Assignment 2

Nonlinear Optimization WS2024/25

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1 Lagrange Multiplier Problem (15 P.)

Given the following constrained optimization problems over $\mathbf{x} = (x_1, x_2)^\top$, compute the optimal solution using Lagrangian formulations.

a)

$$\min_{\mathbf{x}} \; x_2 - x_1 \quad \text{s.t.} \quad x_1 \geq 4x_2, \quad x_2 = \frac{1}{10}x_1^2 - 3$$

b)

$$\min_{\mathbf{x}} \; \|\mathbf{x}\|_2^2 \quad \text{s.t.} \quad x_2 \geq 3 - x_1, \quad x_2 \geq 2$$

c)

$$\min_{\mathbf{x}} \; (x_1 - 1)^2 + x_1 x_2^2 - 2 \quad \text{s.t.} \quad x_1^2 + x_2^2 \leq 4$$

For each example - **In Python**:

1. Plot the level sets of the above functions using a (filled) contour plot[1].
2. Draw the equality-/inequality-constraints on top[2].
3. Add markers to all points which fulfill the KKT-conditions (computed with Pen & Paper). Highlight the overall optimal solution with a point in a different color.

For each example - **With Pen & Paper**:

4. Formulate the Lagrangian equation. As shown in the exercise.
5. Formulate the KKT-conditions.
6. Discuss every case with respect to the given constraints. Compute all possible solutions and identify valid ones.
7. Show which solution is the optimal one.

2 Augmented Lagrangian (10 P.)

Consider an optimization problem with linear constraints of the form

$$\min_{\mathbf{x}} \; f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and the associated Lagrangian function is given as

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}). \tag{2}$$

[1] https://matplotlib.org/stable/api/_as_gen/matplotlib.pyplot.contourf.html
[2] https://matplotlib.org/stable/gallery/images_contours_and_fields/contours_in_optimization_demo.html

Recall from the lecture that the Lagrangian function provides a lower bound to the original constrained optimization problem which can be written in the form of a min-max problem

$$\min_{\mathbf{x}} \{\max_{\boldsymbol{\lambda}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\}. \tag{3}$$

A naive approach to arrive at a saddle point of the Lagrangian is to start with some initial $(\mathbf{x}^0, \boldsymbol{\lambda}^0)$ and perform gradient descent in $\mathbf{x}$ and gradient ascent in $\boldsymbol{\lambda}$ as

$$\begin{cases} \mathbf{x}^{k+1} = \mathbf{x}^k - \tau \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^k, \boldsymbol{\lambda}^k), \\ \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \tau \nabla_{\boldsymbol{\lambda}} \mathcal{L}(\mathbf{x}^{k+1}, \boldsymbol{\lambda}^k), \end{cases} \qquad \text{with } \tau > 0,\ k = 0, 1, 2, \ldots \tag{4}$$

Another way to tackle Eq. (1) is to introduce the augmented Lagrangian function with parameter $\mu > 0$ as

$$\mathcal{L}_\mu(\mathbf{x}, \boldsymbol{\lambda}) = \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) + \frac{\mu}{2} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 . \tag{5}$$

This modification of the original Lagrangian is valid as at optimality we have that $\mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$. The advantage of the augmented Lagrangian is that we can come up with a better update rule for $\mathbf{x}$ and $\boldsymbol{\lambda}$ as

$$\begin{cases} \mathbf{x}^{k+1} \in \operatorname{argmin}_{\mathbf{x}} \mathcal{L}_\mu(\mathbf{x}, \boldsymbol{\lambda}^k), \\ \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \mu(\mathbf{A}\mathbf{x}^{k+1} - \mathbf{b}). \end{cases} \tag{6}$$

Tasks: Given the optimization problem over $\mathbf{x} = (x_1, x_2)^\top$

$$\min_{\mathbf{x}} f(\mathbf{x}) := (x_1 - 1)^2 - x_1 x_2 \quad \text{s.t.} \quad -x_1 + 4 = x_2$$

With Pen & Paper:

1. Identify $\mathbf{A}$ and $\mathbf{b}$ and state the Lagrangian $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$ and the augmented Lagrangian $\mathcal{L}_\mu(\mathbf{x}, \boldsymbol{\lambda})$ for the given problem.
2. State the KKT-conditions with $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$. Solve for $\mathbf{x}^*$ and $\boldsymbol{\lambda}^*$.
3. Come up with the update rule for $\mathbf{x}$ in procedure Eq. (6).
4. Also state the KKT-conditions for $\mathcal{L}_\mu(\mathbf{x}, \boldsymbol{\lambda})$ and justify the update rule for $\boldsymbol{\lambda}$ in Eq. (6) by comparing with the original KKT-conditions.

