CS70: Countability and Uncountability

Alex Psomas

June 30, 2016

Warning!

Warning:

Warning!

Warning: I’m really loud!

Today.

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One idea, from around 130 years ago.

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At the heart of set theory.

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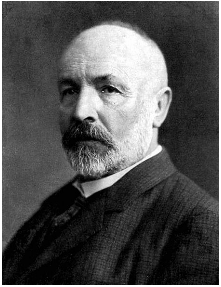

Georg Cantor

Life before Cantor

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How many elements in $\{1, 2, 4\}$?

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The symbol you write after taking a limit....

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Even Gauss: " ... first of all I must protest against the use of an infinite magnitude as a completed quantity, which is never allowed in mathematics. The Infinite is just a mannner of speaking, in which one is really talking in terms of limits, which certain ratios may approach as close as one wishes, while others may be allowed to increase without restriction. "

Cantor's questions

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Is $\mathbb{N} \setminus \{0\}$ smaller than $\mathbb{N}$?

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A hotel with infinite rooms.

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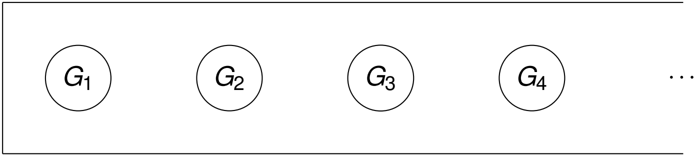

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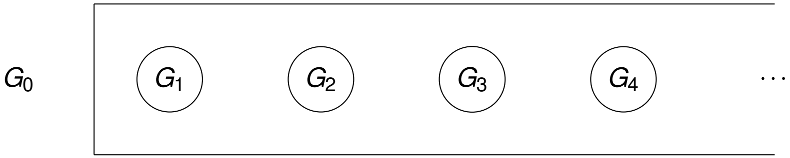

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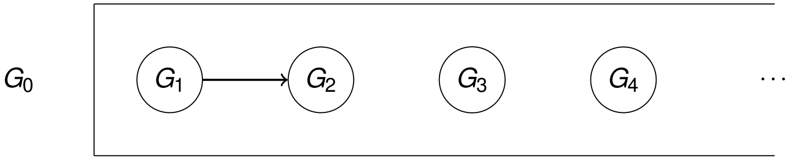

G_0 shows up. What do we do?

Move G_1 to room number 2.

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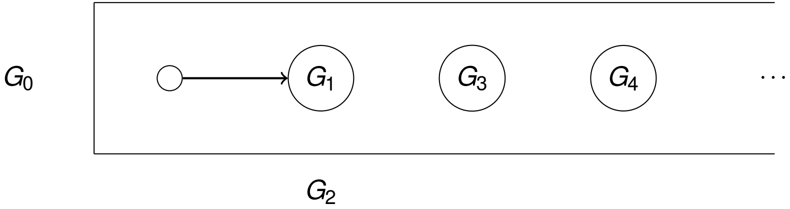

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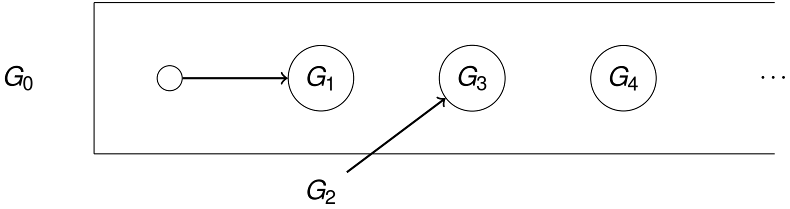

Move G_2 to room number 3.

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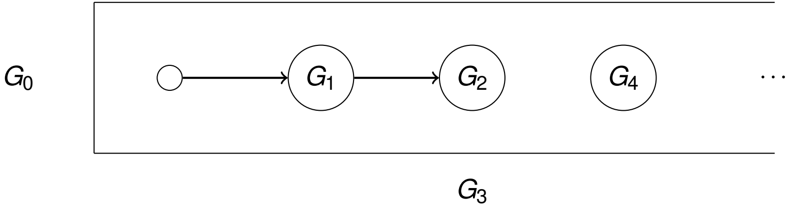

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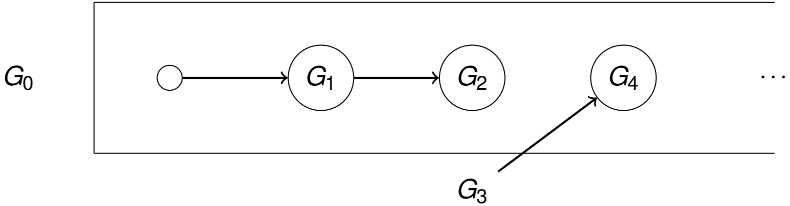

Move G_3 to room number 4.

