Summing angular momentum

Forest Yang

June 2, 2017

Consider the possibility of a product of angular momentum states, with each state being independent from the other, in the sense that if one defines angular momentum operators

$$\begin{aligned} &\hat{J}_{1i} && \text{acting on the first state} \\ &\hat{J}_{2i} && \text{acting on the second state,} \\ &\text{then} && [\hat{J}_{1i}, \hat{J}_{2j}] = 0 \end{aligned}$$

Furthermore, these angular momentum operators are assumed by definition to follow the standard relations

$$[\hat{J}_{ni}, \hat{J}_{nj}] = i\hbar\epsilon_{ijk}\hat{J}_{nk}$$

So the myriad of results proven from this relation still follow. Identifying a state $|j_1m_1\rangle \times |j_2m_2\rangle$ as $|j_1m_1j_2m_2\rangle$

$$\begin{aligned} \hat{J}_{1z}\,|j_1m_1j_2m_2\rangle &= \hbar m_1\,|j_1m_1j_2m_2\rangle & \hat{J}_{2z}\,|j_1m_1j_2m_2\rangle &= \hbar m_2\,|j_1m_1j_2m_2\rangle \\ \hat{\mathbf{J}}_1^2\,|j_1m_1j_2m_2\rangle &= \hbar^2 j_1(j_1+1)\,|j_1m_1j_2m_2\rangle & \hat{\mathbf{J}}_2^2\,|j_1m_1j_2m_2\rangle &= \hbar^2 j_2(j_2+1)\,|j_1m_1j_2m_2\rangle \end{aligned}$$

Now, these states are eigenstates of $\{\hat{J}_{1z}, \hat{\mathbf{J}}_1^2, \hat{J}_{2z}, \hat{\mathbf{J}}_2^2\}$. These operators clearly commute with each other. Somewhat by definition, these states form a complete set (it consists of all combinations of each of the states in the product). It turns out that eigenstates of $\{\hat{\mathbf{J}}^2, \hat{J}_z^2, \hat{J}_1, \hat{J}_2\}$ where $\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$ span the same complete set (Wikipedia "The Compatibility Theorem." This is still confusing but it helps). First step is to verify that $\hat{\mathbf{J}}$ is indeed an angular momentum operator.

$$[\hat{J}_i, \hat{J}_j] = [\hat{J}_{1i} + \hat{J}_{2i}, \hat{J}_{1j} + \hat{J}_{2j}] = [\hat{J}_{1i}, \hat{J}_{1j}] + [\hat{J}_{2i}, \hat{J}_{2j}] = i\hbar\epsilon_{ijk}\hat{J}_{1k} + i\hbar\epsilon_{ijk}\hat{J}_{2k} = i\hbar\epsilon\hat{J}_k$$

The concomitant relations follow. In particular, $[\hat{\mathbf{J}}^2, \hat{J}_z] = 0$. Furthermore, $[\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_n^2] = 0$ because $\hat{\mathbf{J}}_n^2$ essentially commutes with everything. Therefore $\{\hat{\mathbf{J}}^2, \hat{J}_z^2, \hat{J}_1, \hat{J}_2\}$ is a complete set of commuting observables, with eigenstates denoted by $|jmj_1j_2\rangle$. These are generally not eigenstates of the $\hat{J}_{1z}$ and $\hat{J}_{2z}$, since $\hat{\mathbf{J}}^2$ can be rewritten as

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_1 \cdot \hat{J}_2 = \hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+} + \hat{J}_{1z}\hat{J}_{2z}$$

Due to the raising and lowering operators, $|j_1m_1j_2m_2\rangle$ wouldn't be an eigenstate unless m_1 and m_2 were both maximum or both minimum. It is in our interest to consider the possible values of j, the total angular momentum number. In the separate angular momentum basis, there is one state with $m = j_1 + j_2$, it would be the $|j_1j_1j_2j_2\rangle$ state. In the total angular momentum basis, this corresponds to the $|j_1 + j_2, j_1 + j_2, j_1j_2\rangle$ state. The j number must be $j_1 + j_2$ since it must be at least as high as the m number. There also must be a $j = j_1 + j_2 - 1$ number, because there are two $m = j_1 + j_2 - 1$ states in the separate basis and the total basis only has a $j = j_1 + j_2$ that can support an $m = j_1 + j_2 - 1$ so far. (There is actually an implied non-degeneracy assumption here, but it can be proven. If the state $|j, m-1, j_1, j_2\rangle$ is k-fold degenerate it implies the state $|j, m, j_1, j_2\rangle$ is k-fold degenerate because $\hat{J}_+$ maps the latter to the former) Continuing on in this fashion, the number of m states stops increasing when you get to $m = |j_1 - j_2|$, because in a sense we've reached the bottom value of the minimum of j_1, j_2. So, the minimum value of j is actually $|j_1 - j_2|$.

This observation also gives a way to express a $|jmj_1j_2\rangle$ state in terms of $|j_1m_1j_2m_2\rangle$ states. This is done for two spin-$\frac{1}{2}$ particles as follows.

$$
\begin{aligned}
|11\rangle &= |{\uparrow\uparrow}\rangle \\
\hat{J}_- |11\rangle &= (\hat{J}_{1-} + \hat{J}_{2-}) |{\uparrow\uparrow}\rangle \\
\sqrt{(1+1)(1-1+1)}\, |10\rangle &= \sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-\frac{1}{2}+1)}(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle) \\
|10\rangle &= \frac{1}{\sqrt{2}}(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle) \\
|1\text{-}1\rangle &= |{\downarrow\downarrow}\rangle \\
|00\rangle &= \frac{1}{\sqrt{2}}(|{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle)
\end{aligned}
$$

This can be expressed in matrix form as

$$
\begin{pmatrix} |11\rangle \\ |10\rangle \\ |1,-1\rangle \\ |00\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |{\uparrow\uparrow}\rangle \\ |{\uparrow\downarrow}\rangle \\ |{\downarrow\uparrow}\rangle \\ |{\downarrow\downarrow}\rangle \end{pmatrix}
$$

So to get the $|j_1 m_1 j_2 m_2\rangle$ states in terms of the $|j m j_1 j_2\rangle$ states one simply needs to invert the above matrix.