CS70: Countability and Uncountability

Alex Psomas

June 30, 2016

Warning!

Warning:

Warning!

Warning: I’m really loud!

Today.

Today.

One idea, from around 130 years ago.

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!!!!!

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!!!!!

The man who worked on this was described as:

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!!!!!

The man who worked on this was described as:

- Genious?

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!!!!!

The man who worked on this was described as:

- Genious?
- Renegade?

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!!!!!

The man who worked on this was described as:

- Genious?
- Renegade?
- Corrupter of youth?

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!!!!!

The man who worked on this was described as:

- Genious?
- Renegade?
- Corrupter of youth?
- The King in the North?

Today.

One idea, from around 130 years ago.

At the heart of set theory.

Started a crisis in mathematics in the middle of the previous century!!!!!

The man who worked on this was described as:

- Genious?
- Renegade?
- Corrupter of youth?
- The King in the North?

The idea.

The idea.

The idea: **More than one infinities!!!!!!**

The idea.

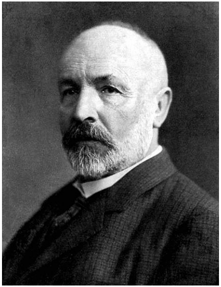
The idea: **More than one infinities!!!!!!**

The man:

The idea.

The idea: **More than one infinities!!!!!!**

The man:



Georg Cantor

Life before Cantor

Life before Cantor

How many elements in $\{1, 2, 4\}$?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$? Infinite!

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$? Infinite!

How many elements in $\mathbb{R}$?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$? Infinite!

How many elements in $\mathbb{R}$? Infinite!

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$? Infinite!

How many elements in $\mathbb{R}$? Infinite!

What is this infinity though?

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$? Infinite!

How many elements in $\mathbb{R}$? Infinite!

What is this infinity though?

The symbol you write after taking a limit....

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$? Infinite!

How many elements in $\mathbb{R}$? Infinite!

What is this infinity though?

The symbol you write after taking a limit....

Don't think about it....

Life before Cantor

How many elements in $\{1, 2, 4\}$? 3

How many elements in $\{1, 2, 4, 10, 13, 18\}$? 6

How many primes? Infinite!

How many elements in $\mathbb{N}$? Infinite!

How many elements in $\mathbb{N} \setminus \{0\}$? Infinite!

How many elements in $\mathbb{Z}$? Infinite!

How many elements in $\mathbb{R}$? Infinite!

What is this infinity though?

The symbol you write after taking a limit....

Don't think about it....

Even Gauss: "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction. "

Cantor's questions

Cantor's questions

Is $\mathbb{N} \setminus \{0\}$ smaller than $\mathbb{N}$?

Cantor's questions

Is $\mathbb{N} \setminus \{0\}$ smaller than $\mathbb{N}$?

Is $\mathbb{N}$ smaller than $\mathbb{Z}$?

Cantor's questions

Is $\mathbb{N} \setminus \{0\}$ smaller than $\mathbb{N}$?

Is $\mathbb{N}$ smaller than $\mathbb{Z}$? What about $\mathbb{Z}^2$?

Cantor's questions

Is $\mathbb{N} \setminus \{0\}$ smaller than $\mathbb{N}$?

Is $\mathbb{N}$ smaller than $\mathbb{Z}$? What about $\mathbb{Z}^2$?

Is $\mathbb{N}$ smaller than $\mathbb{R}$?

Hilbert's hotel

Hilbert's hotel

A hotel with infinite rooms.

Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity.

Hilbert's hotel

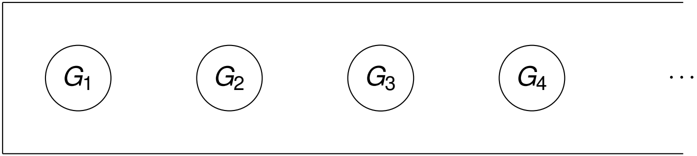
A hotel with infinite rooms. Rooms are numbered from 1 to infinity. Every room is occupied.

Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity.
Every room is occupied. Room i has guest G_i.

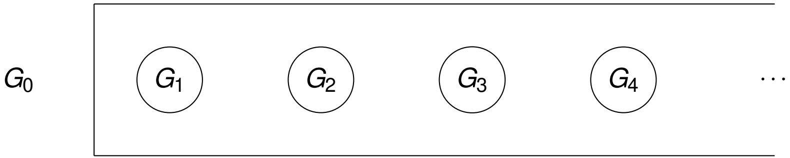
Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity.
Every room is occupied. Room i has guest G_i.



Hilbert's hotel

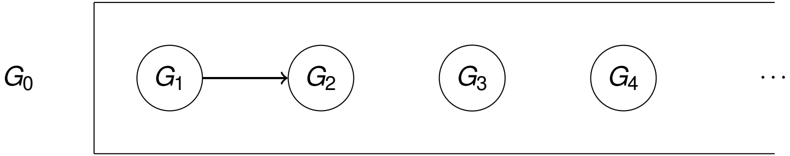
A hotel with infinite rooms. Rooms are numbered from 1 to infinity.
Every room is occupied. Room i has guest G_i.



G_0 shows up. What do we do?

Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity. Every room is occupied. Room i has guest G_i.

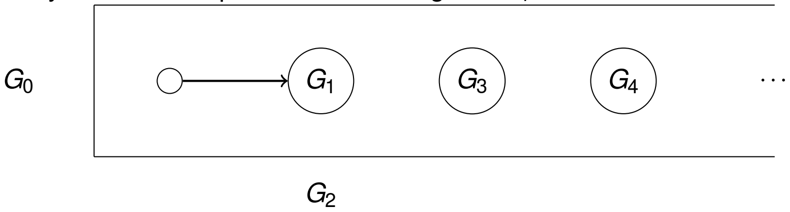


G_0 shows up. What do we do?

Move G_1 to room number 2.

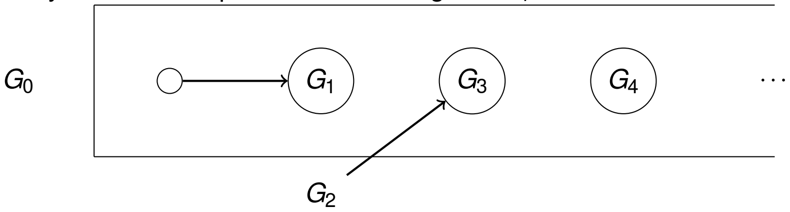
Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity.
Every room is occupied. Room i has guest G_i.



Hilbert's hotel

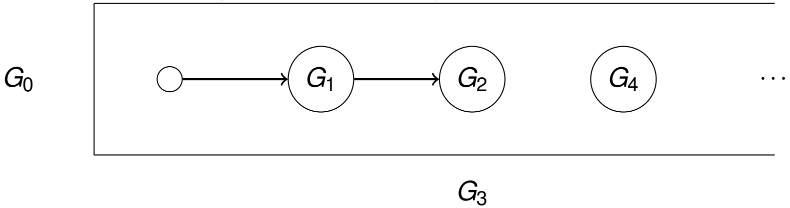
A hotel with infinite rooms. Rooms are numbered from 1 to infinity. Every room is occupied. Room i has guest G_i.



Move G_2 to room number 3.

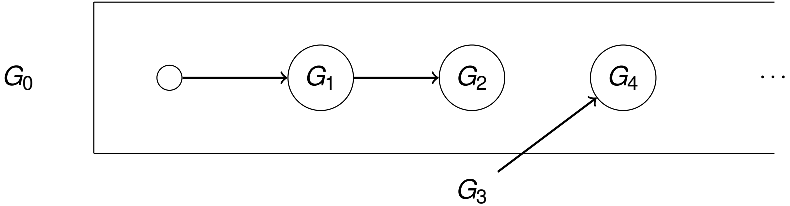
Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity. Every room is occupied. Room i has guest G_i.



