CS70: Counting

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Today:

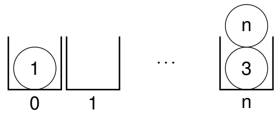
- Balls and bins.
- Sum rule.
- Combinatorial proofs.
- Maybe start review?

What we've learned so far

Sample k items out of n.

	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$

Balls in bins.



"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers or exactly one Joker or exactly two Jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$

$\binom{52}{5} = \frac{52!}{5!47!}$, $\binom{52}{4} = \frac{52!}{4!48!}$, $\binom{52}{3} = \frac{52!}{3!49!}$

$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3} = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!}$

$= \frac{52!(4!48!3!49! + 2*5!47!3!49! + 5!47!4!48!)}{5!47!4!48!3!49!}$

$= \ldots.$

Let's actually go with the other one....

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

$\square$

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....
Foil?? I hate this...

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.
$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elts.
$\implies \binom{n}{k}$
So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

$\square$

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1,\ldots,i,\underline{\ldots,n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.
2 is first element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Add them up to get the total number of subsets of size k which is also $\binom{n}{k}$.

$\square$

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1,\ldots,n\}$?
Construct a subset with sequence of n choices:
element i **is in** or **is not** in the subset: 2 possibilities.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

How many subsets of $\{1,\ldots,n\}$?
$\binom{n}{i}$ = subsets of size i.
A subset has size either 0, or 1, or 2, . . . , or n
Sum over i to get total number of subsets.

$\square$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 0, 1, 2, 3,. . .

Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.$|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Summary.

Inclusion/Exclusion: two sets of objects.
Add number of each subtract intersection of sets.
Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.
Pascal's Triangle Example: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
RHS: Number of subsets of $n+1$ items size k.
LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.
$\binom{n}{k}$ counts subsets of $n+1$ items without first item.
Disjoint – so add!