04 Convexity

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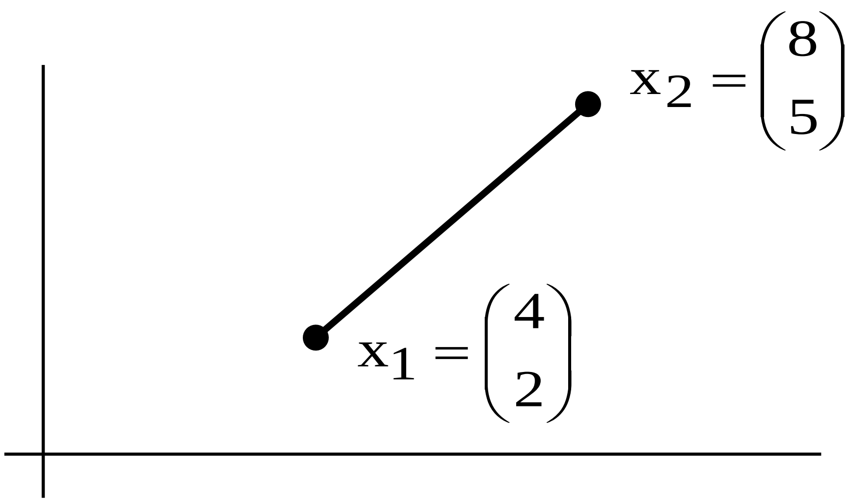
Definition of Convex Combination

DEFINITION: If $x_1, x_2 \in E^n$, then the line segment joining x_1 and x_2, denoted as $L(x_1, x_2)$, can be represented as follows:

$$L(x_1, x_2) = \{\, y \;:\; y = \alpha\, x_1 + (1 - \alpha)\, x_2,\;\; 0 \le \alpha \le 1 \,\}.$$

The elements of $L(x_1, x_2)$ are **convex combinations** of x_1 and x_2.

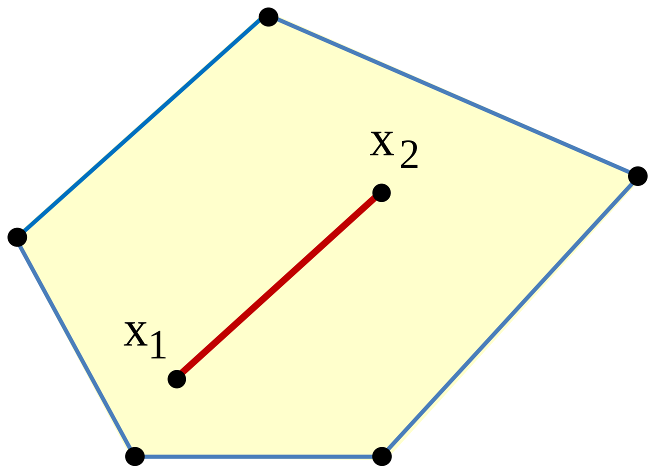
Example



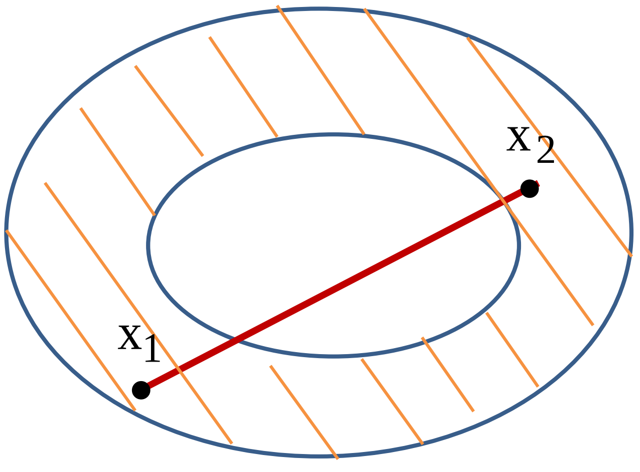
Definition of Convex Set

DEFINITION: A set $S \subset E^n$ is **convex** if, for any $x_1, x_2 \in S$, $L(x_1, x_2) \subseteq S$.

Example



Convex set
(it contains the line segment $L(x_1, x_2)$)



Non-convex set

Convex Combination of Points

DEFINITION: A **convex combination** of points $x_1, x_2, \ldots, x_m \in E^n$ is any point x such that

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$$

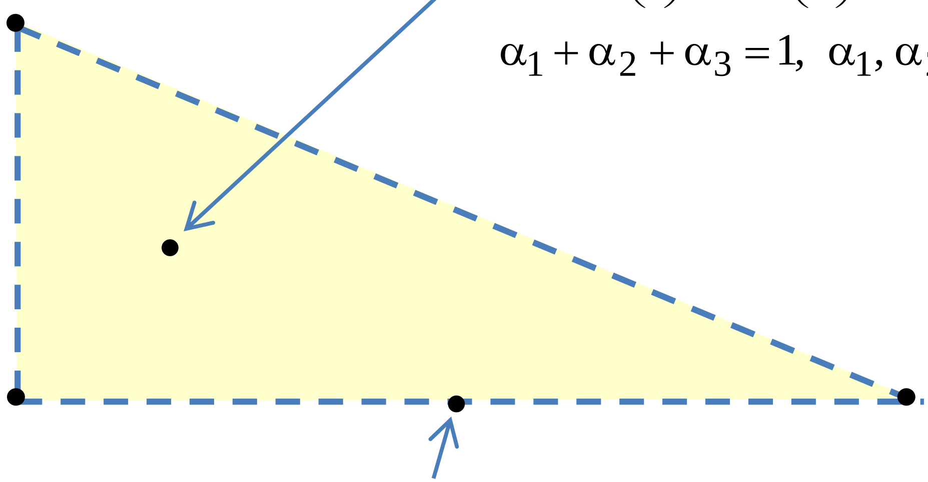
where $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$ and $\alpha_1, \alpha_2, \cdots, \alpha_m \geq 0$.

