Alex Psomas: Lecture 14.

Events, Conditional Probability, Independence, Bayes' Rule

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1. Probability Basics Review
2. Conditional Probability
3. Independence of Events
4. Bayes' Rule

Probability Basics Review

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- Random Experiment.

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  Flip a fair coin twice.

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See next two slides for (a) and (c).

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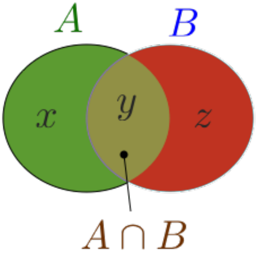
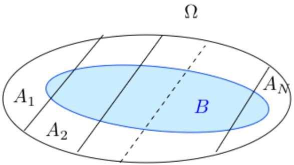


$$\begin{aligned} Pr[A] &= x + y \\ Pr[B] &= y + z \\ Pr[A \cap B] &= y \\ Pr[A \cup B] &= x + y + z \end{aligned}$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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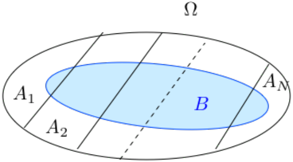

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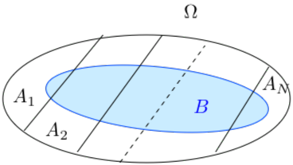

Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, B is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$.

Roll a Red and a Blue Die.

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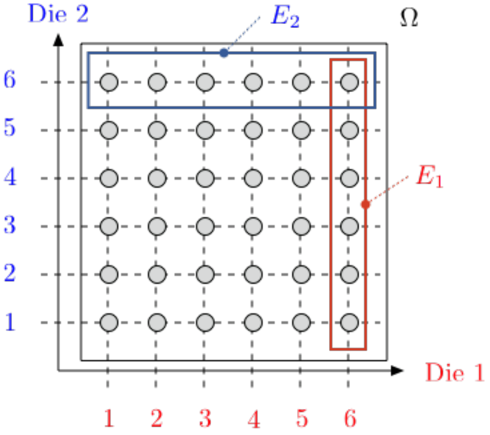

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

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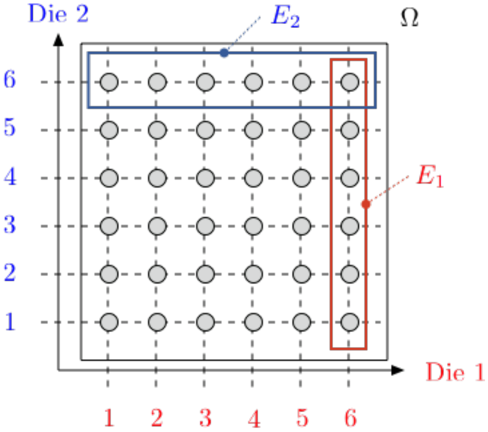

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

E_1 = ‘Red die shows 6’;

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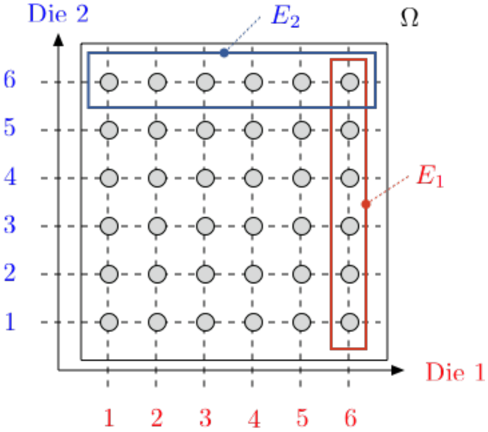

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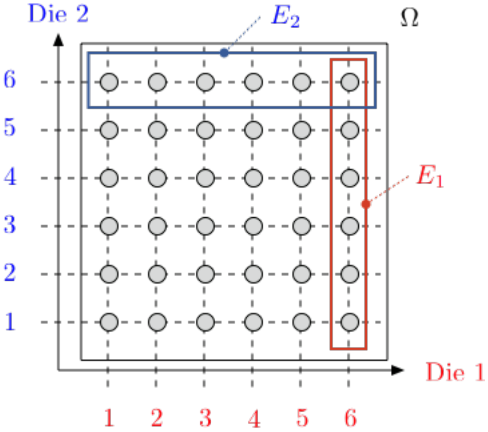

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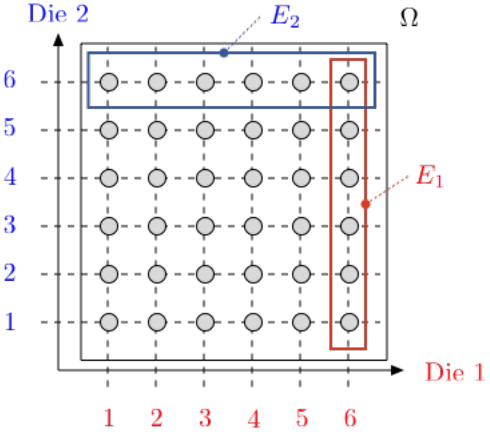

