CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Discussion 2D Sol

1. Stable Marriage

Consider the following list of preferences:

Men	Preferences
A	$4 > 2 > 1 > 3$
B	$2 > 4 > 3 > 1$
C	$4 > 3 > 1 > 2$
D	$3 > 1 > 4 > 2$

Women	Preferences
1	$A > D > B > C$
2	$D > C > A > B$
3	$C > D > B > A$
4	$B > C > A > D$

1. Is $\{(A,4),(B,2),(C,1),(D,3)\}$ a stable pairing?

   No. Rogue pair: $(C,3)$.

2. Find a stable matching by running the Traditional Propose & Reject algorithm.

   A men-optimal pairing: $\{(A,2),(B,4),(C,3),(D,1)\}$.

3. Show that there exist a stable matching where women 1 is matched to men A.

   A pairing can be: $\{(A,1),(B,4),(C,3),(D,2)\}$. This is stable because each woman gets their first preference. In other words, man A is the optimal man for woman 1, the best man can do in a pairing.

2. Objective Preferences Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking.

1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

   If the first woman is not paired with her first choice, then she and her first choice would form a rogue couple, because her first choice prefers her over any other woman, and vice versa.

2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

   If the first and second women have different first choices, then the second woman must be matched to her first choice. Otherwise she and her first choice would form a rogue couple (since her first choice is not matched to the first woman, he would prefer the second woman over his current match).

If the first choices are the same, then the second woman must be paired with her second choice, otherwise she and her second choice would form a rogue couple (neither of them are matched to their first choices, and they are each other's second choice).

3. Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the k-th woman be paired with?

The k-th woman should be paired with the first man on her list who has not been matched yet (with the first $k-1$ women). If she's not matched to him, they would form a rogue couple. This is because the man would have to be matched to a woman ranked worse than k, so she would prefer the k-th woman over his current partner, and the k-th woman obviously prefers him to whoever she's matched with.

4. Prove that there is a unique stable pairing.

In the previous parts, we saw that for each woman, given the pairs for the lower-ranked women, her pair would be determined uniquely. So there is only one stable pairing.
This can be stated and proved more rigorously using induction. Namely that there is a unique pairing for the first k women, assuming stability. An induction on k would prove this.

3. Examples or It's Impossible

Determine if each of the situations below is possible with the traditional propose-and-reject algorithm. If so, give an example of size $n \geq 3$. Otherwise, explain briefly why you think it's impossible.

1. Every man gets his first choice.

One way to construct the answer: Anything that every man has all different first choices. For example,

Men	Preferences
1	$A > C > B$
2	$B > C > A$
3	$C > A > B$

Women	Preferences
A	$3 > 2 > 1$
B	$1 > 3 > 2$
C	$2 > 1 > 3$

2. Every woman gets her first choice, even though her first choice does not prefer her the most.

An example where every woman gets her first choice, and every man his second choice:

Men	Preferences
1	$C > A > B$
2	$A > B > C$
3	$A > C > B$

Women	Preferences
A	$1 > 2 > 3$
B	$2 > 3 > 1$
C	$3 > 1 > 2$

3. Every woman gets her last choice.

One way to construct the answer: Anything that every woman has unique last choice, and her last choice man prefers her the most. For example,

Men	Preferences
1	$A > C > B$
2	$B > C > A$
3	$C > B > A$

Women	Preferences
A	$2 > 3 > 1$
B	$1 > 3 > 2$
C	$1 > 2 > 3$

4. Every man gets his last choice.

Impossible. By contradiction: On the last day, every man proposes to his unique least-favorite woman. So prior to the last day, every man has been rejected by all $n-1$ other women, and every woman has rejected all $n-1$ other men. But then every woman has rejected at least one man, meaning the algorithm should have terminated earlier. Contradiction.

5. A man who is second on every woman's list gets his last choice.

One way to construct the answer: Let M_i prefers W_i the most, and vice versa, for all $1 \leq i < n$. Let M_n be second on every woman's list. Let him prefer W_n the least. For example,

Men	Preferences
1	$A > C > B$
2	$B > C > A$
3	$A > B > C$

Women	Preferences
A	$1 > 3 > 2$
B	$2 > 3 > 1$
C	$1 > 3 > 2$

4. Pairing Up

Prove that for every even $n \geq 2$, there exists an instance of the stable marriage problem with n men and n women such that the instance has at least $2^{n/2}$ distinct stable matchings.

To prove that there exists such a stable marriage instance for any even $n \geq 2$, we just need to show how to construct such an instance.

The idea here is that we can create pairs of men and pairs of women: pair up man i and $i+1$ into a pair and woman i and $i+1$ into a pair (you might come to this idea since we are asked to prove this for *even* n).

For n, we have $n/2$ pairs. Choose the preference lists such that the kth pair of men rank the kth pair of women just higher than the $(k+1)$th pair of women (the pairs wrap around from the last pair to the first pair), and the kth pair of women rank the kth pair of men just higher than the $(k+1)$th pair of men. Within each pair of pairs (m,m') and (w,w'), let m prefer w, let m' prefer w', let w prefer m', and let w' prefer m. It might help to draw out an example on the board with arrows denoting preferences.

Each match will have men in the kth pair paired to women in the kth pair for $1 \leq k \leq n/2$.

A man m in pair k will never form a rogue couple with any woman w in pair $j \neq k$. If $j > k$, then w prefers her current partner in the jth pair to m. If $j < k$, then m prefers his current partner in the kth pair to w. Then a rogue couple could only exist in the same pair - but this is impossible since exactly one of either m or w must be married to their preferred choice in the pair.

Since each man in pair k can be stably married to either woman in pair k, and there are $n/2$ total pairs, the number of stable matchings is $2^{n/2}$.