Tutorial 01 *Solution*

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CPSC 331 - Data Structures, Algorithms, and Their Analysis

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Question 1

CITATION

The format of the proof acts in a similar manner to the proof provided from page 48 and page 49 in L01_intro_and_math_review.pdf[1] in order to providing a completed, clear and professional proof.

Claim For every integer n such that $n \geq 2$,

$$\prod_{i=2}^{n}(1 - \frac{1}{i}) = \frac{1}{n}$$

Proof This will be proved by the standard form of mathematical induction on n. The case that $n = 2$ will be considered in the basis.

Basis (n=2) When $n = 2$

$$\prod_{i=2}^{n}(1 - \frac{1}{i}) = \prod_{i=2}^{2}(1 - \frac{1}{i}) = 1 - \frac{1}{2} = \frac{1}{2} = \frac{1}{n}$$

Thus the result holds when $n = 2$.

Inductive Step: Let $k \geq 2$ be an arbitrarily chosen integer. It is necessary and sufficient to use

Inductive Hypothesis:

$$\prod_{i=2}^{k}(1 - \frac{1}{i}) = \frac{1}{k}$$

to prove

Inductive Claim:

$$\prod_{i=2}^{k+1}(1-\frac{1}{i}) = \frac{1}{k+1}$$

Note that

$$\prod_{i=2}^{k+1}(1-\frac{1}{i}) = (\prod_{i=2}^{k}(1-\frac{1}{i}))(1-\frac{1}{k+1})$$

Thus, by **Inductive Hypothesis**, we have

$$\begin{aligned}\prod_{i=2}^{k+1}(1-\frac{1}{i}) &= \frac{1}{k}(1-\frac{1}{k+1}) \\ &= \frac{1}{k}\cdot\frac{k}{k+1} \\ &= \frac{1}{k+1}\end{aligned}$$

as required to establish the inductive claim and complete the inductive step.

Conclusion: Therefore, by standard form of mathematical induction, we can conclude that, for every integer n such that $n \geq 2$,

$$\prod_{i=2}^{n}(1-\frac{1}{i}) = \frac{1}{n}$$

Question 2

CITATION
The format of the proof acts in a similar manner to the proof provided from page 48 and page 49 in L01_intro_and_math_review.pdf[1] in order to providing a completed, clear and professional proof.

Recall that the **Dumbledore** numbers are a sequence D_0, D_1, D_2, D_3, ... of numbers that are defined as follows: For every integer n such that $n \geq 0$,

$$D_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 4 & \text{if } n = 2 \\ 3D_{n-1} - 3D_{n-2} + D_{n-3} & \text{if } n \geq 3 \end{cases}$$

Claim $D_n = n^2$ for every integer n such that $n \geq 0$.

Proof This will be proved by the strong form of mathematical induction on n. Cases that $n = 0$, $n = 1$, $n = 2$

will be considered in the basis.

Basis (n=0) When $n = 0$, $D_n = D_0 = 0 = 0^2 = n^2$, as required.

Basis (n=1) When $n = 1$, $D_n = D_1 = 1 = 1^2 = n^2$, as required.

Basis (n=2) When $n = 2$, $D_n = D_2 = 4 = 2^2 = n^2$, as required.

Inductive Step: Let $k \geq 2$ be an arbitrarily chosen integer. It is necessary and sufficient to use

Inductive Hypothesis:

For every integer i such that $0 \leq i \leq k$,

$$D_i = i^2$$

to prove

Inductive Claim:

$$D_{k+1} = (k+1)^2$$

By the definition of the sequence since $k \geq 2$, that is $k + 1 \geq 3$, we have

$$D_{k+1} = 3D_k - 3D_{k-1} + D_{k-2}$$

Thus, by **Inductive Hypothesis** which applied since $0 \leq k, k-1, k-2 \leq k$,

$$\begin{aligned} D_{k+1} &= 3k^2 - 3(k-1)^2 + (k-2)^2 \\ &= 3k^2 - 3(k^2 - 2k + 1) + (k^2 - 4k + 4) \\ &= 3k^2 - 3k^2 + 6k - 3 + k^2 - 4k + 4 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

as required to establish the inductive claim and complete the inductive step.

Conclusion: Therefore, by strong form of mathematical induction, we can conclude that, for every integer n such that $n \geq 0$,

$$D_n = n^2$$

References

[1] Wayne Eberly, 2019, CPSC 331: Data Structures, Algorithms, and Their Analysis: Spring, 2019, Introduction and Mathematics Review, Lecture #1: Introduction and Mathematics Review. Retrieved from http://pages.cpsc.ucalgary.ca/~eberly/Courses/CPSC331/2019a/1_Introduction/L01/L01_intro_and_math_review.pdf