Frequency Estimation

ECE565 Class Project:
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Abstract

In this report, we present our final project in Estimation class (ECE565), namely frequency estimation, the report includes our derivations of Maximum Likelihood Estimation (MLE), Method of Moments (MOM) and the Fisher Information Matrix (FIM). The project also present other estimation techniques that can be used for frequency estimation, results show that MLE outperforms every other estimator

I. Introduction and Project Structure

Frequency estimation is a fundamental concept in signal processing and communications, involving the determination of the dominant frequency components within a signal. It plays a pivotal role in analyzing periodic signals, where the ability to accurately estimate the frequency enables a deeper understanding of the underlying phenomena or behaviors driving the signal. At its core, frequency estimation is a mathematical and computational technique used to identify the frequency of a sinusoidal waveform or the frequencies of multiple sinusoidal components within a signal. This process is crucial in scenarios where signals are subject to noise, distortion, or other interferences, requiring robust algorithms for accurate estimation.

A. Motivation

A wide range of engineering fields focus on signal processing and analysis in order to better understand measurements and make decisions. Signals, whether in the form of audio, radio waves, vibrations, or physiological data, often carry critical information encoded in their frequency components. Often frequency components must be understood in order to properly extract information. An example of this is the carrier frequency of a signal which contains no desired information from the perspective of the sender but it is necessary to increase the frequency to physically send the data long distances. It is important to have high precision in predicting the frequency of a signal to ensure the original data is not altered.
Key motivations for estimating signal frequency include:

- *Efficient Signal Analysis*: Frequency estimation helps in isolating key components of a signal, facilitating better noise reduction, filtering, and interpretation.
- *System Identification*: In control systems and communications, knowing the operating or resonance frequency is essential for optimal design and performance tuning.
- *Communication Systems*: In wireless systems, frequency estimation aids in synchronization, demodulation, and channel characterization, ensuring reliable data transmission.

B. Project Structure

The remaining of the project is structured as follows: Section II provides background and motivation behind the need of frequency estimation. Section III explores the case of k frequency signals, how to derive the Fisher Information Matrix (FIM) of the parameters, and how to derive the Maximum Likelihood Estimator (MLE). Section IV derives expression for alternative estimators for frequency. Section V shows the performance of different estimators for frequency compared to the Cramer-Rao Lower Bound in terms of Mean Square Error (MSE). Finally, the conclusion is written in Section VI.

II. Application of Frequency Estimation

Frequency components of a signal are not always obvious in the time domain, transforming the signal into the frequency domain can have benefits in terms of aiding analysis because we are given the magnitude of each component the signal is composed of. This can be helpful in the scenario of filtering out noise from signal because a filter can be created to allow the high magnitude components to remain unaffected while multiplying lower magnitude components, assumed to be noise, by 0. Alternatively if there is a certain undesired frequency present in the data, such as a high pitched noise an audio sample, that frequency can be removed by transforming to the frequency domain and reducing the magnitude of the frequencies in that region.

A second application would be analyzing Carrier Frequency Offset. Ideally, the carrier frequency of the receiver and the transmitter should be the same such that when the signal is received by the receiver it will be able to offset the data to recover the sent message. Unfortunately, this is not always the case and there may be an undesired shift in the signal in the frequency domain which is referred to as Carrier Frequency Offset (CFO). It is important to estimate this frequency with great precision in order to ensure the correct signal is recovered and binary values are not misinterpreted.

Alternatively, CFO can be used to identify devices using an RF Fingerprint. An RF Fingerprint relies on measuring the CFO of different devices after transmitting wirelessly. The CFO will be different for different devices based on unique characteristics in the hardware of the transmitter. Each device will send data so the receiver can measure the CFO to identify which device sent this data in future transmissions. This process would require extreme precision in estimating the frequency in order to not misclassify devices which have similar hardware alterations.

One final example of an application of frequency estimation is an Electrocardiogram (EGC) test. This test is useful for predicting the heartrate of a patient in order to properly diagnose them for certain conditions. This test involves placing electrodes on the skin to capture electrical signals, which are then sent to a computer for processing. It is important these tests have high precision in order to ensure a patient is not misdiagnosed for a medical heart disease, or also not given a negative reading when a dangerious condition is present.

III. Case Study: K tone frequency estimation

In this project, we mainly focused on the multi-tone model where our signal is presented by multiple single-frequency complex exponentials in the presence of additive white gaussian noise. Without loss of generality, we assumed that our measure signal $y(n)$ is given by:

$$y(n) = \sum_{k=1}^{K} c_k e^{j2\pi f_k\left(n-\frac{N+1}{2}\right)} + v(n), \quad n = 1, 2, \ldots, N$$

where $c_k = a_k + jb_k$ is the complex amplitude coefficient of the k-th component, a_k and b_k are the real-valued, real and imaginary components of the complex coefficient c_k. f_k is the frequency of the k-th component and $v(n) \sim \mathcal{CN}(0, \sigma^2)$ for $n = 1, 2, \ldots, N$ are i.i.d complex gaussian noise components. In the subsections III-B the Fisher Information Matrix (FIM) and the Cramer-Rao Lower Bound (CRLB) are derived for this model, additionally, subsection III-C shows the step to derive maximum likelihood estimation of the frequency f_i in the case of single tone.

A. FIM derivation

To simplify the calculation of the Fisher information Matrix, Δ is defined as the difference between f_j and f_i which are the frequencies associated with the indices of the matrix.

$$\Delta = f_j - f_i$$

The following function definition is also used throughout the derivation, as well as its derivatives. The closed form expression for function g can be solved for by using the closed form of a geometric series to get a ratio of sine functions. The derivation of the FIM relies on substituting the summation form with the definition constructed by trigonometric functions in order to have a closed form expression.

$$g(\Delta) = \sum_{n=1}^{N} e^{j2\pi\Delta\left(n-\frac{N+1}{2}\right)} = e^{j2\pi\Delta\left(\frac{N+1}{2}\right)} \sum_{n=1}^{N} (e^{j2\pi\Delta})^n = \frac{\sin(\pi\Delta N)}{\sin(\pi\Delta)}$$

$$\dot{g}(\Delta) = \frac{-\pi(\cos(\pi\Delta)\sin(\pi\Delta N) - N\sin(\pi\Delta)\cos(\pi\Delta N))}{\sin^2(\pi\Delta)}$$

$$\ddot{g}(\Delta) = \frac{-\pi^2(((N^2-1)\sin^2(\pi\Delta) - 2\cos^2(\pi\Delta))\sin(\pi\Delta N) + 2N\cos(\pi\Delta)\sin(\pi\Delta)\cos(\pi\Delta N))}{\sin^3(\pi\Delta)}$$