Hilbert's hotel

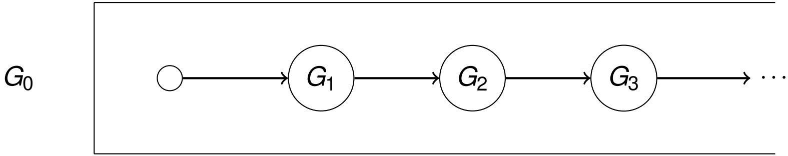
A hotel with infinite rooms. Rooms are numbered from 1 to infinity. Every room is occupied. Room i has guest G_i.



Move G_3 to room number 4.

Hilbert's hotel

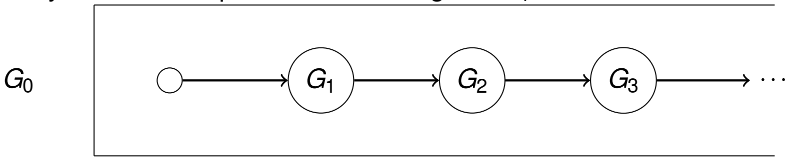
A hotel with infinite rooms. Rooms are numbered from 1 to infinity.
Every room is occupied. Room i has guest G_i.



And so on.

Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity.
Every room is occupied. Room i has guest G_i.

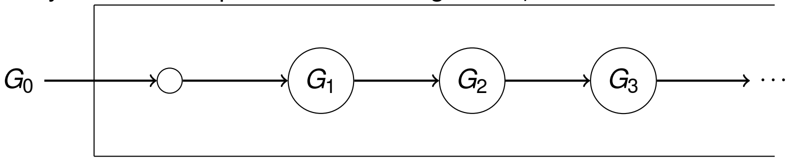


And so on.

Now G_0 can go to room number 1!!

Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity.
Every room is occupied. Room i has guest G_i.

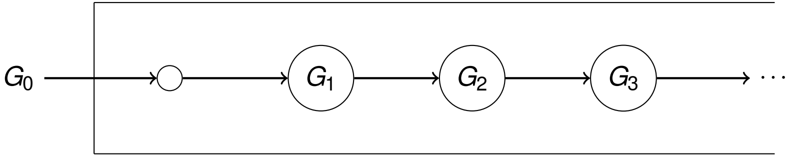


And so on.

Now G_0 can go to room number 1!!

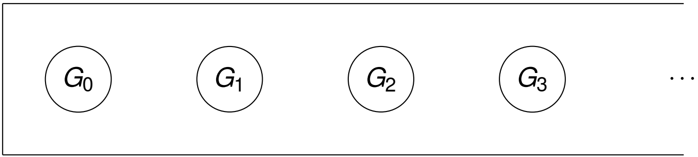
Hilbert's hotel

A hotel with infinite rooms. Rooms are numbered from 1 to infinity. Every room is occupied. Room i has guest G_i.



And so on.

Now G_0 can go to room number 1!!



Moral of the story

Moral of the story

Number of rooms:

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests:

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

$\mathbb{N} \setminus \{0\}$ is **not** smaller than $\mathbb{N}$.

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

$\mathbb{N} \setminus \{0\}$ is **not** smaller than $\mathbb{N}$.

$\mathbb{N} \setminus \{0\}$ is not bigger than $\mathbb{N}$.

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

$\mathbb{N} \setminus \{0\}$ is **not** smaller than $\mathbb{N}$.

$\mathbb{N} \setminus \{0\}$ is not bigger than $\mathbb{N}$. Why?

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

$\mathbb{N} \setminus \{0\}$ is **not** smaller than $\mathbb{N}$.

$\mathbb{N} \setminus \{0\}$ is not bigger than $\mathbb{N}$. Why? Because it's a subset.

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

$\mathbb{N} \setminus \{0\}$ is **not** smaller than $\mathbb{N}$.

$\mathbb{N} \setminus \{0\}$ is not bigger than $\mathbb{N}$. Why? Because it's a subset.

Therefore, $\mathbb{N} \setminus \{0\}$ must have the same number of elements as $\mathbb{N}$.

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

$\mathbb{N} \setminus \{0\}$ is **not** smaller than $\mathbb{N}$.

$\mathbb{N} \setminus \{0\}$ is not bigger than $\mathbb{N}$. Why? Because it's a subset.

Therefore, $\mathbb{N} \setminus \{0\}$ must have the same number of elements as $\mathbb{N}$.

Is this a proof?

Moral of the story

Number of rooms: $\mathbb{N} \setminus \{0\}$
Number of guests: $\mathbb{N}$

$\mathbb{N} \setminus \{0\}$ is **not** smaller than $\mathbb{N}$.

$\mathbb{N} \setminus \{0\}$ is not bigger than $\mathbb{N}$. Why? Because it's a subset.

Therefore, $\mathbb{N} \setminus \{0\}$ must have the same number of elements as $\mathbb{N}$.

Is this a proof? How would we show this formally???

Countable.

Countable.

Definition: S is **countable** if there is a bijection between S and some subset of $\mathbb{N}$.

Countable.

Definition: S is **countable** if there is a bijection between S and some subset of $\mathbb{N}$.

If the subset of $\mathbb{N}$ is finite, S has finite **cardinality**.

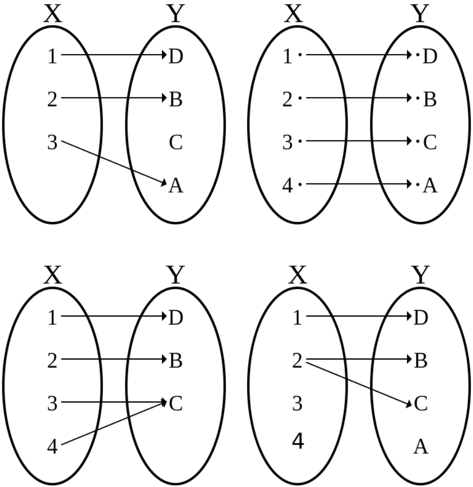
Countable.

Definition: S is **countable** if there is a bijection between S and some subset of $\mathbb{N}$.

If the subset of $\mathbb{N}$ is finite, S has finite **cardinality**.

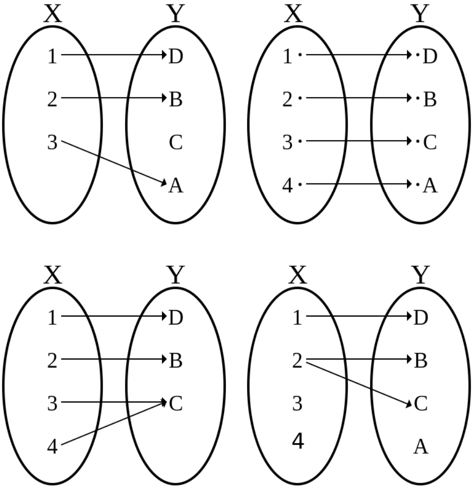
If the subset of $\mathbb{N}$ is infinite, S is **countably infinite**.

Bijections?



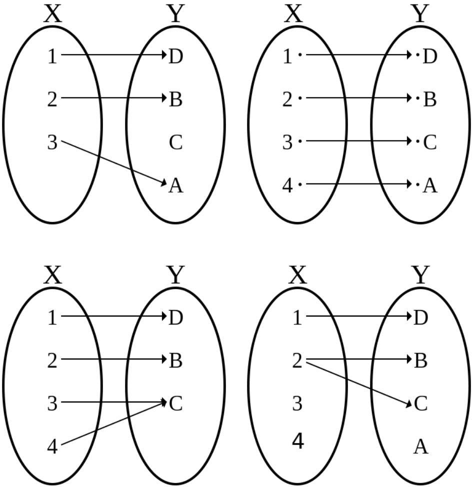
Bijections?

One to one.



Bijections?

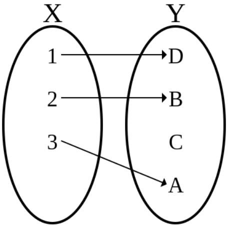
One to one. Bijection: one to one and onto.

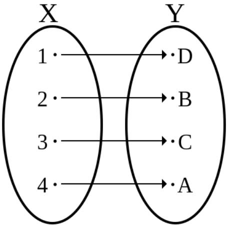


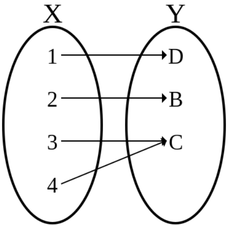
Bijections?

