Due Thursday February 4th at 10PM

1. (5 points)

   Use induction to prove that for all positive integers n, all of the entries in the matrix

   $$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^n$$

   are $\leq 3n$.

   Answer:

   Before starting the proof, writing out the first few powers reveals a telling pattern:

   $$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$
   $$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$$
   $$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix}$$

   It appears (and we shall soon prove) that the upper left and lower right entries are always 1, the upper right entry is always 0, and the lower left entry is $3n$. We shall take this to by our inductive hypothesis. **Proof**: We shall use a proof by induction that the upper left and lower right entries of the matrix are always 1, the upper right entry is always 0, and the lower left entry is $3n$. This will prove that all entries in the matrix are less than or equal to $3n$ for all $n \geq 1$. The base case of $n = 1$ is trivially true. Now suppose that our proposition is true for some $n \geq 1$, meaning

   $$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix}$$

   for some $n \geq 1$. Multiplying both sides of the equation by the original matrix yields

   $$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 0 \\ 3n+3 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3(n+1) & 1 \end{pmatrix}$$

   By the principle of induction, our proposition is therefore true for all $n \geq 1$, so all entries in the matrix will be less than or equal to $3n$. ■

2. (5 points) Divergence of harmonic series

   You may have seen the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ in calculus. This is known as a *harmonic series*, and it diverges, i.e. the sum approaches infinity. We are going to prove this fact using induction.

Let $H_j = \sum_{k=1}^{j} \frac{1}{k}$. Use mathematical induction to show that, for all integers $n \geq 0$, $H_{2^n} \geq 1 + \frac{n}{2}$, thus showing that H_j must grow unboundedly as $j \to \infty$. **Answer:**

Base case: $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$

Inductive Step: Assume that $H_{2^n} \geq 1 + \frac{n}{2}$. Then:

$$
\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \ldots + \frac{1}{2^{k+1}} \\
&= (1 + \frac{1}{2} + \ldots + \frac{1}{2^k}) + (\frac{1}{2^k+1} + \ldots + \frac{1}{2^{k+1}}) \\
&= H_{2^k} + (\frac{1}{2^k+1} + \ldots + \frac{1}{2^{k+1}})
\end{aligned}
$$

By noting that $(\frac{1}{2^k+1} + \ldots + \frac{1}{2^{k+1}})$ has 2^k terms, each of which is at least $\frac{1}{2^{k+1}}$

$$
\geq H_{2^k} + 2^k * \frac{1}{2^{k+1}}
$$

By the inductive hypothesis:

$$
\begin{aligned}
&\geq 1 + \frac{k}{2} + 2^k * \frac{1}{2^{k+1}} \\
&= 1 + \frac{k}{2} + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}
$$

Hence we have proved the statement by induction, and can conclude that H_{2^n} must go to infinity as $n \to \infty$, hence H_n must be diverging as $n \to \infty$.

3. (10 Points)

Prove that for every positive integer k, the following is true:

> For every real number $r > 0$, there are only finitely many solutions in positive integers to $\frac{1}{n_1} + \cdots + \frac{1}{n_k} = r$.

In other words, there exists some number m (that depends on k and r) such that there are at most m ways of choosing a positive integer n_1, and a (possibly different) positive integer n_2, etc., that satisfy the equation.

Hint: You can assume $n_1 \leq n_2 \leq \ldots \leq n_k$ without losing generality. (Why? Think about it)

Answer: **Claim**: $\forall k \in \mathbf{Z}\ \forall r \in \mathbf{R}\ \big((k > 0 \wedge r > 0) \Rightarrow (\text{There are finitely many solutions to } \frac{1}{n_1} + \cdots + \frac{1}{n_k} = r, n_i \in \mathbf{Z}, n_i > 0)\big)$
Proof: We will prove this by induction on k. For our base case, $k = 1$. In the base case, iff r can be written as $\frac{1}{n_1}$ when n_1 is a positive integer, then there is exactly one solution, $n_1 = \frac{1}{r}$. If r cannot be written in that form, then there are exactly zero solutions. In all cases, there is a finite number of solutions. For the inductive hypothesis, assume that there are finitely many solutions for some $k \geq 1$ for all r. Each real number r_1 either can or cannot be written as the sum of $k+1$ integers' inverses. If r_1 cannot be written in that form, then there are exactly zero solutions. If r_1 can be written in that

