CS70: Discrete Math and Probability

Fan Ye

June 28, 2016

Planar non-planar

A finite graph is planar iff it does not contain a subgraph that is (a subdivision of) K_5 or $K_{3,3}$

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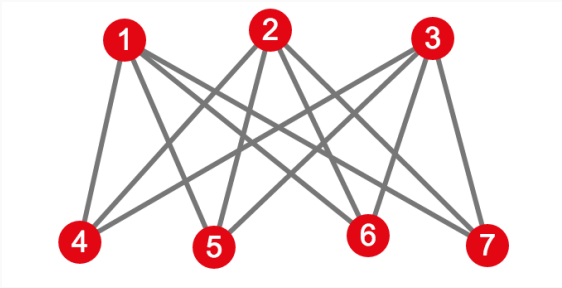

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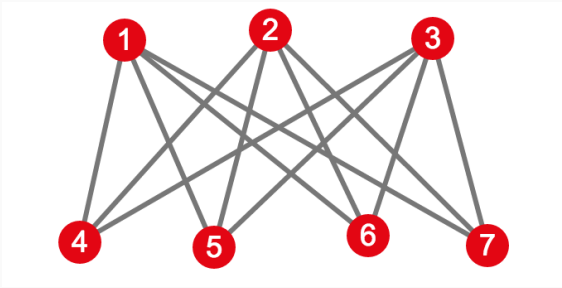
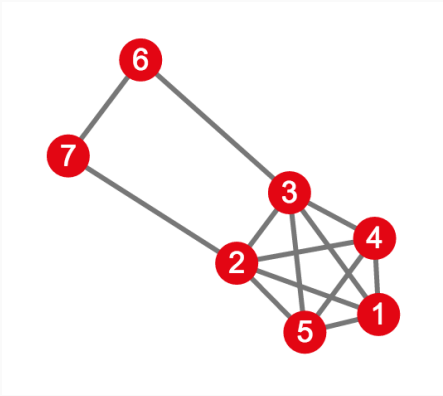
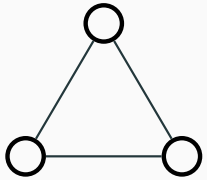
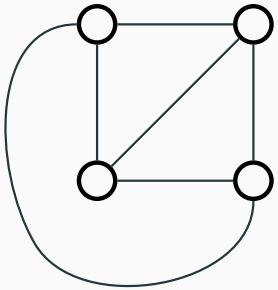




Complete Graph.





K_n complete graph on n vertices.

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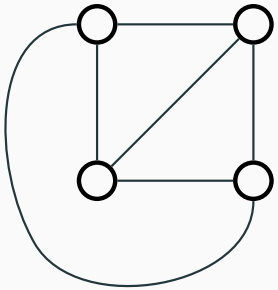

K_n complete graph on n vertices.
All edges are present.

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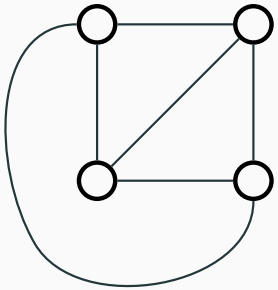

K_n complete graph on n vertices.

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Everyone is my neighbor.

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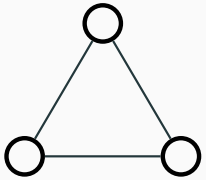

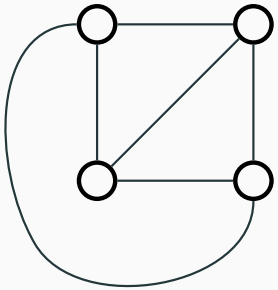

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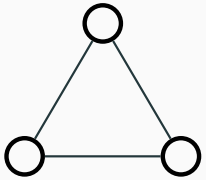

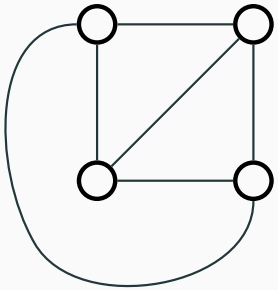

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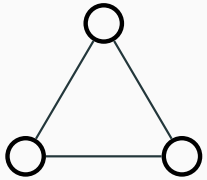

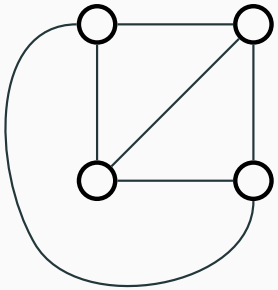

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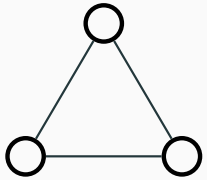

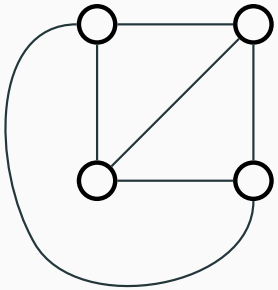

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Each vertex is incident to $n-1$ edges.

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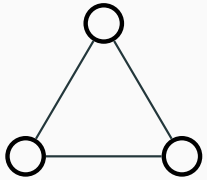

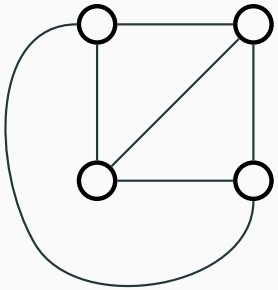

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Sum of degrees is $n(n-1)$.

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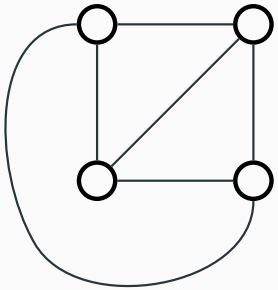

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$\Longrightarrow$ Number of edges is $n(n-1)/2$.

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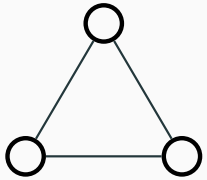

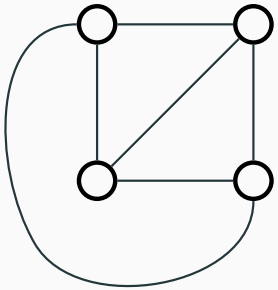

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Remember sum of degree is $2|E|$.

Trees.

Definitions:

Trees.

Definitions:

A connected graph without a cycle.

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A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.

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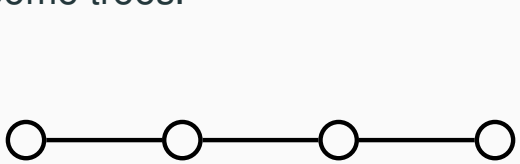

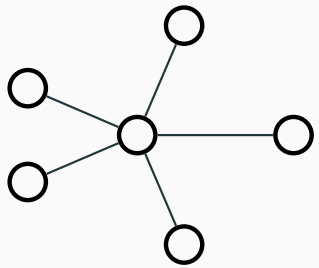

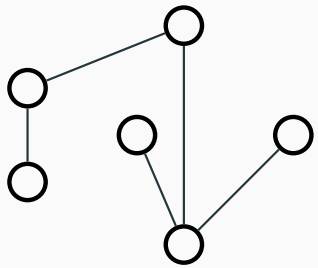

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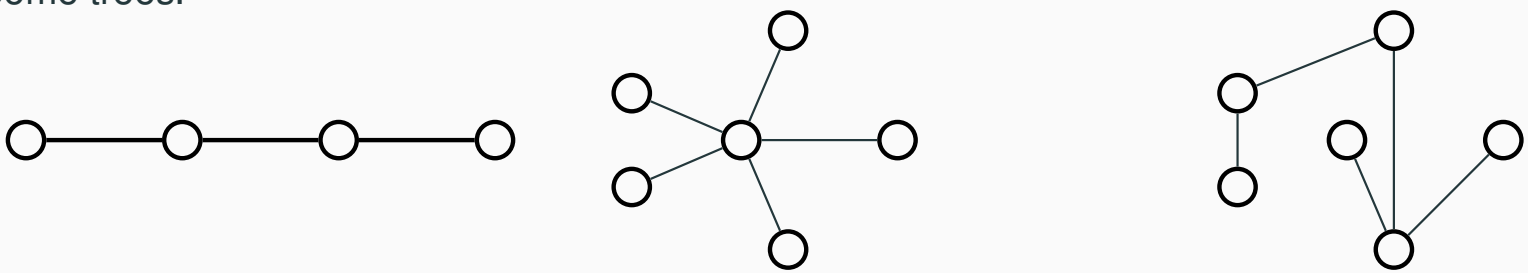

