CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye Discussion 2A

1. Fun Counting Edges

Prove the following claims: In any graph, the number of vertices of odd degree is even.

The sum of all the degrees is equal to twice the number of edges, which means the sum of degrees will be even. Since the sum of the degrees is even and the sum of the degrees of vertices with even degree is even, the sum of the degrees of vertices with odd degree must be even. (Since only even+even = even) If the sum of the degrees of vertices with odd degree is even, there must be an even number of those vertices.

2. Color the graph

Suppose that the degrees of the vertices in a graph are all at most d. Prove, using the well-ordering principle, that one can color the vertices of the graph using at most $d+1$ colors so that no two adjacent vertices end up having the same color.

We prove this by the well-ordering principle. We call a graph colorable with k colors if it can be colored with at most k colors such that no two adjacent vertices end up having the same color. Suppose the statement is not true. Let G be the graph with the smallest number of vertices (all of degree at most d) that is not colorable with $d+1$ colors. We know that G exists by the well-ordering principle. Note that the number of vertices in G is greater than 1 since a graph of 1 vertex is always colorable with 1 color. Now take any vertex v in G and remove it. Then we know that the resulting graph is colorable with $d+1$ colors because G is the smallest graph that is not colorable. However, we can then add v back to the graph and assign it a color. Since v is connected to at most d edges, we can choose a color different from the colors of the vertices it is connected to. But this means that G is colorable with $d+1$ colors. We have arrived at a contradiction, and therefore proved that all graphs with degrees at most d are colorable with $d+1$ colors.

3. Introduction to Trees

Recall that a tree is a connected graph with no cycles, (and so no self-loops, and no multi-edges). Show that any tree with at least 2 nodes must have a node of degree 1.

Use contradiction.

Case 1: All nodes have degree 0 or degree ≥ 2. The nodes which have degree 0 are not connected to the graph, so the graph cannot be a tree (contradiction).

Case 2: All nodes have degree ≥ 2. Pick any node, and pick an edge connected to that node and traverse the graph. All nodes have degree ≥ 2, so every time you enter a node, you also leave it, so there will always be another edge to cover. Eventually, you will have to revisit a node, which means there is a cycle in the graph. Then, the graph cannot be a tree (contradiction).

4. Graph Gardening

Prove that if graph G is a tree with e edges and n nodes, then $e = n - 1$. Use induction on n.

Solution uses the fact that any tree with at least two nodes must have a node of degree 1. If you remove an arbitrary edge in the induction step, the solution needs to use strong induction.

Base Case: $n = 1$. One node, no possible edges, so $e = 0 = n - 1$.

Induction Hypothesis: Assume that any tree G with k nodes nodes has $k - 1$ edges.

Induction Step: Start with tree G which has $k + 1$ nodes. Identify the node u in G of degree 1 (must exist, by above fact). Remove the edge connecting u to the rest of the graph. The result is the disconnected node u and a graph G' with k nodes. Since G was a connected and had no cycles and G' was contained in G, G' is connected and has no cycles as well (any path between two nodes in G' does not use the edge which was removed). Using the induction hypothesis, G' has $k - 1$ edges. Reconnecting u to the graph, we add 1 edge. Therefore, the number of edges in G is $k + 1 - 1 = k$.