Last updated 2016.07.29 00:51

1 T/F

(3 points each) Circle T for True or F for False. We will only grade the answers, and are unlikely to even look at any justifications or explanations.

(a) T F $(P \to (Q \vee R)) \equiv \overline{(P \wedge \overline{Q}) \wedge (P \wedge \overline{R})}$. **(T)** $LHS \equiv \neg P \vee (Q \vee R)$ $RHS \equiv (\neg P \vee Q) \vee (\neg P \vee R) \equiv \neg P \vee Q \vee R \equiv LHS$

(b) T F $\exists x \forall y P(x,y) \to \exists y \exists x P(x,y)$ **(T)** $LHS \equiv \exists x \forall y P(x,y) \to \exists x \exists y P(x,y) \to [\exists y \exists x P(x,y) \equiv RHS]$

(c) T F There exists some n such that for all $k > n$, the complete graph with k vertices is not planar. **(T)** n is 4. All complete graphs with $k > 4$ vertices contain K_5 (take any five vertices in such a graph, and they form K_5, since any pair of vertices within these five vertices are connected to each other), so any complete graph with 5 or more vertices is non-planar.

(d) T F The traditional marriage algorithm (males propose) never produces a female-optimal matching. **(F)** If given a set of preference lists for which only one stable matching exists, then that matching is both female- and male-optimal. Running the stable marriage algorithm with men proposing would produce the same stable matching as when women are proposing.

(e) T F Suppose we have n men and n women, for $n > 2$. Then there exist some preference lists that causes the traditional marriage algorithm to halt in exactly two steps (i.e. everyone is matched by the end of the second day). **(T)** Example for n = 3:

Men	Preferences
1	$A > B > C$
2	$B > C > A$
3	$B > C > A$

Women	Preferences
A	$1 > 2 > 3$
B	$2 > 3 > 1$
C	$2 > 3 > 1$

(f) T F If an undirected graph does not contain cycles, it's a tree. **(F)** Counterexample: A graph with 2 nodes, each of degree 0, is acyclic. However, this is a disconnected graph, so it cannot be a tree.

(g) T F Complete undirected graphs cannot be trees. **(F)** Counterexample: A graph with 2 nodes, each of degree 1, is complete. However, this is a connected, acyclic graph, so therefore it is a tree.

(h) T F It is impossible to develop a general test that determines whether or not a program returns the string "CS70". **(T)** If such a general test exists, then it would solve the Halting Problem, which we know cannot be solved.

(i) T F Suppose you are given program P that takes a single input. No matter what P is, it is impossible to come up with a program M that tells you whether or not P halts on input x. **(F)** You can have a program M that only tells us whether P halts; its existence doesn't preclude the implication of the Halting Problem. For example, a description of a program P could be "Prints 'halts' and then halts when the input is 'Turing'; else, loops forever." M might just return 'halts' when the input is 'Turing', and 'loops' otherwise.

2 Counting

Clearly indicate your correctly formatted answer: this is what is to be graded. You may leave simple mathematical expressions, including binomial coefficients and factorials, un-evaluated. We will only grade the answers, and are unlikely to even look at any justifications or explanations.

(a) **(5 points)** Suppose $x_1 + x_2 + x_3 + .. + x_{10} = 20$ where x_i $(1 \le i \le 10)$ are all natural numbers (that is, non-negative integers). How many distinct solutions are there for this equation?
This is a stars and bars (balls and bins) problem, with 20 stars and 9 bars: $\binom{20+9}{9} = \binom{29}{9} = \binom{29}{20}$

(b) **(5 points)** In a party, we have n men and n women. In how many different ways can we pair them up?
We can line up the men in a certain order, and have each man's partner stand in front of him; we have **n!** ways to line up the women, and thus to match them up with the men.

(c) **(5 points)** We have 3 TAs covering a 3-hour homework party, which is divided into three slots, each one hour long. Each TA can sign up for anywhere from 0 to 3 slots, as long as there is **at least one TA assigned to each slot**. In how many ways can they sign up for this homework party?

