Lecture 9
Floating Point

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3. Arithmetic for Computers

3.5 Floating Point

- **Real numbers**

  $3.14159265\ldots_{ten}$ (pi), $2.71828\ldots_{ten}$ (e),

  0.000000001_{ten} , $0.1_{ten} \times 10^{-8}$ or $1.0_{ten} \times 10^{-9}$,

  $3{,}155{,}760{,}000_{ten}$, $0.00315576 \times 10^{12}$ or 3.15576×10^{9}

- **Scientific notation**
  - A notation that renders numbers with a single digit to the left of the decimal point

  $1.0_{ten} \times 10^{-9}$, 3.15576×10^{9}

- **Normalized number**
  - A number in scientific notation that has no leading 0s

  $1.0_{ten} \times 10^{-9}$, 3.15576×10^{9}

Floating-point Representation

- **Floating point numbers in binary form**

  $\pm 1.\mathrm{xxxxxxxxx}_{\mathrm{two}} \times 2^{\mathrm{yyyy}}$

- **Sign**
  - 1 bit
- **Exponent**
  - 8 bits (including the sign of the exponent)
- **Fraction (=significand=mantissa)**
  - 23 bits, fraction
  - sign and magnitude representation

31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
s	exponent								fraction																						

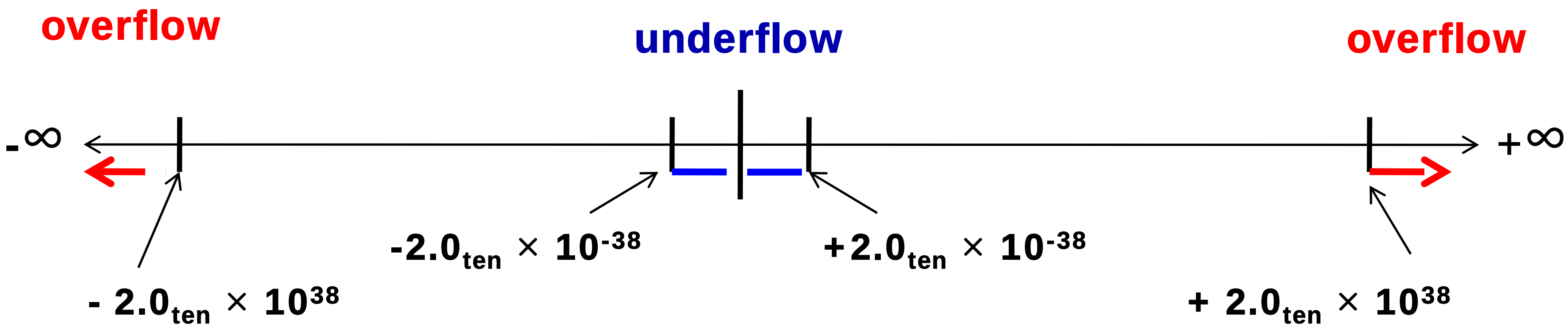
Floating Point Numbers

- **General form of floating-point numbers**

  $(-1)^S \times F \times 2^E$

- **Tradeoff between accuracy and range**
  - Large significand … increased accuracy
  - Large exponent … increased range of numbers
- **Range of floating-point numbers in MIPS**

  $\mathbf{2.0_{ten} \times 10^{-38} \sim 2.0_{ten} \times 10^{38}}$



ANSI / IEEE Std 754-1985

- **IEEE standard for binary floating-point arithmetic**
- **Hidden-bit scheme**

  $(-1)^s \times (1 + \text{fraction}) \times 2^E$

  $= (-1)^s \times (1 + (s1 \times 2^{-1}) + (s2 \times 2^{-2}) + (s3 \times 2^{-3}) + (s4 \times 2^{-4}) + \ldots) \times 2^E$
- In this book,
  - significand … represent the 24/53-bit number that is 1 plus the fraction
  - fraction … represent the 23/52-bit number
- **32-bit single format**
  - 1-bit sign, 8-bit exponent, 23-bit fraction
- **64-bit double format**
  - 1-bit sign, 11-bit exponent, 52-bit fraction

