Assignment 2

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1 Lagrange Multiplier Problem

With Pen & Paper:

1.1 a)

$$\min_{\vec{x}} -x_1 + 2x_2, \quad \text{s.t.} \quad x_1 = -\frac{1}{2}x_2^2 + 3, \quad \frac{1}{2}x_2 \geq -3x_1 + 5 \tag{1}$$

1.1.1 Formulate the Lagrangian equation

$$\begin{aligned}
f(x_1, x_2) &= -x_1 + 2x_2 \\
h(x_1, x_2) &= x_1 + \frac{1}{2}x_2^2 - 3 \\
g(x_1, x_2) &= -3x_1 - \frac{1}{2}x_2 + 5
\end{aligned}$$

$$L(x, \lambda, \mu) = -x_1 + 2x_2 + \lambda(x_1 + \frac{1}{2}x_2^2 - 3) + \mu(-3x_1 - \frac{1}{2}x_2 + 5)$$

1.1.2 Formulate the KKT conditions.

$$\nabla_{x_1} L(x, \lambda, \mu) = -1 + \lambda - 3\mu = 0 \tag{1}$$

$$\nabla_{x_2} L(x, \lambda, \mu) = 2 + \lambda x_2 - \frac{1}{2}\mu = 0 \tag{2}$$

$$\nabla_{\lambda} L(x, \lambda, \mu) = x_1 + \frac{1}{2}x_2^2 - 3 = 0 \tag{3}$$

$$\begin{aligned}
\mu \cdot g(x_1, x_2) &= 0 \\
\mu &\geq 0
\end{aligned}$$

1.1.3 Discuss every case with respect to the given constraints. Compute all possible solutions and identify valid ones.

case 1:

$$\mu > 0$$
$$\mu \cdot g(x_1, x_2) = 0$$
$$g(x_1, x_2) = 0$$
$$-3x_1 - \frac{1}{2}x_2 + 5 = 0$$
$$x_2 = -6x_1 + 10 \text{ (insert to (3))}$$
$$x_1 + \frac{1}{2}(-6x_1 + 10)^2 - 3 = 0$$
$$x_1 + \frac{1}{2}(36x_1^2 - 120x_1 + 100)^2 - 3 = 0$$
$$18x_1^2 - 59x_1 + 47 = 0$$
$$x_{1,1} = 1{,}365 \text{ AND } x_{1,2} = 1{,}9124$$
$$x_{2,1} = 1{,}808 \text{ AND } x_{2,2} = -1{,}475$$

$$(2)\ 2 + \lambda x_2 - \frac{1}{2}\mu = 0$$
$$-6\lambda x_2 + 3\mu - 12 = 0 \tag{4}$$
$$(1) + (4) \Rightarrow -13 - 6\lambda x_2 + \lambda = 0$$
$$\lambda = \frac{13}{-6x_2 + 1}$$
$$\lambda_1 = -1{,}31995 \text{ AND } \lambda_2 = 1{,}31995$$

$$(1)\ \ -1 + \lambda - 3\mu = 0$$
$$\mu = \frac{-1 + \lambda}{3}$$
$$\mu_2 = 0{,}1067 \text{ valid}$$
$$\mu_1 = -0{,}77317 \text{ invalid}$$

$$c_1 = (1{,}365, 1{,}808) \text{ invalid}$$
$$c_2 = (1{,}9124, -1{,}475) \text{ valid}$$

case 2:

$$
\begin{aligned}
\mu &= 0 \\
\mu \cdot g(x_1, x_2) &= 0 \\
(1) \quad -1 + \lambda - 3\mu &= 0 \\
\lambda &= 1 \\
(2) \; 2 + \lambda x_2 - \frac{1}{2}\mu &= 0 \\
x_2 &= -2 \\
(3) \; x_1 + \frac{1}{2}x_2^2 - 3 &= 0 \\
x_1 &= 1 \\
c_3 &= (1, -2) \text{ invalid because it does not satisfy the inequality constraint}
\end{aligned}
$$