form, then the integers' inverses can be ordered. Since r_1 is the sum of $k+1$ integers' inverses, the largest $\frac{1}{n_i}$ must be at least $\frac{r_1}{k+1}$. This means that the smallest n_i must be at most $\frac{k+1}{r_1}$, which means that the smallest n_i has finitely many possible values. For each of the possible smallest n_i values, there is a real number $r_1 - \frac{1}{n_i}$ that can be written as the sum of k integers' inverses in finitely many ways (using the induction hypothesis). This means that there are only finitely many possible solutions for $k+1$ (combining all solutions (finitely many) for each possible smallest n_i values (finitely many)). By the principle of induction, there are finitely many solutions for all k for all r.

4. (12 points: 3 each) Objective Preferences

   Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking.

   (a) Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

   Answer: If the first woman is not paired with her first choice, then she and her first choice would form a rogue couple, because her first choice prefers her over any other woman, and vice versa.

   (b) Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

   Answer: If the first and second women have different first choices, then the second woman must be matched to her first choice. Otherwise she and her first choice would form a rogue couple (since her first choice is not matched to the first woman, he would prefer the second woman over his current match).

   If the first choices are the same, then the second woman must be paired with her second choice, otherwise she and her second choice would form a rogue couple (neither of them are matched to their first choices, and they are each other's second choice).

   (c) Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the k-th woman be paired with?

   Answer: The k-th woman should be paired with the first man on her list who has not been matched yet (with the first $k-1$ women). If she's not matched to him, they would form a rogue couple. This is because the man would have to be matched to a woman ranked worse than k, so she would prefer the k-th woman over his current partner, and the k-th woman obviously prefers him to whoever she's matched with.

   (d) Prove that there is a unique stable pairing.

   Answer: In the previous parts, we saw that for each woman, given the pairs for the lower-ranked women, her pair would be determined uniquely. So there is only one stable pairing.

   This can be stated and proved more rigorously using induction. Namely that there is a unique pairing for the first k women, assuming stability. An induction on k would prove this.

5. (15 points:3/3/4/5)

   You have been asked to assign TAs for the fall semester. Each class has its own method for ranking candidates, and each candidate has their own preferences. An assignment is **unstable** if a class and a candidate prefer each other to their current assignments. Otherwise, it is **stable**.

   Candidate information:

Candidate	CS61C Grade	CS70 Grade	CS61A Grade	Teaching Experience	Overall GPA	Preferences
A	A+	A	A	Yes	3.80	CS61C > CS70 > CS61A
B	A	A	A	No	3.61C	CS61C > CS61A > CS70
C	A	A+	A-	Yes	3.60	CS61C > CS70 > CS61A

Ranking method:

- CS61C: Rank by CS61C grade. Break ties using teaching experience, then overall GPA.
- CS70: Rank by teaching experience. Break ties using CS70 grade, then overall GPA.
- CS61A: Rank by CS61A grade. Break ties using overall GPA, then teaching experience.

a) Find a stable assignment.

b) Can you find another, or is there only one stable assignment (if there is only one, why)?

CS61C is overenrolled and needs two TAs. There is another candidate.

Candidate	CS61C Grade	CS70 Grade	CS61A Grade	Teaching Experience	Overall GPA	Preference
D	A+	A	A+	No	3.90	CS70 > CS61A > CS61C

c) Find a stable assignment.

d) Prove your assignment in Part (c) is stable.

Answer:

(a) Use SMA with the following class rankings and students proposing.

CS61C	A > C > B
70	C > A > B
61A	A > B > C

Day	CS61C	CS70	CS61A
1	A, B, C		
2	A	C	B

Assignment: (CS61C, A), (CS70, C), (CS61A, B).

(b) Run SMA with classes proposing and get the same assignment.

Day	A	B	C
1	61C,61A		70
2	61C	61A	70

For any stable assignment, if a student is paired with a class C', the student must prefer his/her optimal class to C' and prefer C' to his/her pessimal class. When students proposed, SMA outputs the student optimal assignment. When classes proposed, SMA outputs the class optimal assignment, which is the student pessimal assignment. Because the student optimal assignment is the same as the student pessimal assignment, C' can only be the class in the assignment for any student, and thus there can only be one stable assignment.

