

# Today

Review for Midterm.

First there was logic...

**A statement is a true or false.**

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Don't worry about Gödel.

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$(\forall n \in \mathbb{N}), n^2 \geq n$ : Any free variables? No. So it's a statement.

$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})y > x$ .

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$$A \wedge B, A \vee B, \neg A, A \implies B.$$

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Find an example of  $P(x)$  and  $Q(x)$

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**If you think it's not true:**

Find an example of  $P(x)$  and  $Q(x)$  such that one of the above steps fails.

...and then proofs...

**Direct:**  $P \implies Q$

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Example:  $a$  is even  $\implies a^2$  is even.

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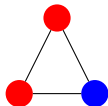


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Given  $G = (V, E)$ , a coloring of a  $G$  assigns colors to vertices  $V$  where for each edge the endpoints have different colors.

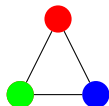
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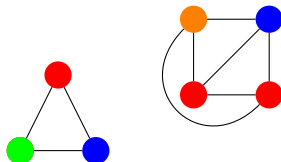
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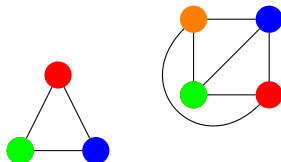
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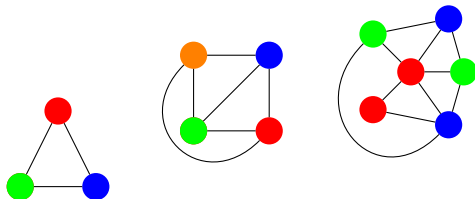
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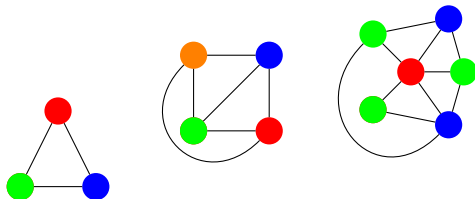
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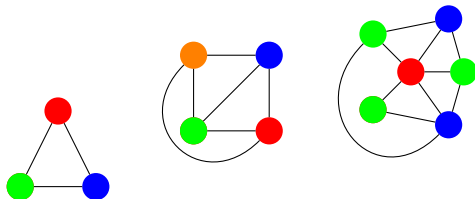
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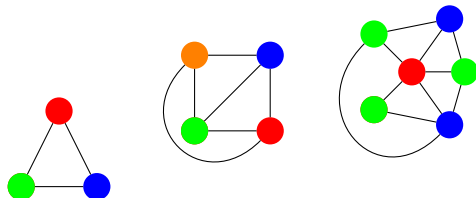
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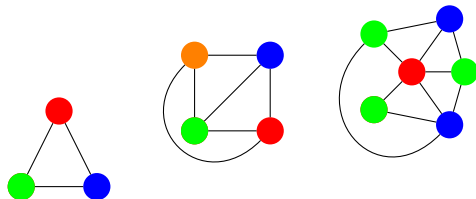
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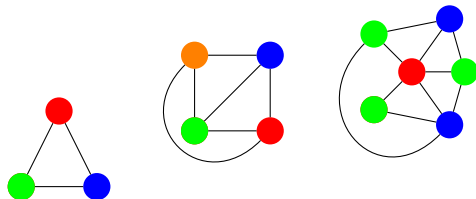
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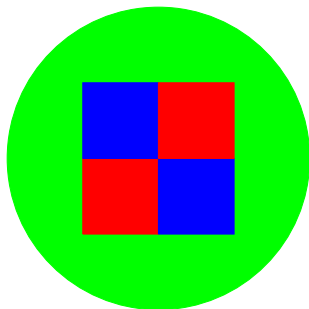
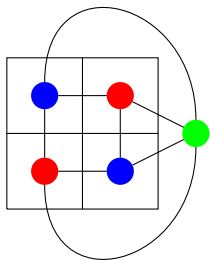
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Fewer colors than number of vertices.

Fewer colors than max degree node.