The Fisher information matrix assumes the following form, derived in Fundamentals of Statistical Signal Processing: Estimation Theory. [1]

$$\text{FIM}(\theta)_{i,j} = \text{Tr}\Big[C_x^{-1}(\theta)\frac{\partial C_x(\theta)}{\partial\theta_i}C_x^{-1}(\theta)\frac{\partial C_x(\theta)}{\partial\theta_j}\Big] + 2\text{Re}\Big[\frac{\partial\mu^H(\theta)}{\partial\theta_i}C_x^{-1}(\theta)\frac{\partial\mu(\theta)}{\partial\theta_j}\Big]$$

The covariance matrix as described in the problem statement is not a function of theta, thus the first term is equal to 0. The inverse of the covariance matrix in the second term can be expressed as the reciprocal of σ^2 which can be added to the coefficient because it is a real quantity. The FIM at index i, j can be simplified to:

$$\text{FIM}(\theta)_{i,j} = \frac{2}{\sigma^2}\text{Re}\Big[\frac{\partial\mu^H(\theta)}{\partial\theta_i}\frac{\partial\mu(\theta)}{\partial\theta_j}\Big]$$

1) FIM$_{f_i,f_j}$

Using Equation III-A, the FIM between these two parameters can be formulated as:

$$\text{FIM}_{f_i,f_j} = \frac{2}{\sigma^2}\text{Re}\{\frac{\partial\mu^H}{\partial f_i}\frac{\partial\mu}{\partial f_j}\}$$

Taking the partial derivative of the mean with respect to the frequency results in the following summation:

$$\text{FIM}_{f_i,f_j} = \frac{2}{\sigma^2}\text{Re}\{\sum_{n=1}^{N} -j2\pi(n-\frac{N+1}{2})c_i^* e^{-j2\pi f_i(n-\frac{N+1}{2})} j2\pi(n-\frac{N+1}{2})c_j e^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

Multiplying terms and simplifying results in:

$$\text{FIM}_{f_i,f_j} = \frac{8\pi^2}{\sigma^2}\text{Re}\{c_i^* c_j \sum_{n=1}^{N}(n-\frac{N+1}{2})^2 e^{j2\pi\Delta(n-\frac{N+1}{2})})\}$$

Note that the second derivative of the function g is equivalent to the following summation:

$$\ddot{g}(\Delta) = \sum_{n=1}^{N}(j2\pi)^2(n-\frac{N+1}{2})^2 e^{j2\pi\Delta(n-\frac{N+1}{2})}$$

Which can be substituted to give:

$$\text{FIM}_{f_i,f_j} = \frac{-2}{\sigma^2}(a_i a_j + b_i b_j)\ddot{g}(\Delta)$$

where $\ddot{g}$ has the closed form shown in Equation III-A.

2) FIM$_{f_i,a_j}$

A similar setup can be achieved using the parameters f_i and a_j:

$$\text{FIM}_{f_i,a_j} = \frac{2}{\sigma^2}\text{Re}\{\frac{\partial\mu^H}{\partial f_i}\frac{\partial\mu}{\partial a_j}\} = \frac{2}{\sigma^2}\text{Re}\{\sum_{n=1}^{N} -j2\pi(n-\frac{N+1}{2})c_i^* e^{-j2\pi f_i(n-\frac{N+1}{2})} e^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

Multiplying terms results in the following simplification:

$$\text{FIM}_{f_i,a_j} = \frac{2}{\sigma^2}\text{Re}\{-c_i\sum_{n=1}^{N} j2\pi(n-\frac{N+1}{2})e^{j2\pi\Delta(n-\frac{N+1}{2})})\}$$

Substituting the summation definition of $\dot{g}$ gives:

$$\text{FIM}_{f_i,a_j} = \frac{2}{\sigma^2}\text{Re}\{-c_i\dot{g}(\Delta)\} = \frac{-2a_i}{\sigma^2}\dot{g}(\Delta)$$

3) FIM$_{f_i,b_j}$

The process for f_i and b_j is:

$$\mathrm{FIM}_{f_i,b_j} = \frac{2}{\sigma^2}\mathrm{Re}\{\frac{\partial\mu^H}{\partial f_i}\frac{\partial\mu}{\partial b_j}\} = \frac{2}{\sigma^2}\mathrm{Re}\{\sum_{n=1}^{N} -j2\pi(n-\frac{N+1}{2})c_i^* e^{-j2\pi f_i(n-\frac{N+1}{2})} j e^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

$$\mathrm{FIM}_{f_i,b_j} = \frac{2}{\sigma^2}\mathrm{Re}\{jc_i^*\sum_{n=1}^{N} j2\pi(n-\frac{N+1}{2})e^{j2\pi\Delta(n-\frac{N+1}{2})})\} = \frac{2b_i}{\sigma^2}\dot{g}(\Delta)$$

4) FIM$_{a_i,a_j}$

The FIM between two a parameters is a single substitution of the function g after multiplying the derivatives together.

$$\mathrm{FIM}_{a_i,a_j} = \frac{2}{\sigma^2}\mathrm{Re}\{\frac{\partial\mu^H}{\partial a_i}\frac{\partial\mu}{\partial a_j}\} = \frac{2}{\sigma^2}\mathrm{Re}\{\sum_{n=1}^{N} e^{-j2\pi f_i(n-\frac{N+1}{2})} e^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

$$\mathrm{FIM}_{a_i,a_j} = \frac{2}{\sigma^2}\mathrm{Re}\{\sum_{n=1}^{N} e^{j2\pi\Delta(n-\frac{N+1}{2})}\} = \frac{2}{\sigma^2}g(\Delta)$$

5) FIM$_{a_i,b_j}$

The FIM of a_i and b_j only relies on substituting the function g after multiplying like terms:

$$\mathrm{FIM}_{a_i,b_j} = \frac{2}{\sigma^2}\mathrm{Re}\{\frac{\partial\mu^H}{\partial a_i}\frac{\partial\mu}{\partial b_j}\} = \frac{2}{\sigma^2}\mathrm{Re}\{\sum_{n=1}^{N} e^{-j2\pi f_i(n-\frac{N+1}{2})} j e^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

