1. **Fun with Binary.**

Prove the following statement:

$$\forall n \in \mathbb{N}, \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Base case: If $n = 0$, then we get $2^0 = 2^1 - 1$ which is true.

Inductive Step: Assume that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$. Then:

$$\begin{aligned}\sum_{k=0}^{n+1} 2^k &= \sum_{k=0}^{n} 2^k + 2^{n+1} \\ &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2 * 2^{n+1} - 1 \\ &= 2^{n+2} - 1\end{aligned}$$

Hence we completed the induction. Note that this answer can be easily seen in binary: $10000 - 1 = 1111$ in binary, and hence realizing this is a good way of believing the statement without induction.

2. **Power Inequality**

Use induction to prove that for all integers $n \geq 1$, $2^n + 3^n \leq 5^n$.

We use induction on n. The base case $n = 1$ is true because $2 + 3 = 5$. Assume the inequality holds for some $n \geq 1$. For $n + 1$, we can write:

$$2^{n+1} + 3^{n+1} \;=\; 2 \cdot 2^n + 3 \cdot 3^n \;<\; 3 \cdot 2^n + 3 \cdot 3^n \;=\; 3(2^n + 3^n) \overset{(*)}{\leq} 3 \cdot 5^n < 5 \cdot 5^n = 5^{n+1},$$

where the inequality (*) follows from the induction hypothesis. This completes the induction.

3. **Triangle Inequality**

Recall the triangle inequality, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Use induction to prove the generalized triangle inequality:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

We use induction on $n \geq 2$. The base case $n = 2$ is the usual triangle inequality. Assume the inequality holds for some $n \geq 2$ (this is the inductive hypothesis). For $n + 1$, we can write:

$$\begin{aligned}|x_1 + x_2 + \cdots + x_n + x_{n+1}| &\leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| && (\text{ by the usual triangle inequality}) \\ &\leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}| && (\text{ by the induction hypothesis}).\end{aligned}$$

This completes the induction.

4. False Proof

What goes wrong in the following "proof"?

Theorem: If n is an even number and $n \geq 2$, then n is a power of two.

Proof:
By induction on the natural number n. Let the induction hypothesis $IH(k)$ be the assertion that "if k is an even number and $k \geq 2$, then $k = 2^i$, where i is a natural number".

Base case: $IH(2)$ states that 2 is a power of two, which it is ($2 = 2^1$).

Inductive step: Assume that k is a number greater than 2, and that $IH(j)$ holds for all $2 \leq j < k$.

Case 1: k is odd, and there is nothing to show.

Case 2: k is even, so $k \geq 4$. Since $k \geq 4$ is an even number, $k = 2l$, with $2 \leq l < k$. Therefore we can use the induction hypothesis $IH(l)$, which asserts that $l = 2^i$ for some integer i. Thus we have $k = 2l = 2^{i+1}$, so k is a power of two. $IH(k)$ holds.

The error in the proof is in the application of the induction hypothesis. The proof states that the induction hypotheses $IH(l)$ asserts that $l = 2i$, but in reality, it asserts that if l is even, then $l = 2i$. Since l may be odd, it is not possible to conclude that $l = 2i$.