In Python:

4. Plot the level sets of the above function using a filled contour plot[3] and draw the constraint on top.
5. Add a marker to the optimal solution which fulfills the original KKT-conditions (computed with Pen & Paper).
6. Implement both procedures in Eqs. (4) and (6) and compute the optimal solution iteratively. Test procedure Eq. (4) also with the augmented Lagrangian. Choose the same $(\mathbf{x}^0, \boldsymbol{\lambda}^0)$ for each method. Pick suitable values for τ and μ. Also choose an adequate number of iterations K.
7. Plot the points $\mathbf{x}_k$ over the iterations.
8. Plot the value of the used Lagrangian over the iterations for the three methods.
9. Discuss the convergence behavior of the three methods by plotting $\|\mathbf{x}^k - \mathbf{x}^*\|$ over the iterations k. Do you observe any problems?
10. Discuss the effects of picking different values for τ and μ. For the discussion on μ, you may also consider the definitness of $\nabla^2_{\mathbf{x}} \mathcal{L}_\mu(\mathbf{x}, \boldsymbol{\lambda})$.

[3] `https://matplotlib.org/stable/api/_as_gen/matplotlib.pyplot.contourf.html`

SOLUTION

1 Lagrange Multiplier Problem

a)

$$\min_{\mathbf{x}} x_2 - x_1 \quad \text{s.t.} \quad x_1 \geq 4x_2, \quad x_2 = \frac{1}{10}x_1^2 - 3.$$

a.1) Formulate the Lagrangian equation

$$\begin{aligned} f(x_1, x_2) &= x_2 - x_1 \\ g_1(x_1, x_2) &= -x_1 + 4x_2 \leq 0 \\ g_2(x_1, x_2) &= x_2 - \frac{1}{10}x_1^2 + 3 = 0 \end{aligned}$$

The Lagrangian for the problem is:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_2 - x_1 + \lambda_1(-x_1 + 4x_2) + \lambda_2\left(x_2 - \frac{1}{10}x_1^2 + 3\right),$$

where:

- $\lambda_1 \geq 0$ is the Lagrange multiplier for the inequality constraint g_1: $x_1 \geq 4x_2$.
- λ_2 is the multiplier for the equality constraint g_2: $x_2 = \frac{1}{10}x_1^2 - 3$.

a.2) Formulate the KKT conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= -1 - \lambda_1 + \lambda_2\left(-\frac{1}{5}x_1\right) = 0 && (1) \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 1 + 4\lambda_1 + \lambda_2 = 0 && (2) \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} &= -x_1 + 4x_2 \leq 0 && (3) \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} &= x_2 - \frac{1}{10}x_1^2 + 3 = 0 && (4) \\ \lambda_1(-x_1 + 4x_2) &= 0 && (5) \\ \lambda_1 &\geq 0 && (6) \end{aligned}$$

a.3) Discuss cases and compute solutions

We analyze the 2 cases based on condition (6):

****Case 1: $\lambda_1 = 0$ (Constraint g_1 is inactive)****
Substitute into condition (2):

$$1 + \lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = -1.$$

Substitute into condition (1):

$$-1 - 0 + (-1)\left(-\frac{1}{5}x_1\right) = 0$$

$$\Rightarrow x_1 = 5$$

Substitute into condition (4):

$$x_2 - \frac{1}{10}5^2 + 3 = 0$$

$$\Rightarrow x_2 = -0.5$$

The point c1 (5, -0.5) is valid because it satisfies the inequality constrain (3), thus satisfies all KKT conditions.