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$$Pr[E_1] = \frac{6}{36},$$

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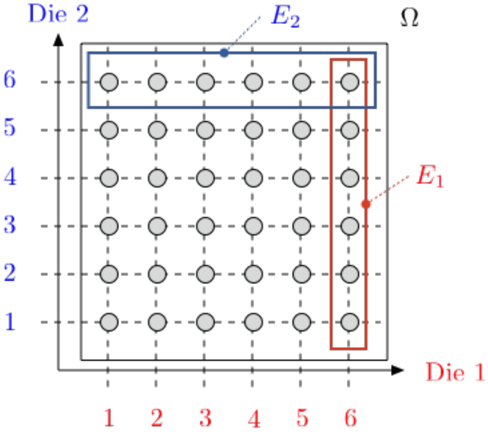

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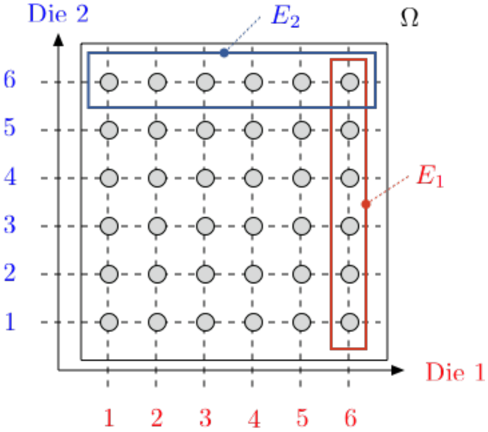

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$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Conditional probability: example.

Two coin flips.

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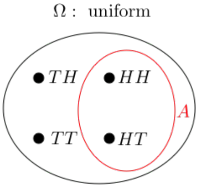

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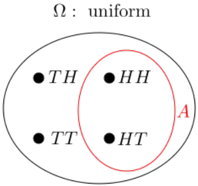

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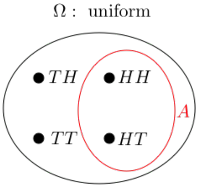

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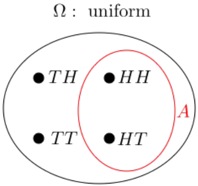

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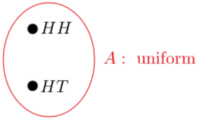

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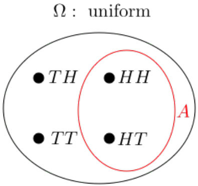

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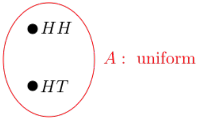

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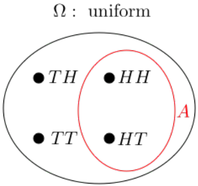

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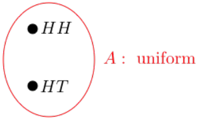

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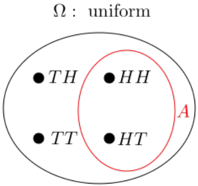

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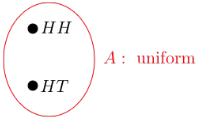

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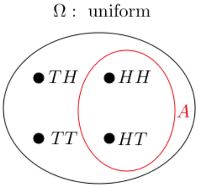

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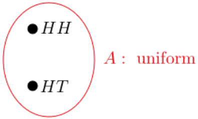

Event $B =$ two heads.

The probability of two heads if the first flip is heads.
The probability of B given A is $1/2$.

A similar example.

Two coin flips.

A similar example.

Two coin flips. At least one of the flips is heads.

A similar example.

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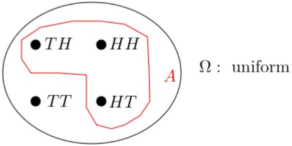

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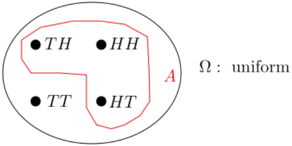

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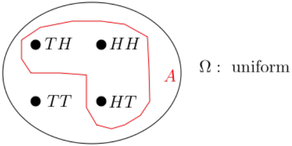

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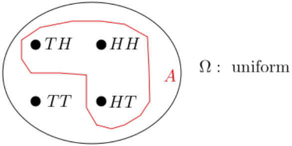

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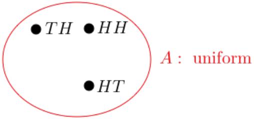

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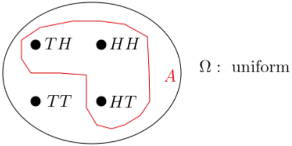

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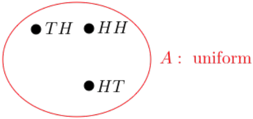

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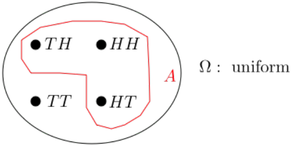

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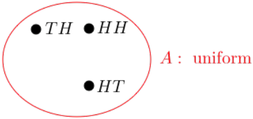

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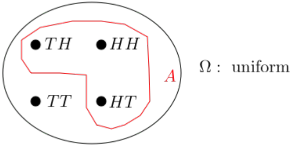

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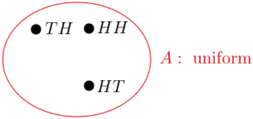

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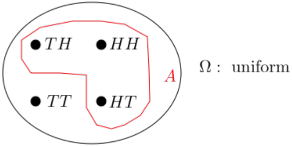

