CS70: Counting

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July 6, 2016

Today:

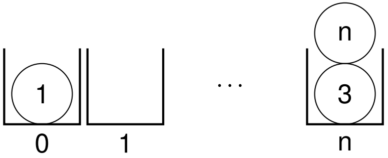
- Balls and bins.
- Sum rule.
- Combinatorial proofs.
- Maybe start review?

What we've learned so far

Sample k items out of n.

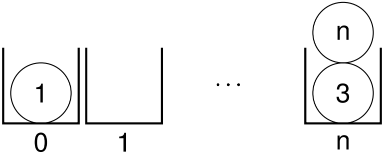
	With Replacement	Without Replacement
Order matters	n^k	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$

Balls in bins.



"k Balls in n bins" $\equiv$ "k samples from n possibilities."

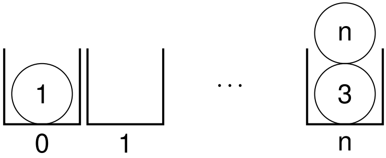
Balls in bins.



"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

Balls in bins.



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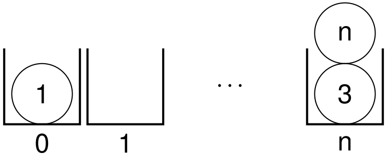
"only one ball in each bin" $\equiv$ "without replacement"

Balls in bins.



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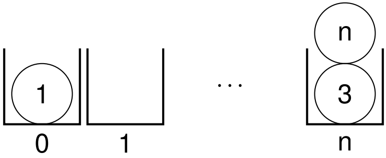
5 balls into 10 bins

Balls in bins.



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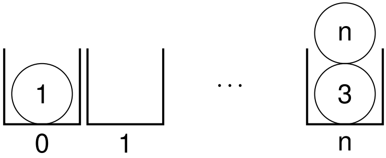
5 balls into 10 bins
5 samples from 10 possibilities with replacement

Balls in bins.



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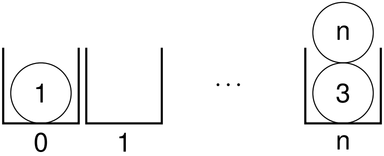
5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

Balls in bins.



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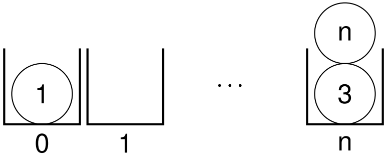
5 indistinguishable balls into 52 bins only one ball in each bin

Balls in bins.



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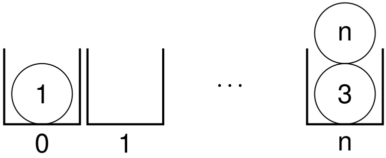
5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement

Balls in bins.



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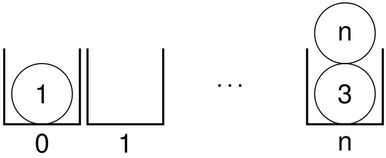
5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

Balls in bins.



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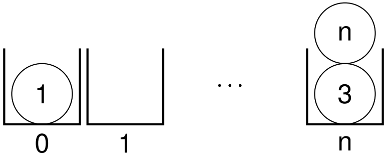
5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins

Balls in bins.



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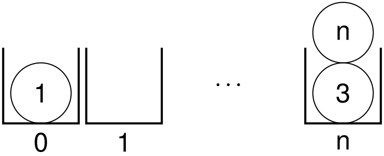
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order

Balls in bins.



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5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

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$$\binom{52}{5} + \binom{52}{4}$$

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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} +$$

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Wait a minute!