Example

$$x = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \ \alpha_1, \alpha_2, \alpha_3 \geq 0$$

$$x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



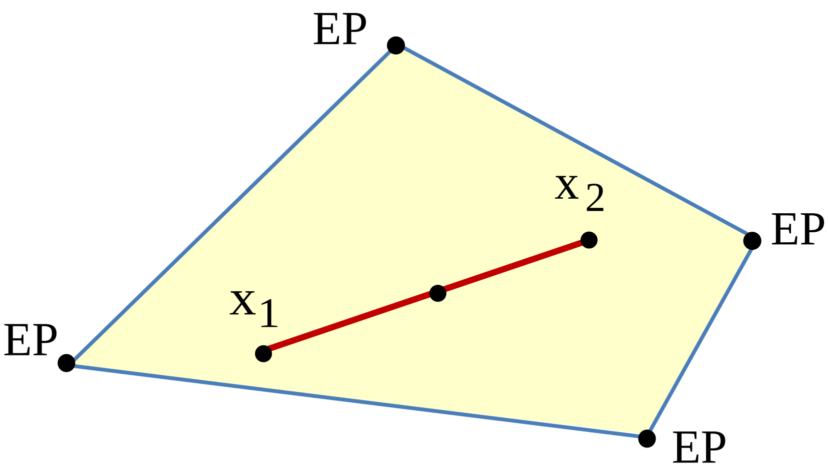
$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

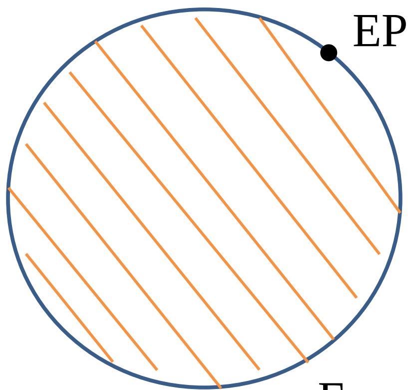
$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Extreme Point

DEFINITION: Given a convex set $S \subset E^n$, a point x is an **extreme point** (EP) of S if x cannot be represented as a strict convex combination of any other two points in S, i.e.,

$$x_1, x_2 \in S, \quad 0 < \alpha < 1, \quad \text{and} \quad x = \alpha\, x_1 + (1-\alpha)\, x_2 \quad \Rightarrow \quad x_1 = x_2 .$$





Every point on the boundary is an EP.

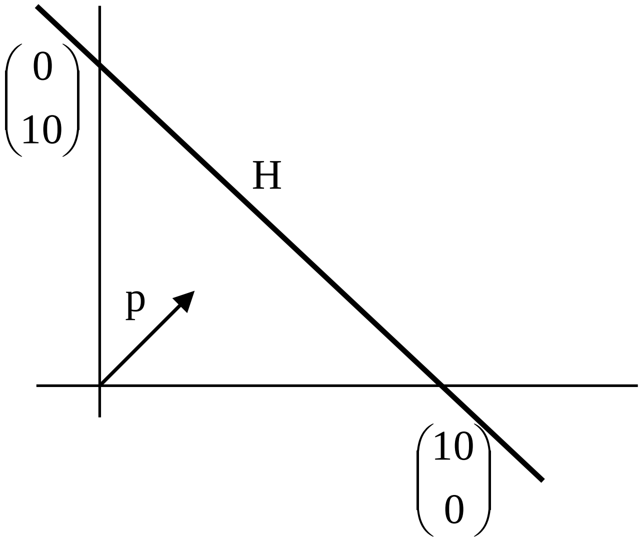
DEFINITION: Two extreme points are **adjacent** if the line segment joining then forms an edge of the convex set.

In an LP, if an extreme point has no adjacent points that are better, then this extreme point is optimal. This property is due to the fact that the feasible region of an LP is convex.

Hyper-plane

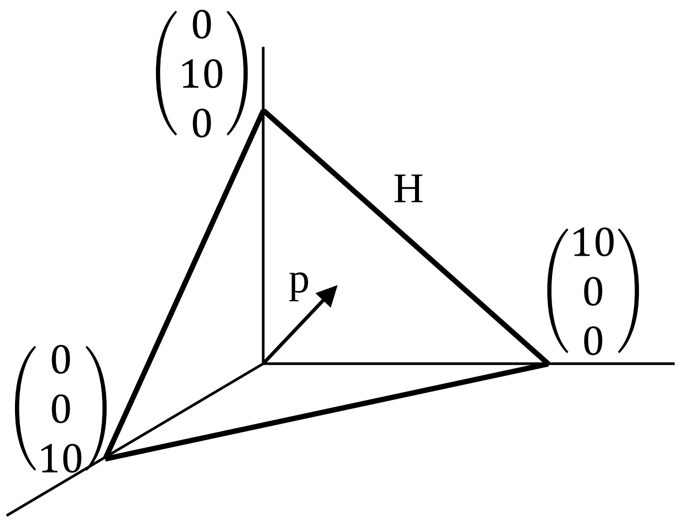
DEFINITION: A **hyper-plane** H in E^n is a set $H = \left\{ x \;:\; x \in E^n, p\,x = k \right\}$, where $p \neq \overline{0}$ and $k \in E$.

H is convex.