(c) Use SMA with students proposing and rule that CS61C can hold 2 proposals, with the following class rankings:

CS61C	A > D > C > B
CS70	C > A > D > B
CS61A	D > A > B > C

Day	CS61C	CS70	CS61A
1	A, B, C	D	
2	A, C	D	B

Assignment: (CS61C, A and C), (CS70, D), (CS61A, B).

(d) Follow the stability proof in the lecture note to prove that the assignment is stable. Suppose some TA T in the assignment prefers some class C^* to their assigned class C. We will argue that C^* prefers their TA(s) to T, so there cannot be a rogue couple (a class and a TA prefer each other to their current assignments). Since C^* occurs before C in T's list, he must have proposed to it before he proposed to C. Therefore, according to the algorithm, C^* must have rejected him for somebody it prefers. If C^* is not CS61C, the Improvement Lemma shows C^* likes its final TA at least as much as T. If C^* is CS61C, then we must now prove an alternate Improvement Lemma to show this.

Prove: If T is rejected by CS61C on the k-th day, then every subsequent day $j \geq k$, CS61C has 2 TAs whom it likes at least as much as T.

- *Base case*: On day k, CS61C rejects T, so it must prefer the two TAs it holds.
- *Induction hypothesis*: Suppose claim is true for $j \geq k$
- *Induction step*: On day $j+1$, by induction hypothesis, CS61C has 2 TAs T' and T'' it prefers to T. Either nobody proposes to CS61C, or T''' proposes. If T''' is accepted, then it must be preferred over T' or T'', which are both at least as good as T, so T''' is preferred over T.

After proving the alternate Improvement Lemma, we can claim C^* likes its final TA at least as much as T. Therefore, no TA T can be involved in a rogue couple, and thus the assignment is stable.

6. (20 points:10/10) **(Better Off Alone)**

In the stable marriage problem, suppose that some men and women have standards and would not just settle for anyone. In other words, in addition to the preference orderings they have, they prefer being alone to being with some of the lower-ranked individuals (in their own preference list). A pairing could ultimately have to be partial, i.e., some individuals would remain single.

The notion of stability here should be adjusted a little bit. A pairing is stable if

- there is no paired individual who prefers being single over being with his/her current partner,
- there is no paired man and paired woman that would both prefer to be with each other over their current partners, and
- there is no single man and single woman that would both prefer to be with each other over being single.

(a) (10 points) Prove that a stable pairing still exists in the case where we allow single individuals. You can approach this by introducing imaginary mates that people "marry" if they are single. How should you adjust the preference lists of people, including those of the newly introduced imaginary ones for this to work?

Answer: Following the hint, we introduce an imaginary mate (let's call it a robot) for each person. Note that we introduce one robot for each individual person, i.e. there are as many robots as there are people. For simplicity let us say each robot is owned by the person we introduce it for.

Each robot is in love with its owner, i.e. it puts its owner at the top of its preference list. The rest of its preference list can be arbitrary. The owner of a robot puts it in his/her preference list exactly after the last person he/she is willing to marry. i.e. owners like their robots more than people they are not willing to marry, but less than people they like to marry. The ordering of people who someone does not like to marry as well as robots he/she does not own is irrelevant as long as they all come after their robot.

To illustrate, consider this simple example: there are three men $1,2,3$ and three women A,B,C. The preference lists for men is given below:

Man	Preference List
1	$A > B$
2	$B > A > C$
3	C

and the following depicts the preference lists for women:

Woman	Preference List
A	1
B	$3 > 2 > 1$
C	$2 > 3 > 1$

In this example, 1 is willing to marry A and B and he likes A better than B, but he'd rather be single than to be with C. On the other side B has a low standard and does not like being single at all. She likes 3 first, then 2, then 1 and if there is no option left she is willing to be forced into singleness. On the other hand, A has pretty high standards. She either marries 1 or remains single.

According to our explanation we should introduce a robot for each person. Let's name the robot owned by person X as R_X. So we introduce male robots R_A, R_B, R_C and female robots R_1, R_2, R_3. Now we should modify the existing preference lists and also introduce the preference lists for robots.