Double and Quad Precision

- **Double precision floating-point number**
  - 11 exponent bits
  - 52 fraction bits

$$2.0_{ten} \times 10^{-308} \sim 2.0_{ten} \times 10^{308}$$

31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
s	exponent											fraction																			

fraction(continued)

- **Quad precision floating-point number**
  - IEEE 754-2008 binary128 standard
  - 15 exponent bits
  - 112 significand bits

IEEE 754 Encoding

Single precision		Double precision		Object represented
Exponent	Fraction	Exponent	Fraction	
0	0	0	0	0.0
0	Nonzero	0	Nonzero	± denormalized number
1~254	Anything	1~2046	Anything	± floating point number
255	0	2047	0	±infinity
255	Nonzero	2047	Nonzero	NAN (Not A Number)

Figure 3.13

Sorting Floating Point Numbers

- Keep encoding that is somewhat compatible with 2’s complement
  - e.g., 0.0 in FP is 0 in two’s complement
  - Can compare two FP numbers in the same way as comparing 2’s complement integers

- Placing the sign in the most significant bit
- Placing exponent before the significand
- But what with the negative exponents ?

Example: 2's complement exponents

- $1.0_{two} \times 2^{-1}$

  0 11111111 00000000000000000000000

- $1.0_{two} \times 2^{+1}$

  0 00000001 00000000000000000000000

- $1.0_{two} \times 2^{-1}$ **looks like a bigger one than** $1.0_{two} \times 2^{+1}$

  Undesirable

Biased Notation

- Can reuse integer comparison hardware
  - If the most negative exponent = 00…000
  - and the most positive exponent = 11...111
- $(-1)^{\text{Sign}} \times (1 + \text{Fraction}) \times 2^{(\text{Exponent - } \textbf{Bias})}$
- **Exponent biases in IEEE 754**
  - 127 for single precision

    $-126 \leq \text{exponent} \leq +127$
  - 1023 for double precision

    $-1022 \leq \text{exponent} \leq +1023$

Biased Exponent with Bias=127

How it is interpreted | **How it is encoded**

	Decimal Exponent	signed 2's complement	Biased Notation	Decimal Value of Biased Notation
∞, NaN	For infinities		11111111	255
	127	01111111	11111110	254
	...	...	...	...
	2	00000010	10000001	129
Getting closer to zero	1	00000001	10000000	128
	0	00000000	01111111	127
	-1	11111111	01111110	126
	-2	11111110	01111101	125
	...	...	...	...
	-126	10000010	00000001	1
Zero	For Denorms	10000001	00000000	0

Example: Floating-Point Representation

- **Show the IEEE 754 single and double precision representations of -0.75_{ten} .**

[Answer]

- $-0.75_{ten} = -0.11_{two} = -1.1_{two} \times 2^{-1}$
- Single: exponent = -1 + bias = -1 + 127 = 126

  $(-1)^1 \times (1 + .1000...00) \times 2^{(126-127)}$

  = 1 01111110 10000000000000000000000
- Double: exponent = -1 + bias = -1 + 1023 = 1022

  $(-1)^1 \times (1 + .1000...00) \times 2^{(1022-1023)}$

  = 1 01111111110 10000000000000000000000000000000000000000000

Example: Converting Binary to Decimal Floating Point

- **What decimal number is represented by**

1 10000001 01000000000000000000000 ?

[Answer]