$p = (1, 1), \; k = 10$

p is the normal of H



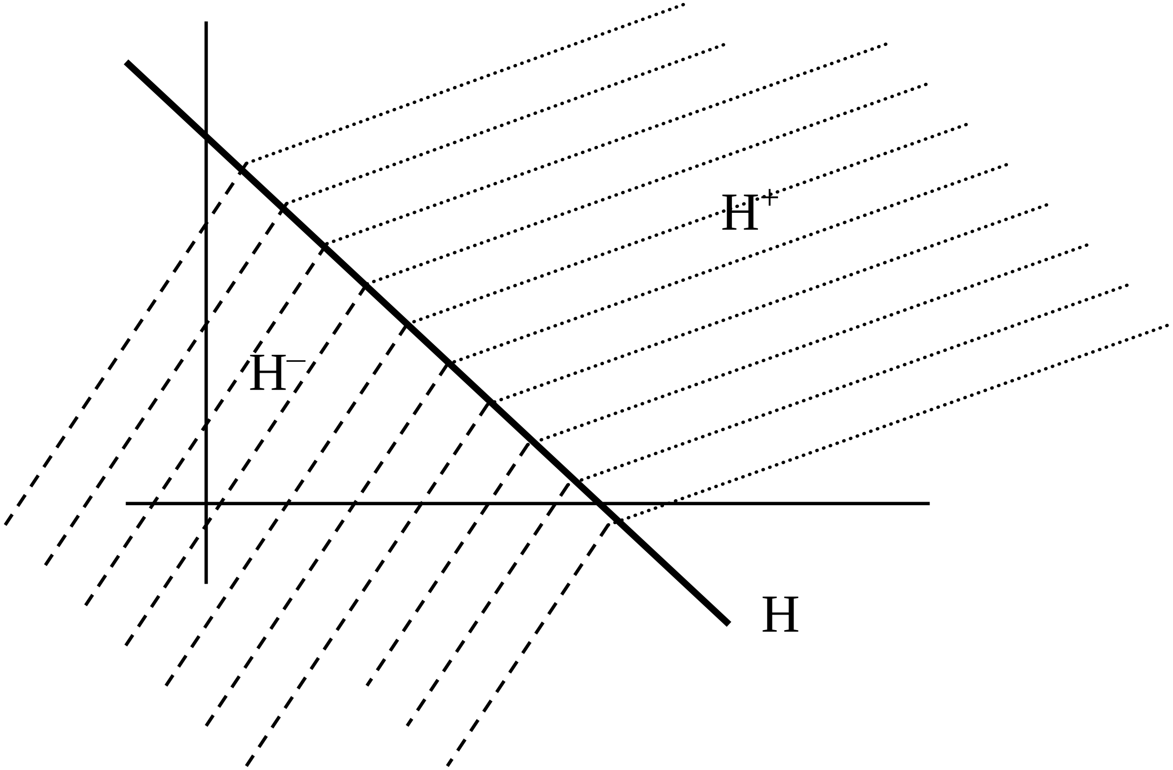
$p = (1, 1, 1), \; k = 10$

Half-space

DEFINITION: If $H = \left\{ x : x \in E^n, p\,x = k \right\}$, where $p \neq \overline{0}$ and $k \in E$, the sets

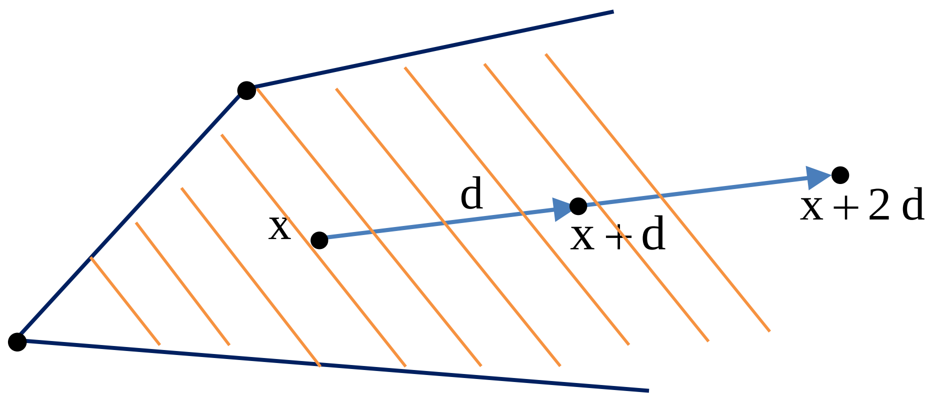
$$H^- = \left\{ x : x \in E^n, p\,x \leq k \right\} \quad \text{and} \quad H^+ = \left\{ x : x \in E^n, p\,x \geq k \right\}$$

are called **half-spaces** and are convex.



Extreme Direction

DEFINITION: A non-zero vector d is a direction of a convex set S if $x + \lambda d \in S$, for all $x \in S$ and $\lambda \geq 0$. Note that **S must be unbounded**.