Case 2: $\lambda_1 > 0$ (Constraint g_1 is active)
From condition (5), we get:

$$\lambda_1(-x_1 + 4x_2) = 0 \quad \Rightarrow \quad x_1 - 4x_2 = 0 \quad \Rightarrow \quad x_1 = 4x_2$$

Substituting into condition (4), we get:

$$x_2 - \frac{1}{10}x_1^2 + 3 = 0$$
$$\Rightarrow x_2 - \frac{1}{10}(4x_2)^2 + 3 = 0$$

Solving this quadratic equation:

$$x_2 = \frac{5 \pm \sqrt{505}}{16}$$
$$\Rightarrow \quad x_{2,1} = 1.717, \quad x_{2,2} = -1.092$$

Substitute back into condition (5) above:

$$\Rightarrow \quad x_{1,1} = 6.868, \quad x_{1,2} = -4.368$$

So we have 2 points c2 (6.868, 1.717) and c3 (-4.368, -1.092). But we still have to check if they satisfies all conditions in order to be valid.

Substitute c2 (6.868, 1.717) into condition (1) and (2):

$$-1 - \lambda_1 + \lambda_2 \left(-\frac{1}{5} 6.868 \right) = 0$$
$$1 + 4\lambda_1 + \lambda_2 = 0$$

This is 2 equations with 2 unknowns. This can easily be solved and have the following result:

$$\lambda_1 = -0.083 \quad and \quad \lambda_2 = -0.667$$

$\lambda_1 < 0$, therefore condition (6) is not fullfilled, thus c2 (6.868, 1.717) is not a valid point.

Similarly, with c3 (-4.368, -1.092), we get the following:

$$\lambda_1 = -0.416 \quad and \quad \lambda_2 = 0.667$$

$\lambda_1 < 0$, therefore condition (6) is not fullfilled, thus c3 (-4.368, -1.092) is a also not valid point.

Summary: we get 3 points: c1 (5, -0.5), c2 (6.868, 1.717) and c3 (-4.368, -1.092), out of which only c1 satisfied all KKT conditions, making it the only valid point.

a.4) Optimal solution

Since there is only 1 valid point, the optimal solution is point c1 (5, -0.5)

b)

$$\min_{\mathbf{x}} \|\mathbf{x}\|_2^2 \quad \text{s.t.} \quad x_2 \geq 3 - x_1, \quad x_2 \geq 2.$$

b.1) Formulate the Lagrangian equation

$$f(x_1, x_2) = x_1^2 + x_2^2$$
$$g_1(x_1, x_2) = -x_2 + 3 - x_1 \leq 0$$
$$g_2(x_1, x_2) = -x_2 + 2 \leq 0$$

The Lagrangian for the problem is:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + \lambda_1\big(- x_2 + 3 - x_1\big) + \lambda_2(-x_2 + 2),$$

where:

- $\lambda_1, \lambda_2 \geq 0$ are the multipliers for the inequality constraints.

b.2) Formulate the KKT conditions

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 2x_1 - \lambda_1 = 0 && (1)\\
\frac{\partial \mathcal{L}}{\partial x_2} &= 2x_2 - \lambda_1 - \lambda_2 = 0 && (2)\\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= -x_2 + 3 - x_1 \leq 0 && (3)\\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= -x_2 + 2 \leq 0 && (4)\\
\lambda_1 \cdot \big(-x_2 + 3 - x_1\big) &= 0 && (5)\\
\lambda_2 \cdot (-x_2 + 2) &= 0 && (6)\\
\lambda_1, \lambda_2 &\geq 0 && (7)
\end{aligned}
$$

b.3) Discuss cases and compute solutions

We analyze the cases based on the activity of the constraints (λ_1, λ_2) (condition (7))

****Case 1:** $\lambda_1 = 0, \lambda_2 = 0$ **(Both constraints inactive)****

Substitute λ_1 and λ_2 into condition (2):

$$2x_2 = 0 \quad \Rightarrow \quad x_2 = 0.$$

This x_2 violates the condition (4) $(0 + 2 \leq 0\ )$. This case is therefore invalid!

