Alex Psomas: Lecture 15.

Bayes’ Rule, Mutual Independence, Collisions and Collecting

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Bayes' Rule, Mutual Independence, Collisions and Collecting

1. Conditional Probability
2. Independence
3. Bayes' Rule
4. Balls and Bins
5. Coupons

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Monty Hall

Balls in bins

Conditional Probability: Pictures

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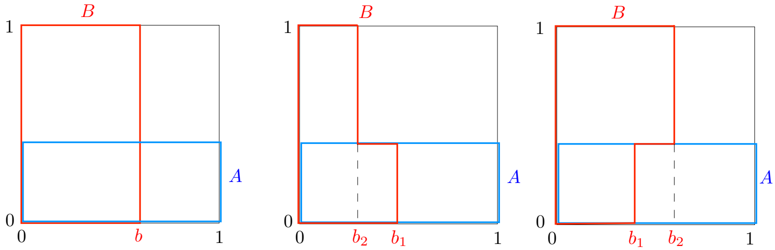
Illustrations: Pick a point uniformly in the unit square



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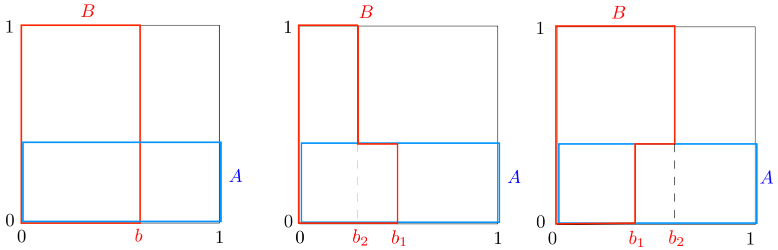

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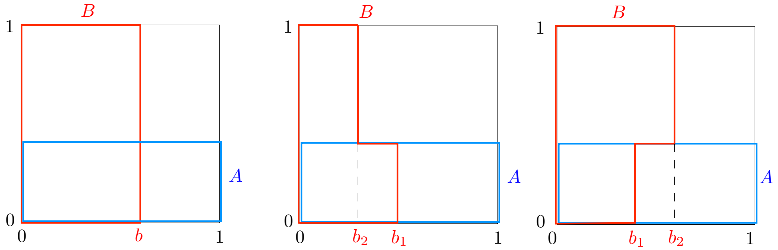

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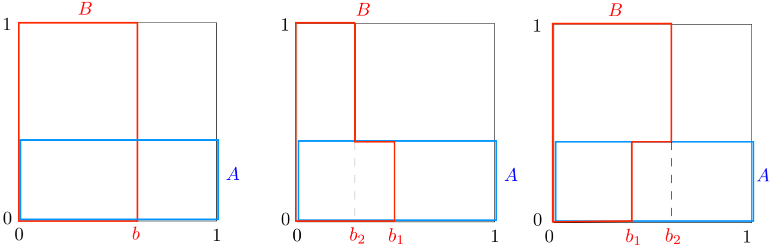

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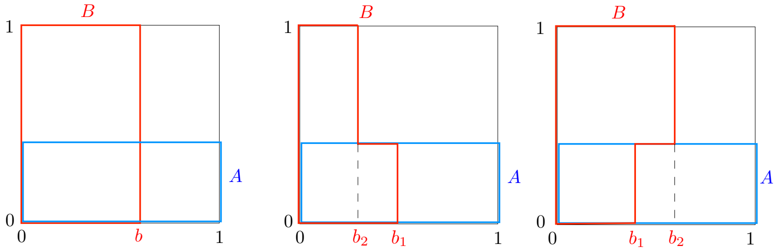

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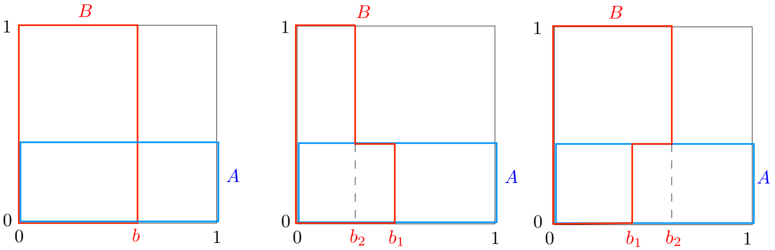

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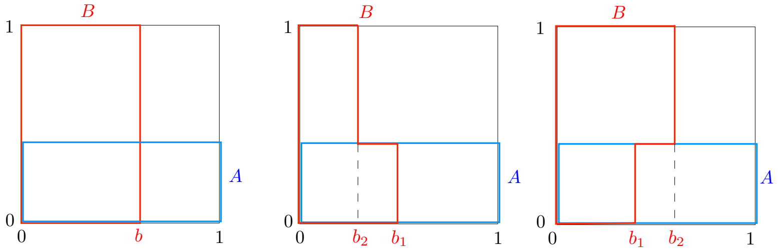