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Theorem: $\binom{54}{5}$

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How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Algebraic proof

$$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$$

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$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$

Proof:

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$

$\binom{52}{5} = \frac{52!}{5!47!}$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}.$

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$

$\binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$

Algebraic proof

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$\binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$, $\quad \binom{52}{3} = \frac{52!}{3!49!}$

$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3} = \frac{52!}{5!47!} + 2\frac{52!}{4!48!} + \frac{52!}{3!49!}$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$

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$= \frac{52!(4!48!3!49! + 2 * 5!47!3!49! + 5!47!4!48!)}{5!47!4!48!3!49!}$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

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$= \ldots.$

Algebraic proof

$\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Proof:

$\binom{54}{5} = \frac{54!}{5!49!}$

$\binom{52}{5} = \frac{52!}{5!47!}$, $\quad \binom{52}{4} = \frac{52!}{4!48!}$, $\quad \binom{52}{3} = \frac{52!}{3!49!}$

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$= \frac{52!(4!48!3!49! + 2*5!47!3!49! + 5!47!4!48!)}{5!47!4!48!3!49!}$

$= \ldots.$

Let's actually go with the other one....

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

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Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

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Choose a subset of size $n-k$
and what's left out is a subset of size k.
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as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

$\square$

Pascal's Triangle

Pascal's Triangle

$$\begin{array}{ccc} & 0 & \\ 1 & & 1 \end{array}$$

Pascal’s Triangle

		0		
	1		1	
1		2		1

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

					0					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Pascal's Triangle

					0					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Pascal's Triangle

					0					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1

Pascal's Triangle

$$\begin{array}{ccccccccccc} &&&&& 0 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 =$

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$.

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 =$

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$.

Pascal's Triangle

$$\begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1

Pascal's Triangle

0

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....

Pascal's Triangle

					0					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
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First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....
Foil??

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.
Zero-th row: $(1+x)^0 = 1$
First row: $(1+x)^1 = x+1$. Coefficients: 1 and 1
Second row: $(1+x)^2 = 1+2x+x^2$. Coefficients: 1,2 and 1
Third row: $(1+x)^3 = 1+3x+3x^2+x^3$. Coefficients: 1,3,3 and 1
.....
Foil?? I hate this...

Pascal's Triangle

					0					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$:

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

Pascal's Triangle

$$\begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$:

Pascal's Triangle

					0					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from:

Pascal's Triangle

					0					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth.

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$\begin{array}{ccc}
 & \binom{0}{0} & \\
\binom{1}{0} & & \binom{1}{1}
\end{array}$$

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$\begin{array}{ccccc}
 & & \binom{0}{0} & & \\
 & \binom{1}{0} & & \binom{1}{1} & \\
\binom{2}{0} & & \binom{2}{1} & & \binom{2}{2}
\end{array}$$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Simplify: collect all terms corresponding to x^k.

Coefficient of x^k is $\binom{n}{k}$:choose k factors where x is in product.

$(1+x)(1+x)(1+x)(1+x)$: Coefficients of x^2 come from: first and second term , first and third term , first and fourth , second and third, second and fourth, third and fourth. 6 of them , aka $\binom{4}{2}$.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.
Proof: How many size k subsets of $n+1$?

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Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elts.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elts.
$\implies \binom{n}{k}$

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elts.
$\implies \binom{n}{k}$
So, $\binom{n}{k-1} + \binom{n}{k}$

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
The ones that contain the first element plus the ones that don't contain the first element.
How many contain the first element?
Need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose k elements from remaining n elts.
$\implies \binom{n}{k}$
So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.

Combinatorial Proof.

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Proof:

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Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i\underline{, \ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen:

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.

2 is first element chosen:

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.

2 is first element chosen: $\binom{n-2}{k-1}$ choices for the rest.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.

2 is first element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.
2 is first element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Add them up to get the total number of subsets of size k

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$.
Proof:
Left Hand Side (LHS): Size k subsets of n.
Consider size k subset where i is the first element chosen.

$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.
$\implies \binom{n-i}{k-1}$ such subsets.

1 is first element chosen: $\binom{n-1}{k-1}$ choices for the rest.
2 is first element chosen: $\binom{n-2}{k-1}$ choices for the rest.
and so on.

Add them up to get the total number of subsets of size k which is also $\binom{n}{k}$.

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

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