07 Branch and Bound for Integer Programming

Sang Jin Kweon (권상진)

School of Management Engineering
UNIST

Office hours: after class (or by APPT.)
Email: sjkweon@unist.ac.kr
Tel: +82 (52) 217-3146
Web: http://or.unist.ac.kr

TA
Seungok Woo (우승옥)
- Email: wso1017@unist.ac.kr
- Office Hours: by APPT.

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Classification of Integer Linear Programs

1. Integer Linear Program (ILP)

$$\text{maximize} \quad z = \sum_{j=1}^{n} c_j x_j,$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, \ldots, m,$$

$$x_j \geq 0, \text{ integer}, \qquad j = 1, \ldots, n.$$

2. Mixed-Integer Linear Program (MILP)

$$\text{maximize} \quad z = \sum_{j=1}^{n} c_j x_j + \sum_{k=1}^{p} d_k y_k,$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{p} g_{ik} y_k = b_i, \qquad i = 1, \ldots, m,$$

$$x_j \geq 0, \text{ integer}, \qquad j = 1, \ldots, n,$$

$$y_k \geq 0, \qquad k = 1, \ldots, p.$$

3. 0-1 Integer Linear Program (0-1 ILP)

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j,$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, \ldots, m,$$

$$x_j \in \{0, 1\}, \qquad j = 1, \ldots, n.$$

4. 0-1 Mixed-Integer Linear Program (0-1 MILP)

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j + \sum_{k=1}^{p} d_k y_k,$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{p} g_{ik} y_k = b_i, \qquad i = 1, \ldots, m,$$

$$x_j \in \{0, 1\}, \qquad j = 1, \ldots, n,$$

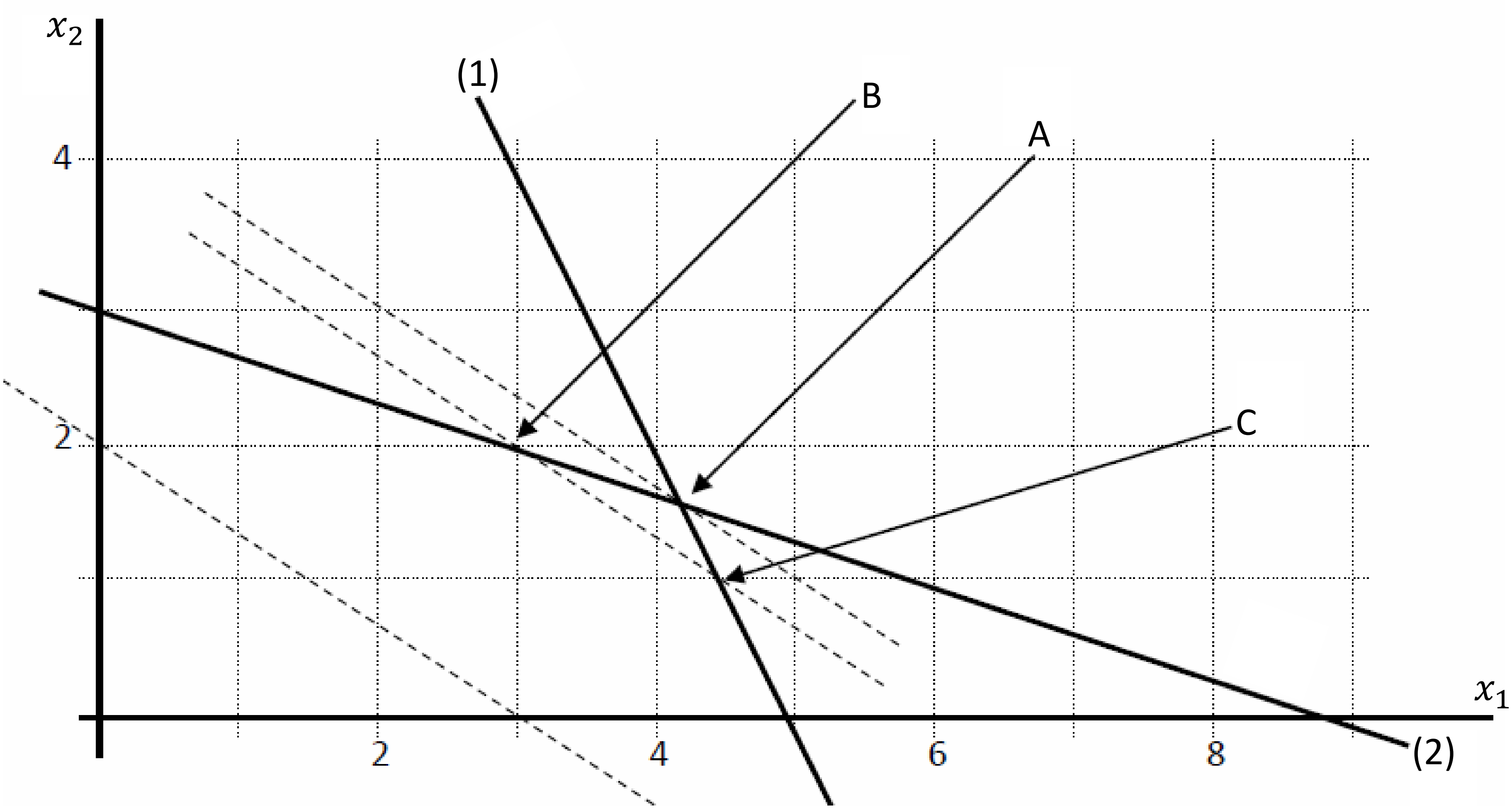
$$y_k \geq 0, \qquad k = 1, \ldots, p.$$