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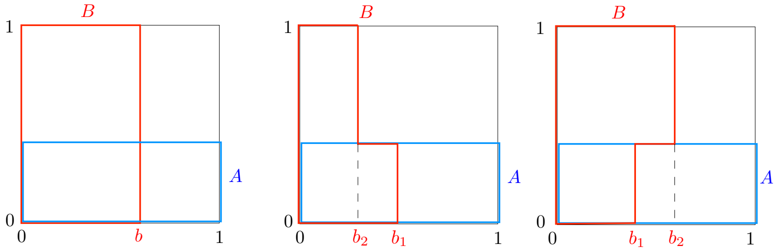

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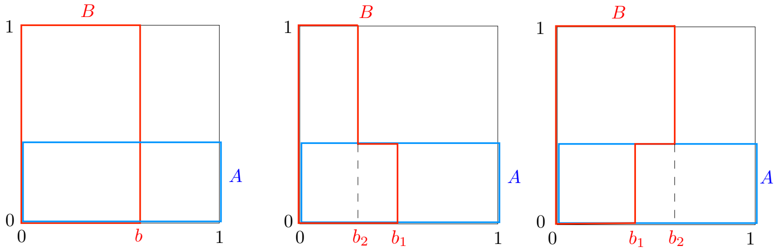

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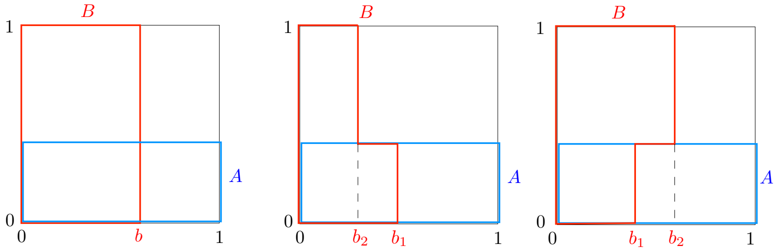

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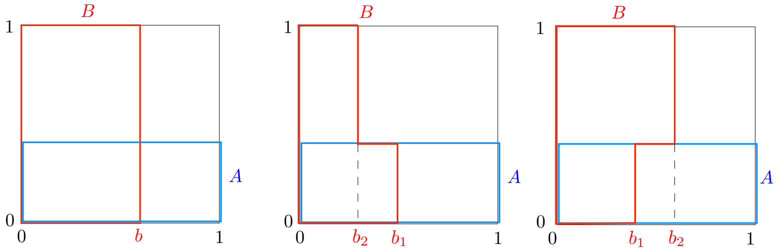

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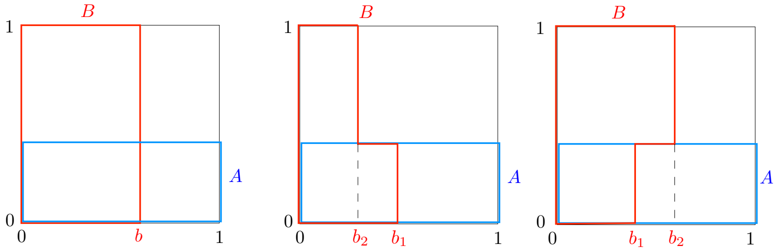

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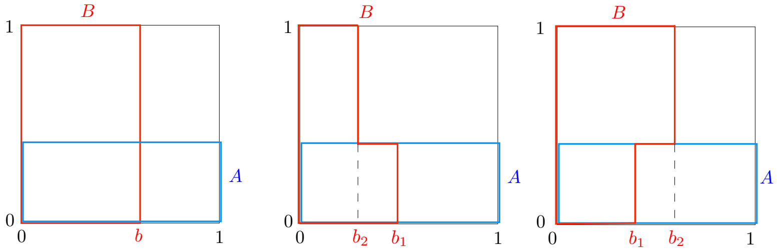

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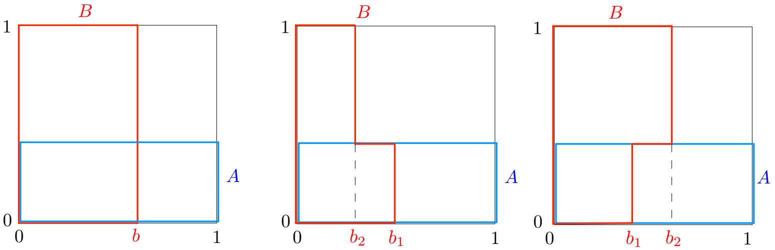