1.1.4 Show which solution is the optimal one

$$
\begin{aligned}
&c_2 \text{ because it is the only valid point.} \\
&\Rightarrow c_2 = (1,9124, -1,475) \text{ is optimal}
\end{aligned}
$$

1.2 b)

$$
\min_{\vec{x}} \frac{1}{3}x_1^2 + 3x_2, \quad \text{s.t.} \quad \frac{1}{x_2}(x_1^2 - 3) = 1, \quad \frac{1}{2}x_1 - 1 \leq x_2 \tag{2}
$$

1.2.1 Formulate the Lagrangian equation

$$
\begin{aligned}
f(x_1, x_2) &= \frac{1}{3}x_1^2 + 3x_2 \\
h(x_1, x_2) &= \frac{1}{x_2}(x_1^2 - 3) - 1 \\
g(x_1, x_2) &= \frac{1}{2}x_1 - x_2 - 1
\end{aligned}
$$

$$
L(x, \lambda, \mu) = \frac{1}{3}x_1^2 + 3x_2 + \lambda(\frac{1}{x_2}(x_1^2 - 3) - 1) + \mu(\frac{1}{2}x_1 - x_2 - 1)
$$

1.2.2 Formulate the KKT conditions.

$$\nabla_{x_1} L(x, \lambda, \mu) = \frac{2}{3}x_1 + \lambda\frac{2x_1}{x_2} + \frac{1}{2}\mu = 0 \tag{1}$$

$$\nabla_{x_2} L(x, \lambda, \mu) = 3 - \lambda(x_1^2 - 3)\frac{1}{x_2^2} - \mu = 0 \tag{2}$$

$$\nabla_{\lambda} L(x, \lambda, \mu) = \frac{1}{x_2}(x_1^2 - 3) - 1 = 0 \tag{3}$$

$$\begin{aligned} \mu \cdot g(x_1, x_2) &= 0 \\ \mu &\geq 0 \end{aligned}$$

1.2.3 Discuss every case with respect to the given constraints. Compute all possible solutions and identify valid ones.

case 1:

$$\mu > 0$$
$$\mu \cdot g(x_1, x_2) = 0$$
$$g(x_1, x_2) = 0$$
$$\frac{1}{2}x_1 - x_2 - 1 = 0$$
$$x_2 = \frac{1}{2}x_1 - 1 \qquad (*)$$

$$(3)\ \frac{1}{x_2}(x_1^2 - 3) - 1 = 0$$
$$x_2 = x_1^2 - 3 \qquad (**)$$
$$(*) = (**) \Leftrightarrow \frac{1}{x_2}(x_1^2 - 3) - 1 = x_1^2 - 3$$
$$x_{1,1} = -1,1861 \text{ AND } x_{1,2} = 1,6861$$
$$x_{2,1} = -1,5931 \text{ AND } x_{2,2} = -0,1569$$

$$2\text{x}(1) + (2) \Leftrightarrow \frac{4}{3}x_1 + 3 + 4\lambda\frac{x_1}{x_2} - \lambda(x_1^2 - 3)\frac{1}{x_2^2} = 0$$
$$\lambda(4\frac{x_1}{x_2} - \frac{x_1^2}{x_2^2} + \frac{3}{x_2^2}) = \frac{-4}{3}x_1 - 3$$
$$\lambda_1 = -0,3934 \text{ AND } \lambda_2 = 0,1434$$

$$\text{From (2) } \mu = 3 - \lambda(x_1^2 - 3)\frac{1}{x_2^2}$$
$$\mu_1 = 2,7531 \text{ valid}$$
$$\mu_2 = 3,9149 \text{ valid}$$

$$c_1 = (-1,1861, -1,5931) \text{ valid}$$
$$c_2 = (1,6861, -0,1569) \text{ valid}$$

