Numerical Analysis:

Due on Oct 4, 2023

Assignment #1

Xm H

Problem 1

Solution
Part a

Proof. For $f \in C[a,b]$ and $x_1, x_2 \in [a,b]$. Assume $f(x_1) \geq f(x_2)$, We have

$$f(x_2) \leq \frac{f(x_1)+f(x_2)}{2} \leq f(x_1)$$

According to Intermediate Value Theorem,there exists a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{f(x_1)+f(x_2)}{2}$$

□

Part b

Proof. Assume $f(x_1) \geq f(x_2)$, For c_1, c_2 are positive constants. we have

$$\frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2} - f(x_1) = \frac{c_2(f(x_2)-f(x_1))}{c_1+c_2} \leq 0$$

$$\frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2} - f(x_2) = \frac{c_1(f(x_1)-f(x_2))}{c_1+c_2} \geq 0$$

Thus,we have

$$f(x_2) \leq \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2} \leq f(x_1)$$

According to Intermediate Value Theorem,there exists a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}$$

□

Part c

Proof. Let $f(x) = x, x_1 = 2, x_2 = 1, c_1 = 2, c_2 = -1$,so that

$$\frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2} = 3$$

But,for $\forall \xi \in [x2, x1]$,

$$f(\xi) \neq \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}$$

□

Problem 2

Solution
Part a
According to Mean Value Theorem,$\exists \xi \in [x_0, x_0+\epsilon]$ make

$$f'(\xi) = \frac{f(x_0+\epsilon) - f(x_0)}{\epsilon}$$

Thus,the absolute error

$$|f(x_0+\epsilon) - f(x_0)| = |f'(\xi)\epsilon|$$

the relative error

$$\left|\frac{f(x_0+\epsilon) - f(x_0)}{f(x_0)}\right| = \left|\frac{f'(\xi)\epsilon}{f(x_0)}\right|$$

Part b
i. The absolute error $|f(x_0+\epsilon) - f(x_0)| = |f'(\xi)\epsilon| = 5 \times 10^-6e^{\xi}$ Due to $\xi \in [x_0, x_0+\epsilon]$, $e \leq e^{\xi} \leq e^{1+5\times10^-6}$
So the bounds of absolute error is

$$\left[5 \times 10^-6e, 5 \times 10^-6e^{1+5\times10^-6}\right]$$

The relative error $\left|\frac{f(x_0+\epsilon)-f(x_0)}{f(x_0)}\right| = \left|\frac{f'(\xi)\epsilon}{f(x_0)}\right| = \frac{f'(\xi)\epsilon}{e}$
So the bounds of relative error is

$$\left[5 \times 10^-6, 5 \times 10^-6e^{5\times10^-6}\right]$$

ii. The absolute error $|f(x_0+\epsilon) - f(x_0)| = |f'(\xi)\epsilon| = 5 \times 10^-6 \cos\xi$
Due to $\xi \in [x_0, x_0+\epsilon]$, $cos(1+5\times10^-6) \leq cos\xi \leq \cos 1$ So the bounds of absolute error is

$$\left[5 \times 10^-6\cos(1+5\times10^-6), 5\times10^-6\cos 1\right]$$

The bounds of relative error is

$$\left[\frac{5\times10^-6cos(1+5\times10^-6)}{\sin 1}, \frac{5\times10^-6\cos 1}{\sin 1}\right]$$

Part c
i. The absolute error $|f(x_0+\epsilon) - f(x_0)| = |f'(\xi)\epsilon| = 5 \times 10^-5e^{\xi}$
Due to $\xi \in [x_0, x_0+\epsilon]$, $e^{10} \leq e^{\xi} \leq e^{10+5\times10^-5}$ So the bounds of absolute error is

$$\left[5 \times 10^-5e^{10}, 5 \times 10^-5e^{10+5\times10^-5}\right]$$

The bounds of relative error is

$$\left[5 \times 10^-5, 5 \times 10^-5e^{5\times10^-5}\right]$$

ii. The absolute error $|f(x_0+\epsilon) - f(x_0)| = |f'(\xi)\epsilon| = -5 \times 10^-5 \cos\xi$
Due to $\xi \in [x_0, x_0+\epsilon]$, $cos10 \leq cos\xi \leq \cos(10+5\times10^-5)$ So the bounds of absolute error is

$$\left[-5 \times 10^-5\cos(10+5\times10^-5), -5\times10^-5\cos 10\right]$$

The bounds of relative error is

$$\left[\frac{5\times10^-5cos(10+5\times10^-5)}{\sin 10}, \frac{5\times10^-5\cos 10}{\sin 10}\right]$$