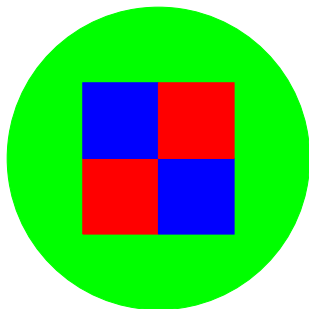
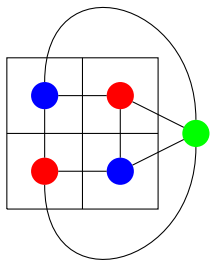
# Planar graphs and maps.

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Four color theorem is about planar graphs!

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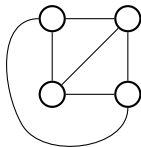
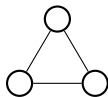
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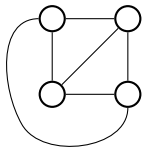
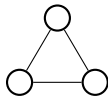
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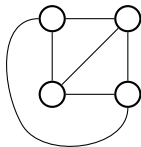
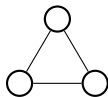
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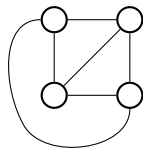
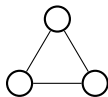
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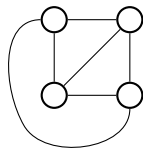
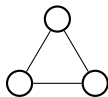


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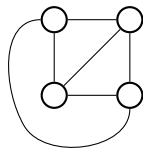
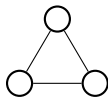


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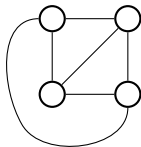
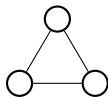
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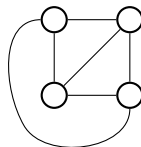
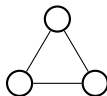
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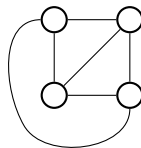
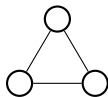
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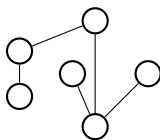
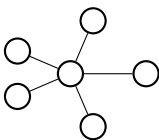
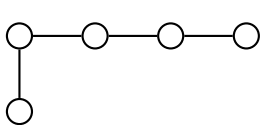
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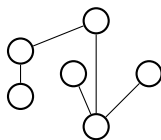
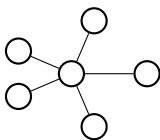
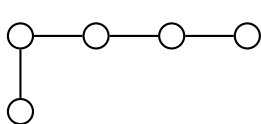
# Trees.



Definitions:



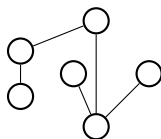
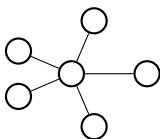
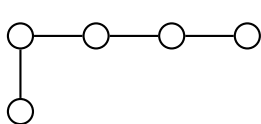
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Definitions:

A connected graph without a cycle.

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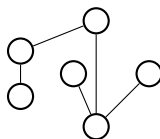
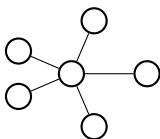
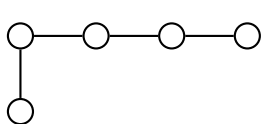


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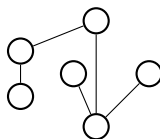
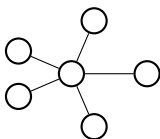
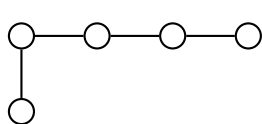
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A connected graph where any edge removal disconnects it.

