

CS70: Countability and Uncountability

Alex Psomas

June 30, 2016

Warning!

Warning:

Warning!

Warning: I'm really loud!

Today.

Today.

One idea, from around 130 years ago.

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At the heart of set theory.

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Started a crisis in mathematics in the middle of the previous century!

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The idea.

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The idea: **More than one infinities!!!!!!**

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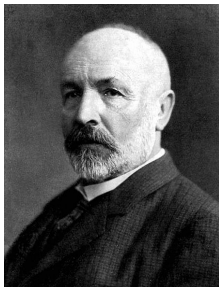
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Georg Cantor

Life before Cantor

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How many elements in $\{1, 2, 4\}$?

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The symbol you write after taking a limit....

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Even Gauss: "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction. "

Cantor's questions

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Is $\mathbb{N} \setminus \{0\}$ smaller than \mathbb{N} ?

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Hilbert's hotel

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A hotel with infinite rooms.

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A hotel with infinite rooms. Rooms are numbered from 1 to infinity.

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Every room is occupied.

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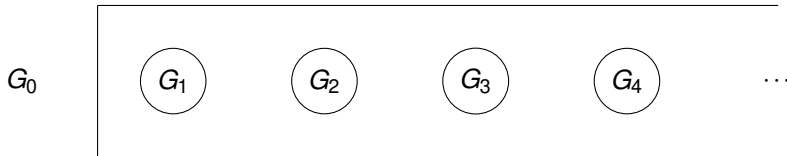
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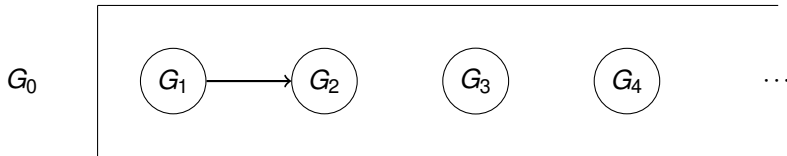
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G_0 shows up. What do we do?

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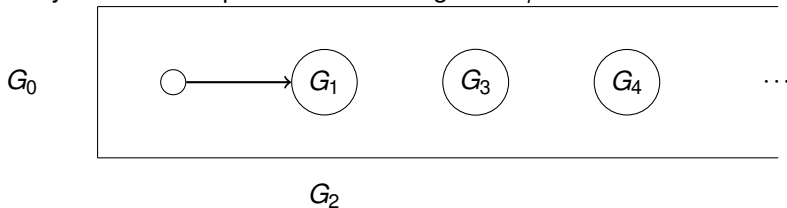


G_0 shows up. What do we do?

Move G_1 to room number 2.

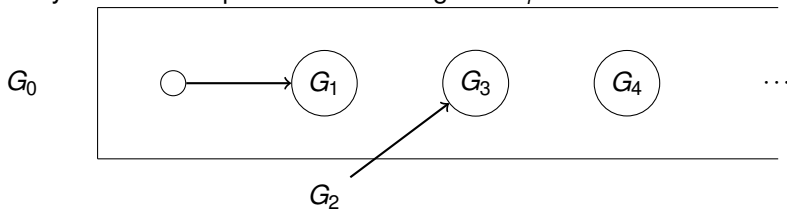
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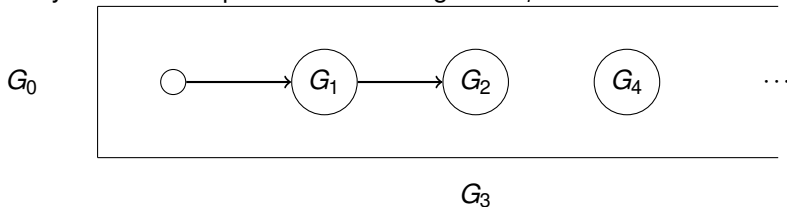
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Move G_2 to room number 3.