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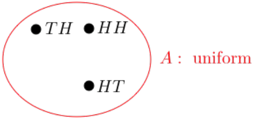

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Conditional Probability: A non-uniform example

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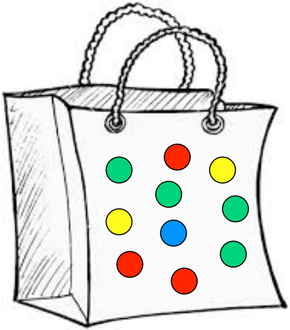
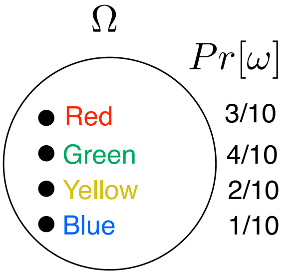


Physical experiment



Probability model

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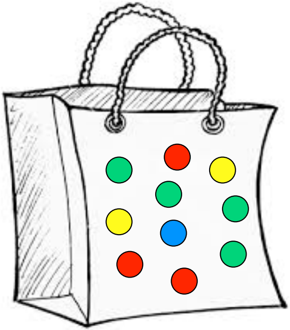

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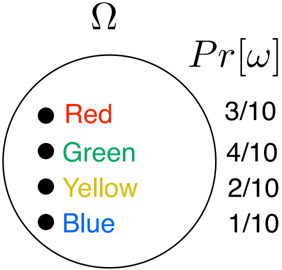

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$\Omega = \{$Red, Green, Yellow, Blue$\}$

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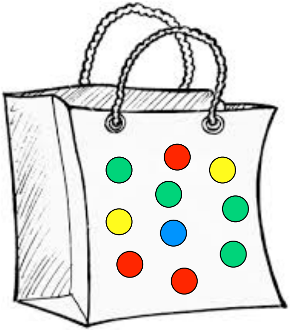

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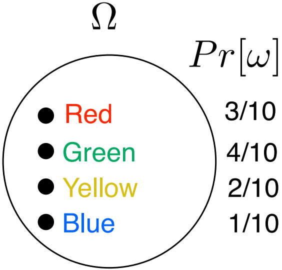

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] =$$

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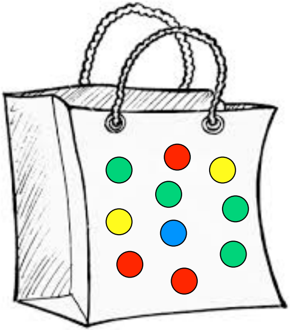

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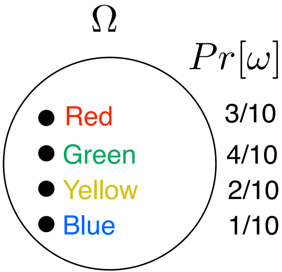

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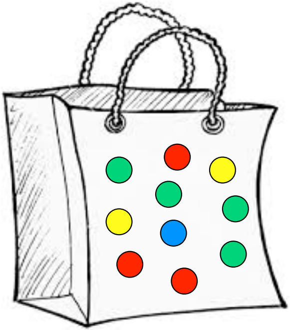

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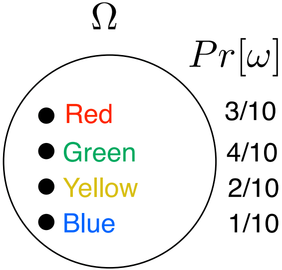

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$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

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Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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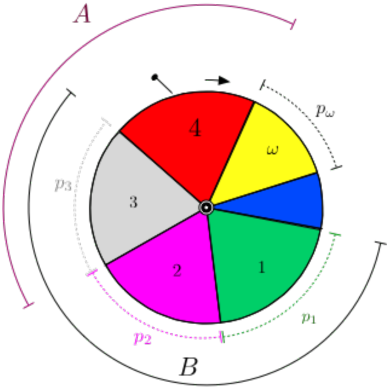

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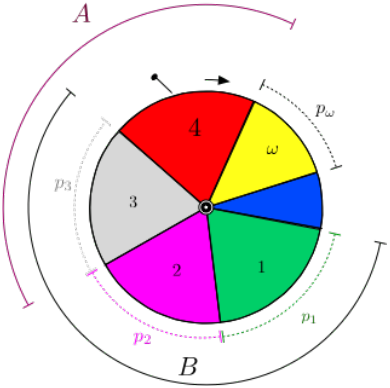

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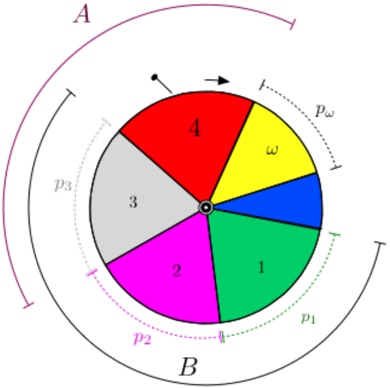

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

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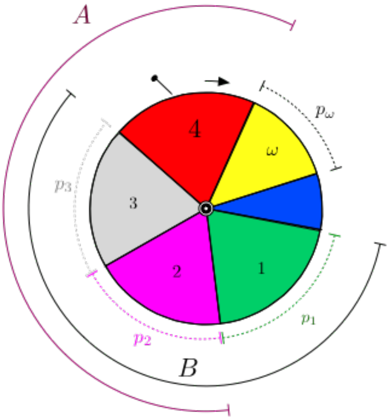

