CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Discussion 4B Sol

1. Birthdays

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

1. What is the probability that the first 3 people do not have the same birthday? $364/365 * 363/365$
2. What is the probability that the first three people have the same birthday? $(1/365)^2$
3. What is the probability that it takes more than 20 people for this to occur? Solution not included because this sub-problem is in Hw 4.
4. What is the probability that it takes exactly 20 people for this to occur? Solution not included because this sub-problem is in Hw 4.
5. Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?
   Solution not included because this sub-problem is in Hw 4.

2. Balls and Bins

You have n empty bins and you throw balls into them one by one randomly. A collision is when a ball is thrown into a bin which already has another ball.

1. What is the probability that the first ball thrown will cause the first collision?
   0
2. What is the probability that the second ball thrown will cause the first collision?
   $\frac{1}{n}$
3. What is the probability that, given the first two balls are not in collision, the third ball thrown will cause the first collision?
   $\frac{2}{n}$
4. What is the probability that the third ball thrown will cause the first collision?
   Basically: $P(\text{Ball 3 collides} \mid \text{Ball 1, 2 do not collide}) \cdot P(\text{Ball 1, 2 do not collide})$, which is $\frac{2}{n} \cdot \frac{n-1}{n}$.
5. What is the probability that, given the first $m-1$ balls are not in collision, the m^{th} ball thrown will cause the first collision?
   $\frac{m-1}{n}$
6. What is the probability that the m^{th} ball thrown will cause the first collision?
   Similar to (d), $\frac{m-1}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-m+2}{n} = \frac{m-1}{n} \cdot \prod_{i=0}^{m-2} \frac{n-i}{n}$.

3. Best choice problem

Three princes are going to stop by Alice's house and invite her to their parties. Alice has not met them before so she does not know who she likes most. There are three strategies:

1. Go with the first prince.
2. Reject the first prince. If she finds she likes the second prince more than the first one, she will go with him. Otherwise, she will go with the third prince.
3. Reject the first prince. If she finds she likes the second prince more than the third one, she will go with him. Otherwise, she will go with the third prince.

Which strategy gives Alice the highest probability to go with the prince she likes the most?

The third one.
If she uses the first strategy, the probability that she goes with the prince she likes the most is $\frac{1}{3}$
We rank the princes from 1 to 3, 1 being the prince Alice likes the most.
There are 6 possible orders of the princes stopping by Alice's house:
$\{123, 132, 213, 231, 312, 321\}$
If she uses the second strategy, she has $\frac{1}{2}$ probability that she can go with the prince she likes the most.
On the other hand, if she uses the third strategy she has $\frac{2}{3}$ probability of going with the prince she likes the most (so this is the best strategy). This situation is analogous to the Monty Hall problem.