CS70: Alex Psomas: Lecture 13.

$\boxed{\text{Modeling Uncertainty: Probability Space}}$

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Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space
4. Events

Key Points

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- Probability
  - Models knowledge about uncertainty
  - Discovers best way to use that knowledge in making decisions

The Magic of Probability

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Uncertainty:

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Uncertainty: vague,

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Uncertainty: vague, fuzzy,

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Your cost: focused attention and practice on examples and problems.

A cool trick

Random Experiment: Flip one Fair Coin

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- Possible outcomes: Heads (H) and Tails (T)
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Flip a fair coin: model

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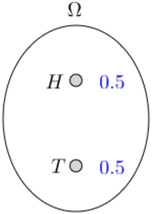
Physical Experiment



Probability Model

Random Experiment: Flip one Fair Coin

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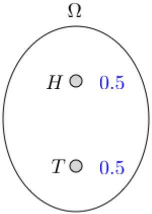

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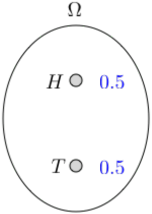

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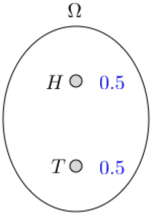

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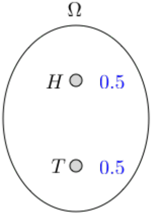

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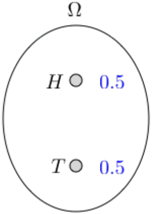

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  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.

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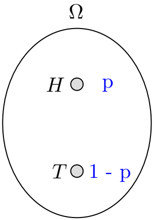

Probability Model

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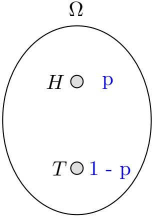

Probability Model

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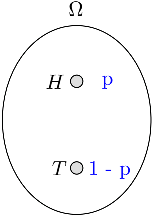

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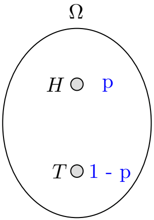

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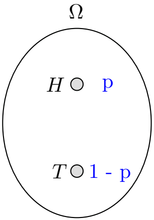

Probability Model

- Same set of outcomes as before!
- Different probabilities!
- The most common mistake in Probability: assuming that outcomes are equally likely.

Flip Two Fair Coins

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- Possible outcomes:

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\}$

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\} \equiv \{H, T\}^2$.

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Flip Glued Coins

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Flips two coins glued together side by side:

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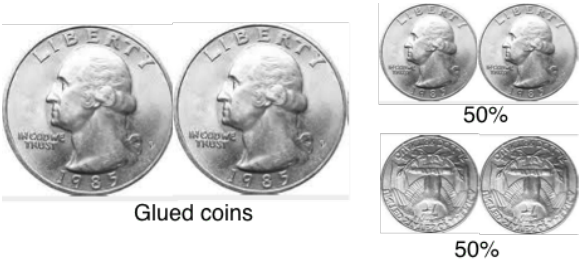

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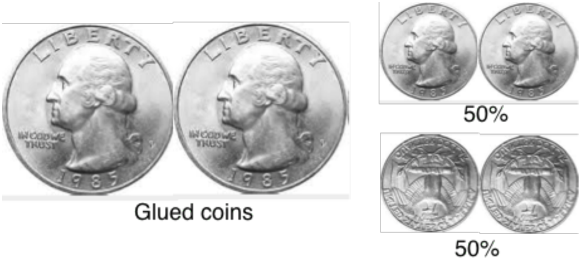

- Possible outcomes: $\{HH, TT\}$.
- Likelihoods: $HH : 0.5, TT : 0.5$.

Flip Glued Coins

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- Possible outcomes: $\{HH, TT\}$.
- Likelihoods: $HH : 0.5, TT : 0.5$.
- Note: Coins are glued so that they show the same face.

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Flips two coins glued together side by side:

Glued coins

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- Possible outcomes: $\{HT, TH\}$.
- Likelihoods: $HT : 0.5$, $TH : 0.5$.
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Flip two Attached Coins

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Flips two coins attached by a spring:

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Flips two coins attached by a spring:

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Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: $\{HH, HT, TH, TT\}$.
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- Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.

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Here is a way to summarize the four random experiments:

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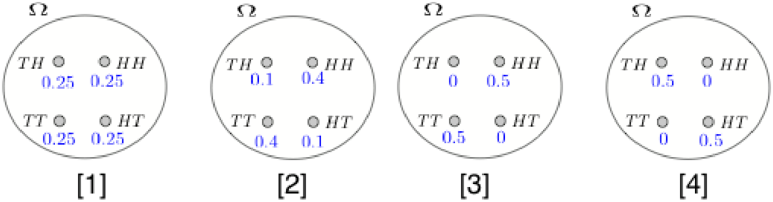

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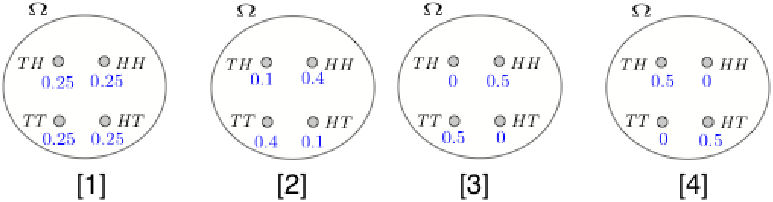

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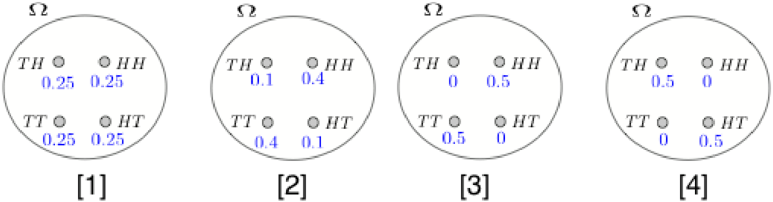