****Case 2:** $\lambda_1 > 0, \lambda_2 = 0$ **(Constraint** g_1 **active,** g_2 **inactive)****
From condition (5):

$$-x_2 + 3 - x_1 = 0 \quad \Rightarrow \quad x_2 = 3 - x_1.$$

Substitute into condition (2):

$$2x_2 - \lambda_1 - \lambda_2 = 0 \quad \Rightarrow \quad 2(3 - x_1) - \lambda_1 - 0 = 0 \quad \Rightarrow \quad x_1 = \frac{6 - \lambda_1}{2}$$

Substitute into condition (1):

$$2x_1 - \lambda_1 = 0 \quad \Rightarrow \quad (6 - \lambda_1) - \lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = 3$$

$$
\begin{aligned}
x_1 = \frac{6 - \lambda_1}{2} \quad &\Rightarrow \quad x_1 = 1.5\\
x_2 = 3 - x_1 \quad &\Rightarrow \quad x_2 = 1.5
\end{aligned}
$$

This x_2 does not fullfill the condition (4) $(-1.5 + 2 \leq 0)$. This case is therefore invalid!

****Case 3:** $\lambda_1 = 0, \lambda_2 > 0$ **(Constraint** g_1 **inactive,** g_2 **active)****
From condition (6):

$$-x_2 + 2 = 0 \quad \Rightarrow \quad x_2 = 2$$

Substitute into condition (2):

$$2(2) - 0 - \lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = 4.$$

Substitute $\lambda_1 = 0$ into condition (1):

$$2x_1 - \lambda_1 = 0 \quad \Rightarrow \quad x_1 = 0$$

Substitute x_1 and x_2 into condition 3:

$$-x_2 + 3 - x_1 \leq 0 \quad \Rightarrow \quad -2 + 3 - 0 \leq 0 \quad \Rightarrow \quad 1 \leq 0$$

So these x_1 and x_2 violates condition (3). This solution is therefore invalid!

****Case 4: Both constraints active ($\lambda_1 > 0, \lambda_2 > 0$)****
From condition (5) and (6):

$$-x_2 + 3 - x_1 = 0 \quad and \quad -x_2 + 2 = 0$$

$$\Rightarrow x_1 = 1 \quad and \quad x_2 = 2$$

Substitute into condition (1) and (2):

$$2x_1 - \lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = 2$$

$$2x_2 - \lambda_1 - \lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = 2$$

These 2 Lambda satisfy the condition (7) (both positive). The point $(x_1, x_2) = (1, 2)$ therefore satisfies all constraints and is valid.

Summary: out of 4 case, only the case, where both Lambda are positive, gives a valid point, making the point (1,2) the only valid point

b.4) Optimal solution

Since there is only 1 valid point, the optimal solution is point (1, 2)

c)

$$\min_{\mathbf{x}} (x_1 - 1)^2 + x_1 x_2^2 - 2 \quad \text{s.t.} \quad x_1^2 + x_2^2 \leq 4.$$

c.1) Formulate the Lagrangian equation

$$f(x_1, x_2) = (x_1 - 1)^2 + x_1 x_2^2 - 2$$
$$g(x_1, x_2) = x_1^2 + x_2^2 - 4 \leq 0$$

The Lagrangian for the problem is:

$$\mathcal{L}(x_1, x_2, \lambda_1) = (x_1 - 1)^2 + x_1 x_2^2 - 2 + \lambda\big(x_1^2 + x_2^2 - 4\big),$$

where:

- $\lambda \geq 0$ is the multiplier for the inequality constraint.

c.2) Formulate the KKT conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2(x_1 - 1) + x_2^2 + 2\lambda x_1 = 0 \qquad (1)$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_1 x_2 + 2\lambda x_2 = 0 \qquad (2)$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_1^2 + x_2^2 - 4 \leq 0 \qquad (3)$$
$$\lambda \cdot (x_1^2 + x_2^2 - 4) = 0 \qquad (4)$$
$$\lambda \geq 0 \qquad (5)$$

c.3) Discuss cases and compute solutions

We analyze the cases based on the activity of the constraint λ (condition (5))

****Case 1: $\lambda = 0$ (Constraint inactive)****

From condition (2):

$$2x_1x_2 + 0 = 0 \quad \Rightarrow \quad x_2 = 0 \quad \text{or} \quad x_1 = 0.$$