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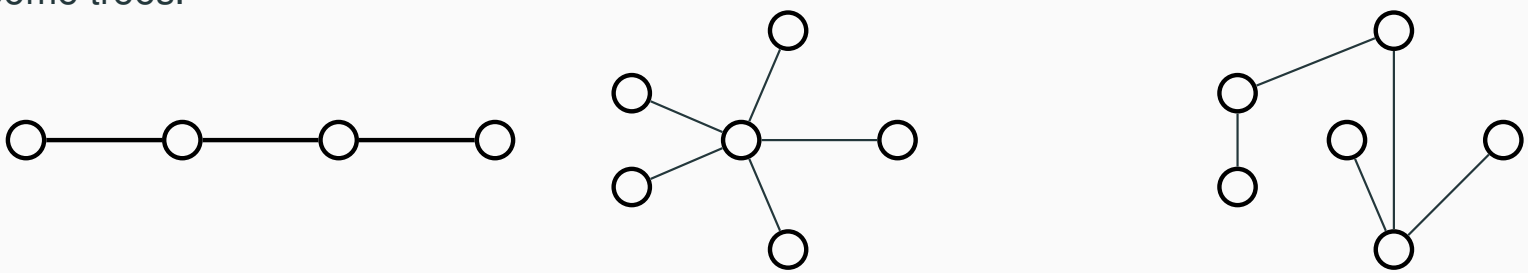

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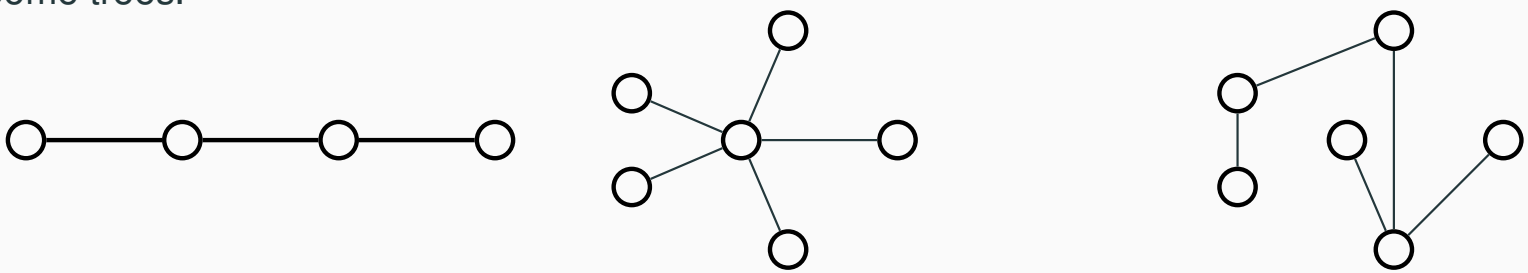

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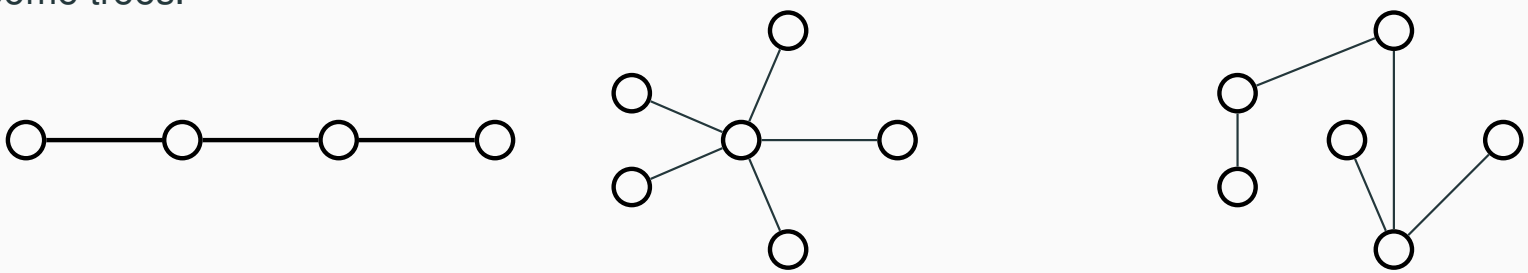

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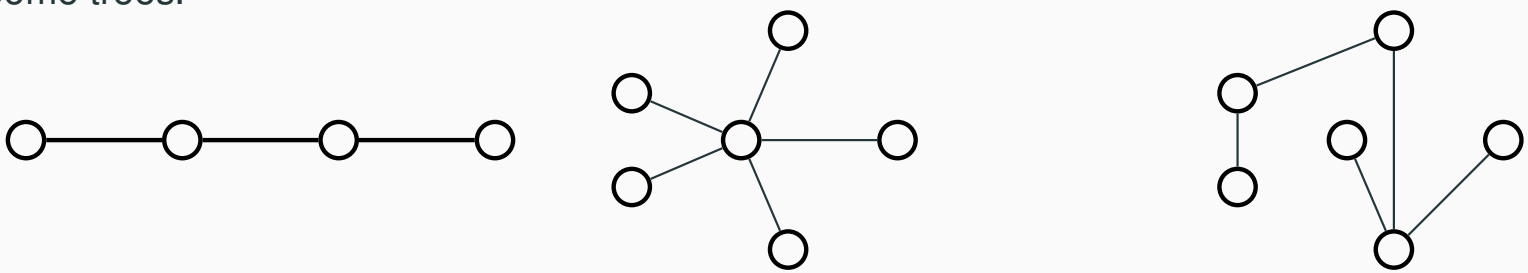

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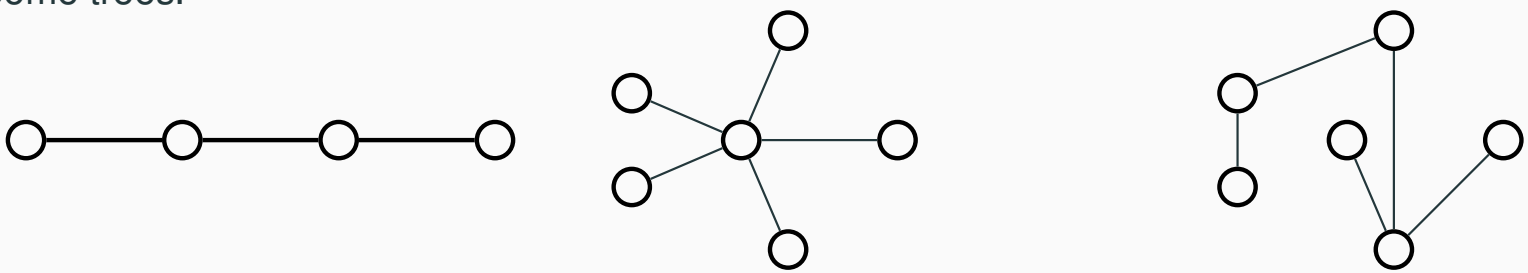

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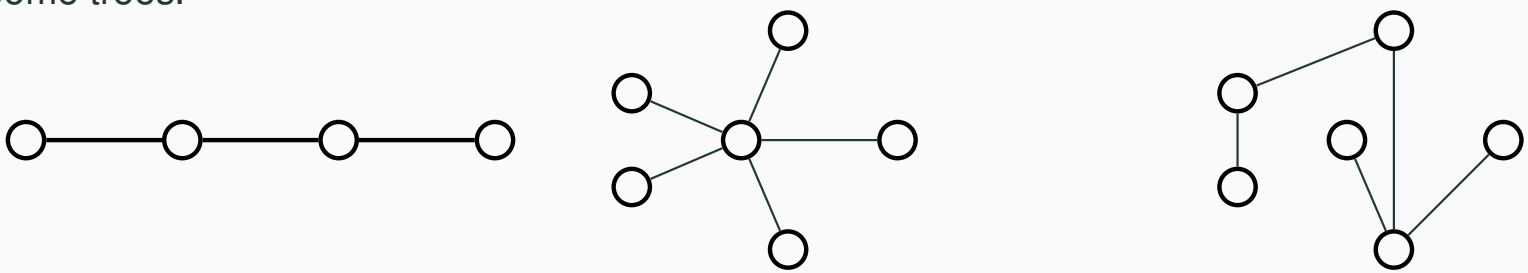