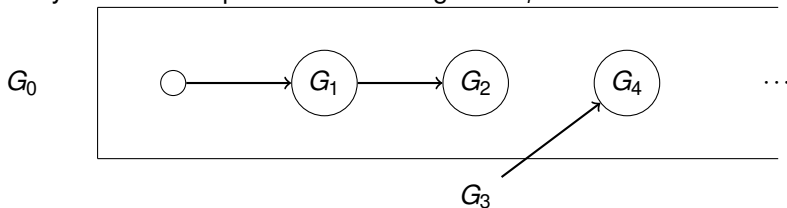
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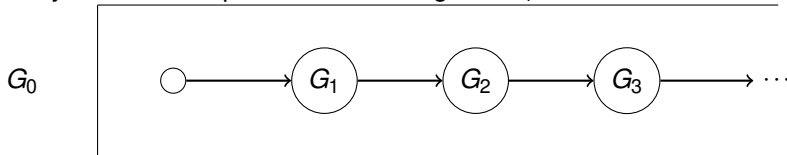
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Move G_3 to room number 4.

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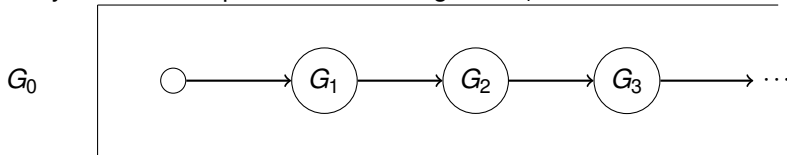
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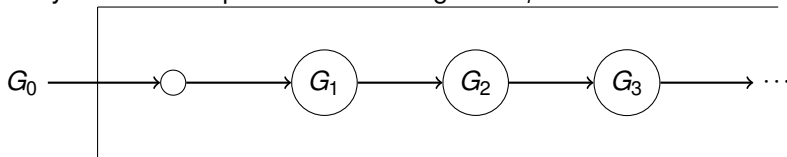


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Now G_0 can go to room number 1!!

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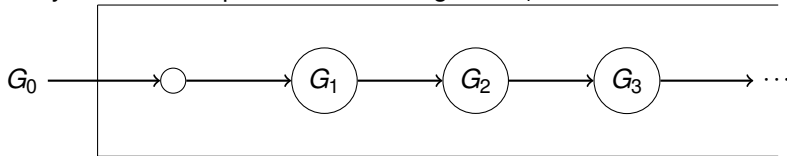


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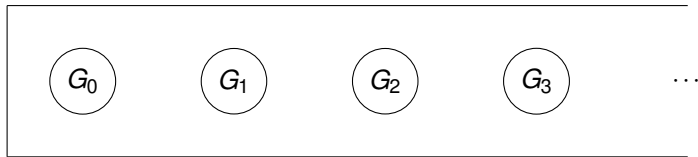
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Is this a proof? How would we show this formally???

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If the subset of \mathbb{N} is finite, S has finite **cardinality**.

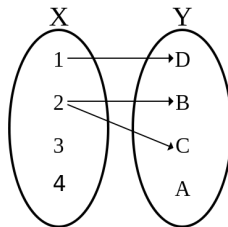
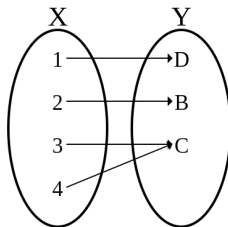
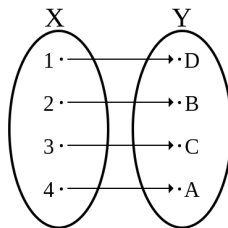
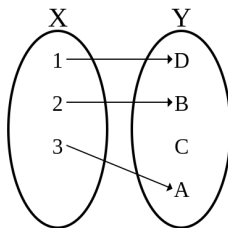
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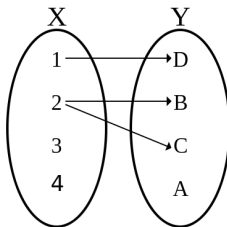
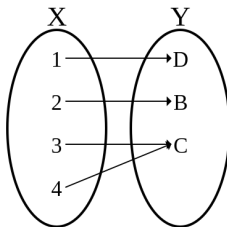
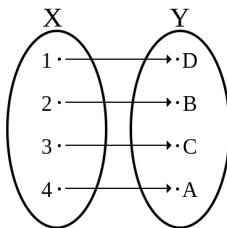
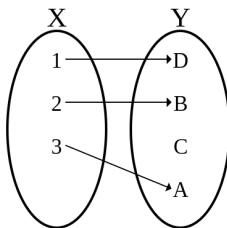
If the subset of \mathbb{N} is infinite, S is **countably infinite**.

