagger e^{-i\int_0^{t'} Uc^\dagger c\,dt}e^{i\mu t}c\right] + \frac{i}{z}\,\mathrm{Tr}\left[e^{-\beta\hat{H}}e^{-i\int_{t'}^{t_0} Uc^\dagger c\,dt}e^{-i\mu t'}c^\dagger e^{i\mu t}c\right] \\
&= -\frac{ie^{i\mu(t-t')}}{z}\left[\,\langle 0|c^\dagger e^{-i\int_0^{t'} Uc^\dagger c\,dt}c|0\rangle + e^{\beta\mu}\,\langle 1|e^\dagger e^{-i\int_0^{t'} Uc^\dagger c\,dt}c|1\rangle - e^{\beta\mu - iU(t_0-t')}\right] \\
&= -\frac{ie^{i\mu(t-t')}}{z}(0 + e^{\beta\mu}(1) - e^{\beta\mu - iU(t_0-t')}) \\
&= -\frac{ie^{\mu(\beta+i(t-t'))}}{z}(1 - e^{-iU(t_0-t')})
\end{aligned}$$