- Case 1.1: $x_1 = 0$: Substitute into condition (1):

$$2(0-1) + x_2^2 + 0 = 0 \quad \Rightarrow \quad x_2 = \pm\sqrt{2}.$$

This gives 2 points: c1 (0, $\sqrt{2}$) and c2 (0, $-\sqrt{2}$). Both these points satisfy condition (3), making them satisfy all conditions of KKT, so they are both valid points

- Case 1.2: $x_2 = 0$: Substitute into condition (1):

$$2(x_1 - 1) + 0 + 0 = 0 \quad \Rightarrow \quad x_1 = 1.$$

This give 1 point c3 (1, 0), which also satisfy condition (3), making it also a valid point

****Case 2: $\lambda > 0$ (Constraint active)****
From condition (4):

$$x_1^2 + x_2^2 - 4 = 0$$

From condition (2):

$$2x_1x_2 + 2\lambda x_2 = 0 \quad \Rightarrow \quad x_2(x_1 + \lambda) = 0.$$

This gives two subcases:
- Case 2.1: $\mathbf{x_2 = 0}$: Substitute into condition (4) above

$$x_1^2 + x_2^2 - 4 = 0 \quad \Rightarrow \quad x_1^2 = 4 \quad \Rightarrow \quad x_1 = \pm 2.$$

This gives 2 points c4 (2, 0) and c5 (-2, 0). Both satisfy condition (3).
Substitute c4 (2, 0) into condition (1):

$$2(2-1) + 0 + 2\lambda 2 = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{2}$$

But this violates condition (5) ($\lambda \geq 0$), so point c4 (2, 0) is invalid.
Substitute c5 (-2, 0) into condition (1):

$$2(-2-1) + 0 + 2\lambda(-2) = 0 \quad \Rightarrow \quad \lambda = -\frac{3}{2}$$

Again, this violates condition (5) ($\lambda \geq 0$), so point c5 (-2, 0) is also invalid.
- Case 2.2: $\mathbf{x_1} + \lambda = \mathbf{0}$: Which means:

$$\lambda = -x_1.$$

Substitute into condition (1):

$$2(x_1 - 1) + x_2^2 + 2(-x_1)x_1 = 0 \quad \Rightarrow \quad x_2^2 = 2x_1^2 - 2x_1 + 2.$$

Substitute this into condition (4) above:

$$x_1^2 + x_2^2 - 4 = 0 \quad \Rightarrow \quad x_1^2 + (2x_1^2 - 2x_1 + 2) - 4 = 0 \quad \Rightarrow \quad 3x_1^2 - 2x_1 - 2 = 0.$$

Solve this quadratic equation:

$$x_1 = \frac{2 \pm \sqrt{4 + 24}}{6} = \frac{1 \pm \sqrt{7}}{3}.$$

$$\Rightarrow x_{1.1} = -0.548 \quad and \quad x_{1.2} = 1.215$$

For $x_{1.2}$, λ is:

$$\lambda = -x_{1.2} = -1.215$$

This violates the condition (5) ($\lambda \geq 0$), therefore $x_{1.2}$ is invalide:

$$\Rightarrow x_1 = x_{1.1} = -0.548$$

$$\Rightarrow \lambda = -x_1 = 0.548$$

Substitute x_1 into condition (4) above:

$$x_1^2 + x_2^2 - 4 = 0 \quad \Rightarrow \quad x_2 = \pm\sqrt{4 - x_1^2}$$

$$\Rightarrow x_{2.1} = 1.923 \quad \textit{and} \quad x_{2.2} = -1.923$$

So we have 2 points c6 (-0.548, 1.923) and c7 (-0.548, -1.923), both of which satisfy all KKT conditions, making them 2 valid points

Summary: we found 5 valid points: c1 (0, $\sqrt{2}$), c2 (0, $-\sqrt{2}$), c3 (1, 0), c6 (-0.548, 1.923) and c7 (-0.548, -1.923),

c.4) Optimal solution

Compare the objective function $f(x_1, x_2)$ at all valid points:

$$f(1, 0) = (1-1)^2 + 1 \cdot 0^2 - 2 = -2.$$

$$f(0, \sqrt{2}) = (0-1)^2 + 0 \cdot (\sqrt{2})^2 - 2 = 1 - 2 = -1.$$

$$f(0, -\sqrt{2}) = (0-1)^2 + 0 \cdot (-\sqrt{2})^2 - 2 = 1 - 2 = -1.$$

$$f(-0.548, 1.923) = (-0.548-1)^2 + (-0.548)(1.923)^2 - 2 = -1.63$$

$$f(-0.548, -1.923) = (-0.548-1)^2 + (-0.548)(-1.923)^2 - 2 = -1.63$$

The optimal solution is $(x_1, x_2) = (1, 0)$ with $f(x_1, x_2) = -2$.