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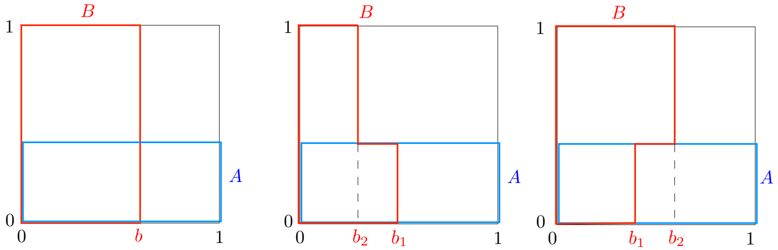

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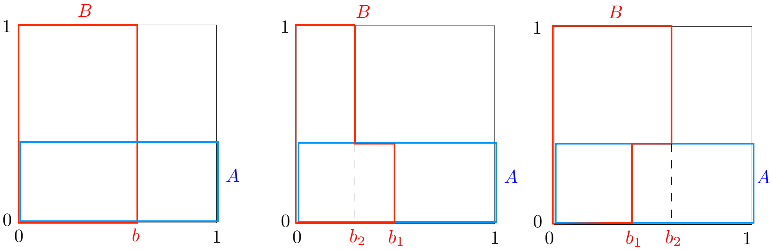

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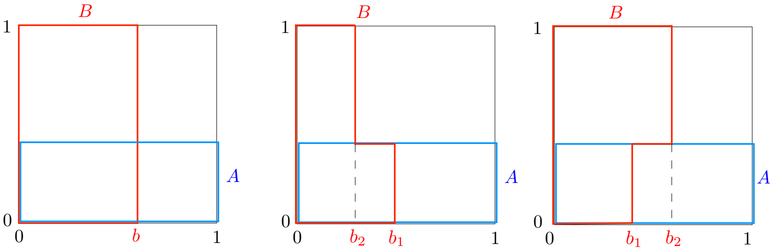

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Bayes and Biased Coin

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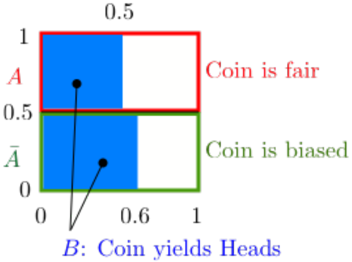

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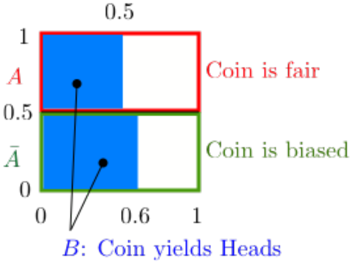

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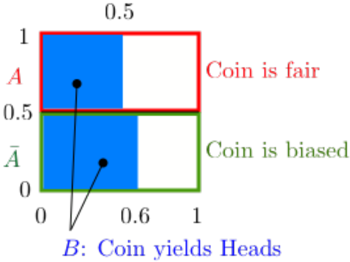

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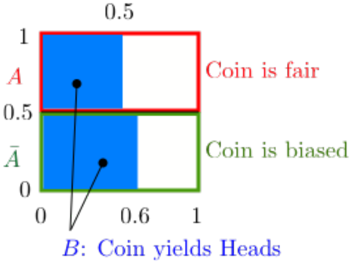

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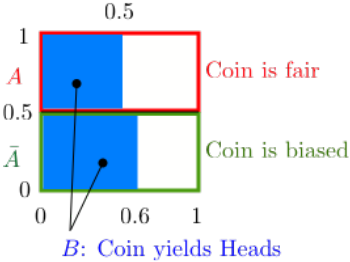

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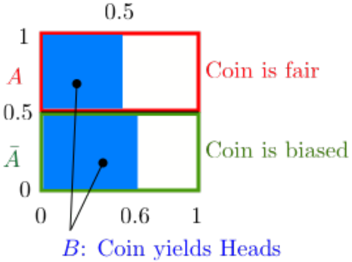

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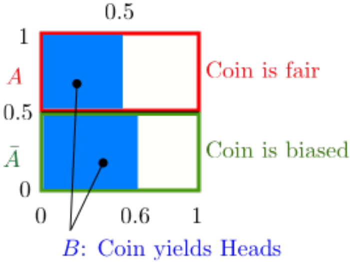