Because the function g is always real, the multiplication with j results in the FIM being equal to 0:

$$\mathrm{FIM}_{a_i,b_j} = \frac{2}{\sigma^2}\mathrm{Im}\{\sum_{n=1}^{N} e^{j2\pi\Delta(n-\frac{N+1}{2})}\} = \frac{2}{\sigma^2}\mathrm{Im}\{g(\Delta)\} = 0$$

6) FIM$_{a_i,f_j}$

A similar setup can be achieved using the parameters a_i and f_j:

$$\mathrm{FIM}_{a_i,f_j} = \frac{2}{\sigma^2}\mathrm{Re}\{\frac{\partial\mu^H}{\partial a_i}\frac{\partial\mu}{\partial f_j}\} = \frac{2}{\sigma^2}\mathrm{Re}\{\sum_{n=1}^{N} e^{-j2\pi f_i(n-\frac{N+1}{2})} j2\pi(n-\frac{N+1}{2})c_j e^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

Multiplying terms results in the following simplification:

$$\mathrm{FIM}_{a_i,f_j} = \frac{2}{\sigma^2}\mathrm{Re}\{c_j\sum_{n=1}^{N} j2\pi(n-\frac{N+1}{2})e^{j2\pi\Delta(n-\frac{N+1}{2})})\}$$

Substituting the summation definition of $\dot{g}$ gives:

$$\mathrm{FIM}_{a_i,f_j} = \frac{2}{\sigma^2}\mathrm{Re}\{c_j\dot{g}(\Delta)\} = \frac{2a_j}{\sigma^2}\dot{g}(\Delta)$$

7) FIM$_{b_i,a_j}$

The FIM of b_i and a_j only relies on substituting the function g after multiplying like terms:

$$\mathrm{FIM}_{b_i,a_j} = \frac{2}{\sigma^2}\mathrm{Re}\{\frac{\partial\mu^H}{\partial b_i}\frac{\partial\mu}{\partial a_j}\} = \frac{2}{\sigma^2}\mathrm{Re}\{\sum_{n=1}^{N} -je^{-j2\pi f_i(n-\frac{N+1}{2})} e^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

Because the function g is always real, the multiplication with j results in the FIM being equal to 0:

$$\mathrm{FIM}_{b_i,a_j} = \frac{-2}{\sigma^2}\mathrm{Im}\{\sum_{n=1}^{N} e^{j2\pi\Delta(n-\frac{N+1}{2})}\} = \frac{-2}{\sigma^2}\mathrm{Im}\{g(\Delta)\} = 0$$

8) FIM$_{b_i,b_j}$

The process for b parameters is similar to that of a_i and a_j as the terms $-j$ and j cancel which yields an identical calculation:

$$\text{FIM}_{b_i,b_j} = \frac{2}{\sigma^2}\text{Re}\{\frac{\partial\mu^H}{\partial b_i}\frac{\partial\mu}{\partial b_j}\} = \frac{2}{\sigma^2}\text{Re}\{\sum_{n=1}^{N}(-j)e^{-j2\pi f_i(n-\frac{N+1}{2})}je^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

$$\text{FIM}_{b_i,b_j} = \frac{2}{\sigma^2}\text{Re}\{\sum_{n=1}^{N}e^{j2\pi\Delta(n-\frac{N+1}{2})}\} = \frac{2}{\sigma^2}g(\Delta)$$

9) FIM$_{b_i,f_j}$

The process for b_i and f_j is similar to that of a_i and f_j:

$$\text{FIM}_{b_i,f_j} = \frac{2}{\sigma^2}\text{Re}\{\frac{\partial\mu^H}{\partial b_i}\frac{\partial\mu}{\partial f_j}\} = \frac{2}{\sigma^2}\text{Re}\{\sum_{n=1}^{N}-je^{-j2\pi f_i(n-\frac{N+1}{2})}j2\pi(n-\frac{N+1}{2})c_je^{j2\pi f_j(n-\frac{N+1}{2})}\}$$

$$\text{FIM}_{b_i,f_j} = \frac{4\pi}{\sigma^2}\text{Re}\{c_j\sum_{n=1}^{N}(n-\frac{N+1}{2})e^{j2\pi\Delta(n-\frac{N+1}{2})})\} = \frac{-2b_i}{\sigma^2}\dot{g}(\Delta)$$

B. CRLB Derivation

If there are k frequencies, the set of parameters can be packed into a vector:

$$\theta = \left[a_1,...a_k,b_1,...,b_k,f_1,...f_k\right]^T$$

where parameters a_i and b_i are the respective real and imaginary parts of the sinusoidal wave coefficients, and f_i is the wave's frequency in radians per second. The Fisher Information Matrix (FIM) can be written as:

$$\text{FIM}(\theta) = \begin{bmatrix} \text{FIM}_{a_1,a_1},\text{FIM}_{a_1,a_2},...,\text{FIM}_{a_1,f_k} \\ \text{FIM}_{a_2,a_1},\text{FIM}_{a_2,a_2},...,\text{FIM}_{a_2,f_k} \\ ... \\ \text{FIM}_{f_k,a_1},\text{FIM}_{f_k,a_2},...,\text{FIM}_{f_k,f_k} \end{bmatrix}$$

Each element in the above matrix can be derived using the above formulas describing the FIM relationships between parameters. In the case of $k=1$, the FIM has the following values:

$$\text{FIM}(\theta) = \begin{bmatrix} \frac{2}{\sigma^2}g(\Delta) & 0 & 0 \\ 0 & \frac{2}{\sigma^2}g(\Delta) & 0 \\ 0 & 0 & \frac{-2}{\sigma^2}(a_ia_j+b_ib_j)\ddot{g}(\Delta) \end{bmatrix}$$

Finally, the Cramer-Rao Lower Bound (CRLB) is equal to the inverse of the FIM:

$$\text{CRLB}(\theta) = \text{FIM}^{-1}(\theta)$$

C. MLE derivation

For this derivation, we assumed the $K=1$ scenario. The received random variable y is then expressed as follows:

$$y(n) = ce^{j2\pi f\left(n-\frac{N+1}{2}\right)} + v(n) \quad \text{Which is equivalent to} \quad y_n \sim \mathcal{CN}\left(ce^{j2\pi f\left(n-\frac{N+1}{2}\right)},\sigma^2\right)$$