2 Augmented Lagrangian

1.Define A, b, and state the Lagrangian $\alpha(x,\lambda)$ and augmented Lagrangian $\alpha_\mu(x,\lambda)$

Given optimization problem:

$$\min_x f(x) = (x_1-1)^2 - x_1x_2 \quad \text{sit. } -x_1+4 = x_2$$

With the constraint written as:

$$\begin{aligned} Ax-b=0 \quad &\Rightarrow -x_1-x_2+4=0 \\ &A = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad b=-4 \end{aligned}$$

The general form of the Lagrangian is:

$$\alpha(x,\lambda) = f(x) + \lambda^\top(Ax-b)$$

with $f(x) = (x_1-1)^2 - x_1x_2$ and $A = \begin{bmatrix} -1 & -1 \end{bmatrix}, b=-4$ we get:

$$\alpha(x,\lambda) = (x_1-1)^2 - x_1x_2 + \lambda(-x_1-x_2+4)$$

The augmented Lagrangian $\alpha_\mu(x,\lambda)$ is:

$$\alpha_\mu(x,\lambda) = \alpha(x,\lambda) + \frac{\mu}{2}\|Ax-b\|_2^2$$

with $Ax-b = -x_1-x_2+4$

$$\alpha_\mu(x,\lambda) = (x_1-1)^2 - x_1x_2 + \lambda(-x_1-x_2+4) + \frac{\mu}{2}(-x_1-x_2+4)^2.$$

2. KKT-conditions

Stationarity (for the general Lagrangian):

$$\begin{gathered} \nabla_x\alpha(x_1\lambda) = 0 \\ \alpha(x,\lambda) = (x_1-1)^2 - x_1x_2 + \lambda(-x_1-x_2+4) \\ \nabla_x\alpha(x,\lambda) = \begin{pmatrix} \frac{\partial\alpha}{\partial x_1} \\ \frac{\partial\alpha}{\partial x_2} \end{pmatrix} \\ \frac{\partial\alpha}{\partial x_1} = 2(x_1-1) - x_2 - \lambda \\ \frac{\partial\alpha}{\partial x_2} = -x_1 - \lambda \end{gathered}$$

- We set $\nabla_x\alpha(x,\lambda) = 0$

$$\begin{aligned} &2(x_1-1) - x_2 - \lambda = 0 \Rightarrow 2x_1 - 2 - x_2 - \lambda = 0 \\ &-x_1 - \lambda = 0 \end{aligned}$$

Primal feasibility:

$$Ax-b=0$$

Solve for x^* and λ^* :

$$\begin{aligned} &2x_1 - 2 - x_2 - \lambda = 0 \\ &-x_1 - \lambda = 0 \Rightarrow \lambda = -x_1 \\ &2x_1 - 2 - x_2 + x_1 = 0 \\ &3x_1 - x_2 = 2 \end{aligned}$$

The constraint also holds:

$$\begin{array}{c} -x_1 - x_2 = -4 \\ 3x_1 - x_2 = 2 \mid \cdot(-1) \\ \hline -x_1 - x_2 = -4 \\ -3x_1 + x_2 = -2 \\ \hline -4x_1 = -6 \\ x_1 = \frac{6}{4} = \frac{3}{2} \\ x_2 = -x_1 + 4 \\ x_2 = -\frac{3}{2} + 4 = \frac{-3+8}{2} = \frac{5}{2} \\ \lambda = -x_1 \Rightarrow \lambda = -\frac{3}{2} \\ x^* = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix} \quad \lambda^* = -\frac{3}{2} \end{array}$$