We can break the 3-hour homework party into 3 time-slots. We can model each time slot as a length-3 bit-string, where the i^{th} bit is 0 if the i^{th} TA does not go to that hour, and 1 if he/she does attend that hour. So there are $2 \cdot 2 \cdot 2 = 2^3$ possible ways for any hour to be filled (this is also an application of the Power Set; see end of Note 6: Infinity and Countability). However, there is one way to have the hour be empty, so we must subtract 1 from this number; there are $2^3 - 1$ ways for each time slot to have at least 1 TA present. Since there are 3 time slots, we have $(2^3 - 1)^3$ total ways to have TA's attend homework party.

3 Proofs

(a) **(7 points)** Show that there exists some numbers $k, m \in \mathbb{R}$ such that for all numbers $n \in \mathbb{R}$: $nk = m$
Let $m = k = 0$, then $n \cdot 0 = 0$.

(b) **(7 points)** Show that during a party where people shake hands with some other guests, the number of guests who shake hands with odd number of guests is even. See the solution to Discussion 2C No. 3; the people can be modeled as the vertices of a graph, and each handshake between person A and person B can be seen as an edge between vertex A and B.

(c) **(7 points)** Suppose you are proving a proposition $P(n)$ by induction on n. You successfully prove the induction step, $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$. But then you notice that $P(2501)$ is false. Can you conclude anything about $P(25)$? Justify your answer. You can conclude that $P(25)$ is false. If $P(25)$ were true, you could use that as the base case of your induction and conclude that $\forall n >= 25, P(n)$. But since $P(2501)$ is false, this is a contradiction.

(d) **(7 points)** For all $n \ge k \ge 0$, $(n-k)\binom{n}{k} = n\binom{n-1}{k}$ *(hint: combinatorial proof)*. [Credit for this question: Identity 151 from Proofs that Really Count: the Art of Combinatorial Proof.]
LHS: First choose k general members ($\binom{n}{k}$) and then choose a leader out of the remaining $n-k$ people ($\binom{n-k}{1} = n-k$). RHS: First choose a leader out of n total people, and then k general members from the remaining $n-1$ people.

4 Knight Rider

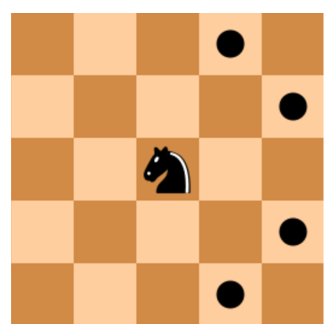
Suppose we have an infinite grid. A knight starts at coordinate $(0,0)$, and each turn, can move ± 2 steps horizontally and ± 1 steps vertically, or ± 1 steps horizontally and ± 2 steps vertically. For instance, in the following diagram, the knight at the center can move into any grid point marked with a black circle.

(a) **(12 points)** Prove that the knight can reach every square (a,b) in the grid, in at most $3(|a|+|b|)$ steps for **full credit** (12 points). Prove that the knight can reach every square in the grid for **partial credit** (7 points). It might be easier to prove the statement by induction, but all proofs will be accepted.

Image credits: `https://en.wikipedia.org/wiki/Knight_(chess)`
We prove that the knight can reach every square (a,b) in the grid in under $3(|a|+|b|)$ steps by inducting on the sum $|a|+|b|$.

Base case: Squares (a,b) such that $|a|+|b|=1$. Let such a square be $(1,0)$ WLOG (since the grid is symmetric by rotation); $(1,0)$ can be reached in three moves $(3(|1|+|0|))$ by the sequence $(1,2),(-1,1),(1,0)$. Thus we have shown the base case.
Inductive hypothesis: All squares (a,b) such that $\forall k \leq n$ $(|a|+|b|)=k$ can be reached from the origin in at most $3k$ moves.
Inductive step: We show that squares (a,b) such that $(|a|+|b|)=k+1$ can be reached from the origin in at most $3(k+1)$ moves. Given such an (a,b) where $a,b>0$ WLOG (again by rotation), it follows from the inductive hypothesis that $(a-2,b-1)$ is reachable by at most $3k$ moves since $(a-2)+(b-1)=a+b-3<k+1$. Since a single move can be used to travel from $(a-2,b-1)$ to (a,b), then (a,b) is reachable by at most $3(k+1)$ moves. Thus, by induction, any square (a,b) is reachable by at most $3(|a|+|b|)$ moves.

