CS70: Discrete Math and Probability

Slides adopted from Satish Rao, CS70 Spring 2016
06/21/2016

Lecture 2: Proofs!

1. Direct proof
2. by Contraposition
3. by Contradiction
4. by Cases

Quick Background and Notation.

Integers closed under addition.

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A natural number $p > 1$, is **prime** if it is divisible only by 1 and itself.

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Direct proof of $P \implies Q$:
Assumed P: $11|a - b + c$. Proved Q: $11|n$.

The Converse

Thm: $\forall n \in D_3, (11 | \text{alt. sum of digits of } n) \implies 11 | n$

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Proof: Assume $\neg Q$: d is even.

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Proof by contradiction:form

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Contrapositive: True $\implies P$.

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Lemma: If x is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in Z$, *then both a and b are even.*

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- $$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

Proof by cases.

Theorem: There exist irrational x and y such that x^y is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.
- $$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}*\sqrt{2}}$$

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Thus, we have irrational x and y with a rational x^y (i.e., 2).

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Question: Which case holds?

Proof by cases.

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Thus, we have irrational x and y with a rational x^y (i.e., 2).

One of the cases is true so theorem holds. $\square$

Question: Which case holds? Don't know!!!

Be careful.

Theorem: $3 = 4$

Be careful.

Theorem: $3 = 4$

Proof: Assume $3 = 4$.

Be careful.

Theorem: $3 = 4$

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Start with $12 = 12$.

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Divide one side by 3 and the other by 4 to get
$$4 = 3.$$

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By commutativity

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By commutativity theorem holds. $\square$

Be careful.

Theorem: $3 = 4$

Proof: Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get

$$4 = 3.$$

By commutativity theorem holds. $\square$

Don't assume what you want to prove!

Be really careful!

Theorem: $1 = 2$
Proof:

Be really careful!

Theorem: $1 = 2$
Proof: For $x = y$, we have

Be really careful!

Theorem: $1 = 2$
Proof: For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$

Be really careful!

Theorem: $1 = 2$
Proof: For $x = y$, we have

$$\begin{aligned}(x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y)\end{aligned}$$

Be really careful!

Theorem: $1 = 2$
Proof: For $x = y$, we have

$$\begin{aligned} (x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y) \\ x &= (x+y) \end{aligned}$$

Be really careful!

Theorem: $1 = 2$
Proof: For $x = y$, we have

$$\begin{aligned}
(x^2 - xy) &= x^2 - y^2 \\
x(x-y) &= (x+y)(x-y) \\
x &= (x+y) \\
x &= 2x
\end{aligned}$$

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$$

□

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Dividing by zero is no good.

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$$

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Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$.

Summary: Note 2.

Direct Proof:

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Direct Proof:
 To Prove: $P \implies Q$.

Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume P.

Summary: Note 2.

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Summary: Note 2.

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Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume P. Prove Q.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$.

Summary: Note 2.

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To Prove: $P \implies Q$. Assume P. Prove Q.

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Summary: Note 2.

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To Prove: P

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Universal: show that statement holds in all cases.
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Summary: Note 2.

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Careful when proving!

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Don't assume the theorem.

Summary: Note 2.

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Don't assume the theorem. Divide by zero.

Summary: Note 2.

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To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

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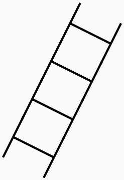
Don't assume the theorem. Divide by zero.Watch converse. ...

CS70: Note 3. Induction!

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.

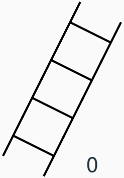
The naturals.

The naturals.



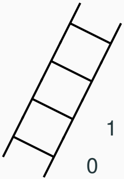
The naturals.

0,



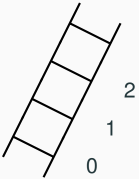
The naturals.

0, 1,



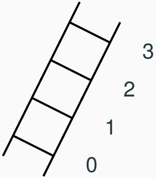
The naturals.

0, 1, 2,



The naturals.

0, 1, 2, 3,



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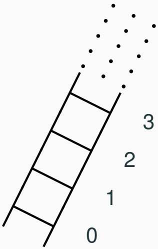
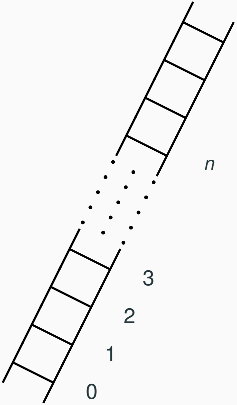


0, 1, 2, 3,

. . . ,

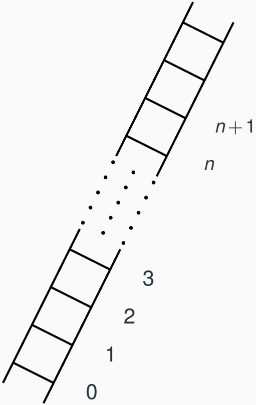
The naturals.



0, 1, 2, 3,

$\ldots$, n,

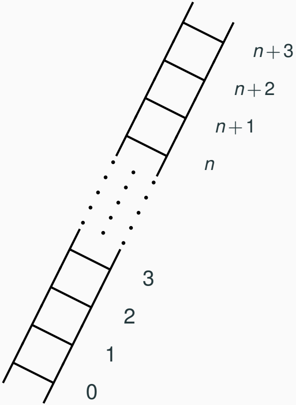
The naturals.



0, 1, 2, 3,

$\ldots$, n, $n+1$,

The naturals.



0, 1, 2, 3,

$\ldots$, n, $n+1$, $n+2$,$n+3$,

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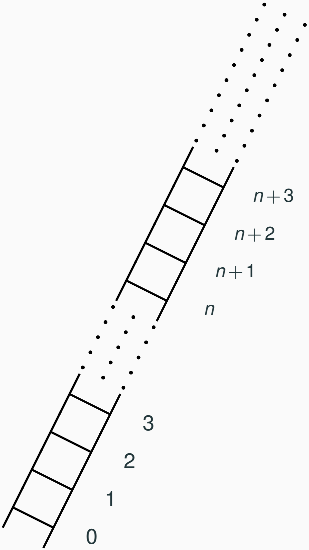

$0, 1, 2, 3,$

$\ldots, n, n+1, n+2, n+3, \ldots$

A formula.

A formula.

Teacher: Hello class.

A formula.

Teacher: Hello class.
Teacher:

A formula.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's

A formula.

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Gauss: It's $\frac{(100)(101)}{2}$

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Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Gauss and Induction

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Idea: assume predicate $P(n)$ for $n = k$.

Gauss and Induction

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Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

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Is predicate, $P(n)$ true for $n = k + 1$?

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$\sum_{i=1}^{k+1} i$

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$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1)$

Gauss and Induction

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$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

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How about $k+2$.

Gauss and Induction

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How about $k+2$. Same argument starting at $k+1$ works!

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Induction Step.

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Gauss and Induction

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Need to start somewhere.

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Statement is true for $n = 0$

Gauss and Induction

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Statement is true for $n = 0$ $P(0)$ is true

Gauss and Induction

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Statement is true for $n = 0$ $P(0)$ is true
plus inductive step

Gauss and Induction

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plus inductive step $\implies$ true for $n = 1$

Gauss and Induction

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Gauss and Induction

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Gauss and Induction

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plus inductive step $\implies$ true for $n = 2$

Gauss and Induction

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Predicate, $P(n)$, True for all natural numbers!

Gauss and Induction

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Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**