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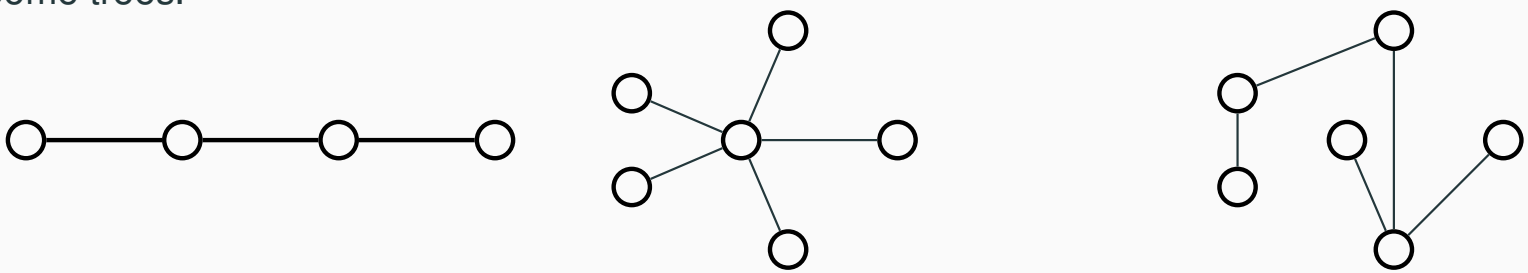

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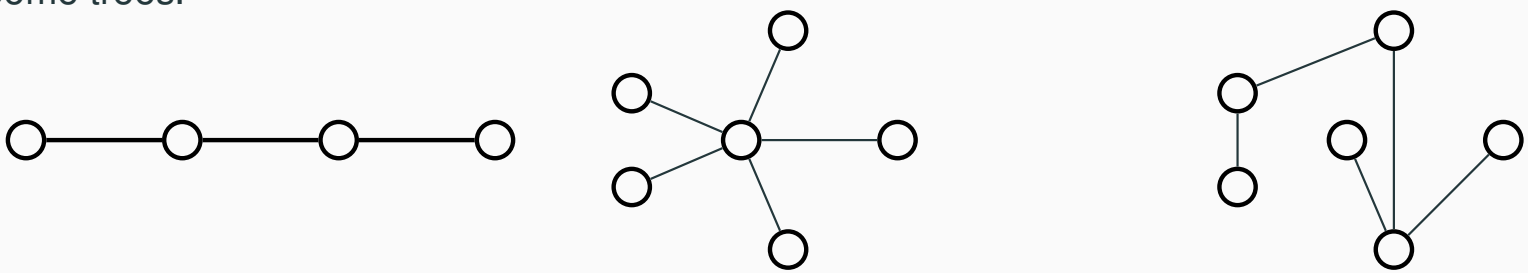

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Theorem:
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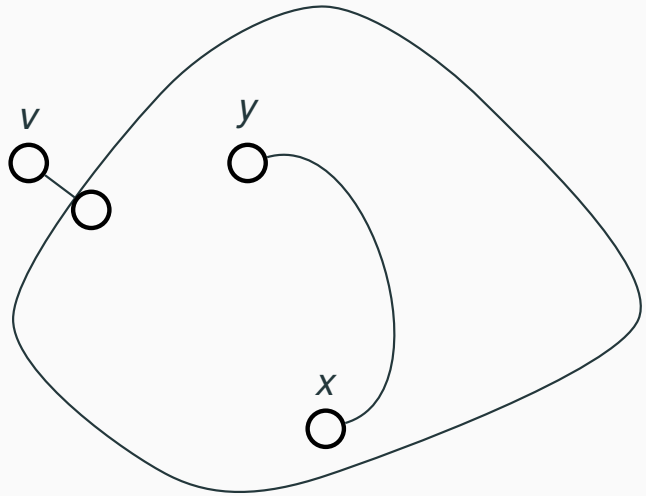

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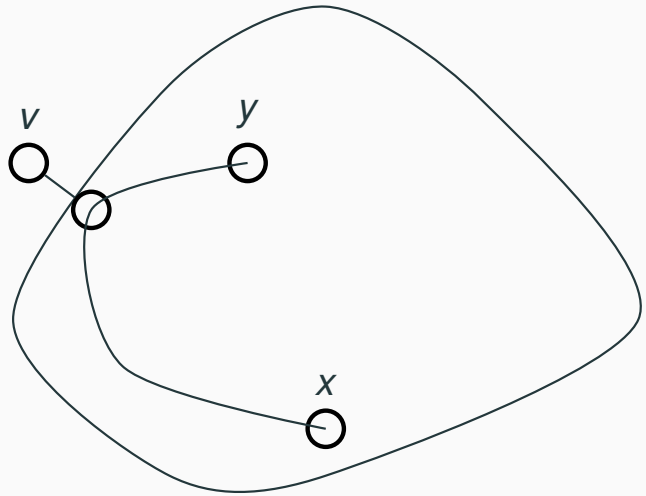

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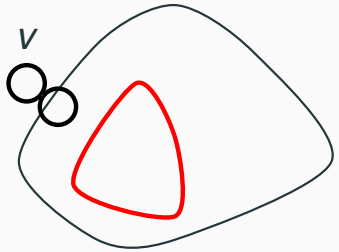

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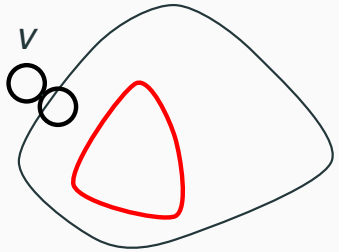

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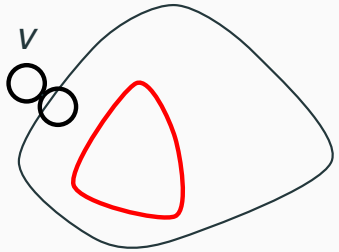

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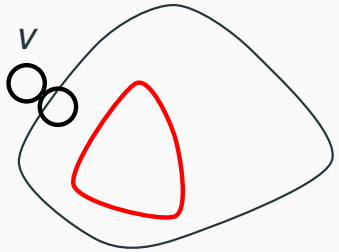

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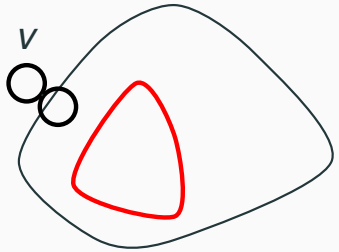

Proof of $\Longrightarrow$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step:

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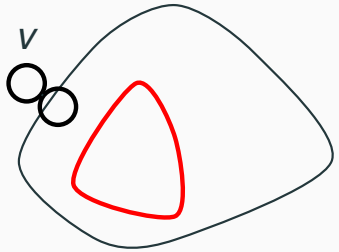

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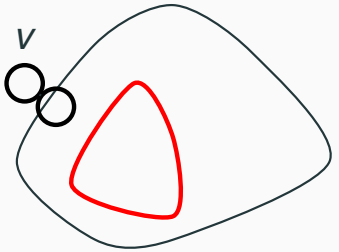

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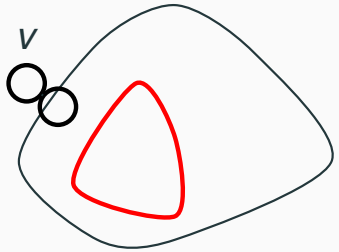

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Claim: There is a degree 1 node.
Proof: First, connected $\Longrightarrow$ every vertex degree ≥ 1.
Sum of degrees is $2|V| - 2$

Proof of only if.

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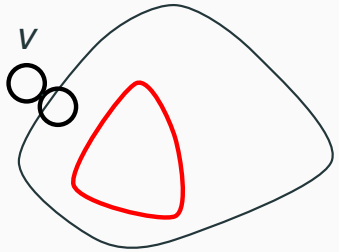

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Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\Longrightarrow$ every vertex degree ≥ 1.
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Average degree $2 - 2/|V|$

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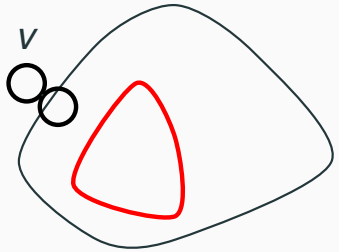

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Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\Longrightarrow$ every vertex degree ≥ 1.
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Not everyone is bigger than average!

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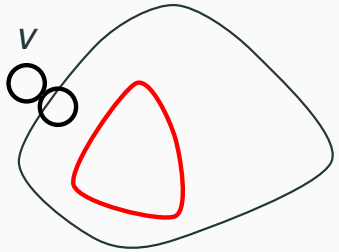

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By degree 1 removal lemma, $G-v$ is connected.

