CS70: Discrete Math and Probability

Fan Ye
June 29, 2016

Stable Marriage Problem

- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of n boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S.

A stable pairing??

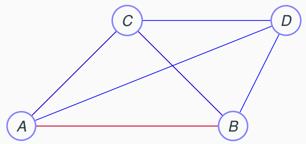
Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do "better"?

Example.

Boys			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C	C	C
2	C	B, ~~C~~	B	A, ~~B~~	A
3					B

Termination.

Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

Terminates in at most n^2+1 steps!

It gets better every day for girls..

Improvement Lemma: It just gets better for girls.
If on day t a girl, g, has a boy b on a string,
any boy, b', on g's string for any day $t' > t$
is at least as good as b.

Proof:
$P(k)$- - "boy on g's string is at least as good as b on day $t+k$"

$P(0)$– true. Girl has b on string.

Assume $P(k)$. Let b' be boy **on string** on day $t+k$.

On day $t+k+1$, boy b' comes back.
Girl can choose b', or do better with another boy, b''

That is, $b \leq b'$ by induction hypothesis.
And b'' is better than b' by algorithm.

$P(k) \implies P(k+1)$. And by principle of induction. □

Pairing when done.

Lemma: Every boy is matched at end.

Proof:
If not, a boy b must have been rejected n times.

Every girl has been proposed to by b,
and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

n girls and n boys. Same number of each.

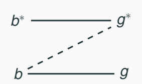
$\implies$ b must be on some girl's string!

Contradiction. □

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:
Assume there is a rogue couple; (b, g^*)



b likes g^* more than g.

g^* likes b more than b^*.

Boy b proposes to g^* before proposing to g.

So g^* rejected b (since he moved on)

By improvement lemma, g^* likes b^* better than b.

Contradiction! □

Good for boys? girls?

Is the TMA better for boys? for girls?

Definition: A **pairing is x-optimal** if x's partner
is its best partner in any stable pairing.

Definition: A **pairing is x-pessimal** if x's partner
is its worst partner in any stable pairing.

Definition: A **pairing is boy optimal** if it is x-optimal for **all** boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing.
As well as you can in a globally stable solution!

Question: Is there a boy or girl optimal pairing?
Is it possible:
b-optimal pairing different from the b'-optimal pairing!
Yes? No?

TMA is optimal!

For boys? For girls?

Theorem: TMA produces a boy-optimal pairing.

Proof:
Assume not: there are boys who do not get their optimal girl.

Let t be first day a boy b gets rejected
by his optimal girl g who he is paired with
in stable pairing S.

b^* - knocks b off of g's string on day t $\implies$ g prefers b^* to b

By choice of t, b^* prefers g to optimal girl.

$\implies$ b^* prefers g to his partner g^* in S.

Rogue couple for S.
So S is not a stable pairing. Contradiction. $\square$

Notes: S - stable. $(b^*, g^*) \in S$. But (b^*, g) is rogue couple!

Used Well-Ordering principle...Induction.

How about for girls?

Theorem: TMA produces girl-pessimal pairing.

T – pairing produced by TMA.

S – worse stable pairing for girl g.

In T, (g, b) is pair.

In S, (g, b^*) is pair.

g likes b^* less than she likes b.

T is boy optimal, so b likes g more than his partner in S.

(g, b) is Rogue couple for S

S is not stable.

Contradiction. $\square$

Notes: Not really induction.
Structural statement: Boy optimality $\implies$ Girl pessimality.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose. $\implies$ optimal for girls.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Summary

Link

Tomorrow Alex starts on Infinity and Countability

Thank you all!