05 The Simplex Method

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Overview

- Review of Linear Algebra (cont.)
  - Example of Degenerate BFS.
  - Theorem: EP $\equiv$ BSF
  - Theorem: Feasible Region of an LP has at least one EP.
- Simplex Method
  - Algorithm
  - Example
- Special Cases of a Linear Program in the Simplex Tableau
  - Alternative
  - Unbounded Optimal Solution
  - Infeasible Problem: Big-M Method

Review of Linear Algebra (cont.)

$\overbrace{A}^{m\times n} x = b$: system of linear equations

where $m \le n$, $\overbrace{A}^{m\times n} = \left(\overbrace{B}^{m\times m} \;\middle|\; \overbrace{N}^{m\times(n-m)} \right)$

and $r(A) = r(B) = m$.

B : basic matrix,
N : nonbasic matrix,

$$A\,x = b \quad \Rightarrow \quad (B \quad N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} = b\,,$$

x_B: vector of basic variables (dependent variables),
x_N: vector of nonbasic variables (free or independent variables),

$$B\,x_B + N\,x_N = b \quad \Rightarrow \quad B^{-1}B\,x_B + B^{-1}N\,x_N = B^{-1}b\,,$$

$$\Rightarrow \quad x_B = B^{-1}b - B^{-1}N\,x_N\,. \qquad \text{(General solution)}$$

In addition, $x_N = \bar{0} \quad \Rightarrow \quad x_B = B^{-1} b$.

$x = \begin{pmatrix} B^{-1} b \\ \bar{0} \end{pmatrix}$: **basic solution** (BS); x is called **basic feasible solution** (BFS) if $B^{-1} b \ge \bar{0}$.

$x = \begin{pmatrix} B^{-1} b \\ \bar{0} \end{pmatrix}$ is called **non-degenerate BFS** if each entry in $B^{-1} b$ is positive. It is called **degenerate BFS** if at least one entry in $B^{-1} b$ is zero.

If there exist two bases in A, $\hat{B} \ne B$, such that $\hat{B}^{-1} b = B^{-1} b \ge 0$, then $\begin{pmatrix} \hat{B}^{-1} b \\ \bar{0} \end{pmatrix} = \begin{pmatrix} B^{-1} b \\ \bar{0} \end{pmatrix}$ is a **degenerate BFS**.

Simplex Method

EXAMPLE OF THE SIMPLEX METHOD

$$\begin{aligned} &\text{maximize } z = 5x_1 + 3x_2, \\ &\text{subject to} \quad 3x_1 + 5x_2 \le 15, \\ &\qquad\qquad\quad 5x_1 + 2x_2 \le 10, \\ &\qquad\qquad\quad x_1, x_2 \ge 0. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\text{maximize } z = 5x_1 + 3x_2, \\ &\text{subject to} \quad 3x_1 + 5x_2 + x_3 = 15, \\ &\qquad\qquad\quad 5x_1 + 2x_2 + x_4 = 10, \\ &\qquad\qquad\quad x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

Also,

$$\text{maximize } z = 5x_1 + 3x_2, \quad \Leftrightarrow \quad \text{minimize } z = -5x_1 - 3x_2$$

Equivalently,

$$\text{minimize } z$$

$$\begin{aligned} \text{subject to} \quad & z + 5x_1 + 3x_2 + 0x_3 + 0x_4 = 0, \\ & x_1\begin{pmatrix}3\\5\end{pmatrix} + x_2\begin{pmatrix}5\\2\end{pmatrix} + x_3\begin{pmatrix}1\\0\end{pmatrix} + x_4\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}15\\10\end{pmatrix}, \\ & \text{all } x_j \ge 0. \end{aligned}$$

TABLEAU FORM

Row #	z	x_1	x_2	x_3	x_4	RHS	B_S	c_B	Θ
0	1	$\underbrace{5}_{z_1-c_1}$	$\underbrace{3}_{z_2-c_2}$	$\underbrace{0}_{z_3-c_3}$	$\underbrace{0}_{z_4-c_4}$	$\underbrace{0}_{z}$			
1	0	$\underbrace{3}_{y_{11}}$	$\underbrace{5}_{y_{12}}$	$\underbrace{1}_{y_{13}}$	$\underbrace{0}_{y_{14}}$	15	a_3	0	$15/3$
2	0	$(\underbrace{5}_{y_{21}})$	$\underbrace{2}_{y_{22}}$	$\underbrace{0}_{y_{23}}$	$\underbrace{1}_{y_{24}}$	10	a_4	0	$10/5$

