CS70: Counting

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July 7, 2016

Today:

- Balls and bins.
- Sum rule.
- Combinatorial proofs.
- Maybe start review?

What we've learned so far

Sample k items out of n.

	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$

A unifying example

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Hats!

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    How many samples?

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    $\binom{10}{7} = 120$

A cool stars and bars application

How many (non-negative) solutions to $x + y = 10$?

A cool stars and bars application

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Easy:

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How many (non-negative) solutions to $x + y = 10$?
Easy: $x = 0, y = 10$, $x = 1, y = 9$,$x = 2, y = 8$, . . . , $x = 10, y = 0$. So 11 solutions.

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Same as 10 stars, and 1 bar.

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$x=3, y=7: \quad \star\star\star|\star\star\star\star\star\star\star$

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Think of a star as the number 1.

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How many ways to make an 8 problem midterm such the total points add up to 100?

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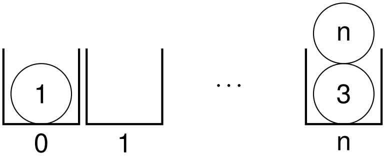
How many ways to make an 8 problem midterm such the total points add up to 100?
100 stars, 7 bars.

Balls in bins.



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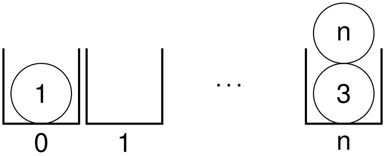
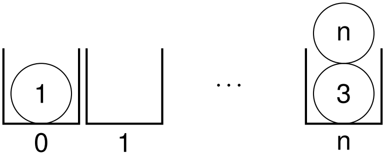


"k Balls in n bins" $\equiv$ "k samples from n possibilities."

Balls in bins.



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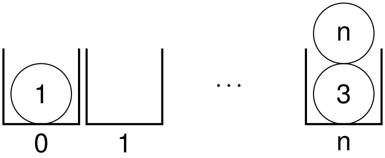
Balls in bins.



"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

Balls and bins

How many 5 digit numbers?

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Throwing 5 numbered balls in 10 (numbered) bins:

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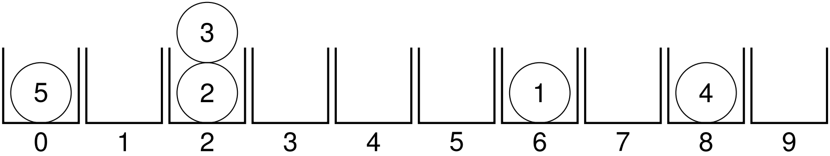

Balls and bins

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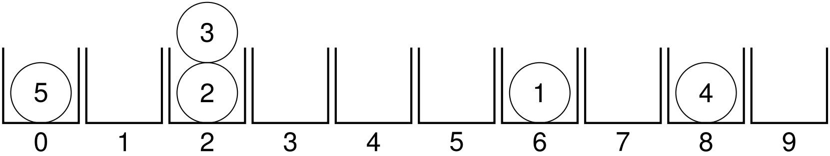

Picture has number 62280.

Balls and bins

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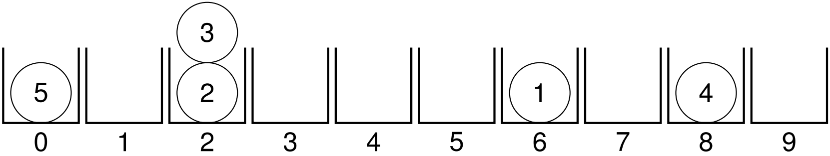

Picture has number 62280.
5 samples from 10 possibilities with replacement (order matters): 10^5

Balls and bins

How many 5 digit numbers without repeating a digit?

