1. Set Operations

- $\mathbb{R}$, the set of real numbers.
- $\mathbb{Q}$, the set of rational numbers: $\{\frac{a}{b} : a,b \in \mathbb{Z} \wedge b \neq 0\}$.
- $\mathbb{Z}$, the set of integers. $\{\ldots,-2,-1,0,1,2,\ldots\}$.
- $\mathbb{N}$, the set of natural numbers: $\{0,1,2,3,\ldots\}$.

(a) $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$

(b) $\mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$

(c) If $S \subseteq T$, what is $S \cap T$? S

(e) $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$

(f) $\mathbb{R} \cup \mathbb{R} = \mathbb{R}$

(g) If $S \subseteq T$, what is $S \cup T$? T

(h) $\mathbb{R} \backslash \mathbb{Q} =$ The set of irrational numbers.

(i) $\mathbb{Z} \backslash \mathbb{Q} = \emptyset$

(j) If $S \subseteq T$, what is $S \backslash T$? $\emptyset$

2. Sums and Products

(a) Evaluate: $\sum_{i=0}^{3} i^2 = 0+1+4+9 = 14$

(b) Evaluate: $\prod_{i=-1}^{2}(2^i) = 1/2 \times 1 \times 2 \times 4 = 4$

(c) True or false? $\sum_{i=0}^{2}\prod_{j=-1}^{1}(ij) = \prod_{j=-1}^{1}\sum_{i=0}^{2}(ij)$ True - both are zero.

(d) True or false? True or false? $\sum_{i=0}^{2}\prod_{j=-1}^{1}(i+j) = \prod_{j=-1}^{1}\sum_{i=0}^{2}(i+j)$ False - the left evaluates to 6 and the right to 0.

3. Boolean Logic

(a) Use truth tables to prove that $A \rightarrow B \equiv \neg(A \wedge \neg B)$.

A	B	$A \rightarrow B$	$\neg(A \wedge \neg B)$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

(b) Use truth tables to show that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$. These two equivalences are known as De Morgan's Law.

A	B	$\neg(A \vee B)$	$\neg A \wedge \neg B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$
0	0	1	1	1	1
0	1	0	0	1	1
1	0	0	0	1	1
1	1	0	0	0	0

4. Propositional Logic

(a) Write the following proposition in logical notation: Every nonzero rational number has a corresponding rational number such that the product of the two numbers is 1. $\forall x \in \mathbb{Q}, (x \neq 0) \rightarrow \exists y \in Q$, xy = 1

(b) There are no integer solutions to the equation $x^2 - y^2 = 10$. $\neg(\exists\, x, y \in \mathbb{Z}, x^2 - y^2 = 10)$

5. Challenge Question: Boolean Matrix Multiply

This is an "extra" question to do if the section has extra time, or if some people finish the earlier questions before section has ended and want to do more. It foreshadows a couple of things that might pop up later in the class, but is not strictly required knowledge.

Define an $m \times n$ *boolean matrix* as a grid of mn boolean variables (m rows, n columns). The variable at the ith row and jth column of boolean matrix A is denoted A_{ij}.

Define the *product* of an $m \times n$ boolean matrix A and and the $n \times p$ boolean matrix B to be the $m \times p$ boolean matrix C whose i, jth element is $C_{ij} = \bigvee_{k=0}^{n} \left(A_{ik} \wedge B_{kj}\right)$ (this notation means taking the boolean OR over a bunch of Boolean terms for $k = 0$ through n, similar to the sum and product notations).

For instance, suppose we define the matrix A as:

$$A = \begin{pmatrix} true & false \\ false & true \\ true & false \end{pmatrix}$$

and the matrix B as:

$$B = \begin{pmatrix} false & false \\ false & true \end{pmatrix}.$$

a) What is AB?

$$A = \begin{pmatrix} (true \wedge false) \vee (false \wedge false) & (true \wedge false) \vee (false \wedge true) \\ (false \wedge false) \vee (true \wedge false) & (false \wedge false) \vee (true \wedge true) \\ (true \wedge false) \vee (false \wedge false) & (true \wedge false) \vee (false \wedge true) \end{pmatrix} = \begin{pmatrix} false & false \\ false & true \\ false & false \end{pmatrix}$$

b) If I tell you that the ith row of A is entirely comprised of $false$ values, what can you tell me about the ith row of AB?

Directly from the definition, notice that $(AB)_{ij}$ must be false, because we know that A_{ik} is false for all k, so the big OR evaluates to false too. Therefore, the ith row of AB is entirely comprised of $false$ values.

c) What if I tell you that the ith row of A is entirely composed of $true$ values?

I can't fully determine the values this time. However, let's look at the definition of $(AB)_{ij}$ if A_{ik} is true for all k: $AB_{ij} = \bigvee_{k=0}^{n} \left(A_{ik} \wedge B_{kj}\right) = \bigvee_{k=0}^{n} \left(true \wedge B_{kj}\right)$. Recall that AND-ing true to any boolean variable x is just x. Therefore, this reduces to $(AB)_{ij} = \bigvee_{k=0}^{n} \left(B_{kj}\right)$, which means that the ijth element of AB is true if and only if there exists some element in column j of B that is true.

We can also define matrices over real numbers similarly (where the values in the grid are real numbers instead of booleans), with products of $W = XY$ of matrices X, Y over real numbers being defined as $W_{ij} = \sum_{k=0}^{n} \left(X_{ik} Y_{kj}\right)$. (If you've been exposed to linear algebra before, this definition should be familiar to you.)

d) Suppose I give you two Boolean matrices A, B. Can you give me two matrices over real numbers, X, Y such that I can obtain AB easily from XY?

Let X be 0 at all locations where A is false and 1 at all locations where A is true, and let Y be defined similarly for B. Then AB is false at locations where XY is zero and true at locations where XY is nonzero. Note that this also works if you replace 1 with an arbitrary positive number.