(b) Now suppose our knight is crippled and can only move to the right. This means that there are only 4 possible moves: 2 steps right and ± 1 step vertically, or ± 2 steps vertically and 1 step right, as shown in the diagram to the right. Consider the (infinite) *directed* graph where vertices are points the knight can reach and edge (u,v) means that the knight can move from u to v in one move. Answer the following questions (with justification):



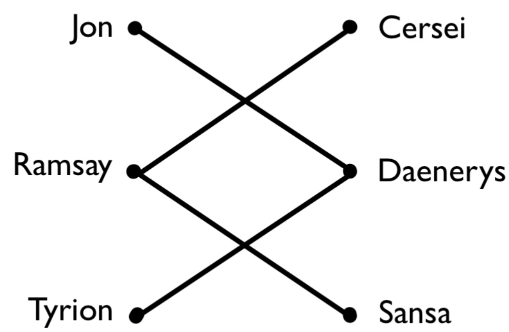
(i) **(6 points)** Is the graph acyclic? Yes. It cannot be the case that the graph is cyclic since visiting a previously visited square requires moving to the left. More formally, for any existing path from u to v in the graph, there does not exist an edge (v,u) since u is left of v as every edge in the u to v path is to a square that is at least one right of the previous and left moves are not allowed.

(ii) **(6 points)** For every vertex v, is there a unique path from the origin to v? No. As a counterexample, consider square $(3,1)$. It can be reached from $(1,2)$ and $(2,-1)$ which are both reachable from $(0,0)$.

5 Stable Matching

In this problem, we are given a bipartite graph: $G = (M,W,E)$ where there are two sets of vertices, M and W, and $E \subseteq M \times W$; that is, each edge is incident to a vertex in M and a vertex in W. We also know that $|M| = |W| = k$. The graph corresponds to an input to the stable marriage problem, with k men (represented by vertices in M) and k women (represented by vertices in W). Every man $v \in M$ prefers every woman in the set S_v that v is adjacent to in G over all women corresponding to vertices in $W \setminus S_v$. Likewise, every woman prefers any man she is adjacent to in G over any man she is not adjacent to.

A *perfect matching* is a set of edges where every vertex is incident to exactly one edge in the matching. Another view is that each vertex is matched to another vertex; similar to a pairing in stable marriage except that the pair must correspond to an edge in the graph. We say that a graph has a perfect matching if a subset of its edges forms a perfect matching.



In the above graph, Cersei prefers Ramsay (whom she is adjacent to) over both Jon and Tyrion (who do not have edges connecting them with Cersei). Note that this graph does not say anything about whether Jon or Tyrion ranks higher in Cersei's preference list, only that Ramsay is higher than both of them. Also, note that this graph has no perfect matching.

(a) **(12 points)** Prove that if G has a perfect matching, then there exists a preference list such that the traditional marriage algorithm results in a perfect matching.

General idea: Find a perfect matching. for each edge (m,w) make m's first choice w and vice-versa.