Graphical Solution to Integer Linear Programs with Two Variables

Example:

maximize $z = 2x_1 + 3x_2$

subject to
$$2x_1 + x_2 \leq 10 \qquad (1)$$
$$x_1 + 3x_2 \leq 9 \qquad (2)$$



Case 1 (LP): $x_1, x_2 \geq 0$.

Unique solution: $x_1^* = \frac{21}{5}$ $x_2^* = \frac{8}{5}$ $z^* = \frac{66}{5} = 13.2$ (A)

Case 2 (ILP): $x_1, x_2 \geq 0$, integer.

Unique solution: $x_1^* = 3$ $x_2^* = 2$ $z^* = 12$ (B)

Case 3 (MILP): $x_1 \geq 0$; $x_2 \geq 0$, integer.

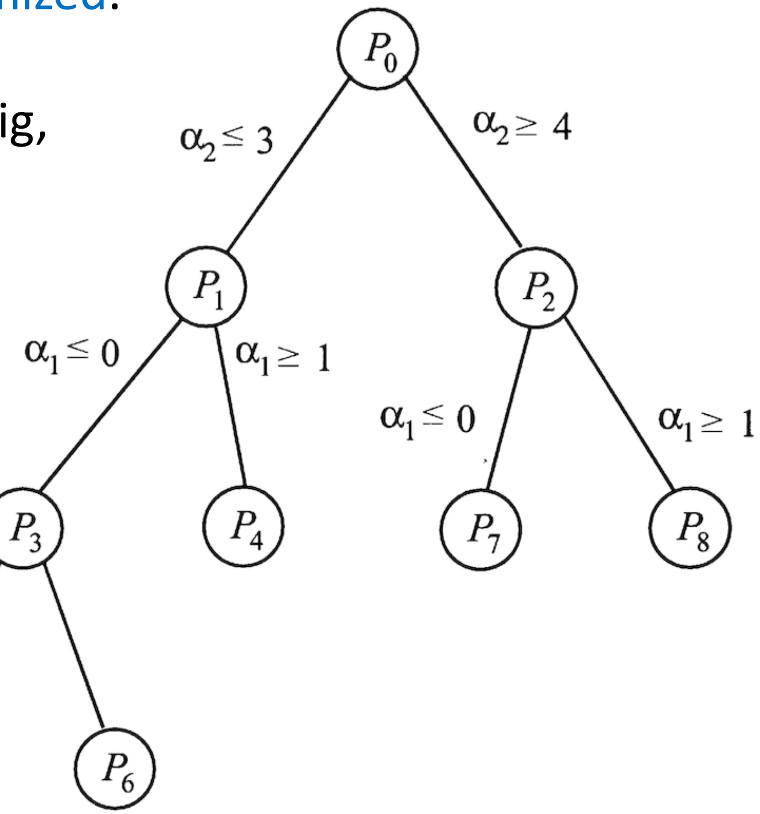
Two solutions: $x_1^* = 3$, $x_2^* = 2$ and $x_1^* = \frac{9}{2}$, $x_2^* = 1$ $z^* = 12$ (B) and (C)

Branch and Bound

- **Branch and Bound (BB)** is a general method for finding optimal solutions to problems, typically discrete problems.

