Assignment 2 – Probabilistic Reasoning over Time

1 – Hidden Markov Model

a) See further down.
b) This is an operation for filtering. This computes the probability of reaching this state t given the evidence/observation from start until day t. Given all observations, what do we think of the world in the last observation? It is in short the best approximation of what we know of the world right now.

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The probabilities by filtering with evidence from day 1 to 6:

The probability at day  1  with evidence; Birds nearby: True  are:   [true, false] = [ 0.82090 0.17910 ]
The probability at day  2  with evidence; Birds nearby: True  are:   [true, false] = [ 0.90197 0.09803 ]
The probability at day  3  with evidence; Birds nearby: False  are:  [true, false] = [ 0.48519 0.51481 ]
The probability at day  4  with evidence; Birds nearby: True  are:   [true, false] = [ 0.81646 0.18354 ]
The probability at day  5  with evidence; Birds nearby: False  are:  [true, false] = [ 0.43135 0.56865 ]
The probability at day  6  with evidence; Birds nearby: True  are:   [true, false] = [ 0.79971 0.20029 ]
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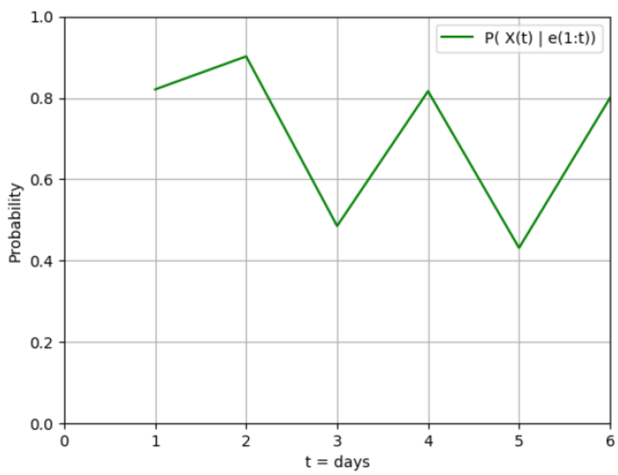

The prior probability <0.5, 0.5> were not mentioned to calculate so it is not included in the plot or answer.

c) This operation is prediction. With this function we try to predict what the future will be, given the evidence/observations we have. When t increases and get further and further away from the last evidence, it gets closer and closer, and in the end converges to the stationary state. This is the same as probability of a state when we have no evidence. The figure is probably wrong, it does converge to a value, but I do not think it is the real stationary state.

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The probabilities by filtering with evidence from day 1 to 6:

The probability at day  1  with evidence  True  are:   [true, false] = [ 0.82090 0.17910 ]
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The probability at day  5  with evidence  False  are:  [true, false] = [ 0.43135 0.56865 ]
The probability at day  6  with evidence  True  are:   [true, false] = [ 0.79971 0.20029 ]
The probability at day  7  with evidence  Prediction  are:   [true, false] = [ 0.89737 0.10263 ]
The probability at day  8  with evidence  Prediction  are:   [true, false] = [ 0.91784 0.08216 ]
The probability at day  9  with evidence  Prediction  are:   [true, false] = [ 0.92191 0.07809 ]
The probability at day  10  with evidence  Prediction  are:  [true, false] = [ 0.92270 0.07730 ]
The probability at day  11  with evidence  Prediction  are:  [true, false] = [ 0.92286 0.07714 ]
The probability at day  12  with evidence  Prediction  are:  [true, false] = [ 0.92289 0.07711 ]
The probability at day  13  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  14  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  15  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  16  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  17  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  18  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  19  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  20  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  21  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  22  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  23  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  24  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  25  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  26  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  27  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  28  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  29  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
The probability at day  30  with evidence  Prediction  are:  [true, false] = [ 0.92290 0.07710 ]
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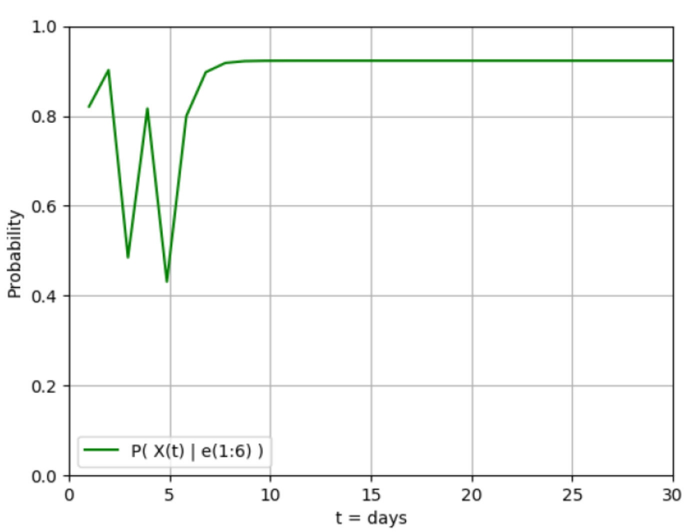