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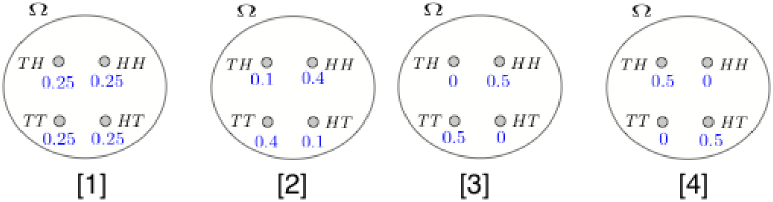

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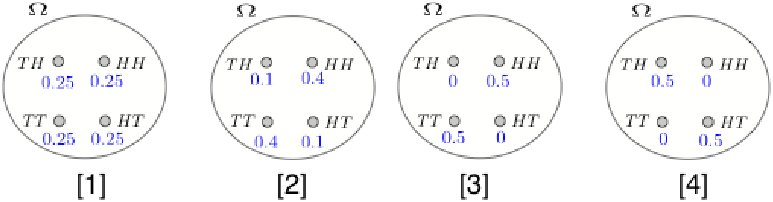

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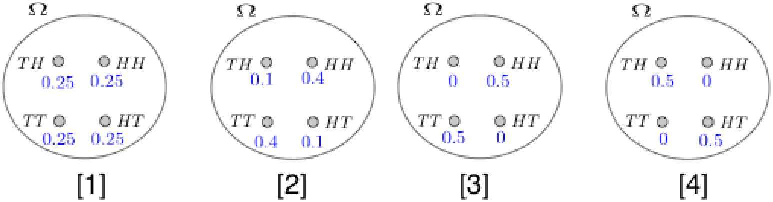

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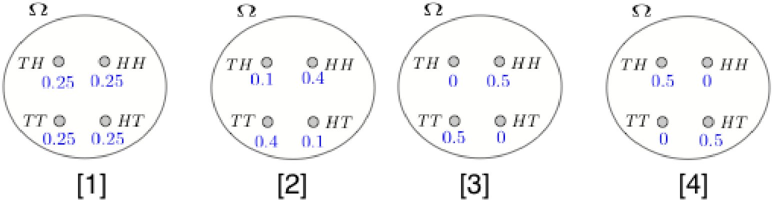

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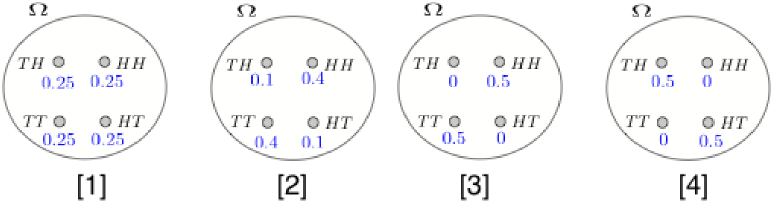

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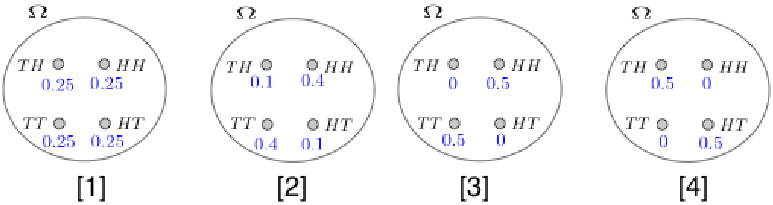

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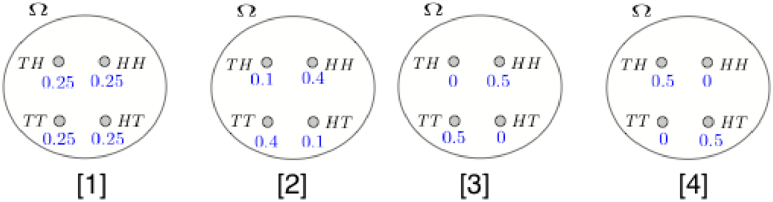

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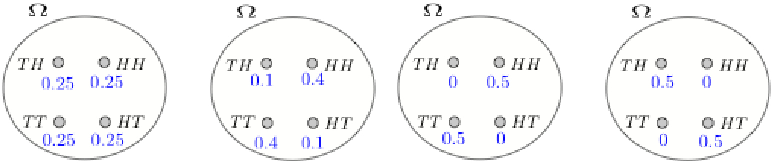

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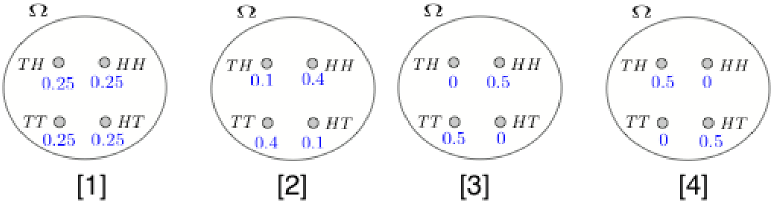

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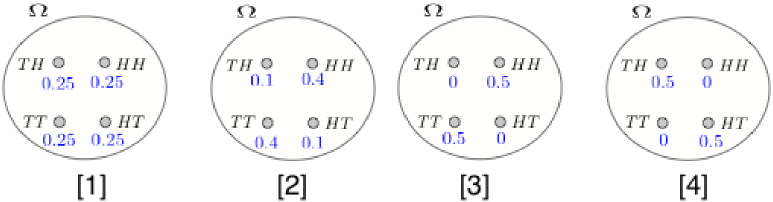