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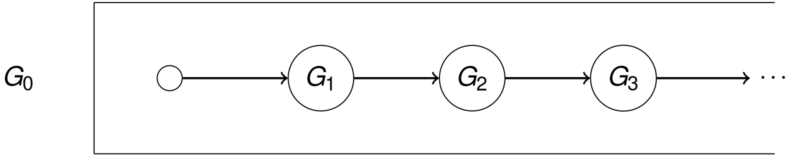

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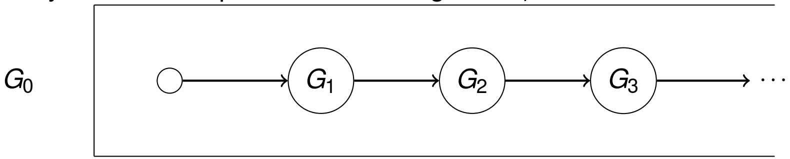

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Now G_0 can go to room number 1!!

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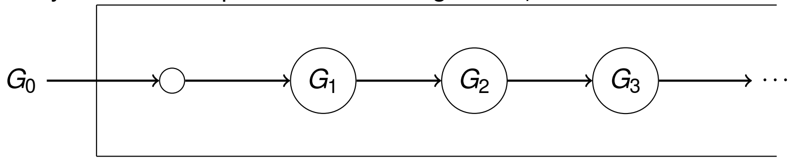

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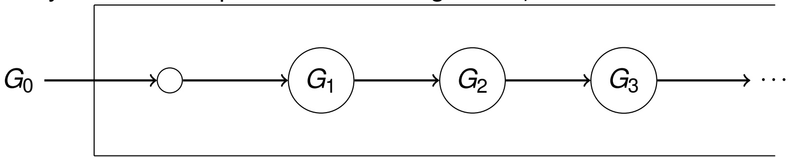

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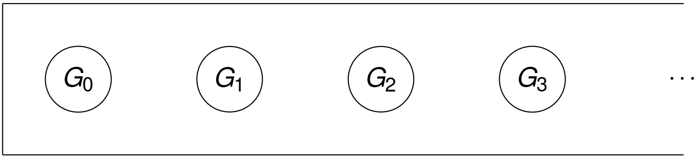

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Is this a proof?

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Is this a proof? How would we show this formally???

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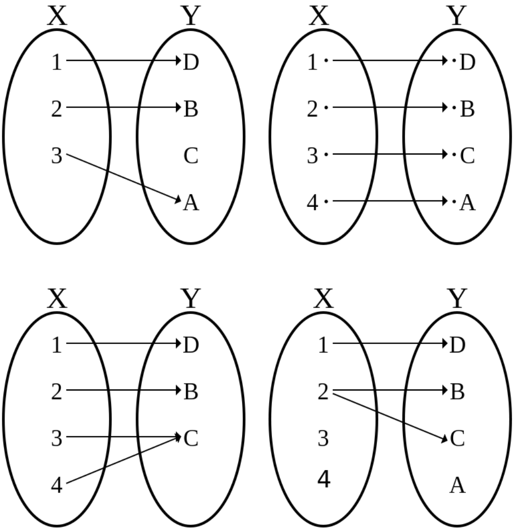
Bijection to or from natural numbers implies countably infinite.

