CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye HW 1

Due Monday Jun 27th 1:59PM

1. (10 points) Classical logic
   Here is an extract from Lewis Carroll's treatise *Symbolic Logic* of 1896:

   (I) No shark ever doubts that he is well fitted out.

   (II) A fish, that cannot dance a minuet, is contemptible.

   (III) No fish is quite certain that it is well fitted out, unless it has three rows of teeth.

   (IV) All fishes, except sharks, are kind to children.

   (V) No heavy fish can dance a minuet.

   (VI) A fish with three rows of teeth is not to be despised.

   (a) Write each of the above six sentences as a quantified proposition over the universe of all fish. You should use the following symbols for the various elementary propositions: $P(x)$ for "x doubts he is well fitted out", $B(x)$ for "x is a shark", $F(x)$ for "x is contemptible" (also despised), $U(x)$ for "x can dance a minuet", $O(x)$ for "x has three rows of teeth", $N(x)$ for "x is kind to children", $K(x)$ for "x is heavy".

   (b) Now rewrite each proposition equivalently using the *contrapositive*.

   (c) You now have twelve propositions in total. What can you conclude from them about a fish who is certain that it is well fitted out? Explain clearly the implications you used to arrive at your conclusion.

   Answer:

   (a) It is useful to rephrase the sentences to be in the form "If X, then Y." Writing sentences as quantified propositions:
   (I) "No shark ever doubts that he is well fitted out."
   This means: "If a fish is a shark, then it does not doubt that it is well fitted out."

   $$\forall x, B(x) \Rightarrow \neg P(x).$$

   (II) "A fish, that cannot dance a minuet, is contemptible."

   $$\forall x, \neg U(x) \Rightarrow F(x).$$

   (III) "No fish is quite certain that it is well fitted out, unless it has three rows of teeth."
   This statement means that "A fish is certain that it is well fitted out only if it has three rows of teeth," or equivalently, "If a fish is certain that it is well fitted out, then it has three rows of teeth."
   Thus we have

   $$\forall x, \neg P(x) \Rightarrow O(x).$$

(IV) "All fishes, except sharks, are kind to children."
This means: "If a fish is not a shark, then it is kind to children."

$$\forall x, \neg B(x) \Rightarrow N(x).$$

(V) "No heavy fish can dance a minuet."
This means: "If a fish is heavy, then it cannot dance a minuet."

$$\forall x, K(x) \Rightarrow \neg U(x).$$

(VI) "A fish with three rows of teeth is not to be despised."

$$\forall x, O(x) \Rightarrow \neg F(x).$$

(b) Writing the contrapositives:

(I) $\forall x, P(x) \Rightarrow \neg B(x)$.

(II) $\forall x, \neg F(x) \Rightarrow U(x)$.

(III) $\forall x, \neg O(x) \Rightarrow P(x)$.

(IV) $\forall x, \neg N(x) \Rightarrow B(x)$.

(V) $\forall x, U(x) \Rightarrow \neg K(x)$.

(VI) $\forall x, F(x) \Rightarrow \neg O(x)$.

(c) Deriving a conclusion:
Starting from $\neg P(x)$ and using various of the above implications and their contrapositives, we have

$$\neg P(x) \Rightarrow O(x) \Rightarrow \neg F(x) \Rightarrow U(x) \Rightarrow \neg K(x)$$

which means that a fish that is certain that it is well fitted out is not heavy!

2. (10 points) Propositional logic
For each the following logical equivalence assertions, either prove it is true or give a counterexample showing it is false (i.e., some choices of P and Q such that one side of the equivalence is true and the other is false), together with a one to two sentence justification that it is indeed a counterexample.

(a) $\forall x P(x) \equiv \neg \exists x \neg P(x)$

(b) $\forall x \exists y P(x,y) \equiv \forall y \exists x P(x,y)$

(c) $P \Rightarrow \neg Q \equiv \neg P \Rightarrow Q$

(d) $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q) \equiv P \Leftrightarrow Q$

Answer:

(a) True. We use the fact that $\neg(\neg A) \equiv A$ for any proposition A. The full proof is:

$$\forall x P(x) \equiv \neg(\neg(\forall x P(x))) \equiv \neg(\exists x \neg P(x)).$$

(b) False. Take the universe of both x,y to be $\mathbb{N}$, and take $P(x,y)$ to be the statement "$x > y$". Then the left hand side $\forall x \exists y P(x,y)$ claims that for every $x \in \mathbb{N}$ we can find another natural number $y \in \mathbb{N}$ that is strictly less than x; this is false, since when $x = 0$ we cannot find such a y. The right hand side $\forall y \exists x P(x,y)$ claims that for all $y \in \mathbb{N}$ we can find $x \in \mathbb{N}$ that us strictly larger than y; this is true, e.g., we can take $x = y+1$.