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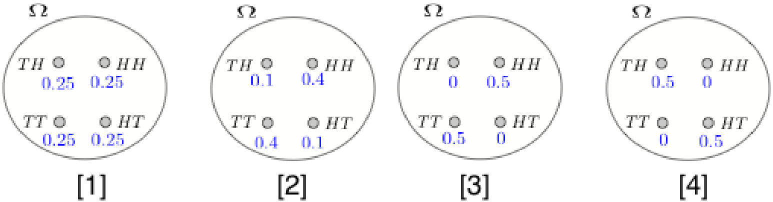

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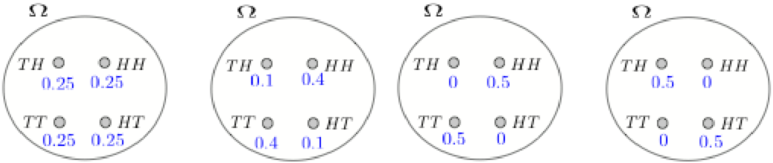

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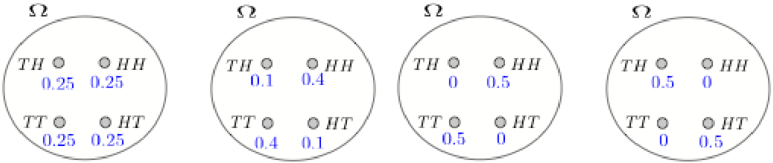

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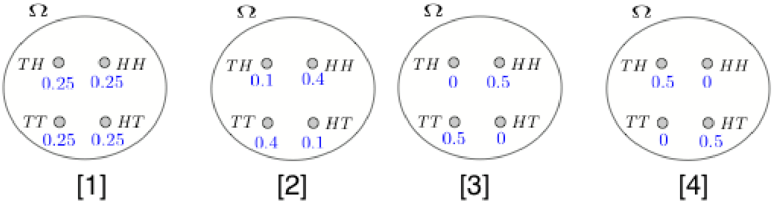

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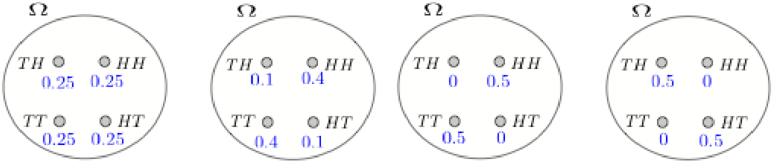

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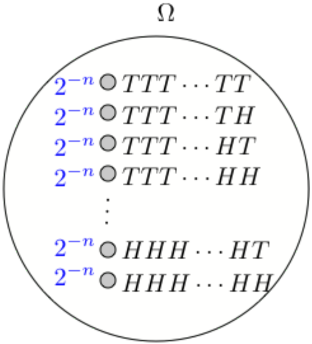

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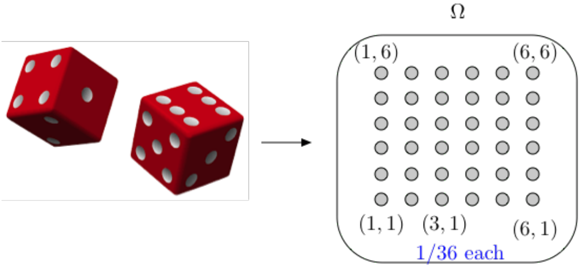

Physical Experiment

Probability Model

Probability Space.

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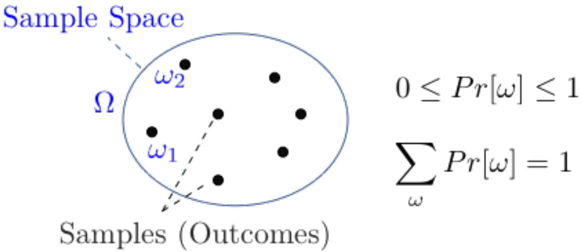

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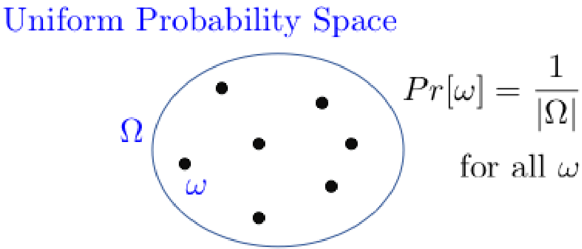

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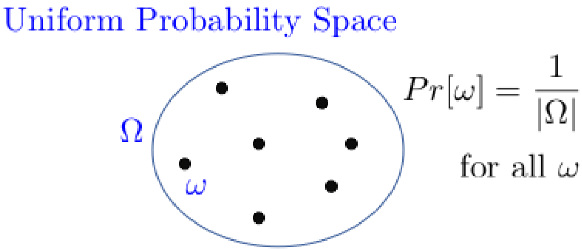

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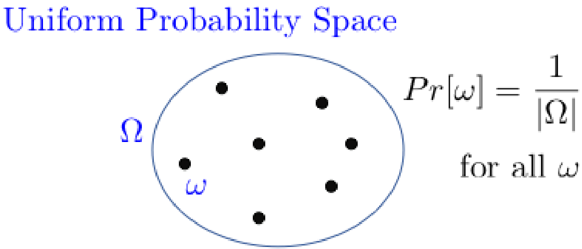

Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
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Simplest physical model of a uniform probability space:

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Simplest physical model of a uniform probability space:

Physical experiment

Ω	$Pr[\omega]$
Red	1/8
Green	1/8
⋮	...
Maroon	1/8

Probability model

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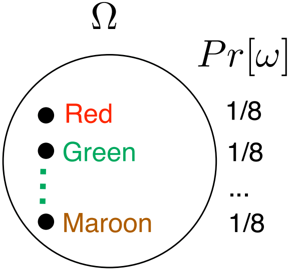