$\Rightarrow$ **PIVOT ROW**

$\Downarrow$

PIVOT COLUMN

$$x_1\begin{pmatrix}3\\5\end{pmatrix}+x_2\begin{pmatrix}5\\2\end{pmatrix}+x_3\begin{pmatrix}1\\0\end{pmatrix}+x_4\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}15\\10\end{pmatrix},$$

$$\begin{pmatrix}x_3\\x_4\end{pmatrix}=\begin{pmatrix}15-3x_1\\10-5x_1\end{pmatrix}\geq\begin{pmatrix}0\\0\end{pmatrix}\quad\Rightarrow\quad\begin{matrix}15-3x_1\geq 0\\10-5x_1\geq 0\end{matrix}\quad\Rightarrow\quad\begin{matrix}x_1\leq 5\\x_1\leq 2\end{matrix}.$$

ALGORITHM for LP minimization

Step 0: Start with an initial BFS (IBFS).

In example, IBSF: $B = (b_1 \quad b_2) = (a_3 \quad a_4) = I$, and $x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = B^{-1}b = b = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$.

Step 1: If all $z_j - c_j \le 0$, stop with an optimum BFS.

Step 2: Else,

$$z_k - c_k = \max_{1 \le j \le n} \{ z_j - c_j \} = \max_{j \text{ nonbasic}} \{ z_j - c_j \} > 0 .$$

Example: $z_1 - c_1 = 5 = \max \{ 3, 5 \}$.

a_k enters the basis if at least one entry in y_k is positive. Otherwise ($y_k \le \overline{0}$), the solution is unbounded.

Example: $y_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and a_1 enters the basis.

Step 3: Compute

$$\Theta = \frac{x_{B_r}}{y_{rk}} = \min\left\{ \frac{x_{B_i}}{y_{ik}} : y_{ik} > 0 \right\},$$

b_r (r^{th} basic column) leaves the basis.

Example: $\Theta = \dfrac{x_{B_2}}{y_{21}} = \min\left\{ \dfrac{15}{3}, \dfrac{10}{5} \right\} = 2$ and $b_2 = a_4$ leaves the basis.

Step 4: Perform elementary row operations on the tableau that transform $\begin{pmatrix} z_k - c_k \\ y_k \end{pmatrix}$ into a unit vector in the next tableau with a one in row r, the pivot row.

Row #	z	x_1	x_2	x_3	x_4	RHS	B_S	c_B	Θ
0	1	0	1	0	-1	-10			
1 2	0 0	0 1	(19/5) 2/5	1 0	-3/5 1/5	9 2	a_3 a_1	0 -5	45/19 5

⇒ **PIVOT ROW**

⇓

PIVOT COLUMN

$$
\begin{array}{lllll}
z & +x_2 & & -x_4 & = -10, \\
& \frac{19}{5}x_2 & +x_3 & -\frac{3}{5}x_4 & = \ \ 9, \\
x_1 & +\frac{2}{5}x_2 & & +\frac{1}{5}x_4 & = \ \ 2.
\end{array}
$$

$$
\begin{pmatrix} x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 9 - \frac{19}{5}x_2 \\ 2 - \frac{2}{5}x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Step 5: Go back to Step 1.

Row #	z	x_1	x_2	x_3	x_4	RHS	B_S	c_B	Θ
0	1	0	0	-5/19	-16/19	-235/19			
1	0	0	1	5/19	−3/19	45/19	a_2	-3	
2	0	1	0	−2/19	5/19	20/19	a_1	-5	

$$\underbrace{\begin{pmatrix} 5/19 & -3/19 \\ -2/19 & 5/19 \end{pmatrix}}_{B^{-1}}$$

Matrix B^{-1} is available in the tableau under the columns corresponding to the slack variables.

$$\begin{aligned} z \qquad\qquad & -\frac{5}{19}x_3 - \frac{16}{19}x_4 = -\frac{235}{19}, \\ x_2 & + \frac{5}{19}x_3 - \frac{3}{19}x_4 = \frac{45}{19}, \\ x_1 \qquad & - \frac{2}{19}x_3 + \frac{5}{19}x_4 = \frac{20}{19}. \end{aligned}$$

$$\left.\begin{aligned} z^* &= -\frac{235}{19} \\ x_1{}^* &= \frac{20}{19} \\ x_2{}^* &= \frac{45}{19} \\ x_3{}^* &= 0 \\ x_4{}^* &= 0 \end{aligned}\right\} \text{Optimal BFS (OBFS)}.$$