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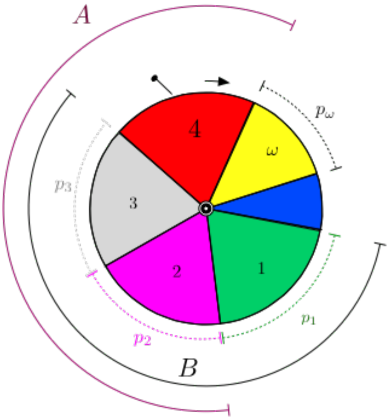

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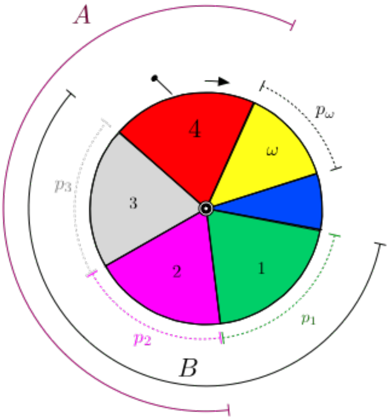

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Conditional Probability.

Definition: The **conditional probability** of B given A is

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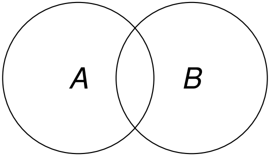

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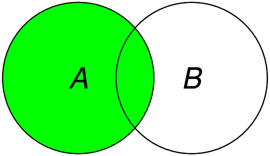

In A!

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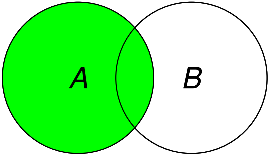

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In B?

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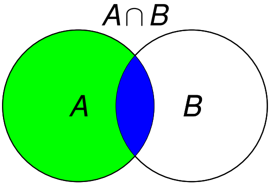

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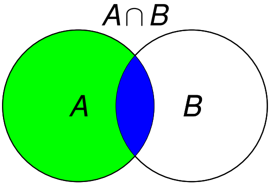

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More fun with conditional probability.

Toss a red and a blue die, sum is 4,

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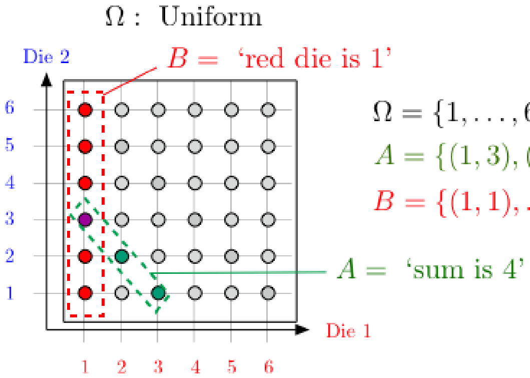

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1,3), (2,2), (3,1)\}$

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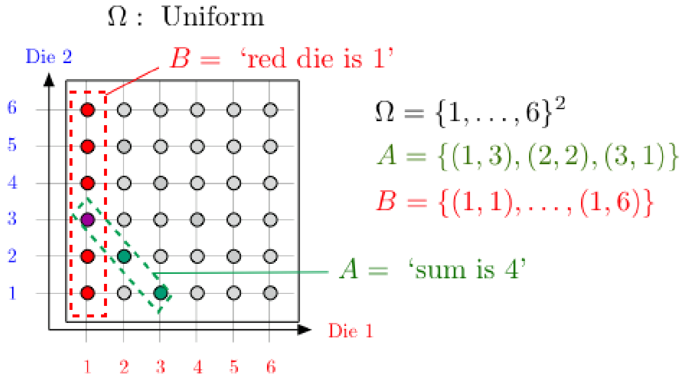

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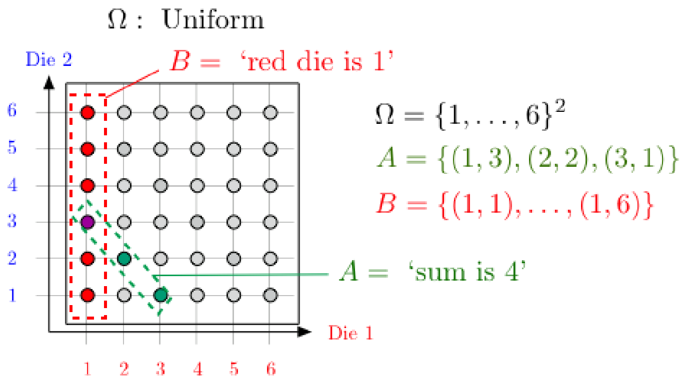

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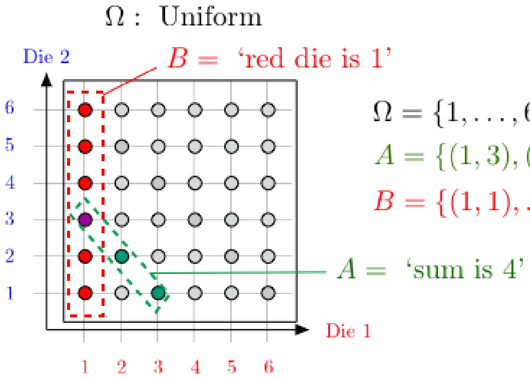

$$\Omega = \{1,\ldots,6\}^2$$
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$$B = \{(1,1),\ldots,(1,6)\}$$

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{3}$; versus $Pr[B] = 1/6$.