- A branch-and-bound algorithm searches the entire space of candidate solutions, with one extra trick: it throws out large parts of the search space by using previous estimates on the quantity being optimized.

- The method is due to Ailsa Land and Alison Doig, who wrote the original paper in 1960:
  "An automatic method of solving discrete programming problems."

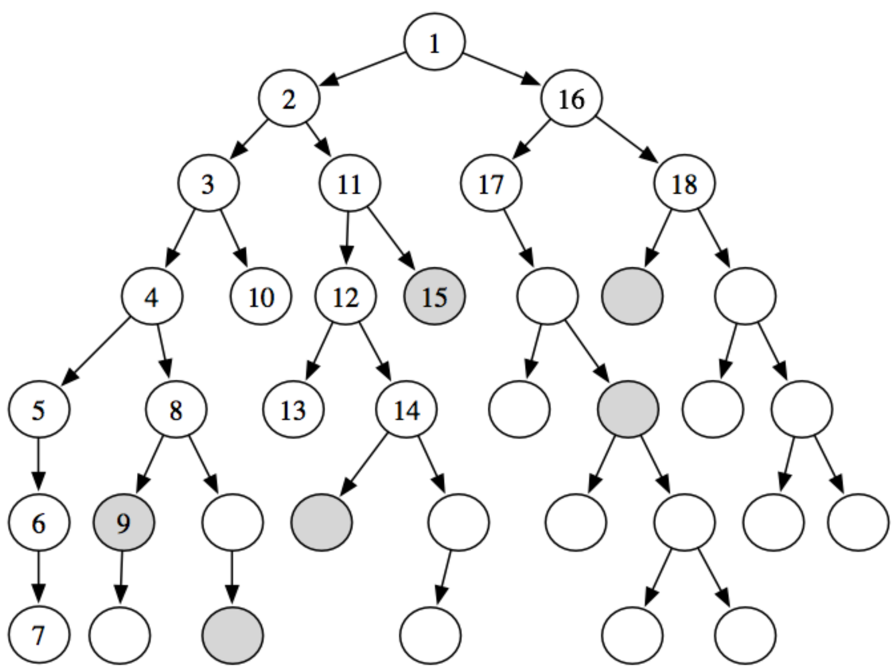


Branch and Bound

- Typically we can view a branch-and-bound algorithm as a tree search.

- At any node of the tree, the algorithm must make a finite decision and set one of the unbound variables.

- For example, suppose a binary integer programming model where each variable is only 0 or 1 valued. Then at a node where the variable x is on-tap, the algorithm must decide whether or not $x = 0$ or $x = 1$ is the right choice. In a brute-force search, both choices would be examined.



Branch and Bound

- In the BB case, the algorithm tries to avoid searches that are "useless."

- Imagine that the algorithm can prove that setting $x = 0$ will lead to a value for the optimization quantity of at most A, while setting $x = 1$ will lead to a value for this quantity of at least B.

- If $A < B$ in a maximization problem, then there is no reason to even consider setting x to 0. The upper bound on A and a lower bound on B allows the algorithm to safely avoid searching a whole chunk of the space: all those cases where x would be set to 0. This is the power of the method.

Reference: https://rjlipton.wordpress.com/2012/12/19/branch-and-bound-why-does-it-work/

Maximization with Branch and Bound

Maximize $3x_1 + 5x_2$

Subject to

$$
\begin{aligned}
2x_1 + 4x_2 &\le 25 \\
x_1 \qquad &\le 8 \\
2x_2 &\le 10 \\
x_1, \quad x_2&: \text{ integers} \\
x_1, \quad x_2 &\ge 0
\end{aligned}
$$

Maximization with Branch and Bound

Maximize $3x_1 + 5x_2$

Subject to $2x_1 + 4x_2 \leq 25$

$x_1 \leq 8$

$2x_2 \leq 10$

~~$x_1, \quad x_2$: integers~~

LP Relaxation
(i.e., we allow x_1 and x_2 to have real values)

$x_1, \quad x_2 \geq 0$

Optimal Solution with LP Relaxation:

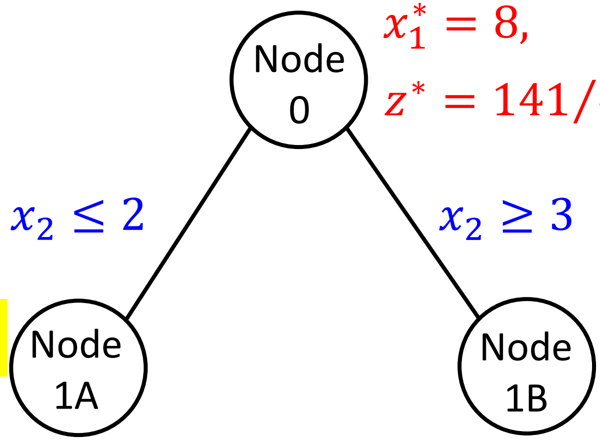
$x_1^* = 8, \qquad x_2^* = 9/4$

$z^* = 141/4$

Maximization with Branch and Bound

$x_1^* = 8, \quad x_2^* = 9/4 = 2.25$

$z^* = 141/4 = 35.25$



$x_2 \leq 2$

$x_2 \geq 3$

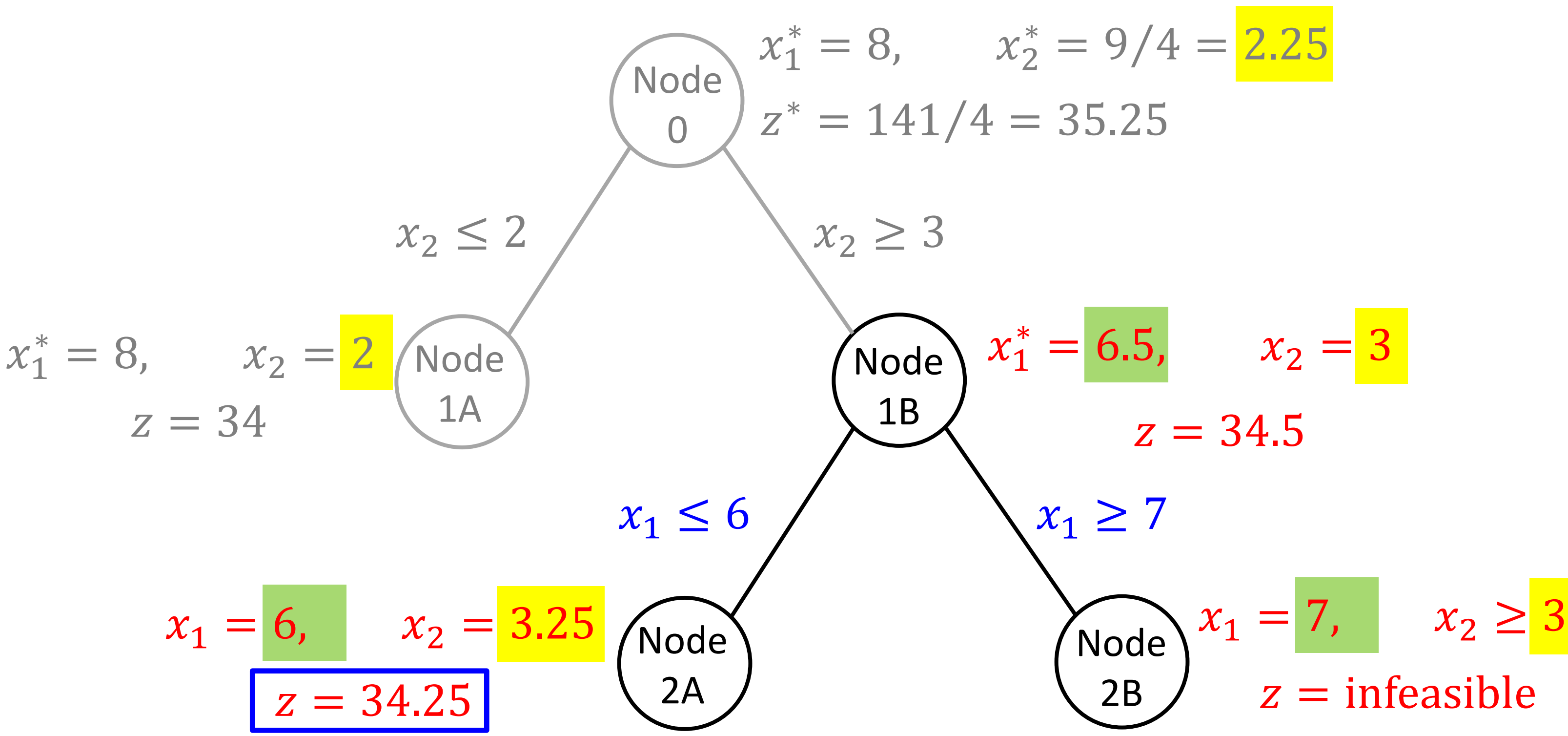
