CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Discussion 4A Sol

1. Counting and Probability Practice

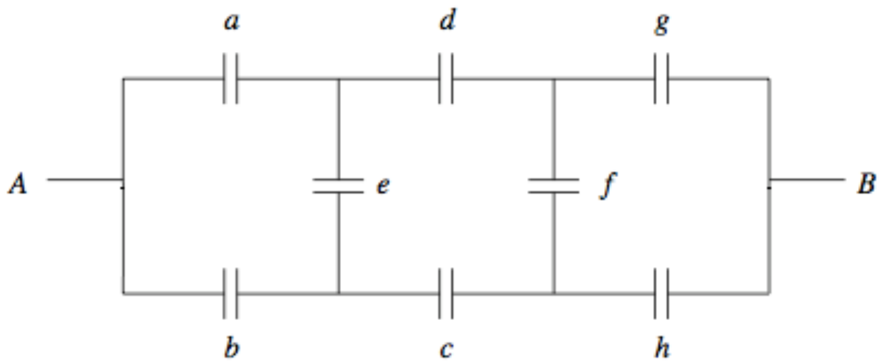
1. A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?
   $\left(\frac{2}{3}\right)^8 = \frac{256}{6561}$ by independence.
2. If five numbers are selected at random from the set $\{1,2,3,\dots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once.)
   $\left(\frac{15}{20}\right)^5 = \frac{243}{1024}$ by independence (every number is between 6 to 20 inclusive).
3. If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?
   $\frac{18!5!}{22!} = \frac{1}{1463}$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.

2. Balls in Bins: Independent? You have k balls and n bins labelled $1,2,\dots,n$, where $n \geq 2$. You drop each ball uniformly at random into the bins.

1. What is the probability that bin n is empty?
   $\left(\frac{n-1}{n}\right)^k$
2. What is the probability that bin 1 is non-empty?
   $1-\left(\frac{n-1}{n}\right)^k$
3. What is the probability that both bin 1 and bin n are empty?
   $\left(\frac{n-2}{n}\right)^k$
4. What is the probability that bin 1 is non-empty and bin n is empty?
   $\left(\frac{n-1}{n}\right)^k - \left(\frac{n-2}{n}\right)^k$
5. What is the probability that bin 1 is non-empty given that bin n is empty?
   $$\frac{\left(\frac{n-1}{n}\right)^k - \left(\frac{n-2}{n}\right)^k}{\left(\frac{n-1}{n}\right)^k} = 1-\left(\frac{n-2}{n-1}\right)^k$$

3. Communication network

In the communication network shown below, link failures are independent, and each link has a probability of failure of p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links.



1. Given that exactly five links have failed, determine the probability that A can still communicate with B.

   There are only two paths of 3 links from A to B. And there are $\binom{8}{5}$ ways of the links messing up.

   So the probability is $\frac{2}{56} = \frac{1}{28}$.

   This is because every single case of exactly 5 links being down have the same probability. So it's a uniform distribution over all possibilities.

2. Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

   Fix g as down and h as working. There are $\binom{6}{4}$ ways to have 4 out of the remaining go down. Symmetric argument for h down and g up.

   So probability is $\frac{30}{56} = \frac{15}{28}$.

3. Given that *a, d* and h have failed (but no information about the information of the other links), determine the probability that A can communicate with B.

   We would just want the 4 on the only remaining path from A to B not to be down.

   The probability of this happening is $(1-p)^4$.