Bijections



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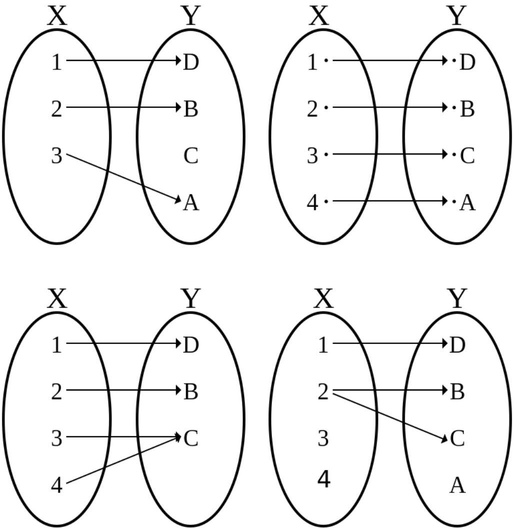
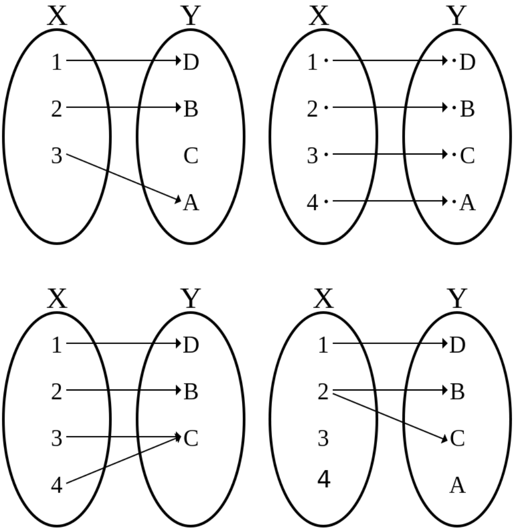


One to one.

Bijections



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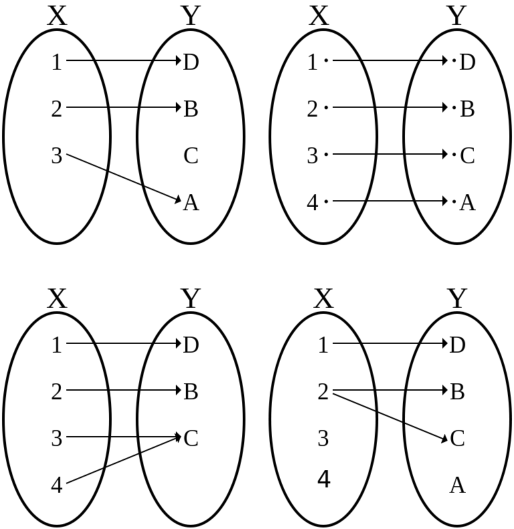

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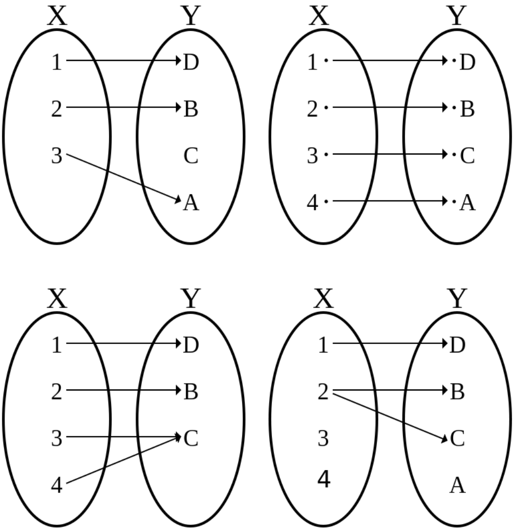

One to one. Bijection: one to one and onto. Onto. Not a function.

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  For example the set $\{14, 54, 5332, 10^{12}+4\}$ is countable. (It has 4 elements) Even numbers are countable. Prime numbers are countable. Multiples of 3 are countable.
- All countably infinite sets are the same cardinality as each other.

Back to Hilbert's hotel

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots$$

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$$G_0 \to G_1 \to G_2 \to G_3 \to \cdots$$

Where's the function?

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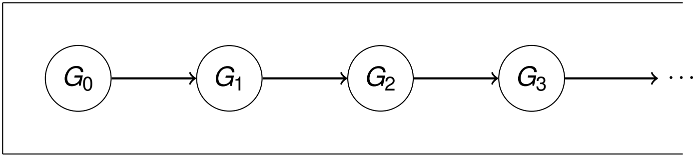
We want a bijection from:

Back to Hilbert's hotel



Where's the function?