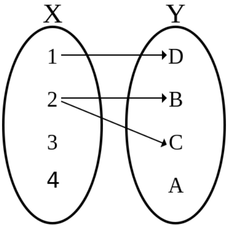
One to one.

Bijection: one to one and onto.





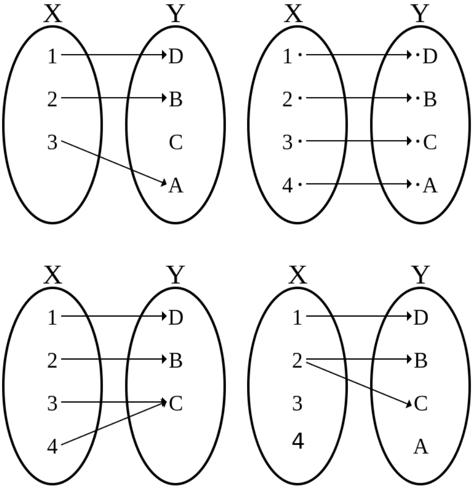




Onto.

Bijections?

One to one. Bijection: one to one and onto.



Onto. Not a function.

Countable.

- Enumerable means countable.

Countable.

- Enumerable means countable.
- Subsets of countable sets are countable.

Countable.

- Enumerable means countable.
- Subsets of countable sets are countable.
  For example the set $\{14, 54, 5332, 10^{12}+4\}$ is countable.

Countable.

- Enumerable means countable.
- Subsets of countable sets are countable.
  For example the set $\{14, 54, 5332, 10^{12} + 4\}$ is countable. (It has 4 elements)

Countable.

- Enumerable means countable.
- Subsets of countable sets are countable.
  For example the set $\{14, 54, 5332, 10^{12} + 4\}$ is countable. (It has 4 elements) Even numbers are countable.

Countable.

- Enumerable means countable.
- Subsets of countable sets are countable.
  For example the set $\{14, 54, 5332, 10^{12} + 4\}$ is countable. (It has 4 elements) Even numbers are countable. Prime numbers are countable.

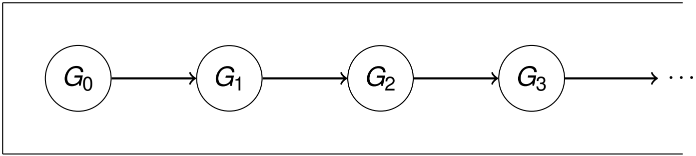
Countable.

- Enumerable means countable.
- Subsets of countable sets are countable.
  For example the set $\{14, 54, 5332, 10^{12} + 4\}$ is countable. (It has 4 elements) Even numbers are countable. Prime numbers are countable. Multiples of 3 are countable.

Countable.

- Enumerable means countable.
- Subsets of countable sets are countable.
  For example the set $\{14, 54, 5332, 10^{12} + 4\}$ is countable. (It has 4 elements) Even numbers are countable. Prime numbers are countable. Multiples of 3 are countable.
- All countably infinite sets have the same cardinality as each other.

Back to Hilbert's hotel



Back to Hilbert's hotel

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots$$

Where's the function?

Back to Hilbert's hotel

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots$$

Where's the function?

We want a bijection from:

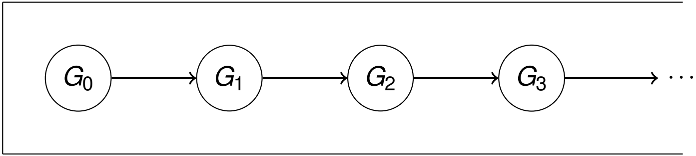
Back to Hilbert's hotel

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots$$

Where's the function?

We want a bijection from: $\mathbb{N}$

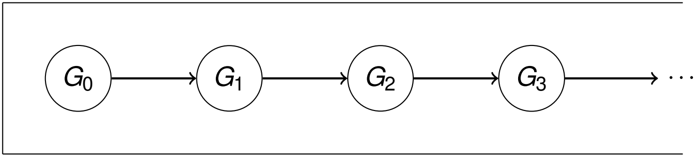
Back to Hilbert's hotel



Where's the function?

We want a bijection from: $\mathbb{N}$ to

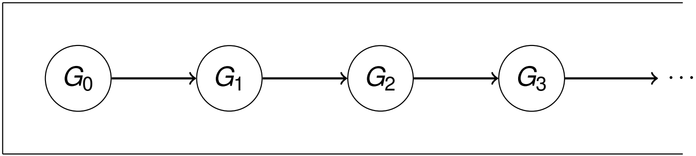
Back to Hilbert's hotel



Where's the function?

We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

Back to Hilbert's hotel



Where's the function?

We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

$f(x) = x + 1$.

Back to Hilbert's hotel

$$G_0 \to G_1 \to G_2 \to G_3 \to \cdots$$

Where's the function?

We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

$f(x) = x + 1$. Maps every number from $\mathbb{N}$ to a number in $\mathbb{N} \setminus \{0\}$, and

Back to Hilbert's hotel

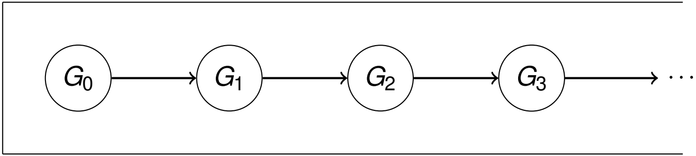
$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots$$

Where's the function?

We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

$f(x) = x + 1$. Maps every number from $\mathbb{N}$ to a number in $\mathbb{N} \setminus \{0\}$, and every number in $x \in \mathbb{N} \setminus \{0\}$ has exactly one number $y \in \mathbb{N}$ such that $f(y) = x$.

Back to Hilbert's hotel



Where's the function?

We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

$f(x) = x + 1$. Maps every number from $\mathbb{N}$ to a number in $\mathbb{N} \setminus \{0\}$, and every number in $x \in \mathbb{N} \setminus \{0\}$ has exactly one number $y \in \mathbb{N}$ such that $f(y) = x$.

What if we had a bijection from $\mathbb{N} \setminus \{0\}$ to $\mathbb{N}$?

Back to Hilbert's hotel

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots$$

Where's the function?

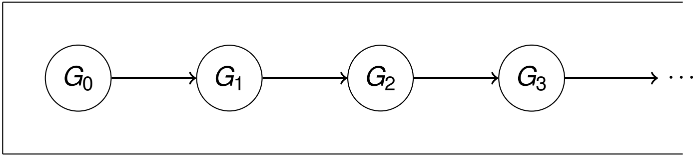
We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

$f(x) = x + 1$. Maps every number from $\mathbb{N}$ to a number in $\mathbb{N} \setminus \{0\}$, and every number in $x \in \mathbb{N} \setminus \{0\}$ has exactly one number $y \in \mathbb{N}$ such that $f(y) = x$.

What if we had a bijection from $\mathbb{N} \setminus \{0\}$ to $\mathbb{N}$?

Same thing! Bijection means that the sets have the same size.

Back to Hilbert's hotel



Where's the function?

We want a bijection from: $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

$f(x) = x + 1$. Maps every number from $\mathbb{N}$ to a number in $\mathbb{N} \setminus \{0\}$, and every number in $x \in \mathbb{N} \setminus \{0\}$ has exactly one number $y \in \mathbb{N}$ such that $f(y) = x$.

What if we had a bijection from $\mathbb{N} \setminus \{0\}$ to $\mathbb{N}$?

Same thing! Bijection means that the sets have the same size. Invert it and you'll get a bijection from $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds?

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate:

Examples

Countably infinite (same cardinality as naturals)

- *E* even numbers.
  Where are the odds? Half as big?
  Enumerate: 0,

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2,

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0,

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 ,

Examples

Countably infinite (same cardinality as naturals)

- *E* even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...

Examples

Countably infinite (same cardinality as naturals)

- *E* even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.

Examples

Countably infinite (same cardinality as naturals)

- *E* even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: 0,

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1,$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2,$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3,$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, ...$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, ...$
  When will we get to -1???

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.

- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, \ldots$
  When will we get to -1???
  New Enumeration: $0,$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, \ldots$
  When will we get to -1???
  New Enumeration: $0, -1,$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, \ldots$
  When will we get to -1???
  New Enumeration: $0, -1, 1,$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, \ldots$
  When will we get to -1???
  New Enumeration: $0, -1, 1, -2,$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, \ldots$
  When will we get to -1???
  New Enumeration: $0, -1, 1, -2, 2 \ldots$

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, ...$
  When will we get to -1???
  New Enumeration: $0, -1, 1, -2, 2...$
  Bijection: $f(z) = 2|z| - sign(z)$.

Examples

Countably infinite (same cardinality as naturals)

- E even numbers.
  Where are the odds? Half as big?
  Enumerate: 0, 2, 4, ...
  0 maps to 0, 2 maps to 1 , 4 maps to 2, ...
  Enumeration naturally corresponds to function.
  No two evens map to the same natural.
  For every natural, there is a corresponding even.
  Bijection: $f(e) = e/2$.
- Z- all integers.
  Twice as big?
  Enumerate: $0, 1, 2, 3, ...$
  When will we get to -1???
  New Enumeration: $0, -1, 1, -2, 2...$
  Bijection: $f(z) = 2|z| - sign(z)$.
  Where $sign(z) = 1$ if $z > 0$ and $sign(z) = 0$ otherwise.

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0)$,

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0),(1,0),$

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0),(1,0),(0,1),$

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0),(1,0),(0,1),(2,0),$

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0, 0), (0, 1), (0, 2), \ldots$???
  Never get to $(1, 1)$!
  Enumerate: $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1),$

Examples: Countable by enumeration

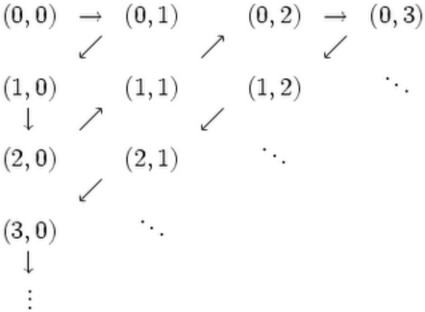
- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\ldots$

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\ldots$ (dovetailing)

Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\ldots$ (dovetailing)



Examples: Countable by enumeration

- $\mathbb{N} \times \mathbb{N}$ - Pairs of integers.
  Square of countably infinite?
  Enumerate: $(0,0),(0,1),(0,2),\ldots$???
  Never get to $(1,1)$!
  Enumerate: $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\ldots$ (dovetailing)

$$
\begin{array}{ccccccc}
(0,0) & \rightarrow & (0,1) & & (0,2) & \rightarrow & (0,3) \\
 & \swarrow & & \nearrow & & \swarrow & \\
(1,0) & & (1,1) & & (1,2) & & \ddots \\
\downarrow & \nearrow & & \swarrow & & & \\
(2,0) & & (2,1) & & \ddots & & \\
 & \swarrow & & & & & \\
(3,0) & & \ddots & & & & \\
\downarrow & & & & & & \\
\vdots & & & & & &
\end{array}
$$

(a,b) at position $(a+b+1)(a+b)/2+b$ in this order.

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative.

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!!

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.

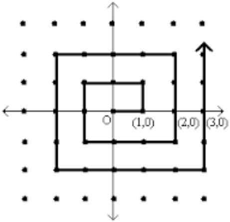
Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.

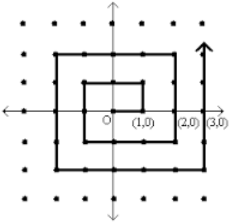
Enumerate: (0,0),



Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.

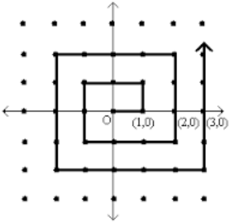
Enumerate: (0,0), (1,0),



Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.

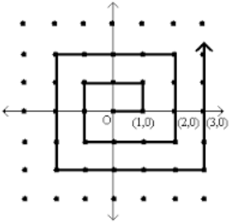
Enumerate: (0,0), (1,0), (1,1),



Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.

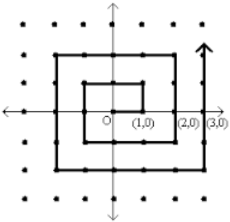
Enumerate: (0,0), (1,0), (1,1), (0,1),



Rationals

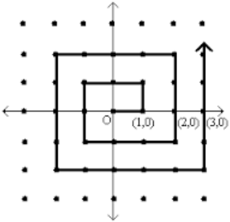
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.

Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...



Rationals

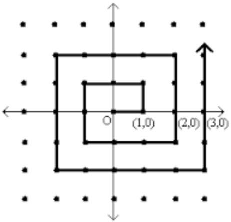
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.

Rationals

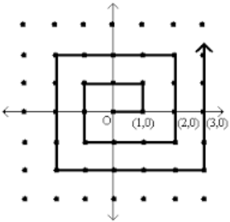
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.

Rationals

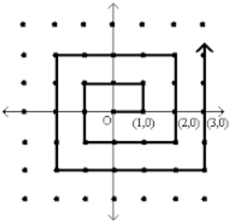
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.
Every natural has a "corresponding" pair.

Rationals

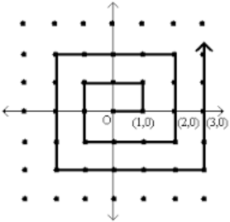
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.
Every natural has a "corresponding" pair.
Where's my bijection???

Rationals

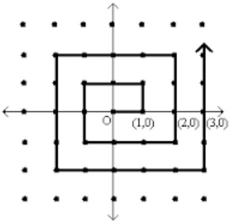
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.
Every natural has a "corresponding" pair.
Where's my bijection??? Too complicated!

Rationals

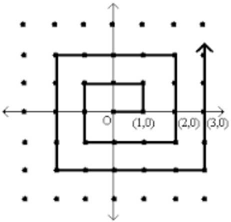
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.
Every natural has a "corresponding" pair.
Where's my bijection??? Too complicated! Enumeration is good enough:
A set S is countable if it can be enumerated in a sequence, i.e., if all of its elements can be listed as a sequence $a_1, a_2, \ldots$.

Rationals

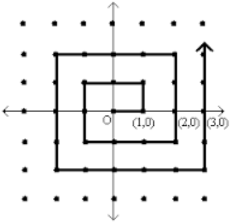
All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is "bigger" than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.
Every natural has a "corresponding" pair.
Where's my bijection??? Too complicated! Enumeration is good enough:
A set S is countable if it can be enumerated in a sequence, i.e., if all of its elements can be listed as a sequence $a_1, a_2, \ldots$. Make sure that (1) different elements map to different naturals.

Rationals

All rational numbers $\mathbb{Q}$: $\frac{a}{b}$, such that $a, b \in \mathbb{Z}$, and $b \neq 0$.
Enumerate: list 0, positive and negative. How?
Same as $\mathbb{Z}^2$!!!! In fact, $\mathbb{Z}^2$ is ”bigger” than $\mathbb{Q}$.
So let's show $\mathbb{Z}^2$ is countable.



Enumerate: (0,0), (1,0), (1,1), (0,1), (-1,1)...
Will eventually get to any pair.
Two different pairs cannot map to the same natural number/same position in the spiral.
Every natural has a ”corresponding” pair.
Where's my bijection??? Too complicated! Enumeration is good enough:
A set S is countable if it can be enumerated in a sequence, i.e., if all of its elements can be listed as a sequence $a_1, a_2, \ldots$. Make sure that (1) different elements map to different naturals. (2) every natural gets an element.

Let’s get real

Is the set of Reals countable?

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000...

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$

Let’s get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162...

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... (1/2)
.785398162... $\pi/4$
.367879441...

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.
.500000000... (1/2)
.785398162... $\pi/4$
.367879441... $1/e$
.632120558...

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312...

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312... Some real number

Let's get real

Is the set of Reals countable?

Lets consider the reals $[0,1]$.

Each real has a decimal representation.
.500000000... $(1/2)$
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1-1/e$
.345212312... Some real number

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$.

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0, 1]$. For example

0: .500000000...