Special Cases of a Linear Program in the Simplex Tableau

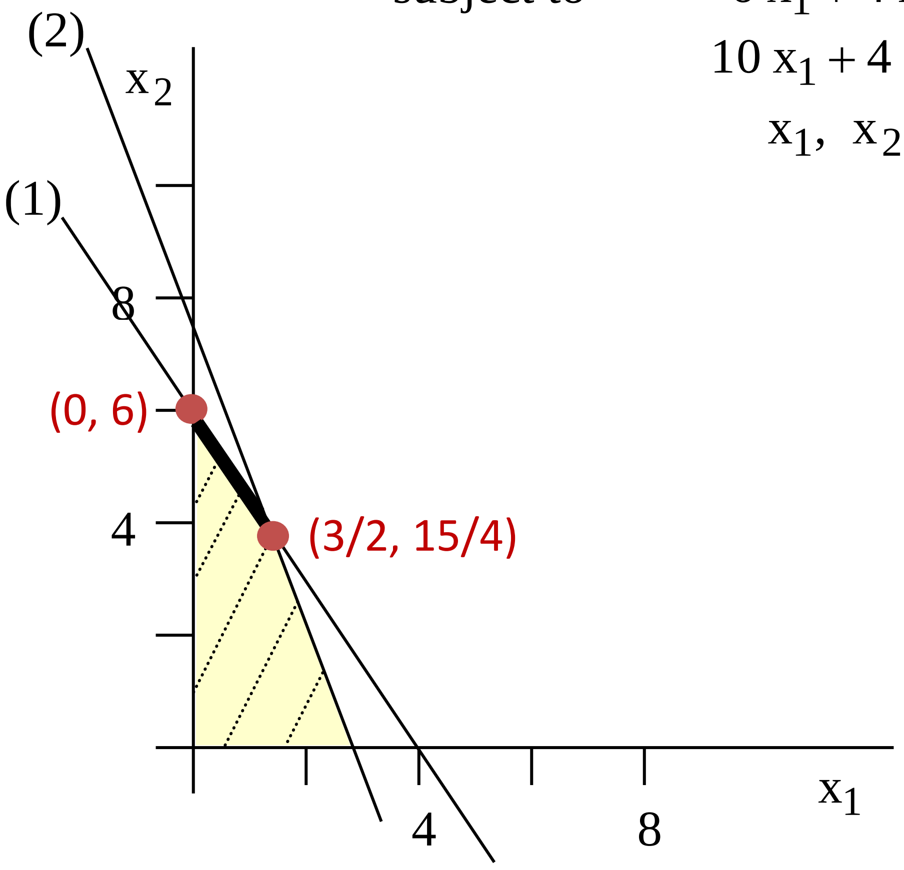
ALTERNATIVE (MULTIPLE) OPTIMA

$$\text{minimize} \quad z = -3x_1 - 2x_2,$$

$$\text{subject to} \quad 6x_1 + 4x_2 \le 24, \qquad (1)$$

$$10x_1 + 4x_2 \le 30, \qquad (2)$$

$$x_1,\ x_2 \ge 0.$$



Optimal tableau: **PIVOT COLUMN** $\Downarrow$ (x_1)

Row #	z	x_1	x_2	x_3	x_4	RHS	B_S	c_B	Θ
0	1	0	0	-1/2	0	-12			
1	0	3/2	1	1/4	0	6	a_2	-2	6/(3/2) = 4
2	0	(4)	0	-1	1	6	a_4	0	6/4 $\Rightarrow$ **PIVOT ROW**

$z_j - c_j \le 0$, for all nonbasic x_j $\Rightarrow$ this tableau provides an optimum BFS.

Since $z_1 - c_1 = 0$, multiple optimal solutions exist.