# Trees.



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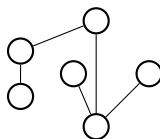
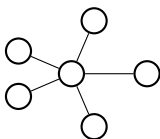
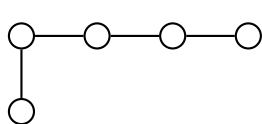
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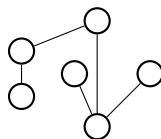
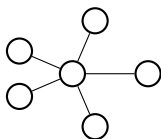
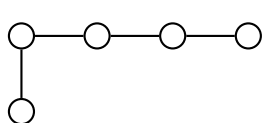
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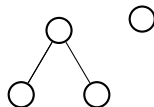
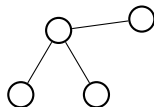
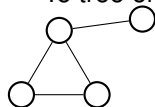
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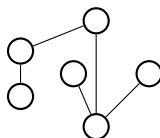
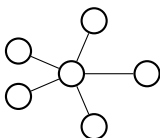
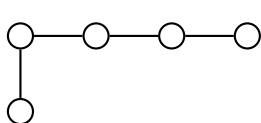
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To tree or not to tree!



# Trees.



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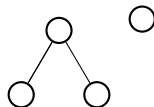
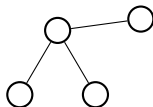
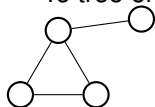
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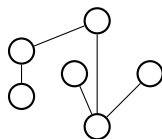
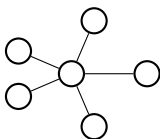
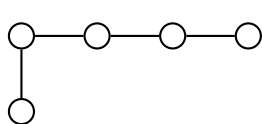
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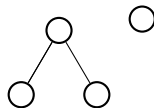
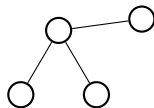
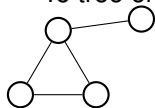
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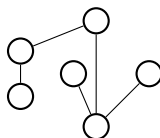
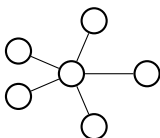
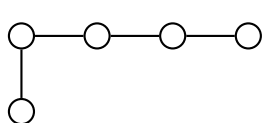


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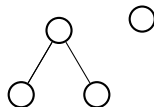
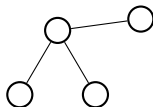
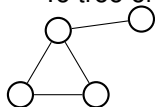
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Can remove a single node and break into components of size at most  $|V|/2$ .

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Hypercubes.

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Hypercubes. Really connected.

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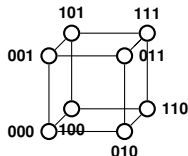
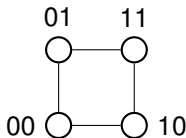
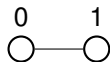
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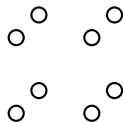
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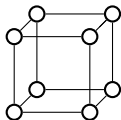
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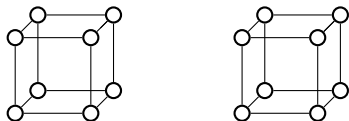
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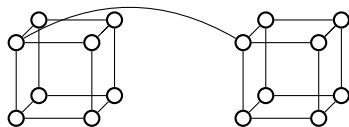
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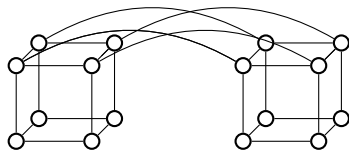
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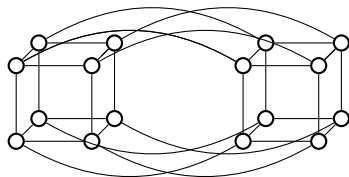
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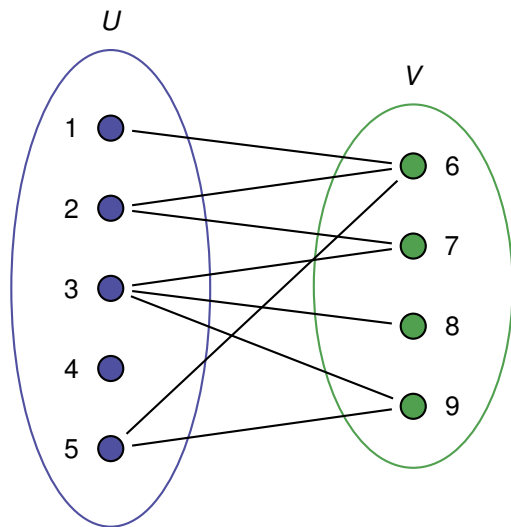
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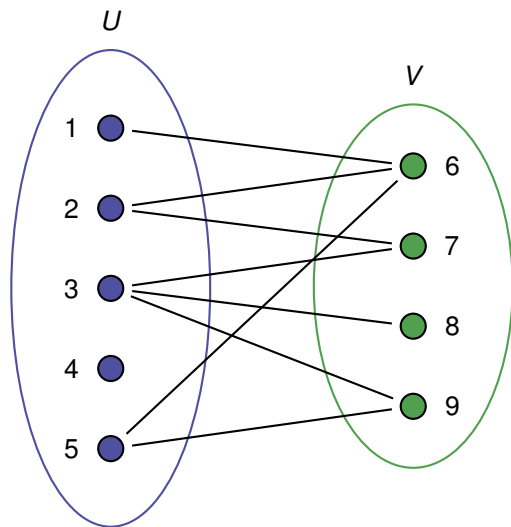
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Good communication network!

