CS70: Discrete Math and Probability

June 22, 2016

Induction

Principle of Induction.

Induction

Principle of Induction.

$$P(0) \land (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

Induction

Principle of Induction.

$$P(0) \land (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

Induction

Principle of Induction.

$$P(0) \land (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

...Yes for 0,

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

...Yes for 0, and we can conclude

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

...Yes for 0, and we can conclude Yes for 1...

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

...Yes for 0, and we can conclude Yes for 1...
and we can conclude

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

...Yes for 0, and we can conclude Yes for 1...
and we can conclude Yes for 2...

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

...Yes for 0, and we can conclude Yes for 1...
and we can conclude Yes for 2.......

Induction

Principle of Induction.

$$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$$

And we get...

$$(\forall n \in \mathbb{N}) P(n).$$

...Yes for 0, and we can conclude Yes for 1...
and we can conclude Yes for 2.......

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1)$$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done?

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!
Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k \implies$ true for $n = k+1$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k \implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

...

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k \implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$
...

Predicate, $P(n)$, True for all natural numbers!

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$
...

Predicate, $P(n)$, True for all natural numbers!
Is this a proof?

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$
...

Predicate, $P(n)$, True for all natural numbers!
Is this a proof? Not really. Just an idea, not formal enough to be a proof

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Are we done? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = 1 = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$
...

Predicate, $P(n)$, True for all natural numbers!
Is this a proof? Not really. Just an idea, not formal enough to be a proof yet

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. "Base Case".

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
    - Assume $P(k)$, “Induction Hypothesis”

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
    - Assume $P(k)$, “Induction Hypothesis”
    - Prove $P(k+1)$. “Induction Step.”

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, "Induction Hypothesis"
  - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers n!!!

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
    - Assume $P(k)$, “Induction Hypothesis”
    - Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers n!!!

Get to use $P(k)$ to prove $P(k+1)$!

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
    - Assume $P(k)$, "Induction Hypothesis"
    - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers n!!!

Get to use $P(k)$ to prove $P(k+1)$! !

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers n!!!

Get to use $P(k)$ to prove $P(k+1)$! ! !

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

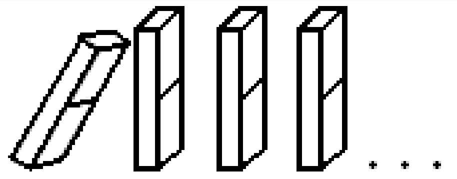
- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
    - Assume $P(k)$, "Induction Hypothesis"
    - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers n!!!

Get to use $P(k)$ to prove $P(k+1)$! ! ! !

Notes visualization

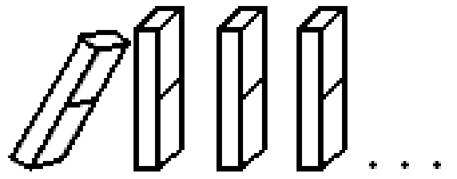
Note's visualization: an infinite sequence of dominos.



Prove they all fall down;

Notes visualization

Note's visualization: an infinite sequence of dominos.

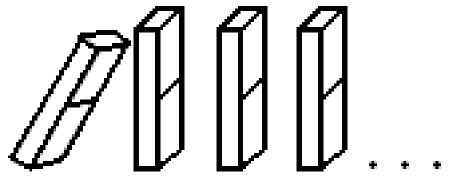


Prove they all fall down;

- $P(0)$ = "First domino falls"

Notes visualization

Note's visualization: an infinite sequence of dominos.

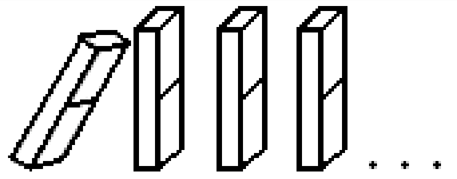


Prove they all fall down;

- $P(0)$ = "First domino falls"
- $(\forall k)\ P(k) \implies P(k+1)$:

Notes visualization

Note's visualization: an infinite sequence of dominos.

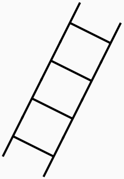


Prove they all fall down;

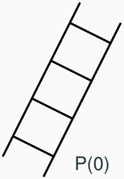
- $P(0)$ = "First domino falls"
- $(\forall k)\ P(k) \implies P(k+1)$:
  "kth domino falls implies that $k+1$st domino falls"

Climb an infinite ladder?

Climb an infinite ladder?



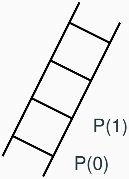
Climb an infinite ladder?



$P(0)$

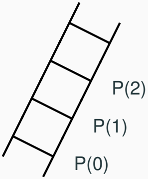
Climb an infinite ladder?

$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$



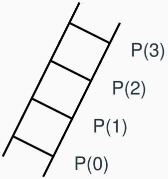
Climb an infinite ladder?

$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2)$$

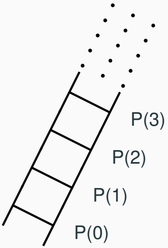


Climb an infinite ladder?