Bijections?



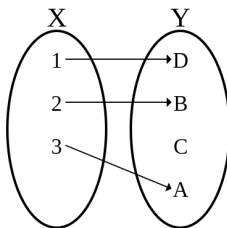
Bijections?

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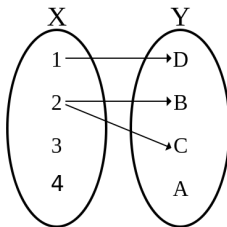
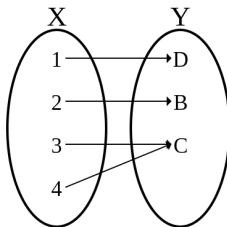
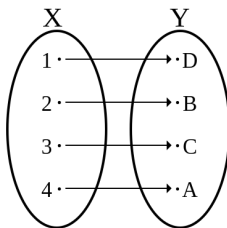


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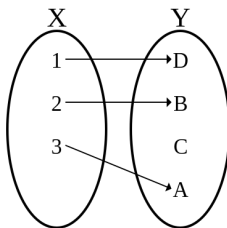


Bijection: one to one and onto.

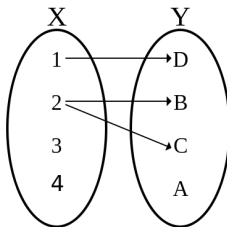
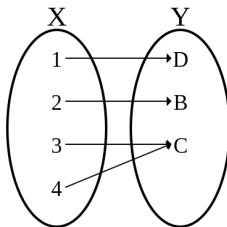
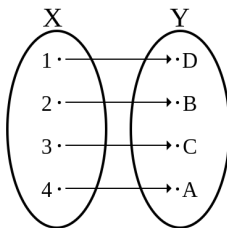


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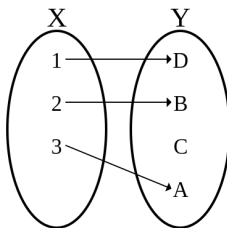
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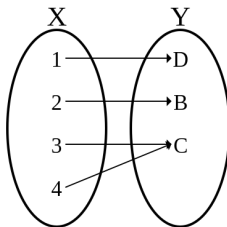
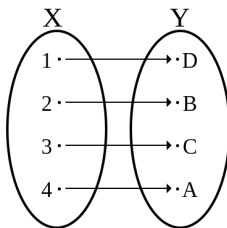
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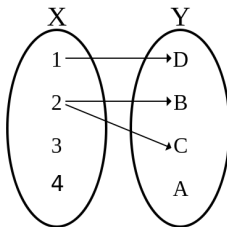
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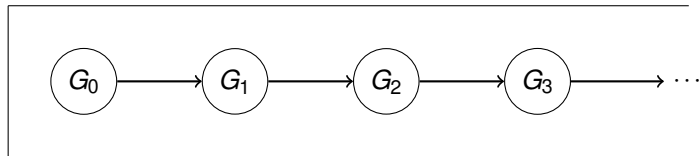
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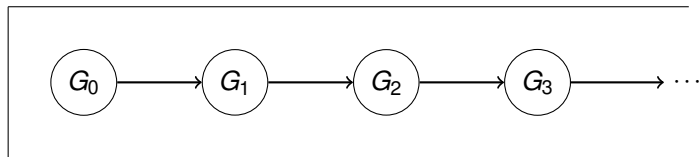
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For example the set $\{14, 54, 5332, 10^{12} + 4\}$ is countable. (It has 4 elements) Even numbers are countable. Prime numbers are countable. Multiples of 3 are countable.
- ▶ All countably infinite sets have the same cardinality as each other.

Back to Hilbert's hotel

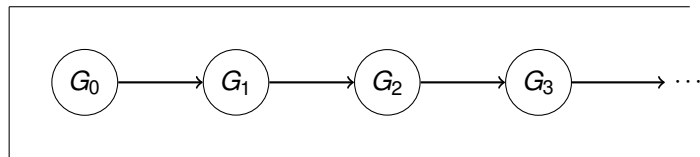


Back to Hilbert's hotel



Where's the function?