According to our method, 1's preference list should begin with his original preference list, i.e. $A > B$. Then comes the robot owned by 1, i.e. R_1. The rest of the ordering, which should include C and R_2, R_3 does not matter, and can be arbitrary.

For B, the preference list should begin with $3 > 2 > 1$ and continue with R_B, but the ordering between the remaining robots (R_A and R_C) does not matter.

What about robots' preference lists? They should begin with their owners and the rest does not matter. So for example R_A's list should begin with A, but the rest of the humans/robots (B, C, R_1, R_2, and R_3) can come in any arbitrary order.

So the following is a list of preference lists that adhere to our method. There are arbitrary choices which are shown in bold (everything in bold can be reordered within the bold elements).

Man	Preference List
1	$A > B > R_1 > \mathbf{3} > \mathbf{R_3} > \mathbf{R_2}$
2	$B > A > C > R_2 > \mathbf{R_1} > \mathbf{R_3}$
3	$C > R_3 > \mathbf{R_1} > \mathbf{R_3} > \mathbf{A} > \mathbf{B}$
R_A	$A > \mathbf{B} > \mathbf{C} > \mathbf{R_1} > \mathbf{R_2} > \mathbf{R_3}$
R_B	$B > \mathbf{R_1} > \mathbf{R_2} > \mathbf{R_3} > \mathbf{A} > \mathbf{C}$
R_C	$C > \mathbf{A} > \mathbf{R_2} > \mathbf{B} > \mathbf{R_1} > \mathbf{R_3}$

and the following depicts the preference lists for women and female robots:

Woman	Preference List
A	$1 > R_A > \mathbf{3} > \mathbf{R_B} > \mathbf{2} > \mathbf{R_C}$
B	$3 > 2 > 1 > R_B > \mathbf{R_C} > \mathbf{R_A}$
C	$2 > 3 > 1 > R_C > \mathbf{R_A} > \mathbf{R_B}$
R_1	$1 > \mathbf{R_B} > \mathbf{2} > \mathbf{R_C} > \mathbf{3} > \mathbf{R_A}$
R_2	$2 > \mathbf{R_A} > \mathbf{R_C} > \mathbf{1} > \mathbf{3} > \mathbf{R_B}$
R_3	$3 > \mathbf{2} > \mathbf{1} > \mathbf{R_A} > \mathbf{R_C} > \mathbf{R_B}$

Now let us prove that a stable pairing between robots and owners actually corresponds to a stable pairing (with singleness as an option). This will finish the proof, since we know that in the robots and owners case, the propose and reject algorithm will give us a stable matching.

It is obvious that to extract a pairing without robots, we should simply remove all pairs in which there is at least one robot (two robots can marry each other, yes). Then each human who is not matched is declared to be single. It remains to check that this is a stable matching (in the new, modified sense). Before we do that, notice that a person will never be matched with another person's robot, because if that were so he/she and his/her robot would form a rogue couple (the robot's love is there, and the owner actually likes his/her robot more than other robots).

i. No one who is paired would rather break out of his/her pairing and be single. This is because if that were so, that person along with its robot would have formed a rogue couple in the original pairing. Remember, the robot loves its owner more than anything, so if the owner likes it more than his/her mate too, they would be a rogue couple.

ii. There is no rogue couple. If a rogue couple m and w existed, they would also be a rogue couple in the pairing which includes robots. If neither m nor w is single, this is fairly obvious. If one or both of them are single, they prefer the other person over being single, which in the robots scenario means they prefer being with each other over being with their robot(s) which is their actual match.

This shows that each stable pairing in the robots and humans setup gives us a stable pairing in the humans-only setup. It is noteworthy that the reverse direction also works. If there is a stable pairing in the humans-only setup, one can extend it to a pairing for robots and humans setup by first creating pairs of owners who are single and their robots, and then finding an arbitrary stable matching between the unmatched robots (i.e. we exclude everything other than the unmatched robots and find a stable pairing between them). To show why this works, we have to refute the possibility of a rogue pair. There are three cases:

i. A human-human rogue pair. This would also be rogue pair in the humans-only setup. The humans prefer each other over their current matches. If their matches are robots, that translates to them preferring each other over being single in the humans-only setup.

ii. A human-robot rogue pair. If the human is matched to his/her robot, our pair won't be a rogue pair since a human likes his/her robot more than any other robot. On the other hand if the human is matched to another human, he/she prefers being with that human over being single which places that human higher than any robot. Again this refutes the human-robot pair being rogue.

iii. A robot-robot rogue pair. If both robots are matched to other robots, then by our construction, this won't be a rogue couple (we explicitly selected a stable matching between left-alone robots). On the other hand, if either robot is matched to a human, that human is its owner, and obviously a robot loves its owner more than anything, including other robots. So again this cannot be a rogue pair.

