

CS70: Counting

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Today:

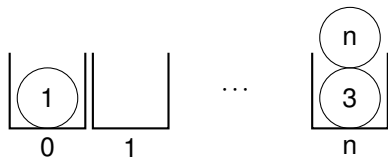
- ▶ Balls and bins.
- ▶ Sum rule.
- ▶ Combinatorial proofs.
- ▶ Maybe start review?

What we've learned so far

Sample k items out of n .

	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$

Balls in bins.



“ k Balls in n bins” \equiv “ k samples from n possibilities.”

“indistinguishable balls” \equiv “order doesn’t matter”

“only one ball in each bin” \equiv “without replacement”

5 balls into 10 bins

5 samples from 10 possibilities with replacement

Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin

5 samples from 52 possibilities without replacement

Example: Poker hands.

5 indistinguishable balls into 3 bins

5 samples from 3 possibilities with replacement and no order

Dividing 5 dollars among Alice, Bob and Eve.

Sum Rule

Two indistinguishable jokers in 54 card deck.

How many 5 card poker hands?

Sum rule: Can sum over disjoint sets.

No jokers or exactly one Joker or exactly two Jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.

How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$

Algebraic proof

$$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$$

Proof:

$$\binom{54}{5} = \frac{54!}{5!49!}$$

$$\binom{52}{5} = \frac{52!}{5!47!}, \quad \binom{52}{4} = \frac{52!}{4!48!}, \quad \binom{52}{3} = \frac{52!}{3!49!}$$

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3} = \frac{52!}{5!47!} + 2 \frac{52!}{4!48!} + \frac{52!}{3!49!}$$

$$= \frac{52!(4!48!3!49! + 2*5!47!3!49! + 5!47!4!48!)}{5!47!4!48!3!49!}$$

=

Let's actually go with the other one....

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k ? $\binom{n}{k}$

How many subsets of size k ?

Choose a subset of size $n - k$

and what's left out is a subset of size k .

Choosing a subset of size k is same

as choosing $n - k$ elements to not take.

$\implies \binom{n}{n-k}$ subsets of size k .



Pascal's Triangle

		0				
	1		1			
	1	2	1			
	1	3	3	1		
	1	4	6	4	1	
	1	5	10	10	5	1

Row n : coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Zero-th row: $(1+x)^0 = 1$

First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1

Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1

Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1

.....

Foil?? I hate this...

Pascal's Triangle

		0			
	1		1		
	1	2	1		
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

Simplify: collect all terms corresponding to x^k .

Coefficient of x^k is $\binom{n}{k}$: choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term, first and third term, first and fourth, second and third, second and fourth, third and fourth. 6 of them, aka $\binom{4}{2}$.

		$\binom{0}{0}$		
	$\binom{1}{0}$		$\binom{1}{1}$	
	$\binom{2}{0}$	$\binom{2}{1}$	$\binom{2}{2}$	
$\binom{3}{0}$	$\binom{3}{1}$	$\binom{3}{2}$	$\binom{3}{3}$	

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?

The ones that contain the first element plus the ones that don't contain the first element.

How many contain the first element?

Need to choose $k-1$ more from remaining n elements.

$$\implies \binom{n}{k-1}$$

How many don't contain the first element ?

Need to choose k elements from remaining n elts.

$$\implies \binom{n}{k}$$

So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.



Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.

Proof:

Left Hand Side (LHS): Size k subsets of n .

Consider size k subset where i is the first element chosen.

$$\{1, \dots, \underline{i}, \dots, n\}$$

Must choose $k - 1$ elements from $n - i$ remaining elements.

$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.

2 is first element chosen: $\binom{n-2}{k-1}$ choices for the rest.

and so on.

Add them up to get the total number of subsets of size k which is also $\binom{n}{k}$.



Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1, \dots, n\}$?

Construct a subset with sequence of n choices:

element i **is in** or **is not** in the subset: 2 possibilities.

First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

How many subsets of $\{1, \dots, n\}$?

$\binom{n}{i}$ = subsets of size i .

A subset has size either 0, or 1, or 2, \dots , or n

Sum over i to get total number of subsets.



Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T , $|S \cup T| = |S| + |T|$

Used to reason about all subsets

by adding number of subsets of size 0, 1, 2, 3,...

Also reasoned about subsets that contained

or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule: For any S and T ,

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit. $|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Summary.

Inclusion/Exclusion: two sets of objects.

Add number of each subtract intersection of sets.

Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.

Pascal's Triangle Example: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

RHS: Number of subsets of $n+1$ items size k .

LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.

$\binom{n}{k}$ counts subsets of $n+1$ items without first item.

Disjoint – so add!