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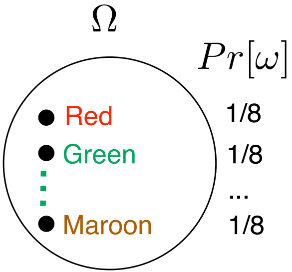

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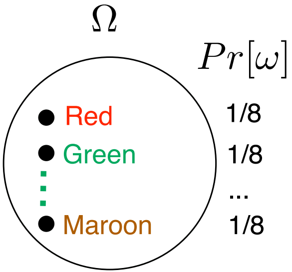

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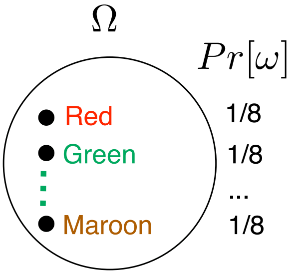

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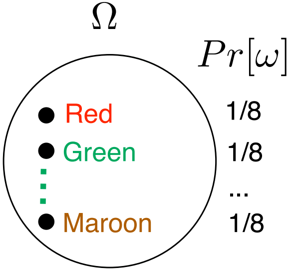

Probability model

A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$

$$Pr[\text{blue}] = \frac{1}{8}.$$

Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

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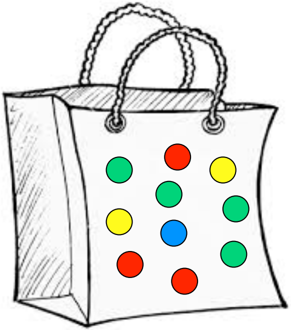

Physical experiment

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Probability model

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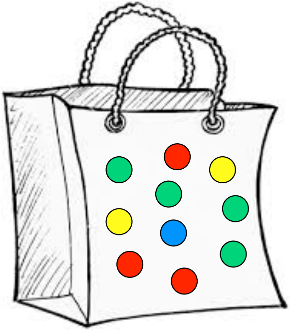

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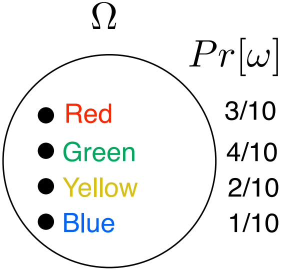

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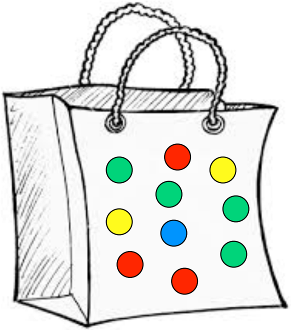

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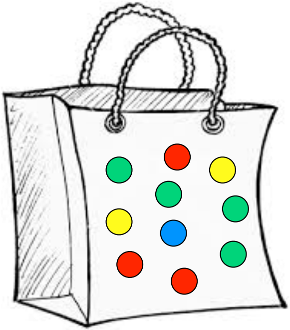

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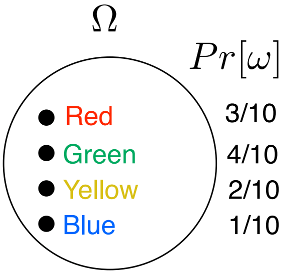

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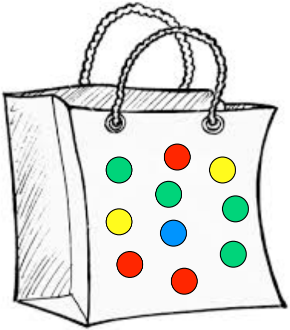

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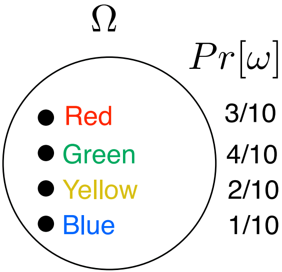

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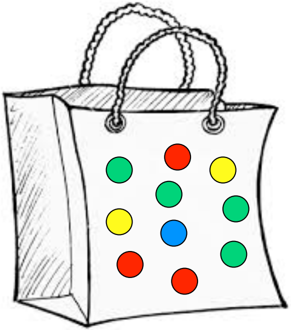

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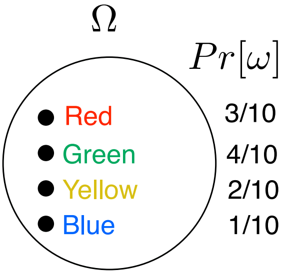

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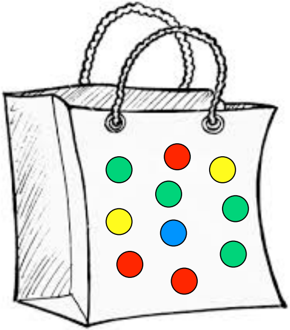

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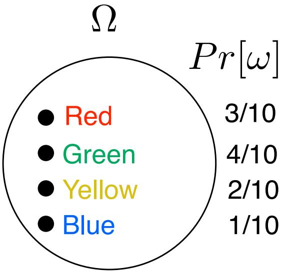

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Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

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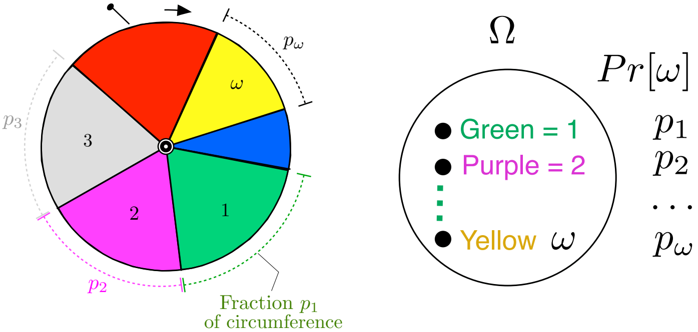