$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3)$$

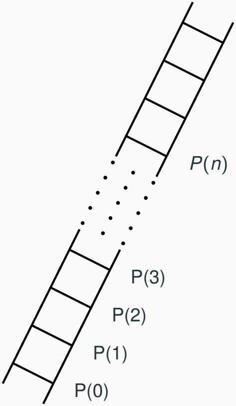


Climb an infinite ladder?



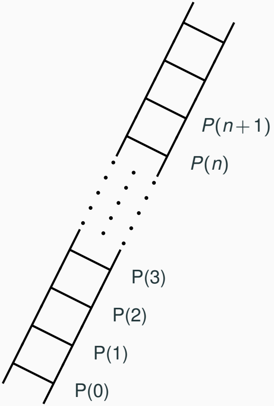
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



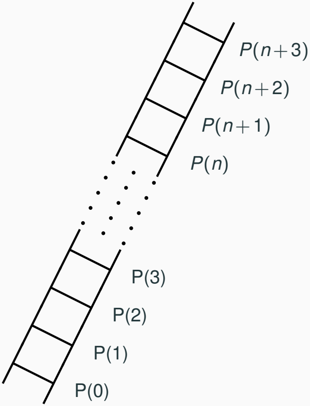
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



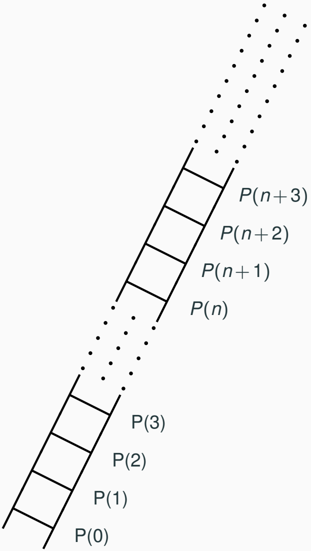
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$
$$(\forall n \in N) P(n)$$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does $0 = \frac{0(0+1)}{2}$?

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does $0 = \frac{0(0+1)}{2}$? Yes.

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does $0 = \frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n=\frac{n(n+1)}{2}$

Base Case: Does $0 = \frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k=\frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$1+\cdots+k+(k+1) \quad = \quad \frac{k(k+1)}{2}+(k+1)$$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \end{aligned}$$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \\ &= \frac{k^2+3k+2}{2} \end{aligned}$$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$
\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

$P(k+1)$!.

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

$P(k+1)$!. By principle of induction...

Again: Simple induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $P(k)$ is true: $1+\cdots+k = \frac{k(k+1)}{2}$
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

$P(k+1)$!. By principle of induction... $\square$

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case:

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{N}$

Base case: For $n=0$,

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case: For $n = 0$, $0^2 = \frac{1}{6} * 0 * 1 * 1 = 0$,

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case: For $n = 0$, $0^2 = \frac{1}{6} * 0 * 1 * 1 = 0$,$p(0)$ is true.

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case: For $n = 0$, $0^2 = \frac{1}{6} * 0 * 1 * 1 = 0$,$p(0)$ is true.

Induction hypothesis:

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case: For $n = 0$, $0^2 = \frac{1}{6} * 0 * 1 * 1 = 0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Try it yourself!

For all natural numbers n, $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2 + 1^2 + 2^2 \cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case: For $n = 0$, $0^2 = \frac{1}{6} * 0 * 1 * 1 = 0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k) \implies p(k+1)$

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{N}$

Base case: For $n=0$, $0^2=\frac{1}{6}*0*1*1=0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k)\implies p(k+1)$

$$0^2+1^2+2^2\cdots+k^2+(k+1)^2 \quad = \quad (0^2+1^2+2^2\cdots+k^2)+(k+1)^2$$

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case: For $n=0$, $0^2 = \frac{1}{6}*0*1*1 = 0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k) \implies p(k+1)$

$$\begin{aligned} 0^2+1^2+2^2\cdots+k^2+(k+1)^2 &= (0^2+1^2+2^2\cdots+k^2)+(k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2 \end{aligned}$$

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$ for $n \in \mathbb{N}$

Base case: For $n=0$, $0^2 = \frac{1}{6}*0*1*1 = 0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k) \implies p(k+1)$

$$
\begin{aligned}
0^2+1^2+2^2\cdots+k^2+(k+1)^2 &= (0^2+1^2+2^2\cdots+k^2)+(k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2 \\
&= (k+1)(\frac{1}{6}k(2k+1)+(k+1))
\end{aligned}
$$

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{N}$

Base case: For $n=0$, $0^2=\frac{1}{6}*0*1*1=0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k)\implies p(k+1)$

$$\begin{aligned}
0^2+1^2+2^2\cdots+k^2+(k+1)^2 &= (0^2+1^2+2^2\cdots+k^2)+(k+1)^2\\
&= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2\\
&= (k+1)(\frac{1}{6}k(2k+1)+(k+1))\\
&= \frac{1}{6}(k+1)(2k^2+k+6k+6)
\end{aligned}$$