(c) False. Take P and Q to be any true propositions (e.g., P is "$1+1=2$" and Q is "$1+2=3$"). Then $P \Rightarrow \neg Q$ is false while $\neg P \Rightarrow Q$ is true.

(d) True. Recall that $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Since $Q \Rightarrow P$ is equivalent to its contraposition $\neg P \Rightarrow \neg Q$, we conclude that $P \Leftrightarrow Q$ is also equivalent to $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$.

Note: You can use any counterexamples for (b) and (c).

3. (10 points) If you show up on time, you won't have to work this hard!

You show up late to CS 70 lecture and come in the middle of a complex derivation involving the propositions P, Q, and R. From what you can see on the board, you're able to deduce that the following three propositions are true: $P \implies \neg P$, $Q \implies R$, $P \vee Q \vee \neg R$. Unfortunately, it looks like the definition of the propositions P, Q, R has already been erased.

(a) Do you have enough information to deduce the truth value of P? If yes, what is the truth value of P? **Answer:** P is false. Using the first proposition, we have

$$P \implies \neg P,$$

which is equivalent to $\neg P \vee \neg P$, which is in turn equivalent to $\neg P$.
Thus, $\neg P$ is true, and therefore P is false.

(b) Do you have enough information to deduce the truth value of Q? If yes, what is the truth value of Q? **Answer:** We do not have enough information to deduce the value of Q. Propositions Q and R could either be both true or both false. (This could be seen by constructing a truth table; or more easily, by plugging in what we know about P into the third proposition on the board to get $Q \vee \neg R$, which is equivalent to $R \implies Q$, so we have both $Q \implies R$ and $R \implies Q$.)

(c) David asks the class whether $(\neg Q \wedge R) \vee (Q \wedge \neg R)$ is true. Do you have enough information to deduce the truth value of this proposition? If yes, what is its truth value? **Answer:** Yes, we have enough information. The proposition is false. From the second proposition on the board, we have $Q \implies R$, which is equivalent to $\neg Q \vee R$. From the third proposition on the board, combined with our answer to part 1, we have $Q \vee \neg R$. Since they are both true, the following proposition must be true:

$$(\neg Q \vee R) \wedge (Q \vee \neg R).$$

Applying DeMorgan's law gives

$$\begin{aligned}(\neg Q \vee R) \wedge (Q \vee \neg R) &\equiv \neg\big(\neg(\neg Q \vee R) \vee \neg(Q \vee \neg R)\big) \\ &\equiv \neg\big((Q \wedge \neg R) \vee (\neg Q \wedge R)\big).\end{aligned}$$

This says precisely that the original proposition is false.
A different way to see that David's proposition is false is to draw a truth table showing all 4 combinations of possible truth values for Q and R, circling only the rows that are consistent

with the propositions on the board. You will find that the circled rows correspond to the case where both Q and R are false, or the case where both are true. In either case, the proposition $(\neg Q \wedge R) \vee (Q \wedge \neg R)$ evaluates to false.

4. (20 points) Grade these answers

   You be the grader. Students have submitted the following answers to several exam questions. Assign each student answer either an A (correct yes/no answer, valid justification), a D (correct yes/no answer, invalid justification), or an F (incorrect answer). As always, $\pi = 3.14159\ldots$

   (a) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
   Student answer: Yes. $2\pi = 6.283\ldots$, which is less than 100. Also $\pi = 3.1459\ldots$ is less than 50. Therefore the proposition is of the form True $\implies$ True, which is true. **Answer:** *A.*

   (b) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
   Student answer: Yes. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$.

   Answer: *A. One valid way to justify a proposition $P \implies Q$ is to demonstrate that if P is true then Q is true too. (Technically, you could view this as the student proving a more general statement, namely: for all x, if $2x < 100$, then $x < 50$.)*

   Some folks suggested a grade of D, because they felt the student should also have justified why one can always divide both sides of an inequality by 2. This view is not egregiously unreasonable, so it will receive full credit, though we probably wouldn't hold you to such a strict standard on an exam.

