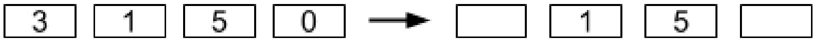
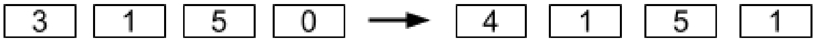
CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Note 23

Error Correcting Codes

We will consider two situations in which we wish to transmit information on an unreliable channel. The first is exemplified by the internet, where the information (say a file) is broken up into packets, and the unreliability is manifest in the fact that some of the packets are lost (or erased) during transmission. Moreover the packets are labeled so that the recipient knows exactly which packets were received and which were dropped. We will refer to such errors as erasure errors. See the figure below:



In the second situation, some of the packets are corrupted during transmission due to channel noise. Now the recipient has no idea which packets were corrupted and which were received unmodified:



In the above example, packets 1 and 4 are corrupted. These types of errors are called general errors. We will discuss methods of encoding messages, called error correcting codes, which are capable of correcting both erasure and general errors.

Assume that the information consists of n packets. We can assume without loss of generality that the contents of each packet is a number modulo q (denoted by $GF(q)$), where q is a prime. For example, the contents of the packet might be a 32-bit string and can therefore be regarded as a number between 0 and $2^{32}-1$; then we could choose q to be any prime larger than 2^{32}. The properties of polynomials over $GF(q)$ (i.e., with coefficients and values reduced modulo q) are the backbone of both error-correcting schemes. To see this, let us denote the message to be sent by $m_1,\ldots,m_n$ and make the following crucial observations:

1) There is a unique polynomial $P(x)$ of degree $n-1$ such that $P(i)=m_i$ for $1\leq i\leq n$ (i.e., $P(x)$ contains all of the information about the message, and evaluating $P(i)$ gives the contents of the *i-th* packet).

2) The message to be sent is now $m_1=P(1),\ldots,m_n=P(n)$. We can generate additional packets by evaluating $P(x)$ at additional points $n+1,n+2,\ldots,n+j$ (remember, our transmitted message must be redundant, i.e., it must contain more packets than the original message to account for the lost or corrupted packets). Thus the transmitted message is $c_1=P(1),c_2=P(2),\ldots,c_{n+j}=P(n+j)$. Since we are working modulo q, we must make sure that $n+j\leq q$, but this condition does not impose a serious constraint since q is very large.

Erasure Errors

Here we consider the setting of packets being transmitted over the internet. In this setting, the packets are labeled and so the recipient knows exactly which packets were dropped during transmission. One additional observation will be useful:

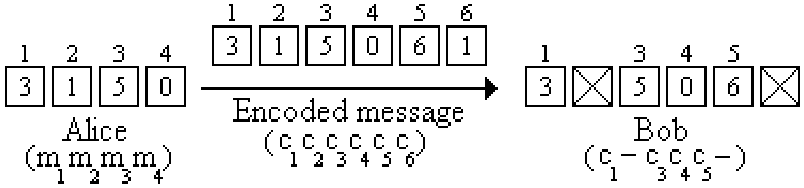
3) By Property 2 in Note 8, we can uniquely reconstruct $P(x)$ from its values at any n distinct points, since it has degree $n-1$. This means that $P(x)$ can be reconstructed from any n of the transmitted packets. Evaluating this reconstructed polynomial $P(x)$ at $x = 1,\ldots,n$ yields the original message $m_1,\ldots,m_n$.

Recall that in our scheme, the transmitted message is $c_1 = P(1), c_2 = P(2), \ldots, c_{n+j} = P(n+j)$. Thus, if we hope to be able to correct k errors, we simply need to set $j = k$. The encoded message will then consist of $n+k$ packets.

Example

Suppose Alice wants to send Bob a message of $n = 4$ packets and she wants to guard against $k = 2$ lost packets. Then, assuming the packets can be coded up as integers between 0 and 6, Alice can work over $GF(7)$ (since $7 \geq n+k = 6$). Suppose the message that Alice wants to send to Bob is $m_1 = 3$, $m_2 = 1$, $m_3 = 5$, and $m_4 = 0$. She interpolates to find the unique polynomial of degree $n-1 = 3$ described by these 4 points: $P(x) = x^3 + 4x^2 + 5$ (verify that $P(i) = m_i$ for $1 \leq i \leq 4$).