□

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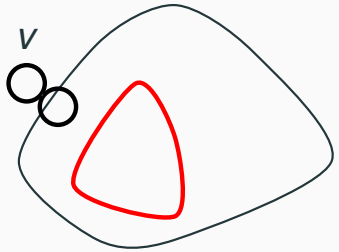

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$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction

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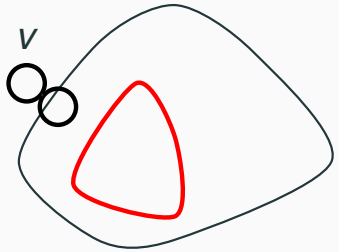

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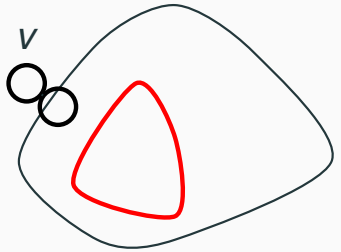

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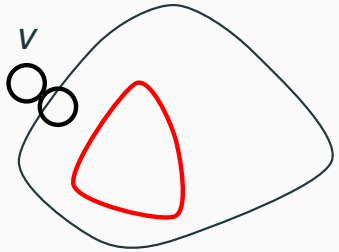

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"G is connected and has no cycles" $\implies$ "G connected and has $|V| - 1$ edges"

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Proof:
Walk from a vertex using untraversed edges.

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Walk from a vertex using untraversed edges.
Until get stuck.

Proof of if

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Claim: Must stuck at a degree 1 vertex.

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Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V|-1$ edges"

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Must stuck at a degree 1 vertex.
Proof of Claim:
Can't visit any vertex more than once since no cycle.

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Proof:
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Can't visit any vertex more than once since no cycle.
Entered.

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Walk from a vertex using untraversed edges.
Until get stuck.
Claim: Must stuck at a degree 1 vertex.
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Can't visit any vertex more than once since no cycle.
Entered. Didn't leave.

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Entered. Didn't leave. Only one incident edge.

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New graph is connected.

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G has one more or $|V|-1$ edges.

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Thm: Can always find a node such that the largest connected component we get by removing it has size at most $|V|/2$

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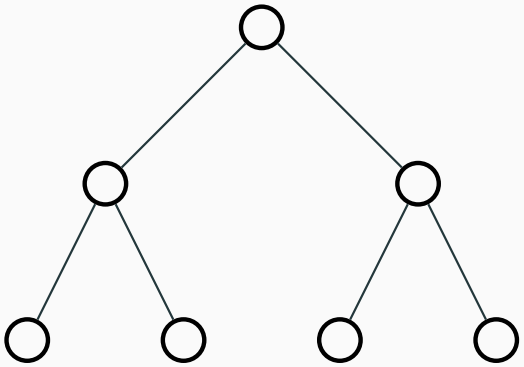

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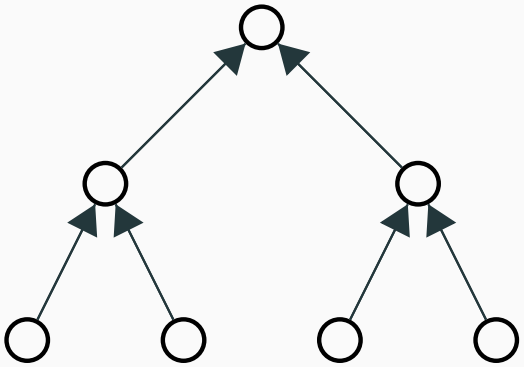

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Point edge toward bigger side.

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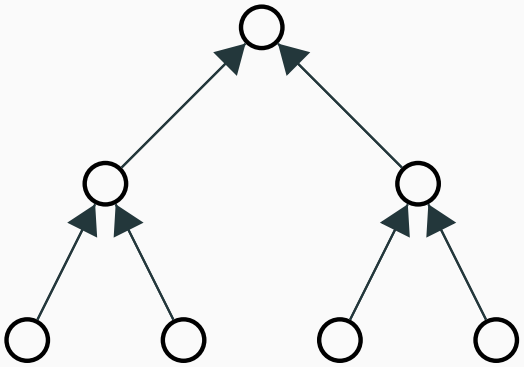

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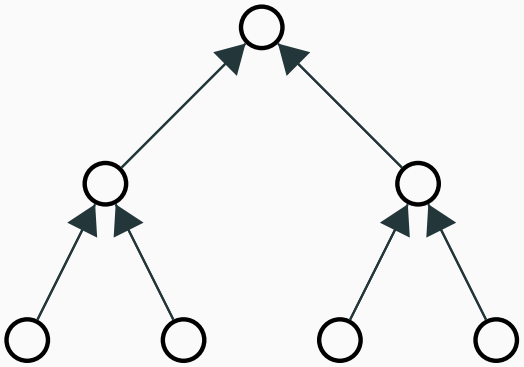

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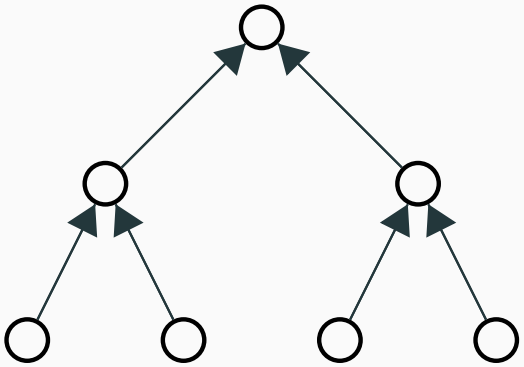

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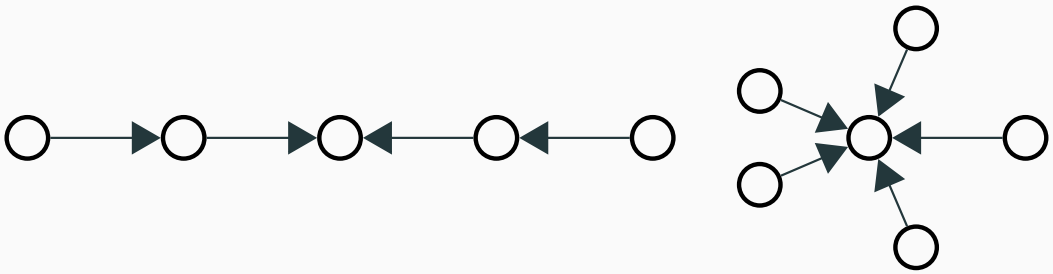

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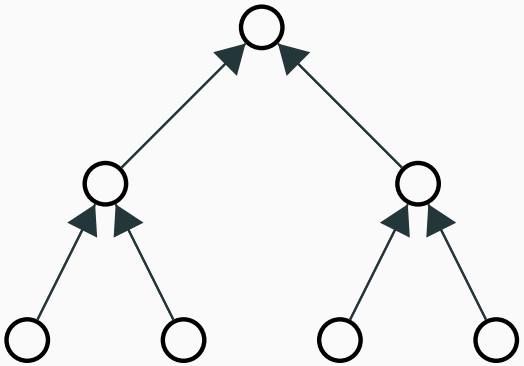

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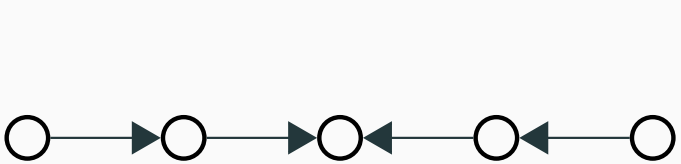

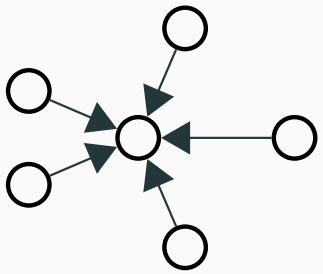

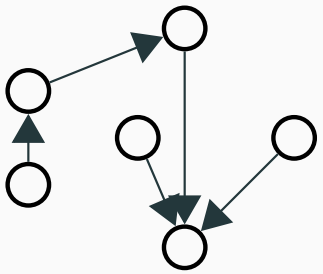

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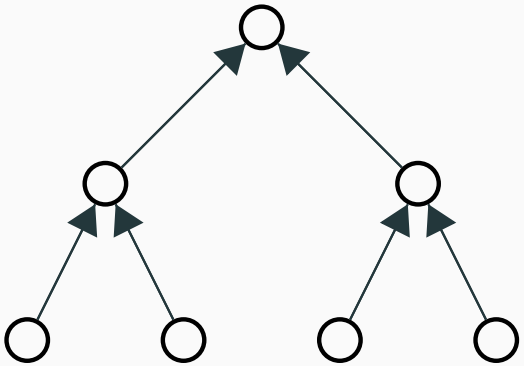

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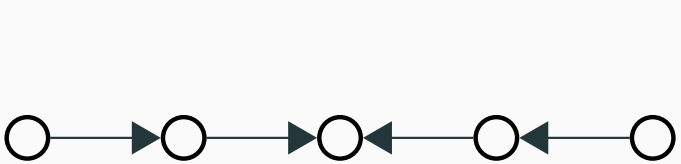

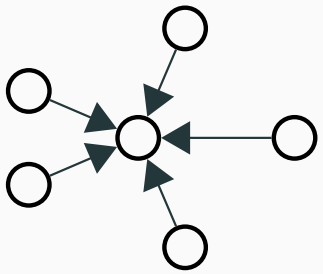

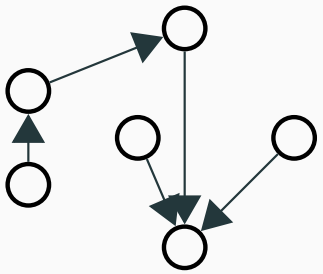

Hypercubes.