d) This operation is smoothing. The function for this is that we try to estimate the earlier state is really true based on the evidence/observations we have now. In short, given the observations we have now, were the observed state really correct?
e) This operation is for finding MLS, or the most likely sequence. This computes the most likely sequence to X(t) given the most likely path to state X(t)

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1) Hidden Markov Model (HMM)

Dynamics

Curious if fish in lake. X_t = fish present at day t

Unable to observe directly staring — observed

Can observe if birds nearby. E_t = Birds present at day t

Domain Theory: 1) Prior probability for fish nearby; $p = 0.5 \rightarrow P(X_0) = 0.5$

2) Probability of fish nearby at day t is 0.8 given fish nearby at t-1. 0.3 if not.

X_{t-1}	$P(X_t)$
T	0.8
F	0.3

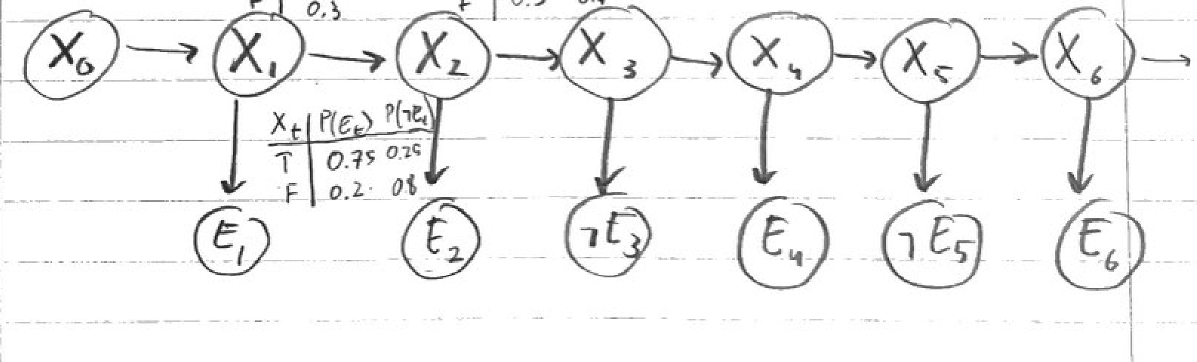
3) Probability of birds nearby at day t if fish nearby is 0.75. 0.2 if not.

X_t	$P(E_t)$
T	0.75
F	0.2

$P(X_0) = \langle 0.5, 0.5 \rangle$

X_{t-1}	$P(X_t)$
T	0.8
F	0.3

X_{t-1}	$P(X_t)$	$P(\neg X_t)$
T	0.8	0.2
F	0.3	0.7



X_t	$P(E_t)$	$P(\neg E_t)$
T	0.75	0.25
F	0.2	0.8

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2) Dynamic Bayesian Network

Tourists at cabin, want to find out if animals nearby
Observes if tracks outside OR food gone
Believe tracks and food are conditional independent given animals nearby

$$AnimalTracks_t \perp\!\!\!\perp FoodGone_t \mid AnimalsNearby_t$$

Domain Theory:

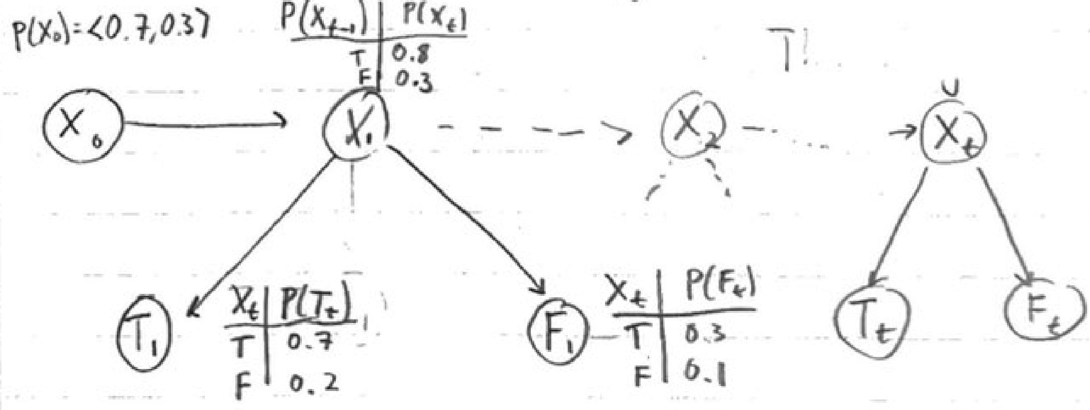
1) Prior Prob animals nearby is 0.7
2) Prob animal nearby day t is 0.8 given animals near t-1
   0.3 if not
3) Prob animal tracks day t if animals nearby same day 0.7
   0.2 if not
4) Prob food gone day t if animals nearby same day 0.3
   0.1 if not.

$e_1 = \{animalTracks, foodGone\}$ $e_3 = \{\neg animalTracks, \neg foodGone\}$

$e_2 = \{\neg animalTracks, foodGone\}$ $e_4 = \{animalTracks, \neg foodGone\}$

X_t = Animals nearby at day t

a)



This can just be duplicated as many times as days.
The probabilities are the same for all nodes and edges.

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b) Compute $P(X_t | e_{1:t})$ $t = 1,2,3,4$

$E_t = [[\text{Tracks}, \text{Food}], ..]$

$= [[1,1],[0,1],[0,0],[1,0]] = e_{1:t}$

$E_2 = e_{1:2}$

$P(X_0) = \langle 0.7, 0.3 \rangle$

$P(X_1) = \sum_{x_0} P(X_1|x_0)P(x_0) = \langle 0.8, 0.2\rangle \cdot 0.7 + \langle 0.3, 0.7\rangle \cdot 0.3$

$= \langle 0.56 + 0.09, 0.35 \rangle$

$= \underline{\langle 0.65, 0.35 \rangle}$

$P(X_1 | E_1) = P(X_1 | T_1, F_1) = \alpha P(T_1, F_1 | X_1) P(X_1)$ Legal bc Because $T_1 \perp\!\!\!\perp F_1 | X_1$

$= \alpha P(T_1|X_1) P(F_1|X_1) P(X_1)$

$= \alpha \langle 0.7, 0.2\rangle \langle 0.3, 0.1\rangle \langle 0.65, 0.35\rangle$

$= \alpha \cdot \langle 0.1365, 0.007 \rangle$

$= \underline{\langle 0.9512, 0.0488 \rangle}$

$P(X_2 | E_1) = \sum_{x_1} P(X_2|x_1)P(x_1|E_1) = \langle 0.8, 0.2 \rangle \cdot 0.9512 + \langle 0.3, 0.7\rangle \cdot 0.0488$

$= \langle 0.76096 + 0.01464, 0.19024 + 0.03416 \rangle$

$= \underline{\langle 0.7756, 0.2244 \rangle}$

$P(X_2 | \overset{[0,1]}{E_2}) = P(X_2 | \neg T_2, F_2) = \alpha P(\neg T_2, F_2 | X_2) P(X_2 | E_1)$

$= \alpha P(\neg T_2|X_2) P(F_2|X_2) P(X_2|E_1)$

$= \alpha \langle 0.3, 0.8 \rangle \langle 0.3, 0.1 \rangle \langle 0.7756, 0.2244\rangle$

$= \alpha \langle 0.0698, 0.01795 \rangle$

$= \underline{\langle 0.795, 0.205 \rangle}$

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$$P(X_3 \mid E_2) = \sum_{x_2} P(X_3 \mid x_2) P(x_2 \mid E_2) = \langle 0.8, 0.2 \rangle \cdot 0.795 + \langle 0.3, 0.7 \rangle \cdot 0.205$$
$$= \underline{\langle 0.6975, 0.3025 \rangle}$$

$$P(X_3 \mid E_3) = P(X_3 \mid \neg T_3, \neg F_3) = \alpha\, P(\neg T_3, \neg F_3 \mid X_3) P(X_3 \mid E_2)$$
$$= \alpha\, P(\neg T_3 \mid X_3) P(\neg F_3 \mid X_3) P(X_3 \mid E_2)$$
$$= \alpha \langle 0.3, 0.8 \rangle \langle 0.7, 0.9 \rangle \langle 0.6975, 0.3025 \rangle$$
$$= \alpha \langle 0.14595, 0.2178 \rangle$$
$$= \underline{\langle 0.4012?, 0.5988 \rangle}$$