Solution: Assume that $G = (M,W,E)$ has the following perfect matching

$$(m_1,w_1),(m_2,w_2),(m_3,w_3),\ldots,(m_k,w_k)$$

We can construct a preference list in the following way:
For each man m_i, we use S_{m_i} to denote the set of all vertices that are adjacent to m_i. Then, make woman w_i (the woman who is matched with m_i in the perfect matching) the first choice of man m_i. Following w_i on the list are women in the set $S_{m_i} \setminus w_i$. The rest of the women are attached to the end of m_i's preference list. In general, we need to make sure that in m_i's list

$$\forall w_j \in S_{m_i} \setminus w_i, w_l \in W \setminus S_{m_i}, w_i > w_j > w_l$$

For each woman w_i, we use S_{w_i} to denote the set of all vertices that are adjacent to w_i. w_i's preference list starts with all the men in S_{w_i}, and ends with all the men in $M \setminus S_{w_i}$. In general, we need to make sure that in w_i's list

$$\forall m_j \in S_{w_i}, m_l \in M \setminus S_{w_i}$$

Run traditional stable marriage algorithm with these preference list, the algorithm would terminate after the first day, because every man proposes to a different woman, and the woman is exactly the woman who is matched with him in the perfect matching.
Therefore, such a preference list always exists.

(b) **(12 points)** Disprove by counterexample: If G has a perfect matching, then for all preference lists consistent with G, the traditional marriage algorithm results in a perfect matching. That is, find a graph G, with a perfect matching, and a preference list consistent with G (every person prefers people that he/she is adjacent to in the graph over people that he/she is not adjacent to), such that if we run the traditional marriage algorithm on the preference list, the output matching of men and women will not be a perfect matching in G.

Figure 2 shows an counter example. A, B, C, and D stand for Adam, Bob, Cathy, and Daisy respectively.

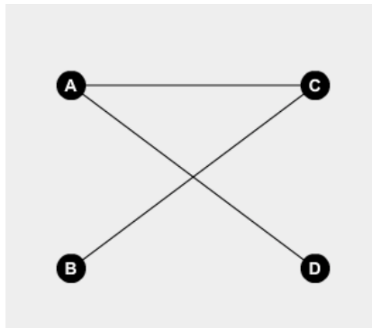


Figure 1: G

The only perfect matching in G is

$$(Adam, Daisy), (Bob, Cathy)$$

However, if we have the following preference list

Cathy	Adam > Bob
Daisy	Bob > Adam
Adam	Cathy > Daisy
Bob	Cathy > Daisy

Run traditional stable marriage algorithm with this preference list, we get the following matching

$$(Adam, Cathy), (Bob, Daisy)$$

This output is not a perfect matching in G.

6 Good Proof, Bad Proof

For each of the following propositions and proofs, indicate which of the following cases apply:

1. Correct proposition with correct proof. No further explanation is needed for this case.
2. Correct proposition but incorrect proof. In this case, identify what the error in the proof is and provide a correct proof.
3. Incorrect proposition (therefore the proof is clearly incorrect). In this case, identify what the error in the proof is and provide a counterexample to the proposition.

(a) **(9 points) If $n \geq 2$, n can be expressed as the sum of distinct prime numbers.**

Proof. By induction on n. For the base case, $n = 2$, the proposition immediately follows from the fact that 2 is prime. Now suppose that the statement is true for all $2 < n' < n$, that is n' can be written as the sum of distinct prime numbers. Therefore $n-2$ can be written as the sum of distinct primes. So we can write n as the sum of 2 and the distinct primes that sum up to $n-2$ (which we know exist by the inductive hypothesis); therefore, the proposition also holds for n. $\square$

(3) Counterexample: consider 4. There are two errors here: the prime-sum decomposition of $n-2$ might contain 2, so the combination isn't distinct, and the inductive step is applied for $n = 3$, but $n-2 = 1$ where the proposition doesn't apply.

(b) **(9 points) A graph G with n vertices and $n-1$ edges contains no cycles.**

Proof. Let G be a graph with n vertices and $n-1$ edges. We show that G contains no cycles. Assume to the contrary that G contains cycles. Remove an edge from a cycle. This leaves a connected graph on n vertices with $n-2$ edges which is impossible as a connected graph on n vertices must at least have $n-1$ edges. $\square$

(3) Incorrect statement: the graph doesn't have to be connected. Mistake in the proof: assuming that the graph is connected to begin with.

