CS70: Discrete Math and Probability

Fan Ye

June 28, 2016

Planar non-planar

A finite graph is planar iff it does not contain a subgraph that is (a subdivision of) K_5 or $K_{3,3}$

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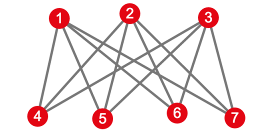

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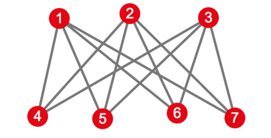
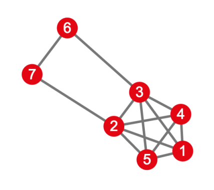
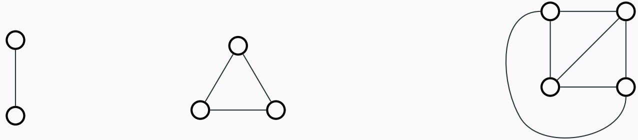




Complete Graph.



K_n complete graph on n vertices.

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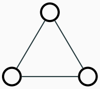

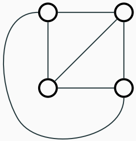

K_n complete graph on n vertices.
All edges are present.

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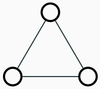
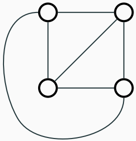


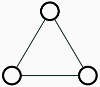
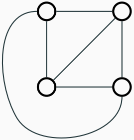


K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Complete Graph.

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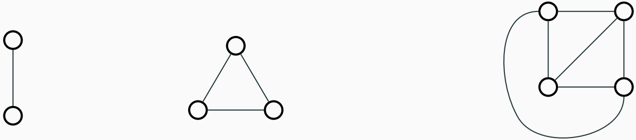

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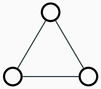

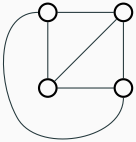

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How many edges?

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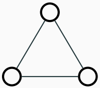

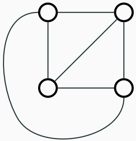

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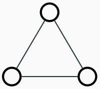

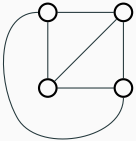

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- Sum of degrees is $n(n-1)$.

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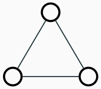
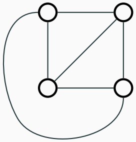

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$\implies$ Number of edges is $n(n-1)/2$.

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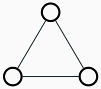

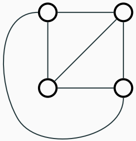

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Remember sum of degree is $2|E|$.

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A connected graph without a cycle.

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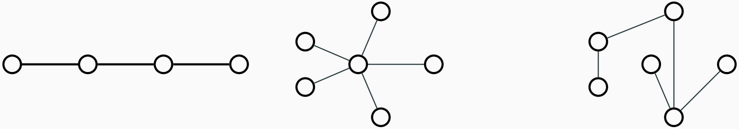

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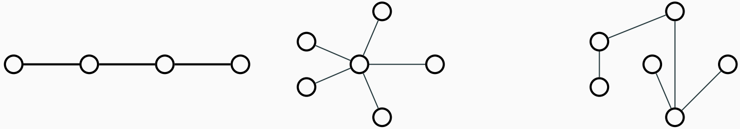

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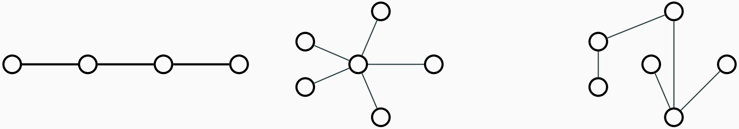

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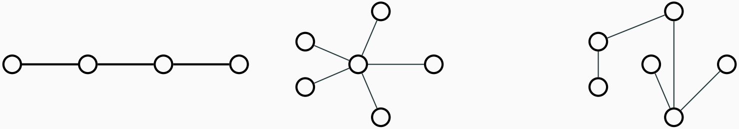

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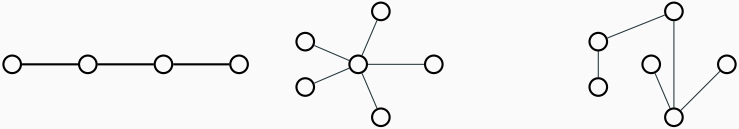