$$P(X_4 \mid E_3) = \sum_{x_3} P(X_4 \mid x_3) P(x_3 \mid E_3) = \langle 0.8, 0.2 \rangle \cdot 0.4012 + \langle 0.3, 0.7 \rangle \cdot 0.5988$$
$$= \underline{\langle 0.5006, 0.4994 \rangle}$$

$$P(X_4 \mid E_4) = P(X_4 \mid T_4, \neg F_4) = \alpha\, P(T_4, \neg F_4 \mid X_4) P(X_4 \mid E_3)$$
$$= \alpha\, P(T_4 \mid X_4) P(\neg F_4 \mid X_4) P(X_4 \mid E_3)$$
$$= \alpha \langle 0.7, 0.2 \rangle \langle 0.7, 0.9 \rangle \langle 0.5006, 0.4994 \rangle$$
$$= \alpha \langle 0.2453, 0.0899 \rangle$$
$$= \underline{\underline{\langle 0.7318, .2682 \rangle}}$$

c) Compute $P(X_t | e_{1:4})$ for $t = 5, 6, 7, 8$ $| e_{1:4} = E_4$

$$P(X_5 | e_{1:4}) = \sum_{x_4} P(X_5 | x_4) P(x_4 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.7318 + \langle 0.3, 0.7 \rangle \cdot 0.2682$$

$$= \underline{\langle 0.6659, 0.3341 \rangle}$$

$$P(X_6 | e_{1:4}) = \sum_{x_5} P(X_6 | x_5) P(x_5 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.6659 + \langle 0.3, 0.7 \rangle \cdot 0.3341$$

$$= \underline{\langle 0.63295, 0.36705 \rangle}$$

$$P(X_7 | e_{1:4}) = \sum_{x_6} P(X_7 | x_6) P(x_6 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.63295 + \langle 0.3, 0.7 \rangle \cdot 0.3341$$

$$= \underline{\langle 0.60659, 0.36046 \rangle}$$

$$P(X_8 | e_{1:4}) = \sum_{x_7} P(X_8 | x_7) P(x_7 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.60659 + \langle 0.3, 0.7 \rangle \cdot 0.36046$$

$$= \underline{\underline{\langle 0.5934, 0.3736 \rangle}}$$

Probably something wrong here, does not sum up to 1...

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d) When $\lim_{t \to \infty} P(X_t | e_{1:4}) = \langle 0.6, 0.4 \rangle$ when far enough into future.

Verify the convergence.

Assume true for $t = 100$.
Verify by calculating for $t = 101$, if $\langle 0.6, 0.4 \rangle$ we know it converges for all valuse larger than 100. (Probably we can use a much smaller value than $t = 100$...)

$$P(X_{100} | e_{1:4}) = \underline{\langle 0.6, 0.4 \rangle}$$

$$P(X_{101} | e_{1:4}) = \sum_{x_{100}} P(X_{101} | x_{100}) P(x_{100} | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.6 + \langle 0.3, 0.7 \rangle \cdot 0.4$$

$$= \underline{\underline{\langle 0.6, 0.4 \rangle}}$$

We have now verified that when $t \to \infty$, it converges towards $\langle 0.6, 0.4 \rangle$ by showing it is true for $t = 101$ when assuming true for $t = 100$. (Induction) □.

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e) Compute $P(X_t | e_{1:4})$ $t = 0,1,2,3$

Smoothing is using backward and forward-algorithm.
The forward-part is already calculated from the filtering (in 2b).
From slides, the formula of smoothing are

$$P(X_k | e_{1:t}) = \alpha \underbrace{P(X_k | e_{1:k})}_{f_{1:k} = \text{From filtering}} \cdot \underbrace{P(e_{k+1:t} | X_k)}_{= b_{k+1:t}}$$

$$= \alpha f_{1:k} b_{k+1:t}$$

Have that
$f_{1:0} = P(X_0) = \langle 0.7, 0.3 \rangle$ $t=0$
$f_{1:1} = P(X_1 | e_1) = \langle 0.9512, 0.0488 \rangle$
$f_{1:2} = P(X_2 | e_{1:2}) = \langle 0.795, 0.205 \rangle$
$f_{1:3} = P(X_3 | e_{1:3}) = \langle 0.4012, 0.5988 \rangle$ $t=3$