Since $k = 2$, Alice must evaluate $P(x)$ at 2 extra points: $P(5) = 6$ and $P(6) = 1$. Now, Alice can transmit the encoded message which consists of $n+k = 6$ packets, where $c_j = P(j)$ for $1 \leq j \leq 6$. So $c_1 = P(1) = 3$, $c_2 = P(2) = 1$, $c_3 = P(3) = 5$, $c_4 = P(4) = 0$, $c_5 = P(5) = 6$, and $c_6 = P(6) = 1$. Suppose packets 2 and 6 are dropped, in which case we have the following situation:



From the values that Bob received (3, 5, 0, and 6), he uses Lagrange interpolation and computes the following delta functions:

$$\begin{aligned}
\Delta_1(x) &= \frac{(x-3)(x-4)(x-5)}{-24} \\
\Delta_3(x) &= \frac{(x-1)(x-4)(x-5)}{4} \\
\Delta_4(x) &= \frac{(x-1)(x-3)(x-5)}{-3} \\
\Delta_5(x) &= \frac{(x-1)(x-3)(x-4)}{8}.
\end{aligned}$$

He then reconstructs the polynomial $P(x) = (3)\Delta_1(x) + (5)\Delta_3(x) + (0)\Delta_4(x) + (6)\Delta_5(x) = x^3 + 4x^2 + 5$. Bob then evaluates $m_2 = P(2) = 1$, which is the packet that was lost from the original message. More generally, no matter which two packets were dropped, following the same method Bob could still have reconstructed $P(x)$ and thus the original message.

Let us consider what would happen if Alice sent one fewer packet. If Alice only sent c_j for $1 \leq j \leq n+k-1$, then with k erasures, Bob would only receive c_j for $n-1$ distinct values j. Thus, Bob would not be able to reconstruct $P(x)$ (since there are exactly q polynomials of degree at most $n-1$ that agree with the $n-1$ packets which Bob received). This error-correcting scheme is therefore optimal: it can recover the n characters of the transmitted message from any n received characters, but recovery from any fewer characters is impossible.

Polynomial Interpolation

Let us take a brief digression to discuss another method of polynomial interpolation which will be useful in handling general errors. The goal of the algorithm will be to take as input $d+1$ pairs $(x_1,y_1),\cdots,(x_{d+1},y_{d+1})$, and output the polynomial $p(x)=a_dx^d+\cdots+a_1x+a_0$ such that $p(x_i)=y_i$ for $i=1$ to $d+1$.

The first step of the algorithm is to write a system of $d+1$ linear equations in $d+1$ variables: the coefficients of the polynomial $a_0,\ldots,a_d$. Each equation is obtained by fixing x to be one of $d+1$ values: $x_1,\cdots,x_{d+1}$. Note that in $p(x)$, x is a variable and $a_0,\ldots,a_d$ are fixed constants. In the equations below, these roles are swapped: x_i is a fixed constant and $a_0,\ldots,a_d$ are variables. For example, the *i-th* equation is the result of fixing x to be x_i: $a_dx_i^d+a_{d-1}x_i^{d-1}+\ldots+a_0=y_i$.

Now solving these equations gives the coefficients of the polynomial $p(x)$. For example, given the 3 pairs $(-1,2)$, $(0,1)$, and $(2,5)$, we will construct the degree 2 polynomial $p(x)$ which goes through these points. The first equation says $a_2(-1)^2+a_1(-1)+a_0=2$. Simplifying, we get $a_2-a_1+a_0=2$. Similarly, the second equation says $a_2(0)^2+a_1(0)+a_0=1$, or $a_0=1$. And the third equation says $a_2(2)^2+a_1(2)+a_0=5$ So we get the following system of equations:

$$\begin{aligned}
a_2-a_1+a_0&=2\\
a_0&=1\\
4a_2+2a_1+a_0&=5
\end{aligned}$$

Substituting for a_0 and multiplying the first equation by 2 we get:

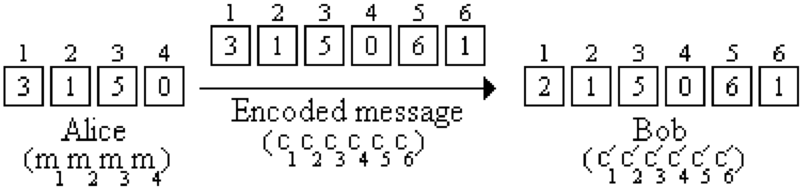
$$\begin{aligned}
2a_2-2a_1&=2\\
4a_2+2a_1&=4
\end{aligned}$$

Then, adding the two equations we find that $6a_2=6$, so $a_2=1$, and plugging back in we find that $a_1=0$. Thus, we have determined the polynomial $p(x)=x^2+1$. To justify this method more carefully, we must show that the equations always have a solution and that it is unique. This involves showing that a certain determinant is non-zero, which we will leave as an exercise.

