CPSC 449 — Principles of Programming Languages
Theory Components of Assignment 3

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Tutorial: 03

Problem 2

A *polynomial with one variable* is represented using the following algebraic type.

```
data Polynomial
= PConst Integer
| PVar
| PAdd Polynomial Polynomial
| PMul Polynomial Polynomial
```

The *degree* of a *Polynomial* can be obtained by the function *degree* :: *Polynomial* -> *Integer*, which is defined by the following equations:

```
degree :: Polynomial -> Integer
degree (PConst n) = 0                          -- (degree.1)
degree PVar = 1                                -- (degree.2)
degree (PAdd p1 p2) = max (degree p1) (degree p2) -- (degree.3)
degree (PMul p1 p2) = (degree p1) + (degree p2) -- (degree.4)
```

The *first derivative* of a *Polynomial* can be computed using the function *d* :: *Polynomial* -> *Polynomial*, the definition of which is given by the following equations.

```
d (PConst n) = PConst 0 -- (d.1)
d PVar = PConst 1 -- (d.2)
d (PAdd p1 p2) = PAdd (d p1) (d p2) -- (d.3)
d (PMul p1 p2) = PAdd (PMul p1 (d p2)) (PMul (d p1) p2) -- (d.4)
```

These four equations are the direct encoding of the well-known formulas in standard calculus textbooks.

$$\frac{dC}{dx} = 0$$
$$\frac{dx}{dx} = 1$$
$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$
$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Prove, by structural induction, that the following inequality holds for all finite *Polynomial* p.

$$\text{degree p} \geq \text{degree (d p)} \tag{1}$$

(a) State the Principle of Structural Induction for *Polynomial*s.

Solution. From the description above we have the theorem $P(p)$ as

$$\text{degree p} \geq \text{degree (d p)}$$

Thus to prove that $P(p)$ holds for all finite *Polynomial* p, prove the following:

i. P(Pconst Integer)
ii. P(PVar)
iii. $P(\text{p1}) \wedge P(\text{p2}) \implies P(\text{PAdd p1 p2})$
iv. $P(\text{p1}) \wedge P(\text{p2}) \implies P(\text{PMul p1 p2})$

Therefore, the proof goals are:

Proof Goals

- **Base Cases**

$$\text{degree (PConst n)} \geq \text{degree (d (PConst n))} \tag{base.1}$$
$$\text{degree PVar} \geq \text{degree (d PVar)} \tag{base.2}$$

- **Induction Steps**
  - **Assume:**

$$\text{degree p1} \geq \text{degree (d p1)} \tag{hyp.1}$$
$$\text{degree p2} \geq \text{degree (d p2)} \tag{hyp.2}$$

  - **Prove:**

$$\text{degree (PAdd p1 p2)} \geq \text{degree (d (PAdd p1 p2))} \tag{ind.1}$$
$$\text{degree (PMul p1 p2)} \geq \text{degree (d (PMul p1 p2))} \tag{ind.2}$$

(b) Prove the base case(s).

Firstly, we prove the first base case:

- **Want:**

$$\text{degree (PConst n)} \geq \text{degree (d (PConst n))}$$

- **Left-hand side:**

$$\text{degree (PConst n)} = 0 \tag{by degree.1}$$

- **Right-hand side:**

$$\begin{aligned}
\text{degree (d (PConst n))} &= \text{degree (PConst 0)} && \text{(by d.1)}\\
&= 0 && \text{(by degree.1)}\\
&\leq \text{L.H.S} && \text{(by arith.)}
\end{aligned}$$

Thus it shows that the left-hand-side is greater than or equal to the right-hand-side, which completes the proof of the first base case.

Secondly, we prove the second base case:

- **Want:**

$$\text{degree PVar} \geq \text{degree (d PVar)}$$

- **Left-hand side:**

$$\text{degree PVar} = 1 \qquad \text{(by degree.2)}$$

- **Right-hand side:**

$$\begin{aligned} \text{degree (d PVar)} &= \text{degree (PConst 1)} && \text{(by d.2)} \\ &= 0 && \text{(by degree.1)} \\ &\leq \text{L.H.S} && \text{(by arith.)} \end{aligned}$$

  Thus it shows that the left-hand-side is greater than or equal to the right-hand-side, which completes the proof of the second base case.

Since all base cases are proved, we completed the proof of the base case of the statement.

(c) Prove the induction step(s).

Firstly, we prove the first induction step:

- **Assume:**

$$\begin{aligned} \text{degree p1} &\geq \text{degree (d p1)} && \text{(hyp.1)} \\ \text{degree p2} &\geq \text{degree (d p2)} && \text{(hyp.2)} \end{aligned}$$

- **Want:**

$$\text{degree (PAdd p1 p2)} \geq \text{degree (d (PAdd p1 p2))}$$

- **Left-hand side:**

$$\text{degree (PAdd p1 p2)} = \text{max (degree p1) (degree p2)} \qquad \text{(by degree.3)}$$

- **Right-hand side:**

$$\begin{aligned} \text{degree (d (PAdd p1 p2))} &= \text{degree (PAdd (d p1) (d p2))} && \text{(by d.3)} \\ &= \text{max (degree (d p1)) (degree (d p2))} && \text{(by degree.3)} \end{aligned}$$

  Thus, we can split the possible situations of the left-hand-side into 2 cases:

  - **Case 1:** If degree p1 $\geq$ degree p2, then we continue the deduction in the left-hand-side and we have:

$$\begin{aligned} \text{L.H.S} &= \text{max (degree p1) (degree p2)} \\ &= \text{degree p1} && \text{(by defn. of max)} \\ &\geq \text{degree (d p1)} && \text{(by hyp.1)} \end{aligned}$$

