CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Discussion 1B Sol

1. Logic

Decide whether each of the following is true or false and justify your answer:

a) $\forall x\big(P(x) \wedge Q(x)\big) \equiv \forall x P(x) \wedge \forall x Q(x)$

True

$\forall x P(x)$	$\forall x Q(x)$	$\forall x P(x) \wedge \forall x Q(x)$	$\forall x\big(P(x) \wedge Q(x)\big)$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

b) $\forall x\big(P(x) \vee Q(x)\big) \equiv \forall x P(x) \vee \forall x Q(x)$

False. If $P(1)$ is true, $Q(1)$ is false, $P(2)$ is false and $Q(2)$ is true, the left-hand side will be true, but the right-hand side will be false.

2. (Proof)

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2+4l+1 = 4l(l+1)+1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

3. Contradiction

Prove that $2^{1/n}$ is not rational for any integer $n > 3$. [Hint : Fermat's Last Theorem and the method of contradiction]

If not, then there exists an integer $n > 3$ such that $2^{1/n} = \frac{p}{q}$ where p, q are positive integers. Thus, $2q^n = p^n$, and this implies,

$$q^n + q^n = p^n$$

, which is a contradiction to the Fermat's Last Theorem.

4. Problem solving

Prove that if you put $n+1$ apples into n boxes, any way you like, then at least one box must contain at least 2 apples. This is known as the *pigeonhole principle*.

Suppose this is not the case. Then all the boxes would contain at most 1 apple. Then the maximum number of apples we could have would be n, but this is a contradiction since we have $n+1$ apples.

5. Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. **Answer:** Suppose the contrary that everyone has a different number of friends at the party.

Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \dots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.