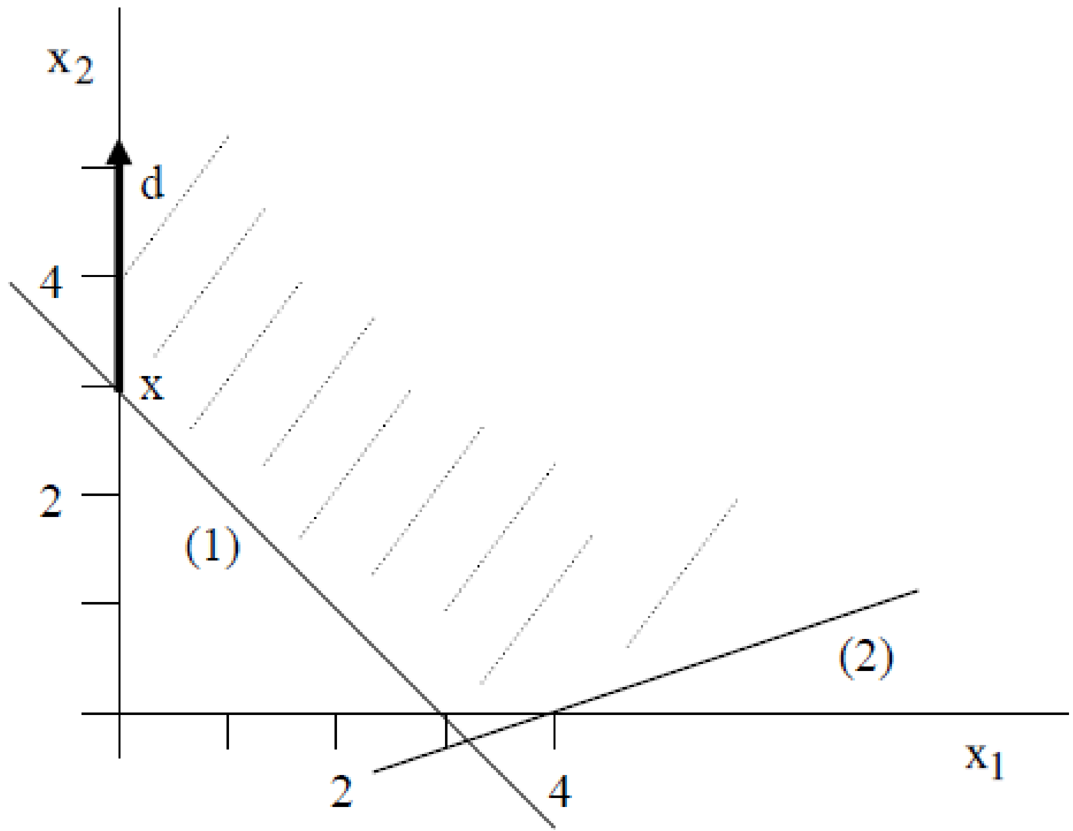
Row #	z	x_1	x_2	x_3	x_4	RHS	B_S	c_B
0	1	0	0	-1/2	0	-12		
1	0	0	1	5/8	-3/8	15/4	a_2	-2
2	0	1	0	-1/4	1/4	3/2	a_1	-3

$$\text{Set of optimal solutions} = \left\{ \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 6 \\ 0 \\ 6 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 3/2 \\ 15/4 \\ 0 \\ 0 \end{pmatrix} \ : \ 0 \le \alpha \le 1 \right\}.$$

NO OPTIMAL SOLUTION (UNBOUNDED SOLUTION)

$$\text{minimize} \quad z = -2x_1 - 3x_2,$$

$$\begin{aligned} \text{subject to} \quad & x_1 + x_2 \geq 3, && (1) \\ & x_1 - 2x_2 \leq 4, && (2) \\ & x_1,\ x_2 \geq 0. \end{aligned}$$



Intermediate tableau:

Row #	z	x_1	x_2	x_3	x_4	RHS	B_S	c_B
0	1	-1	0	3	0	-9		
1	0	1	1	-1	0	3	a_2	-3
2	0	3	0	-2	1	10	a_4	0

$$z_3 - c = 3 > 0 \text{ and } y_3 = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \leq \bar{0} \quad \Rightarrow \quad \text{solution is unbounded.}$$

$$\text{Current EP: } x = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 10 \end{pmatrix}, \qquad \text{Decreasing ED: } d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

INFEASIBLE PROBLEM

The infeasible solution in the following example is identified by using the Big-M Method. The Big-M method is an approach to find an initial BFS when the none is given and simple initial solutions, such as the origin, $x = \overline{0}$, are infeasible.

$$\text{minimize} \quad z = -4x_1 + 3x_2,$$

$$\begin{array}{llr} \text{subject to} & x_1 + x_2 \le 3, & (1) \\ & 2x_1 - x_2 \le 3, & (2) \\ & x_1 \ge 4, & (3) \\ & x_1, x_2 \ge 0. & \end{array}$$