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{N}$

Base case: For $n=0$, $0^2=\frac{1}{6}*0*1*1=0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k)\implies p(k+1)$

$$\begin{aligned}0^2+1^2+2^2\cdots+k^2+(k+1)^2 &= (0^2+1^2+2^2\cdots+k^2)+(k+1)^2\\ &= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2\\ &= (k+1)(\frac{1}{6}k(2k+1)+(k+1))\\ &= \frac{1}{6}(k+1)(2k^2+k+6k+6)\\ &= \frac{1}{6}(k+1)(k+2)(2k+3)\end{aligned}$$

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{N}$

Base case: For $n=0$, $0^2=\frac{1}{6}*0*1*1=0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k)\implies p(k+1)$

$$
\begin{aligned}
0^2+1^2+2^2\cdots+k^2+(k+1)^2 &= (0^2+1^2+2^2\cdots+k^2)+(k+1)^2\\
&= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2\\
&= (k+1)(\frac{1}{6}k(2k+1)+(k+1))\\
&= \frac{1}{6}(k+1)(2k^2+k+6k+6)\\
&= \frac{1}{6}(k+1)(k+2)(2k+3)\\
&= \frac{1}{6}(k+1)(k+2)(2(k+1)+1)
\end{aligned}
$$

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{N}$

Base case: For $n=0$, $0^2=\frac{1}{6}*0*1*1=0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k)\implies p(k+1)$

$$\begin{aligned}
0^2+1^2+2^2\cdots+k^2+(k+1)^2 &= (0^2+1^2+2^2\cdots+k^2)+(k+1)^2\\
&= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2\\
&= (k+1)(\frac{1}{6}k(2k+1)+(k+1))\\
&= \frac{1}{6}(k+1)(2k^2+k+6k+6)\\
&= \frac{1}{6}(k+1)(k+2)(2k+3)\\
&= \frac{1}{6}(k+1)(k+2)(2(k+1)+1)
\end{aligned}$$

$p(k+1)$ is true.

Try it yourself!

For all natural numbers n, $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$

Define predicate $p(n)$ as $0^2+1^2+2^2\cdots n^2=\frac{1}{6}n(n+1)(2n+1)$ for $n\in\mathbb{N}$

Base case: For $n=0$, $0^2=\frac{1}{6}*0*1*1=0$,$p(0)$ is true.

Induction hypothesis: assume $p(k)$ is true for some natural number k.

Inductive steps: need to prove $p(k)\implies p(k+1)$

$$\begin{aligned}
0^2+1^2+2^2\cdots+k^2+(k+1)^2 &= (0^2+1^2+2^2\cdots+k^2)+(k+1)^2\\
&= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2\\
&= (k+1)(\frac{1}{6}k(2k+1)+(k+1))\\
&= \frac{1}{6}(k+1)(2k^2+k+6k+6)\\
&= \frac{1}{6}(k+1)(k+2)(2k+3)\\
&= \frac{1}{6}(k+1)(k+2)(2(k+1)+1)
\end{aligned}$$

$p(k+1)$ is true.By principle of induction... $\square$

Homework, Exam

We will use some problems from homework in our exams,

Homework, Exam

We will use some problems from homework in our exams,

with some modifications like the question we just saw.

Homework, Exam

We will use some problems from homework in our exams,

with some modifications like the question we just saw.

Take homework seriously,

Homework, Exam

We will use some problems from homework in our exams,

with some modifications like the question we just saw.

Take homework seriously, and study the solutions carefully after we release them.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof:

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)\ )$.

Proof: By induction.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)\ )$.

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
   or $k^3 - k = 3q$ for some integer q.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$(k+1)^3 - (k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
    or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
 or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k)\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
    or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.}
\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.}\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3-n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\ &= 3(q + k^2 + k)\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
    or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive} + \text{over} \times
\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
    or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive } + \text{ over } \times
\end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\ &= 3(q + k^2 + k) \qquad \text{(Un)Distributive + over} \times\end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \qquad \text{Induction Hyp. Factor.} \\ &= 3(q + k^2 + k) \qquad \text{(Un)Distributive } + \text{ over } \times\end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
  $\implies (k+1)^3 - (k+1)$ is divisible by 3.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\ &= 3(q + k^2 + k) \qquad \text{(Un)Distributive} + \text{over} \times\end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
$\implies (k+1)^3 - (k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
 or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive } + \text{ over } \times
\end{aligned}
$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
 $\implies (k+1)^3 - (k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$
Thus, theorem holds by induction.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

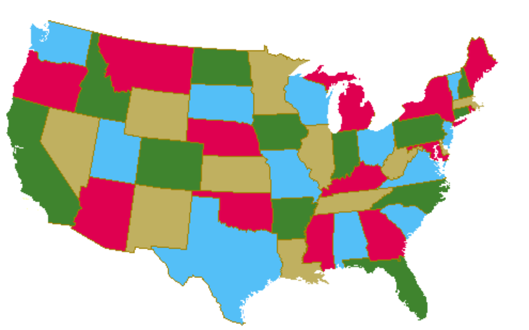
$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive + over} \times
\end{aligned}
$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
  $\implies (k+1)^3 - (k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$
Thus, theorem holds by induction. $\square$

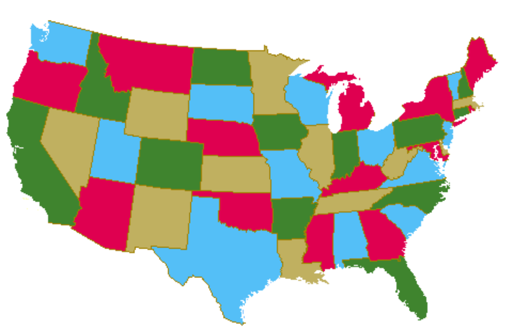
Four Color Theorem.