Complete graphs, really connected!

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Complete graphs, really connected! But lots of edges.

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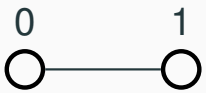

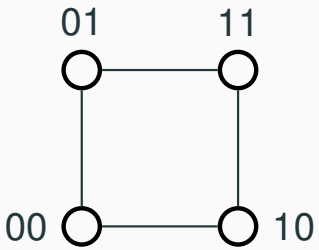

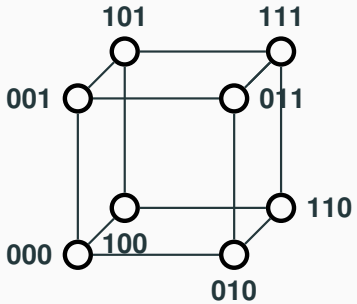

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Complete graphs, really connected! But lots of edges.
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Trees, But few edges. $(|V|-1)$
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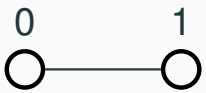

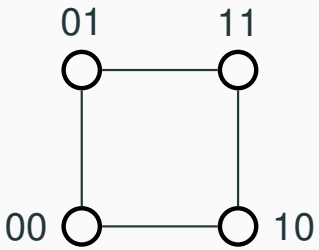

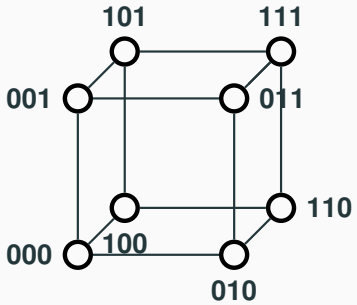

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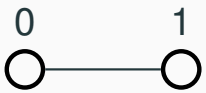

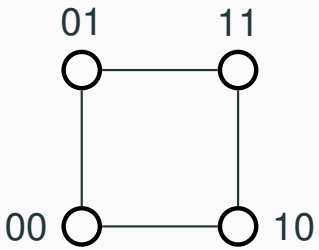

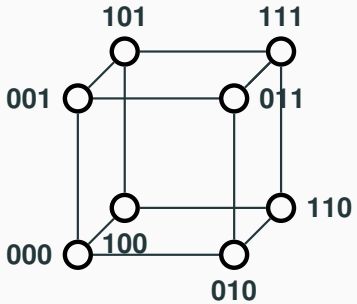

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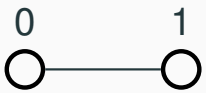

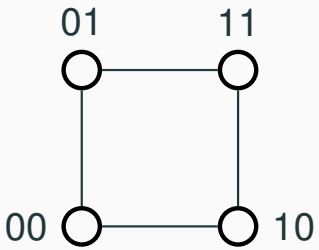

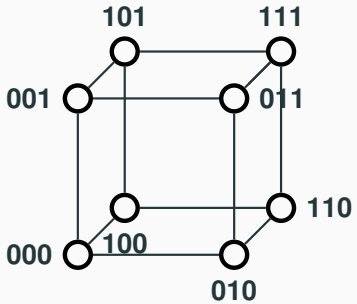

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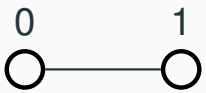

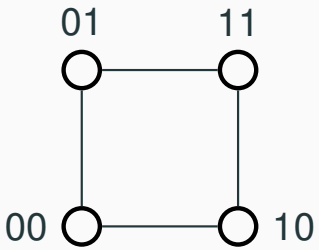

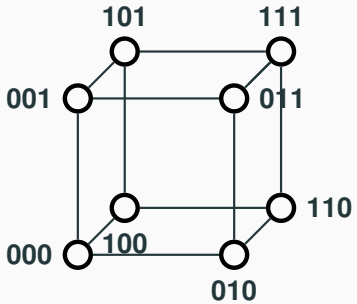

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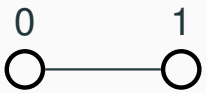

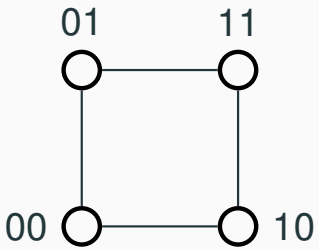

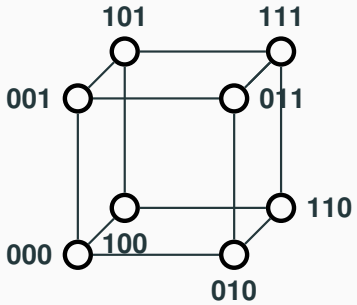

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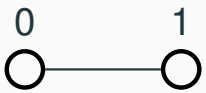

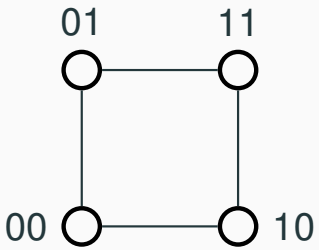

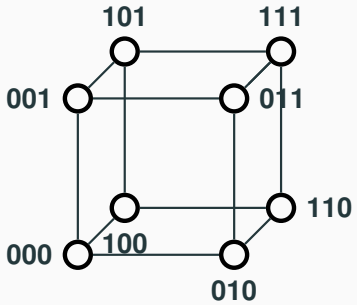

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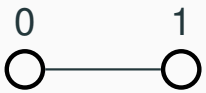

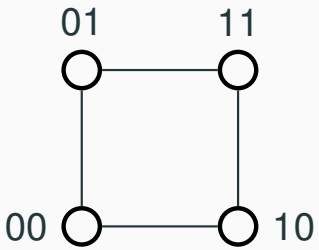

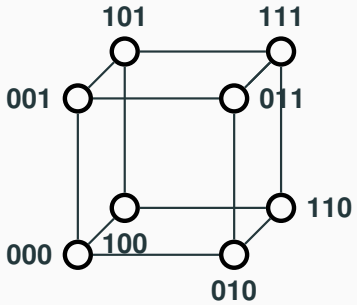

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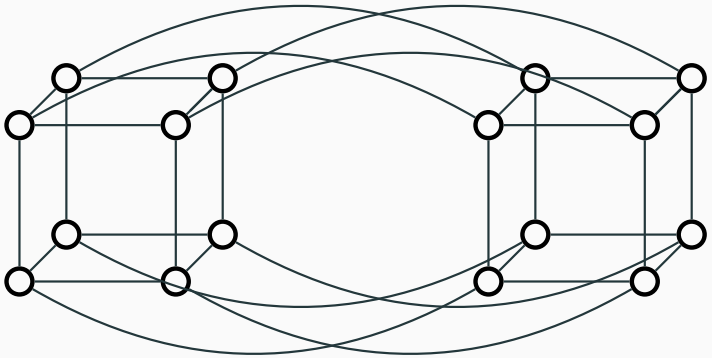

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

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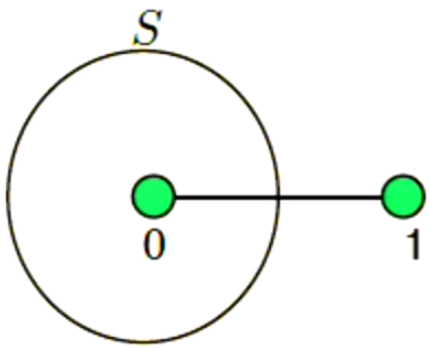

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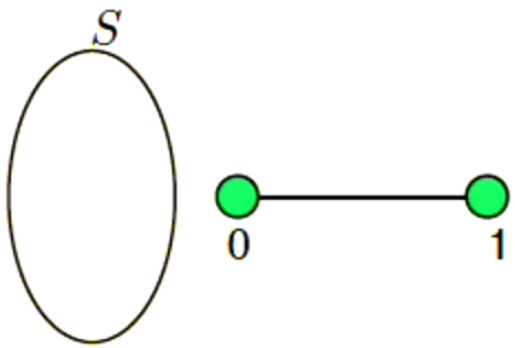