(c) **(9 points) $\sum_{i=1}^{n} \frac{1}{i}$ goes to infinity as n grows**

(2) Incorrect proof; instead of saying the upper bound tends toward infinity, we prove the lower bound tends toward infinity: $1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>=\frac{1}{4}+\frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}} + ... >= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$ This shows that the sum goes to infinity as n grows. If one wants to make the claim more rigorous, recall from solution to homework 1, we proved that $\forall n \geq 0, \sum_{i=1}^{2^n} \frac{1}{i} \geq 1 + \frac{n}{2}$ and as $n \to \infty$, our lower bound clearly goes to ∞ and thus our series diverges.

7 Countability

Which of the following are countable, and which are uncountable? Justify your answer.

(a) **(7 points)** The set of all graphs.

Countable. Graphs can be enumerated; give any n (number of vertices), there are at most $\frac{n \cdot (n-1)}{2}$ edges possible, so we can represent each graph as an $\frac{n \cdot (n-1)}{2}$ - long bit-string, where each bit represents an edge, and is 1 if that edge exists, or 0 if it does not exist. Since finite-length bit strings can be enumerated, they are countable.

(b) **(7 points)** The set of all subsets of $\mathbb{N} \setminus \{0\}$.

Uncountable. We establish a bijection between the power set of $\mathbb{N}$, denoted as $\mathscr{P}(\mathbb{N})$, and the set of infinite bit string which has been shown to be uncountable. The most natural way to think of power set is simply to represent them as a bitstring with 1's on the positions where we have the element in our set and 0 otherwise. For example, if we have, in $\mathscr{P}(\mathbb{N})$, element $\{1, 2, 3\}$, then it will be represented by a string $1110000\cdots000$ and the argument for converting an infinite bit string into an element of $\mathscr{P}(\mathbb{N})$ works similarly.

In case you cannot find the proof of why infinite bit strings are uncountable, we attached a sketch of proof below.

Proof of why infinite bit string is uncoutable:

We can show that the set of all infinite-length binary strings is uncountable using a diagonalization argument, similar to the one in the notes. Suppose the contrary that the set of all infinite-length bitstrings is countable, so we can enumerate it as follows:

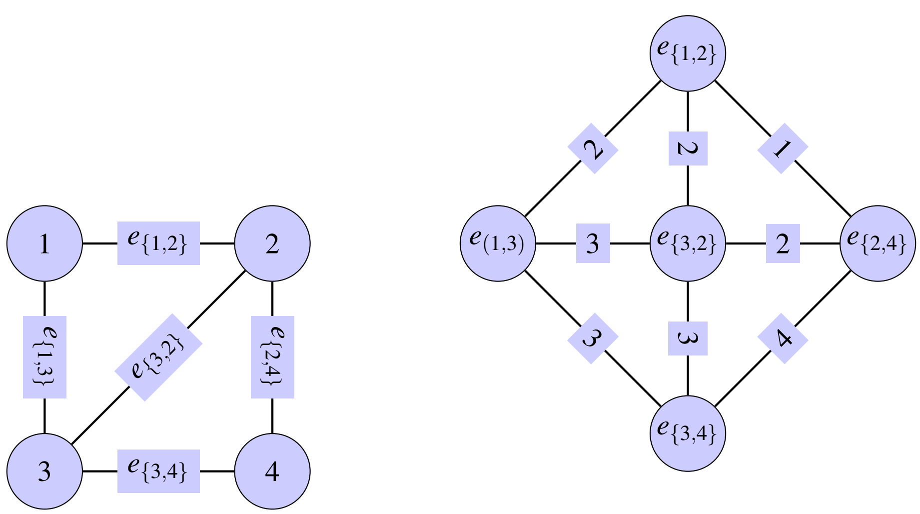
$$\begin{array}{llllll}
1 & \longleftrightarrow 0 & 1 & 1 & 0 & 1\ldots \\
2 & \longleftrightarrow 0 & 1 & 0 & 0 & 1\ldots \\
3 & \longleftrightarrow 1 & 1 & 1 & 0 & 0\ldots \\
4 & \longleftrightarrow 0 & 0 & 1 & 0 & 1\ldots \\
5 & \longleftrightarrow 1 & 0 & 1 & 1 & 1\ldots
\end{array}$$

We can construct a new infinite-length bitstring by flipping each bit down the diagonal, giving us a new infinite-length bitstring that is not in our enumeration, a contradiction. Therefore, the number of infinite-length binary strings is uncountable.

