1. Injection, Surjection, or Bijection?

For each of the following functions from $\mathbb{R}$ to $\mathbb{R}$, determine whether it is an injection, surjection, bijection, or none of the above.

1. $f(x) = 2^x$
   Injection. $f(x)$ cannot take on non-positive values.
2. $f(x) = x^2$
   None. Not an injection since every non-zero $f(x)$ occurs twice. Not a surjection because $f(x)$ cannot take on negative values.
3. $f(x) = 2x+1$
   Injection, Surjection, and Bijection. There is exactly one x that maps to any given value, namely $f^{-1}(y) = (y-1)/2$.

2. Union of Countable Sets

Prove that if A is countable and B is countable, then $A \cup B$ is countable.
Proof: Direct proof. Since A is countable, there exists a bijection from A to a subset of $\mathbb{N}$. Since B is countable, there exists a bijection from B to a subset of $\mathbb{N}$. Consider the bijection from $\mathbb{N}$ to nonnegative even numbers. Using that bijection and the bijection from A to $\mathbb{N}$, there exists a bijection from A to a subset of nonnegative even numbers. Using the bijection from $\mathbb{N}$ to positive odd numbers and the bijection from B to $\mathbb{N}$, there exists a bijection from B to a subset of positive odd numbers. This means that $A \cup B$ has a bijection onto a subset of the union of nonnegative even numbers and positive odd numbers, which is just $\mathbb{N}$. This means that $A \cup B$ is countable.

3. A city of n people must elect its city council. The council has a president, a vice president, a secretary, and k general members (the k general member positions are identical). How many ways are there to choose the city council from among the n residents?

There are a number of ways to count this. One such way is to first pick the president, then pick the vice president, then pick the secretary, and then finally pick the k general members:
$\binom{n}{1}\binom{n-1}{1}\binom{n-2}{1}\binom{n-3}{k} = \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)!}{k!(n-3-k)!} = \frac{n!}{k!(n-(k+3))!} = \frac{nPr(n,k+3)}{k!}$.
The simplified expression actually gives us a second way to count things: we first pick the $k+3$ people we need to be on the council, then we get rid of our overcounting by addressing the fact that the order of the general members don't matter (hence the division by $k!$).

There are even more ways to count this! We could first pick the $k+3$ people, but this time discount the order (this prevents the overcounting issue from before). Then, from those $k+3$ people, pick the president, vice president, and secretary: $\binom{n}{k+3}\binom{k+3}{3}$.
But now we've undercounted, because the titles can be given to any of the three people, and so we need another factor of $3!$, and our expression becomes:
$\binom{n}{k+3}\binom{k+3}{3}3!$.

Confused yet? There's more! We can first pick the three titles (disregarding order), then pick the k general members:
$3!\binom{n}{3}\binom{n-3}{k}$.

Note: As you can see, counting can be tricky (and confusing) business! There are often many ways to count the same thing, so if you ever get an answer that differs from another person's solution, either do some combinatorial reasoning or algebra to make sure that they're equivalent! Typically, we will only show one or two solutions; there may be many more!

4. A license plate contains 5 characters (order matters). Each character may either be an upper-case letter A-Z or a number 0-9. How many license plates…

1. contain only letters?
   26^5
2. have exactly three letters and two numbers?
   $\binom{5}{3}26^3 10^2$
3. contain the string ABC?
   $\binom{3}{1}36^2$
4. have at least two of the same character?
   This is the total number of license plates minus the ones that have all different characters: $36^5 - \frac{36!}{(36-5)!}$