   (c) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 49$. Explain your answer.
   Student answer: No. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$, which does not imply $\pi < 49$.

   Answer: *F. $2\pi < 100$ is true and so is $\pi < 50$, so the implication is of the form $T \implies T$. The correct answer is "Yes", not "No."*

   (d) **Exam question:** Is the following proposition true? $\pi^2 < 5 \implies \pi < 5$. Explain your answer.
   Student answer: No, it is false. $\pi^2 = 9.87\ldots$, which is not less than 5, so the premise is false. You can't start from a faulty premise.

   Answer: *F. Any implication of the form $F \implies ?$ is true.*

5. (20 points) Liars and Truthtellers

   You find yourself on a desert island inhabited by two types of people: the Liars and the Truthtellers. Liars always lie, and Truthtellers always tell the truth. In all other respects, the two types are indistinguishable.

   (a) You meet a very attractive local and ask him/her on a date. The local responds, "I will go on a date with you if and only if I am a Truthteller." Is this good news? Explain your answer with reference to logical notation.

   (b) You are trying to find your way to the lagoon and encounter a local inhabitant on the road. Which of the following questions could you ask him in order to reliably deduce whether you are on the correct path? In each case, explain your answer with reference to logical notation.

      i. If I were to ask you "If this is the way to the lagoon, what would you say?"

ii. If I were to ask you, “If this is the way to the lagoon and you say “yes”, can I believe you?”

iii. If I were to ask somebody of the other type than yours, “If this is the way to the lagoon, what would that person say?”

iv. “Is at least one of the following true? You are a Liar and this is the way to the lagoon; or you are a Truthteller and this is not the way to the lagoon.”

Answer:

(a) This is good news. Depending on whether the local is a Truthteller or a Liar, there are two possibilities:

- If he/she is a Truthteller, the statement “I will go on a date with you $\Leftrightarrow$ I am a Truthteller” is true. Now since the “I am a Truthteller” part is true in this case, we deduce that “I will go on a date with you” is also true.
- Otherwise, he/she is a Liar, and the “if and only if" statement is false. Since the “I am a Truthteller” part is false in this case, the only way that the “if and only if" can be false is if “I will go on a date with you” is true.

Either way, you get a date with the attractive local.

(b) For brevity, we use the following shorthands (where x refers to a person):

- W means “this is the way to the lagoon”;
- $S(x)$ means “x says this is the way to the lagoon”, or “x says W”;
- $T(x)$ means “x is a Truthteller”.

Note that $S(x) \equiv (T(x) \Leftrightarrow W)$, in words, “$x$ would say this is the way to the lagoon” is logically equivalent to “x is a Truthteller if and only if this is the way to the lagoon.” In general, given any proposition P (quantified or not), we have “x says P” is logically equivalent to $T(x) \Leftrightarrow P$.

i. **Can deduce.** We further use $A(x)$ to denote “x says ‘yes’ in answer to the question ‘If I were to ask you if this is the way to the lagoon, what would you say?’”. Since $S(x)$ is “x says this is the way to the lagoon”, we have

$$A(x) \equiv (T(x) \Leftrightarrow S(x)).$$

Plugging in $S(x) \equiv (T(x) \Leftrightarrow W)$, we get

$$A(x) \equiv \big(T(x) \Leftrightarrow (T(x) \Leftrightarrow W)\big),$$

which after simplification gives $A(x) \equiv W$. So the answer of the local tells you whether you are on the correct path.

ii. **Cannot deduce.** We use $G(x)$ to denote “If I were to ask x if this is the way to the lagoon and x says ‘yes’, I can believe x.”

- Note that “I ask x if this is the way to the lagoon and x say ‘yes’” is logically equivalent to “x says this is the way to the lagoon”, which is $S(x)$ by definition.
- Note also that “I can believe $S(x)$” is logically equivalent to $S(x) \Leftrightarrow W$.

