

CS70: Discrete Math and Probability

Fan Ye

June 29, 2016

Stable Marriage Problem

- Small town with n boys and n girls.

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- Each girl has a ranked preference list of boys.

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- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.

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- Maximize number of first choices.

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- Maximize worse off.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

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Brad prefers Angelina to Jennifer.

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The best laid plans..

Consider the couples..

- Jennifer and Brad
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Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

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Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of n boy-girl pairs.

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Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

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 b and g^* prefer each other to their partners in S

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Example: Brad and Angelina are a rogue couple in S .

A stable pairing??

Given a set of preferences.

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Is there a stable pairing?

How does one find it?

A stable pairing??

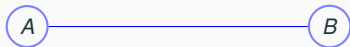
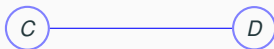
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Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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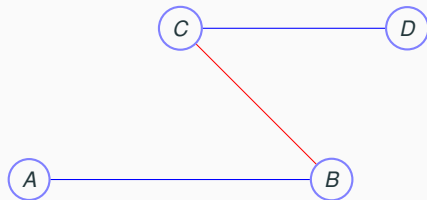
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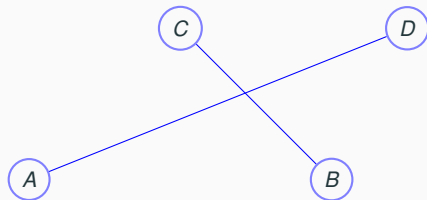
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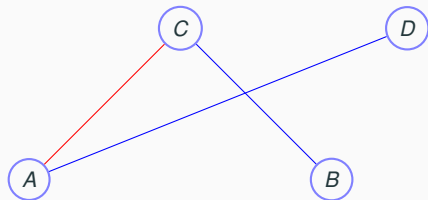
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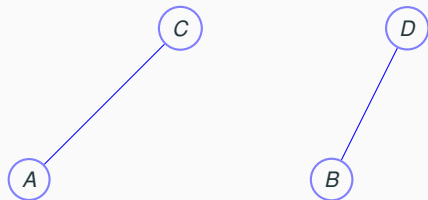
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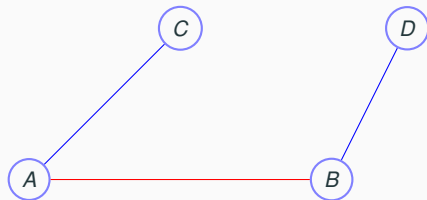
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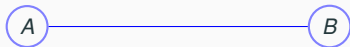
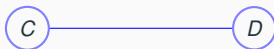
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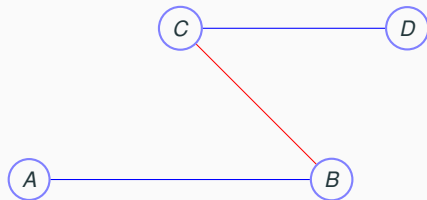
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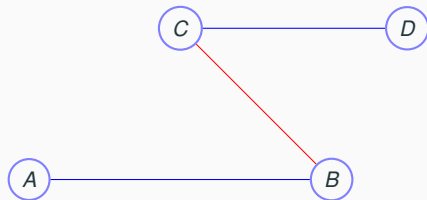
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Does this terminate?

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Do boys or girls do "better"?

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Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

Boys				
A	1	2	3	
B	1	2	3	
C	2	1	3	

Girls				
1	C	A	B	
2	A	B	C	
3	A	C	B	

Example.

Boys				Girls			
A	1	2	3	1	C	A	B
B	1	2	3	2	A	B	C
C	2	1	3	3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

Boys			
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B	1	2	3
C	2	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Boys				Girls			
A	1	2	3	1	C	A	B
B	X	2	3	2	A	B	C
C	2	1	3	3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Boys				Girls			
A	1	2	3	1	C	A	B
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1	A, B	A			
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1	A, B	A	A, C		
2	C	B, C	B		
3					

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	Day 1	Day 2	Day 3	Day 4	Day 5
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3					B

Example.

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1	A, B	A	A, C	C	C
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3					B

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Every non-terminated day a boy **crossed** an item off the list.

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Total size of lists?

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Total size of lists? n boys, n length list.

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Total size of lists? n boys, n length list. n^2

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Terminates in at most $n^2 + 1$ steps!

It gets better every day for girls..

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Improvement Lemma: It just gets better for girls.

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If on day t a girl, g , has a boy b on a string,
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$P(k)$ - - "boy on g 's string is at least as good as b on day $t + k$ "

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$P(0)$ — true. Girl has b on string.

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On day $t + k + 1$, boy b' comes back.

Girl can choose b' ,

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Girl can choose b' , or do better with another boy, b''

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Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

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Is the TMA better for boys?

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Definition: A **pairing is x -optimal** if x 's partner is its best partner in any **stable** pairing.

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Claim: The optimal partner for a boy must be first in his preference list.

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Theorem: TMA produces a boy-optimal pairing.

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Used Well-Ordering principle...Induction.

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Structural statement: Boy optimality

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Structural statement: Boy optimality \implies Girl pessimality.

Quick Questions.

How does one make it better for girls?

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SMA - stable marriage algorithm. One side proposes.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.

TMA - boys propose.

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How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.

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Girls could propose.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.

TMA - boys propose.

Girls could propose. \implies optimal for girls.

Residency Matching..

The method was used to match residents to hospitals.

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Hospital optimal....

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Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

▶ [Link](#)

▶ [Link](#)

Tomorrow Alex starts on Infinity and Countability

▶ [Link](#)

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Thank you all!