We want a bijection from: $\mathbb{N}$

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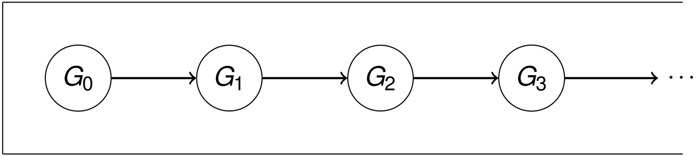

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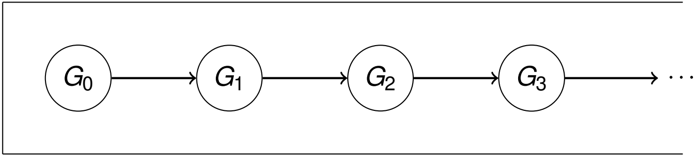

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$f(x) = x + 1$.

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We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

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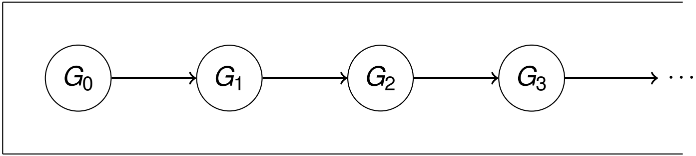
Back to Hilbert's hotel



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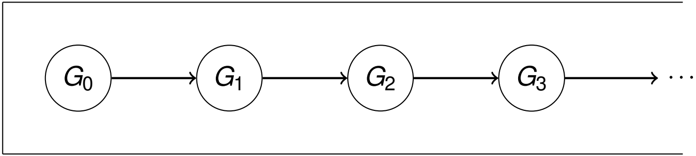
$f(x) = x + 1$. Maps every number from $\mathbb{N}$ to a number in $\mathbb{N} \setminus \{0\}$, and every number in $x \in \mathbb{N} \setminus \{0\}$ has exactly one number $y \in \mathbb{N}$ such that $f(y) = x$.

Back to Hilbert's hotel



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Same thing! Bijection means that the sets have the same size.

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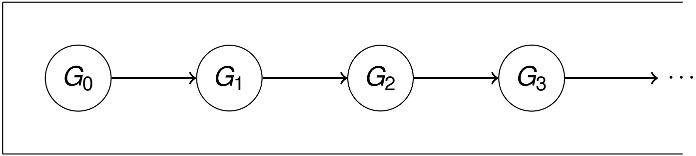

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Same thing! Bijection means that the sets have the same size. Invert it and you'll get a bijection from $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

Examples

Countable infinite (same cardinality as naturals)

- E even numbers.

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  Where are the odds?

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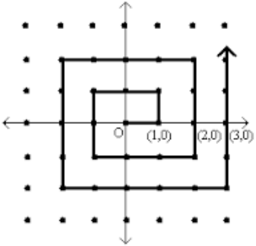

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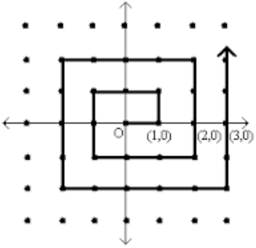

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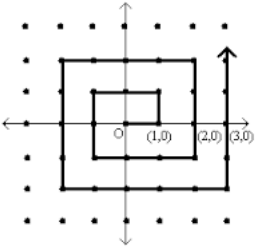

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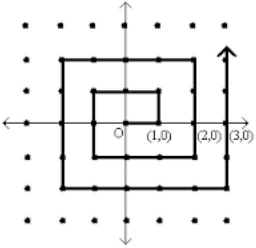

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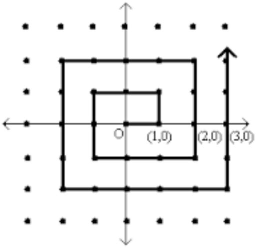

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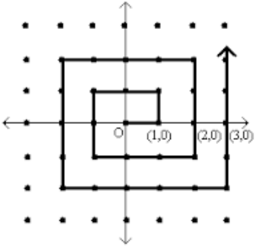