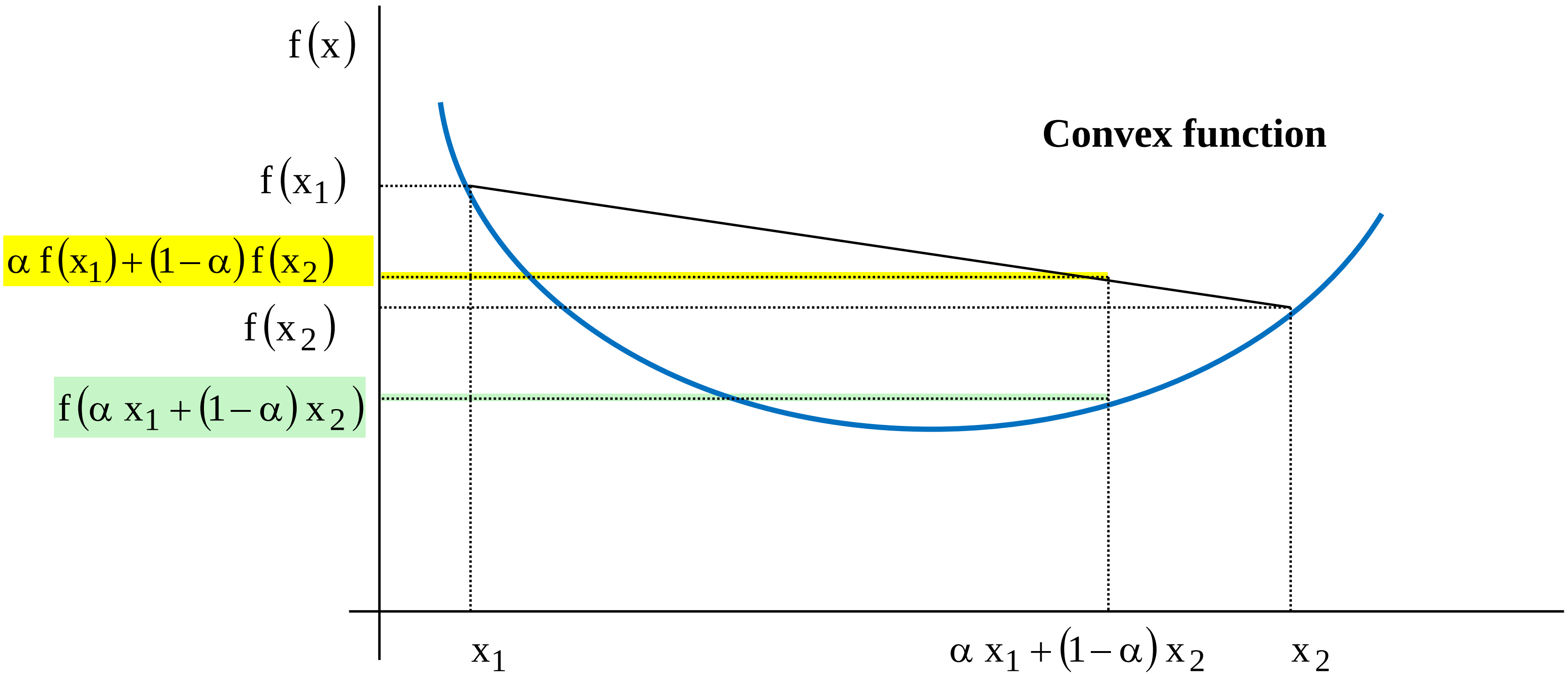


Set $R = \{ x + \lambda d \in S \; : \; \lambda \geq 0 \}$ defines a **ray** of S with point $x \in S$ and direction d of S.

DEFINITION: A direction d of a set S is an **extreme direction** (ED) if d cannot be represented as a positive combination of any other two directions of S.

Set $R = \{ x + \lambda d \in S \; : \; \lambda \geq 0 \}$ defines an **extreme ray** of S with EP x and ED d.

Convex Function

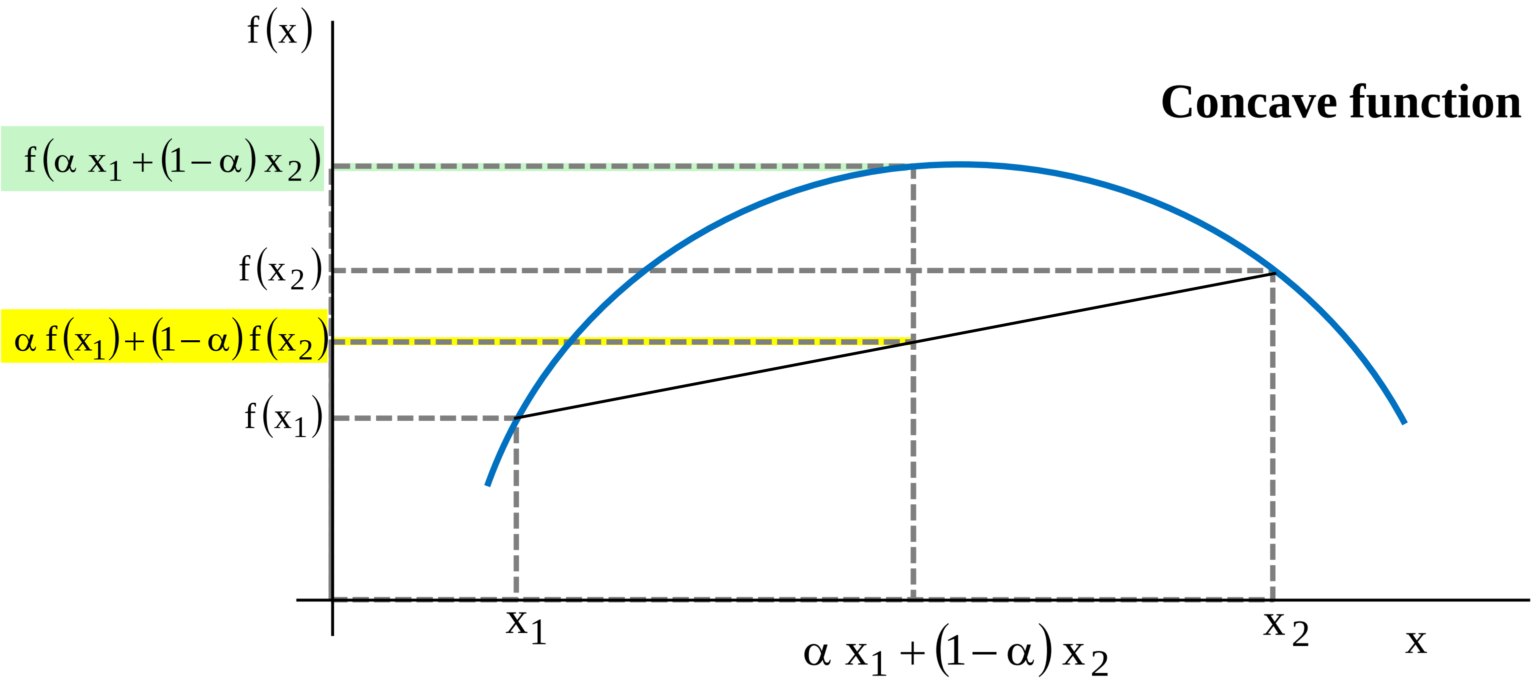


DEFINITION: Given a convex set S in E^n with $f(x)$ defined for all $x \in S$, f is **convex** in S if

$$f(\alpha x_1+(1-\alpha)x_2) \le \alpha f(x_1)+(1-\alpha)f(x_2),$$

for each choice of $x_1, x_2 \in S$ and $0 \le \alpha \le 1$.