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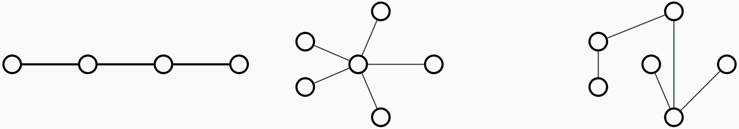

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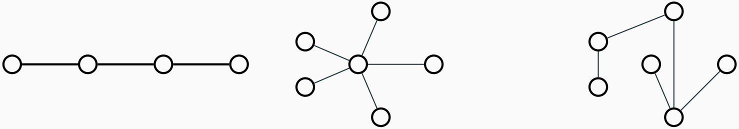

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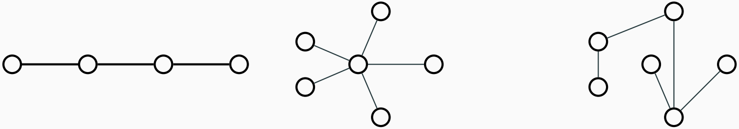

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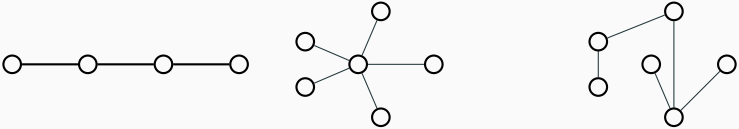

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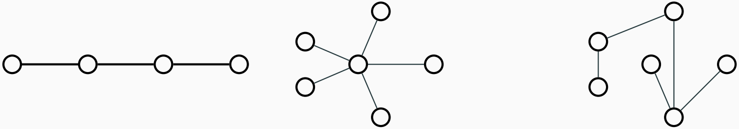

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Theorem:
"G connected and has $|V| - 1$ edges" $\equiv$
"G is connected and has no cycles."

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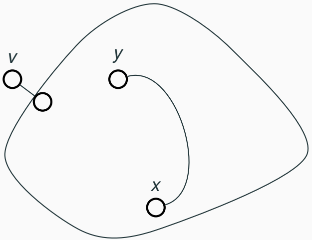

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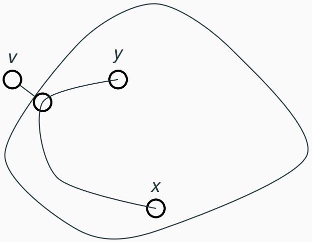

Proof of only if.

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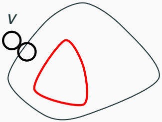

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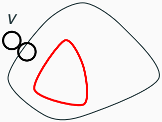

Proof of $\implies$: By induction on $|V|$.

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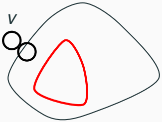

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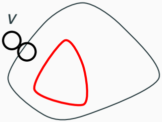

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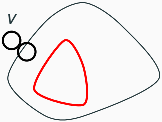

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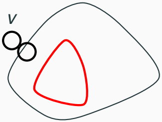
Induction Step:

Proof of only if.



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Claim: There is a degree 1 node.

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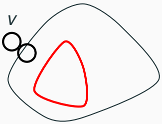

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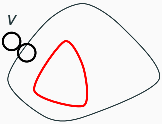

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Sum of degrees is $2|V|-2$

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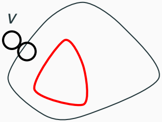

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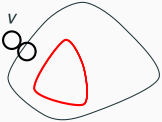

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Not everyone is bigger than average!

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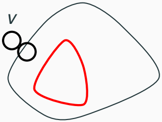

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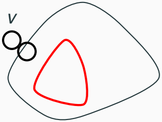
Proof of $\implies$: By induction on $|V|$.
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.

Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|V|-2$
Average degree $2-2/|V|$
Not everyone is bigger than average! $\square$
By degree 1 removal lemma, $G-v$ is connected.

Proof of only if.



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$G-v$ has $|V|-1$ vertices and $|V|-2$ edges so by induction

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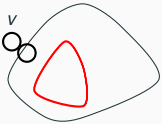

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By degree 1 removal lemma, $G-v$ is connected.
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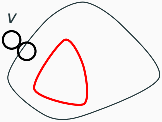
Proof of only if.



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By degree 1 removal lemma, $G-v$ is connected.
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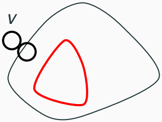
And no cycle in G since degree 1 cannot participate in cycle.

Proof of only if.



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"G is connected and has no cycles" $\implies$ "G connected and has $|V|-1$ edges"

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"G is connected and has no cycles" $\implies$ "G connected and has $|V|-1$ edges"

Proof:
Walk from a vertex using untraversed edges.