B is more likely given *A*.

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Toss a red and a blue die, sum is 7,
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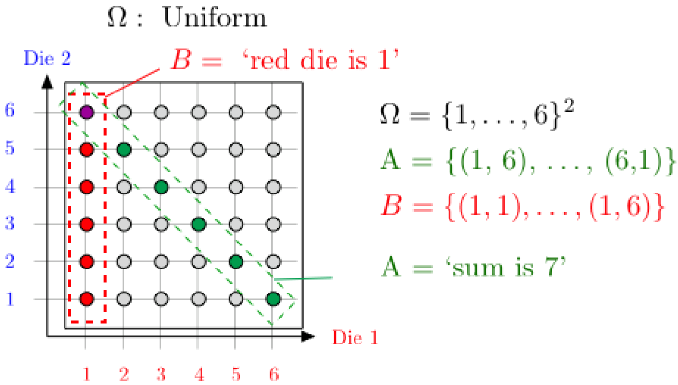

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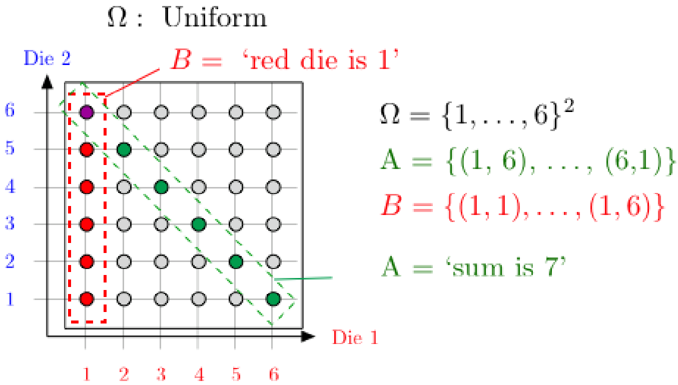

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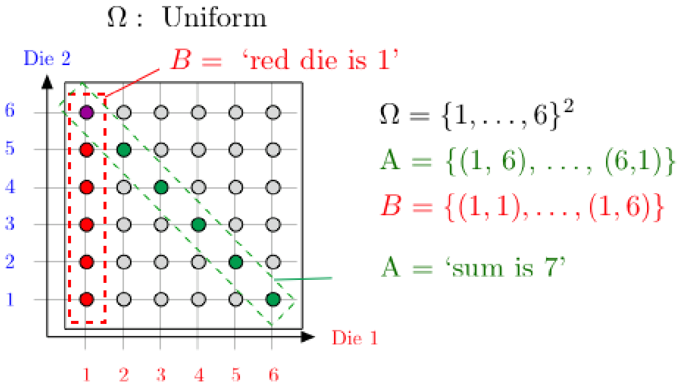

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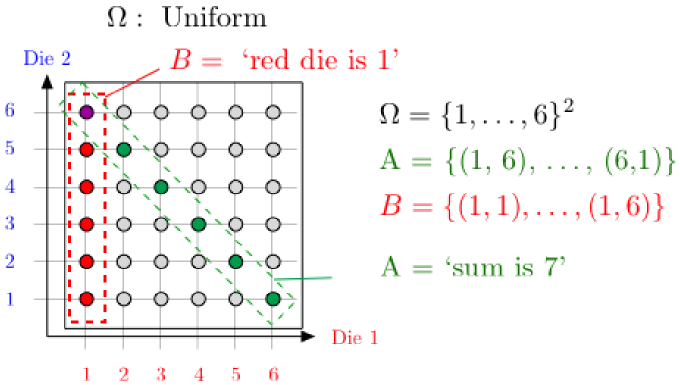

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Observing A does not change your mind about the likelihood of B.

Emptiness..

Suppose I toss 3 balls into 3 bins.

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Emptiness..

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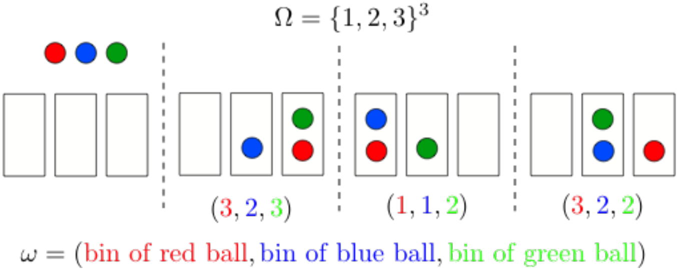

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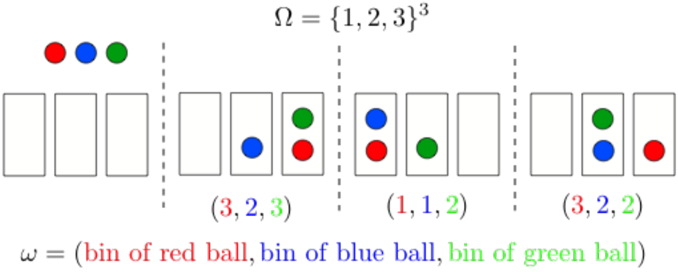