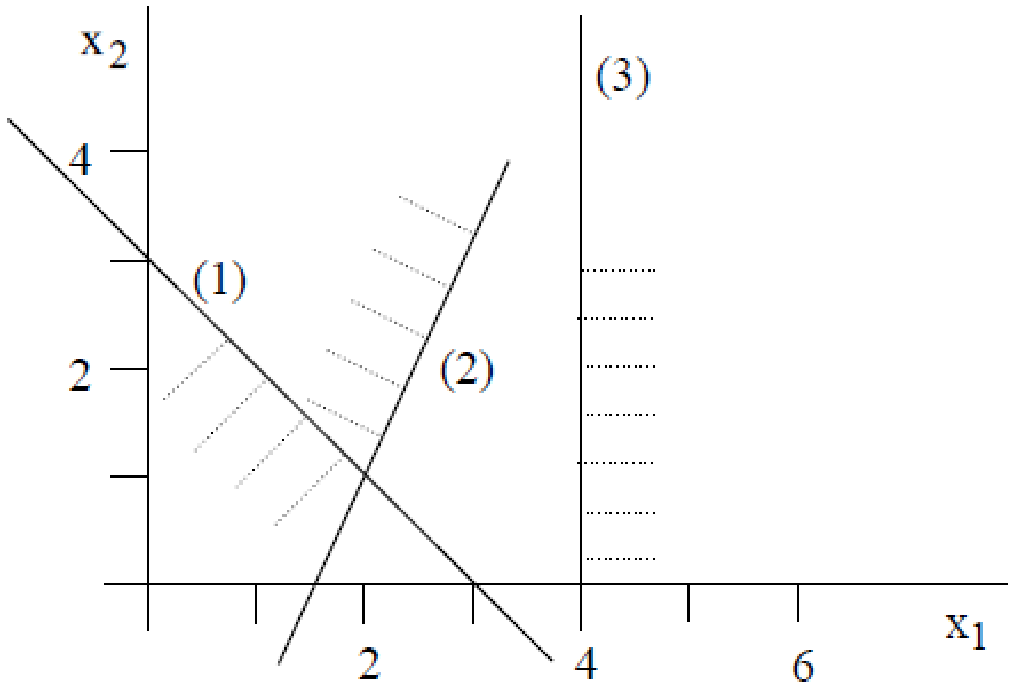
In the Big-M Method, the problem is modified by introducing artificial variables that can be used to construct an initial BFS. This initial solution is not feasible to the original LP.

$$
\begin{array}{llr}
\text{minimize} & z = -4x_1 + 3x_2 \qquad\qquad + M x_6, & \\
\text{subject to} & x_1 + x_2 + x_3 & = 3, \\
 & 2x_1 - x_2 \quad + x_4 & = 3, \\
 & x_1 \qquad\qquad - x_5 + x_6 & = 4, \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. &
\end{array}
$$

Given that $b \geq \overline{0}$, artificial variables are added to the left-hand-side of equality constraints and inequality constraints of the form "≥", after they are transformed to equalities with surplus variables.

In our example, artificial variable x_6 has been introduced in the third constraint. Now, the initial basis will be $B = (a_3, a_4, a_6) = I$. In order to force the artificial variable out-of the basis, x_6 has been penalized in the objective function (M is a very large number).

The graphical representation of the original LP is provided below. Note that the feasible region is empty. Thus, the LP is infeasible.



The initial and final tableaus for the Big M Method are given below.

Initial tableau:

Row #	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	B_S	c_B
0		4	-3	0	0	0	-M	0		
	1	(4+M)	-3	0	0	-M	0	4M		
1	0	1	1	1	0	0	0	3	a_3	0
2	0	2	-1	0	1	0	0	3	a_4	0
3	0	1	0	0	0	-1	1	4	a_6	M

Final tableau:

Row #	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	B_S	c_B
0	1	0	0	(2-M)/3	(-7-M)/3	-M	0	-5+2M		
1	0	0	1	2/3	-1/3	0	0	1	a_2	3
2	0	1	0	1/3	1/3	0	0	2	a_1	-4
3	0	0	0	-1/3	-1/3	-1	1	2	a_6	M

Generally, when all artificial variables are equal to zero in the optimal tableau, the optimal BFS is also feasible and optimal to the original problem. Note that some artificial variables may still remain in the basis, but their value must be zero. When the artificial variables are equal to zero, the objective function value does not depend on M. If at least one artificial variable is positive, then the original LP is infeasible.

In our example, artificial variable x_6 remains in the basis with a positive value, $x_6 = 2$. Thus, the original LP is infeasible.

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