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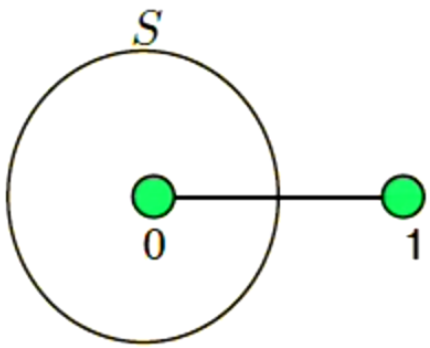

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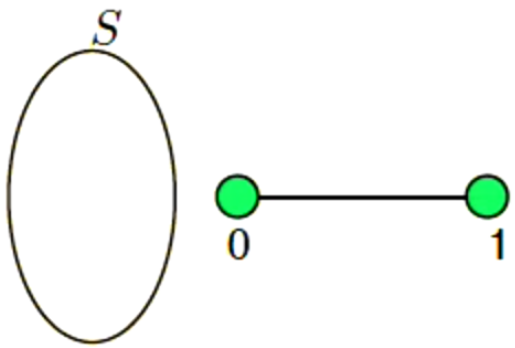

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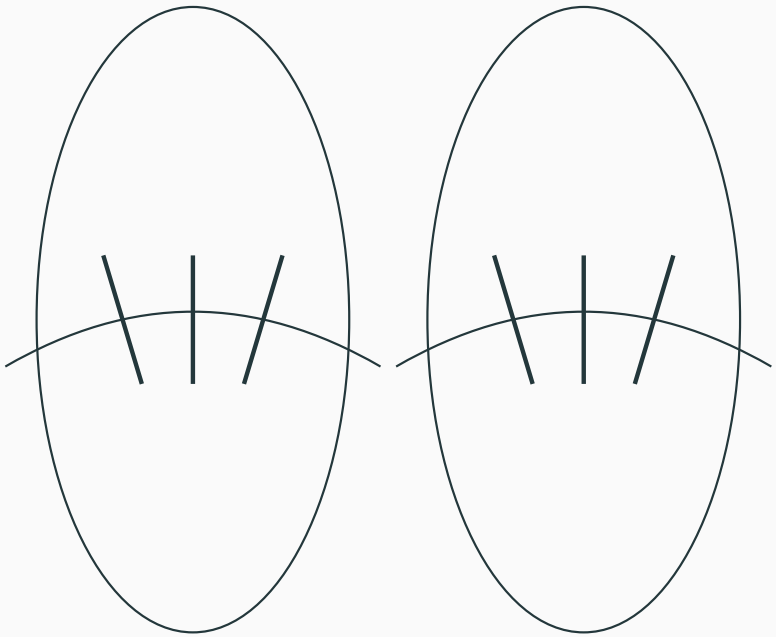

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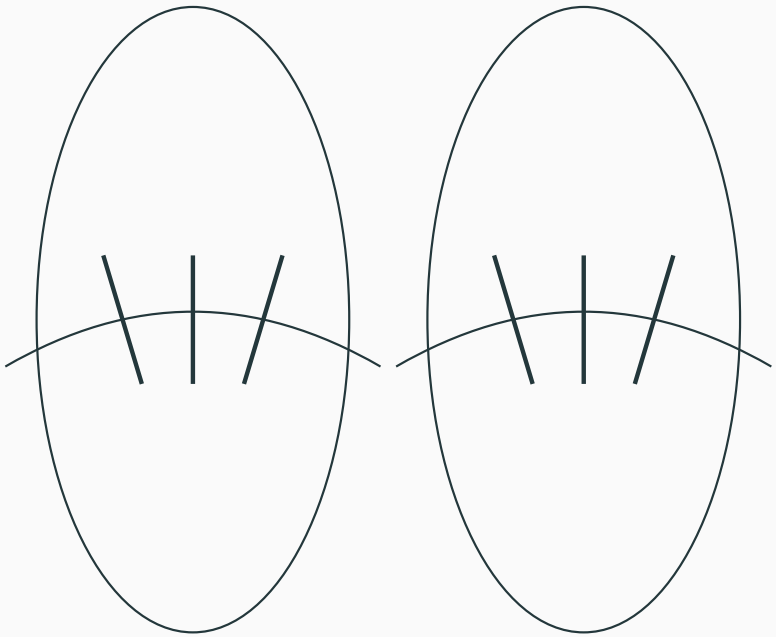

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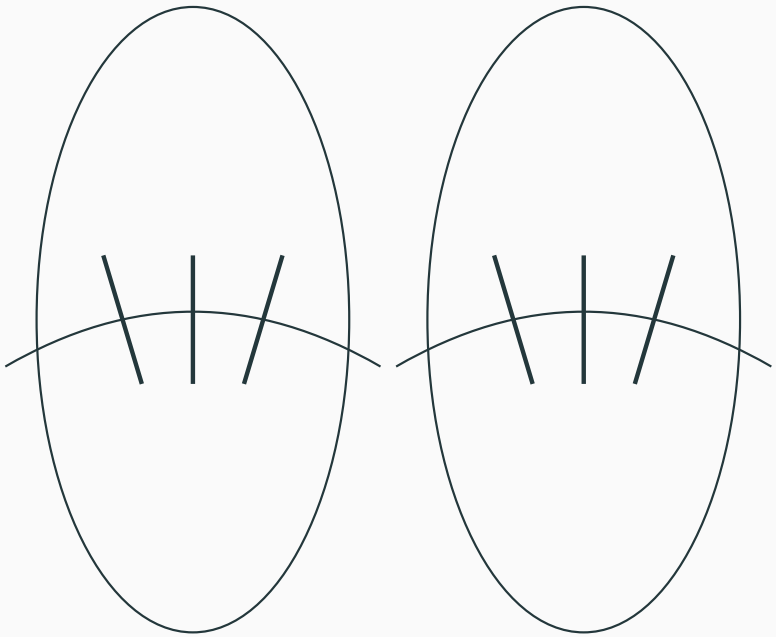

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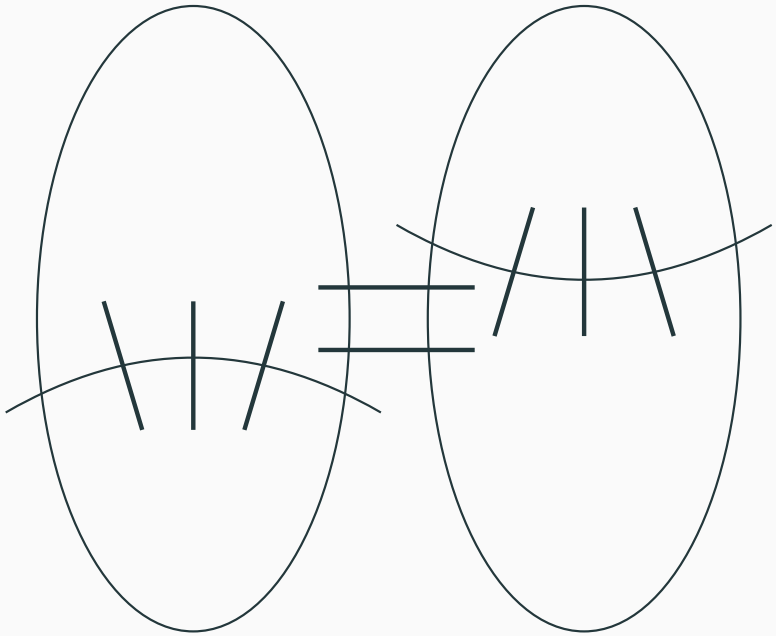

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$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

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$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$

Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:

$H_0 = (V_0, E_0)$,$H_1 = (V_1, E_1)$, edges E_x that connect them.
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Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides.

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Edges cut in $H_0 \geq |S_0|$.

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Proof: Induction Step.
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$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
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Both S_0 and S_1 are small sides. So by induction.
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Edges cut in $H_0 \geq |S_0|$.
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Total cut edges $\geq |S_0| + |S_1| = |S|$.

Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

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Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
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Edges cut in $H_0 \geq |S_0|$.
Edges cut in $H_1 \geq |S_1|$.

Total cut edges $\geq |S_0| + |S_1| = |S|$. □

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

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$\implies \geq |S_1|$ edges cut in E_1.

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$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

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Edges in E_x connect corresponding nodes.

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Edges in E_x connect corresponding nodes.

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Total edges cut:

Induction Step. Case 2.