$Pr[B]$

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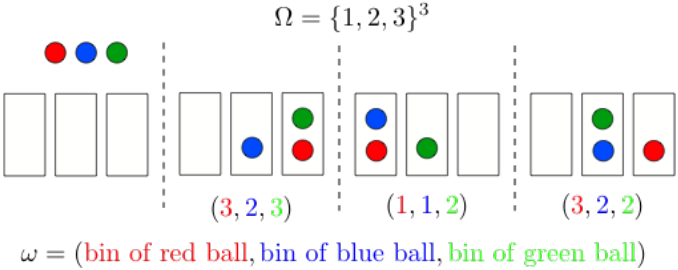

$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] =$

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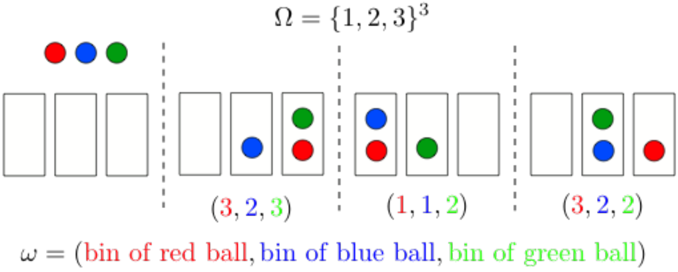

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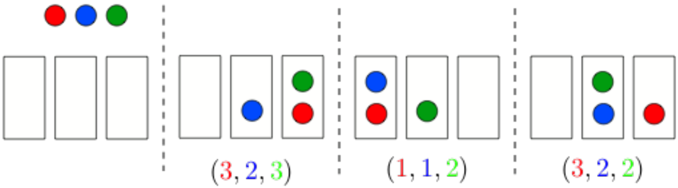

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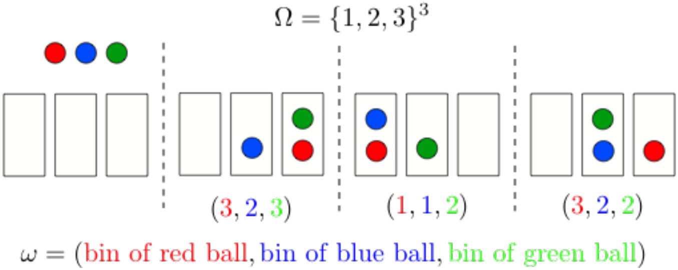

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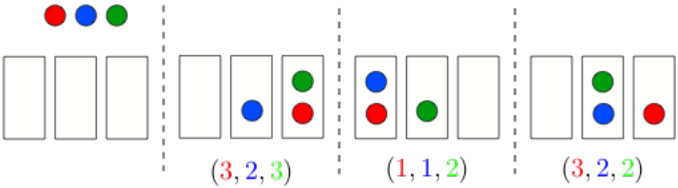

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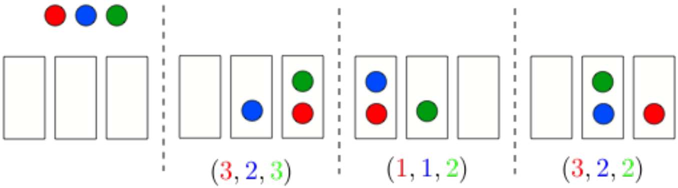

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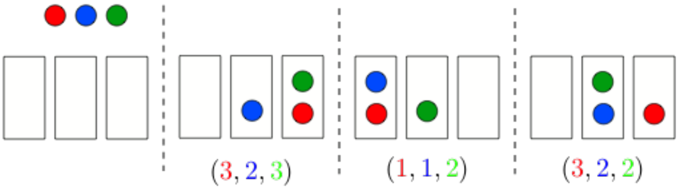

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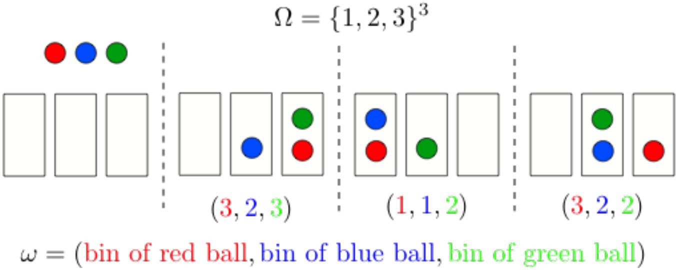

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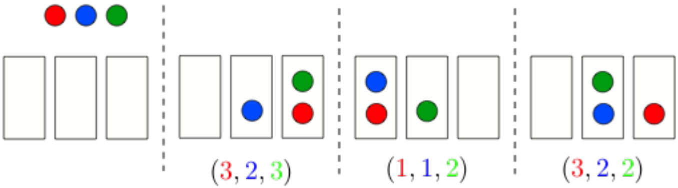

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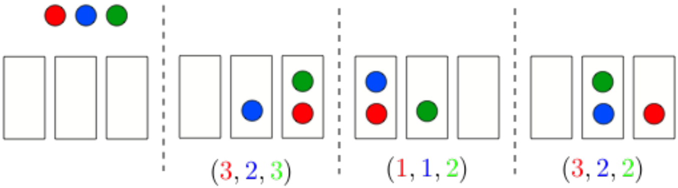

