
CS 70

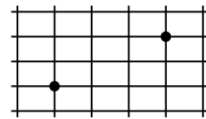
Summer 2016

Discrete Mathematics and Probability Theory

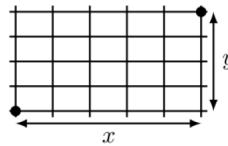
Dinh, Psomas, and Ye

Discussion 3B Sol

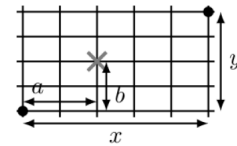
1. Manhattan is well-known for its grid layout and busy traffic, so you are interested in evaluating different shortest paths from one point to another.



Part (b)



Part (c)



Part (d)

1. Smile at your neighbor (:
2. Consider two locations that are 3 blocks by 2 blocks away from each other. If you take a cab from one point to the other, how many shortest paths are there?

There are five different blocks in the shortest path, of which any two can be streets. This means there are $\binom{5}{2} = \frac{5!}{3!2!} = 20$ shortest paths.

3. How many shortest paths are there connecting two points that are x blocks by y blocks away from each other? (x and y are non-negative integers.)

We have $x + y$ different blocks in the shortest path, of which any y can be streets. This means there are $\binom{x+y}{y}$ shortest paths.

4. Assume that an intersection, a blocks by b blocks away from the starting point, is under road work and is not usable. How many shortest paths are there connecting the starting point and the destination without using that intersection (which is assumed to be between the starting point and the destination, see figure above)?

To find this, we can find the total number of shortest paths for travelling x by y blocks and subtract any of the shortest paths that use the unusable intersection. We found in the previous part that there are $\binom{x+y}{y}$ total shortest paths, including those that use the bad intersection. The number of shortest paths that use the bad intersection must first be a shortest path from the start to that intersection, then a shortest path to the end. There are $\binom{a+b}{b}$ different shortest paths from the start to that intersection, and $\binom{x+y-a-b}{y-b}$ shortest paths from that intersection to the destination. Therefore, there are $\binom{x+y}{y} - \binom{a+b}{b} \cdot \binom{x+y-a-b}{y-b}$ shortest paths that avoid the unusable intersection.

2. How many solutions does $x_1 + \dots + x_k = n$ have, if we have the additional constraint that $x_i \geq a_i$, with $a_i \in \mathbf{N}$, for $1 \leq i \leq k$?

$\binom{(n - \sum_{i=1}^k a_i) + k - 1}{k - 1}$. By subtracting a_i s from all the corresponding x_i s, and thus $\sum_{i=1}^k a_i$ from the total required, we reduce it to the familiar problem where all variables are just non-negative. Once we have a solution to that we reverse the process, by adding a_i to all the corresponding non-negative variables ($1 \leq i \leq k$).

3. Prove the following using combinatorial arguments.

$$1. \binom{m+n}{r} = \sum_{k=0}^r \binom{m}{k} \cdot \binom{n}{r-k}$$

The LHS counts the number of ways to choose a committee of r people from a group of m men and n women. The RHS counts the same thing according to cases depending on the number of men on the committee, which can range from 0 to r . If there are k men, then there must be $r - k$ women. Since in such a case there are $\binom{m}{k}$ ways to select the men and $\binom{n}{r-k}$ ways to select the women, the number of such committees is $\binom{m}{k} \cdot \binom{n}{r-k}$. The result now follows from the Rule of Sum.

$$2. k \binom{n}{k} = n \binom{n-1}{k-1}$$

The LHS counts the number of pairs (x, S) , where S is a k -subset of $\{1, 2, \dots, n\}$ and $x \in S$. There are $\binom{n}{k}$ choices for k , and for each of these there are k choices for x . The RHS counts the same thing in a different order. First choose x - there are n choices - and then choose the other $k - 1$ elements of S from the remaining $n - 1$ elements of $\{1, 2, \dots, n\}$.