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number:

Diagonalization.

If countable, there exists a listing (enumeration), *L* contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .7

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .776

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .7767

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77677

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: .77677 . . .

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number:

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0, 1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0, 1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Contradiction!

Diagonalization.

If countable, there exists a listing (enumeration), L contains all reals in $[0,1]$. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...
$\vdots$

Construct "diagonal" number: $.77677\ldots$

Diagonal Number: Digit i is 7 if number i's ith digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
Uncountable.

All reals?

Subset $[0,1]$ is not countable!!

What about all reals?
Uncountable.

Any subset of a countable set is countable.

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
Uncountable.

Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies$ t is not in the list.
5. Show that t is in S.

Diagonalization.

1. Assume that a set S can be enumerated.
2. Consider an arbitrary list of all the elements of S.
3. Use the diagonal from the list to construct a new element t.
4. Show that t is different from all elements in the list
   $\implies t$ is not in the list.
5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds,

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes,

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, *L*, that contains all subsets of *N*.
- Define a diagonal set, *D*:
  If *i*th set in *L* does not contain *i*, $i \in D$.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.
- D is different from ith set in L for every i.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.
- D is different from ith set in L for every i.
  $\implies$ D is not in the listing.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.
- D is different from ith set in L for every i.
  $\Longrightarrow$ D is not in the listing.
- D is a subset of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.
- D is different from ith set in L for every i.
  $\implies$ D is not in the listing.
- D is a subset of N.
- L does not contain all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.
- D is different from ith set in L for every i.
  $\implies$ D is not in the listing.
- D is a subset of N.
- L does not contain all subsets of N.

  Contradiction.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.
- D is different from ith set in L for every i.
  $\implies$ D is not in the listing.
- D is a subset of N.
- L does not contain all subsets of N.
  Contradiction.

Theorem: The set of all subsets of N is not countable.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
evens, odds, primes, multiples of 10

- Assume is countable.
- There is a listing, L, that contains all subsets of N.
- Define a diagonal set, D:
  If ith set in L does not contain i, $i \in D$.
  otherwise $i \notin D$.
- D is different from ith set in L for every i.
  $\Longrightarrow$ D is not in the listing.
- D is a subset of N.
- L does not contain all subsets of N.

  Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Another diagonalization.

$$
\boxed{\begin{array}{ll}
s_1 & = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,\ldots \\
s_2 & = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,\ldots \\
s_3 & = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,\ldots \\
s_4 & = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,\ldots \\
s_5 & = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\,\ldots \\
s_6 & = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\,\ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,\ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\,\ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,\ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\,\ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\,\ldots \\
\vdots & \quad \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots
\end{array}}
$$

$$
\boxed{s = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1\,\ldots}
$$

Countable or uncountable??

- Binary strings?

Countable or uncountable??

- Binary strings?
- Trees?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?
- Mathematical proofs?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?
- Mathematical proofs?
- Programs in Java?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?
- Mathematical proofs?
- Programs in Java?
- All possible endings to Game of Thrones?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?
- Mathematical proofs?
- Programs in Java?
- All possible endings to Game of Thrones?
- All subsets of Reals?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?
- Mathematical proofs?
- Programs in Java?
- All possible endings to Game of Thrones?
- All subsets of Reals?
- Functions from $\mathbb{N}$ to $\mathbb{N}$?

Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?
- Mathematical proofs?
- Programs in Java?
- All possible endings to Game of Thrones?
- All subsets of Reals?
- Functions from $\mathbb{N}$ to $\mathbb{N}$?

You already know some of these.....

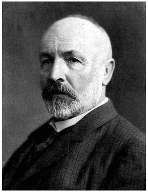
Countable or uncountable??

- Binary strings?
- Trees?
- Weighted trees?
- Inputs to the stable marriage algorithm?
- Mathematical proofs?
- Programs in Java?
- All possible endings to Game of Thrones?
- All subsets of Reals?
- Functions from $\mathbb{N}$ to $\mathbb{N}$?

You already know some of these..... Think about induction!

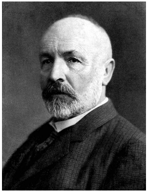
What happened with Cantor?

What happened with Cantor?



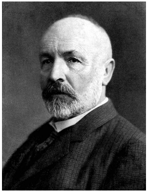
Cantor's work between 1874 and 1884 is the origin of set theory.

What happened with Cantor?



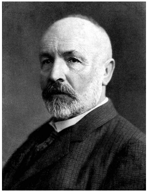
Cantor’s work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.

What happened with Cantor?



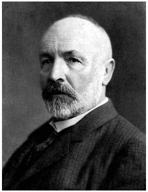
Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite

What happened with Cantor?



Cantor’s work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite

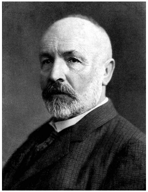
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable

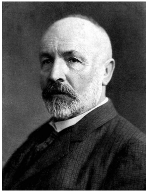
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable.

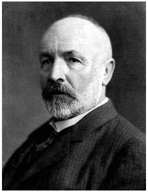
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$

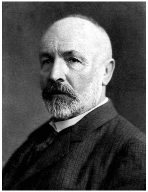
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable.

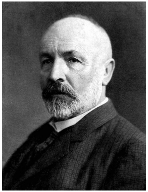
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...

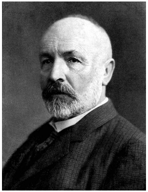
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable.

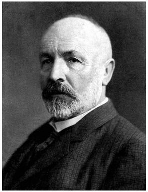
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...

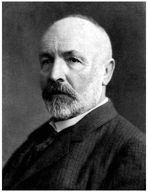
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable!

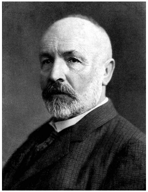
What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable! (Math 135, Math 136, Math 227A ...)

What happened with Cantor?

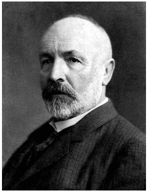


Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable! (Math 135, Math 136, Math 227A ...)

Everyone was upset!

What happened with Cantor?

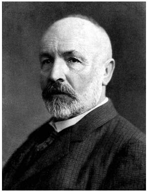


Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable! (Math 135, Math 136, Math 227A ...)

Everyone was upset! Many puzzled...

What happened with Cantor?

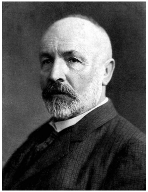


Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable! (Math 135, Math 136, Math 227A ...)

Everyone was upset! Many puzzled... Many openly hostile to Cantor...

What happened with Cantor?

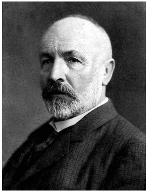


Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable! (Math 135, Math 136, Math 227A ...)

Everyone was upset! Many puzzled... Many openly hostile to Cantor... Cantor was clinically depressed.

What happened with Cantor?

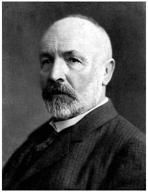


Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable! (Math 135, Math 136, Math 227A ...)

Everyone was upset! Many puzzled... Many openly hostile to Cantor... Cantor was clinically depressed. In and out of hospitals until the end of his life.

What happened with Cantor?



Cantor's work between 1874 and 1884 is the origin of set theory.
No one had realized that set theory had any nontrivial content.
Before Cantor: Finite , Infinite
After Cantor:

- Countable
  - Finite and countable. For example $\{1, 2, 3\}$
  - Infinite and countable. For example $\mathbb{N}$, $\mathbb{Z}$, ...
- Uncountable. For example $[0, 1]$, $\mathbb{R}$...
- Bigger than uncountable! (Math 135, Math 136, Math 227A ...)

Everyone was upset! Many puzzled... Many openly hostile to Cantor... Cantor was clinically depressed. In and out of hospitals until the end of his life. Died in poverty...

Cantor's legacy

Gottlob Frege:

Cantor’s legacy

Gottlob Frege: Let’s look at the foundations!

Cantor’s legacy

Gottlob Frege: Let’s look at the foundations!
Clear ambition:

Cantor's legacy

Gottlob Frege: Let's look at the foundations!
Clear ambition: Become the new Euclid.