case 2:

$$\mu = 0$$

$$3\text{x}(3)\ 2x_1 + 2\lambda\frac{x_1}{x_2} = 0$$

$$(2)\ 3 - \lambda(\frac{x_1^2}{x_2^2} - \frac{3}{x_2^2}) = 0$$

$$\text{Insert 3x(3) into (1) AND (2)}$$

$$\text{I: } 2x_1 + 6\lambda\frac{x_1}{x_1^2 - 3} = 0$$

$$\text{II: } 3 - \lambda(\frac{x_1}{(x_1^2-3)^2} - \frac{3}{(x_1^2-3)^2}) = 0$$

$$\lambda = 3(x_1^2 - 3) \text{ Insert to I}$$

$$2x_1 + 6(3(x_1^2-3))\frac{x_1}{x_1^2-3} = 0$$

$$x_1 = 0 \text{ AND } x_2 = -3$$

$$\lambda = 3(0-3) = -9$$

$$c_3 = (0, -3)$$

invalid because it does not satisfy the inequality constraint

1.2.4 Show which solution is the optimal one

$$f(x_1, x_2) = \frac{1}{3}x_1^2 + 3x_2$$

$$f(-1,1861, -1,5931) = -4,3104$$

$$f(1,6861, -0,1569) = -0,4769$$

$$\Rightarrow c_1 = (-1,1861, -1,5931) \text{ is optimal}$$

1.3 c)

$$\min_{\vec{x}} \frac{1}{2}(2x_1^2 + x_1x_2 + 4x_2^2), \quad \text{s.t.} \quad x_2 = \frac{1}{2}x_1 - 1, \quad x_1^2 \leq -x_2 + 3. \qquad (3)$$

1.3.1 Formulate the Lagrangian equation

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2}(2x_1^2 + x_1x_2 + 4x_2^2) \\ &= x_1^2 + \frac{1}{2}x_1x_2 + 2x_2^2 \\ h(x_1, x_2) &= -x_2 + \frac{1}{2}x_1 - 1 \\ g(x_1, x_2) &= x_1^2 + x_2 - 3 \end{aligned}$$

$$L(x, \lambda, \mu) = x_1^2 + \frac{1}{2}x_1x_2 + 2x_2^2 + \lambda(-x_2 + \frac{1}{2}x_1 - 1) + \mu(x_1^2 + x_2 - 3)$$

1.3.2 Formulate the KKT conditions.

$$\nabla_{x_1} L(x, \lambda, \mu) = 2x_1 + \frac{1}{2}x_2 + \frac{1}{2}\lambda + 2\mu x_1 = 0 \tag{1}$$

$$\nabla_{x_2} L(x, \lambda, \mu) = \frac{1}{2}x_1 + 4x_2 - \lambda + \mu = 0 \tag{2}$$

$$\nabla_{\lambda} L(x, \lambda, \mu) = \frac{1}{2}x_1 - x_2 - 1 = 0 \tag{3}$$

$$\begin{aligned} \mu g(x_1, x_2) &= 0 \\ \mu &\geq 0 \end{aligned}$$

1.3.3 Discuss every case with respect to the given constraints. Compute all possible solutions and identify valid ones.

case 1:

$$\mu > 0$$
$$\mu \cdot g(x_1, x_2) = 0$$
$$g(x_1, x_2) = 0$$
$$x_1^2 + x_2 - 3 = 0$$
$$x_2 = -x_1^2 + 3 \text{ Insert to (3)}$$

$$\frac{1}{2}x_1 + x_1^2 - 3 - 1 = 0$$
$$x_{1,1} = -2,2655 \text{ AND } x_{1,2} = 1,7655$$
$$x_{2,1} = -2,132 \text{ AND } x_{2,2} = -0,1172$$