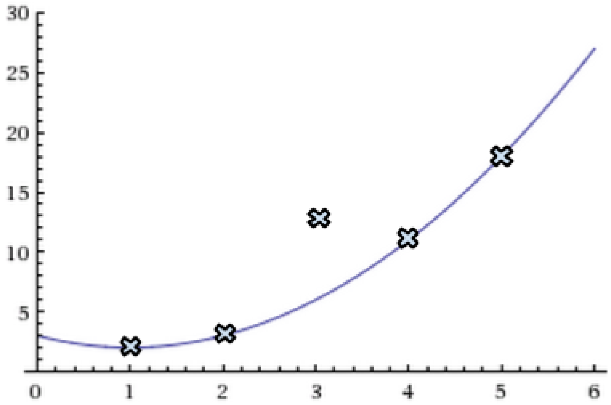
General Errors

Now let us return to general errors. General errors are much more challenging to correct than erasure errors. This is because packets are corrupted, not erased and Bob no longer knows which packets are correct. As we shall see shortly, Alice can still guard against k general errors, at the expense of transmitting only $2k$ additional packets or characters (only twice as many as in the erasure case). Thus the encoded message is $c_1,\ldots,c_{n+2k}$ where $c_j=P(j)$ for $1\le j\le n+2k$. This means that at least $n+k$ of these characters are received uncorrupted by Bob.

For example, if Alice wishes to send $n=4$ characters to Bob via a modem in which $k=1$ of the characters is corrupted, she must redundantly send an encoded message consisting of 6 characters. Suppose she wants to transmit the same message as above, and that c_1 is corrupted and changed to $r_1=2$. This scenario can be visualized in the following figure:



Bob's goal is to reconstruct $P(x)$ from the $n+2k$ received characters $r_1, \cdots, r_{n+2k}$. He knows that $P(i)$ must equal r_i on at least $n+k$ points (since only k points are corrupted), but he does not know which of the $n+k$ values are correct. As an example, consider a possible scenario depicted in the picture below- the points represent the message received from Alice, and the line represents $P(x)$. In this example, $n=3$, $k=1$, and the third packet is corrupted. Bob does not know the index at which the message and the polynomial deviate:



Bob attempts to construct $P(x)$ by searching for a polynomial $P'(x)$ with the following property: $P'(i) = r_i$ for at least $n+k$ distinct values of i between 1 and $n+2k$. Of course, $P(x)$ is one such polynomial. It turns out that $P(x)$ is actually the only polynomial with the desired property. Therefore, $P'(x)$ must equal $P(x)$.

Finding $P(x)$ efficiently requires a remarkable idea, which is just about simple enough to be described here. Suppose packets $e_1, \ldots, e_k$ are corrupted. Define the degree k polynomial $E(x)$ to be $(x-e_1)\cdots(x-e_k)$. Let us make a simple but crucial observation: $P(i)E(i) = r_iE(i)$ for $1 \leq i \leq n+2k$ (this is true at points i at which no error occurred since $P(i) = r_i$, and trivially true at points i at which an error occurred since $E(i) = 0$).

This observation forms the basis of a very clever algorithm invented by Berlekamp and Welch. Looking more closely at these equalities, we will show that they are $n+2k$ linear equations in $n+2k$ unknowns. The unknowns correspond to the coefficients of $E(x)$ and $Q(x)$ (where we define $Q(x) = P(x)E(x)$). Once $Q(x)$ and $E(x)$ are known, we can divide $Q(x)$ by $E(x)$ to obtain $P(x)$.

Since $Q(x)$ is a polynomial of degree $n+k-1$, it can be described by $n+k$ coefficients. $E(x)$ is a degree k polynomial, but its definition implies that its first coefficient must be 1. It can therefore be described by k coefficients:

$$Q(x) = a_{n+k-1}x^{n+k-1} + \cdots + a_1x + a_0$$
$$E(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0$$

As seen in the interpolation method above, once we fix a value i for x, $Q(i)$ and $E(i)$ are linear functions of $a_{n+k-1}, \cdots, a_0$ and $b_{k-1}, \cdots, b_0$ respectively. The received value r_i is also fixed. Therefore the equation $Q(i) = r_iE(i)$ is a linear equation in the $n+2k$ unknowns $a_{n+k-1}, \ldots, a_0$ and $b_{k-1}, \ldots, b_0$. We thus have $n+2k$ linear equations, one for each value of i, and $n+2k$ unknowns. We can solve these equations and get $E(x)$ and $Q(x)$. We can then compute the ratio $\frac{Q(x)}{E(x)}$ to obtain $P(x)$.