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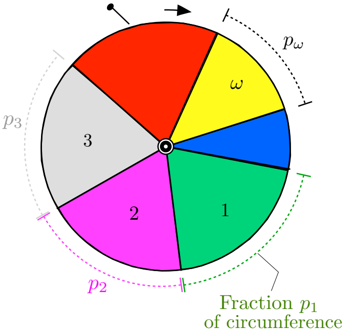

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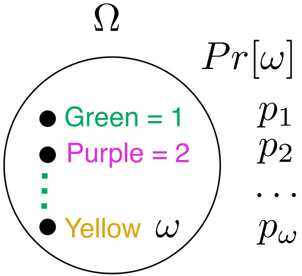

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The roulette wheel stops in sector ω with probability p_ω.

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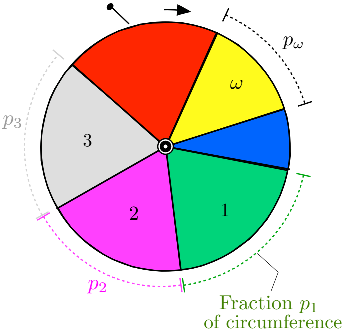

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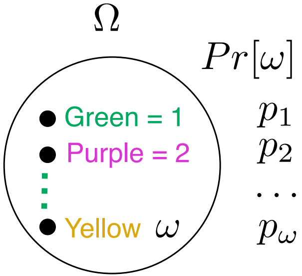

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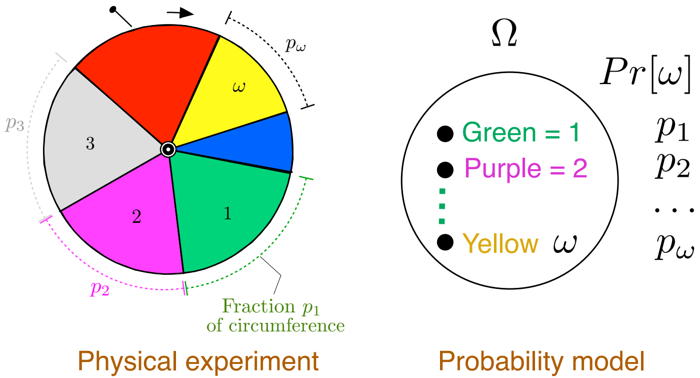

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Events

Next idea: an event!

Set notation review

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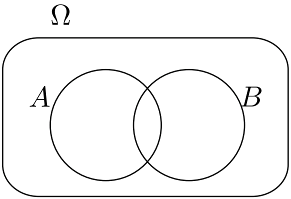

Figure : Two events

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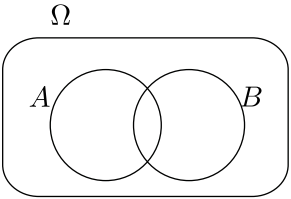

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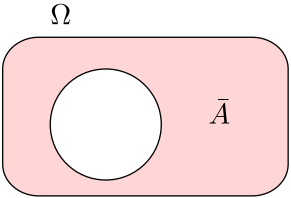

Figure : Complement (not)

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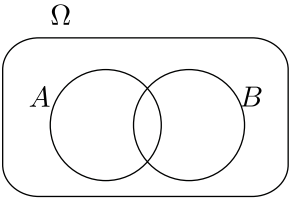

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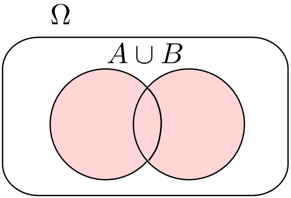

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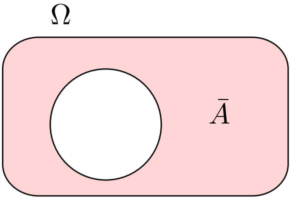

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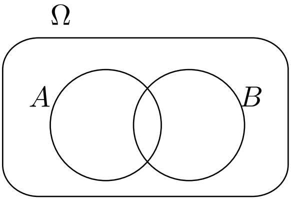

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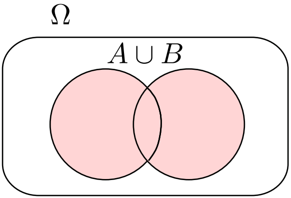

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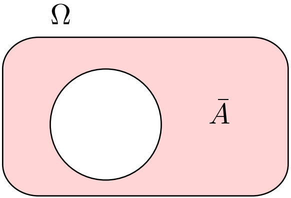

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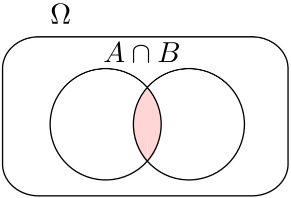

Figure : Intersection (and)

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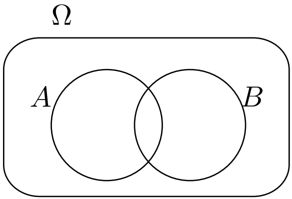

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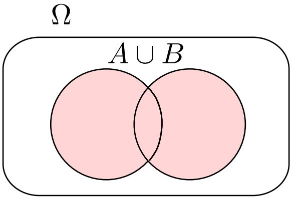

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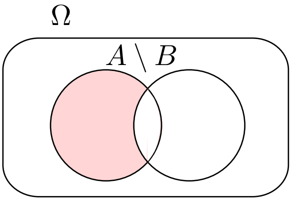
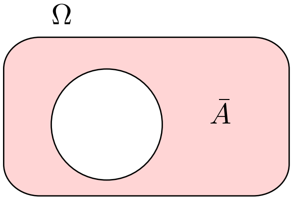