Theorem: Any map can be 4-colored so that those regions that share an edge have different colors.



Four Color Theorem.

Theorem: Any map can be 4-colored so that those regions that share an edge have different colors.



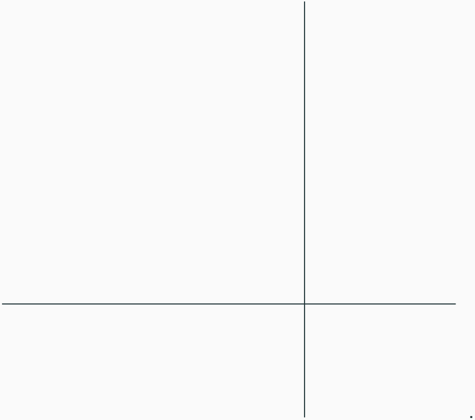
Not gonna prove it.

Two color theorem: example.

Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.

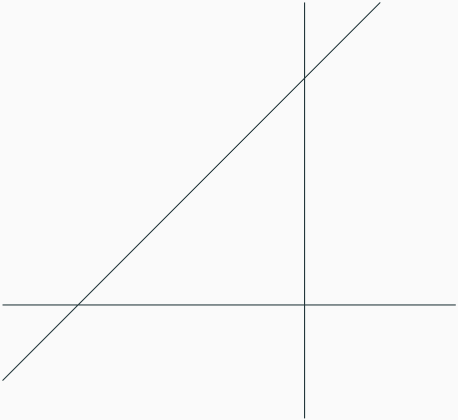
Two color theorem: example.

Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.



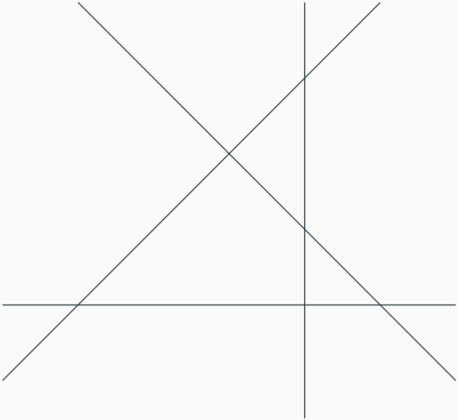
Two color theorem: example.

Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.



Two color theorem: example.

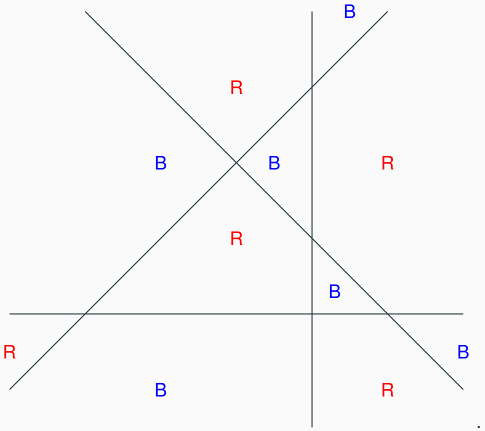
Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.



.

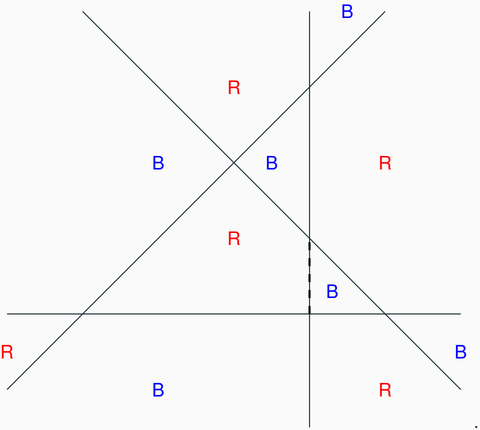
Two color theorem: example.

Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.



Two color theorem: example.

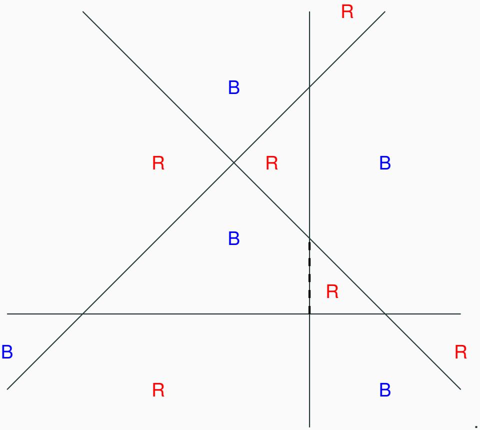
Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.