# Bipartite graphs

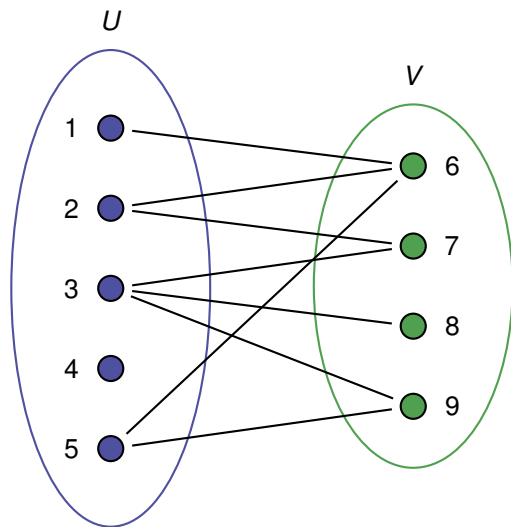


# Bipartite graphs



There is a cut with all the edges.

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Cycles have length 4 or more edges.

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**Every man proposes to his favorite woman from the ones that haven't already rejected him.**

**Every woman rejects all but best man who proposes.**

Useful Algorithmic Definitions:

Man **crosses off** woman who rejected him.

Woman's current proposer is "**on string.**"

Key Property: Improvement Lemma:

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And then countability

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**More than one infinities**

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Why? **Diagonalization:** Well, assume there is a list. Can construct a diagonal element  $x$ .  $x$  is not in the list! Contradiction.

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Like: Will this program  $P$  even print "Hello World"?

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Like: Will this program  $P$  even print "Hello World"?

Or "Is there an input for this program  $P$  that will give an attacker admin access?"

# Counting!

Sample  $k$  items out of  $n$ .

	With Replacement	Without Replacement
Order matters	$n^k$	$\frac{n!}{(n-k)!}$
Order doesn't matter	$\binom{n+k-1}{n-1}$	$\binom{n}{k}$

Stars and bars!

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Confusion yesterday: 10 hats.

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Confusion yesterday: 10 hats. 7 days.

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Didn't wear hats 1 and 2. Wore hat 3 for 1 day, hat 4 for 2 days, hat 5 days. Didn't wear hats 6 and 7. Hat 8 for 3 days. Didn't wear hats 9 and 10.

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Add them up. (**Sum rule**)

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Know material well: fast, correct.

Know material medium:

# Midterm format

Time: 110 minutes.

Some short answers.

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Remember that a problem from hw and/or discussions is in the midterm! (identical or almost identical)

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So study those!

# FAQ

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- ▶ Will this proof from the notes that I don't like be in the midterm?

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Understanding a complex proof is a useful skill.

Also, big proofs are usually a bunch of little proofs put together.  
And every proof is a new trick. And we like tricks!

Wrapup.

## Wrapup.

If you sent us an email about Midterm conflicts

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Other arrangements.



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