Figure : Difference (*A*, not *B*)



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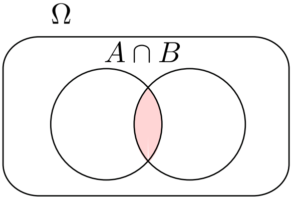

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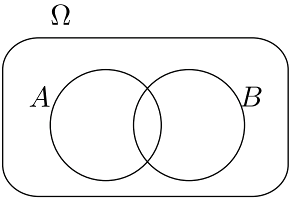

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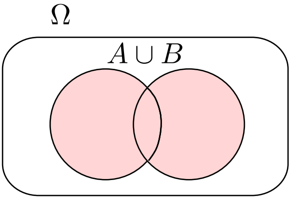

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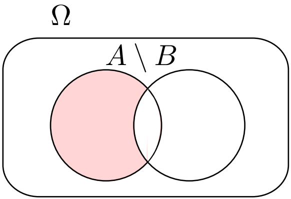

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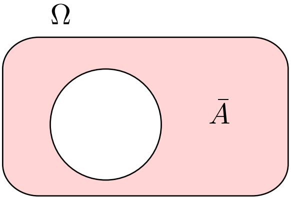

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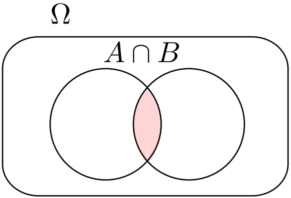

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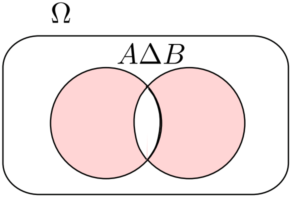

Figure : Symmetric difference (only one)

Probability of exactly one ‘heads’ in two coin flips?

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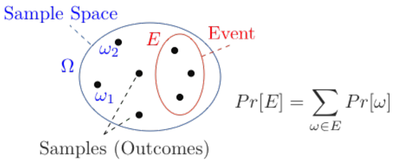

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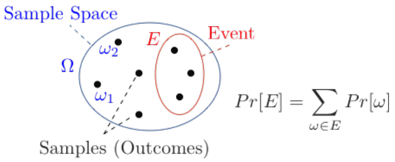

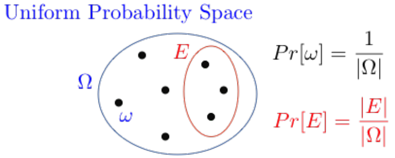

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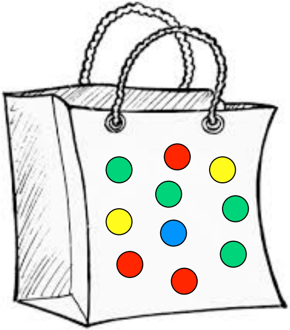

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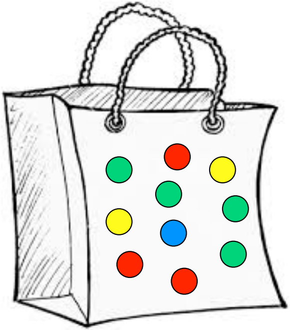

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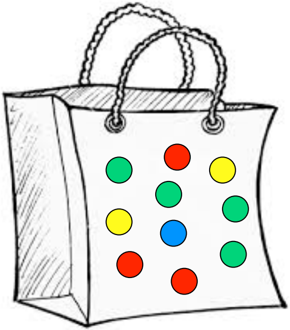

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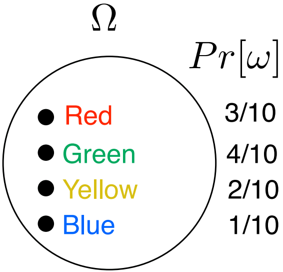

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$$Pr[\text{Red}] =$$

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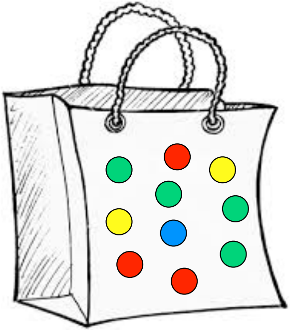

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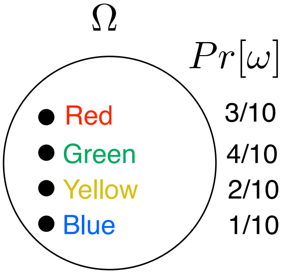

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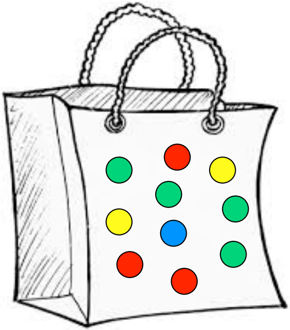

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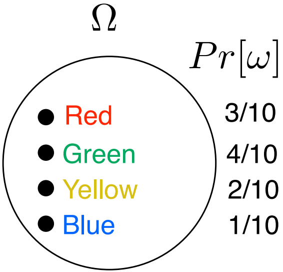

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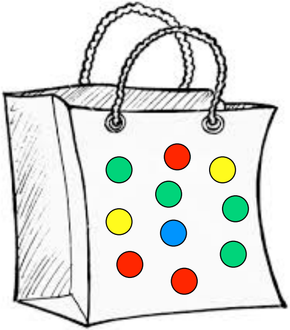
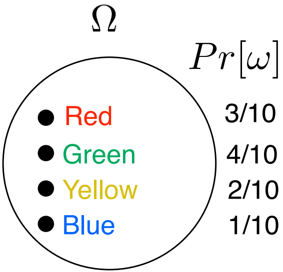