Thus, the pdf of y is expressed as:

$$f_Y(y|f) = \frac{1}{\pi\sigma^2}e^{-\frac{1}{\sigma^2}\left|y-ce^{j2\pi f\left(n-\frac{N+1}{2}\right)}\right|^2}$$

and the pdf of joint N iid samples distribution:

$$f_Y(\mathbf{y}|f) = \frac{1}{\pi^N\sigma^{2N}}e^{-\frac{1}{\sigma^2}\sum_{n=1}^{N}\left|y_n-ce^{j2\pi f\left(n-\frac{N+1}{2}\right)}\right|^2}$$

next step is obtaining $logf_Y(\mathbf{y}/f)$:

$$logf_Y(\mathbf{y}|f) = -Nlog(\pi\sigma^2) - \frac{1}{\sigma^2}\sum_{n=1}^{N}\left|y_n - ce^{j2\pi f\left(n-\frac{N+1}{2}\right)}\right|^2$$

Where our optimization problem can be written as:

$$\max_{c,f}\ logf_Y(\mathbf{y}|f)$$

or equivalently:

$$\min_{c,f}\ \sum_{n=1}^{N}\left|y_n - ce^{j2\pi f\left(n-\frac{N+1}{2}\right)}\right|^2$$

which can be simplified further to:

$$\min_{c,f}\ \sum_{n=1}^{N}\left(y_n^* - c^*e^{-j2\pi f\left(n-\frac{N+1}{2}\right)}\right)\left(y_n - ce^{j2\pi f\left(n-\frac{N+1}{2}\right)}\right)$$

Denoted by $L(c,f)$, we can simply this expression and express c as $a+jb$ as follows:

$$\begin{aligned}
L(c,f) &= \sum_{n=1}^{N}|y_n|^2 + |c|^2 - y_nc^*e^{-j2\pi f\left(n-\frac{N+1}{2}\right)} - y_n^*ce^{j2\pi f\left(n+1-\frac{N+1}{2}\right)} \\
&= \sum_{n=1}^{N}|y_n|^2 + |c|^2 - 2Re\{c^*y_ne^{-j2\pi f\left(n-\frac{N+1}{2}\right)}\} \\
&= \sum_{n=1}^{N}|y_n|^2 + a^2 + b^2 - 2Re\{(a-jb)y_ne^{-j2\pi f\left(n+1-\frac{N+1}{2}\right)}\}
\end{aligned}$$

To find the optimum values for a, b we need to take the first derivative and set to zero:

$$\frac{dL(c,f)}{da} = \sum_{n=1}^{N}2a - 2Re\{y_ne^{-j2\pi f\left(n-\frac{N+1}{2}\right)}\} = 0$$

Now solving for a, we get:

$$\begin{aligned}
\sum_{n=1}^{N}2a &= 2\sum_{n=1}^{N}Re\{y_ne^{-j2\pi f\left(n-\frac{N+1}{2}\right)} \\
2Na &= 2\sum_{n=1}^{N}y_n\ cos\left(2\pi f\left(n-\frac{N+1}{2}\right)\right)
\end{aligned}$$

$$\boxed{\bar{a} = \frac{\sum_{n=1}^{N}y_n\cos\left(2\pi f\left(n-\frac{N+1}{2}\right)\right)}{N}}$$

and similarly we can find the optimum value for b, the final result for b is given by:

$$\boxed{\bar{b} = -\frac{\sum_{n=1}^{N}y_n\ sin\left(2\pi f\left(n-\frac{N+1}{2}\right)\right)}{N}}$$

Therefore, the complex amplitude $\bar{c}^* = \bar{b} - j\bar{b}$ can we written as:

$$\bar{c}^* = \frac{\sum_{n=1}^{N}y_n\ cos\left(2\pi f\left(n-\frac{N+1}{2}\right)\right)}{N} + j\frac{\sum_{n=1}^{N}y_n\ sin\left(2\pi f\left(n-\frac{N+1}{2}\right)\right)}{N}$$

$$\boxed{\bar{c}^* = \frac{\sum_{n=1}^{N} y_n \ e^{j2\pi f\left(n-\frac{N+1}{2}\right)}}{N}}$$

and to find $\bar{f}$, we need to substitute $\bar{c}$ into $L(c, f)$ to get:

$$L(f) = \sum_{n=1}^{N} |y_n|^2 + N|\bar{c}|^2 - 2Re\{\bar{c}^* \sum_{n=1}^{N} y_n e^{-j2\pi f\left(n-\frac{N+1}{2}\right)}\}$$
$$= \sum_{n=1}^{N} |y_n|^2 + N|\frac{\sum_{n=1}^{N} y_n \ e^{-j2\pi f\left(n-\frac{N+1}{2}\right)}}{N}|^2 - 2Re\{\frac{\sum_{n=1}^{N} y_n \ e^{j2\pi f\left(n-\frac{N+1}{2}\right)}}{N} \sum_{n=1}^{N} y_n e^{-j2\pi f\left(n-\frac{N+1}{2}\right)}\}$$

Which can be written as:

$$L(f) = \sum_{n=1}^{N} |y_n|^2 + N|\frac{Y(f)e^{j2\pi f\left(\frac{N+1}{2}\right)}}{N}|^2 - 2Re\{\frac{Y(f^*)e^{-j2\pi f\left(\frac{N+1}{2}\right)}}{N} Y(f)e^{j2\pi f\left(\frac{N+1}{2}\right)}\}$$
$$= \sum_{n=1}^{N} |y_n|^2 + \frac{|Y(f)|^2}{N} - 2Re\{\frac{|Y(f)|^2}{N}\}$$
$$= \sum_{n=1}^{N} |y_n|^2 - \frac{|Y(f)|^2}{N}$$

where $Y(f)$, represent the discrete time Fourier transform (DTFT) of $y(n)$ and satisfy the relationship:

$$Y(f)e^{j2\pi f\left(\frac{N+1}{2}\right)} = \sum_{n=1}^{N} y_n e^{-j2\pi f\left(n-\frac{N+1}{2}\right)}$$