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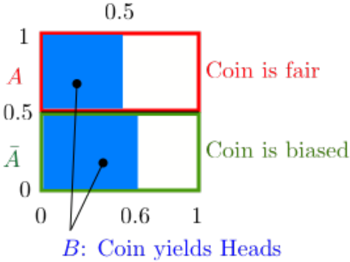

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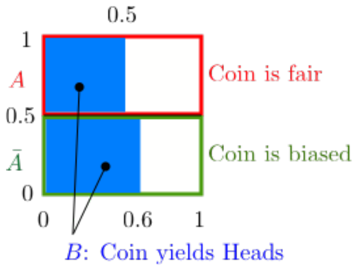

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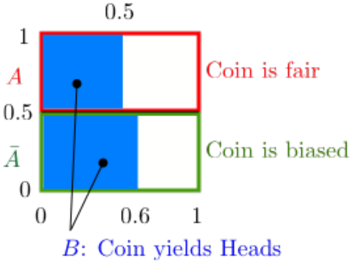

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$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
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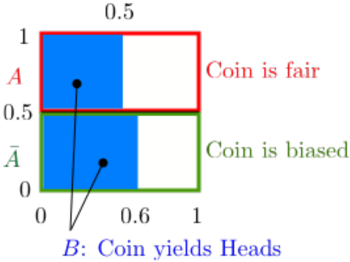
Bayes and Biased Coin



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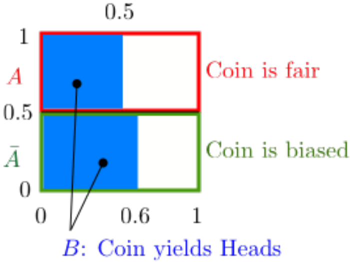

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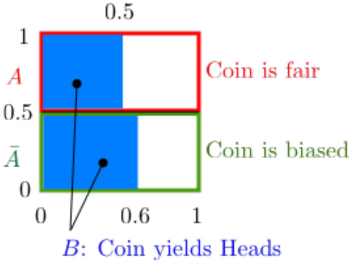

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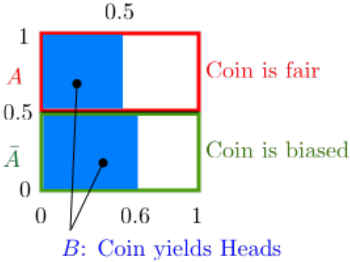

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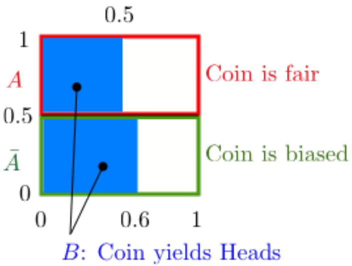
Bayes and Biased Coin



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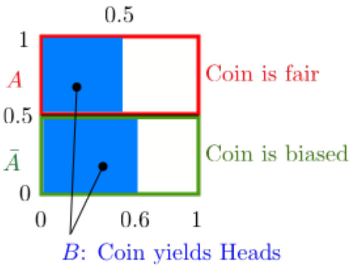

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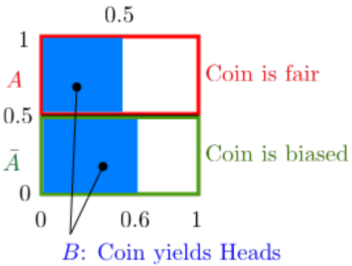

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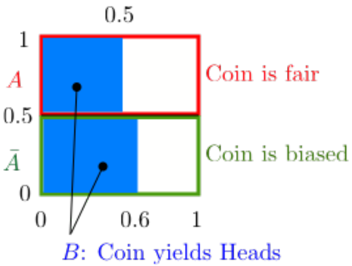

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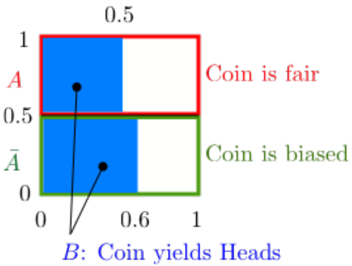
Bayes and Biased Coin



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$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

Bayes: General Case

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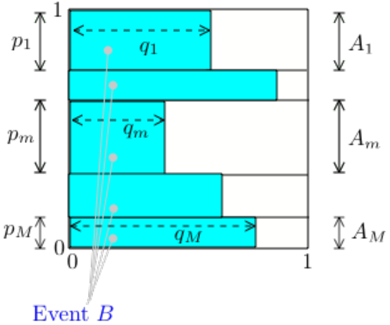