Proof of if

Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V|-1$ edges"

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.

Proof of if

Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V| - 1$ edges"

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Must stuck at a degree 1 vertex.

Proof of if

Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V| - 1$ edges"

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Must stuck at a degree 1 vertex.
Proof of Claim:
Can't visit any vertex more than once since no cycle.

Proof of if

Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V| - 1$ edges"

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Must stuck at a degree 1 vertex.
Proof of Claim:
Can't visit any vertex more than once since no cycle.
Entered.

Proof of if

Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V| - 1$ edges"

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Must stuck at a degree 1 vertex.
Proof of Claim:
Can't visit any vertex more than once since no cycle.
Entered. Didn't leave.

Proof of if

Thm:
"G is connected and has no cycles" $\Longrightarrow$ "G connected and has $|V|-1$ edges"

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.

Claim: Must stuck at a degree 1 vertex.

Proof of Claim:
Can't visit any vertex more than once since no cycle.
Entered. Didn't leave. Only one incident edge.

Proof of if

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Removing node doesn't create cycle.

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Removing node doesn't create cycle.
New graph is connected.

Proof of if

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Removing node doesn't create cycle.
New graph is connected.
Removing degree 1 node doesn't disconnect from Degree 1 lemma.

Proof of if

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Removing node doesn't create cycle.
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Removing degree 1 node doesn't disconnect from Degree 1 lemma.
By induction $G - v$ has $|V| - 2$ edges.

Proof of if

Thm:
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Proof:
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Proof of Claim:
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Entered. Didn't leave. Only one incident edge. $\square$

Removing node doesn't create cycle.
New graph is connected.
Removing degree 1 node doesn't disconnect from Degree 1 lemma.
By induction $G-v$ has $|V|-2$ edges.
G has one more or $|V|-1$ edges.

Proof of if

Thm:
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Walk from a vertex using untraversed edges.
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Proof of Claim:
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Tree's fall apart.

Thm: Can always find a node such that the largest connected component we get by removing it has size at most $|V|/2$

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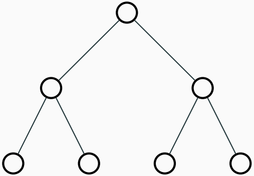

Idea of proof.

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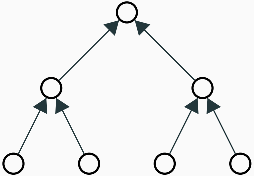

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Point edge toward bigger side.

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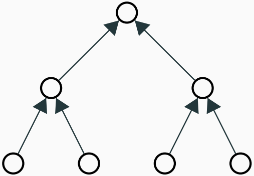

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Point edge toward bigger side.
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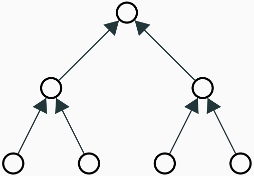

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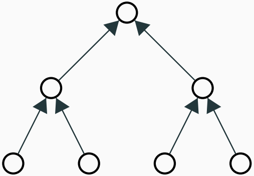

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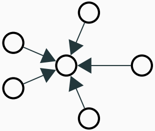

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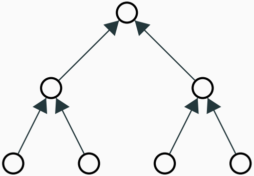

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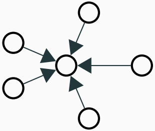

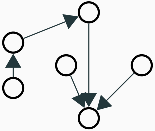

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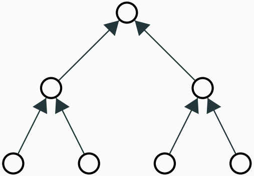

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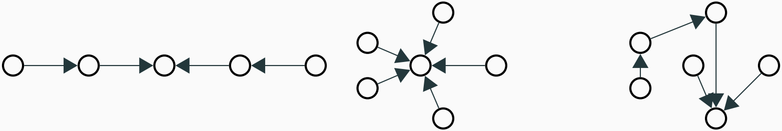

Hypercubes.

Complete graphs, really connected!

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Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

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$G=(V,E)$

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$G = (V,E)$

$|V| = \{0,1\}^n$,

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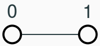

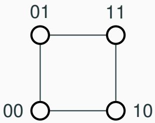

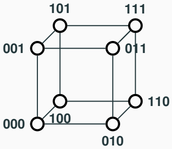

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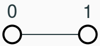

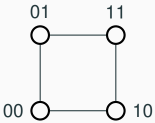

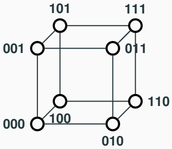

2^n vertices.