$$(2) \Leftrightarrow \lambda = \frac{1}{2}x_1 + 4x_2 + \mu$$
$$\lambda_1 = -11,7369 \text{ AND } \lambda_2 = -0,7999$$

$$2(1) + (2) \Rightarrow \frac{9}{2}x_1 + 5x_2 + \mu(4x_1 + 1) = 0$$
$$\mu = \frac{-\frac{9}{2}x_1 - 5x_1}{4x_1 + 1}$$
$$\mu_1 = -2,5868 \text{ invalid}$$
$$\mu_2 = -0,918 \text{ invalid}$$

$$c_1 = (-2,2655, -2,132) \text{ invalid}$$
$$c_2 = (1,7655, -0,1172) \text{ invalid}$$

case 2:

$$\mu = 0$$

$$2(1) + (2) \Rightarrow \frac{9}{2}x_1 + 5x_2 = 0$$

$$x_2 = \frac{-9}{10}x_1 \text{ Insert to (3)}$$

$$\frac{1}{2}x_1 + \frac{9}{10}x_1 - 1 = 0$$

$$x_1 = \frac{10}{14} \text{ AND } x_2 = \frac{-9}{14}$$

$$(2) \Leftrightarrow \lambda = \frac{1}{2}x_1 + 4x_2$$

$$\lambda = -2,214$$

$$c_3 = \left(\frac{10}{14}, \frac{-9}{14}\right) \text{ valid}$$

1.3.4 Show which solution is the optimal one

$$c_3 \text{ is our optimal solution since it is the only valid point.}$$

$$\Rightarrow c_3 = \left(\frac{10}{14}, \frac{-9}{14}\right) \text{ is optimal}$$

2 Optimization of Glider's Trajectory

With Pen & Paper:

$$\max_{x_1 \in \mathbb{R}^+ \setminus \{0\}, x_2 \in \mathbb{R}^+} f(x_1, x_2) := \frac{x_2}{x_1}. \tag{4}$$

$$h(x_1, x_2) = a + b(x_2 - c)^2 - x_1. \tag{5}$$

2.1 Formulate the Lagrangian equation

$$\begin{aligned} f(x_1, x_2) &= \frac{x_2}{x_1} \\ h(x_1, x_2) &= x_1 + a + b(x_2 - c)^2 - x_1 \end{aligned}$$

$$L(x, \lambda) = \frac{x_2}{x_1} + \lambda(x_1 + a + b(x_2 - c)^2 - x_1)$$

We did not include the two inequality constraints $x_1 > 0$ and $x_2 \geq 0$ in the Lagrangian equation since it is easy to check manually.

2.2 Analytically compute the gradient of the Lagrangian

$$\begin{aligned} \nabla_{x_1} L(x, \lambda) &= \frac{-x_2}{x_1^2} - \lambda = 0 \\ \nabla_{x_2} L(x, \lambda) &= \frac{1}{x_1} + 2\lambda b(x_2 - c) = 0 \\ \nabla_{\lambda} L(x, \lambda) &= a + b(x_2 - c)^2 - x_1 \end{aligned}$$

2.3 Using $a = 1, b = \frac{1}{2}, c = 0$, solve the system for x_1^*, x_2^*, λ^*

$$a = 1, b = \frac{1}{2}, c = 0$$

$$\frac{-x_2}{x_1^2} - \lambda = 0 \tag{1}$$

$$\frac{1}{x_1} + 2\lambda\frac{1}{2}(x_2) = 0 \tag{2}$$

$$1 + \frac{1}{2}(x_2)^2 - x_1 = 0 \tag{3}$$

$$\Rightarrow x_1 = 1 + \frac{1}{2}(x_2)^2 \tag{*}$$

$$\text{x2(1) + (2)} \Leftrightarrow -\frac{x_2^2}{x_1^2} + \frac{1}{x_1} = 0$$

$$x_1 = x_2^2 \tag{**}$$

$$(*) = (**) \Leftrightarrow x_2^2 = 1 + \frac{1}{2}(x_2)^2$$

$$x_2^2 = 2$$

$$x_2 = \pm\sqrt{2}$$

$$x_1 = 2$$

$$\lambda = \frac{-x_2}{x_1}$$

$$\lambda_1 = -\frac{1}{2\sqrt{2}} \text{ AND } \lambda_2 = \frac{1}{2\sqrt{2}}$$

