$Assignment\ 1$

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MATH 271 - Discrete Mathematics

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Question 1

(a)

- $\mathcal{P}$ is true.
- **Proof** Suppose x and y are real numbers such that $x - \lfloor x \rfloor < \frac{1}{2}$ and $y - \lfloor y \rfloor < \frac{1}{2}$. We show that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

  Let

  $$a = x - \lfloor x \rfloor < \frac{1}{2}$$

  $$b = y - \lfloor y \rfloor < \frac{1}{2}$$

  Then

  $$a + b < \frac{1}{2} + \frac{1}{2} = 1$$

  Thus

  $$\lfloor a + b \rfloor = 0$$

  Since $\lfloor x \rfloor + \lfloor y \rfloor \in \mathbb{Z}$, we have

  $$\begin{aligned} \lfloor x + y \rfloor &= \lfloor a + \lfloor x \rfloor + b + \lfloor y \rfloor \rfloor \\ &= \lfloor (a + b) + (\lfloor x \rfloor + \lfloor y \rfloor) \rfloor \\ &= \lfloor a + b \rfloor + \lfloor \lfloor x \rfloor + \lfloor y \rfloor \rfloor \\ &= \lfloor \lfloor x \rfloor + \lfloor y \rfloor \rfloor \\ &= \lfloor x \rfloor + \lfloor y \rfloor \end{aligned}$$

as required.

(b)

- The converse of $\mathcal{P}$ is: For all real numbers x and y, if $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, then $x - \lfloor x \rfloor < \frac{1}{2}$ and $y - \lfloor y \rfloor < \frac{1}{2}$.
- The converse of $\mathcal{P}$ is false.
- **Proof** (of the converse): Let $x = 0.5 \in \mathbb{R}$ and $y = 0.4 \in \mathbb{R}$ such that

$$\lfloor x+y \rfloor = \lfloor 0.5+0.4 \rfloor = \lfloor 0.9 \rfloor = 0 = 0+0 = \lfloor 0.5 \rfloor + \lfloor 0.4 \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$

But

$$x - \lfloor x \rfloor = 0.5 - \lfloor 0.5 \rfloor = 0.5 - 0 = 0.5 = \frac{1}{2} \not< \frac{1}{2}$$

Thus the converse of $\mathcal{P}$ is false.

(c)

- The contrapositive of $\mathcal{P}$ is: For all real numbers x and y, if $\lfloor x+y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$, then $x - \lfloor x \rfloor \geq \frac{1}{2}$ or $y - \lfloor y \rfloor \geq \frac{1}{2}$.
- The contrapositive of $\mathcal{P}$ is true because it is logically equivalent to $\mathcal{P}$ which is proven to be true in **(a)**.

(d)

- The negation of $\mathcal{P}$ is: There exist real numbers x and y such that $x - \lfloor x \rfloor < \frac{1}{2}$ and $y - \lfloor y \rfloor < \frac{1}{2}$ but $\lfloor x+y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$.
- The negation of $\mathcal{P}$ is false because its truth value is logically opposite to the truth value of $\mathcal{P}$ which is proven to be true in **(a)**.

Question 2

(a) Solution

Since

$$\begin{aligned}
2019 &= 271 \times 7 + 122 \\
271 &= 122 \times 2 + 27 \\
122 &= 27 \times 4 + 14 \\
27 &= 14 \times 1 + 13 \\
14 &= 13 \times 1 + 1 \\
13 &= 1 \times 13 + 0
\end{aligned}$$

We have

$$\begin{aligned}\gcd(2019,271) &= \gcd(271,122)\\ &= \gcd(122,27)\\ &= \gcd(27,14)\\ &= \gcd(14,13)\\ &= \gcd(13,1)\\ &= \gcd(1,0)\\ &= 1\end{aligned}$$

Using the table method, we have

	2019	271	
2019	1	0	R_1
271	0	1	R_2
122	1	-7	$R_3 \leftarrow R_1 - 7R_2$
27	-2	15	$R_4 \leftarrow R_2 - 2R_3$
14	9	-67	$R_5 \leftarrow R_3 - 4R_4$
13	-11	82	$R_6 \leftarrow R_4 - R_5$
1	20	-149	$R_7 \leftarrow R_5 - R_6$

Thus when $x = 20$ and $y = -149$,

$$\begin{aligned}2019x + 271y &= 2019 \times 20 + 271 \times (-149)\\ &= 40380 - 40379\\ &= 1\\ &= \gcd(2019,271)\end{aligned}$$

(b) Solution Let $m = 291 \in \mathbb{Z}$, $n = -2168 \in \mathbb{Z}$ such that

$$2019m + 271n = 2019 \times 291 + 271 \times (-2168) = 587529 - 587528 = 1 = \gcd(2019,271)$$

In this case $m = 291 \neq 20 = x$, $n = -2168 \neq -149 = y$.