    Also, we have

$$\begin{aligned} \text{L.H.S} &= \text{max (degree p1) (degree p2)} \\ &= \text{degree p1} && \text{(by defn. of max)} \\ &\geq \text{degree p2} && \text{(by cond. of the case)} \\ &\geq \text{degree (d p2)} && \text{(by hyp.2)} \end{aligned}$$

Since L.H.S $\geq$ degree (d p1) and L.H.S $\geq$ degree (d p2), we have

$$\text{L.H.S} \geq \text{max (degree (d p1)) (degree (d p2))} \qquad \text{(by defn. of max)}$$

Since R.H.S = max (degree (d p1)) (degree (d p2)), we have L.H.S $\geq$ R.H.S. Thus in this case, the left-hand-side is greater than or equal to the right-hand-side.

– **Case 2:** If degree p2 $\geq$ degree p1, then we continue the deduction in the left-hand-side and we have:

$$\begin{aligned} \text{L.H.S} &= \text{max (degree p1) (degree p2)} \\ &= \text{degree p2} && \text{(by defn. of max)} \\ &\geq \text{degree (d p2)} && \text{(by hyp.2)} \end{aligned}$$

Also, we have

$$\begin{aligned} \text{L.H.S} &= \text{max (degree p1) (degree p2)} \\ &= \text{degree p2} && \text{(by defn. of max)} \\ &\geq \text{degree p1} && \text{(by cond. of the case)} \\ &\geq \text{degree (d p1)} && \text{(by hyp.1)} \end{aligned}$$

Since L.H.S $\geq$ degree (d p1) and L.H.S $\geq$ degree (d p2), we have

$$\text{L.H.S} \geq \text{max (degree (d p1)) (degree (d p2))} \qquad \text{(by defn. of max)}$$

Since R.H.S = max (degree (d p1)) (degree (d p2)), we have L.H.S $\geq$ R.H.S. Thus in this case, the left-hand-side is greater than or equal to the right-hand-side.

Therefore, in both cases we have proved that the left-hand-side is greater than or equal to the right-hand-side, on the assumption that the induction hypothesis holds, which completes the induction step.

Secondly, we prove the second induction step:

- **Assume:**

$$\begin{aligned} \text{degree p1} &\geq \text{degree (d p1)} && \text{(hyp.1)} \\ \text{degree p2} &\geq \text{degree (d p2)} && \text{(hyp.2)} \end{aligned}$$

- **Want:**

$$\text{degree (PMul p1 p2)} \geq \text{degree (d (PMul p1 p2))}$$

- **Left-hand side:**

$$\text{degree (PMul p1 p2)} = \text{(degree p1)} + \text{(degree p2)} \qquad \text{(by degree.4)}$$

- **Right-hand side:**

$$\text{degree (d (PMul p1 p2))} = \text{degree (PAdd (PMul p1 (d p2)) (PMul (d p1) p2))} \qquad \text{(by d.4)}$$

Let $s1$ = PMul p1 (d p2), $s2$ = PMul (d p1) p2, then we have

$$\begin{aligned}\text{R.H.S} &= \text{degree (PAdd s1 s2)} && \text{(by replacement)}\\ &= \text{max (degree s1) (degree s2)} && \text{(by degree.3)}\end{aligned}$$

Since

$$\begin{aligned}\text{degree s1} &= \text{degree (PMul p1 (d p2))} && \text{(by replacement)}\\ &= \text{(degree p1) + (degree (d p2))} && \text{(by degree.4)}\\ &\leq \text{(degree p1) + (degree p2)} && \text{(by hyp.2)}\end{aligned}$$

Also

$$\begin{aligned}\text{degree s2} &= \text{degree (PMul (d p1) p2)} && \text{(by replacement)}\\ &= \text{(degree (d p1)) + (degree p2)} && \text{(by degree.4)}\\ &\leq \text{(degree p1) + (degree p2)} && \text{(by hyp.1)}\end{aligned}$$

Thus, we can split the possible situations of the right-hand-side into 2 cases:

- **Case 1:** If degree s1 $\geq$ degree s2, then we continue the deduction in the right-hand-side and have

$$\begin{aligned}\text{R.H.S} &= \text{max (degree s1) (degree s2)}\\ &= \text{degree s1} && \text{(by defn. of max)}\\ &\leq \text{(degree p1) + (degree p2)} && \text{(from the deduction in RHS)}\end{aligned}$$

  Since L.H.S = (degree p1) + (degree p2), we have L.H.S $\geq$ R.H.S. Thus in this case, the left-hand-side is greater than or equal to the right-hand-side.

- **Case 2:** If degree s2 $\geq$ degree s1, then we continue the deduction in the right-hand-side and have

$$\begin{aligned}\text{R.H.S} &= \text{max (degree s1) (degree s2)}\\ &= \text{degree s2} && \text{(by defn. of max)}\\ &\leq \text{(degree p1) + (degree p2)} && \text{(from the deduction in RHS)}\end{aligned}$$

  Since L.H.S = (degree p1) + (degree p2), we have L.H.S $\geq$ R.H.S. Thus in this case, the left-hand-side is greater than or equal to the right-hand-side.

Therefore, in both cases we have proved that the left-hand-side is greater than or equal to the right-hand-side, on the assumption that the induction hypothesis holds, which completes the induction step.

Since all induction steps are completed, we can say the proof itself is also completed. □