This completes the proof.

(b) (10 points) As you saw in the lecture, we may have different stable pairings. But interestingly, if a person remains single in one stable pairing, s/he must remain single in any other stable pairing as well (there really is no hope for some people!). Prove this fact by contradiction.

Answer: We will perform proof by contradiction. Assume that there exists some man m_1 who is paired with a woman w_1 in stable pairing S and unpaired in stable pairing T. Since S is a stable pairing and m_1 is unpaired, w_1 must be paired in T with a man m_2 whom she prefers over m_1. (If w_1 were unpaired or paired with a man she does not prefer over m_1, then (m_1, w_1) would be a rouge couple, which is a contradiction.)

Since m_2 is paired with w_1 in T, he must be paired in S with some woman w_2 whom m_2 prefers over w_1. This process continues (w_2 must be paired with some m_3 in T, m_3 must be paired with some w_3 in S, etc.) until all persons are paired. Since this requires m_1 to be paired in T, where he is known to be unpaired, we have reached a contradiction. Therefore, our assumption must be false, and there cannot exist some man who is paired in a stable pairing S and unpaired in a stable pairing T. A similar argument can be used for women.

Since no man or woman can be paired in one stable pairing and unpaired in another, every man or woman must be either paired in all stable pairings or unpaired in all stable pairings.

Here is another possible proof:

We know that some male-optimal stable pairing exists. Call this pairing M. We first establish two lemmas.

Lemma 1. If a man is single in male-optimal pairing M, then he is single in all other stabling pairings.

Proof. Assume there exists a man that is single in M but not single in some other stable pairing M'. Then M would not be a male-optimal pairing, so this is a contradiction.

Lemma 2. If a woman is paired in male-optimal pairing M, she is paired in all other stable pairings.

Proof. Assume there exists a woman that is paired in M but single in some other stable pairing M'. Then M would not be female-pessimal, so this is a contradiction.

Let there be k single men in M. Let M' be some other stable pairing. Then by Lemma 1, we know single men in M' will be greater than or equal to k. We also know that there are $n-k$ paired men and women in M. Then by Lemma 2, we know that the number of paired women in M' will be greater than or equal to $n-k$.

Now, we want to prove that if a man is paired in M, then he is paired in every other stable pairing. We prove this by contradiction. Assume that there exists a man m that is paired in M but is single in some other stable pairing M'. Then there must be strictly greater than k single men in M', and thus strictly greater than k single women in M'. Since there are strictly greater than k single women in M', there must be strictly less than $n-k$ paired women in M'. But this contradicts that the number of paired women in M' will be greater than or equal to $n-k$.

We also have to prove that if a woman is single in M, then she must be single every other stable pairing. We again prove this by contradiction. Assume that there exists a woman w that is single in M and paired in some other stable pairing M'. Then there are strictly greater than $n-k$ paired women in M', which means there are strictly greater than $n-k$ paired men in M'. This means there must be strictly less than k single men in M'. But this contradicts that the number of single men in M' will be greater than or equal to k.

Since we have proved both 1) If a man is single in M then he is single in every other stable pairing and 2) If a man is paired in M then he is paired in every other stable pairing (note that the contrapositive of this is if a man is single in any other stable pairing, then this man is single in M), we know that a man is single in M if and only if he is single in every other stable pairing. Similarly, since we have proved both 1) If a woman is single in M then she is single in every other stable pairing and 2) If a woman is paired in M then she is paired in every other stable pairing, we know that a woman is single in M if and only if she is single in every stable pairing. Thus we have proved that if a person is single in one stable pairing, s/he is single in every stable pairing.