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Total edges cut:

$\geq$

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Total edges cut:

$\geq |S_1|$

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Total edges cut:

$\geq |S_1| + |V_0| - |S_0|$

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$|V_0|$

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$|V_0| = |V|/2 \geq |S|$.

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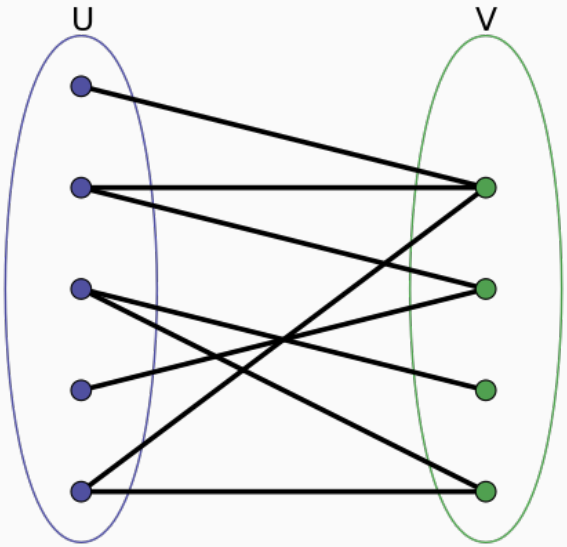
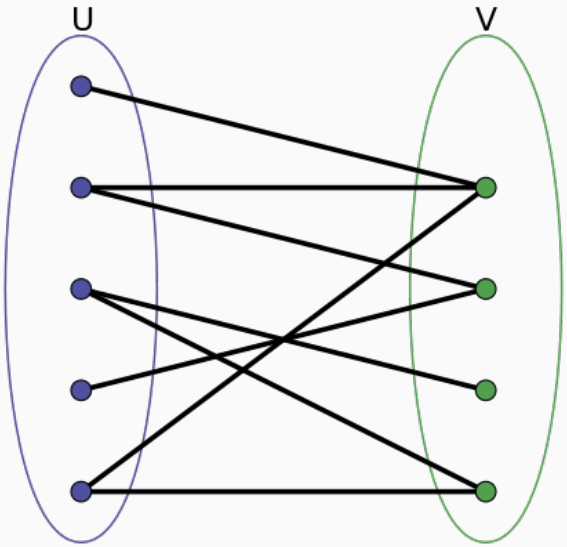
Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

Bipartite graph

Bipartite graph



Bipartite graph



Bipartite graph:

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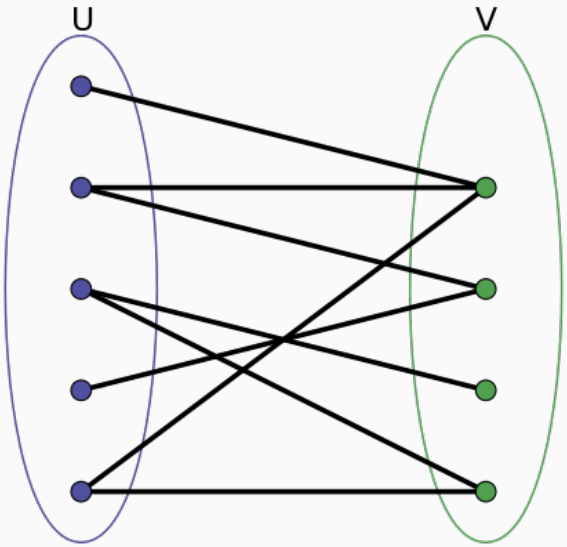
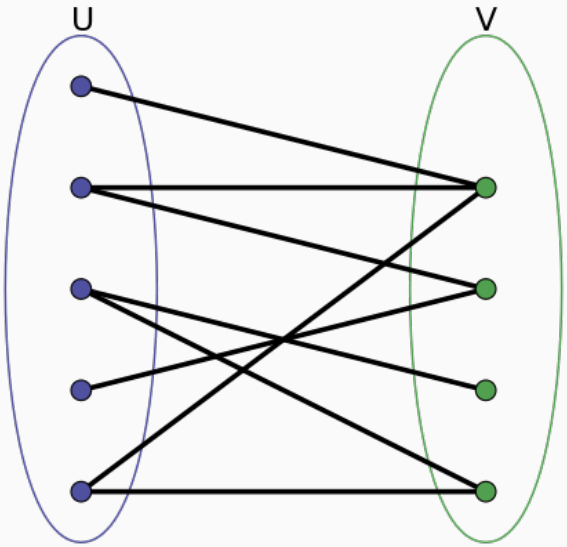


Bipartite graph: a bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.

Bipartite graph



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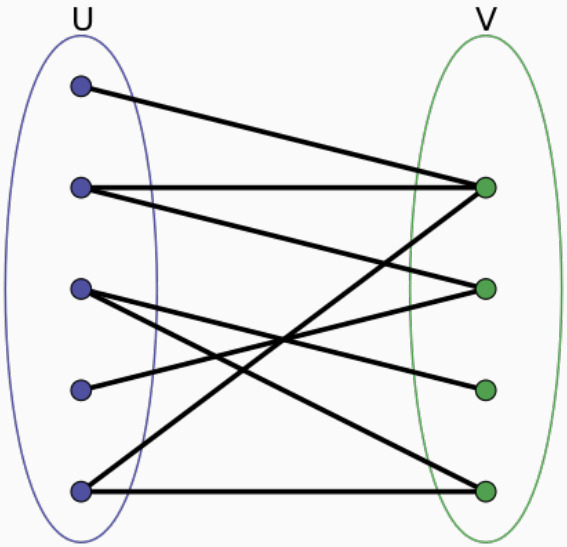
U and V are sometimes called the parts of the graph.

Bipartite graph



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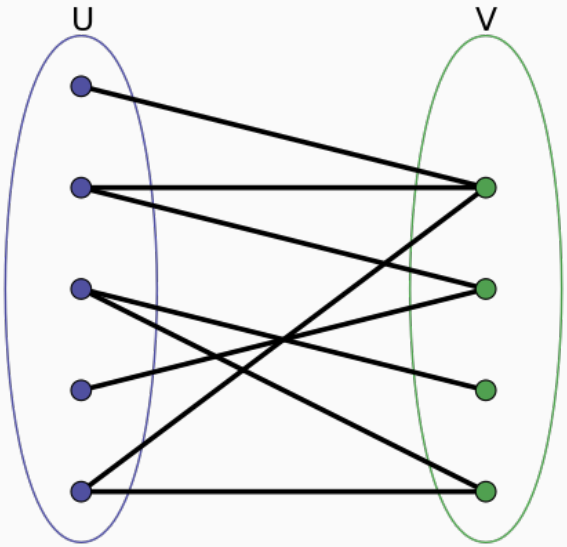
Coloring?

Bipartite graph



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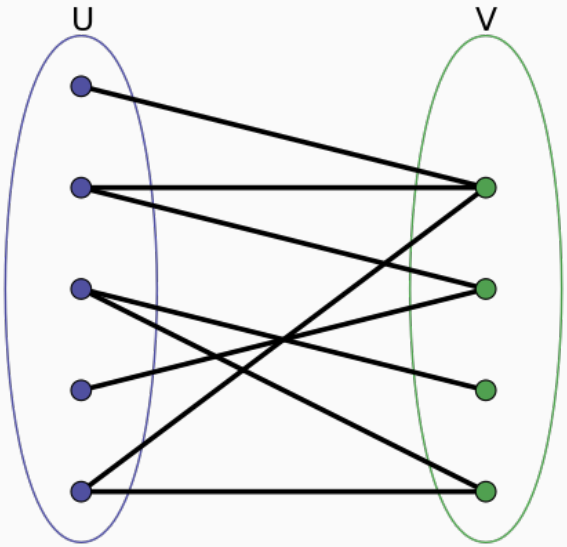
Coloring? How many colors do we need?

Bipartite graph



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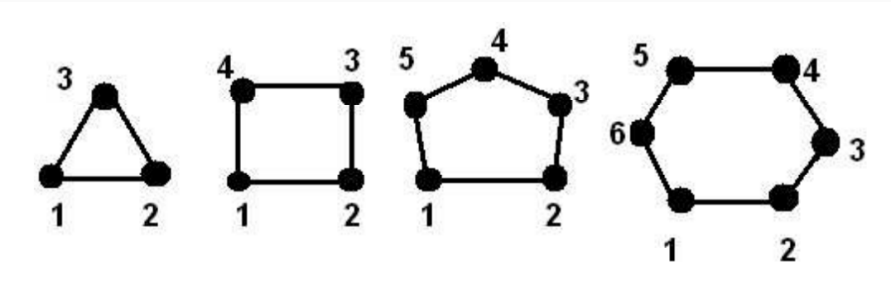
Coloring? How many colors do we need? 2!