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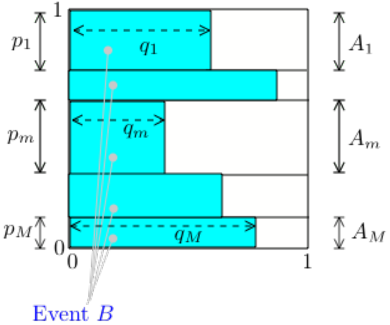

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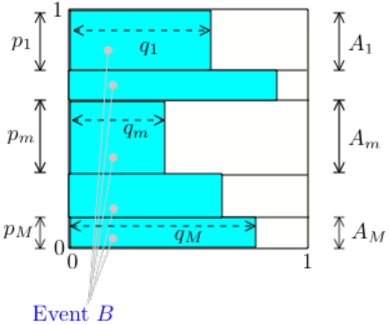

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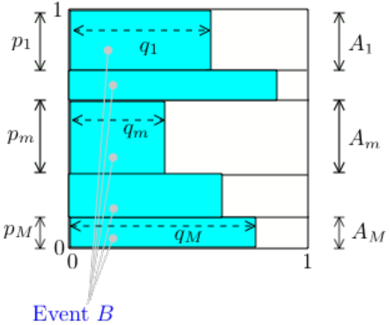

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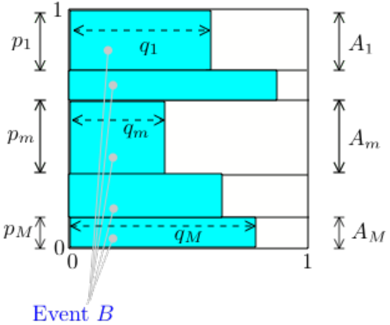

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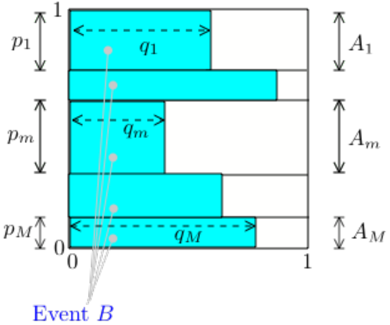

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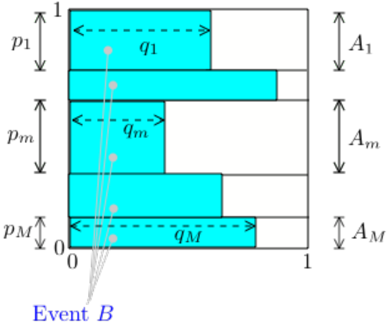

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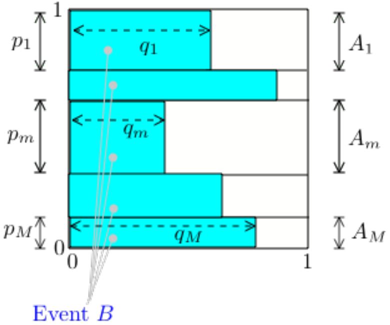

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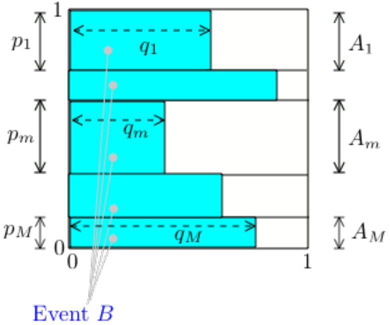

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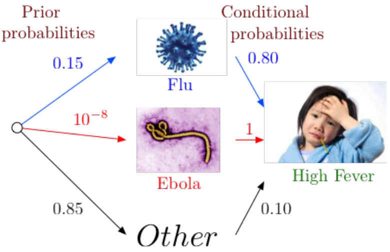

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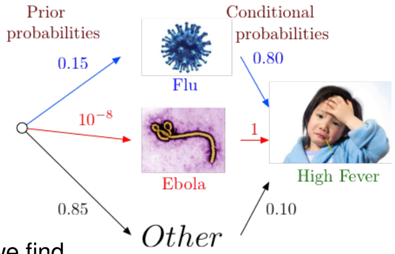

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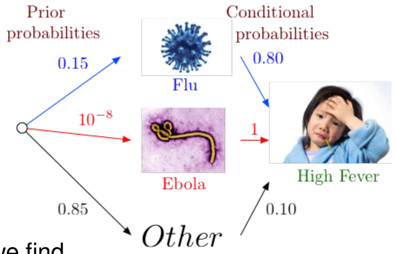

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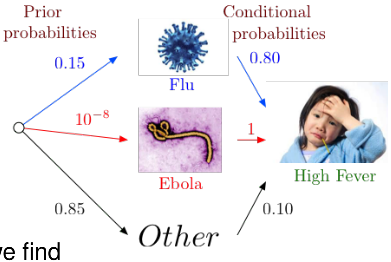

