

Due Monday Jun 27th 1:59PM

1. (10 points) Classical logic

Here is an extract from Lewis Carroll's treatise *Symbolic Logic* of 1896:

- (I) No shark ever doubts that he is well fitted out.
- (II) A fish, that cannot dance a minuet, is contemptible.
- (III) No fish is quite certain that it is well fitted out, unless it has three rows of teeth.
- (IV) All fishes, except sharks, are kind to children.
- (V) No heavy fish can dance a minuet.
- (VI) A fish with three rows of teeth is not to be despised.

- (a) Write each of the above six sentences as a quantified proposition over the universe of all fish. You should use the following symbols for the various elementary propositions:  $P(x)$  for “ $x$  doubts he is well fitted out”,  $B(x)$  for “ $x$  is a shark”,  $F(x)$  for “ $x$  is contemptible” (also despised),  $U(x)$  for “ $x$  can dance a minuet”,  $O(x)$  for “ $x$  has three rows of teeth”,  $N(x)$  for “ $x$  is kind to children”,  $K(x)$  for “ $x$  is heavy”.
- (b) Now rewrite each proposition equivalently using the *contrapositive*.
- (c) You now have twelve propositions in total. What can you conclude from them about a fish who is certain that it is well fitted out? Explain clearly the implications you used to arrive at your conclusion.

2. (10 points) Propositional logic

For each the following logical equivalence assertions, either prove it is true or give a counterexample showing it is false (i.e., some choices of  $P$  and  $Q$  such that one side of the equivalence is true and the other is false), together with a one to two sentence justification that it is indeed a counterexample.

- (a)  $\forall x P(x) \equiv \neg \exists x \neg P(x)$
- (b)  $\forall x \exists y P(x, y) \equiv \forall y \exists x P(x, y)$
- (c)  $P \Rightarrow \neg Q \equiv \neg P \Rightarrow Q$
- (d)  $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q) \equiv P \Leftrightarrow Q$

3. (10 points) If you show up on time, you won't have to work this hard!

You show up late to CS 70 lecture and come in the middle of a complex derivation involving the propositions  $P$ ,  $Q$ , and  $R$ . From what you can see on the board, you're able to deduce that the following three propositions are true:  $P \Rightarrow \neg P$ ,  $Q \Rightarrow R$ ,  $P \vee Q \vee \neg R$ . Unfortunately, it looks like the definition of the propositions  $P, Q, R$  has already been erased.

- (a) Do you have enough information to deduce the truth value of  $P$ ? If yes, what is the truth value of  $P$ ?
- (b) Do you have enough information to deduce the truth value of  $Q$ ? If yes, what is the truth value of  $Q$ ?
- (c) David asks the class whether  $(\neg Q \wedge R) \vee (Q \wedge \neg R)$  is true. Do you have enough information to deduce the truth value of this proposition? If yes, what is its truth value?

4. (20 points) Grade these answers

You be the grader. Students have submitted the following answers to several exam questions. Assign each student answer either an A (correct yes/no answer, valid justification), a D (correct yes/no answer, invalid justification), or an F (incorrect answer). As always,  $\pi = 3.14159\dots$

- (a) **Exam question:** Is the following proposition true?  $2\pi < 100 \implies \pi < 50$ . Explain your answer.  
**Student answer:** Yes.  $2\pi = 6.283\dots$ , which is less than 100. Also  $\pi = 3.1459\dots$  is less than 50. Therefore the proposition is of the form True  $\implies$  True, which is true.
- (b) **Exam question:** Is the following proposition true?  $2\pi < 100 \implies \pi < 50$ . Explain your answer.  
**Student answer:** Yes. If  $2\pi < 100$ , then dividing both sides by two, we see that  $\pi < 50$ .
- (c) **Exam question:** Is the following proposition true?  $2\pi < 100 \implies \pi < 49$ . Explain your answer.  
**Student answer:** No. If  $2\pi < 100$ , then dividing both sides by two, we see that  $\pi < 50$ , which does not imply  $\pi < 49$ .
- (d) **Exam question:** Is the following proposition true?  $\pi^2 < 5 \implies \pi < 5$ . Explain your answer.  
**Student answer:** No, it is false.  $\pi^2 = 9.87\dots$ , which is not less than 5, so the premise is false. You can't start from a faulty premise.

5. (20 points) Liars and Truthtellers

You find yourself on a desert island inhabited by two types of people: the Liars and the Truthtellers. Liars always lie, and Truthtellers always tell the truth. In all other respects, the two types are indistinguishable.

- (a) You meet a very attractive local and ask him/her on a date. The local responds, "I will go on a date with you if and only if I am a Truthteller." Is this good news? Explain your answer with reference to logical notation.
- (b) You are trying to find your way to the lagoon and encounter a local inhabitant on the road. Which of the following questions could you ask him in order to reliably deduce whether you are on the correct path? In each case, explain your answer with reference to logical notation.
  - i. If I were to ask you "If this is the way to the lagoon, what would you say?"
  - ii. If I were to ask you, "If this is the way to the lagoon and you say "yes", can I believe you?"
  - iii. If I were to ask somebody of the other type than yours, "If this is the way to the lagoon, what would that person say?"
  - iv. "Is at least one of the following true? You are a Liar and this is the way to the lagoon; or you are a Truthteller and this is not the way to the lagoon."

6. (10 points) Proof by?

Prove that if  $x, y \in \mathbb{Z}$ , if 6 does not divide  $xy$ , then 6 does not divide  $x$  and 6 does not divide  $y$ . In notation:  $(\forall x, y \in \mathbb{Z}) \ 6 \nmid xy \implies (6 \nmid x \wedge 6 \nmid y)$ . What proof technique did you use?

7. (10 points) **Inductions!!**

- (a) Prove that  $3 + 11 + 19 + \dots + (8n - 5) = 4n^2 - n$  for all integers  $n \geq 1$ .
- (b) Prove that  $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$  for all integers  $n \geq 1$ .
- (c) Let  $a_0 = 1$  and  $a_n = 2a_{n-1} + 7$ . Prove that there is a constant  $C > 0$ , which does not depend on  $n$ , such that  $a_n \leq C \cdot 2^n$  for all  $n \in \mathbb{N}$ . *Hint:* Strengthen the induction hypothesis into  $a_n \leq C \cdot 2^n - D$  for some constant  $D$ . What value of  $D$  should you choose to make the proof easiest?

8. (5 points) Divergence of harmonic series

You may have seen the series  $1 + \frac{1}{2} + \frac{1}{3} + \dots$  in calculus. This is known as a *harmonic series*, and it diverges, i.e. the sum approaches infinity. We are going to prove this fact using induction.

Let  $H_j = \sum_{k=1}^j \frac{1}{k}$ . Use mathematical induction to show that, for all integers  $n \geq 0$ ,  $H_{2^n} \geq 1 + \frac{n}{2}$ , thus showing that  $H_j$  must grow unboundedly as  $j \rightarrow \infty$ .