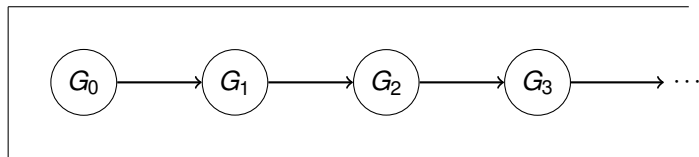
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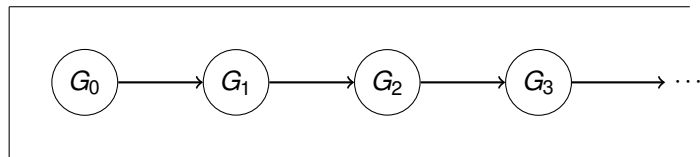
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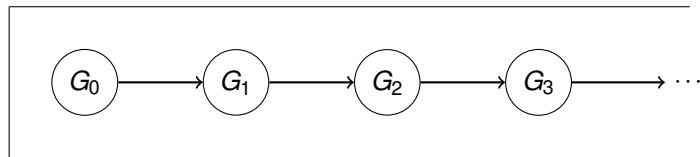
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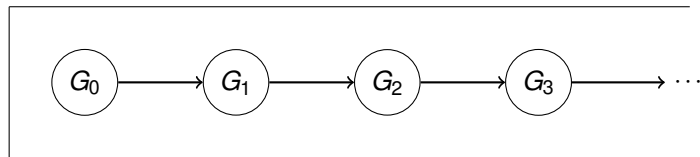
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Where's the function?

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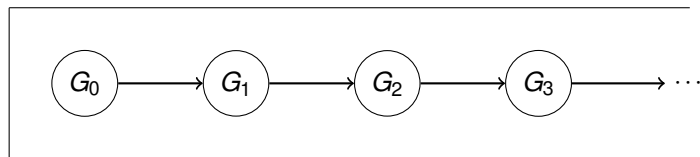


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$$f(x) = x - 1.$$

Back to Hilbert's hotel

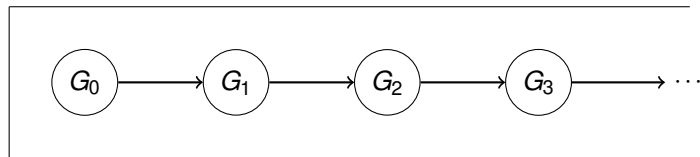


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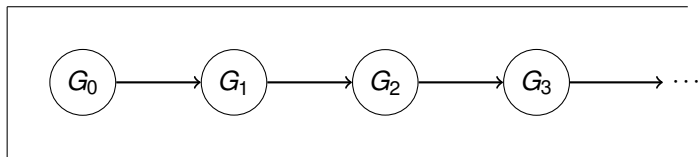


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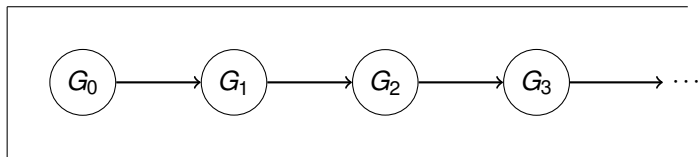
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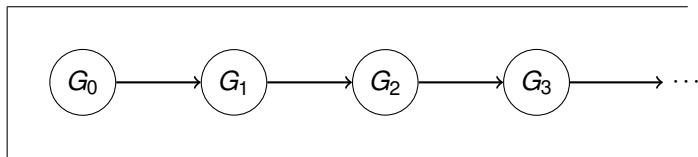
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Countably infinite (same cardinality as naturals)

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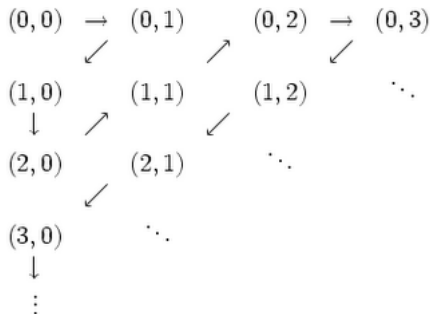
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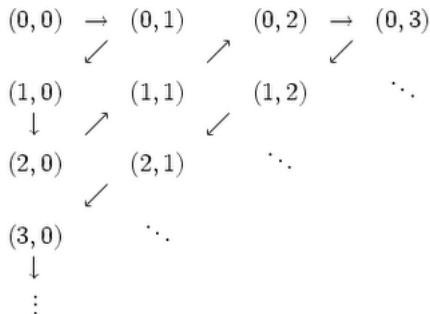
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(a, b) at position $(a+b+1)(a+b)/2 + b$ in this order.