Problem 3

Solution
Part a
(i) $\frac{4}{5} + \frac{1}{3} = \frac{17}{15}$
(ii) $\frac{4}{5} + \frac{1}{3} = 0.800 + 0.333 = 1.13$
(iii) $\frac{4}{5} + \frac{1}{3} = 0.800 + 0.333 = 1.13$
(iv) the relative error in(ii) is

$$\frac{\frac{17}{15} - 1.13}{\frac{17}{15}} \simeq 0.003$$

the relative error in(iii) is

$$\frac{\frac{17}{15} - 1.13}{\frac{17}{15}} \simeq 0.003$$

Part b
(i) $\frac{3}{11} + \frac{1}{3} - \frac{3}{20} = \frac{301}{660}$
(ii) $\frac{3}{11} + \frac{1}{3} - \frac{3}{20} = 0.272 + 0.333 - 0.150 = 0.455$
(iii) $\frac{3}{11} + \frac{1}{3} - \frac{3}{20} = 0.273 + 0.333 - 0.150 = 0.456$
(iv) the relative error in(ii) is

$$\frac{\frac{301}{660} - 0.455}{\frac{301}{660}} \simeq 0.002326$$

the relative error in(iii) is

$$\frac{\frac{301}{660} - 0.456}{\frac{301}{660}} \simeq 0.000133$$

Problem 4

Solution

Part a

Proof.

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} c_1 F_1(x) + c_2 F_2(x) = \lim_{x\to 0} c_1 L_1 + c_2 L_2 + c_1 O(x^\alpha) + c_2 O(x^\beta)$$

Suppose $\alpha \leq \beta$, so that $\gamma = \alpha$ We have

$$\lim_{x\to 0} \frac{c_1 O(x^\alpha) + c_2 O(x^\beta)}{x^\gamma} = 0$$

So

$$c_1 O(x^\alpha) + c_2 O(x^\beta) = O(x^\gamma) \qquad (x \to 0)$$

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} c_1 L_1 + c_2 L_2 + c_1 O(x^\alpha) + c_2 O(x^\beta) = \lim_{x\to 0} c_1 L_1 + c_2 L_2 + O(x^\gamma)$$

□

Part a

Proof.

$$\lim_{x\to 0} G(x) = \lim_{x\to 0} F_1(c_1 x) + F_2(c_2 x) = \lim_{x\to 0} L_1 + L_2 + O(c_1^\alpha x^\alpha) + O(c_2^\beta x^\beta)$$

Suppose $\alpha \leq \beta$, so that $\gamma = \alpha$ We have

$$\lim_{x\to 0} \frac{O(c_1^\alpha x^\alpha) + O(c_2^\beta x^\beta)}{x^\gamma} = 0$$

So

$$O(c_1^\alpha x^\alpha) + O(c_2^\beta x^\beta) = O(x^\gamma) \qquad (x \to 0)$$

$$\lim_{x\to 0} G(x) = \lim_{x\to 0} L_1 + L_2 + O(c_1^\alpha x^\alpha) + O(c_2^\beta x^\beta) = \lim_{x\to 0} L_1 + L_2 + O(x^\gamma)$$

□

Problem 5

Refer to the attached code

Problem 6

Refer to the attached code

Problem 7

Proof. According to the problem condition,we have

$$g(p_0) = p_1, g(p) = p, |g'(p)| > 1$$

Due to the local monotonicity of limits

$$\exists \delta > 0, st \ \forall \ \xi \ that \ |\xi - p| < \delta, |g'(\xi)| > 1$$

If $|p_0 - p| < \delta$ By Lagrange's mean value theorem

$$g(p_0) - g(p) = g'(\xi)(p_0 - p)$$

ξ is between p and p_0,Thus $|\xi - p| < \delta, |g'(\xi)| > 1$,we have

$$g(p_0) - g(p) = p_1 - p = g'(\xi)(p_0 - p)$$
$$|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$$

□