Concave Function

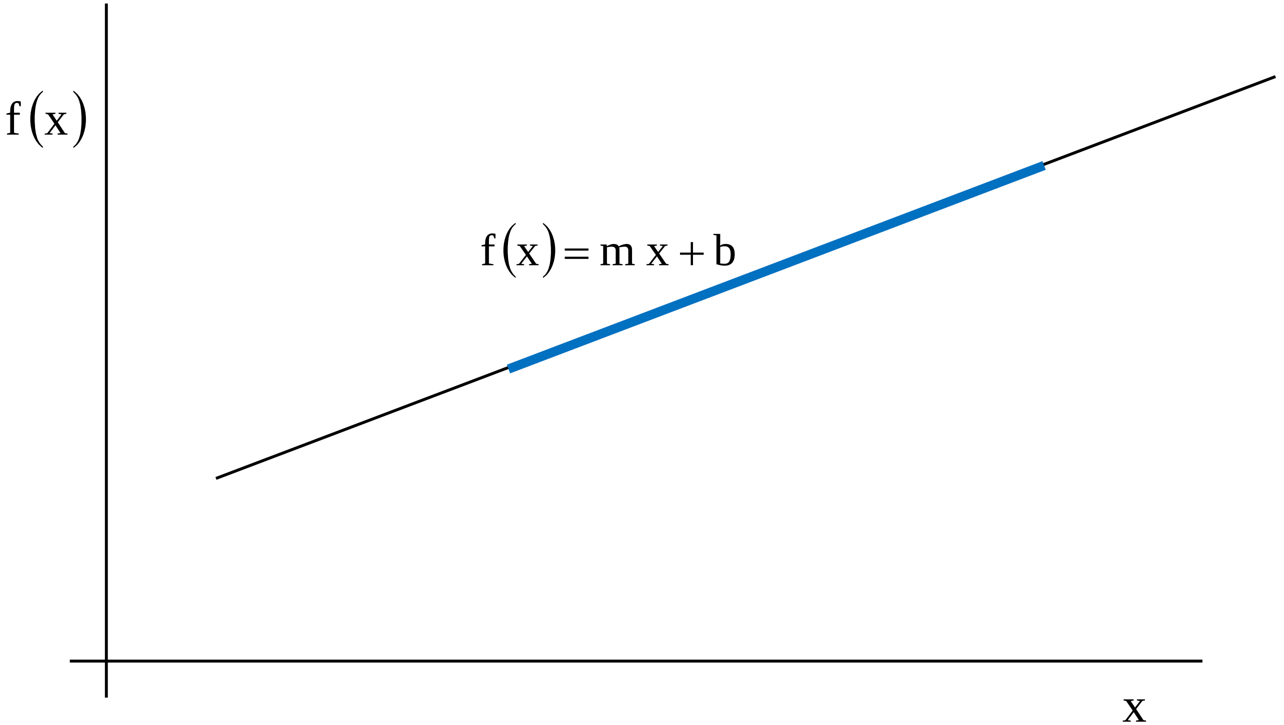


DEFINITION: Given a convex set S in E^n with $f(x)$ defined for all $x \in S$, f is **concave** in S if

$$f(\alpha x_1 + (1-\alpha) x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2),$$

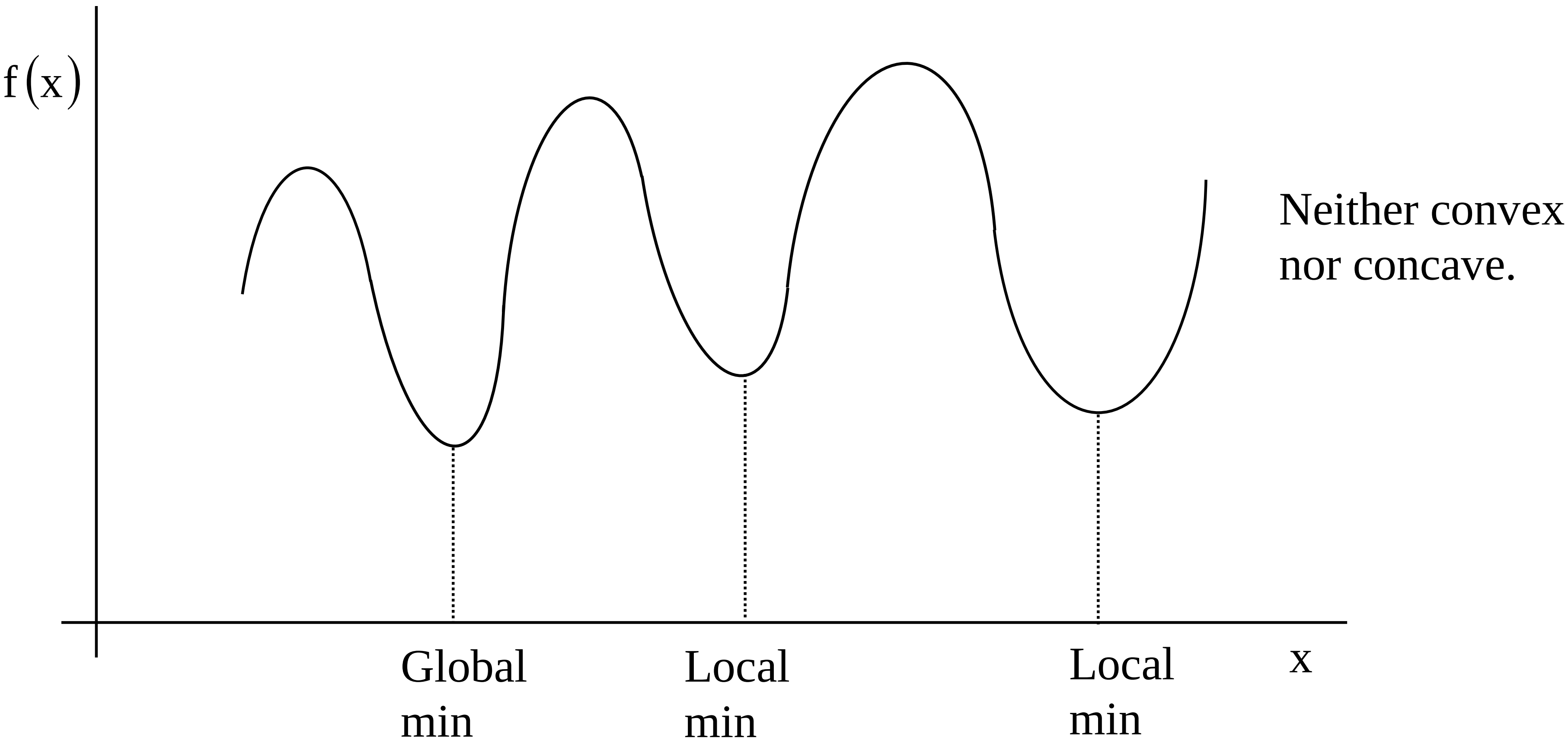
for each choice of $x_1, x_2 \in S$ and $0 \leq \alpha \leq 1$.

