

1. Leaves in a tree

A *leaf* in a tree is a vertex with degree 1.

- (a) Prove that every tree on $n \geq 2$ vertices has at least two leaves.
- (b) What is the maximum number of leaves in a tree with $n \geq 3$ vertices?

Solution:

- (a) We give a direct proof. Consider the longest path $\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-1}, v_k\}$ between two vertices $x = v_0$ and $y = v_k$ in the tree (here the length of a path is how many edges it uses, and if there are multiple longest paths then we just pick one of them). We claim that x and y must be leaves. Suppose the contrary that x is not a leaf, so it has degree at least two. This means x is adjacent to another vertex z different from v_1 . Observe that z cannot appear in the path from x to y that we are considering, for otherwise there would be a cycle in the tree. Therefore, we can add the edge $\{z, x\}$ to our path to obtain a longer path in the tree, contradicting our earlier choice of the longest path. Thus, we conclude that x is a leaf. By the same argument, we conclude y is also a leaf.

The case when a tree has only two leaves is called the *path graph*, which is the graph on $V = \{1, 2, \dots, n\}$ with edges $E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$.

- (b) We claim the maximum number of leaves is $n - 1$. This is achieved when there is one vertex that is connected to all other vertices (this is called the *star graph*).

We now show that a tree on $n \geq 3$ vertices cannot have n leaves. Suppose the contrary that there is a tree on $n \geq 3$ vertices such that all its n vertices are leaves. Pick an arbitrary vertex x , and let y be its unique neighbor. Since x and y both have degree 1, the vertices x, y form a connected component separate from the rest of the tree, contradicting the fact that a tree is connected.

2. Edge-disjoint paths in hypercube

Prove that between any two distinct vertices x, y in the n -dimensional hypercube graph, there are at least n edge-disjoint paths from x to y (i.e., no two paths share an edge, though they may share vertices).

Solution: We use induction on $n \geq 1$. The base case $n = 1$ holds because in this case the graph only has two vertices $V = \{0, 1\}$, and there is 1 path connecting them. Assume the claim holds for the $(n - 1)$ -dimensional hypercube. Let $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_n$ be distinct vertices in the n -dimensional hypercube; we want to show there are at least n edge-disjoint paths from x to y . To do that, we consider two cases:

1. Suppose $x_i = y_i$ for some index $i \in \{1, \dots, n\}$. Without loss of generality (and for ease of explanation), we may assume $i = 1$, because the hypercube is symmetric with respect to the indices. Moreover, by interchanging the bits 0 and 1 if necessary, we may also assume $x_1 = y_1 = 0$. This means x and y both lie in the 0-subcube, where recall the 0-subcube (respectively, the 1-subcube) is the $(n - 1)$ -dimensional hypercube with vertices labeled $0z$ (respectively, $1z$) for $z \in \{0, 1\}^{n-1}$.

Applying the inductive hypothesis, we know there are at least $n - 1$ edge-disjoint paths from x to y , and moreover, these paths all lie within the 0-subcube. Clearly these $n - 1$ paths will still be edge-disjoint

in the original n -dimensional hypercube. We have an additional path from x to y that goes through the 1-subcube as follows: go from x to x' , then from x' to y' following any path in the 1-subcube, and finally go from y' back to y . Here $x' = 1x_2 \dots x_n$ and $y' = 1y_2 \dots y_n$ are the corresponding points of x and y in the 1-subcube. Since this last path does not use any edges in the 0-subcube, this path is edge-disjoint to the $n - 1$ paths that we have found. Therefore, we conclude that there are at least n edge-disjoint paths from x to y .

2. Suppose $x_i \neq y_i$ for all $i \in \{1, \dots, n\}$. This means x and y are two opposite vertices in the hypercube, and without loss of generality, we may assume $x = 00 \dots 0$ and $y = 11 \dots 1$. We explicitly exhibit n paths P_1, \dots, P_n from x to y , and we claim they are edge-disjoint.

For $i \in \{1, \dots, n\}$, the i -th path P_i is defined as follows: start from the vertex x (which is all zeros), flip the i -th bit to a 1, then keep flipping the bits one by one moving rightward from position $i + 1$ to n , then from position 1 moving rightward to $i - 1$. For example, the path P_1 is given by

$$000 \dots 0 \rightarrow 100 \dots 0 \rightarrow 110 \dots 0 \rightarrow 111 \dots 0 \rightarrow \dots \rightarrow 111 \dots 1$$

while the path P_2 is given by

$$000 \dots 0 \rightarrow 010 \dots 0 \rightarrow 011 \dots 0 \rightarrow \dots \rightarrow 011 \dots 1 \rightarrow 111 \dots 1$$

Note that the paths P_1, \dots, P_n don't share vertices other than $x = 00 \dots 0$ and $y = 11 \dots 1$, so in particular they must be edge-disjoint.

3. (Odd degree vertices)

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

- (i) Direct proof (e.g., counting the number of edges in G)
- (ii) Induction on $m = |E|$ (number of edges)
- (iii) Induction on $n = |V|$ (number of vertices)
- (iv) Well-ordering principle

Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

- (i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^c$, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\text{odd}}(G)} d_v.$$

Both terms in the righthand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the lefthand side $\sum_{v \in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

(ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G , then all vertices have degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m + 1$ edges. Remove an arbitrary edge $\{u, v\}$ from G , so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u, v\}$ to get back the original graph G . Note that u has one more edge in G than it does in G' , so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G . Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u, v\} & \text{if } u, v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u, v\} & \text{if } u, v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n + 1$ vertices. Remove a vertex v and all edges adjacent to it from G . The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G . Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v , the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G' , so they now have even degree in G . On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^c$ had even degree in G' , and they now have odd degree in G . The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v , so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (i), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

- (iv) Here we give a well-ordering proof using the number of edges m as the notion of “size” of G , so this is equivalent to the proof in part (i) using induction on m . (You can also try to give a well-ordering proof using n as the size of G .)

Suppose the contrary that the claim is false for some graphs. This means the set M is not empty, where M is the set of $m \in \mathbb{N}$ for which there exists a graph G with m edges that is a counterexample to the claim. Thus, we have a nonempty subset M of \mathbb{N} , so by the well-ordering principle, M has a smallest element m' . Note that $m' > 0$, since the claim is true for all graphs with 0 edges.

Let G be a graph with m' edges for which the claim is false, i.e., $|V_{\text{odd}}(G)|$ is odd (here we know such a G must exist from the definition of $m' \in M$). Remove one edge from G to obtain a smaller graph G' with $m' - 1$ edges (here we need $m' \geq 1$, which we have seen above). By our choice of m' as the smallest element of M , we know that $m' - 1 \notin M$, so the claim holds for G' , namely, $|V_{\text{odd}}(G')|$ is even. Now add the removed edge to get back G . By the same argument as in the inductive step in part (i), this implies that $|V_{\text{odd}}(G)|$ is also even, a contradiction.