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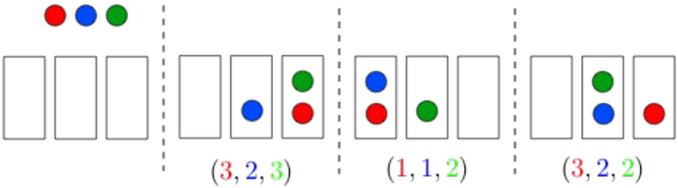

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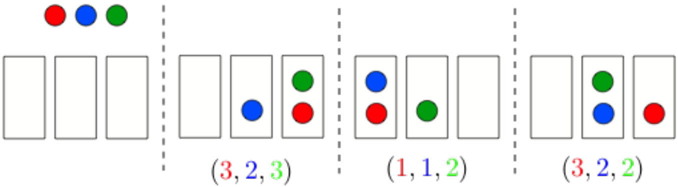

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The likelihood of 51st heads does not depend on the previous flips.

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so that the result holds for $n + 1$. $\square$

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- A and B may be positively correlated because they have a common cause. (E.g., being a rabbit.)
- If B precedes A, then B is more likely to be the cause. (E.g., smoking.) However, they could have a common cause that induces B before A.

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Proving causality is generally difficult. One has to eliminate external causes of correlation and be able to test the cause/effect relationship (e.g., randomized clinical trials).

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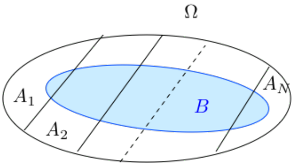
More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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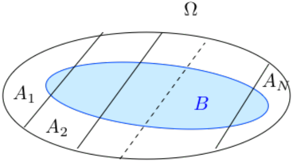

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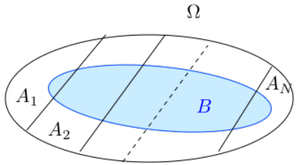

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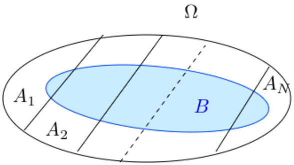

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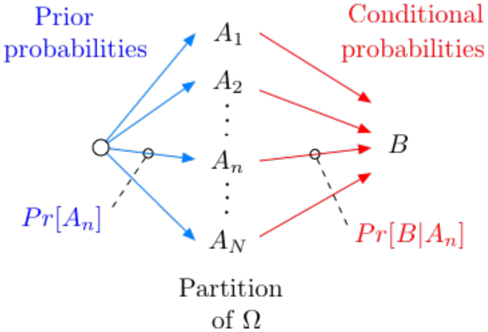

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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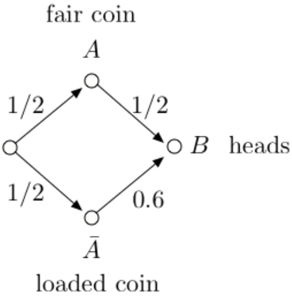

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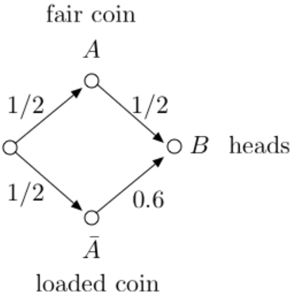

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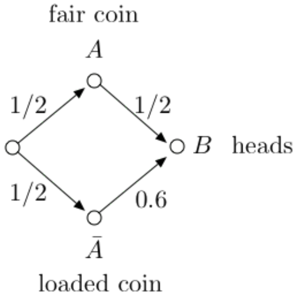

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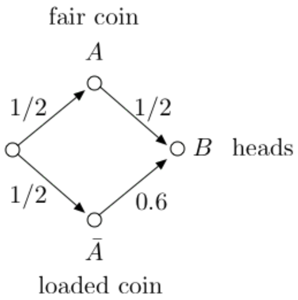

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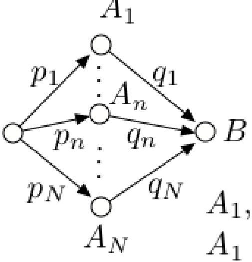

$$p_n = Pr[A_n]$$

$$q_n = Pr[B|A_n]$$

$A_1, \ldots, A_N$ disjoint

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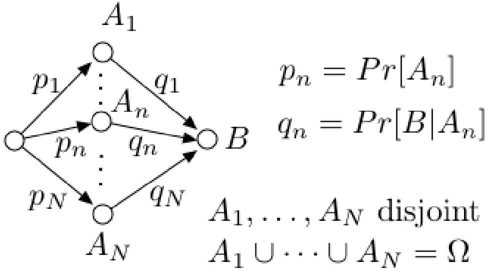

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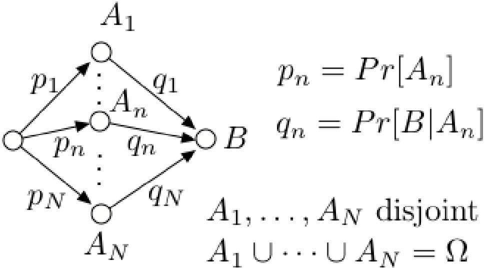

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Hence,

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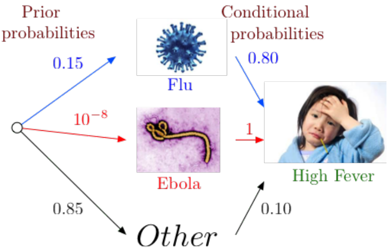
Why do you have a fever?