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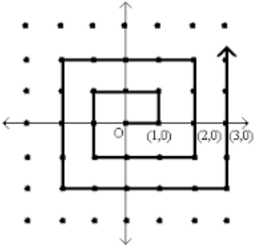

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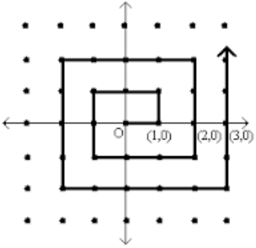

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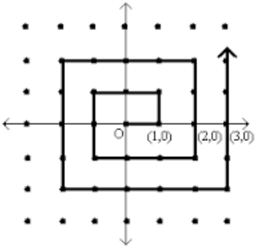

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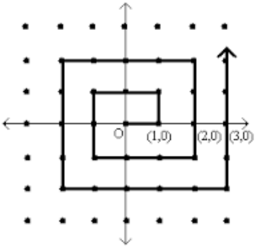

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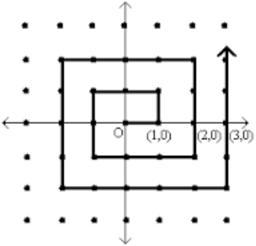

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Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
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Construct "diagonal" number:

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Construct "diagonal" number: .7767

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Any subset of a countable set is countable.

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Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

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Another diagonalization.

The set of all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0,\ldots,7\}$,

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- Assume is countable.

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Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
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Another diagonalization.

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Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes, multiples of 10

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Another diagonalization.

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  Contradiction.

Another diagonalization.

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Theorem: The set of all subsets of N is not countable.

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Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Countable or uncountable??

- Binary strings?

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- Trees?

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- Functions from $\mathbb{N}$ to $\mathbb{N}$?

What happened with Cantor?

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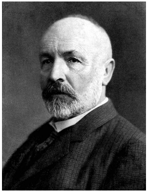

Cantor's work between 1874 and 1884 is the origin of set theory.

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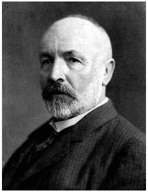

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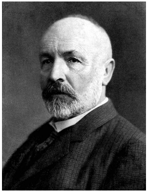

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Before Cantor: Finite

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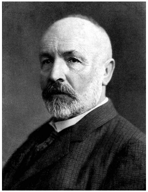

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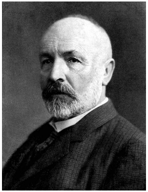

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Before Cantor: Finite , Infinite
After Cantor:

- Countable

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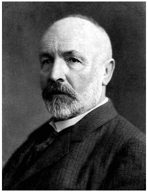

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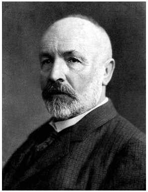

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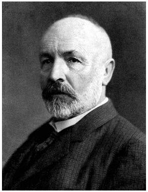

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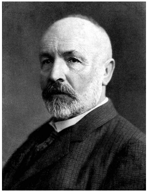

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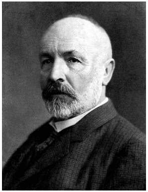

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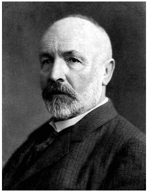

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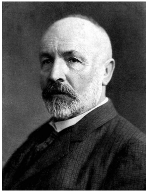

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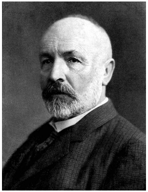

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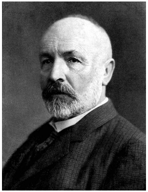

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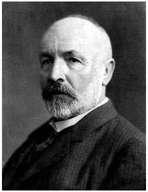

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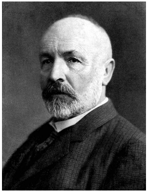

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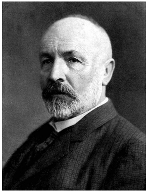

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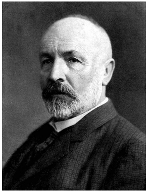

Cantor's work between 1874 and 1884 is the origin of set theory.
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Before Cantor: Finite , Infinite
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- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
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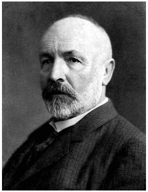

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Cantor’s legacy

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Disaster!!

