CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Discussion 1D Sol

1. **(Induction) Prove that, for any positive integer n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.**

   - Base case: when $n = 1$, $\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$.
   - Inductive hypothesis: assume for $n = k \geq 1$ that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.
   - Inductive step:

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}
$$

   By the principle of induction, the claim is proved.

2. **Dividing n-gon**

   Assume that any simple (but not necessarily convex) n-gon ($n > 3$) has a diagonal (line between two non-adjacent vertices) that lies completely within the n-gon. Show that any such n-gon ($n \geq 3$) can be divided into $n-2$ triangles such that all vertices of each triangle are vertices of the n-gon.

   We run strong induction over n:

   Base Case $n = 3$. This is a triangle.

   Inductive Hypothesis Assume that the claim is true for all n-gons, $n \geq 3$.

   Inductive Step For a $(n+1)$-gon, a diagonal divides it into two smaller polygons. Suppose one of them is a k-gon, then the other is a $(n-k+3)$-gon, where $k \geq 3$. (The two vertices at either end of the diagonal are repeated, so there are a total of $n+3$ vertices between the two polygons.) By the inductive hypothesis, the first polygon can be divided into $k-2$ triangles, and the second into $(n-k+3)-2$ triangles. The total number of triangles is $k-2+(n-k+3)-2 = n-1$. Thus, an $(n+1)$-gon can be divided into $(n+1)-2$ triangles.

3. Convergence of Series

Use induction to prove that for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2.$$

Hint: Strengthen the induction hypothesis to $\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2 - \frac{1}{\sqrt{n}}$.

We use induction on n. The base case $n = 1$ is true because $1/3 < 1$. Assume the inequality holds for some $n \geq 1$. For $n+1$, by the inductive hypothesis, we have that

$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}}.$$

Thus, to prove our claim, it suffices to show that

$$-\frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}} \leq -\frac{1}{\sqrt{n+1}}. \tag{1}$$

This is a purely arithmetic problem and there are multiple ways to proceed.

Notice that to prove the inequality (3), it suffices to show that

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \geq \frac{1}{3(n+1)^{3/2}} = \frac{1}{3(n+1)\sqrt{n+1}},$$

which is equivalent to showing that

$$\frac{\sqrt{n+1}}{\sqrt{n}} - 1 = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} \geq \frac{1}{3(n+1)}.$$

So we want to show

$$\frac{\sqrt{n+1}}{\sqrt{n}} \geq \frac{1}{3(n+1)} + 1 = \frac{3n+4}{3n+3},$$

and squaring both sides means this is equivalent to

$$\frac{n+1}{n} \geq \frac{(3n+4)^2}{(3n+3)^2}.$$

At this point we cross-multiply, so we just need to show that

$$(n+1)(3n+3)^2 \geq n(3n+4)^2.$$

This is something that can be easily seen by expanding both sides and canceling terms, so we have shown Equation (3).

This computation allows us to conclude that

$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}} \overset{(1)}{\leq} 2 - \frac{1}{\sqrt{n+1}},$$

where we have used equation (3) for the last inequality. This concludes the induction.