Physical experiment



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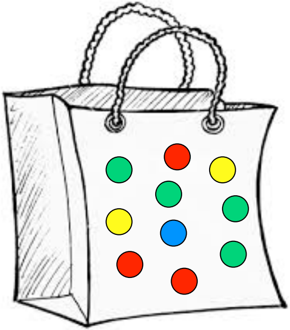

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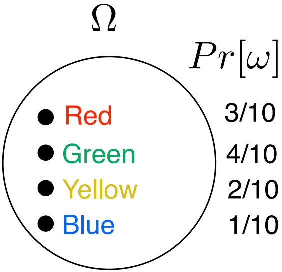

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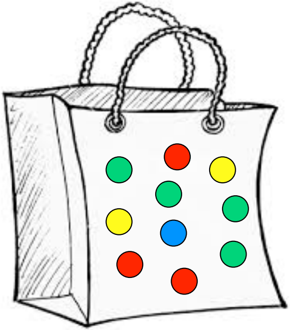

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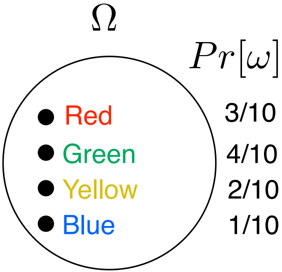

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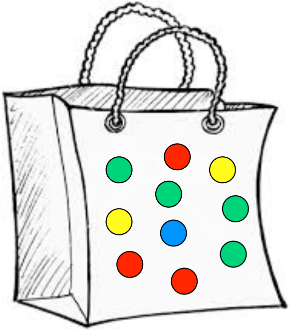

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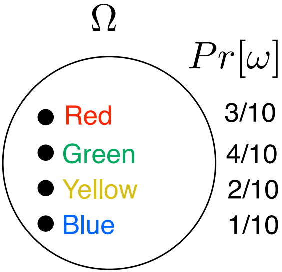

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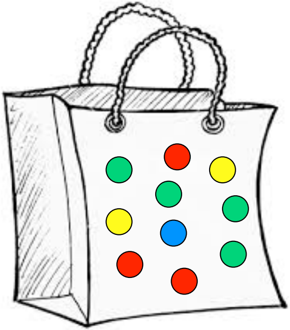

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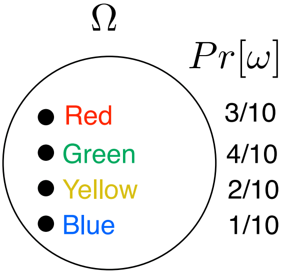

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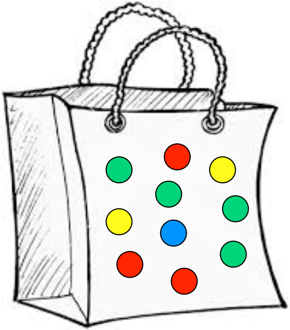

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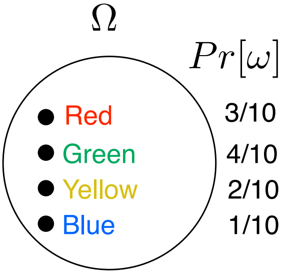

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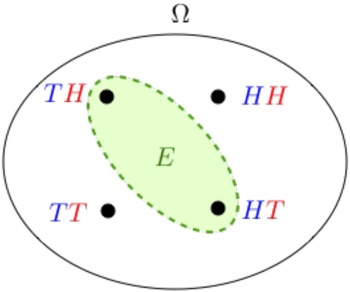

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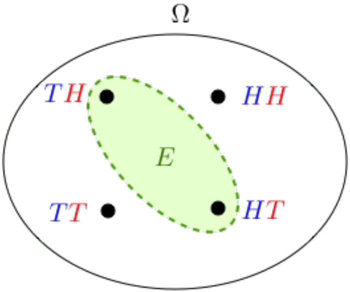

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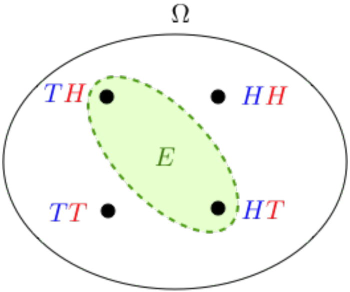

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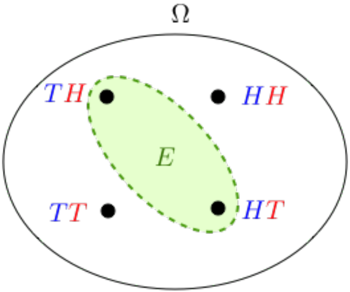

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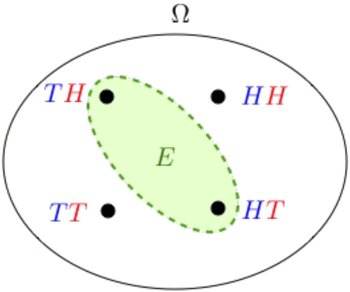

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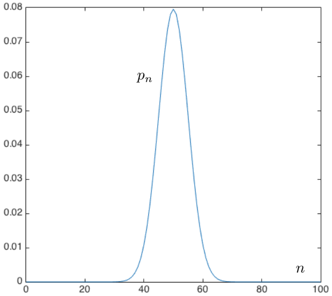

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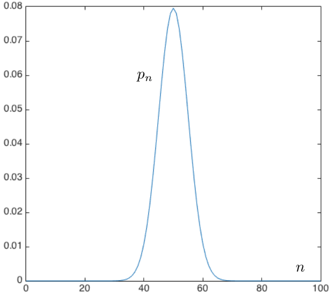