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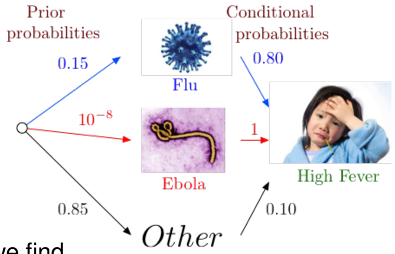

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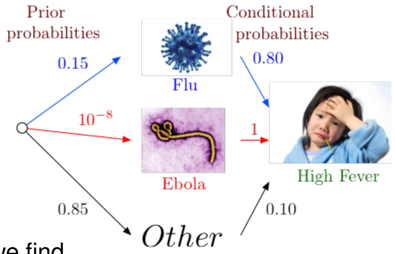

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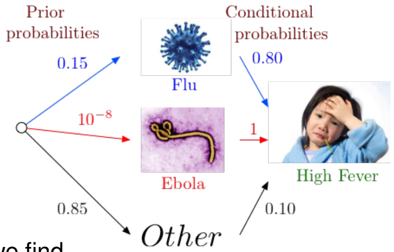

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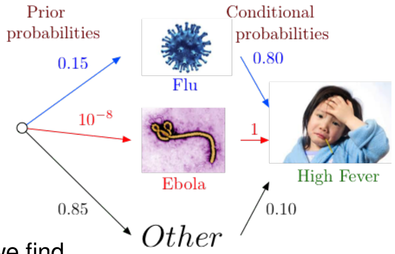

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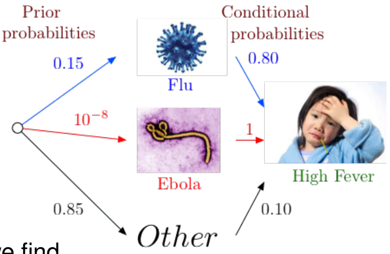

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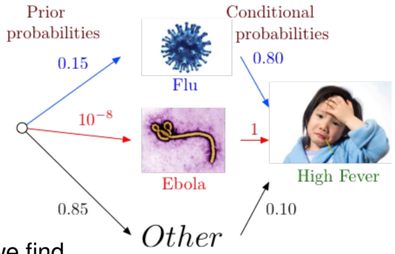

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These are the posterior probabilities. One says that 'Flu' is the Most Likely a Posteriori (MAP) cause of the high fever.

Bayes' Rule Operations

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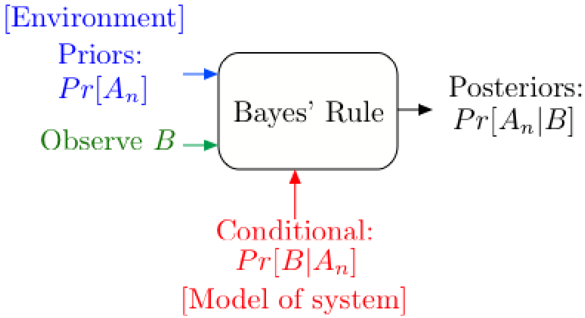

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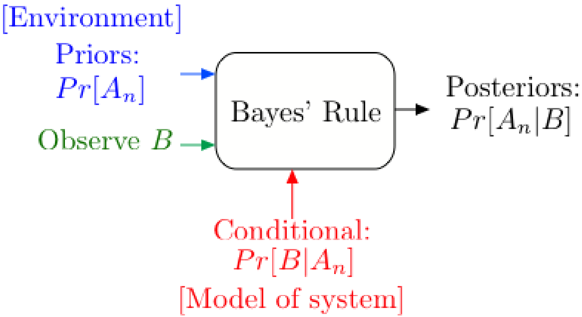

Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$$\xrightarrow{Pr[B|A]}$$

$\text{Max over } B$

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Let’s watch TV!!

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Random Experiment: Pick a random male.

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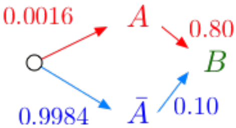
$$Pr[A|B]???$$

Bayes Rule.



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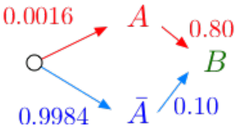

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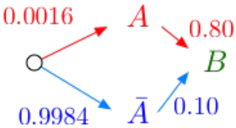

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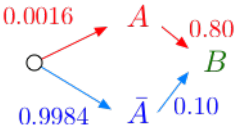

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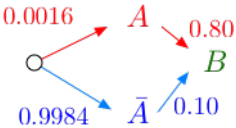

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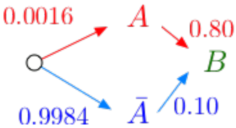

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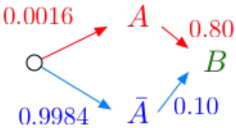

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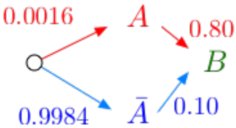

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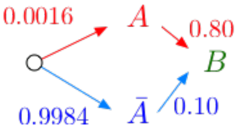

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- All these are possible:

  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A].$