$$(-1)^{S} \times (1 + \text{Significand}) \times 2^{(\text{Exponent - Bias})}$$

$$= (-1)^{1} \times (1 + 0.25) \times 2^{(129-127)}$$

$$= -1 \times 1.25 \times 2^{2}$$

$$= -1.25 \times 4$$

$$= -5.0$$

Floating-Point Addition

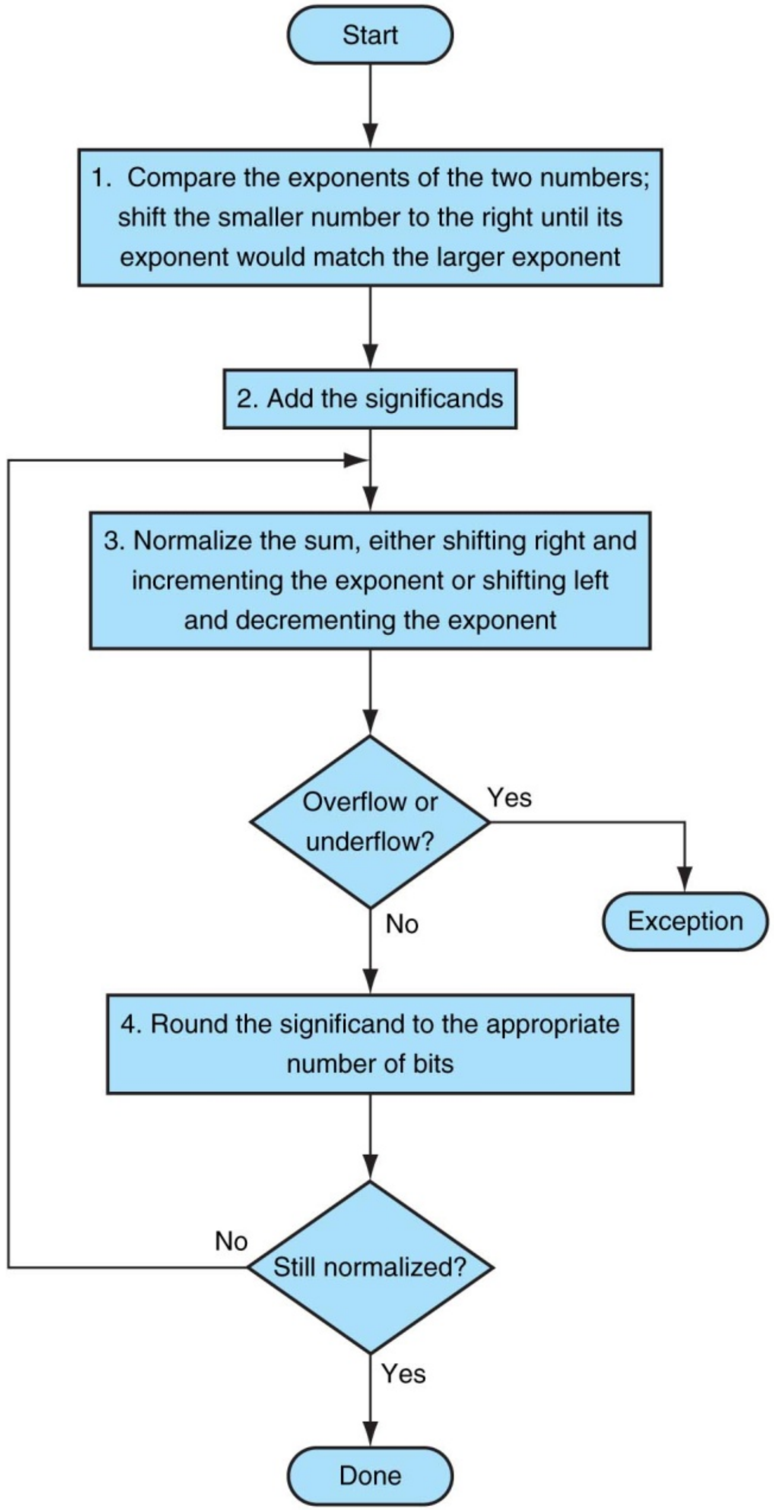


Figure 3.14

Floating Point Addition

- Addition (and subtraction)

$$(\pm F1 \times 2^{E1}) + (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

- Step 0: Restore the hidden bit in F1 and in F2
- Step 1: Align fractions by right shifting F2 by E1 - E2 positions (assuming $E1 \geq E2$) keeping track of (three of) the bits shifted out in G R and S
- Step 2: Add the resulting F2 to F1 to form F3
- Step 3: Normalize F3 (so it is in the form 1.XXXXX …)
  - If F1 and F2 have the same sign $\rightarrow$ F3 $\in [1,4) \rightarrow$ 1 bit right shift F3 and increment E3 (check for overflow)
  - If F1 and F2 have different signs $\rightarrow$ F3 may require *many* left shifts each time decrementing E3 (check for underflow)
- Step 4: Round F3 and possibly normalize F3 again
- Step 5: Rehide the most significant bit of F3 before storing the result