So we know $G(x) \equiv \big(S(x) \Leftrightarrow W\big)$. Inserting $S(x) \equiv T(x) \Leftrightarrow W$, we get

$$G(x) \equiv \Big(\big(T(x) \Leftrightarrow W\big) \Leftrightarrow W\Big),$$

which simplifies to $G(x) \equiv T(x)$. That is, we can believe x if and only if x is a Truthteller. We use $B(x)$ to denote "x says 'yes' in answer to the question 'If I were to ask you if this is the way to the lagoon and you say 'yes', can I believe you?'", or just "x says $G(x)$". As a result, we have

$$B(x) \equiv T(x) \Leftrightarrow G(x).$$

Since $G(x) \equiv T(x)$, we get $B(x) \equiv \mathsf{true}$. That is, the local always answers 'yes', regardless of whether you are on the correct path, and regardless of whether he/she is a Truthteller.

iii. **Can deduce.** Assume that y is a person of a different type than x, we have

$$T(y) \equiv \neg T(x).$$

We know that "somebody of the other type than x would say this is the way to the lagoon" is $S(y)$, by definition. If we use $C(x)$ to denote "x says 'yes' in answer to the question 'If I were to ask somebody of the other type than yours if this is the way to the lagoon, what would that person say?'", we have

$$C(x) \equiv T(x) \Leftrightarrow S(y).$$

Since we know $S(y) \equiv T(y) \Leftrightarrow W$, we get

$$C(x) \equiv T(x) \Leftrightarrow \big(\neg T(x) \Leftrightarrow W\big),$$

which simplifies to $C(x) \equiv \neg W$. So the *negation* of the answer of the local tells you whether you are on the correct path.

iv. **Can deduce.** We use $H(x)$ to denote "x is a Liar and this is the way to lagoon; or x is a Truthteller and this is not the way to the lagoon." We have

$$H(x) \equiv \big(\neg T(x) \wedge W\big) \vee \big(T(x) \wedge \neg W\big)$$

by definition. Simplifying, we get $H(x) \equiv T(x) \Leftrightarrow \neg W$. Finally we write $D(x)$ for "x says $H(x)$", so

$$D(x) \equiv T(x) \Leftrightarrow H(x).$$

Plugging in $H(x) \equiv T(x) \Leftrightarrow \neg W$ and cleaning up, we conclude that $D(x) \equiv \neg W$, so you can deduce whether you are on the correct path by negating the answer, as in Part (iii).

6. (10 points) Proof by?
Prove that if $x, y \in \mathbb{Z}$, if 6 does not divide xy, then 6 does not divide x and 6 does not divide y. In notation: $(\forall x, y \in \mathbb{Z})\ \ 6 \nmid xy \implies (6 \nmid x \ \wedge\ 6 \nmid y)$. What proof technique did you use? **Answer:** We will use proof by contraposition. For any arbitrary given x and y, the statement $6 \nmid xy \implies (6 \nmid x \wedge 6 \nmid y)$ is equivalent using contraposition to $\neg(6 \nmid x \wedge 6 \nmid y) \implies \neg(6 \nmid xy)$. Moving the negations inside, this becomes equivalent to $(6 \mid x \vee 6 \mid y) \implies 6 \mid xy$.

Now for this part, we give a proof by cases. Assuming that $6 \mid x \vee 6 \mid y$, one of the two cases must be true.

(a) $6 \mid x$: in this case $x = 6k$ for some $k \in \mathbb{Z}$. Therefore $xy = 6ky$ which is a multiple of 6. So $6 \mid xy$.

(b) $6 \mid y$: in this case $y = 6k$ for some $k \in \mathbb{Z}$. Therefore $xy = 6kx$ which is a multiple of 6. So $6 \mid xy$.

Therefore assuming $6 \mid x \vee 6 \mid y$ we proved $6 \mid xy$.

We used proof by cases and proof by contraposition.

7. (10 points)**Inductions!!**

   (a) Prove that $3 + 11 + 19 + \cdots + (8n-5) = 4n^2 - n$ for all integers $n \geq 1$.

   (b) Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$ for all integers $n \geq 1$.

   (c) Let $a_0 = 1$ and $a_n = 2a_{n-1} + 7$. Prove that there is a constant $C > 0$, which does not depend on n, such that $a_n \leq C \cdot 2^n$ for all $n \in \mathbb{N}$. *Hint:* Strengthen the induction hypothesis into $a_n \leq C \cdot 2^n - D$ for some constant D. What value of D should you choose to make the proof easiest?