Fact: Swapping red and blue gives another valid colors.

Two color theorem: example.

Any map formed by dividing the plane M into regions by drawing straight lines can be colored with two colors so that those regions share an edge have different colors.

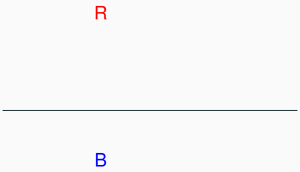


Fact: Swapping red and blue gives another valid colors.

Two color theorem: proof illustration.

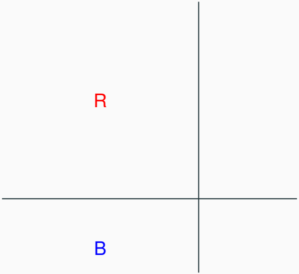
Base Case.

Two color theorem: proof illustration.



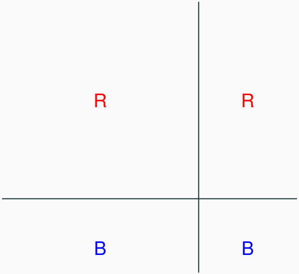
Base Case.

Two color theorem: proof illustration.



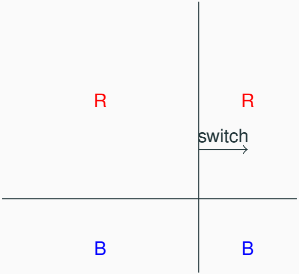
1. Add line.

Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions

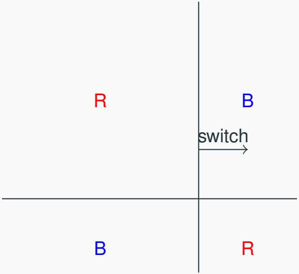
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

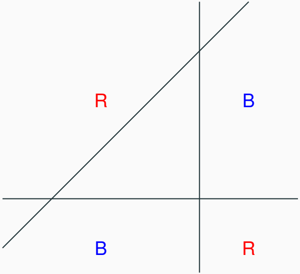
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

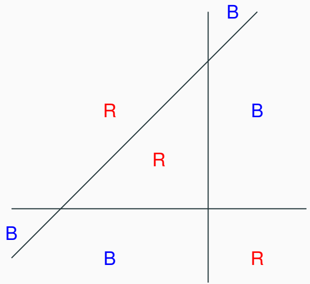
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

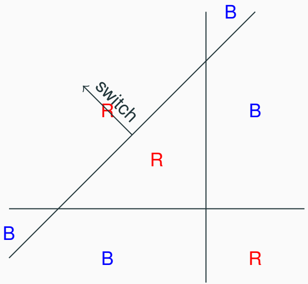
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

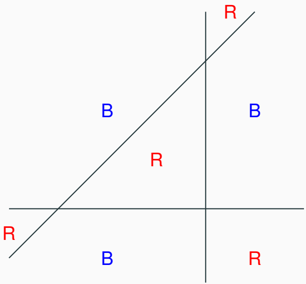
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

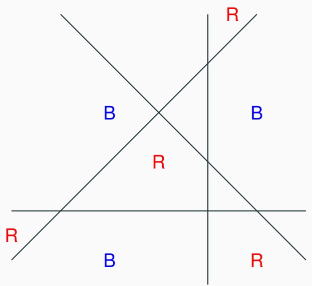
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

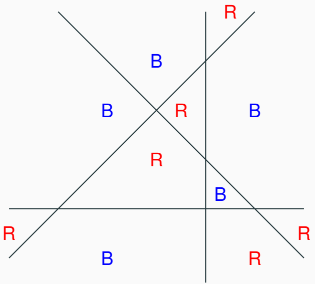
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

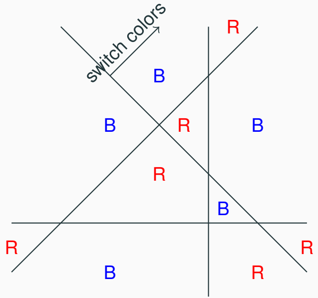
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

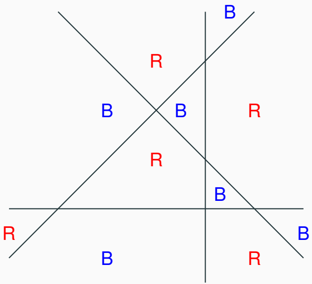
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

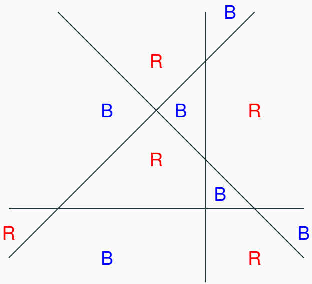
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

Two color theorem: proof illustration.

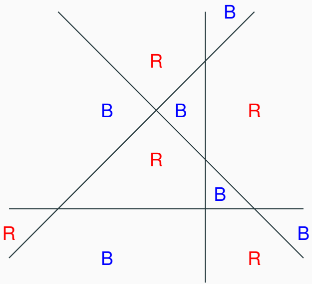


1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$.

Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$. $\square$

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square a^2

Induction Step
1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$

$\square$

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

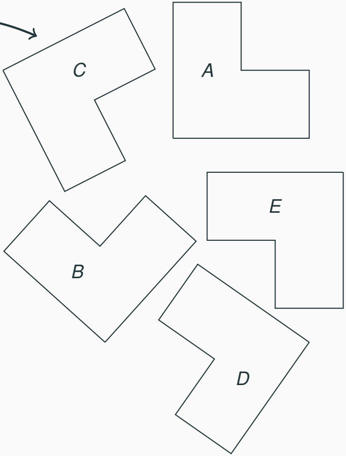
Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2 + 2k + 1 = k^2 + 2k + 1$
3. $k^2 + 2k + 1 = (k+1)^2$
   ... P(k+1)!

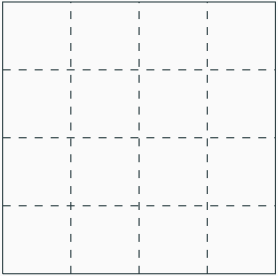
□

Tiling Cory Hall Courtyard.

Use these L-tiles.

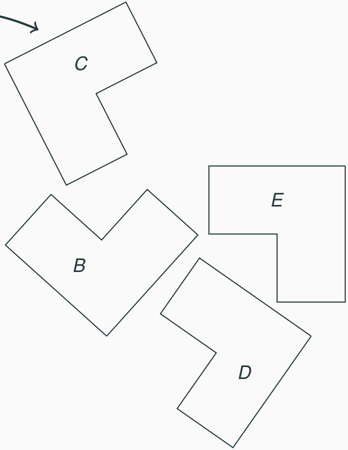


To Tile this 4×4 courtyard.

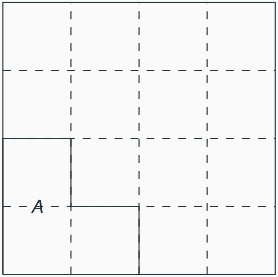


Tiling Cory Hall Courtyard.

Use these L-tiles.



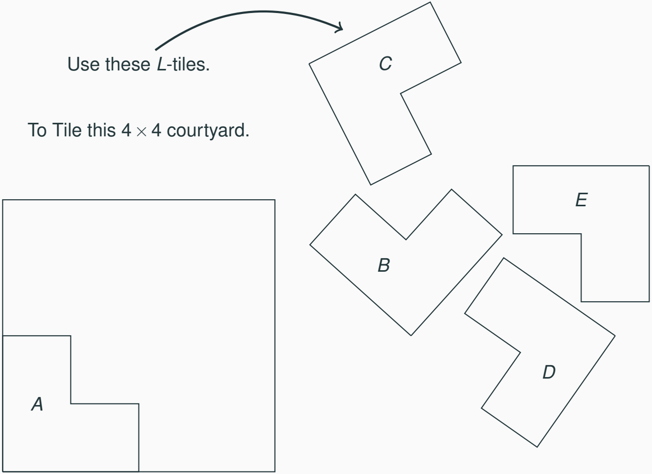
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

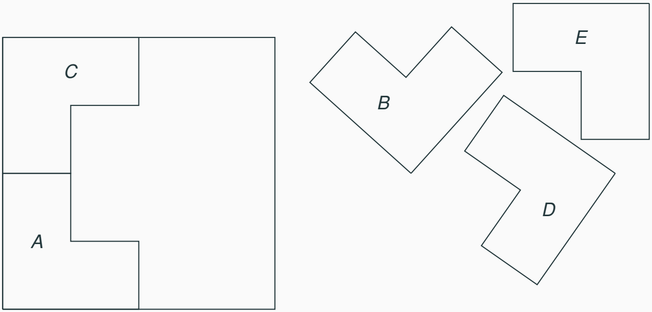
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

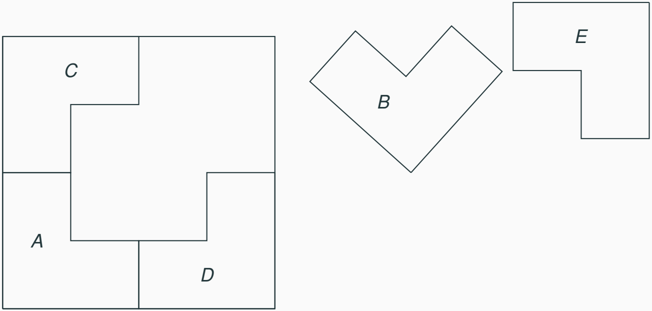
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

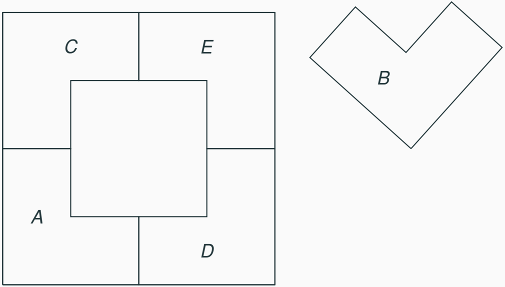
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

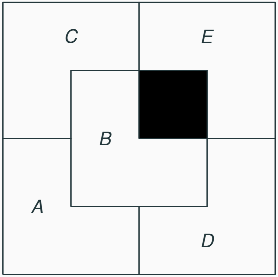
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

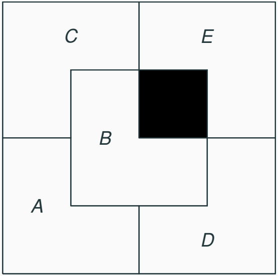
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.

