

CS70: Discrete Math and Probability

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Trees.

Definitions:

- A connected graph without a cycle.
- A connected graph with $|V| - 1$ edges.
- A connected graph where any edge removal disconnects it.
- A connected graph where any edge addition creates a cycle.

Some trees.

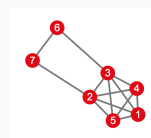
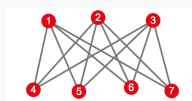


no cycle and connected? Yes.
 $|V| - 1$ edges and connected? Yes.
 removing any edge disconnects it. Harder to check. but yes.
 Adding any edge creates cycle. Harder to check. but yes.

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Planar non-planar

A finite graph is planar iff it does not contain a subgraph that is (a subdivision of) K_5 or $K_{3,3}$



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Equivalence of Definitions.

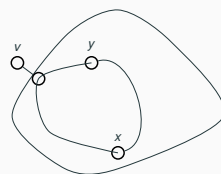
Theorem:

"G connected and has $|V| - 1$ edges" \equiv
 "G is connected and has no cycles."

Lemma: If v is a degree 1 in connected graph G , $G - v$ is connected.

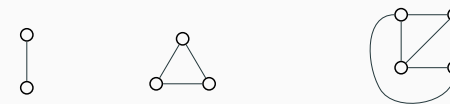
Proof:

For $x \neq v, y \neq v \in V$,
 there is path between x and y in G since connected.
 and does not use v (degree 1)
 $\Rightarrow G - v$ is connected. \square



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Complete Graph.



K_n complete graph on n vertices.
 All edges are present.
 Everyone is my neighbor.
 Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n - 1$ edges.

Sum of degrees is $n(n - 1)$.

\Rightarrow Number of edges is $n(n - 1)/2$.

Remember sum of degree is $2|E|$.

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Proof of only if.

Thm:

"G connected and has $|V| - 1$ edges" \equiv
 "G is connected and has no cycles."



Proof of \Rightarrow : By induction on $|V|$.

Base Case: $|V| = 1$, $0 = |V| - 1$ edges and has no cycles.

Induction Step:

Claim: There is a degree 1 node.

Proof: First, connected \Rightarrow every vertex degree ≥ 1 .

Sum of degrees is $2|V| - 2$

Average degree $2 - 2/|V|$

Not everyone is bigger than average! \square

By degree 1 removal lemma, $G - v$ is connected.

$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction

\Rightarrow no cycle in $G - v$.

And no cycle in G since degree 1 cannot participate in cycle. \square

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Proof of if

Thm:

"G is connected and has no cycles" \implies "G connected and has $|V| - 1$ edges"

Proof:

Walk from a vertex using untraversed edges.
Until get stuck.

Claim: Must stuck at a degree 1 vertex.

Proof of Claim:

Can't visit any vertex more than once since no cycle.
Entered. Didn't leave. Only one incident edge.

Removing node doesn't create cycle.

New graph is connected.

Removing degree 1 node doesn't disconnect from Degree 1 lemma.

By induction $G - v$ has $|V| - 2$ edges.

G has one more or $|V| - 1$ edges.

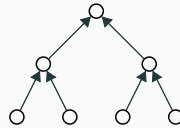
□

□

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Tree's fall apart.

Thm: Can always find a node such that the largest connected component we get by removing it has size at most $|V|/2$



Idea of proof.

Point edge toward bigger side.

Remove center node.

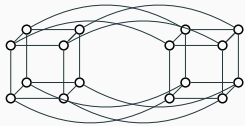


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Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An n -dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$ -dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.



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Hypercube: Can't cut me!

Thm: Any subset S of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

Terminology:

$(S, V - S)$ is cut.

a partition of the vertices of a graph into two disjoint subsets.

$(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

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Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V| - 1)/2$

Trees, But few edges. $(|V| - 1)$

just falls apart!

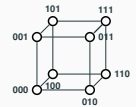
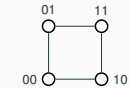
Hypercubes. Really connected. $|V| \log |V|/2$ edges!

Also represents bit-strings nicely.

$G = (V, E)$

$|V| = \{0, 1\}^n$,

$|E| = \{(x, y) | x \text{ and } y \text{ differ in one bit position.}\}$



2^n vertices. number of n -bit strings!

$n2^{n-1}$ edges.

2^n vertices each of degree n
total degree is $n2^n$ and half as many edges!

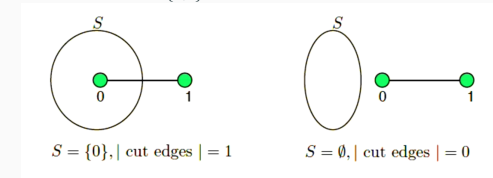
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Proof of Large Cuts.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

Proof:

Base Case: $n = 1$ $V = \{0, 1\}$.



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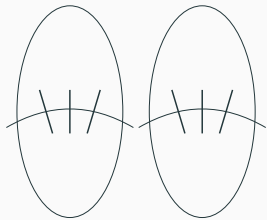
Induction Step Idea

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

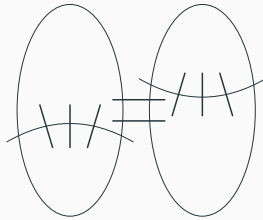
Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

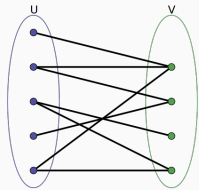


Case 2: Count inside and across.



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Bipartite graph



Bipartite graph: a bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V .

U and V are sometimes called the parts of the graph.

Coloring? How many colors do we need? 2!

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Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.

Recursive definition:

$H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges E_x that connect them.

$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$

$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$

Both S_0 and S_1 are small sides. So by induction.

Edges cut in $H_0 \geq |S_0|$.

Edges cut in $H_1 \geq |S_1|$.

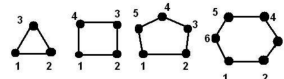
Total cut edges $\geq |S_0| + |S_1| = |S|$.

□

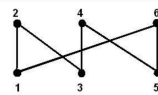
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Bipartite?

Which of the following graphs are bipartite?



No Yes No Yes



A graph is a bipartite graph if and only if it does not contain any odd-length cycles.

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Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies \geq |S_1|$ edges cut in E_1 .

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in E_0 .

Edges in E_x connect corresponding nodes.

$\implies \geq |S_0| - |S_1|$ edges cut in E_x .

Total edges cut:

$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$

$|V_0| = |V|/2 \geq |S|$.

□

Also, case 3 where $|S_1| \geq |V_1|/2$ is symmetric.

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Proof

Only if: trivial

Start at a node v in one part, say V , the cycle must be like leaving V , entering V , ... Also the cycle must end at v , so the cycle must end with "entering V ". All paired up, even length.

No odd-length cycle \implies bipartite:

Different connected components does not influence each other, just look at one first

Pick one arbitrary vertex v , split all vertices into two groups

$A = \{u \in V \mid \exists \text{ odd length path from } v \text{ to } u\}$

$B = \{u \in V \mid \exists \text{ even length path from } v \text{ to } u\}$

We have a bipartite graph if A and B are disjoint.

What if a vertex in both sets? Odd length cycle! Contradiction

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What have we done?!

Graphs!

Eulerian tour: DNA sequence reconstructing

Coloring: Cellular tower frequency assignment

Trees: Immense applications.....

Modeling reality:

Internet? Giant directed graph

Dark net? A separate connect component!

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