$x_1^* = 8, \quad x_2 = 2$

$z = 34$

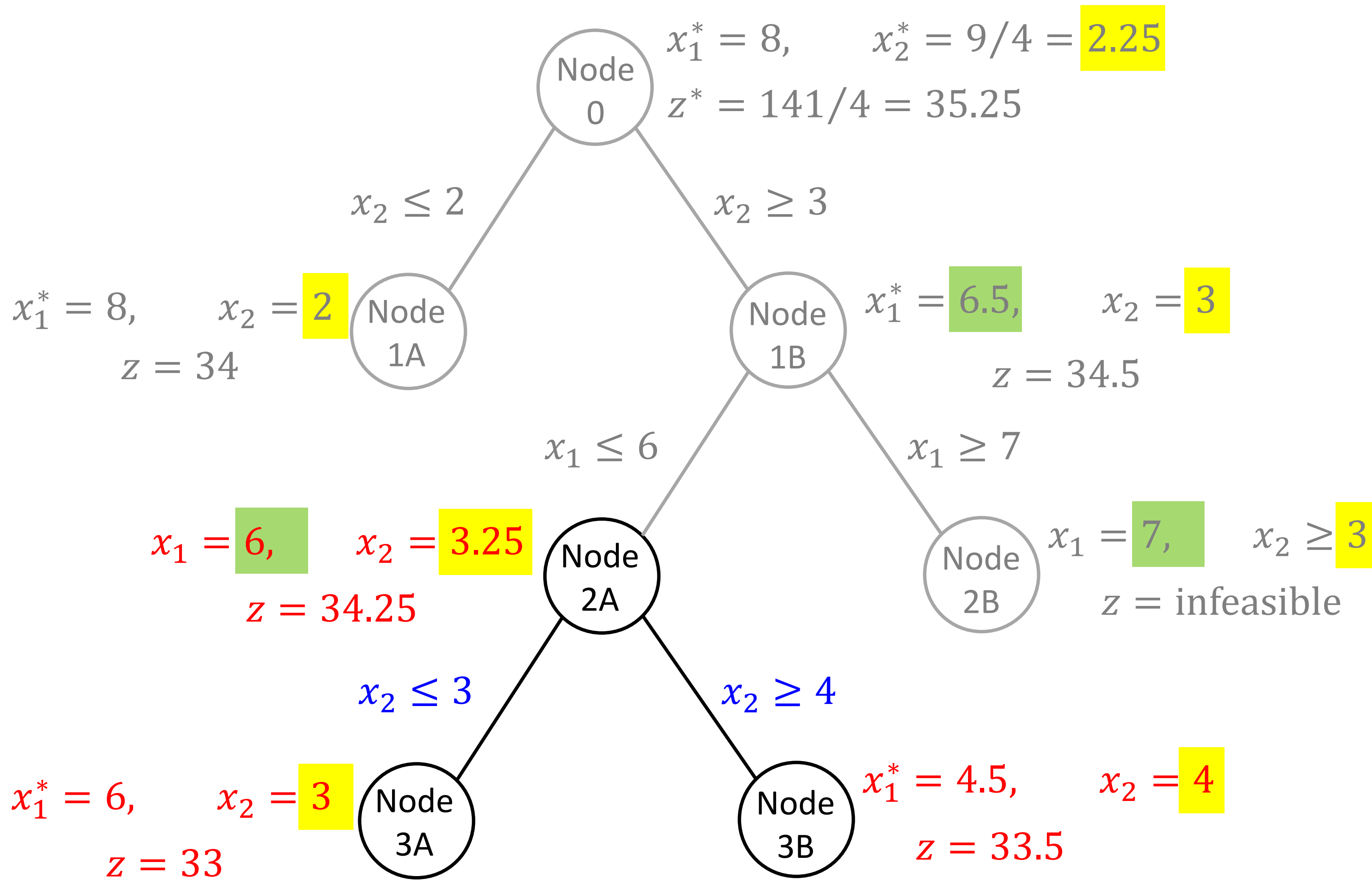
$x_1^* = 6.5, \quad x_2 = 3$

$\boxed{z = 34.5}$

Maximization with Branch and Bound

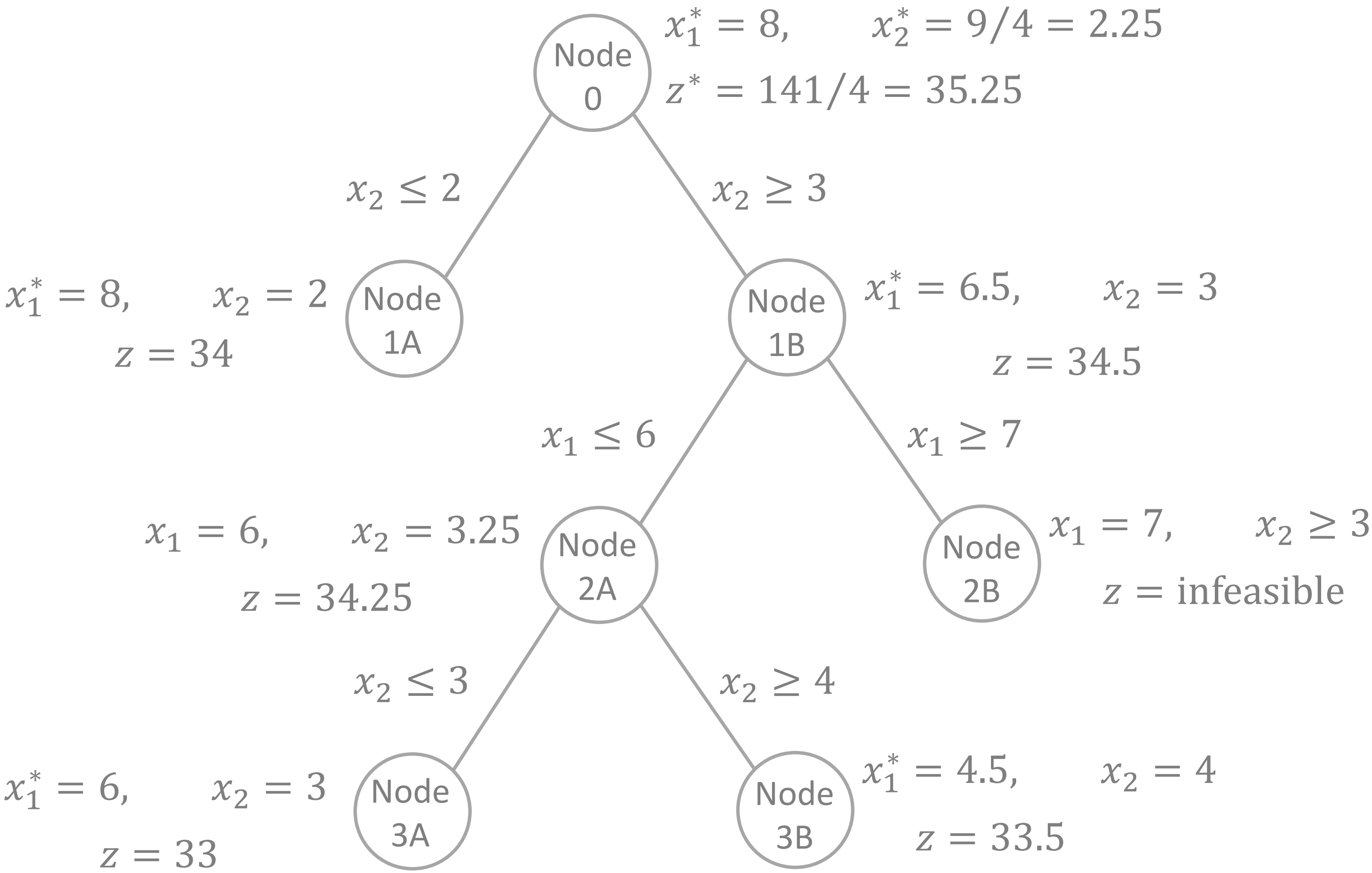


Maximization with Branch and Bound



We stop here because the solutions in Nodes 3A and 3B **are dominated by** the Solution in Node 1A. → **Optimal Solution** exists at **Node 1A**.

Maximization with Branch and Bound



What about **the computational complexity**?

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