(c)

- The statement is false.
- Its negation is: $\exists a, b, c \in \mathbb{Z}^+, \gcd(a, c) + \gcd(b, c) \neq \gcd(a + b, c)$. We show its negation is true.
- **Proof** (of negation) Let $a = 2 \in \mathbb{Z}$, $b = 5 \in \mathbb{Z}$ and $c = 1 \in \mathbb{Z}$. Then

$$\gcd(a, c) + \gcd(b, c) = \gcd(2, 1) + \gcd(5, 1) = 1 + 1 = 2 \neq 1 = \gcd(7, 1) = \gcd(2 + 5, 1) = \gcd(a + b, c)$$

as required.

(d)

- The statement is true.
- **Proof** Let $a = 3 \in \mathbb{Z}$, $b = 3 \in \mathbb{Z}$ and $c = 2 \in \mathbb{Z}$. Then

$$\gcd(a, c) + \gcd(b, c) = \gcd(3, 2) + \gcd(3, 2) = 1 + 1 = 2 = \gcd(6, 2) = \gcd(3 + 3, 2) = \gcd(a + b, c)$$

as required.

Question 3

(a)

- The statement is true.
- **Proof** Suppose $x \in \mathbb{Z}$, we show that $x^3 + x$ is even, and we can split the value of x into two cases.

  Case 1 If x is odd, then $\exists k \in \mathbb{Z}$ such that $x = 2k + 1$. Thus

$$\begin{aligned} x^3 + x &= (2k+1)^3 + (2k+1) \\ &= (2k+1)((2k+1)^2 + 1) \\ &= (2k+1)(4k^2 + 4k + 2) \\ &= 2(2k+1)(2k^2 + 2k + 1) \end{aligned}$$

  Since $(2k + 1)(2k^2 + 2k + 1) \in \mathbb{Z}$, we have $x^3 + x$ is even by the definition.

  Case 2 If x is even, then $\exists k \in \mathbb{Z}$ such that $x = 2k$. Thus

$$\begin{aligned} x^3 + x &= (2k)^3 + 2k \\ &= 8k^3 + 2k \\ &= 2(4k^3 + k) \end{aligned}$$

  Since $4k^3 + k \in \mathbb{Z}$, we have $x^3 + x$ is even by the definition.

Conclusion Since in both cases we have $x^3 + x$ is even, thus we can conclude that, for all integers x, $x^3 + x$ is even.

(b)

- The statement is false.
- Its negation is: There exist an integer y such that for all integers x, $x^3 + x \neq y$. We prove its negation is true.
- **Proof** (of negation) Let $y = 1 \in \mathbb{Z}$, then $y = 1 = 2 \cdot 0 + 1$, thus y is odd, since from **(a)** we have proven that for all integers x, $x^3 + x$ is even, thus it is impossible that there is an $x \in \mathbb{Z}$ so that $x^3 + x = y$.

(b)

- The statement is true.
- **Proof** Suppose $x, y \in \mathbb{Z}$ such that $x^3 + x = y^3 + y$, we show that $x = y$.

  Since

$$x^3 + x = y^3 + y$$
$$x^3 + x - y - y^3 = 0$$
$$(x^3 - y^3) + (x - y) = 0$$
$$(x - y)(x^2 + xy + y^2) + (x - y) = 0$$
$$(x - y)(x^2 + xy + y^2 + 1) = 0$$
$$(x - y)((x + \frac{y}{2})^2 + \frac{3}{4}y^2 + 1) = 0$$

  Since

$$(x + \frac{y}{2})^2 + \frac{3}{4}y^2 + 1 \geq 0 + 0 + 1 = 1$$

  It must be the case that

$$x - y = 0$$

  Thus

$$x = y$$

  as required.