So the maximum likelihood objective will be finally written as:

$$\min_f L(f) = \min_f \ \sum_{n=1}^{N} |y_n|^2 - \frac{|Y(f)|^2}{N} = \max_f \frac{|Y(f)|^2}{N}$$

or equivalently, the estimated value of f, $\hat{f}$ is the frequency value that satisfy the following objective:

$$\boxed{\hat{f} = \text{arg} \max_f \ |Y(f)|^2}$$

IV. Other Frequency Estimation Methods

In this section, we will list other methods used to estimate the operation frequency f and compare them with the solution we obtained from MLE. The estimators we chose are *Mean Phase Derivative* (MPD), *Method of Moments* (MOM) based estimator and a robust version of MOM, we referred to as (MOM+), in which will be explained in the following subsections:

A. Methods of Moments (MOM)

In this technique, we first obtain the dot product between $y^*(n)$ and a shifted version $y(n+1)$ and then compute the expectation, which is equivalent to evaluating the auto correlation function $R_y(k) = \mathbb{E}\left[y^*(n) \cdot y(n+k)\right] = \sum_{n=1}^{N-k} y^*(n) \cdot y(n+k)$ at $k = 1$. In other words:

$$R_y(1) = \mathbb{E}\left[y^*(n) \cdot y(n+1)\right] = \frac{1}{N-1} \sum_{n=1}^{N-1} y^*(n) \cdot y(n+1)$$
$$R_y(1) = \mathbb{E}\left[c^* e^{-j2\pi f\left(n-\frac{N+1}{2}\right)} \cdot c e^{j2\pi f\left(n+1-\frac{N+1}{2}\right)}\right] + \mathbb{E}\left[v^*(n)v(n+1)\right]$$
$$+ \mathbb{E}\left[c e^{-j2\pi f\left(n-\frac{N+1}{2}\right)} v(n+1)\right] + \mathbb{E}\left[v^*(n) c * e^{j2\pi f\left(n+1-\frac{N+1}{2}\right)}\right]$$

And since the random variables $v(n)$ are independent from $y(n)$, and there is no correlation between $v^*(n)$ and $v(n+1)$, the last three terms are equal to zero. And thus the final formula can be written as:

$$R_y(1) = \mathbb{E}\left[c^* e^{-j2\pi f\left(n-\frac{N+1}{2}\right)} \cdot c e^{j2\pi f\left(n+1-\frac{N+1}{2}\right)}\right] = |c|^2 e^{j2\pi f} \tag{1}$$

Where $\hat{f}$ will be given as

$$\hat{f} = \frac{1}{2\pi} \angle R_y(1) \tag{2}$$

B. Robust Methods of Moments (MOM+)

Based on the simple MOM with a single delay $k = 1$, we can obtain the auto correlation function for all values of k and then differentiate the angle with respect to n (taken the difference in this case, since discrete signals). Assume that the auto correlation function $R_y(k) = \mathbb{E}\left[y^*(n) \cdot y(n+k)\right] = \sum_{n=1}^{N-k} y^*(n) \cdot y(n+k)$, for $k = 1, \ldots N-1$. Using the same approach as before, we can we that the phase of $R_y(k)$ will be a linear function of f, and we can estimate the frequency by :

$$\hat{f} = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{2\pi} \frac{d}{dk} \angle R_y(k) \tag{3}$$

Where $\frac{d}{dk} \angle R_y(k)$ can be obtained using either:

- $\frac{d}{dk} \angle R_y(k) \equiv R(k+1) - R(k)$ that represents the difference in discrete signals which is the derivative equivalent in continuous signals.
- $\frac{d}{dk} \angle R_y(k) \equiv A$, where A represent the parameter (slope) of a linearly fitted function $AR(k) + B$.

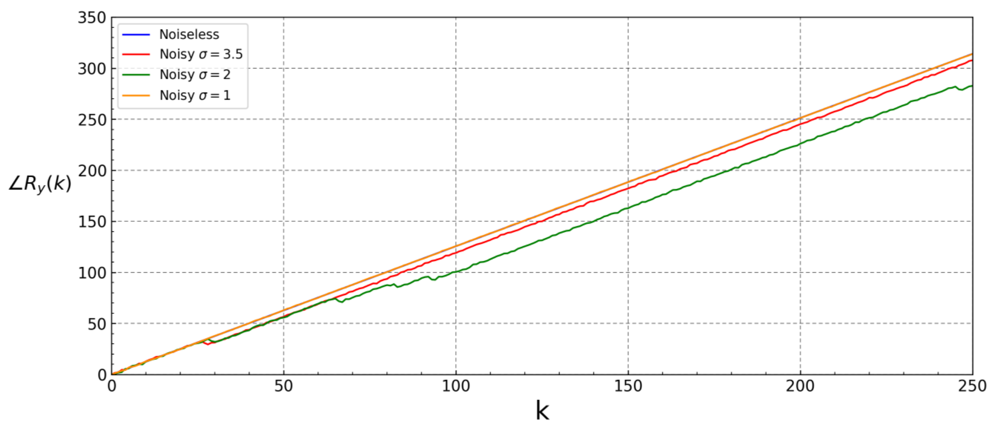


Fig. 1: Autocorrelation function phase for two different values of standard derivation

Figure 1 shows the phase of autocorrelation function, $\angle R_y(k)$, the slope here represent the operating frequency multiplied by 2π, we can clearly see that the slope is more robust to noise.

C. Mean Phase Derivative (MPD)

For any noiseless complex exponential signal $y(n) = e^{j2\pi f n+\phi}$, the frequency can be estimated as:

$$\hat{f} = \frac{1}{N} \sum_{n=1}^{N-1} \frac{1}{2\pi} \frac{d}{dn} \angle y(n) \tag{4}$$

Where $\frac{d}{dn} \angle y(n)$ can be obtained using the mentioned methods in Subsection IV-B. This estimator can be highly sensitive to noise since the noise effect wasn't minimize as in the autocorrelation based estimator as we can see in figure 2