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How many 5 digit numbers without repeating a digit?
Throwing 5 numbered balls in 10 (numbered) bins, at most one ball per bin:

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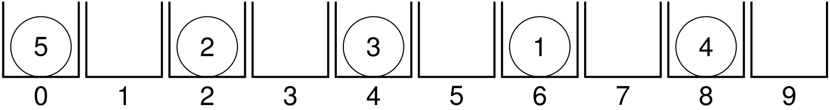

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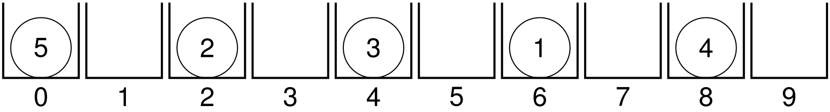

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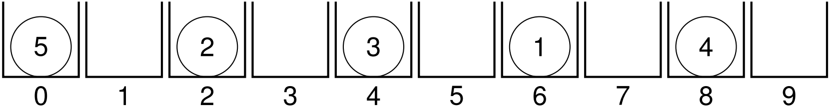

Picture has number 62480.
5 samples from 10 possibilities without replacement (order matters): $\frac{10!}{5!}$

Balls and bins

How many 3 card poker hands?

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Throwing 3 indistinguishable balls in 52 (numbered) bins. At most 1 ball per bin:

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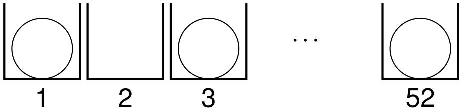

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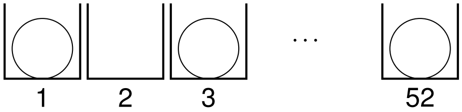

Picture has cards 1,3 and 52.

Balls and bins

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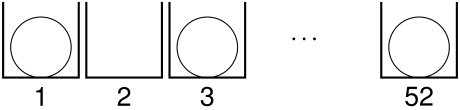

Picture has cards 1,3 and 52.
3 samples from 52 possibilities without replacement (order doesn't matter): $\binom{52}{3}$

Balls and bins

Dividing 5 dollars among Alice, Bob and Eve.

Balls and bins

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5 indistinguishable balls into 3 (numbered) bins:

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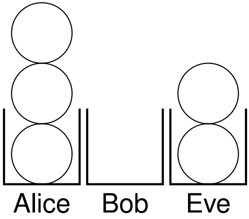

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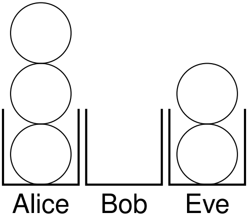
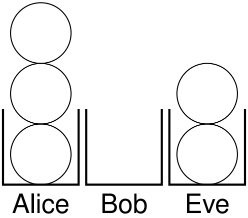


Picture: Alice 3, Bob 0, Eve 2.

Balls and bins

Dividing 5 dollars among Alice, Bob and Eve.
5 indistinguishable balls into 3 (numbered) bins:



Picture: Alice 3, Bob 0, Eve 2.
5 samples from 3 possibilities with replacement (order doesn't matter): $\binom{7}{2}$

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

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How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.

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$$\binom{52}{5}$$

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$$\binom{52}{5} + \binom{52}{4}$$

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Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
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$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Sum Rule

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How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers or exactly one joker or exactly two jokers

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Two distinguishable jokers in 54 card deck.

Sum Rule

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How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands (distinguishable jokers)?

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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands (distinguishable jokers)?
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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands (distinguishable jokers)?
No jokers or exactly one of two jokers or exactly two jokers

$$\binom{52}{5}+$$

Sum Rule

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Sum rule: Can sum over disjoint sets.
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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands (distinguishable jokers)?
No jokers or exactly one of two jokers or exactly two jokers

$$\binom{52}{5} + 2 * \binom{52}{4} +$$

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers or exactly one joker or exactly two jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands (distinguishable jokers)?
No jokers or exactly one of two jokers or exactly two jokers

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

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Wait a minute! Same as

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Wait a minute! Same as choosing 5 cards from 54

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$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5}$

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers or exactly one joker or exactly two jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands (distinguishable jokers)?
No jokers or exactly one of two jokers or exactly two jokers

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers or exactly one joker or exactly two jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

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Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.
Proof:

Sum Rule

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How many 5 card poker hands?
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$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.
Proof: Above is combinatorial proof.