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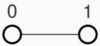

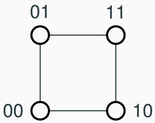

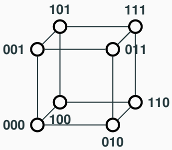

2^n vertices. number of n-bit strings!

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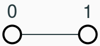

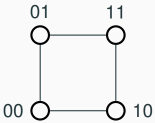

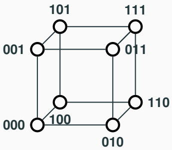

2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.

Hypercubes.

Complete graphs, really connected! But lots of edges.
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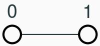

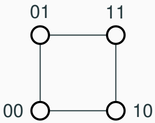

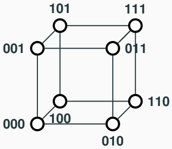

2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.
  2^n vertices each of degree n

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, But few edges. $(|V|-1)$

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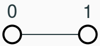

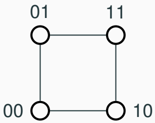

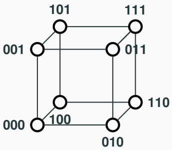

2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.

2^n vertices each of degree n

total degree is $n2^n$

Hypercubes.

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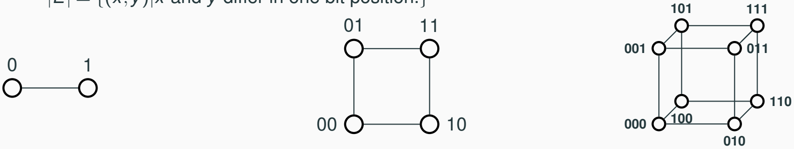

2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.
  2^n vertices each of degree n
   total degree is $n2^n$ and half as many edges!

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, But few edges. $(|V|-1)$

just falls apart!

Hypercubes. Really connected.

Also represents bit-strings nicely.

$G = (V, E)$

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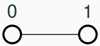

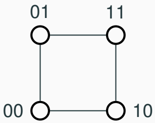

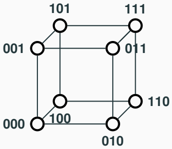

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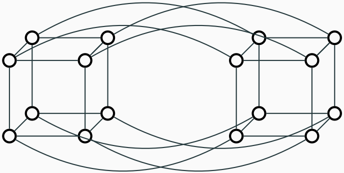

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Proof of Large Cuts.

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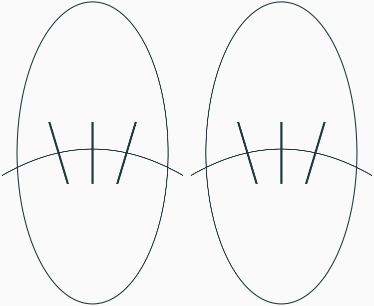

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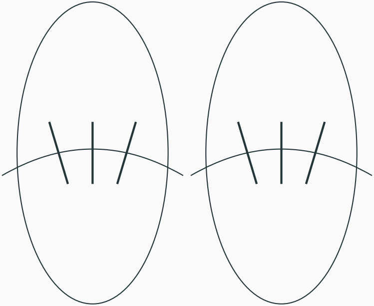

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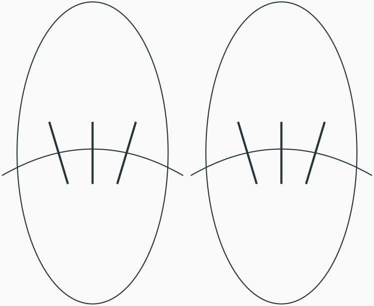

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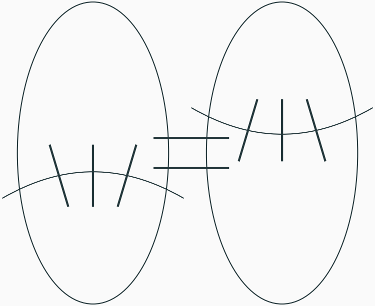

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Total edges cut:

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Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