Example: Floating Point Addition

- Add

$$(0.5 = 1.0000 \times 2^{-1}) + (-0.4375 = -1.1100 \times 2^{-2})$$

  - Step 0:
  - Step 1:
  - Step 2:
  - Step 3:
  - Step 4:
  - Step 5:

Example: Floating Point Addition

- Add

$$(0.5 = 1.0000 \times 2^{-1}) + (-0.4375 = -1.1100 \times 2^{-2})$$

- Step 0: Hidden bits restored in the representation above
- Step 1: Shift significand with the smaller exponent (1.1100) right until its exponent matches the larger exponent (so once)
- Step 2: Add significands

  $1.0000 + (-0.111) = 1.0000 – 0.111 = 0.001$
- Step 3: Normalize the sum, checking for exponent over/underflow

  $0.001 \times 2^{-1} = 0.010 \times 2^{-2} = .. = 1.000 \times 2^{-4}$
- Step 4: The sum is already rounded, so we're done
- Step 5: Rehide the hidden bit before storing

  0 01111011 00000000000000000000000

Floating-Point Adder

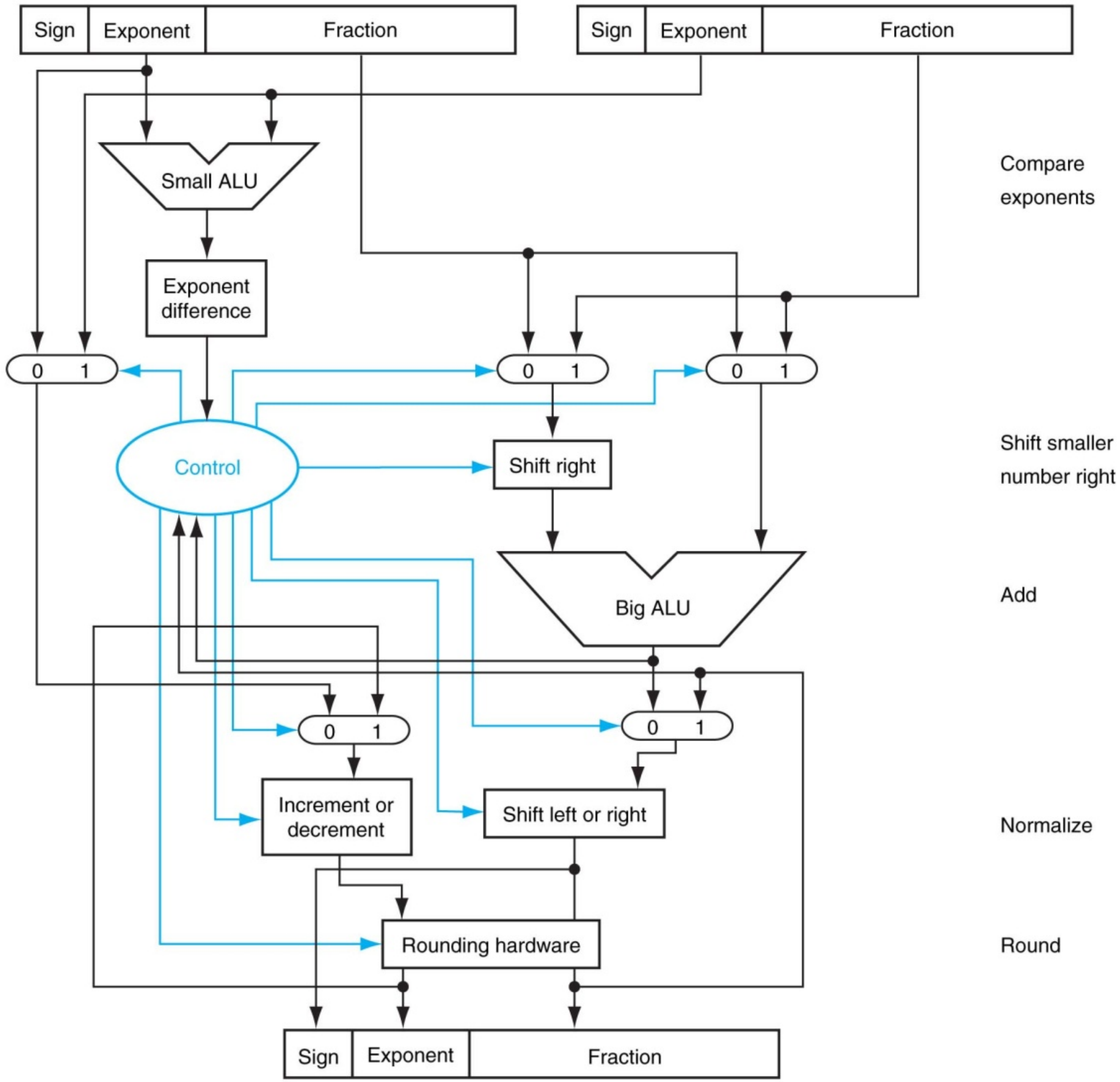


Figure 3.15

Floating-Point Multiplication

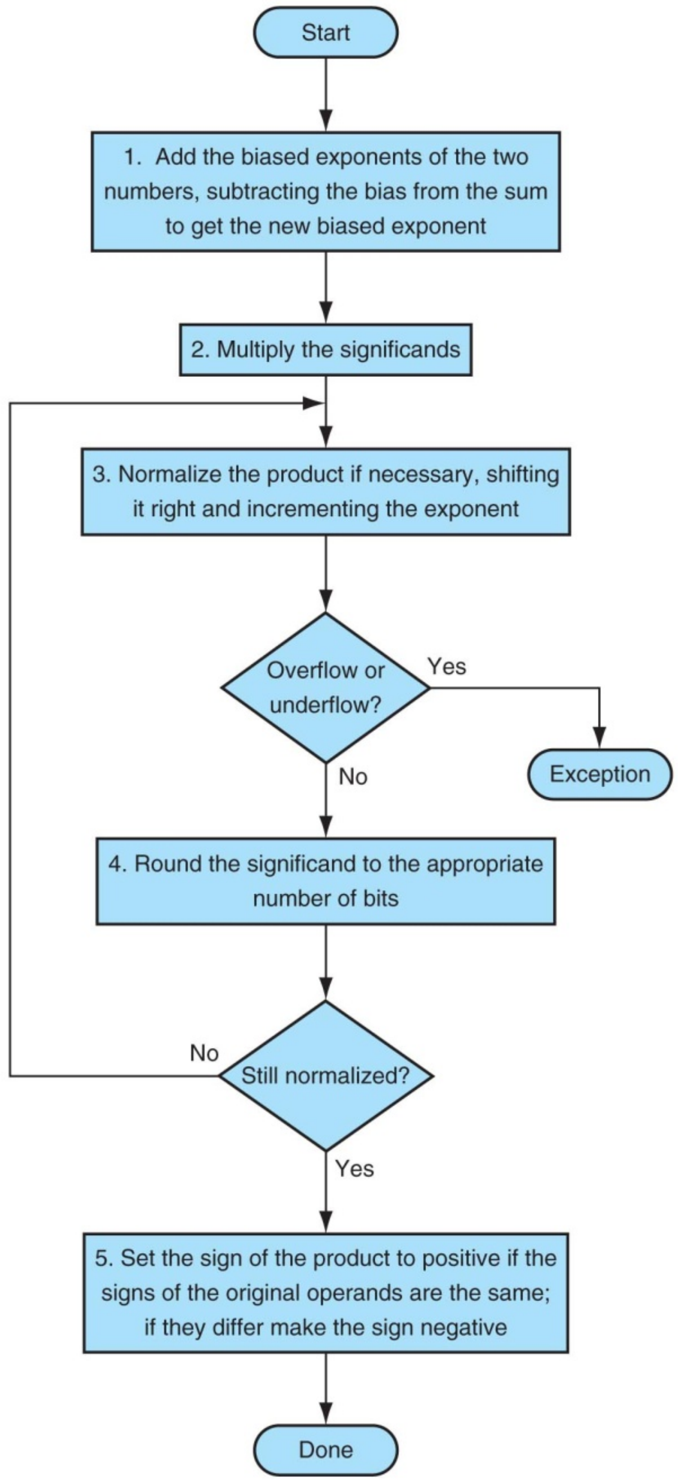


Figure 3.16

Floating Point Multiplication

- Multiplication

$$(\pm F1 \times 2^{E1}) \text{ x } (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