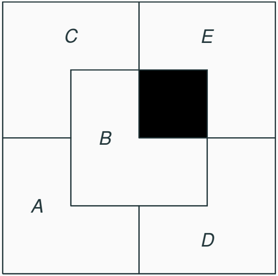


Alright!

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



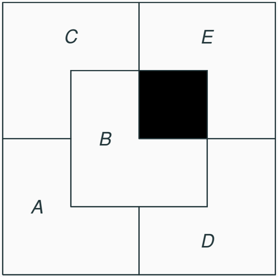
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

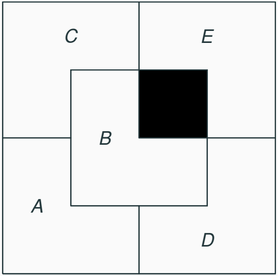
Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

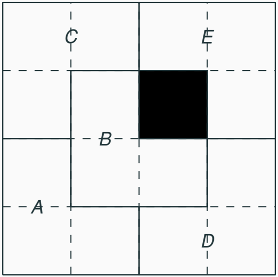
with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$2^{2(k+1)}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$2^{2(k+1)} \quad = \quad 2^{2k} * 2^2$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \end{aligned}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a + 1) \end{aligned}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a + 1) \\ &= 12a + 3 + 1 \end{aligned}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$
\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}
$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$
\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}
$$

a integer

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$
\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}
$$

a integer $\implies$ $(4a + 1)$ is an integer.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a+1) \\ &= 12a + 3 + 1 \\ &= 3(4a+1) + 1 \end{aligned}$$

a integer $\implies$ $(4a+1)$ is an integer. $\square$

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

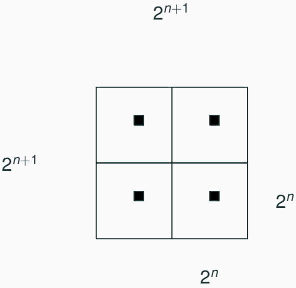
Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.



Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

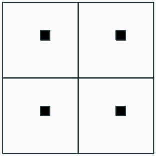
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

2^{n+1}

2^{n+1}



2^n

2^n

What to do now???

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

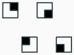
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



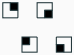
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



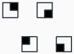
Induction Hypothesis:

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.



Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.

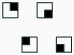
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

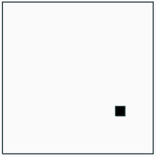
Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.



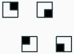
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

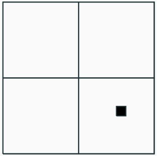
Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

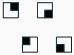
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

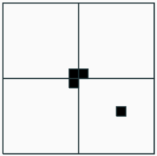
Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.



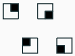
Use induction hypothesis in each.

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

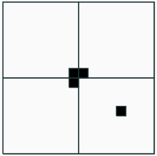
Base case: Sure. A tile is fine.



Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ...

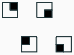
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

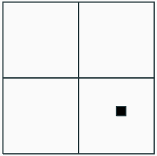
Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done.

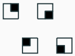
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

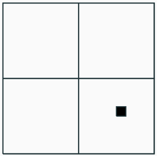
Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done. $\square$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"
$n+1$ can be written as the product of the prime factors!

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:
$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"
$n+1$ can be written as the product of the prime factors!

$\square$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$(\forall k \in N)(Q(k) \implies Q(k+1))$
$\quad \equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$$
\begin{aligned}
&(\forall k \in N)(Q(k) \implies Q(k+1)) \\
&\quad \equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))
\end{aligned}
$$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$$
\begin{aligned}
&(\forall k \in N)(Q(k) \implies Q(k+1)) \\
&\quad \equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1))) \\
&\quad \equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))
\end{aligned}
$$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$$\begin{aligned}
&(\forall k \in N)(Q(k) \implies Q(k+1)) \\
&\quad \equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1))) \\
&\quad \equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))
\end{aligned}$$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$$(\forall k \in N)(Q(k) \implies Q(k+1))$$
$$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$$
$$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))$$

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$$(\forall k \in N)(Q(k) \implies Q(k+1))$$
$$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$$
$$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))$$

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$(\forall k \in N)(Q(k) \implies Q(k+1))$
$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$
$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))$

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\neg\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\neg \forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\neg \forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\neg\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\neg \forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$$(\neg\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

E.g. Reduced form is “smallest” representation of a rational number a/b.