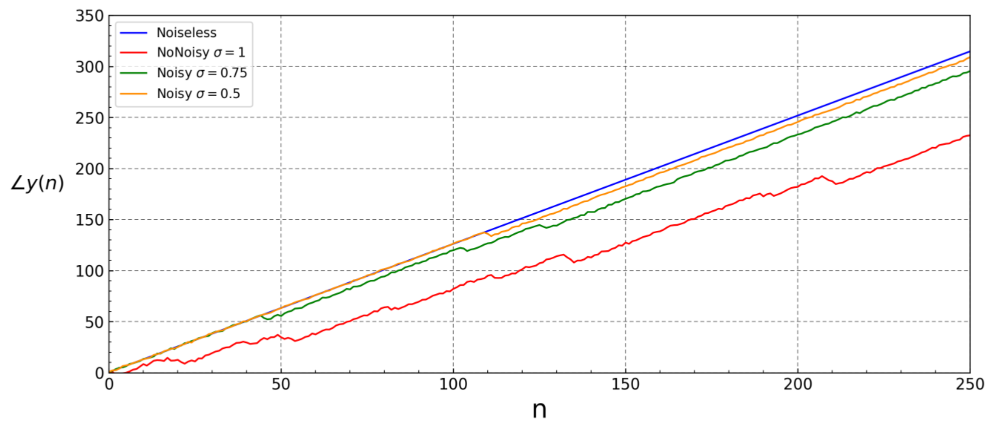


Fig. 2: The phase of the generated signals at different levels of noise

V. Evaluation and Analysis

In this section we will present our simulation results. In the case of $k = 1$, the FIM is diagonal as shown in Equation III-B. Thus the CRLB for the frequency component, f_1, can be calculated by taking the reciprocal of index $(2, 2)$ of the FIM. This is used to plot the CRLB of frequency, shown in blue in the plots. The MSE of the MLE is found through a grid search over the frequency domain in order to find the frequency with the most magnitude. The estimators using MOM, MOM+, and MPD, rely on taking the discrete time derivative of the given sequences in order to estimate the frequency.

A. Noise Effect

The plots displayed in Figure 3 show that as the variance of the noise increases, so does the MSE of each estimator and the CRLB. The MLE estimator, shown in red, demonstrates the best results as it achieves an MSE close to that of the CRLB. The MOM+ estimator achieves second best results, shown in orange(yellow) using the difference (Linear regression slope) approximation, and the MPD estimator achieve the worst results, shown in black and purple for the difference approximation and the linear regression (LR) approximation, respectively. In general, LR approximation based estimators tend to perform slightly better in the lower level of noise. In addition to that, As frequency increases from one plot to another, there is little variation in the performance for each estimator in terms of MSE.

B. Number of Samples Effect

In this subsection, we plots the error of our defined estimators vs the number of samples under different levels of noise. The plots displayed in Figure 4 show how as the number of samples increases, the MSE of the CRLB and the estimators monotonically decrease which is expected. The relative performance of the estimators is similar to that of the previous experiments as the MLE, closely followed by the MOM+ estimator, outperforms the MOM and MPD estimators. We can also notice that MSE values for both MLE and MOM+ are inversely proportional to N^3 while MPD and MOM to N. We can also highlight $f = 0.4$ case where TPD failed significantly, a further investigation of the frequency effect is presented in the next subsection.

C. Frequency Effect

In this subsection the effect of sampling rate is investigated. Since we assumed that the sampling frequency is fixed and set to be 1, the only way to study the effect of the sampling rate is to vary the operating frequency f, where $f \in [0, 0.5]$, since 0.5 represent the maximum frequency that satisfy Nyquist criterion. Figure 5 shows the effect of the operating frequency on the achieved MSE for the different techniques, the results highlight that MLE and MOM do not get affected by the operating frequency, on the other hand, MPD is highly dependent on the operating frequency, where the lower the frequency (hence higher sampling rate), the better the performance. In the addition to that, MOM+ seems to be affected by the operating frequency in case of extremely high noise or extremely low number of samples N.

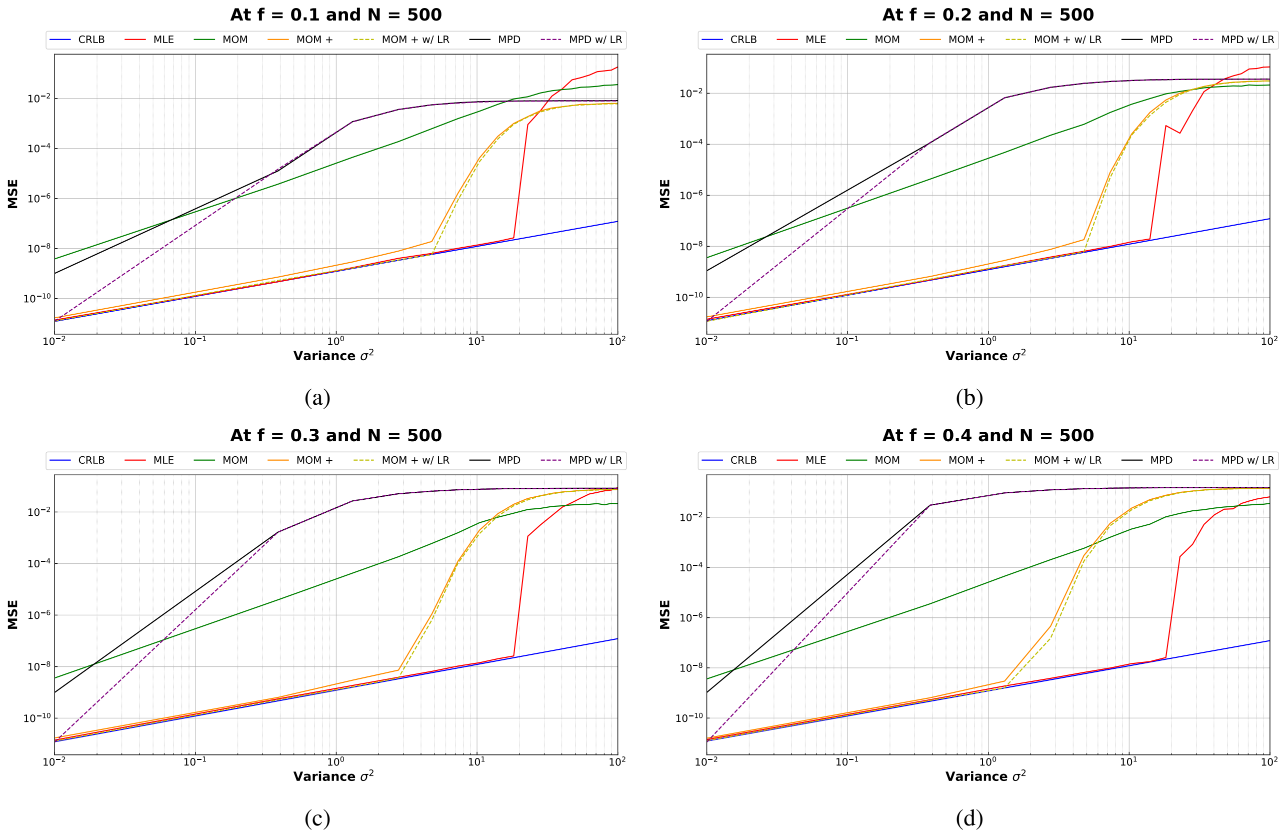
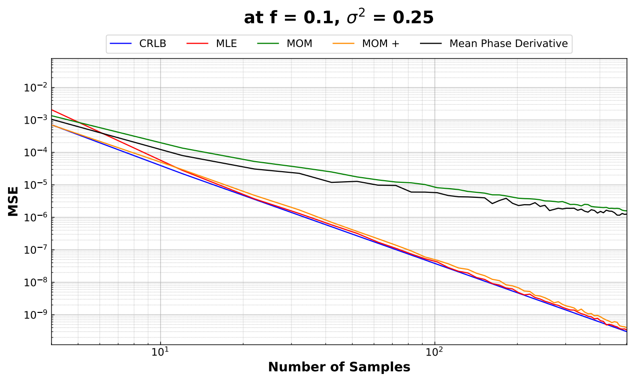


