CS70: Discrete Math and Probability

June 21, 2016

Lecture 2: Proofs!

1. Direct proof
2. by Contraposition
3. by Contradiction
4. by Cases

Quick Background and Notation.

Integers closed under addition.

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A natural number $p > 1$, is **prime** if it is divisible only by 1 and itself.

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Add $99a+11b$ to both sides.

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Assumed P: $11|a-b+c$. Proved Q: $11|n$.

The Converse

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Contrapositive: True $\implies P$.

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Theorem: $x^5 - x + 1 = 0$ has no solution in the rationals.

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Lemma: If x is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in Z$, *then both a and b are even.*

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Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.
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Thus, we have irrational x and y with a rational x^y (i.e., 2).

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Question: Which case holds?

Proof by cases.

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Question: Which case holds? Don't know!!!

Be careful.

Theorem: $3 = 4$

Be careful.

Theorem: $3 = 4$

Proof: Assume $3 = 4$.

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Start with $12 = 12$.

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Divide one side by 3 and the other by 4 to get

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By commutativity

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Be careful.

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Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get

$4 = 3$.

By commutativity theorem holds. $\square$

Don't assume what you want to prove!

Be really careful!

Theorem: $1 = 2$
Proof:

Be really careful!

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Proof: For $x = y$, we have

Be really careful!

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Proof: For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$

Be really careful!

Theorem: $1 = 2$
Proof: For $x = y$, we have

$$
\begin{aligned}
(x^2 - xy) &= x^2 - y^2 \\
x(x - y) &= (x + y)(x - y)
\end{aligned}
$$

Be really careful!

Theorem: $1 = 2$
Proof: For $x = y$, we have

$$\begin{aligned} (x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y) \\ x &= (x+y) \end{aligned}$$

Be really careful!

Theorem: $1 = 2$
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$$\begin{aligned}(x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y) \\ x &= (x+y) \\ x &= 2x\end{aligned}$$

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$P \implies Q$ does not mean $Q \implies P$.

Summary: Note 2.

Direct Proof:

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Direct Proof:
To Prove: $P \implies Q$.

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Direct Proof:
To Prove: $P \implies Q$. Assume P.

Summary: Note 2.

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By Contraposition:

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To Prove: $P \implies Q$ Assume $\neg Q$.

Summary: Note 2.

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To Prove: P

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To Prove: P Assume $\neg P$. Prove False .

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Don't assume the theorem.

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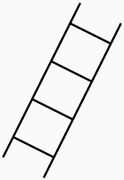
CS70: Note 3. Induction!

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.

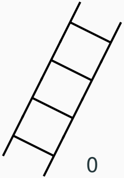
The naturals.

The naturals.



The naturals.

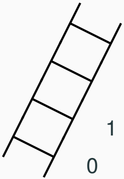
0,



0

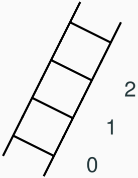
The naturals.

0, 1,



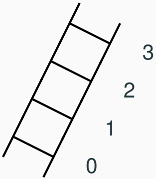
The naturals.

0, 1, 2,



The naturals.

0, 1, 2, 3,



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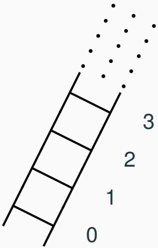
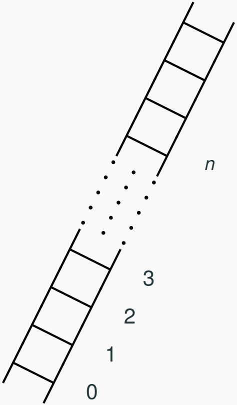


0, 1, 2, 3,

. . . ,

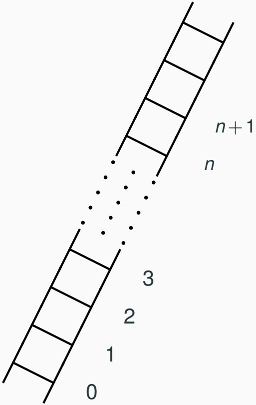
The naturals.



0, 1, 2, 3,

…, n,

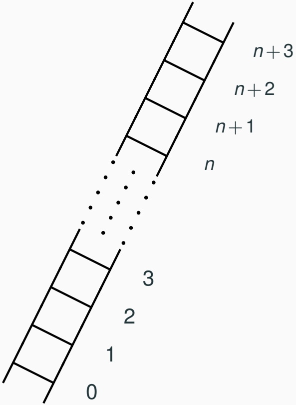
The naturals.



0, 1, 2, 3,

$\ldots, n, n+1,$

The naturals.



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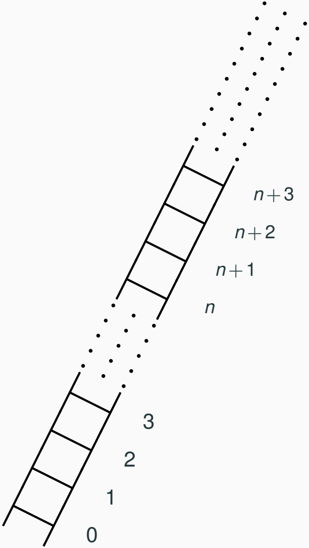

$0, 1, 2, 3,$

$\ldots, n, n+1, n+2, n+3, \ldots$

A formula.

A formula.

Teacher: Hello class.

A formula.

Teacher: Hello class.
Teacher:

A formula.

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Teacher: Please add the numbers from 1 to 100.

A formula.

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Gauss: It's

A formula.

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Gauss: It's $\frac{(100)(101)}{2}$

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Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

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Idea: assume predicate $P(n)$ for $n = k$.

Gauss and Induction

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$\sum_{i=1}^{k+1} i$

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$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1)$

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$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

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How about $k+2$.

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How about $k+2$. Same argument starting at $k+1$ works!

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$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

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Induction Step.

Gauss and Induction

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Gauss and Induction

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Statement is true for $n = 0$

Gauss and Induction

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Statement is true for $n = 0$ $P(0)$ is true

Gauss and Induction

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plus inductive step

Gauss and Induction

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plus inductive step $\implies$ true for $n = 1$

Gauss and Induction

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Gauss and Induction

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Gauss and Induction

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plus inductive step $\implies$ true for $n = 2$

Gauss and Induction

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Gauss and Induction

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Predicate, $P(n)$, True for all natural numbers!

Gauss and Induction

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Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**