다변수 미분 적분학.

다음시간 (3/14 목) Quiz
(~13장까지.)

일변수 함수 $\xrightarrow[\text{확장}]{}$ 벡터 함수 < 미분 / 적분 → "벡터함수에 관한 미분적분학의 기본정리"

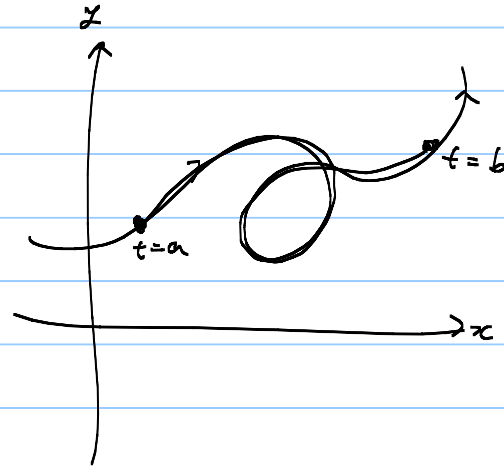
$$\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$$

$$\left(\vec{R}'(t) = \vec{r}(t)\right)$$

vector 함수의 적분의 응용.

→ 곡선의 길이

우선 : 평면곡선의 경우



계!: $\vec{r}(t) = (f(t), g(t))$

방향이존재

속도: $\vec{r}'(t) = (f'(t), g'(t))$

방향이없음

→ 속력 : $|\vec{r}'(t)|$

→ 곡선의 길이 $= \int_a^b |\vec{r}'(t)|\,dt = \int_a^b \sqrt{\{f'(t)\}^2 + \{g'(t)\}^2}\,dt$

공간곡선 (3차원 벡터 함수)의 경우 : $\vec{r}(t) = (f(t), g(t), h(t))$

→ $\vec{r}'(t) = (f'(t), g'(t), h'(t))$: 속도

→ $|\vec{r}'(t)| = \sqrt{\{f'(t)\}^2 + \{g'(t)\}^2 + \{h'(t)\}^2}$: 속력

→
$$\boxed{\text{길이} = \int_a^b |\vec{r}'(t)|\,dt = \int_a^b \sqrt{\{f'(t)\}^2 + \{g'(t)\}^2 + \{h'(t)\}^2}\,dt}$$

예) $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k} = (\cos t, \sin t, t)$

$0 \le t \le 2\pi$

미분

$$\text{길이} = \int_0^{2\pi} |\vec{r}'(t)|\,dt = \int_0^{2\pi} \left| -\sin t\,\vec{i} + \cos t\,\vec{j} + 1\cdot\vec{k} \right| dt$$

$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,dt = \int_0^{2\pi} \sqrt{2}\,dt = \sqrt{2}\cdot 2\pi$$

$$= 2\sqrt{2}\pi$$

14장. Partial Derivatives (편미분)

$\mathbb{R} \to \mathbb{R}$ $\xrightarrow{\text{확장}}$ 13장 : $\mathbb{R} \to \mathbb{R}^n$ vector functions

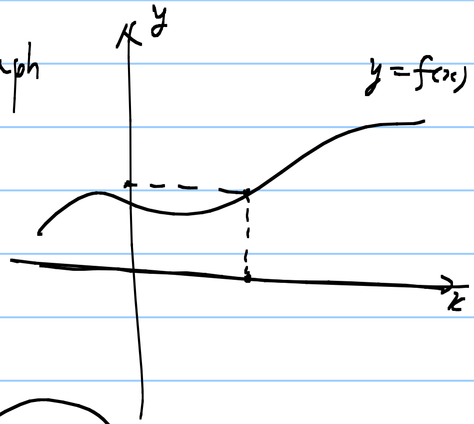
확장

14장 : $\mathbb{R}^n \to \mathbb{R}$ 다변수 함수

예) $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$; 2변수 함수.