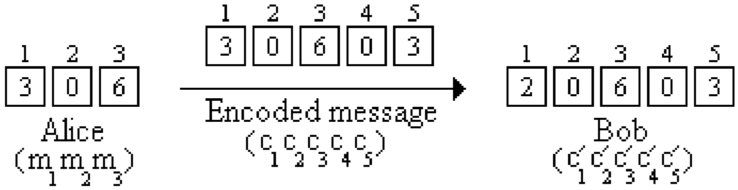
Example

Suppose we are working over $GF(7)$ and Alice wants to send Bob the $n = 3$ characters "3," "0," and "6" over a modem. Turning to the analogy of the English alphabet, this is equivalent to using only the first 7 letters of the alphabet, where $a = 0, b = 1, \ldots, g = 6$. So the message which Alice wishes for Bob to receive is "dag". Then Alice interpolates to find the polynomial

$$P(x) = x^2 + x + 1,$$

which is the unique polynomial of degree 2 such that $P(1) = 3$, $P(2) = 0$, and $P(3) = 6$.

Suppose that $k = 1$ character is corrupted, so she needs to transmit the $n + 2k = 5$ characters $P(1) = 3$, $P(2) = 0$, $P(3) = 6$, $P(4) = 0$, and $P(5) = 3$ to Bob. Suppose $P(1)$ is corrupted, so he receives 2 instead of 3 (i.e., Alice sends the encoded message "dagad" but Bob instead receives "cagad"). Summarizing, we have the following situation:



Let $E(x) = x + b_0$ be the error-locator polynomial—remember, Bob doesn't know what b_0 is yet since he doesn't know where the error occurred. Let $Q(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Now Bob just substitutes $x = 1, x = 2, \cdots, x = 5$ into $Q(x) = r_xE(x)$ and simplifies to get five linear equations in five unknowns. Recall that we are working modulo 7 and that $r_i = c'_i$ is the value Bob received for the *i-th* character.

The first equation will be $a_3 + a_2 + a_1 + a_0 = 2(1 + b_0)$, which simplifies to $a_3 + a_2 + a_1 + a_0 + 5b_0 = 2$. Bob can determine the remaining equations in the same manner, obtaining:

$$\begin{aligned}
a_3 + a_2 + a_1 + a_0 + 5b_0 &= 2 \\
a_3 + 4a_2 + 2a_1 + a_0 &= 0 \\
6a_3 + 2a_2 + 3a_1 + a_0 + b_0 &= 4 \\
a_3 + 2a_2 + 4a_1 + a_0 &= 0 \\
6a_3 + 4a_2 + 5a_1 + a_0 + 4b_0 &= 1
\end{aligned}$$

Bob then solves this linear system and finds that $a_3 = 1$, $a_2 = 0$, $a_1 = 0$, $a_0 = 6$, and $b_0 = 6$ (all mod 7). (As a check, this implies that $E(x) = x+6 = x-1$, so the location of the error is position $e_1 = 1$, which is correct since the first character was corrupted from a "d" to a "c".) This gives him the polynomials $Q(x) = x^3 + 6$ and $E(x) = x - 1$. He can then find $P(x)$ by computing the quotient $P(x) = \frac{Q(x)}{E(x)} = \frac{x^3+6}{x-1} = x^2 + x + 1$. Bob notices that the first character was corrupted (since $e_1 = 1$), so now that he has $P(x)$, he just computes $P(1) = 3 =$ "d" and obtains the original, uncorrupted message "dag".

Finer points

Two points need further discussion. How do we know that the $n + 2k$ equations are consistent? What if they have no solution? This is simple. The equations must be consistent since $Q(x) = P(x)E(x)$ together with the error locator polynomial $E(x)$ gives a solution.

A more interesting question is this: how do we know that the $n+2k$ equations are independent, i.e., how do we know that there aren't other spurious solutions in addition to the real solution that we are looking for? Put more mathematically, how do we know that the solution $Q'(x)$ and $E'(x)$ that we reconstruct satisfies the property that $E'(x)$ divides $Q'(x)$ and that $\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x)$?

We claim that $Q(x)E'(x) = Q'(x)E(x)$ for $1 \leq x \leq n+2k$. Since the degree of both $Q(x)E'(x)$ and $Q'(x)E(x)$ is $n+2k-1$ and they are equal at $n+2k$ points, it follows from Property 2 of Note 7 that they are the same polynomial. Rearranging, we get $\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x)$.

Why is the claim above true? Based on our method of obtaining $Q'(x)$ and $E'(x)$, we know that $Q'(i) = r_iE'(i)$ and $Q(i) = r_iE(i)$. Now assume $E(i)$ is 0. Then $Q(i)$ is also 0, so both $Q(i)E'(i)$ and $Q'(i)E(i)$ are 0 and the claim holds. The same reasoning applies when $E'(i)$ is 0. If both $E(i)$ and $E'(i)$ are not 0, we can rearrange the above equality to obtain $\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)}$, which implies the claim.