(c) **(7 points)** The set of all programs that halt.

Countable. Programs are finite-length strings over some alphabet, and the set of all finite-length strings is countable.

8 Edge complement



The **edge complement** graph of a graph $G = (V,E)$ is a graph $G' = (V',E')$, such that $V' = E$, and $(i, j) \in E'$ if and only if i and j had a common vertex in G. In the above picture, the graph on the right is the edge complement of the graph on the left: for every edge $e_{\{i,j\}}$ in the graph on the left there is a vertex in the graph on the right. If two edges $e_{\{i,j\}}$ and $e_{\{j,k\}}$ share a vertex j on the left, then the corresponding vertices on the right have an edge j connecting them.

(a) **(12 points)** Prove or disprove: if a graph G has an Eulerian tour, then its **edge complement** graph has an Eulerian tour. **True.** Using the same notation as above, we let an edge in G be $e_{\{i,j\}}$, with endpoints i and j. Then in G', $e_{\{i,j\}}$ is a vertex with edges between itself and only those vertices whose edge-representations in G shared the same vertex as itself. In other words, $e_{\{i,j\}}$ in G' will be neighbors with vertices of the form $e_{\{a,i\}}$ for some $a \neq i$ and $e_{\{j,b\}}$ for some $b \neq j$. If G had an Eulerian tour, then both i and j were incident to an even number of edges; this means that besides $e_{\{i,j\}}$, there were an odd number of other edges in G which were also incident to i, and likewise an odd number of other edges also incident to j. Thus in G', $e_{\{i,j\}}$ has an $odd + odd = even$ number of neighbors and thus is incident on an even number of edges. This is true for all vertices in G'; therefore, there is an Eulerian tour in G'.

(b) **(12 points)** Prove or disprove: if a graph's **edge complement** graph G' has an Eulerian tour, then graph G has an Eulerian tour.

False. We will again use the same notation as above. If there is an Eulerian tour in G', any vertex $e_{\{i,j\}}$ will have an even number of neighbors. This means that in G, there are either an odd number of other edges besides $e_{\{i,j\}}$ incident to both i and j, or there are an even number of other edges incident to both i and j; i.e. in order for $e_{\{i,j\}}$ to in total have an even number of adjacent edges in G, the size of its two groups of neighbors must be either both an even number or both an odd number. If both groups of neighbors are odd, then both i and j have even degrees, since we add $e_{\{i,j\}}$ to the each group to make up the set of incident edges to i and j. However, if the number of neighbors in both groups are even, then both i and j will have odd degrees, since again we must add $e_{\{i,j\}}$ to both groups. If this is the case, then G will not have an Eulerian Tour. Also, a counterexample is a star with 3 vertices.

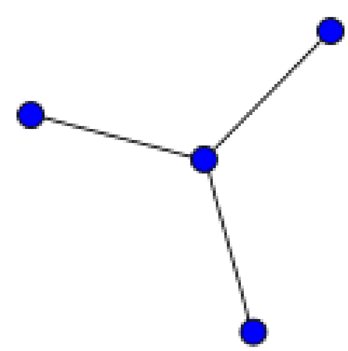


Figure 2: star graph with 3 vertices

9 Empty page for extra work, doodles, etc.

If you use this page as extra space for answers to problems, please indicate clearly which problem(s) you are answering here, and indicate **in the original space for the problem** that you are continuing your work on an extra sheet. You can also use this page to give us feedback or suggestions, report cheating or other suspicious activity, or to draw doodles.

More extra paper. If you fill this sheet up you can request extra sheets from a proctor (just make sure to write your SID on each one, and to staple the extra sheets to your exam when you submit it).