<u>다변수 함수의 graph</u>

일변수 함수의 경우 : $y = f(x)$ → graph

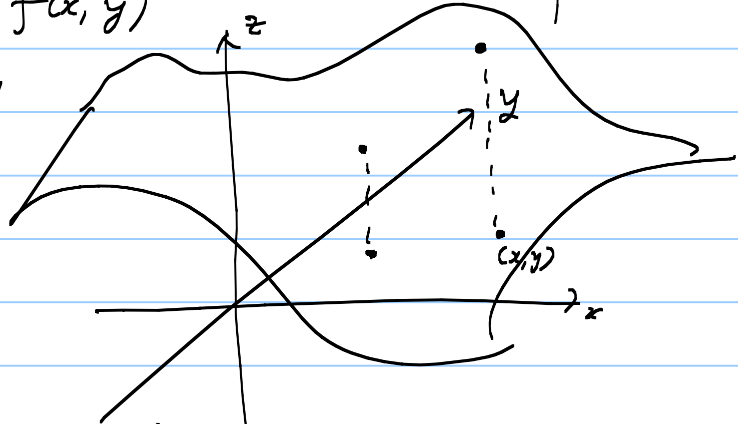


" $\{(x, y) \mid y = f(x)\}$ " : graph

2변수 함수의 경우 : $z = f(x, y)$

↓

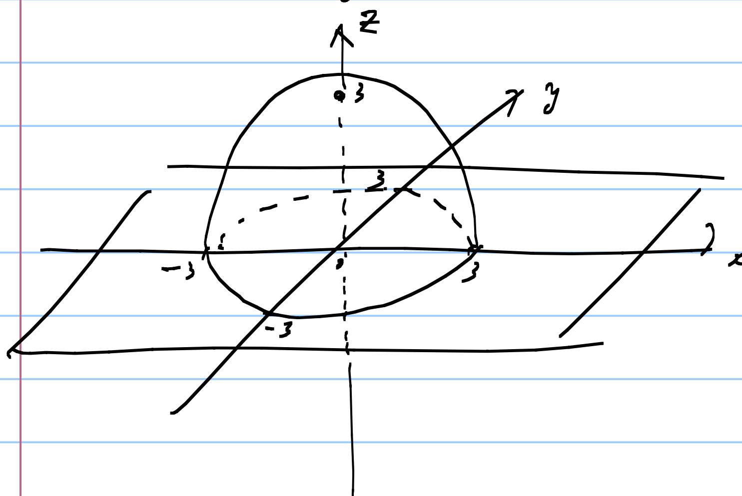
graph



" $\{(x, y, z) \mid z = f(x, y)\}$ " $\subset \mathbb{R}^3$

예) $g(x, y) = \sqrt{9 - x^2 - y^2}$

→ graph : $\{(x, y, z) \mid \underbrace{z = \sqrt{9 - x^2 - y^2}}_{x^2 + y^2 + z^2 = 3^2}\}$

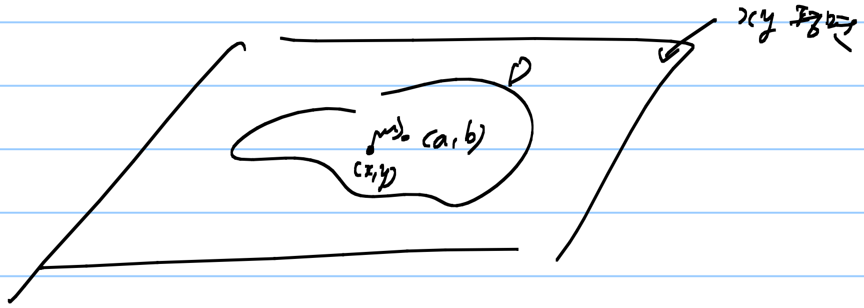


정의역 :