Algebraic proof

$$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$

Algebraic proof

$$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$$

Proof:

$$\binom{54}{5} = \frac{54!}{5!49!}\,, \qquad \binom{52}{5} = \frac{52!}{5!47!}$$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}\ , \quad \binom{52}{5} = \frac{52!}{5!47!}\ , \quad \binom{52}{4} = \frac{52!}{4!48!}$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$, $\binom{52}{5} = \frac{52!}{5!47!}$, $\binom{52}{4} = \frac{52!}{4!48!}$, $\binom{52}{3} = \frac{52!}{3!49!}$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$, $\quad \binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$, $\quad \binom{52}{3} = \frac{52!}{3!49!}$

RHS

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$, $\quad \binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$, $\quad \binom{52}{3} = \frac{52!}{3!49!}$

$RHS = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!}$

Algebraic proof

$$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$$

Proof:

$$\binom{54}{5} = \frac{54!}{5!49!}, \quad \binom{52}{5} = \frac{52!}{5!47!}, \quad \binom{52}{4} = \frac{52!}{4!48!}, \quad \binom{52}{3} = \frac{52!}{3!49!}$$

$$RHS = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!} = \frac{52!(4!48!3!49! + 2 * 5!47!3!49! + 5!47!4!48!)}{5!47!4!48!3!49!}$$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$, $\quad \binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$, $\quad \binom{52}{3} = \frac{52!}{3!49!}$

$RHS = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!} = \frac{52!(4!48!3!49! + 2*5!47!3!49! + 5!47!4!48!)}{5!47!4!48!3!49!} \stackrel{?}{=} \frac{54!}{5!49!}$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$, $\quad \binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$, $\quad \binom{52}{3} = \frac{52!}{3!49!}$

$RHS = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!} = \frac{52!(4!48!3!49!+2*5!47!3!49!+5!47!4!48!)}{5!47!4!48!3!49!} \stackrel{?}{=} \frac{54!}{5!49!}$

49! and 5! cancel out.

Algebraic proof

$$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$$

Proof:

$$\binom{54}{5} = \frac{54!}{5!49!}\ , \quad \binom{52}{5} = \frac{52!}{5!47!}\ , \quad \binom{52}{4} = \frac{52!}{4!48!}\ , \quad \binom{52}{3} = \frac{52!}{3!49!}$$

$$RHS = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!} = \frac{52!(4!48!3!49!+2*5!47!3!49!+5!47!4!48!)}{5!47!4!48!3!49!} \stackrel{?}{=} \frac{54!}{5!49!}$$

49! and 5! cancel out. Cross multiply and get:

$$54!47!4!48!3! \stackrel{?}{=} 52!(4!48!3!49! + 2 * 5!47!3!49! + 5!47!4!48!)$$

I tried this for a while...

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$, $\quad \binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$, $\quad \binom{52}{3} = \frac{52!}{3!49!}$

$RHS = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!} = \frac{52!(4!48!3!49!+2*5!47!3!49!+5!47!4!48!)}{5!47!4!48!3!49!} \stackrel{?}{=} \frac{54!}{5!49!}$

49! and 5! cancel out. Cross multiply and get:

$54!47!4!48!3! \stackrel{?}{=} 52!(4!48!3!49! + 2 * 5!47!3!49! + 5!47!4!48!)$

I tried this for a while.......

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

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Proof: How many subsets of size k?

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Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

$\square$

Pascal’s Triangle

Pascal's Triangle

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

Pascal's Triangle

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

Pascal's Triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

Pascal’s Triangle

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$

Pascal's Triangle

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$.

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
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Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 =$

Pascal's Triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

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Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

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Zero-th row: $(1+x)^0 = 1$
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Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
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Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 =$

Pascal's Triangle

$$\begin{array}{ccccccccccc}
&&&&& 1 &&&&& \\
&&&& 1 && 1 &&&& \\
&&& 1 && 2 && 1 &&& \\
&& 1 && 3 && 3 && 1 && \\
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1 && 5 && 10 && 10 && 5 && 1
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Third row: $(1+x)^3 = 1+3x+3x^2+x^3$.