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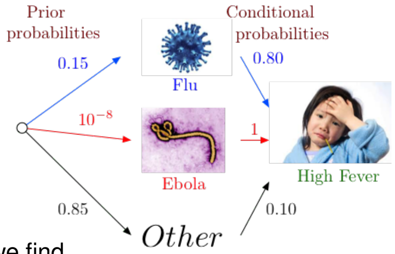

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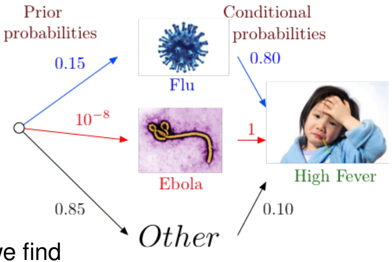

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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

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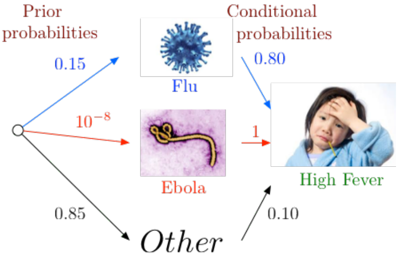

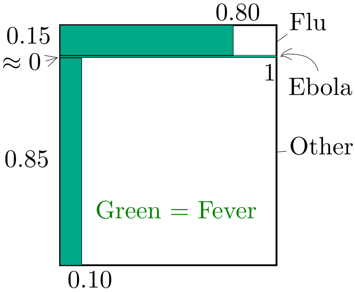

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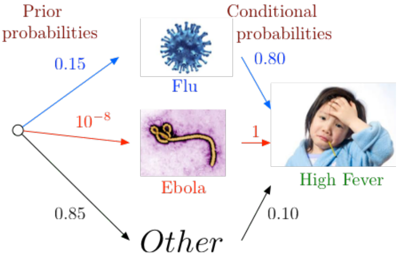

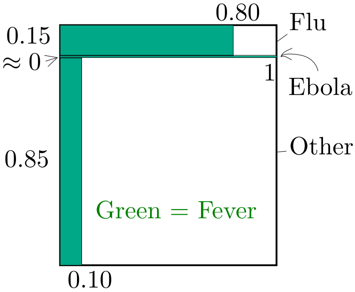

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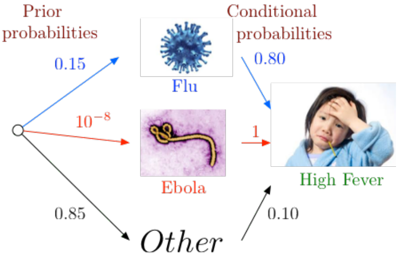

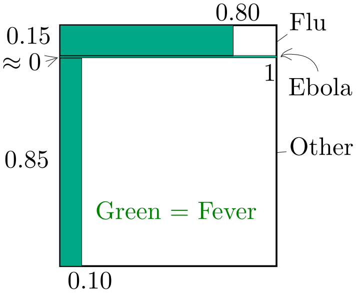

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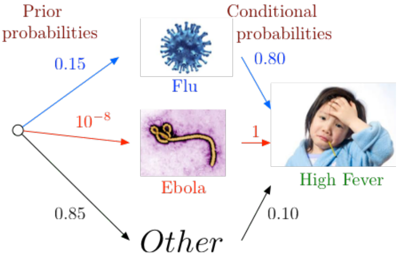

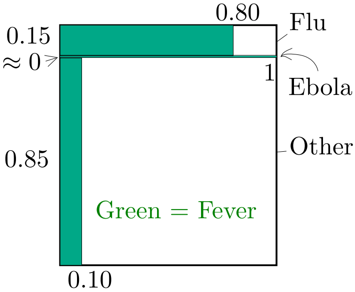

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This example shows the importance of the prior probabilities.

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Bayes' Rule Operations

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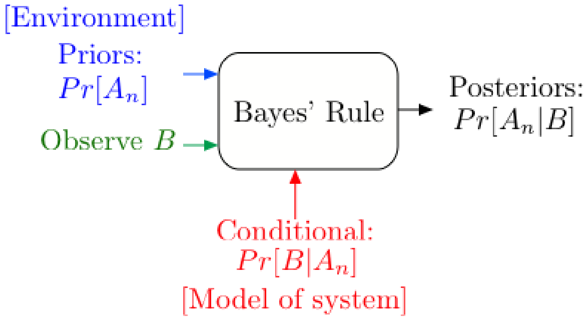

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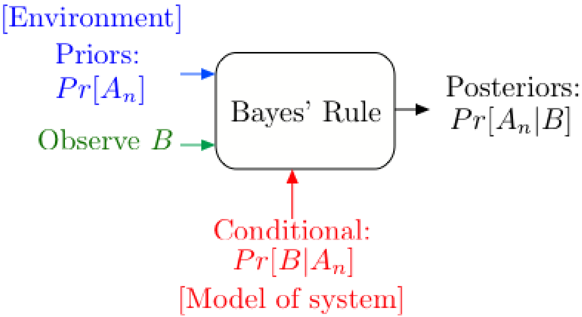

Bayes' Rule is the canonical example of how information changes our opinions.