Bipartite?

Which of the following graphs are bipartite?

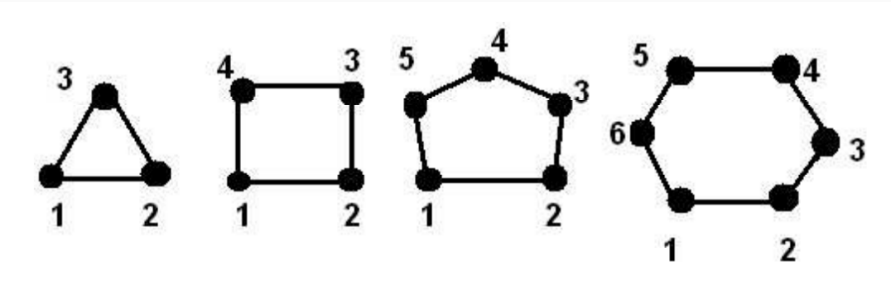
Bipartite?

Which of the following graphs are bipartite?



Bipartite?

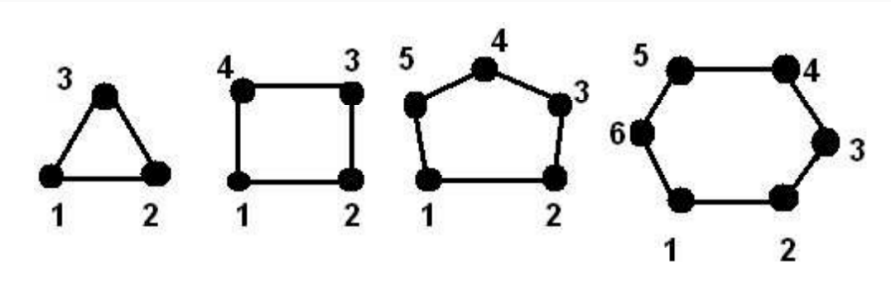
Which of the following graphs are bipartite?



No

Bipartite?

Which of the following graphs are bipartite?



No Yes

Bipartite?

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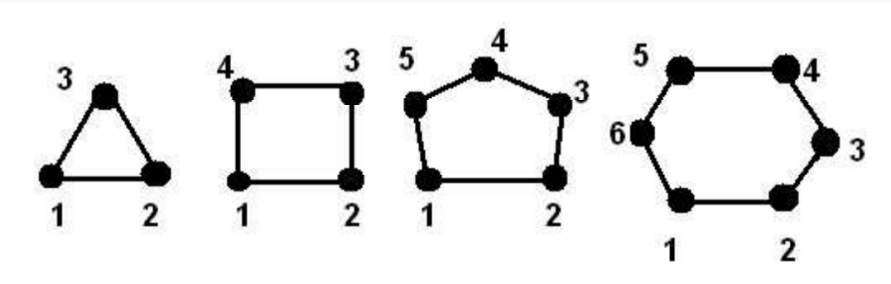

No Yes No

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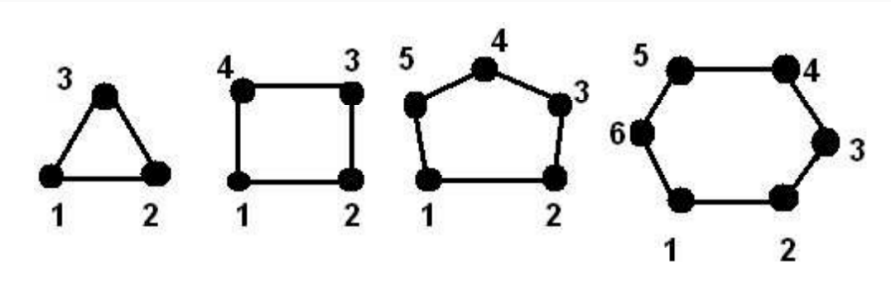

No Yes No Yes

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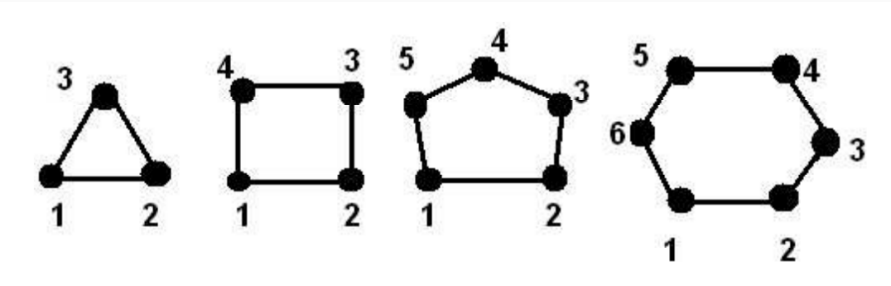
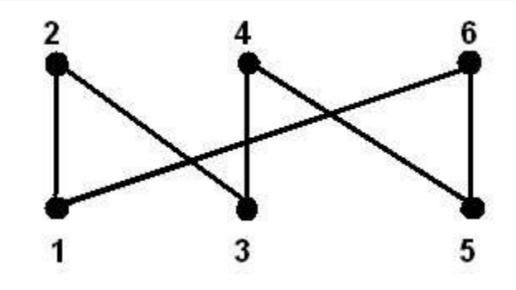
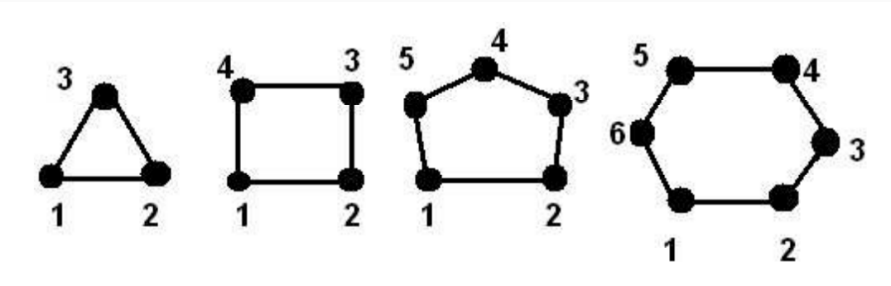


No Yes No Yes



Bipartite?

Which of the following graphs are bipartite?



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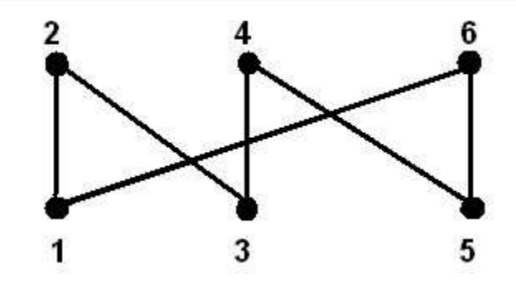

A graph is a bipartite graph if and only if it does not contain any odd-length cycles.

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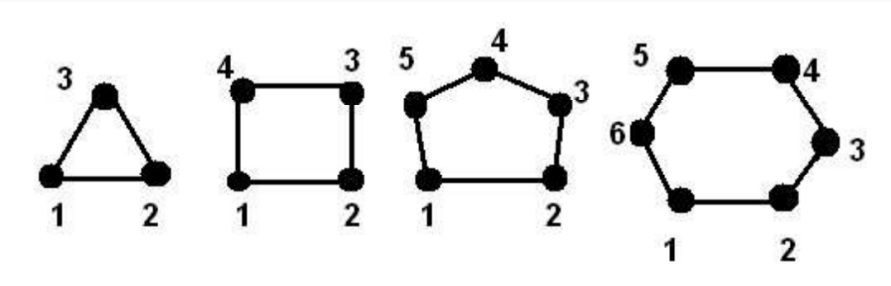

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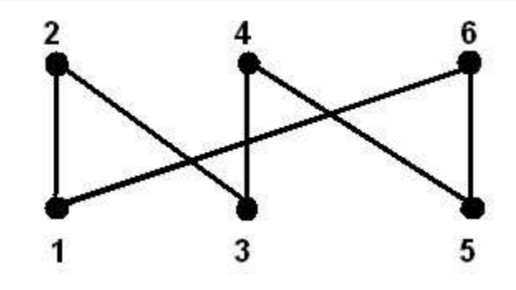

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What have we done?!

Graphs!

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Eulerian tour:

What have we done?!

Graphs!

Eulerian tour: DNA sequence reconstructing

What have we done?!

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Coloring: Cellular tower frequency assignment

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Trees: Immense applications.....

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Trees: Immense applications.....

Modeling reality:

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Trees: Immense applications.....

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