  - Step 0: Restore the hidden bit in F1 and in F2
  - Step 1: Add the two (biased) exponents and subtract the bias from the sum, so E1 + E2 – 127 = E3

    also determine the sign of the product (which depends on the sign of the operands (most significant bits))
  - Step 2: Multiply F1 by F2 to form a double precision F3
  - Step 3: Normalize F3 (so it is in the form 1.XXXXX …)
    - Since F1 and F2 come in normalized $\rightarrow$ F3 $\in [1,4)$ $\rightarrow$ 1 bit right shift F3 and increment E3
    - Check for overflow/underflow
  - Step 4: Round F3 and possibly normalize F3 again
  - Step 5: Rehide the most significant bit of F3 before storing the result

Example: Floating Point Multiplication

- Multiply

$$(0.5 = 1.0000 \times 2^{-1}) \text{ x } (-0.4375 = -1.1100 \times 2^{-2})$$

  - Step 0:
  - Step 1:
  - Step 2:
  - Step 3:
  - Step 4:
  - Step 5:

Example: Floating Point Multiplication

- Multiply

$$(0.5 = 1.0000 \times 2^{-1}) \text{ x } (-0.4375 = -1.1100 \times 2^{-2})$$

- Step 0: Hidden bits restored in the representation above
- Step 1: Add the exponents (not in bias would be -1 + (-2) = -3 and in bias would be (-1+127) + (-2+127) – 127 = (-1 -2) + (127+127-127) = -3 + 127 = 124