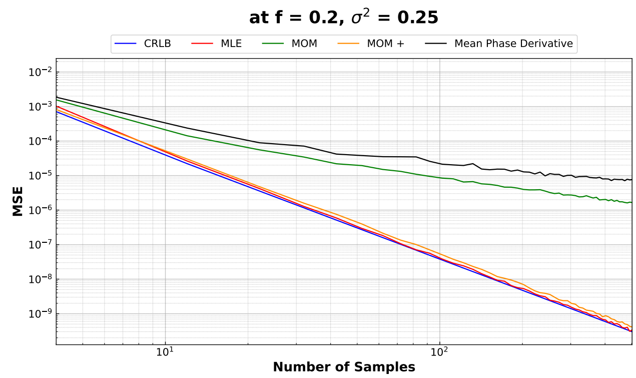
Fig. 3: Comparison of studied methods' MSE vs the noise level

D. *Bias Analysis*

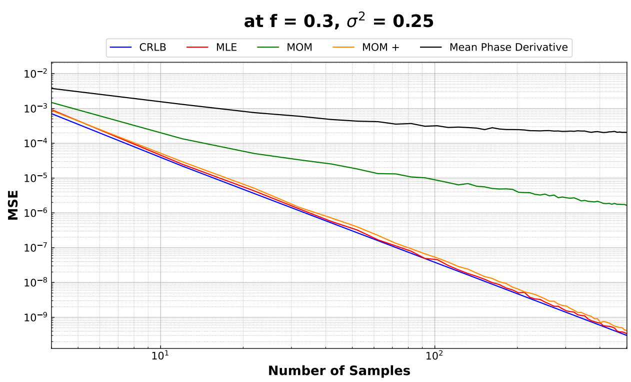
Figure 6 shows how the bias against the noise variance σ^2, in general, the studied estimator seems unbiased until the level of noise exceed a certain threshold, we can see that the MLE, MOM and MOM+ outperform MPD; meaning that MPD is more sensitive to noise. we can also see that the bias can be reduced for all of these techniques by increasing the number of measures N and the sampling rate (reducing f).



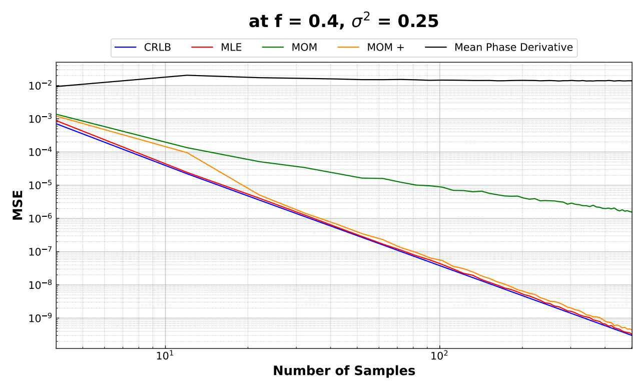
(a)



(b)