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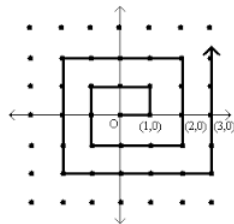
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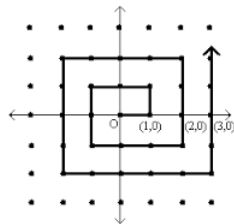
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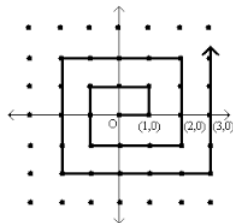
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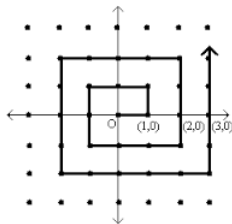
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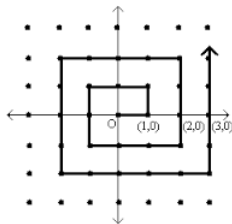
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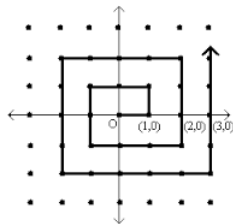
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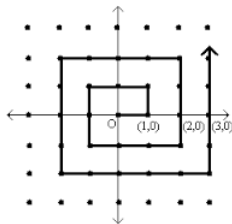
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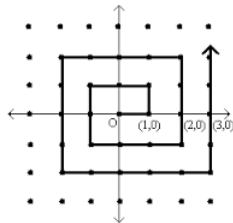
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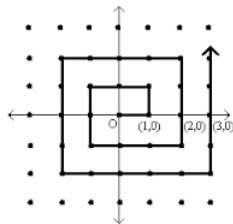
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Will eventually get to any pair.

Two different pairs cannot map to the same natural number/same position in the spiral.

Every natural has a "corresponding" pair.

Where's my bijection???

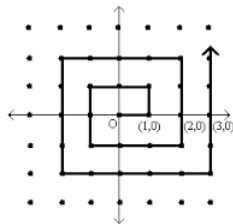
Rationals

All rational numbers \mathbb{Q} : $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.

Enumerate: list 0, positive and negative. **How?**

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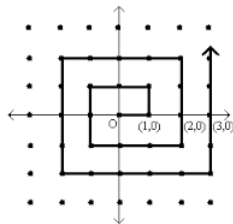
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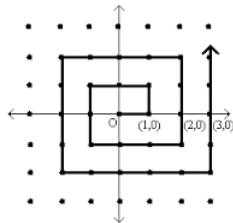
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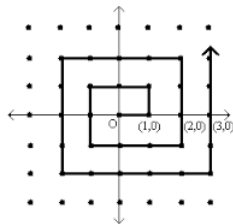
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Lets consider the reals $[0, 1]$.

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Construct “diagonal” number: .7

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\vdots

Construct “diagonal” number: .77

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⋮

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Contradiction!

Diagonalization.

If **countable**, there exists a listing (enumeration), **L contains all reals in $[0, 1]$** . For example

0: .**5**00000000...

1: .7**8**5398162...

2: .36**7**879441...

3: .632**1**20558...

4: .3452**1**2312...

\vdots

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Subset $[0, 1]$ is not countable!!

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What about all reals?

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Any subset of a countable set is countable.

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If reals are countable then so is $[0, 1]$.

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1. Assume that a set S can be enumerated.

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Another diagonalization.

The set of all subsets of N .

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Example subsets of N : $\{0\}$,

Another diagonalization.

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- Assume is countable.

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Example subsets of N : $\{0\}$, $\{0, \dots, 7\}$,
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- ▶ Assume is countable.
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- ▶ L does not contain all subsets of N .
Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S , is the **powerset** of N .)