- Step 2: Multiply the significands

  1.0000 x 1.110 = 1.110000
- Step 3: Normalized the product, checking for exp over/underflow

  1.110000 x 2^{-3} is already normalized
- Step 4: The product is already rounded, so we're done
- Step 5: Rehide the hidden bit before storing

  1 01111100 11000000000000000000000

Floating-Point Instructions

Category	Instruction	MIPS	ARM
Arithmetic	FP add single	add.s $f2,$f4,$f6	FADDS s2,s4,s6
	FP subtract single	sub.s $f2,$f4,$f6	FSUBS s2,s4,s6
	FP multiply single	mul.s $f2,$f4,$f6	FMULS s2,s4,s6
	FP divide single	div.s $f2,$f4,$f6	FDIVS s2,s4,s6
	FP add double	add.d $f2,$f4,$f6	FADDD d2,d4,d6
	FP subtract double	sub.d $f2,$f4,$f6	FSUBD d2,d4,d6
	FP multiply double	mul.d $f2,$f4,$f6	FMULD d2,d4,d6
	FP divide double	div.d $f2,$f4,$f6	FDIVD d2,d4,d6
Data transfer	FP load single	lwc1 $f1,4($s2)	FLDS S1,[R1,#100]
	FP store single	swc1 $f1,4($s2)	FSTS S1,[R1,#100]
	FP load double		FLDD d1,[R1,#100]
	FP store double		FSTD d1,[R1,#100]
Compare	FP compare single	c.x.s	FCMPS s2,s4
	FP compare double	c.x.d	FCMPD d2,d4
	FP move status		FMSTAT
	conditional branch	bclt 100, bclf 100	