A bug

Bertrand Russell finds a bug!

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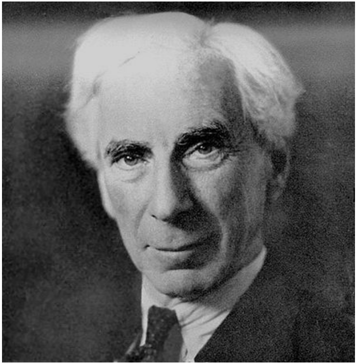

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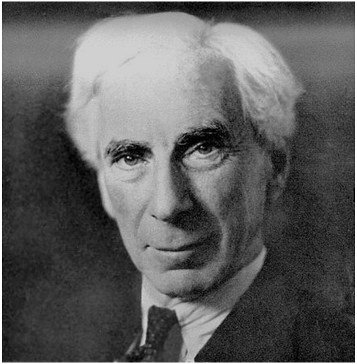

Frege's reaction:

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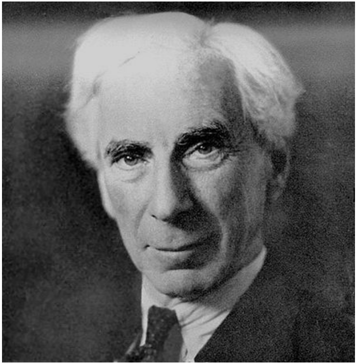

Frege's reaction: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion."

A poem

Zisimos Lorentzatos.
”Beware of systems grandiose, of mathematically strict causalities as you’re trying, stone by stone, to found the goldenwoven tower of the logical, castle and fort immune to contradiction. Designed in two volumes, the foundational laws of arithmetic, or Grundgesetze of der arithmetic in 1893, the first, 1903 the second. A life’s work. Hammer on chisel blows for years and years. So far, so good. But as Frege Gottlob was correcting, content, the printer’s proofs already of the second volume, one cursed logic paradox, one not admitting refutation, question by Russell Bertrand, forced, without delay, the great thinker of Mecklemburg to add a last paragraph to his system, show me a great thinker who would resist the truth, accepting the reversible disaster. His foundations in ruin, his logic flawed, his work wasted, and his two volumes imagine the colossal set back, odd load and ballast for the refuge cart.”

Russell's Paradox.

- ”This statement is false”

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- A barber says ”I shave all and only those men who do not shave themselves.”

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- Self reference.........

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Change Axioms!

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(We must know. We will know.) ...

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Until 1931.

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Concrete example:
Continuum hypothesis (see official notes if interested)

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- Gödel ..starved himself out of fear of being poisoned..

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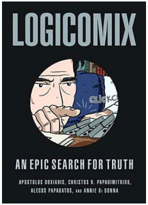
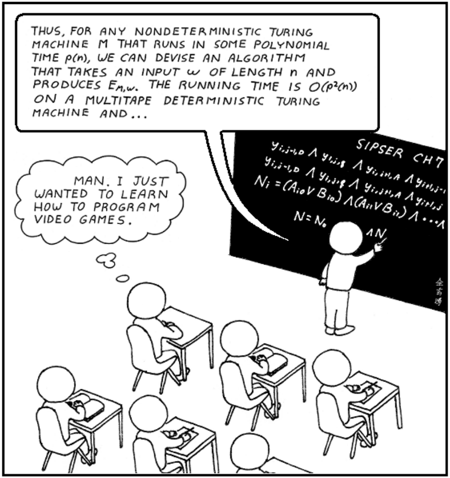
- See Logicomix by Doxiadis, Papadimitriou (my advisor!), Papadatos, Di Donna.



Next Topic: Undecidability.

- Undecidability. A happy ending?



Turing

Is it actually useful?

Turing: Write me a program checker!

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A program that checks that the compiler works!

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Something about infinity here, maybe?

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Turing(P)

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Can run Turing on Turing!

Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALTS(Turing, Turing) $=$ halts
$\implies$ Turing(Turing) loops forever.

Turing(Turing) loops forever
$\implies$ then HALTS(Turing, Turing) $\neq$ halts
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Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

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Another view of proof: diagonalization.