Another diagonalization.

$$\begin{array}{lcl} s_1 & = & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \dots \\ s_2 & = & 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \dots \\ s_3 & = & 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ \dots \\ s_4 & = & 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ \dots \\ s_5 & = & 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ \dots \\ s_6 & = & 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ \dots \\ s_7 & = & 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \dots \\ s_8 & = & 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \dots \\ s_9 & = & 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ \dots \\ s_{10} & = & 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ \dots \\ s_{11} & = & 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \dots \\ & \vdots & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\end{array}$$

$$s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ \dots$$

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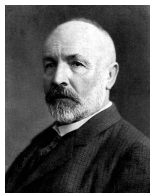
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You already know some of these..... Think about induction!

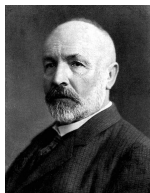
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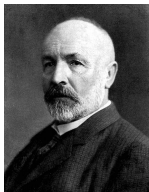
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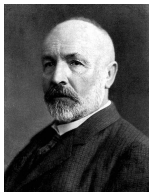
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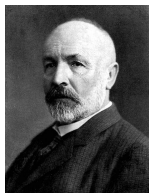
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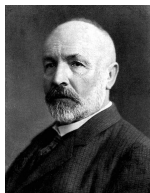
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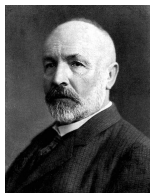
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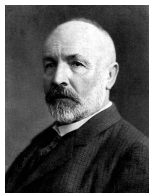
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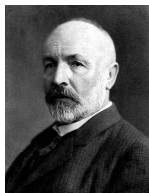
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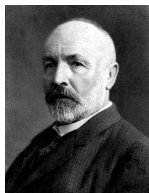
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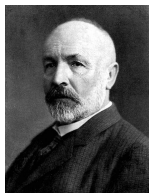
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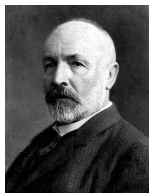
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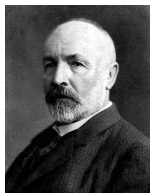
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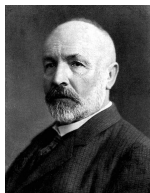
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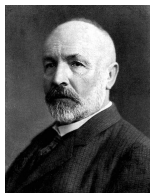
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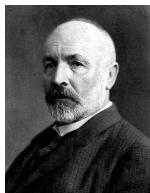
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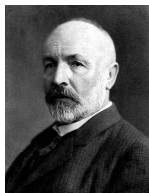
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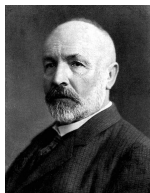
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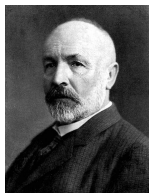
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Cantor's legacy



Gottlob Frege:

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Gottlob Frege: Let's look at the foundations!

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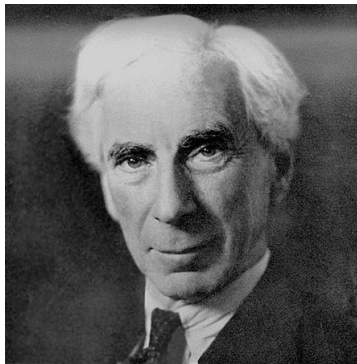
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A bug

Bertrand Russell finds a bug!

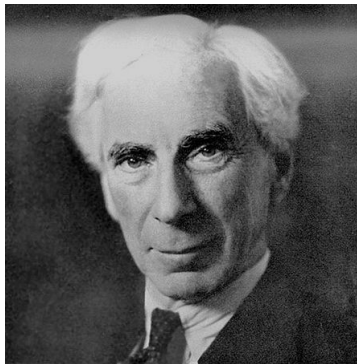
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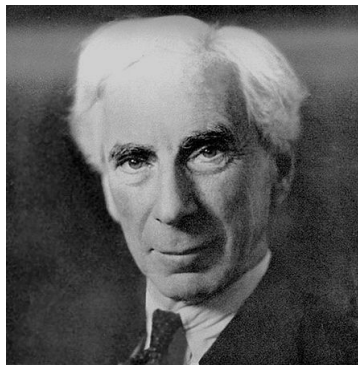
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Frege's reaction:

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Frege's reaction: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion."