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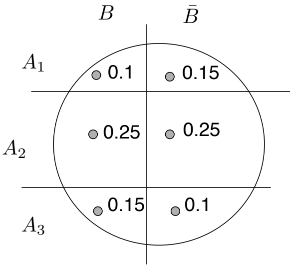

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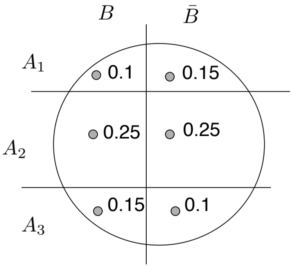

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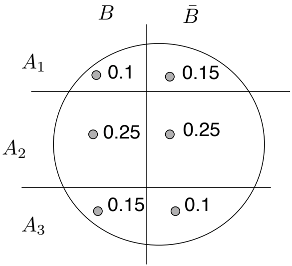

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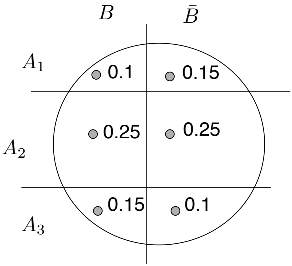

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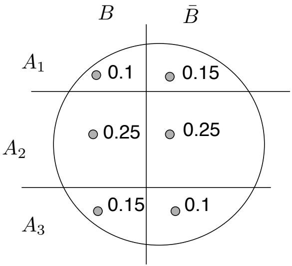

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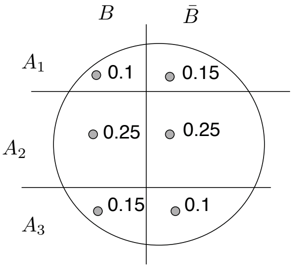

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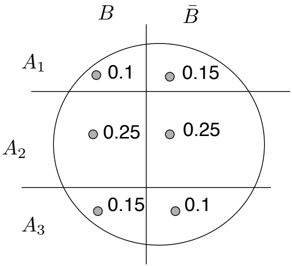

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Flip two fair coins. Let

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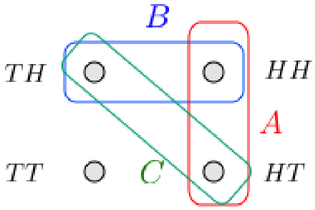

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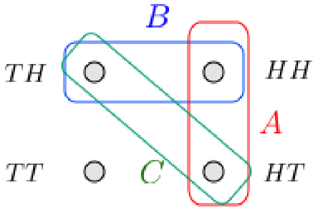

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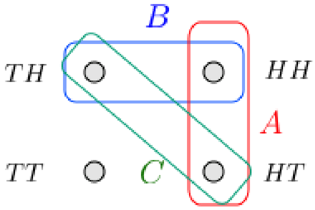

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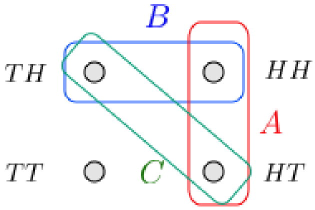

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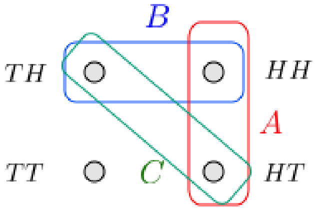

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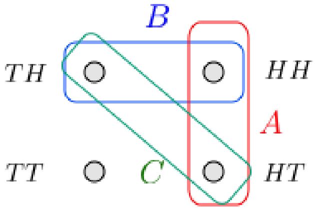

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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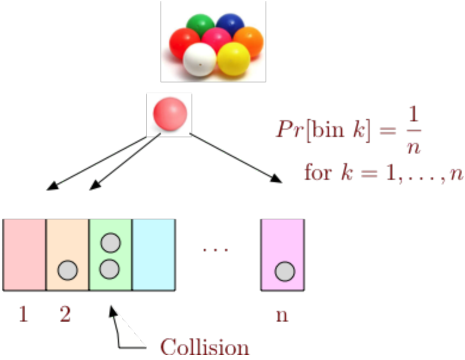