Cantor's legacy

Gottlob Frege: Let's look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory.

Cantor’s legacy

Gottlob Frege: Let’s look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory. (In the case of geometry ”A straight line segment can be drawn joining any two points” etc)

Cantor’s legacy

Gottlob Frege: Let’s look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory. (In the case of geometry ”A straight line segment can be drawn joining any two points” etc)
Everything that is true in number theory can be inferred from the axioms.

Cantor's legacy

Gottlob Frege: Let's look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory. (In the case of geometry "A straight line segment can be drawn joining any two points" etc)
Everything that is true in number theory can be inferred from the axioms.
Writes Basic Laws of Arithmetic vol. 1.

Cantor's legacy

Gottlob Frege: Let's look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory. (In the case of geometry "A straight line segment can be drawn joining any two points" etc)
Everything that is true in number theory can be inferred from the axioms.
Writes Basic Laws of Arithmetic vol. 1. 680 pages (Amazon).

Cantor’s legacy

Gottlob Frege: Let’s look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory. (In the case of geometry ”A straight line segment can be drawn joining any two points” etc)
Everything that is true in number theory can be inferred from the axioms.
Writes Basic Laws of Arithmetic vol. 1. 680 pages (Amazon).
About to publish vol. 2.

Cantor's legacy

Gottlob Frege: Let's look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory. (In the case of geometry ”A straight line segment can be drawn joining any two points” etc)
Everything that is true in number theory can be inferred from the axioms.
Writes Basic Laws of Arithmetic vol. 1. 680 pages (Amazon).
About to publish vol. 2. And then......

Cantor's legacy

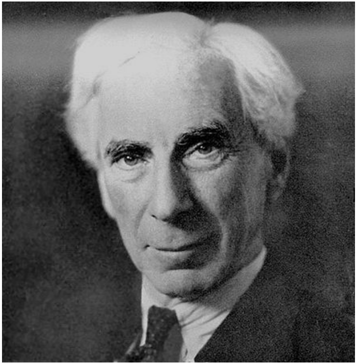
Gottlob Frege: Let's look at the foundations!
Clear ambition: Become the new Euclid.
Make up a bunch of axioms for number theory. (In the case of geometry ”A straight line segment can be drawn joining any two points” etc)
Everything that is true in number theory can be inferred from the axioms.
Writes Basic Laws of Arithmetic vol. 1. 680 pages (Amazon).
About to publish vol. 2. And then......
Disaster!!

A bug

Bertrand Russell finds a bug!

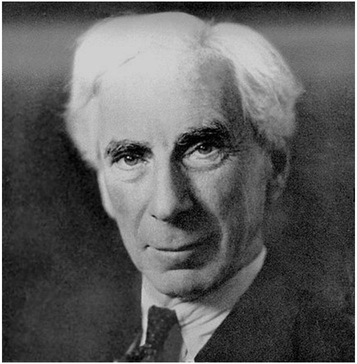
A bug

Bertrand Russell finds a bug!



A bug

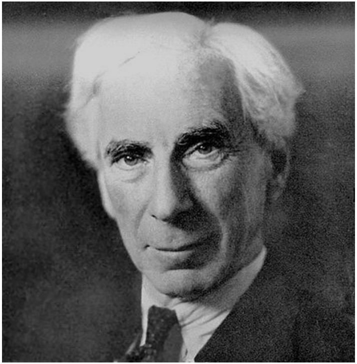
Bertrand Russell finds a bug!



Frege's reaction:

A bug

Bertrand Russell finds a bug!



Frege's reaction: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion."

A poem

Zisimos Lorentzatos.
"Beware of systems grandiose, of mathematically strict causalities as you're trying, stone by stone, to found the goldenwoven tower of the logical, castle and fort immune to contradiction. Designed in two volumes, the foundational laws of arithmetic, or Grundgesetze of der arithmetic in 1893, the first, 1903 the second. A life's work. Hammer on chisel blows for years and years. So far, so good. But as Frege Gottlob was correcting, content, the printer's proofs already of the second volume, one cursed logic paradox, one not admitting refutation, question by Russell Bertrand, forced, without delay, the great thinker of Mecklemburg to add a last paragraph to his system, show me a great thinker who would resist the truth, accepting the reversible disaster. His foundations in ruin, his logic flawed, his work wasted, and his two volumes imagine the colossal set back, odd load and ballast for the refuge cart."

Russell’s Paradox.

- ”This statement is false”

Russell's Paradox.

- "This statement is false"
  Is the statement above true?
- A barber says "I shave all and only those men who do not shave themselves."

Russell’s Paradox.

- ”This statement is false”
  Is the statement above true?
- A barber says ”I shave all and only those men who do not shave themselves.”
  Who shaves the barber??

Russell’s Paradox.

- ”This statement is false”
  Is the statement above true?
- A barber says ”I shave all and only those men who do not shave themselves.”
  Who shaves the barber??
- Self reference.........

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

Let's think about the set of all sets that don't contain themselves.

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

Let's think about the set of all sets that don't contain themselves. Call it A.

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

Let's think about the set of all sets that don't contain themselves. Call it A.

Does A contain itself?

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

Let's think about the set of all sets that don't contain themselves. Call it A.

Does A contain itself?

Oops!

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

Let's think about the set of all sets that don't contain themselves. Call it A.

Does A contain itself?

Oops!

What type of object is a set that contain sets?

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

Let's think about the set of all sets that don't contain themselves. Call it A.

Does A contain itself?

Oops!

What type of object is a set that contain sets?

Change Axioms!

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent:

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete:

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete: Everything true can be proven.

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete: Everything true can be proven.

Other people in this story:

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete: Everything true can be proven.

Other people in this story: Russell

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete: Everything true can be proven.

Other people in this story: Russell , Whitehead

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete: Everything true can be proven.

Other people in this story: Russell , Whitehead , Wittgenstein

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete: Everything true can be proven.

Other people in this story: Russell , Whitehead , Wittgenstein , Hilbert
(We must know. We will know.) ...

Changing Axioms?

They did keep trying to put all of mathematics on a firm basis...
Trying to find a set of axioms such that is

- Consistent: You can't prove false statements
- Complete: Everything true can be proven.

Other people in this story: Russell , Whitehead , Wittgenstein , Hilbert (We must know. We will know.) ...
Until 1931.

Changing Axioms?

Changing Axioms?

Kurt Gödel:

Changing Axioms?

Kurt Gödel:
Any set of axioms is either

Changing Axioms?

Kurt Gödel:
Any set of axioms is either
inconsistent (can prove false statements) or

Changing Axioms?

Kurt Gödel:
Any set of axioms is either
inconsistent (can prove false statements) or
incomplete (true statements cannot be proven.)

Changing Axioms?

Kurt Gödel:
Any set of axioms is either
inconsistent (can prove false statements) or
incomplete (true statements cannot be proven.)

Concrete example:
Continuum hypothesis (see official notes if interested)

- Gödel

- Gödel ..starved himself out of fear of being poisoned..

- Gödel ..starved himself out of fear of being poisoned..
- Russell

- Gödel ..starved himself out of fear of being poisoned..
- Russell .. was fine...

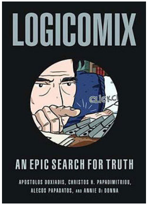
- Gödel ..starved himself out of fear of being poisoned..
- Russell .. was fine.....but for two schizophrenic children..

- Gödel ..starved himself out of fear of being poisoned..
- Russell .. was fine.....but for two schizophrenic children..
- Wittgenstein

- Gödel ..starved himself out of fear of being poisoned..
- Russell .. was fine.....but for two schizophrenic children..
- Wittgenstein ... multiple tragedies in his family.