3.9 Concluding Remarks

	SPECint	SPECfp
addu	5.2%	3.5%
addiu	9.0%	7.2%
or	4.0%	1.2%
sll	4.4%	1.9%
lui	3.3%	0.5%
lw	18.6%	5.8%
sw	7.6%	2.0%
lbu	3.7%	0.1%
beq	8.6%	2.2%
bne	8.4%	1.4%
slt	9.9%	2.3%
slti	3.1%	0.3%
sltu	3.4%	0.8%

	SPECint	SPECfp
add.d	0.0%	10.6%
sub.d	0.0%	4.9%
mul.d	0.0%	15.0%
add.s	0.0%	1.5%
sub.s	0.0%	1.8%
mul.s	0.0%	2.4%
l.d	0.0%	17.5%
s.d	0.0%	4.9%
l.s	0.0%	4.2%
s.s	0.0%	1.1%
lhu	1.3%	0.0%

Homework #1

- **Posted on e-Campus.**
- **Due**
  - (가) 10/7 (월) 1:30 PM
  - (나)

Supplement

Accurate Arithmetic

- **Guard bit**
  - Used to provide one fraction bit when shifting left to normalize a result
  - e.g., when normalizing fraction after division or subtraction
- **Round bit**
  - Used to improve rounding accuracy
- **Sticky bit**
  - Used to support Round to nearest even
  - $0.50...00_{ten}$ vs. $0.50...01_{ten}$
  - Is set to a 1 whenever a 1 bit shifts (right) through it
  - e.g., when aligning fraction during addition/subtraction

F = 1 . xxxxxxxxxxxxxxxxxxxxxxx G R S

Example: Rounding with Guard Digits

- Add $2.56_{ten} \times 10^0$ to $2.34_{ten} \times 10^2$.
- Significant digit = 3 decimal digits
- **Round to the nearest number with and without guard and round digits.**

[Answer]

1) Without guard and round digits

$$2.34 \times 10^2 + 0.02 \times 10^2 = 2.36 \times 10^2$$

2) With guard and round digits

2.56×10^0 -> 0.0256×10^2 (guard = 5, round = 6)

$$2.3400 \times 10^2 + 0.0256 \times 10^2 = 2.3656 \times 10^2$$

=> rounded to 2.37×10^2

Example: Prob. 7 of 2010-1 Midterm Exam

- **Add $x=1.110\ 0000\ 0000\ 0000\ 0011\ 0001_{two}$ X 2^{-1} to $y=1.01_{two}$ X 2^5. Use G, R and S bits.**

[Answer]

- x = 0.000 0011 1000 0000 0000 0000 11 0001 X 2^5
  = 0.000 0011 1000 0000 0000 0000 X 2^5 (G=1, R=1, S=1)

	fraction	G	R	S
x	0.000 0011 1000 0000 0000 0000	1	1	1
y	1.010 0000 0000 0000 0000 0000	0	0	0
X+y	1.010 0011 1000 0000 0000 0000	1	1	1

- **x+y =** 1.010 0011 1000 0000 0000 0001 X 2^5

0	**1000 0100**	**010 0011 1000 0000 0000 0001**

Elaboration

- **Rounding modes in IEEE 754**
  1) Always round up
     - Toward +∞
  2) Always round down (toward -∞)
     - Toward -∞
  3) Truncate
     - Toward 0
     - Round down if positive, up if negative
  4) Round to nearest even
     - When the Guard || Round || Sticky are 100
     - Always creates a 0 in the least significant bit of fraction

Examples

	7.3	7.5	8.5	7.7	-7.3	-7.5	-8.5	-7.7
Round up	8	8	9	8	-7	-7	-8	-7
Round down	7	7	8	7	-8	-8	-9	-8
Truncate	7	7	8	7	-7	-7	-8	-7
Round to nearest even	7	8	8	8	-7	-8	-8	-8

	+0001.01	-0001.01	+0101.10	+0100.10	-0011.10
Round up	+0010	-0001	+0110	+0101	-0011
Round down	+0001	-0010	+0101	+0100	-0100
Truncate	+0001	-0001	+0101	+0100	-0011
Round to nearest even	+0001	-0010	+0110	+0100	-0100