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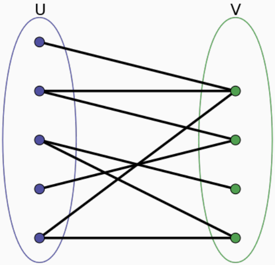
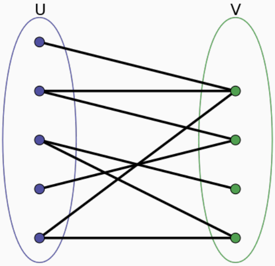
Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

Bipartite graph

Bipartite graph



Bipartite graph



Bipartite graph:

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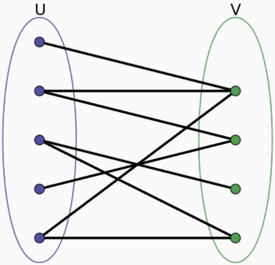

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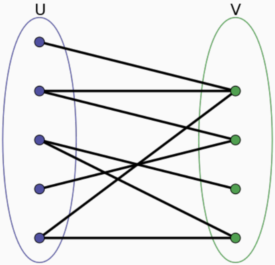

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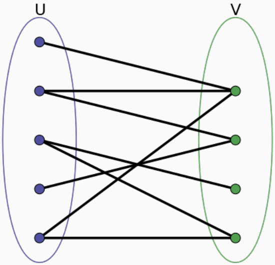

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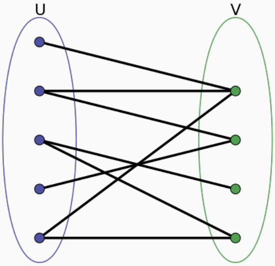
Coloring?

Bipartite graph



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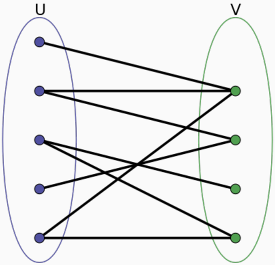
Coloring? How many colors do we need?

Bipartite graph



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Coloring? How many colors do we need? 2!

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Which of the following graphs are bipartite?

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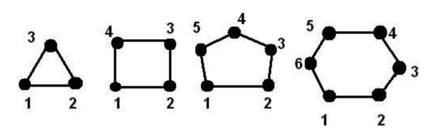

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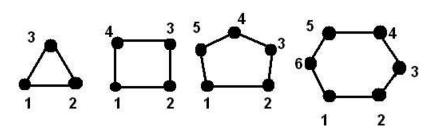

No

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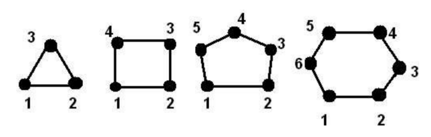

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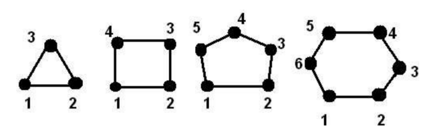

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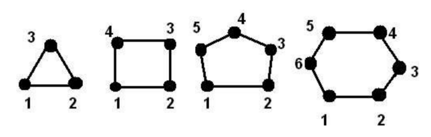

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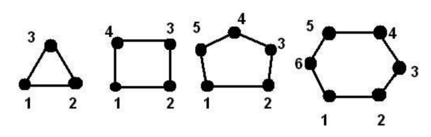

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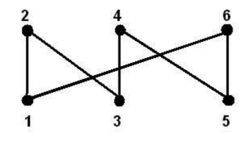

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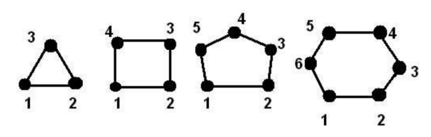

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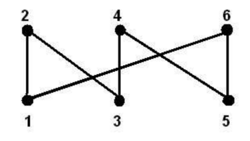

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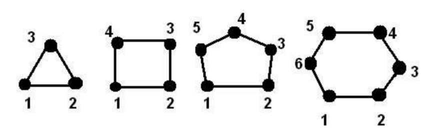

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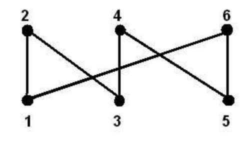

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Also the cycle must end at v, so the cycle must end with ”entering V”.

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What have we done?!

Graphs!

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Eulerian tour:

What have we done?!

Graphs!

Eulerian tour: DNA sequence reconstructing

What have we done?!

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Eulerian tour: DNA sequence reconstructing

Coloring:

What have we done?!

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Trees:

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Trees: Immense applications.........

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Modeling reality:

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Trees: Immense applications.........

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Internet?

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Modeling reality:

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Modeling reality:

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- Dark net?

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