Tournaments have short cycles

Def: A **round robin tournament on n players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Tournaments have short cycles

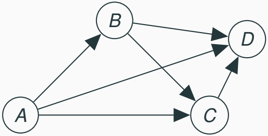
Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

Tournaments have short cycles

Def: A **round robin tournament on *n* players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

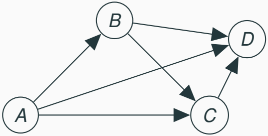
Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.



Tournaments have short cycles

Def: A **round robin tournament on** *n* **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

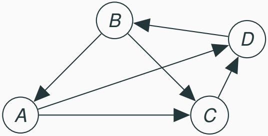


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

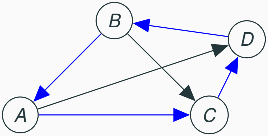


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on n players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

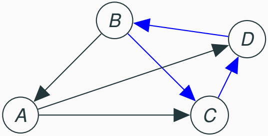


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on n players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.



Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

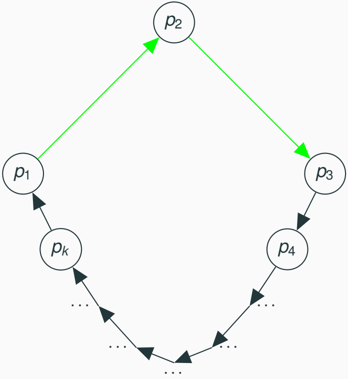
Case 1: Of length 3. Done.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



"$p_3 \to p_1$" $\implies$ 3 cycle

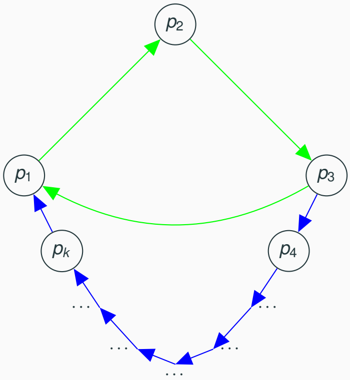
Contradiction.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



“$p_3 \to p_1$” $\implies$ 3 cycle

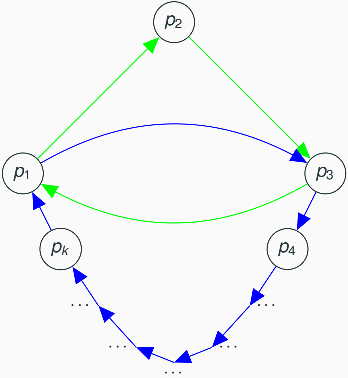
Contradiction.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



"$p_3 \to p_1$" $\implies$ 3 cycle

Contradiction.

"$p_1 \to p_3$" $\implies$ $k-1$ length cycle!

Contradiction!

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square a^2

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$
3.

$\square$

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2+2k+1 = k^2+2k+1$
3.
4. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

$\square$

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2+2k+1 = k^2+2k+1$
3.
4. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

□

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$
3.
4. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

$\square$

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step
1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2 + 2k + 1 = k^2 + 2k + 1$
3.
4. $k^2 + 2k + 1 = (k+1)^2$
   ... P(k+1)!

$\square$

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2 + 2k + 1 = k^2 + 2k + 1$
3.
4. $k^2 + 2k + 1 = (k+1)^2$
   ... P(k+1)!

$\square$

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (1th odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2+2k+1 = k^2+2k+1$
3. ???
4. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

$\square$

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
```

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
```

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
```

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
```

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases:

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12)

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13)

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14)

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15).

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:
 Recursive call is correct: $P(n-4)$

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:
 Recursive call is correct: $P(n-4) \implies P(n)$.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
     (x',y') = find-x-y(n-4)
     return(x'+1,y')
```

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.
$n - 4 = 4x' + 5y' \implies n = 4(x'+1) + 5(y')$

Summary: principle of induction.

Summary: principle of induction.

$(P(0)$

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1))))$$

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Statement to prove: $P(n)$ for n starting from n_0

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove.

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$,

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1))))$

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
  Sum of first n odds is n^2.
  Hole anywhere.
Not same as strong induction.

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

Induction $\equiv$ Recursion.

Summary: principle of induction.

$(P(0)$

Summary: principle of induction.

$$(P(0) \land ((\forall k \in N)(P(k) \implies P(k+1))))$$

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Variations:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Variations:

$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1))))$

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

$$\begin{aligned}&(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))) \\ &\qquad \implies (\forall n \in N)((n \geq 1) \implies P(n))\end{aligned}$$

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

$$\begin{aligned}&(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))) \\ &\qquad \implies (\forall n \in N)((n \geq 1) \implies P(n))\end{aligned}$$

Statement to prove: $P(n)$ for n starting from n_0

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

$$\begin{aligned}&(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))) \\ &\qquad \implies (\forall n \in N)((n \geq 1) \implies P(n))\end{aligned}$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

$$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))) \\ \implies (\forall n \in N)((n \geq 1) \implies P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove.

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

$$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1))))$$
$$\implies (\forall n \in N)((n \geq 1) \implies P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$,

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

$$\begin{aligned}(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))) \\ \implies (\forall n \in N)((n \geq 1) \implies P(n))\end{aligned}$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Variations:

$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

$$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1))))$$
$$\implies (\forall n \in N)((n \geq 1) \implies P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!