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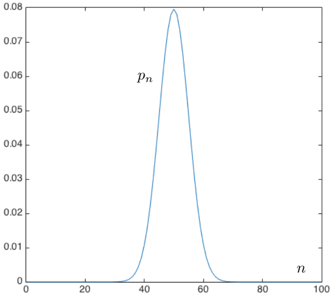

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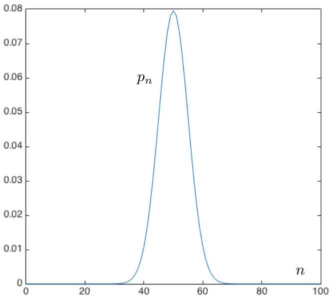

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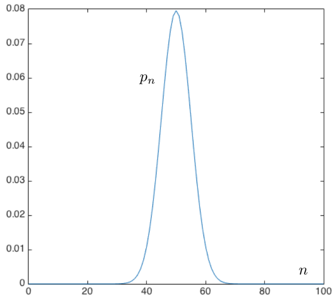

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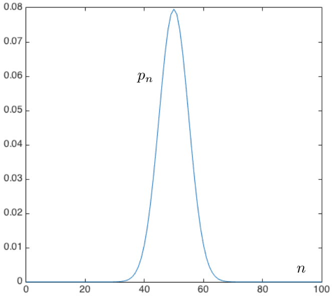

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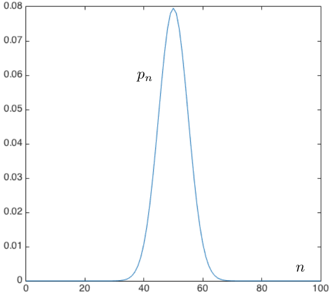

Event E_n = 'n heads'; $|E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

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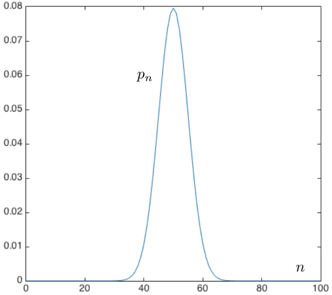

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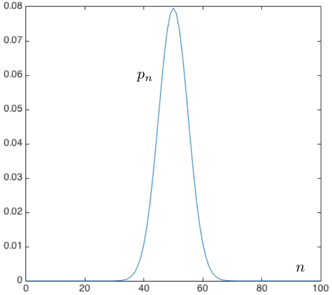

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Observe:

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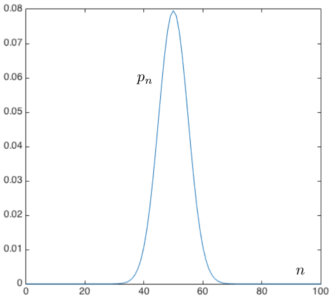

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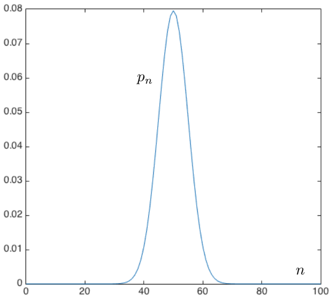

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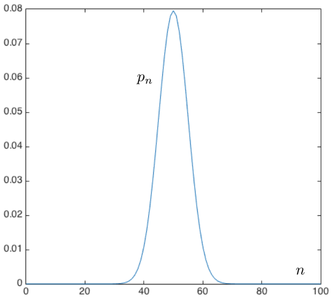

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Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape: Central Limit Theorem.

Roll a red and a blue die.

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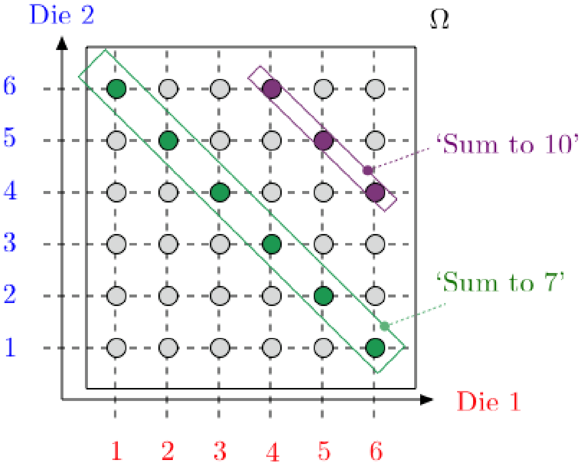

$Pr[\text{Sum to } 7] = \frac{6}{36}$ $Pr[\text{Sum to } 10] = \frac{3}{36}$

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses

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Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.
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Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.
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Event $E =$ "100 coin tosses with exactly 50 heads"

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Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H,T\}^{100}$.
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

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Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$

Calculation.

Stirling formula (for large n):

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}.$$

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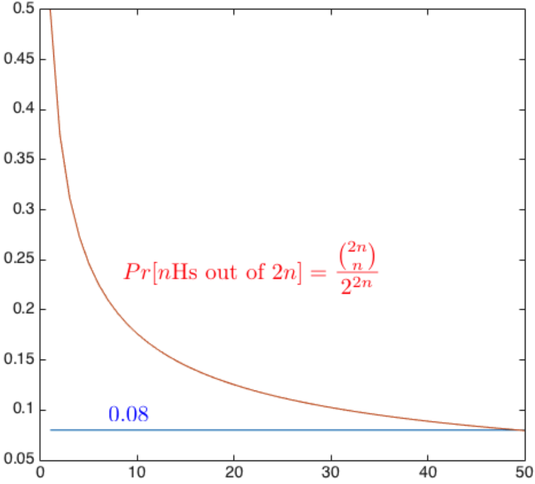
$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$

Exactly 50 heads in 100 coin tosses.



Lecture 13: Summary

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_{\omega} Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.
4. Events: subsets of Ω.
   $Pr[E] = \sum_{\omega \in E} Pr[\omega]$.