A poem

Zisimos Lorentzatos.

"Beware of systems grandiose, of mathematically strict causalities as you're trying, stone by stone, to found the goldenwoven tower of the logical, castle and fort immune to contradiction. Designed in two volumes, the foundational laws of arithmetic, or Grundgesetze of der arithmetic in 1893, the first, 1903 the second. A life's work. Hammer on chisel blows for years and years. So far, so good. But as Frege Gottlob was correcting, content, the printer's proofs already of the second volume, one cursed logic paradox, one not admitting refutation, question by Russell Bertrand, forced, without delay, the great thinker of Mecklemburg to add a last paragraph to his system, show me a great thinker who would resist the truth, accepting the reversible disaster. His foundations in ruin, his logic flawed, his work wasted, and his two volumes imagine the colossal set back, odd load and ballast for the refuge cart."

Russell's Paradox.

- ▶ "This statement is false"

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Is the statement above true?
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Change Axioms!

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(We must know. We will know.) ...

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Concrete example:

Continuum hypothesis (see official notes if interested)

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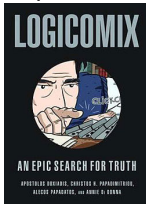
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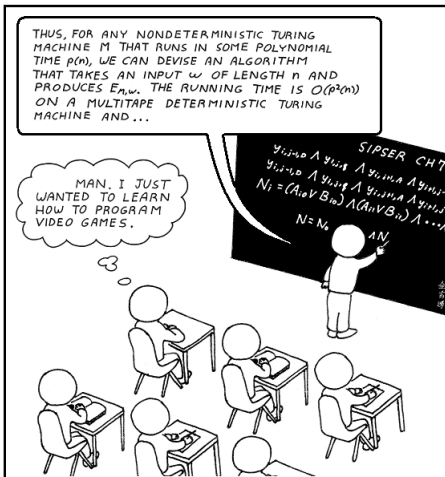
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- ▶ See Logicomix by Doxiadis, Papadimitriou (my advisor!), Papadatos, Di Donna.

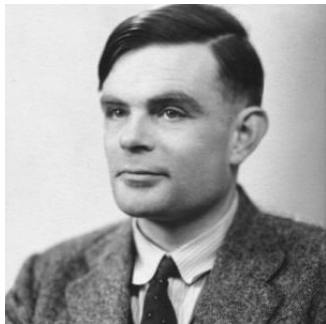


Next Topic: Undecidability.

► Undecidability. A happy ending?



Turing



Is it actually useful?

Turing: Write me a program checker!

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A program that checks that the compiler works!

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Text string can be an input to a program.

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Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:

Need a computer

...with the notion of a stored program!!!!

Program is a text string.

Text string can be an input to a program.

Program can be an input to a program.

Implementing HALT.

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HALT(*P*, *I*)

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$HALT(P, I)$

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Theorem: There is no program HALT.

Halt does not exist.

Proof:

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Proof: Assume there is a program $HALT(\cdot, \cdot)$.

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Wow, that was easy!

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We should be famous!

No computers for Turing!

In Turing's time.

No computers for Turing!

In Turing's time.

No computers.

No computers for Turing!

In Turing's time.

No computers.

Concept of program as data wasn't really there.

Undecidable problems.

Does a program ever print “Hello World”?

Undecidable problems.

Does a program ever print “Hello World”?

Find exit points and add statement: **Print** “Hello World.”

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Is there program that makes other programs faster?

Is there program that decides if two other programs are equivalent?

Does this computer program have any security vulnerabilities?

More about Alan Turing.

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Turing: personal.

Tragic ending...

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(A bite from the apple....) accident?

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So on behalf of the British government, and all those who live freely thanks to Alan's work I am very proud to say: we're sorry, you deserved so much better."

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2013. Granted Royal pardon.

Summary

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Programming is a super power.

HOW MATH WORKS:

STEP 1: INSIGHT



STEP 4: ADDITIONAL DECADES OF DEBATE.



STEP 2: RESISTANCE



STEP 5: CHANGING OF THE GUARD.



STEP 3: DEBATE



STEP 6: TRANSMISSION TO STUDENTS.