$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 3\}$

다변수 함수의 연속과 극한

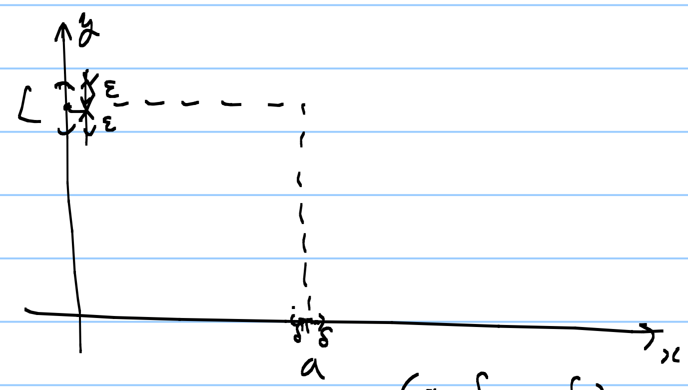
f : 2변수 함수, 정의역 $D \subset \mathbb{R}^2$. Fix $(a,b) \in \cancel{D}$



$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$\overset{\text{정의}}{\Longleftrightarrow}$?

※ 일변수 함수의 경우 : $\lim_{x\to a} f(x) = L$



" $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $x \in (a-\delta, a+\delta) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$ "

(such that)

$\Updownarrow$

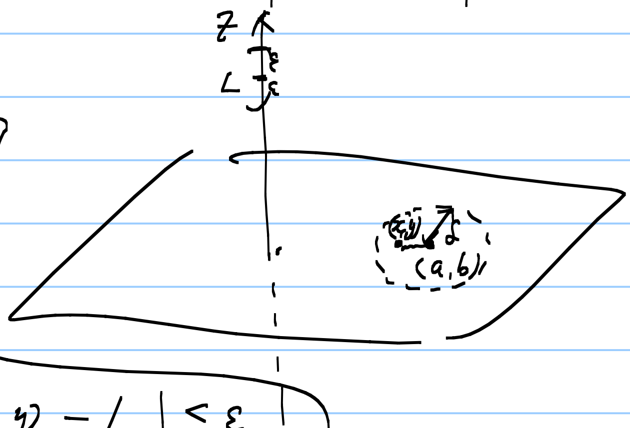
$|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

2변수 함수의 경우 : $f(x,y)$

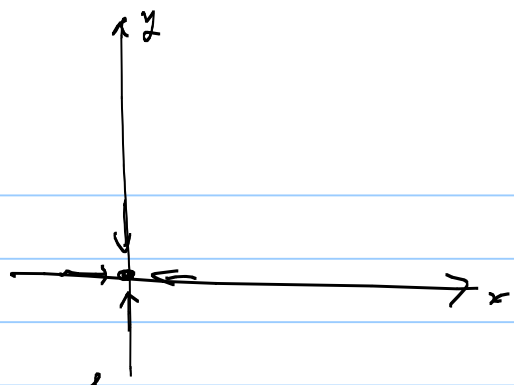
$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

$\overset{\text{정의}}{\Longleftrightarrow}$ $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t.

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$$



예) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = ?$



x축을 따라서 (0,0)으로 갈 때 :

$(x,0) \to (0,0)$

$$\lim_{(x,0)\to(0,0)} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0}\frac{x^2}{x^2} = 1.$$

y축을 따라서 (0,0)으로 갈 때 :

$(0,y) \to (0,0)$

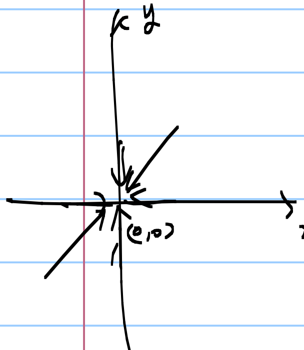
$$\lim_{(0,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0}\frac{-y^2}{y^2} = -1$$

∴ 이 극한은 존재하지 않는다.

예) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = ?$

x축을 따라서 : $(x,0)\to(0,0) \Rightarrow \lim\limits_{x\to 0} \dfrac{x\cdot 0}{x^2+0^2} = 0$

y축을 〃 : $(0,y)\to(0,0) \Rightarrow \lim\limits_{y\to 0} \dfrac{0\cdot y}{0^2+y^2} = 0$

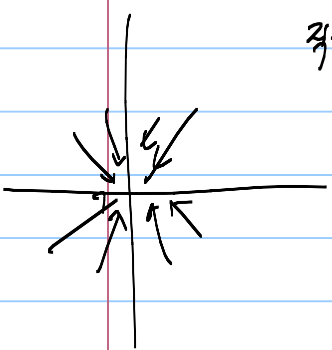


$y=x$를 따라서 ; $(t,t)\to(0,0)$

$$\Rightarrow \lim_{t\to 0} \frac{t\cdot t}{t^2+t^2} = \frac{1}{2}$$

∴ 극한이 존재하지 않는다.

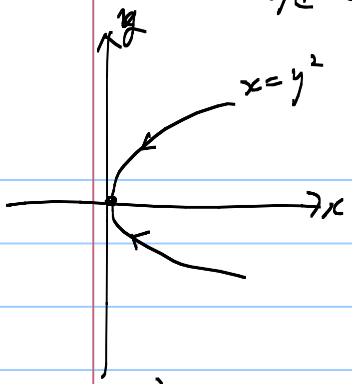
예) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = !$



직선 $y=mx$를 따라서 : $(t,\ mt)\to(0,0)$

$$\Rightarrow \lim_{t\to 0} \frac{t\cdot(mt)^2}{t^2+(mt)^4} = \lim_{t\to 0}\frac{m^2t^3}{t^2+m^4t^4}$$

$$= \lim_{t\to 0}\frac{m^2 t}{1+m^4t^2} = 0$$



곡선 $x = y^2$ 을 따라서 : $(t^2, t) \to (0,0)$

$$\Rightarrow \lim_{t \to 0} \frac{t^2 \cdot t^2}{(t^2)^2 + t^4} = \lim_{t \to 0} \frac{t^4}{2t^4} = \frac{1}{2}$$

∴ 극한이 존재하지 않는다.

예) $$\lim_{(x,y) \to (0,0)} \frac{3x^2 y}{x^2 + y^2} = 0 \ ?$$

$$\text{거리}\big((x,y), (0,0)\big) < \delta \Rightarrow \left| \frac{3x^2 y}{x^2+y^2} - 0 \right| < \varepsilon$$

$$\text{거리}\big((x,y), (0,0)\big) = \sqrt{x^2+y^2}, \qquad \left| \frac{3x^2 y}{x^2+y^2} - 0 \right| = \frac{3x^2|y|}{|x^2+y^2|}$$

Note $x^2 \le x^2 + y^2 \Rightarrow$

$$\frac{3x^2|y|}{x^2+y^2} = 3|y| \cdot \frac{x^2}{x^2+y^2} \le 3|y| \le 3 \cdot \sqrt{x^2+y^2}$$

$$y^2 \le x^2 + y^2 \Rightarrow |y| \le \sqrt{x^2+y^2}$$

w) $$\sqrt{x^2+y^2} < \frac{\varepsilon}{3} \Rightarrow \frac{3x^2|y|}{x^2+y^2} \le 3\sqrt{x^2+y^2} < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

즉, 임의의 $\varepsilon > 0$ 에 대하여, $\delta = \frac{\varepsilon}{3}$ 으로 잡으면.

$$\sqrt{x^2+y^2} < \delta \Rightarrow \frac{3x^2|y|}{x^2+y^2} < \varepsilon$$

$$\therefore \lim_{(x,y) \to (0,0)} \frac{3x^2 y}{x^2+y^2} = 0$$