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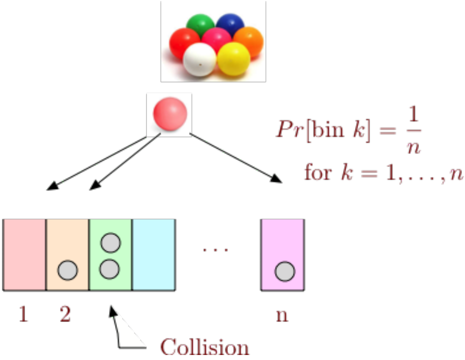

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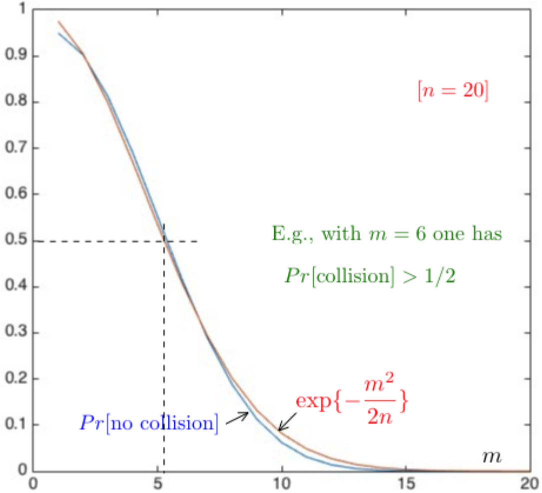

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$^{(\dagger)}$ $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

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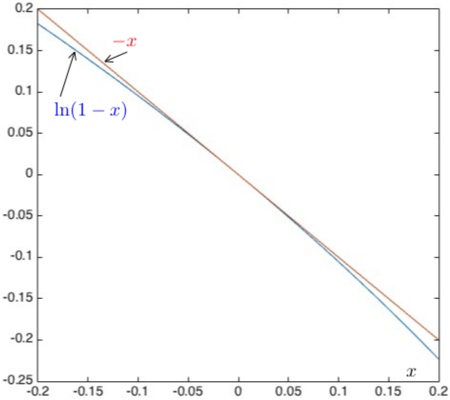

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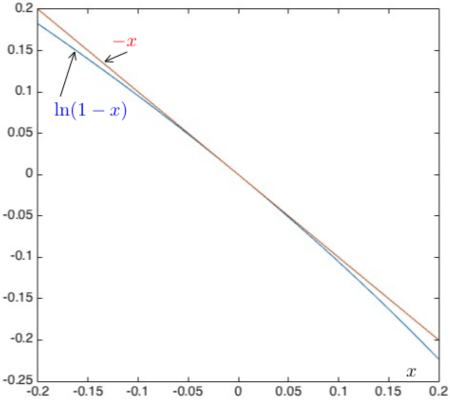

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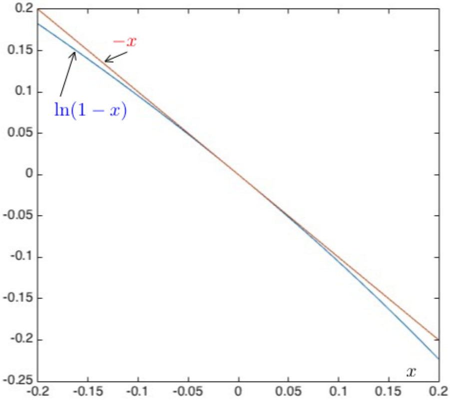

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

The birthday paradox

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Probability that m people all have different birthdays?

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If $m = 366$, then $Pr[\text{no collision}] =$

Today's your birthday, it's my birthday too..

Probability that m people all have different birthdays?
With $n = 365$, one finds

$Pr[\text{collision}] \approx 1/2$ if $m \approx 1.2\sqrt{365} \approx 23$.

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If $m = 366$, then $Pr[\text{no collision}] = 0$. (No approximation here!)

The birthday paradox

n	$p(n)$
1	0.0%
5	2.7%
10	11.7%
20	41.1%
23	50.7%
30	70.6%
40	89.1%
50	97.0%
60	99.4%
70	99.9%
100	99.99997%
200	99.9999999999999999999999999998%
300	$(100 - (6\times10^{-80}))\%$
350	$(100 - (3\times10^{-129}))\%$
365	$(100 - (1.45\times10^{-155}))\%$
366	100%
367	100%

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Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

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Plug in and get

$$p \leq ne^{-\frac{m}{n}}.$$

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Main results:

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