3. Update rule for x

$$x^{k+1} \in \arg\min_x \alpha_\mu\left(x, \lambda^k\right)$$

with

$$\alpha_\mu\left(x, \lambda^k\right) = (x_1 - 1)^2 - x_1 x_2 + \lambda^k\left(-x_1 - x_2 + 4\right) + \frac{\mu}{2}\left(-x_1 - x_2 + 4\right)^2$$
$$(-x_1 - x_2 + 4)^2 = (x_1 + x_2 - 4)^2 = x_1^2 + 2x_1x_2 + x_2^2 - 8x_1 - 8x_2 + 16$$
$$\alpha_\mu\left(x, \lambda^k\right) = (x_1 - 1)^2 - x_1 x_2 + \lambda^k\left(-x_1 - x_2 + 4\right) + \frac{\mu}{2}\left(x_1^2 + 2x_1x_2 + x_2^2 - 8x_1 - 8x_2 + 16\right)$$

With $\nabla_x \alpha_\mu\left(x, \lambda^k\right) = 0$

$$\nabla_x \alpha_\mu\left(x, \lambda^k\right) = \begin{pmatrix} \frac{\partial \alpha_\mu}{\partial x_1} \\ \frac{\partial \alpha_\mu}{\partial x_2} \end{pmatrix}$$
$$\frac{\partial \alpha_\mu}{\partial x_1} = 2\left(x_1 - 1\right) - x_2 - \lambda^t + \mu x_1 + \mu x_2 - 4\mu$$
$$\frac{\partial \alpha_\mu}{\partial x_2} = -x_1 - \lambda^k + \mu x_1 + \mu x_2 - 4\mu$$

Stationarity Conditions:

$$\frac{\partial \alpha_\mu}{\partial x_1} = 0; \quad \frac{\partial \alpha_\mu}{\partial x_2} = 0$$

$$\begin{aligned}
&2\left(x_1 - 1\right) - x_2 - \lambda^k + \mu x_1 + \mu x_2 - 4\mu = 0 \\
&-x_1 - \lambda^k + \mu x_1 + \mu x_2 - 4\mu = 0 \\
&x_1(\mu - 1) + \mu x_2 = 4\mu + \lambda^k \\
&x_2 = \tfrac{4\mu + \lambda^k - x_1(\mu - 1)}{\mu} \\
&2x_1 - 2 + x_2(\mu - 1) - \lambda^k - \mu x_1 - 4\mu = 0 \\
&2x_1 - 2 + \tfrac{4\mu + \lambda^k - x_1(\mu - 1)}{\mu^\mu}(\mu - 1) - \lambda^k - \mu x_1 - 4\mu = 0 \\
&2x_1 - 2 + \left(4 + \tfrac{\lambda^k}{\mu} - x_1 + \tfrac{x_1}{\mu}\right)(\mu - 1) - \lambda^k - \mu x_1 - 4\mu = 0 \\
&2x_1 - 2 + 4\mu - 4 + \lambda^k - \tfrac{\lambda^k}{\mu} - \mu x_1 + x_1 + x_1 - \tfrac{x_1}{\mu} - \lambda^k - \mu x_1 - 4\mu = 0 \\
&3x_1 - 6 - \tfrac{\lambda^k}{\mu} - 2\mu x_1 - \tfrac{x_1}{\mu} = 0
\end{aligned}$$

$$x_1\left(3 - 2\mu - \tfrac{1}{\mu}\right) - 6 - \tfrac{\lambda^k}{\mu} = 0$$

for simplicity we use: $\mu = 1, \lambda^0 = 0$ [the first iteration]

$$\text{in } \frac{\partial \alpha_\mu}{\partial x_1} = 0 : 2x_1 - 2 - x_2 + (x_1 + x_2 - 4) = 0$$
$$3x_1 = 6 \Rightarrow x_1 = 2$$

into the constraint

$$
\begin{aligned}
-x_1 - x_2 + 4 &= 0 \\
-2 - x_2 + 4 &= 0 \\
x_2 &= 2
\end{aligned}
$$

$$
\begin{aligned}
x^{k+1} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ for } k = 0 \\
\lambda^{k+1} &= \lambda^k + \mu \left(-x_1^{k+1} - x_2^{k+1} + 4\right) \\
\lambda^{k+1} &= 0 + 1(-2 - 2 + 4) \\
\lambda^{k+1} &= 0
\end{aligned}
$$