$$c_1 = (2, \sqrt{2}) \text{ valid}$$

$$c_2 = (2, -\sqrt{2}) \text{ invalid because } x_2 \leq 0$$

2.4 Graphically add the analytic (exact) optimum (x_1^*, x_2^*) in your plot. State the analytically determined optimum (x_1^*, x_2^*) in your report, evaluate its function value and check whether the constraint is fulfilled.

$$x_1^* = 2$$

$$x_2^* = \sqrt{2}$$

$$\lambda^* = \frac{1}{2\sqrt{2}}$$

$$\nabla_{\mathbf{x}} f(x_1, x_2) = \begin{pmatrix} \frac{-x_2}{x_1} \\ \frac{1}{x_1} \end{pmatrix} \tag{6}$$

$$\nabla_{\mathbf{x}} h(x_1, x_2) = \begin{pmatrix} -1 \\ x_2 \end{pmatrix} \tag{7}$$

$$\begin{aligned} \nabla_x h(x_1, x_2) &= 1 + \frac{1}{2} x_2^2 - x_1 \\ \nabla_x h(x_1^*, x_2^*) &= 1 + \frac{1}{2} (\sqrt{2})^2 - 2 \\ &= 0 \end{aligned}$$

2.5 Show that the gradient of the objective function in (4) is orthogonal to the tangent line of $h(x_1^*, x_2^*) = 0$

In order to do that we need to calculate the tangent of $h(x_1^*, x_2^*)$ first. Therefore we use the standard formula for a tangent:

$$y = f'(x_0) \cdot (x - x_0) + f(x_0) \tag{8}$$

In our case we convert that to the tangent calculated at the point $\mathbf{x}^* = (x_1^*, x_2^*) = (2, \sqrt{2})$.

$$\begin{aligned} T(x_1, x_2) =& \nabla_{\mathbf{x}} h(x_1^*, x_2^*) \cdot (\mathbf{x} - \mathbf{x}^*) + h(x_1^*, x_2^*) = \\ =& \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} x_1 - 2 \\ x_2 - \sqrt{2} \end{pmatrix} + \left(1 + \frac{1}{2} \cdot 2 - 2\right) = \\ =& \begin{pmatrix} -x_1 + 2 \\ \sqrt{2} x_2 - 1 \end{pmatrix} \end{aligned} \tag{9}$$

Showing that the tangent $T(x_1, x_2)$ is orthogonal to the gradient of the objective function $\nabla_{\mathbf{x}} f(x_1, x_2)$ can be easily done by calculating the dot product.

$$\begin{aligned} T(x_1, x_2) \cdot \nabla_x f(x_1, x_2) =& \begin{pmatrix} -x_1 + 2 \\ \sqrt{2} x_2 - 2 \end{pmatrix} \cdot \begin{pmatrix} -\frac{x_2}{x_1^2} \\ \frac{1}{x_1} \end{pmatrix} = \\ =& \frac{x_2}{x_1} - 2\frac{x_2}{x_1^2} + \sqrt{2}\frac{x_2}{x_1} - \frac{2}{x_1} \end{aligned} \tag{10}$$

Next we are going to insert the optimal point $\mathbf{x}^*$.

$$\begin{aligned} \frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{4} + \sqrt{2}\frac{\sqrt{2}}{2} - \frac{2}{2} = \frac{2\sqrt{2}}{4} - \frac{2\sqrt{2}}{4} + 1 - 1 = 0 \\ \Rightarrow \text{orthogonal} \end{aligned}$$