Answer:

(a) We use induction on $n \geq 1$. The base case $n = 1$ is true because $3 = 4 \cdot 1^2 - 1$. Now assume the claim holds for some $n \geq 1$; we want to show that the claim also holds for $n+1$. We can write

$$\sum_{m=1}^{n+1}(8m-5) \;=\; \sum_{m=1}^{n}(8m-5) + (8(n+1)-5) \;=\; (4n^2-n)+(8n+3) \;=\; 4(n+1)^2-(n+1),$$

where in the calculation above we have used the inductive hypothesis to replace the first n terms in the summation with $4n^2 - n$. This completes the inductive step.

(b) Recall the formula $\sum_{m=1}^{n} m = \frac{1}{2}n(n+1)$ from Theorem 3.1 in Note 3. We use induction to prove that $\sum_{m=1}^{n} m^3 = \frac{1}{4}n^2(n+1)^2$. The base case $n = 1$ holds because $1 = \frac{1}{4} \cdot 1^2 \cdot 2^2$. Now assume the claim holds for some $n \geq 1$; we want to show that the claim also holds for $n+1$. We can write

$$\begin{aligned}\sum_{m=1}^{n+1} m^3 \;=\; \sum_{m=1}^{n} m^3 + (n+1)^3 \;&=\; \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ &=\; \frac{1}{4}(n+1)^2\left(n^2+4(n+1)\right) \;=\; \frac{1}{4}(n+1)^2(n+2)^2,\end{aligned}$$

where in the calculation above we have used the inductive hypothesis to replace the first n terms in the summation with $\frac{1}{4}n^2(n+1)^2$. This completes the inductive step.

(c) We use induction to prove a stronger statement that $a_n \leq 8 \cdot 2^n - 7$ for all $n \in \mathbb{N}$. The base case $n = 0$ is true because $a_0 = 1 = 8 \cdot 2^0 - 7$. Suppose the claim holds for some $n \in \mathbb{N}$; we want to show the claim also holds for $n+1$. Using the recursion, we can write

$$a_{n+1} \;=\; 2a_n + 7 \;\leq\; 2\big(8 \cdot 2^n - 7\big) + 7 \;=\; 8 \cdot 2^{n+1} - 7,$$

where we have applied the inductive hypothesis to obtain the inequality above. This completes the induction step.

Note: You can use any value of $C \geq 8$, but note that $C = 8$ is the smallest choice that makes the inequality true for all $n \in \mathbb{N}$. You can also use any value for D, as long as your proof is correct.

8. (5 points) Divergence of harmonic series

You may have seen the series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ in calculus. This is known as a *harmonic series*, and it diverges, i.e. the sum approaches infinity. We are going to prove this fact using induction.

Let $H_j = \sum_{k=1}^{j} \frac{1}{k}$. Use mathematical induction to show that, for all integers $n \geq 0$, $H_{2^n} \geq 1+\frac{n}{2}$, thus showing that H_j must grow unboundedly as $j \to \infty$. **Answer:**

Base case: $H_{2^0} = H_1 = 1 \geq 1+\frac{0}{2}$

Inductive Step: Assume that $H_{2^n} \geq 1+\frac{n}{2}$. Then:

$$\begin{aligned}
H_{2^{k+1}} &= 1+\frac{1}{2}+\ldots+\frac{1}{2^{k+1}} \\
&= (1+\frac{1}{2}+\ldots+\frac{1}{2^k})+(\frac{1}{2^k+1}+\ldots+\frac{1}{2^{k+1}}) \\
&= H_{2^k}+(\frac{1}{2^k+1}+\ldots+\frac{1}{2^{k+1}})
\end{aligned}$$

By noting that $(\frac{1}{2^k+1}+\ldots+\frac{1}{2^{k+1}})$ has 2^k terms, each of which is at least $\frac{1}{2^{k+1}}$

$$\geq H_{2^k}+2^k * \frac{1}{2^{k+1}}$$

By the inductive hypothesis:

$$\begin{aligned}
&\geq 1+\frac{k}{2}+2^k * \frac{1}{2^{k+1}} \\
&= 1+\frac{k}{2}+\frac{1}{2} \\
&= 1+\frac{k+1}{2}
\end{aligned}$$

Hence we have proved the statement by induction, and can conclude that H_{2^n} must go to infinity as $n \to \infty$, hence H_n must be diverging as $n \to \infty$.