Both Convex & Concave Function



Linear functions are both convex and concave.

Neither Convex nor Concave Function



CLAIM: $f(x) = |x|$ is convex for all $x \in E^1$.

Proof: Let $x_1, x_2 \in E^1$ and $0 \le \alpha \le 1$. Then,

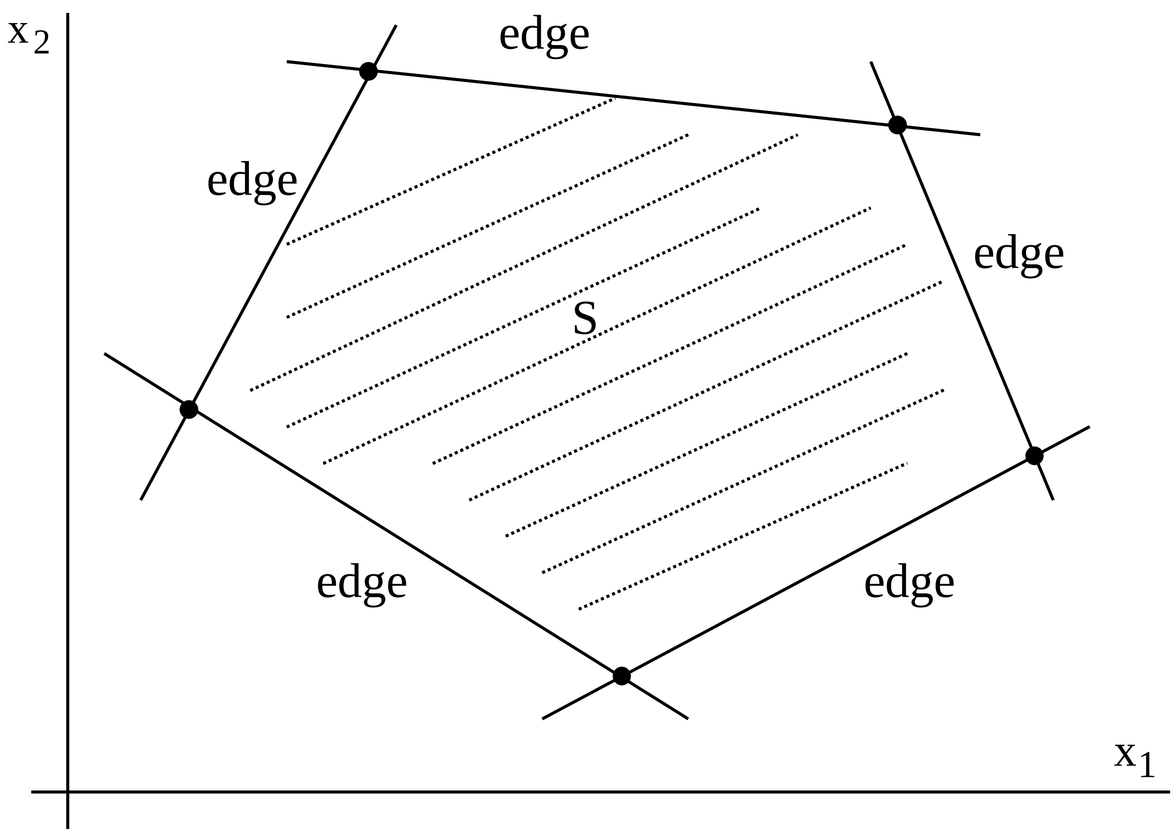
$$f(\alpha x_1 + (1-\alpha)x_2) = |\alpha x_1 + (1-\alpha)x_2| \le |\alpha x_1| + |(1-\alpha)x_2| = \alpha |x_1| + (1-\alpha)|x_2| = \alpha f(x_1) + (1-\alpha) f(x_2).$$

$\Rightarrow$ $f(x)$ is convex for all $x \in E^1$.

Polyhedral Set

DEFINITION: The set S is a **polyhedral set** if S is the intersection of a finite collection of half-spaces, i.e.,