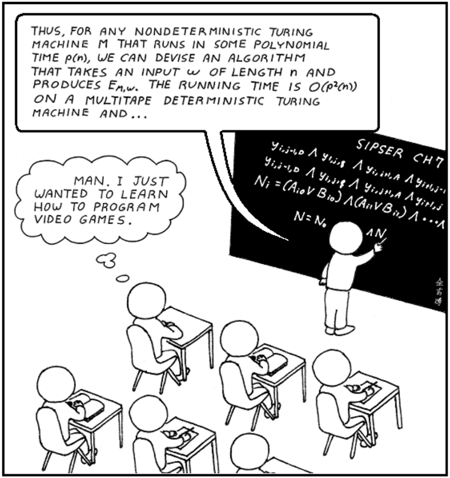
- Gödel ..starved himself out of fear of being poisoned..
- Russell .. was fine.....but for two schizophrenic children..
- Wittgenstein ... multiple tragedies in his family.
- Dangerous work?

- Gödel ..starved himself out of fear of being poisoned..
- Russell .. was fine.....but for two schizophrenic children..
- Wittgenstein ... multiple tragedies in his family.
- Dangerous work?
- See Logicomix by Doxiadis, Papadimitriou (my advisor!), Papadatos, Di Donna.



Next Topic: Undecidability.

- Undecidability. A happy ending?



Turing

Is it actually useful?

Turing: Write me a program checker!

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
    P - program

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
  P - program
  I - input.

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
 P - program
 I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:
Need a computer

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
P - program
I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:
Need a computer
...with the notion of a stored program!!!!

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
    P - program
    I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:
Need a computer
...with the notion of a stored program!!!!
Program is a text string.

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:
Need a computer
...with the notion of a stored program!!!!
Program is a text string.
Text string can be an input to a program.

Is it actually useful?

Turing: Write me a program checker!

A program that checks that the compiler works!

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:
Need a computer
...with the notion of a stored program!!!!
Program is a text string.
Text string can be an input to a program.
Program can be an input to a program.

Implementing HALT.

Implementing HALT.

$HALT(P, I)$

Implementing HALT.

$HALT(P, I)$
  P - program

Implementing HALT.

$HALT(P, I)$
P - program
I - input.

Implementing HALT.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Implementing HALT.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Run P on I and check!

Implementing HALT.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Run P on I and check!

How long do you wait?

Implementing HALT.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Run P on I and check!

How long do you wait?

Something about infinity here, maybe?

Implementing HALT.

$HALT(P, I)$
  P - program
  I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Run P on I and check!

How long do you wait?

Something about infinity here, maybe?

Theorem: There is no program HALT.

Halt does not exist.

Proof:

Halt does not exist.

Proof: Assume there is a program $\mathit{HALT}(\cdot, \cdot)$.

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:
import HALT;

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:
import HALT;
function Turing(Program P) $\{$

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT(  P,  P.toString()  ) ==“halts” ):
        while(true);
```

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT(  P,  P.toString()  ) ==“halts” ):
        while(true); (go in an infinite loop)
```

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
Run Turing(Turing).
```

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT(  P,  P.toString()  ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).

Does Turing(Turing) halt?

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) =="halts" ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).

Does Turing(Turing) halt?

Turing(Turing) halts

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).
Does Turing(Turing) halt?

Turing(Turing) halts
$\Longrightarrow$ then HALT(Turing, Turing.toString()) $=$ halts

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).
Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALT(Turing, Turing.toString()) $=$ halts
$\implies$ Turing(Turing) loops forever.

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).
Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALT(Turing, Turing.toString()) $=$ halts
$\implies$ Turing(Turing) loops forever.

Turing(Turing) loops forever

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).
Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALT(Turing, Turing.toString()) $=$ halts
$\implies$ Turing(Turing) loops forever.

Turing(Turing) loops forever
$\implies$ then HALT(Turing, Turing.toString()) $\neq$ halts

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) =="halts" ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).
Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALT(Turing, Turing.toString()) $=$ halts
$\implies$ Turing(Turing) loops forever.

Turing(Turing) loops forever
$\implies$ then HALT(Turing, Turing.toString()) $\neq$ halts
$\implies$ Turing(Turing) halts. (goes to system.exit())

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) =="halts" ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).
Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALT(Turing, Turing.toString()) $=$ halts
$\implies$ Turing(Turing) loops forever.

Turing(Turing) loops forever
$\implies$ then HALT(Turing, Turing.toString()) $\neq$ halts
$\implies$ Turing(Turing) halts. (goes to system.exit())

Contradiction.

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) =="halts" ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).
Does Turing(Turing) halt?

Turing(Turing) halts
$\implies$ then HALT(Turing, Turing.toString()) $=$ halts
$\implies$ Turing(Turing) loops forever.

Turing(Turing) loops forever
$\implies$ then HALT(Turing, Turing.toString()) $\neq$ halts
$\implies$ Turing(Turing) halts. (goes to system.exit())

Contradiction. Program HALT does not exist!

Halt does not exist.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Code:

```
import HALT;
function Turing( Program P ) {
    if ( HALT( P, P.toString() ) ==“halts” ):
        while(true); (go in an infinite loop)
    else:
        system.exit();
}
```

Run Turing(Turing).

Does Turing(Turing) halt?

Turing(Turing) halts

$\Longrightarrow$ then HALT(Turing, Turing.toString()) $=$ halts

$\Longrightarrow$ Turing(Turing) loops forever.

Turing(Turing) loops forever

$\Longrightarrow$ then HALT(Turing, Turing.toString()) $\neq$ halts

$\Longrightarrow$ Turing(Turing) halts. (goes to system.exit())

Contradiction. Program HALT does not exist! $\square$

Another view of proof: diagonalization.

Any program is a fixed length string.

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

	"P_1"	"P_2"	"P_3"	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input "P_1" and "P_2", doesn't halt on input "P_3", and so on...

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

	$"P_1"$	$"P_2"$	$"P_3"$	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input $"P_1"$ and $"P_2"$, doesn't halt on input $"P_3"$, and so on...
Turing is different from every P_i on the diagonal.

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

	"P_1"	"P_2"	"P_3"	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input "P_1" and "P_2", doesn't halt on input "P_3", and so on...
Turing is different from every P_i on the diagonal.
Turing is not on list.

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

	$''P_1''$	$''P_2''$	$''P_3''$	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input $''P_1''$ and $''P_2''$, doesn't halt on input $''P_3''$, and so on...
Turing is different from every P_i on the diagonal.
Turing is not on list. But, Turing is a program.

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

	"P_1"	"P_2"	"P_3"	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input "P_1" and "P_2", doesn't halt on input "P_3", and so on...
Turing is different from every P_i on the diagonal.
Turing is not on list. But, Turing is a program.
Turing can be constructed from Halt.

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

	"P_1"	"P_2"	"P_3"	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input "P_1" and "P_2", doesn't halt on input "P_3", and so on...
Turing is different from every P_i on the diagonal.
Turing is not on list. But, Turing is a program.
Turing can be constructed from Halt.
Halt does not exist!

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
If HALT existed, we could use it to make the following table:

	"P_1"	"P_2"	"P_3"	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Program P_1 halts on input "P_1" and "P_2", doesn't halt on input "P_3", and so on...
Turing is different from every P_i on the diagonal.
Turing is not on list. But, Turing is a program.
Turing can be constructed from Halt.
Halt does not exist! $\square$

Wow, that was easy!

Wow, that was easy!

We should be famous!

No computers for Turing!

In Turing's time.

No computers for Turing!

In Turing's time.

No computers.

No computers for Turing!

In Turing's time.

No computers.

Concept of program as data wasn't really there.

Undecidable problems.

Does a program ever print “Hello World”?

Undecidable problems.

Does a program ever print “Hello World”?
Find exit points and add statement: **Print** “Hello World.”

Undecidable problems.

Does a program ever print “Hello World”?
Find exit points and add statement: **Print** “Hello World.”

Is there program that makes other programs faster?

Undecidable problems.

Does a program ever print “Hello World”?
Find exit points and add statement: **Print** “Hello World.”

Is there program that makes other programs faster?

Is there program that decides if two other programs are equivalent?

Undecidable problems.

Does a program ever print “Hello World”?
Find exit points and add statement: **Print** “Hello World.”

Is there program that makes other programs faster?

Is there program that decides if two other programs are equivalent?

Does this computer program have any security vulnerabilities?

More about Alan Turing.

- Brilliant codebreaker during WWII, helped break German Enigma Code (which probably shortened war by 1 year).

More about Alan Turing.

- Brilliant codebreaker during WWII, helped break German Enigma Code (which probably shortened war by 1 year).
- Seminal paper in numerical analysis:

More about Alan Turing.

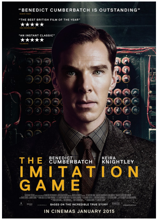
- Brilliant codebreaker during WWII, helped break German Enigma Code (which probably shortened war by 1 year).
- Seminal paper in numerical analysis: Condition number.

More about Alan Turing.

- Brilliant codebreaker during WWII, helped break German Enigma Code (which probably shortened war by 1 year).
- Seminal paper in numerical analysis: Condition number.
- Seminal paper in mathematical biology.

More about Alan Turing.

- Brilliant codebreaker during WWII, helped break German Enigma Code (which probably shortened war by 1 year).
- Seminal paper in numerical analysis: Condition number.
- Seminal paper in mathematical biology.
- Movie:



Turing: personal.

Tragic ending...

Turing: personal.

Tragic ending...

- Arrested as a homosexual

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.
- lost security clearance...

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.
- lost security clearance...
- denied entry into the United States...

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.
- lost security clearance...
- denied entry into the United States...
- suffered from depression;

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.
- lost security clearance...
- denied entry into the United States...
- suffered from depression;
- suicided with cyanide at age 42.

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.
- lost security clearance...
- denied entry into the United States...
- suffered from depression;
- suicided with cyanide at age 42.
  (A bite from the apple....)

Turing: personal.

Tragic ending...

- Arrested as a homosexual
- given choice of prison or (quackish) injections to eliminate sex drive;
- took injections.
- lost security clearance...
- denied entry into the United States...
- suffered from depression;
- suicided with cyanide at age 42.
  (A bite from the apple....) accident?

British Apology.

Gordon Brown. 2009.

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated.

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated. While Turing was dealt with under the law of the time and we can’t put the clock back,

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated. While Turing was dealt with under the law of the time and we can’t put the clock back, his treatment was of course utterly unfair and I am pleased to have the chance to say how deeply sorry I and we all are for what happened to him.

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated. While Turing was dealt with under the law of the time and we can’t put the clock back, his treatment was of course utterly unfair and I am pleased to have the chance to say how deeply sorry I and we all are for what happened to him. Alan and the many thousands of other gay men who were convicted as he was convicted under homophobic laws were treated terribly.

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated. While Turing was dealt with under the law of the time and we can’t put the clock back, his treatment was of course utterly unfair and I am pleased to have the chance to say how deeply sorry I and we all are for what happened to him. Alan and the many thousands of other gay men who were convicted as he was convicted under homophobic laws were treated terribly. Over the years millions more lived in fear of conviction.

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated. While Turing was dealt with under the law of the time and we can’t put the clock back, his treatment was of course utterly unfair and I am pleased to have the chance to say how deeply sorry I and we all are for what happened to him. Alan and the many thousands of other gay men who were convicted as he was convicted under homophobic laws were treated terribly. Over the years millions more lived in fear of conviction. ....

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated. While Turing was dealt with under the law of the time and we can’t put the clock back, his treatment was of course utterly unfair and I am pleased to have the chance to say how deeply sorry I and we all are for what happened to him. Alan and the many thousands of other gay men who were convicted as he was convicted under homophobic laws were treated terribly. Over the years millions more lived in fear of conviction. ....
So on behalf of the British government, and all those who live freely thanks to Alan’s work I am very proud to say: we’re sorry, you deserved so much better.”

British Apology.

Gordon Brown. 2009. “Thousands of people have come together to demand justice for Alan Turing and recognition of the appalling way he was treated. While Turing was dealt with under the law of the time and we can’t put the clock back, his treatment was of course utterly unfair and I am pleased to have the chance to say how deeply sorry I and we all are for what happened to him. Alan and the many thousands of other gay men who were convicted as he was convicted under homophobic laws were treated terribly. Over the years millions more lived in fear of conviction. ....
So on behalf of the British government, and all those who live freely thanks to Alan’s work I am very proud to say: we’re sorry, you deserved so much better.”
2013. Granted Royal pardon.

Summary

Infinity is interesting!

Summary

Infinity is interesting!
And mind boggling

Summary

Infinity is interesting!

And mind boggling

Computer Programs are an interesting thing.

Summary

- Infinity is interesting!
  - And mind boggling
- Computer Programs are an interesting thing.
  - Like Math.

Summary

Infinity is interesting!

And mind boggling

Computer Programs are an interesting thing.

Like Math.

Deep connection between mathematical proofs and computer programs.

Summary

Infinity is interesting!

- And mind boggling

Computer Programs are an interesting thing.

- Like Math.

Deep connection between mathematical proofs and computer programs.

Computer Programs cannot completely “understand” computer programs.

Summary

Infinity is interesting!
  And mind boggling
Computer Programs are an interesting thing.
  Like Math.
Deep connection between mathematical proofs and computer programs.
Computer Programs cannot completely “understand” computer programs.

Example: no computer program can tell if any other computer program HALTS.

Summary

Infinity is interesting!

- And mind boggling

Computer Programs are an interesting thing.

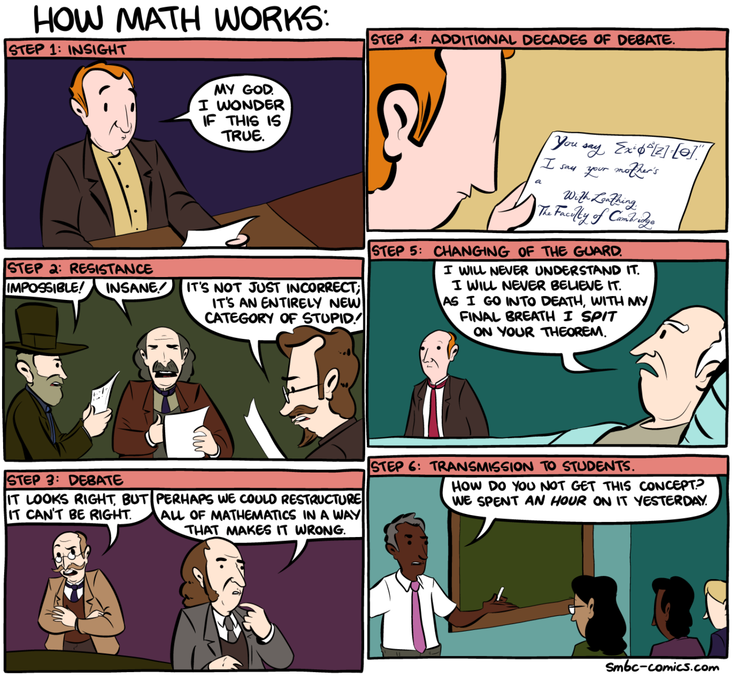
- Like Math.

Deep connection between mathematical proofs and computer programs.

Computer Programs cannot completely “understand” computer programs.

Example: no computer program can tell if any other computer program HALTS.

Programming is a super power.


HOW MATH WORKS:
STEP 1: INSIGHT
MY GOD. I WONDER IF THIS IS TRUE.
STEP 2: RESISTANCE
IMPOSSIBLE!
INSANE!
IT'S NOT JUST INCORRECT; IT'S AN ENTIRELY NEW CATEGORY OF STUPID!
STEP 3: DEBATE
IT LOOKS RIGHT, BUT IT CAN'T BE RIGHT.
PERHAPS WE COULD RESTRUCTURE ALL OF MATHEMATICS IN A WAY THAT MAKES IT WRONG.
STEP 4: ADDITIONAL DECADES OF DEBATE.
You say $\Sigma x^i \phi^\Delta[z] = [\Theta]$."
I say your mother's a
With Loathing,
The Faculty of Cambridge
STEP 5: CHANGING OF THE GUARD.
I WILL NEVER UNDERSTAND IT. I WILL NEVER BELIEVE IT. AS I GO INTO DEATH, WITH MY FINAL BREATH I SPIT ON YOUR THEOREM.
STEP 6: TRANSMISSION TO STUDENTS.
HOW DO YOU NOT GET THIS CONCEPT? WE SPENT AN HOUR ON IT YESTERDAY.
smbc-comics.com