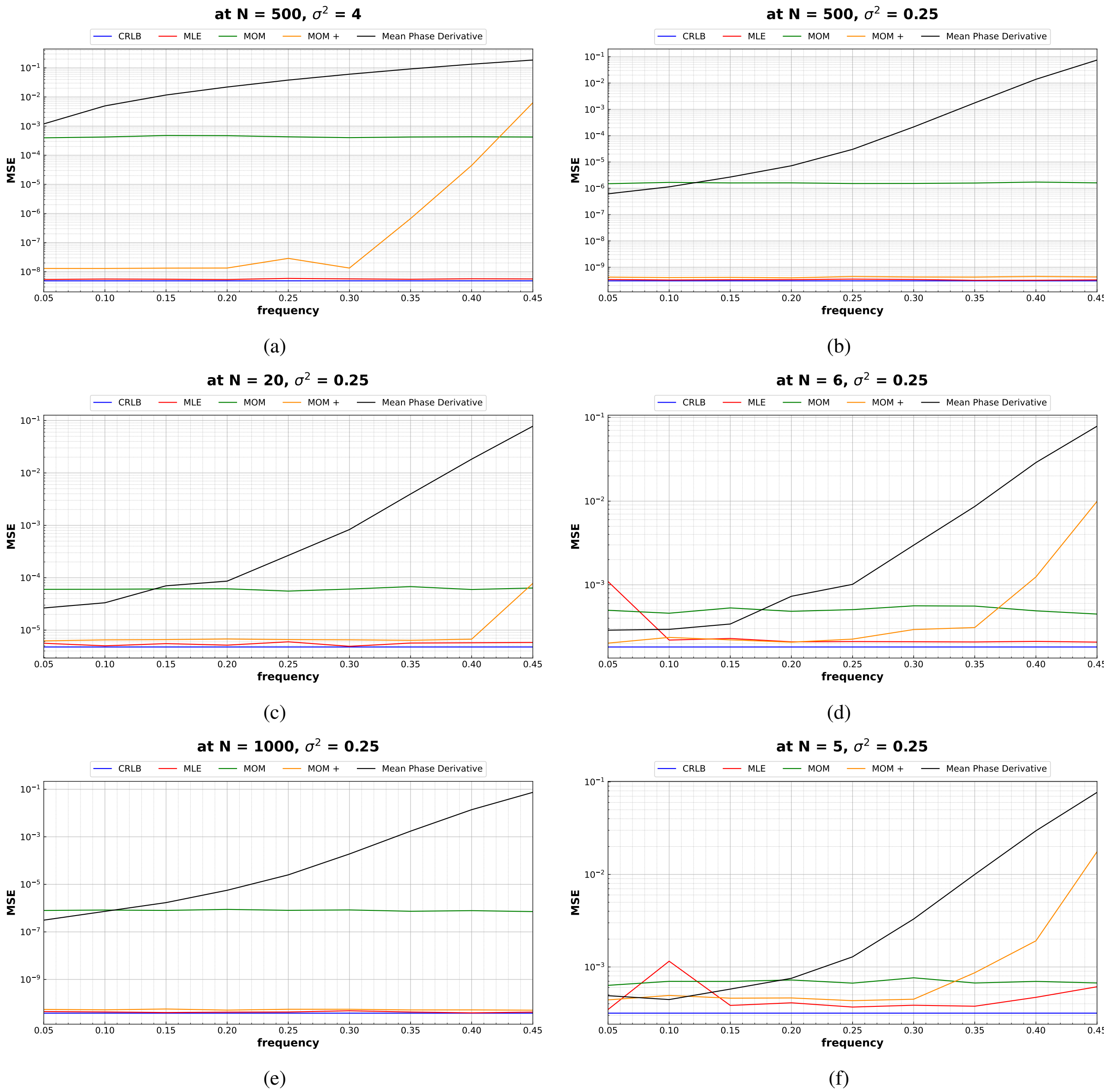


(c)



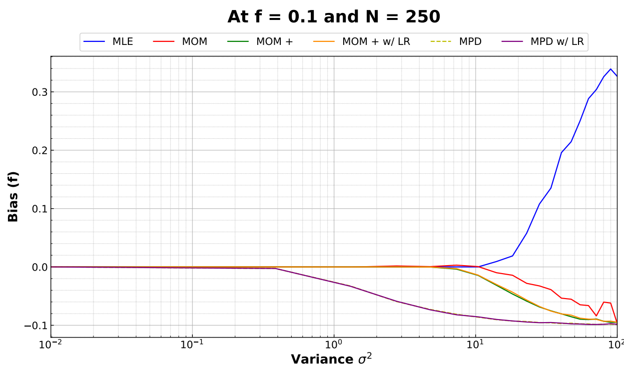
(d)

Fig. 4: Comparison of studied methods' MSE vs the number of samples

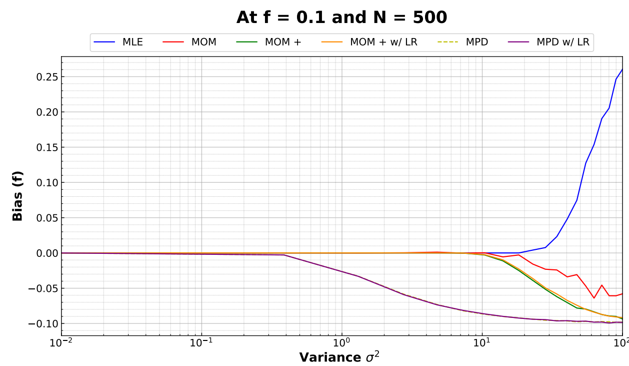


(a) (b) (c) (d) (e) (f)

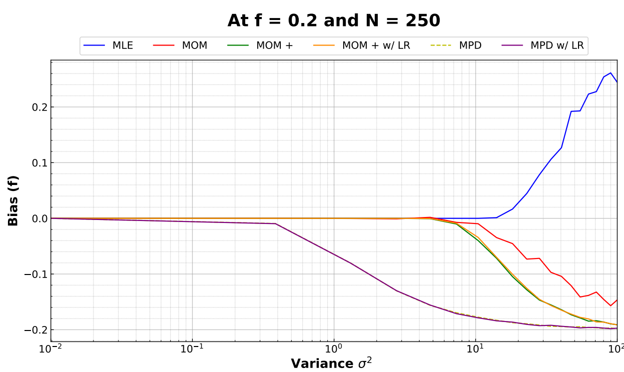
Fig. 5: Comparison of studied methods' MSE vs the operating frequency



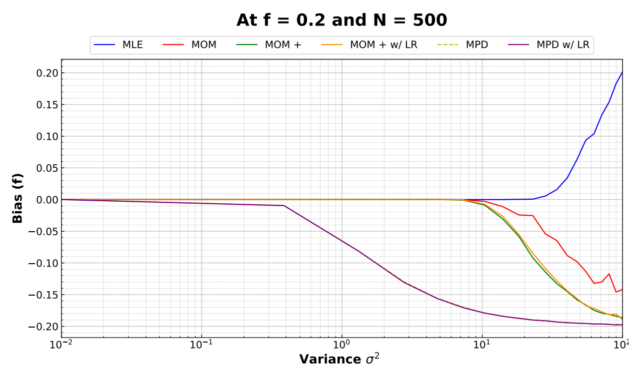
(a)



(b)



(c)



(d)

Fig. 6: Comparison of studied methods' Bias vs the noise level

VI. Conclusion

For this frequency estimation problem, the Maximum Likelihood Estimator achieved the best performance, in terms of Mean Squared Error, compared to alternatives: Method of Moments, Robust Method of Moments, and Mean Phase Derivative. As shown in our experiments, the Mean Squared Error decreases as the variance of noise decreases, and as the number of measurements increases, which is to be expected. This work also includes the derivation for the Fisher Information Matrix of the parameters a_i, b_i, and f_i which are the real coefficients, imaginary coefficients, and frequency of sinusoids, respectively. The Cramer-Rao Lower Bound (CRLB) found by taking the inverse of the derived FIM results in lower MSE compared to each estimator.

References

[1] S. M. Kay, *Fundamentals of statistical signal processing: estimation theory*. USA: Prentice-Hall, Inc., 1993.