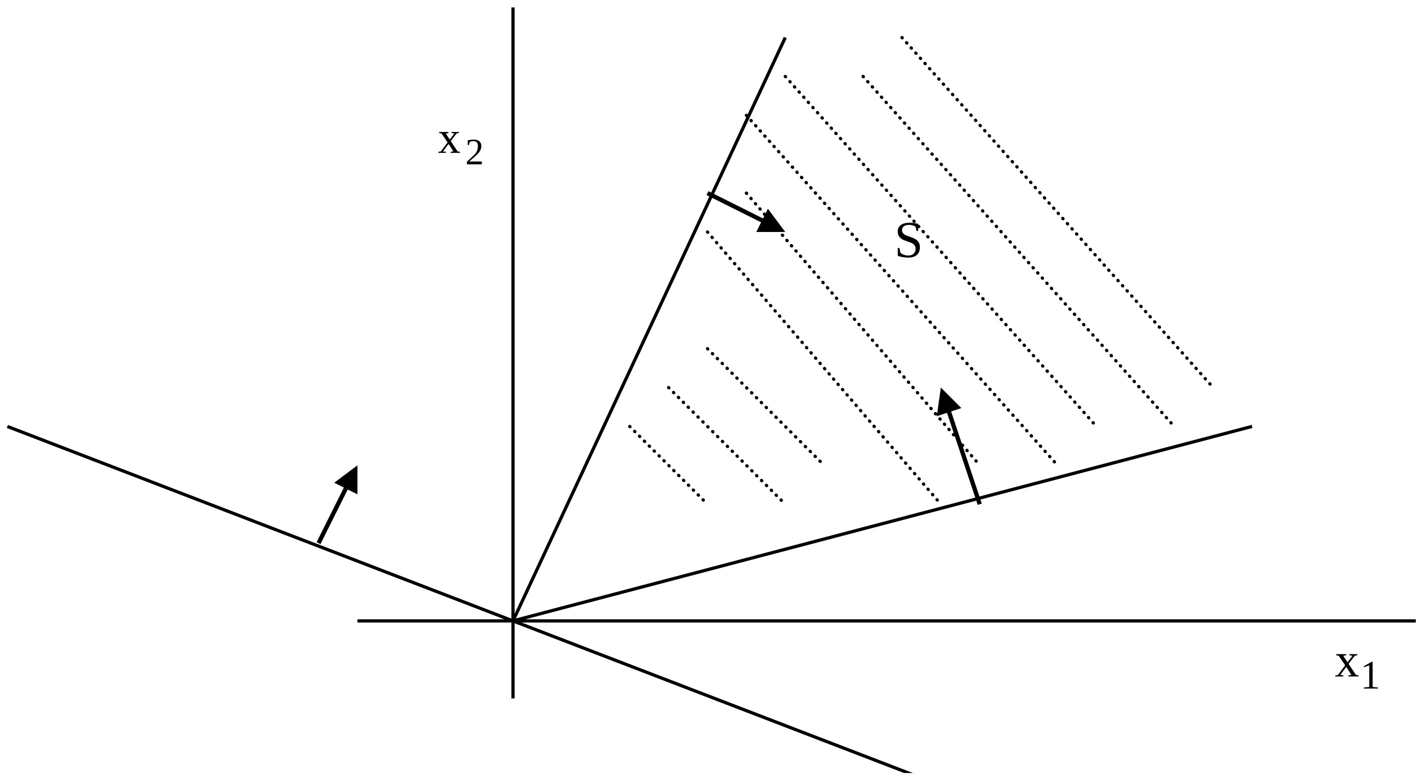
$$S = \bigcap_{i=1}^{k} \left\{ x \in E^n : a^i x \le b_i \right\} = \left\{ x \in E^n : \begin{pmatrix} a^1 \\ a^2 \\ \dots \\ a^k \end{pmatrix} x \le \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix} \right\}.$$



Polyhedral Cone

DEFINITION: Set S is a **polyhedral cone** if S is the intersection of a finite collection of half-spaces, whose hyper-planes pass through the origin i.e.,

$$S = \bigcap_{i=1}^{k} \left\{ x \in E^n : a^i x \le 0 \right\} = \left\{ x \in E^n : \begin{pmatrix} a^1 \\ a^2 \\ \dots \\ a^k \end{pmatrix} x \le \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} \right\}.$$