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$$
\begin{array}{c|cccc}
 & "P_1" & "P_2" & "P_3" & \cdots \\
\hline
 & & & & \\
P_1 & \text{H} & \text{H} & \text{L} & \cdots \\
P_2 & \text{L} & \text{L} & \text{H} & \cdots \\
P_3 & \text{L} & \text{H} & \text{H} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
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Program P_1 halts on input $"P_1"$ and $"P_2"$, doesn't halt on input $"P_3"$, and so on...

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	$''P_1''$	$''P_2''$	$''P_3''$	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Wow, that was easy!

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We should be famous!

No computers for Turing!

In Turing’s time.

No computers for Turing!

In Turing's time.

No computers.

No computers for Turing!

In Turing's time.

No computers.

Concept of program as data wasn't really there.

Undecidable problems.

Does a program ever print "Hello World"?

Undecidable problems.

Does a program ever print “Hello World”?
Find exit points and add statement: **Print** “Hello World.”

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Is there program that makes other programs faster?

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Does this computer program have any security vulnerabilities?

More about Alan Turing.

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Turing: personal.

Tragic ending...

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- Arrested as a homosexual, (not particularly closeted)
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- suffered from depression;

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- Arrested as a homosexual, (not particularly closeted)
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- suicided with cyanide at age 42.

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  (A bite from the apple....) accident?

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2013. Granted Royal pardon.

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And mind boggling

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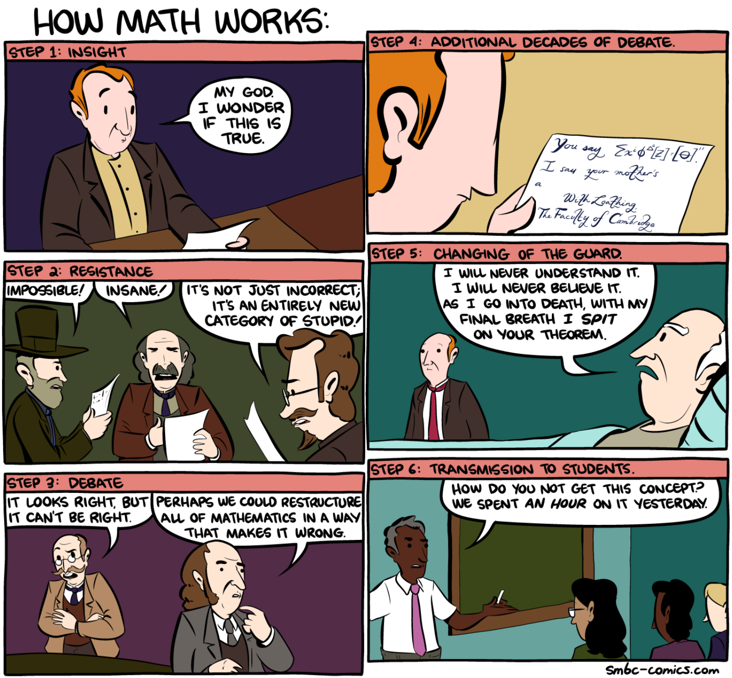
Programming is a super power.


HOW MATH WORKS:
STEP 1: INSIGHT
MY GOD. I WONDER IF THIS IS TRUE.
STEP 2: RESISTANCE
IMPOSSIBLE!
INSANE!
IT'S NOT JUST INCORRECT; IT'S AN ENTIRELY NEW CATEGORY OF STUPID!
STEP 3: DEBATE
IT LOOKS RIGHT, BUT IT CAN'T BE RIGHT.
PERHAPS WE COULD RESTRUCTURE ALL OF MATHEMATICS IN A WAY THAT MAKES IT WRONG.
STEP 4: ADDITIONAL DECADES OF DEBATE.
You say
I say your mother's
a
With Loathing
The Faculty of Cambridge
STEP 5: CHANGING OF THE GUARD.
I WILL NEVER UNDERSTAND IT. I WILL NEVER BELIEVE IT. AS I GO INTO DEATH, WITH MY FINAL BREATH I SPIT ON YOUR THEOREM.
STEP 6: TRANSMISSION TO STUDENTS.
HOW DO YOU NOT GET THIS CONCEPT? WE SPENT AN HOUR ON IT YESTERDAY.
smbc-comics.com