Compute $b_{k+1:t} = P(e_{k+1:t} | X_k) = \sum_{x_{k+1}} P(e_{k+1} | x_{k+1}) b_{k+2:t}(x_{k+1}) P(x_{k+1} | X_k)$

$t=3$ $b_{4:3} = P(e_{4:3} | X_3) = \langle 1, 1 \rangle$

$P(X_3 | e_{1:3}) = \alpha \cdot f_{1:3} \cdot b_{4:3} = \alpha \cdot \langle 0.4012, 0.5988 \rangle$

$= \underline{\langle 0.4012, 0.5988 \rangle}$

$k=2$

$b_{3:3} = P(e_{3:3} | X_2) = \sum_{x_3} P(e_3 | x_3) b_{4:3}(x_3) P(x_3 | X_2)$

$e_3 = [0,0]$

$= \sum P(\neg T_3 | x_3) P(\neg F_3 | x_3) b_{4:3}(x_3) P(x_3 | X_2)$

$= (0.3 \cdot 0.7 \cdot 1 \cdot \langle 0.8, 0.2 \rangle + 0.8 \cdot 0.9 \langle 0.3, 0.7 \rangle)$

$= \underline{\langle 0.384, 0.546 \rangle}$

Should sum up to 1...
Dont know where is wrong :(

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$$P(X_2|e_{1:3}) = \alpha f_{1:2} b_{3:3} = \alpha \langle 0.795, 0.205\rangle \langle 0.384, 0.546\rangle$$

$$= \alpha \langle 0.3053, 0.1119\rangle$$
$$= \underline{\langle 0.7318, 0.2682\rangle}$$

Okay, normalized it here but ~~also~~ wrong cause last $b_{3:3}$ did not seem up to 1.

$t=2$, $h=2$: $b_{3:2} = P(e_{3:2}|X_2) = \langle 1, 1\rangle$

$$P(X_2|e_{1:2}) = \alpha f_{1:2} b_{3:2} = \underline{\langle 0.795, 0.205\rangle}$$

$h=1$: $b_{2:2} = P(e_{2:2}|X_1) = \sum_{x_2} P(e_2|x_2) b_{3:2}(x_2) P(x_2|x_1)$

$e_2 = [0,1]$

$$= \sum P(\neg T_2|x_2) P(F_2|x_2) b_{3:2}(x_2) P(X_2|x_1)$$
$$= (0.3 \cdot 0.3 \cdot 1 \cdot \langle 0.8, 0.2\rangle + 0.8 \cdot 0.1 \cdot 1 \cdot \langle 0.3, 0.7\rangle)$$
$$= \underline{\langle 0.096, 0.074\rangle}$$

$$P(X_1|e_{1:3}) = \alpha f_{1:1} \cdot b_{2:2} = \alpha \cdot \langle 0.9512, 0.0488\rangle \langle 0.096, 0.074\rangle$$

$$= \alpha \langle 0.09132, 0.0036112\rangle$$
$$= \underline{\langle 0.962, 0.038\rangle}$$

$t=1$: $b_{2:1} = P(e_{2:1}|X_1) = \langle 1, 1\rangle$

$$P(X_1|e_{1:1}) = \alpha f_{1:1} b_{2:1} = \alpha \cdot \langle 0.9512, 0.0488\rangle \langle 1, 1\rangle$$
$$= \underline{\langle 0.9512, 0.0488\rangle}$$

$$b_{1:1} = P(e_{1:1}|X_0) = \sum_{x_1} P(e_1|x_1) b_{2:1}(x_1) P(x_1|x_0)$$

$e_1 = [1,1]$

$$= \sum P(T_1|x_1) P(F_1|x_1) b_{2:1}(x_1) P(x_1|x_0)$$
$$= (0.7 \cdot 0.3 \cdot 1 \cdot \langle 0.8, 0.2\rangle + 0.2 \cdot 0.1 \cdot \langle 0.3, 0.7\rangle)$$
$$= \underline{\langle 0.174, 0.056\rangle}$$

$$P(X_0|e_{1:2}) = \alpha f_{1:0} b_{1:1} = \alpha \cdot \langle 0.7, 0.3\rangle \langle 0.174, 0.056\rangle$$
$$= \alpha \langle 0.1218, 0.0168\rangle$$
$$= \underline{\langle 0.8788, .1212\rangle}$$

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Okay, I admit I did not really got the last task, I think we should have bringed something when mooving down/backwards. !But I think the calculations except from that were not so bad!

Would be nice with comments and explanations :)