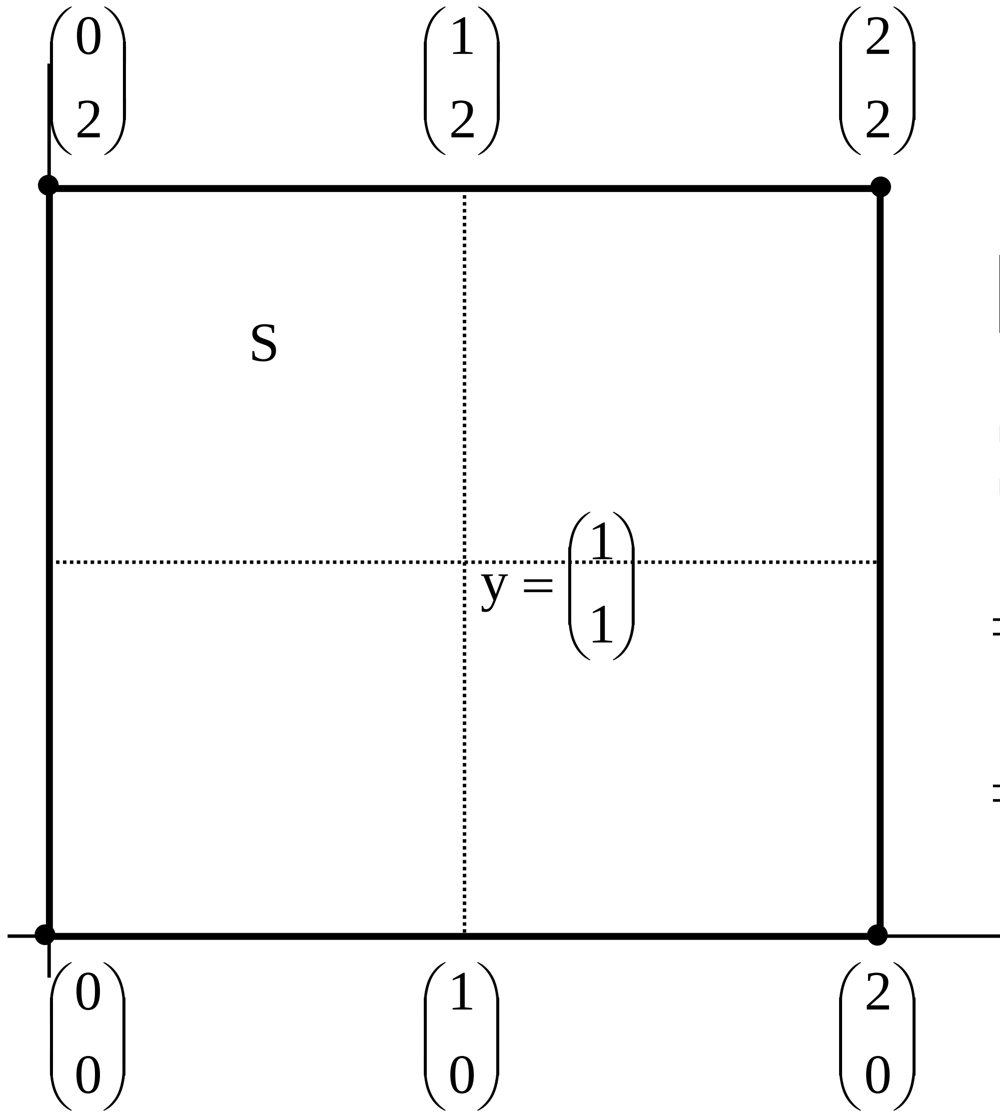
Representation Theorem

REPRESENTATION THEOREM: Let $S = \left\{ x \in E^n \ : \ A\,x = b,\ x \geq 0 \right\}$ be a nonempty and bounded (contained in a hyper-sphere of finite radius) polyhedral set. Then,

a) The collection of all extreme points of S, $E = \left\{ x_1, x_2, \ldots, x_k \right\}$, is finite and nonempty.

b) $x \in S \iff x = \sum_{j=1}^{k} \alpha_j\, x_j$ with all $\alpha_j \geq 0$ and $\sum_{j=1}^{k} \alpha_j = 1$, i.e., x is a convex combination of the extreme points.

Example:



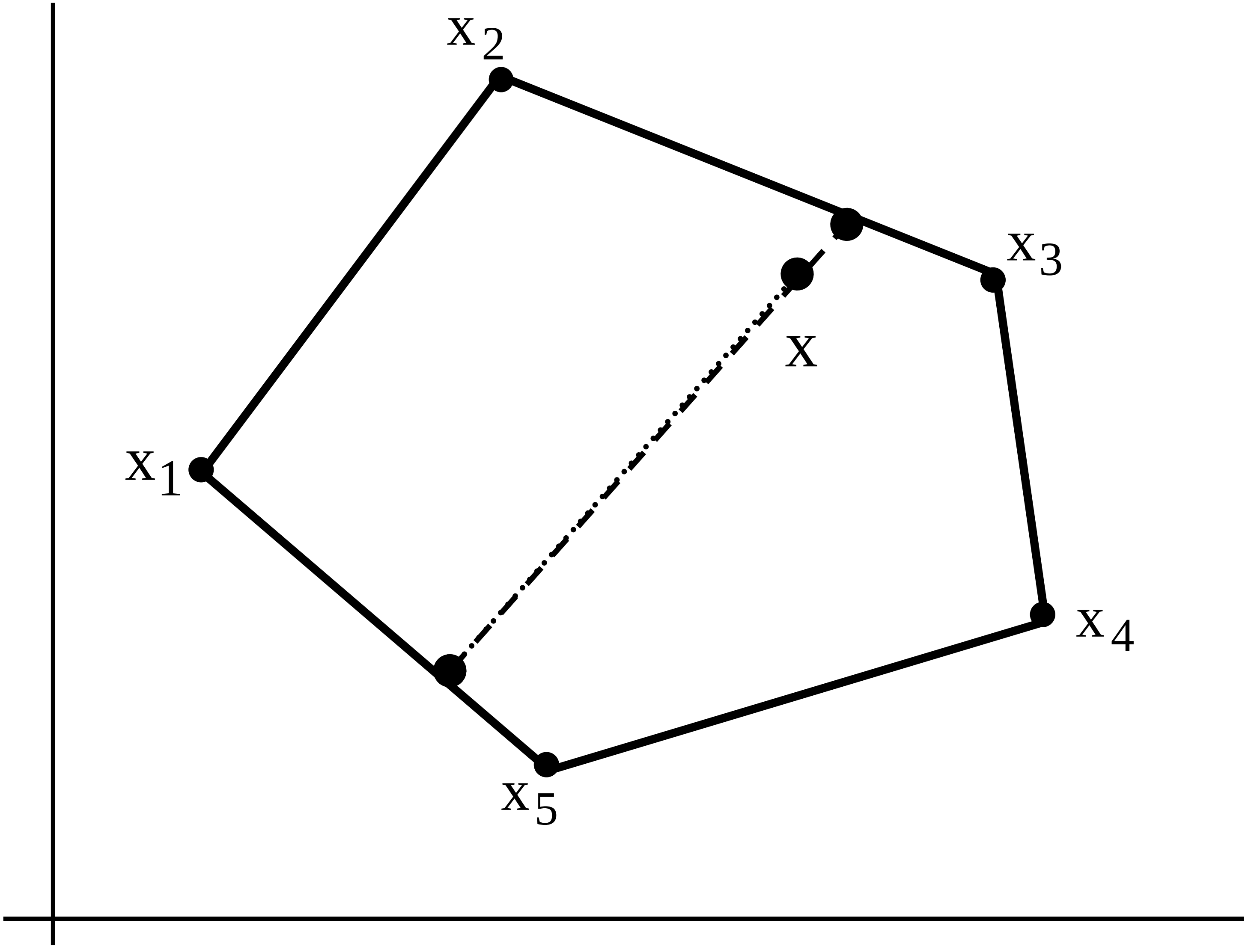
$$\begin{pmatrix}1\\2\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}0\\0\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}2\\0\end{pmatrix}$$

$$\Rightarrow\quad y=\tfrac{1}{2}\begin{pmatrix}1\\2\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\Rightarrow\quad y=\tfrac{1}{4}\begin{pmatrix}0\\2\end{pmatrix}+\tfrac{1}{4}\begin{pmatrix}2\\2\end{pmatrix}+\tfrac{1}{4}\begin{pmatrix}0\\0\end{pmatrix}+\tfrac{1}{4}\begin{pmatrix}2\\0\end{pmatrix}.$$

In general,



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