Pascal's Triangle

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\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
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Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
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\end{array}
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Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
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Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....
Foil??

Pascal's Triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

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First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....
Foil?? I hate this already...

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.
$(1+x)(1+x)(1+x)(1+x)$:

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from:

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.
$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

Pascal's Triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.
$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$\begin{array}{ccc} & \binom{0}{0} & \\ \binom{1}{0} & & \binom{1}{1} \end{array}$$

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$\begin{array}{ccccc}
 & & \binom{0}{0} & & \\
 & \binom{1}{0} & & \binom{1}{1} & \\
\binom{2}{0} & & \binom{2}{1} & & \binom{2}{2}
\end{array}$$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Pascal's Triangle

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \end{array}$$

Simplify: collect all terms corresponding to x^k.
 Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.
$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$\begin{array}{ccccccc} &&& \binom{0}{0} &&& \\ && \binom{1}{0} && \binom{1}{1} && \\ & \binom{2}{0} && \binom{2}{1} && \binom{2}{2} & \\ \binom{3}{0} && \binom{3}{1} && \binom{3}{2} && \binom{3}{3} \end{array}$$

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

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Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.Then I need to choose $k-1$ more from remaining n elements.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.Then I need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.Then I need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.Then I need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elements.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.
Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.Then I need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elements.
$\implies \binom{n}{k}$

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first.Then I need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elements.
$\implies \binom{n}{k}$
So, $\binom{n}{k-1} + \binom{n}{k}$

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.
Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Pick the first. Then I need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elements.
$\implies \binom{n}{k}$
So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.

Combinatorial Proof.

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Proof:

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Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i\underline{, \ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
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Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen:

Combinatorial Proof.

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Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen: $\binom{n-1}{k-1}$ choices for the rest.

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Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen: $\binom{n-1}{k-1}$ choices for the rest.

2 is smallest element chosen:

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen: $\binom{n-1}{k-1}$ choices for the rest.

2 is smallest element chosen: $\binom{n-2}{k-1}$ choices for the rest.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen: $\binom{n-1}{k-1}$ choices for the rest.
2 is smallest element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen: $\binom{n-1}{k-1}$ choices for the rest.
2 is smallest element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Add them up to get the total number of subsets of size k

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the smallest element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is smallest element chosen: $\binom{n-1}{k-1}$ choices for the rest.
2 is smallest element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Add them up to get the total number of subsets of size k
which is also $\binom{n}{k}$.

$\square$

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

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Proof: How many subsets of $\{1, \ldots, n\}$?

Binomial Theorem: $x = 1$

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Proof: How many subsets of $\{1, \ldots, n\}$?
Construct a subset with sequence of n choices:

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1, \ldots, n\}$?
Construct a subset with sequence of n choices:
  element i **is in** or **is not** in the subset: 2 possibilities.

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1, \ldots, n\}$?
Construct a subset with sequence of n choices:
    element i **is in** or **is not** in the subset: 2 possibilities.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

Binomial Theorem: $x = 1$

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How many subsets of $\{1, \ldots, n\}$?
$\binom{n}{i}$ =

Binomial Theorem: $x = 1$

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    $\binom{n}{i}$ = subsets of size i.

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How many subsets of $\{1, \ldots, n\}$?
  $\binom{n}{i}$ = subsets of size i.
  A subset has size either 0, or 1, or 2, ..., or n

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1, \ldots, n\}$?
Construct a subset with sequence of n choices:
element i **is in** or **is not** in the subset: 2 possibilities.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

How many subsets of $\{1, \ldots, n\}$?
$\binom{n}{i}$ = subsets of size i.
A subset has size either 0, or 1, or 2, ..., or n
Sum over i to get total number of subsets. $\square$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$

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Used to reason about all subsets
by adding number of subsets of size 0, 1, 2, 3,$\ldots$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
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Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first i elements.)

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 0, 1, 2, 3,...

Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S **and** T**,** $|S \cup T| = |S| + |T|$
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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

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      Disjoint – so add!