# $Assignment\ 2$

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MATH 271 - Discrete Mathematics

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## Question 1

### (a) Solution

$$\begin{aligned} a^{n+1} - b^{n+1} &= a^{n+1} - b^{n+1} + (ab^n - ab^n) \\ &= a^{n+1} - b^{n+1} + ab^n - ab^n \\ &= (a^{n+1} - ab^n) + (ab^n - b^{n+1}) \\ &= (a \cdot a^n - a \cdot b^n) + (b^n \cdot a - b^n \cdot b) \\ &= a(a^n - b^n) + b^n(a - b) \end{aligned}$$

**(b) Proof** We prove the claim by the standard form of mathematical induction on $n$. The case that $n = 1$ will be used in the basis.

*Basis (n=1)* When $n = 1$,

$$a^n - b^n = a^1 - b^1 = a - b = (a - b) \cdot 1$$

thus $(a - b)|(a^n - b^n)$ by its definition.

*Inductive Step:* Let $k \geq 1$ be an integer. It is necessary and sufficient to use

**Inductive Hypothesis:**

$$(a - b)|(a^k - b^k)$$

to prove

**Inductive Claim:**

$$(a - b)|(a^{k+1} - b^{k+1})$$

Since $(a-b)|(a^k-b^k)$. $\exists m \in \mathbb{Z}$ such that $m(a-b)=a^k-b^k$ by its definition. Thus, from **(a)**, we have

$$a^{k+1}-b^{k+1}=a(a^k-b^k)+b^k(a-b)$$

Thus, by the **inductive hypothesis**, we have

$$\begin{aligned} a^{k+1}-b^{k+1} &= am(a-b)+b^k(a-b) \\ &= (am+b^k)(a-b) \end{aligned}$$

Since $am+b^k \in \mathbb{Z}$, we have $(a-b)|(a^{k+1}-b^{k+1})$ by its definition, as required.

**Conclusion:** Therefore, by the standard form of mathematical induction, we can conclude that, for all integers $n \geq 1$, $(a-b)|(a^n-b^n)$.

**(c) Proof**

For example, we choose $n=271 \geq 1, a=7 \in \mathbb{Z}, b=-4 \in \mathbb{Z}$ such that $a \neq b$. Thus

$$a-b=7-(-4)=11$$

$$a^n-b^n=7^{271}-(-4)^{271}=7^{271}-(-1)^{271}\cdot 4^{271}=7^{271}+4^{271}$$

From **(b)**, we can conclude that

$$11|(7^{271}+4^{271})$$

## Question 2

**(a) Solution**

$$\begin{aligned} a_2 &= a_1+2\cdot 1+1=0+2+1=3 \\ a_3 &= a_2+2\cdot 2+1=3+4+1=8 \\ a_4 &= a_3+2\cdot 3+1=8+6+1=15 \\ a_5 &= a_4+2\cdot 4+1=15+8+1=24 \end{aligned}$$

**(b) Solution** Guess for all positive integers $n$, $a_n=n^2-1$.

**(c) Proof** We prove the claim by the strong form of mathematical induction on $n$. Cases that $n=1$ and $n=2$ will be used in the basis.

*Basis (n=1)* When $n=1$,

$$a_n=a_1=0=1^2-1=n^2-1$$

as required.

*Basis (n=2)* When $n = 2$,

$$a_n = a_2 = 3 = 2^2 - 1 = n^2 - 1$$

as required.

Thus the result holds when $n = 1$ and $n = 2$.

*Inductive Step:* Suppose for some integer $k > 2$. It is necessary and sufficient to use

**Inductive Hypothesis:**

For all integers $m$ such that $1 \leq m < k$, $a_m = m^2 - 1$.

to prove

**Inductive Claim:**

$$a_k = k^2 - 1$$

Since $k > 2$, $k - 1 > 2 - 1 = 1 \geq 1$. Thus, by the definition of $a_k$ when $k \geq 2$, we have

$$a_k = a_{(k-1)+1} = a_{k-1} + 2(k-1) + 1$$

Thus, by the **inductive hypothesis** which applies since $1 \leq k - 1 < k$, we have

$$\begin{aligned} a_k &= a_{k-1} + 2(k-1) + 1 \\ &= ((k-1)^2 - 1) + 2(k-1) + 1 \\ &= (k^2 - 2k + 1 - 1) + 2k - 2 + 1 \\ &= k^2 - 2k + 2k - 2 + 1 \\ &= k^2 - 1 \end{aligned}$$

as required.

**Conclusion:** Therefore, by the strong form of mathematical induction, we can conclude that, for all positive integers $n$, $a_n = n^2 - 1$.

**(c) Proof** We prove that $a_n$ is composite for all integers $n \geq 3$ by its definition.

Suppose $n \in \mathbb{Z}$ and $n \geq 3$, from **(c)**, we have

$$a_n = n^2 - 1 = (n-1)(n+1)$$

Since $n - 1 \in \mathbb{Z}$, $n - 1 \geq 3 - 1 = 2$ and $n + 1 \in \mathbb{Z}$, $n + 1 \geq 3 + 1 = 4$, we have

$$\begin{cases} 1 < n-1 < (n-1)(n+1) \\ 1 < n+1 < (n-1)(n+1) \end{cases}$$

Thus from $a_n = n^2 - 1 = (n-1)(n+1)$, we can conclude that $a_n$ is composite for all integers $n \geq 3$ by its definition.

## Question 3

**(a) Solution**

$$\begin{aligned}
b_2 &= \sqrt{b_1 b_0} + \frac{3 \cdot 2}{2} - 1 = \sqrt{\frac{1}{2} \cdot 0} + \frac{3 \cdot 2}{2} - 1 = 0 + 3 - 1 = 2 \\
b_3 &= \sqrt{b_2 b_1} + \frac{3 \cdot 3}{2} - 1 = \sqrt{2 \cdot \frac{1}{2}} + \frac{3 \cdot 3}{2} - 1 = 1 + \frac{9}{2} - 1 = \frac{9}{2} \\
b_4 &= \sqrt{b_3 b_2} + \frac{3 \cdot 4}{2} - 1 = \sqrt{\frac{9}{2} \cdot 2} + \frac{3 \cdot 4}{2} - 1 = 3 + 6 - 1 = 8 \\
b_5 &= \sqrt{b_4 b_3} + \frac{3 \cdot 5}{2} - 1 = \sqrt{8 \cdot \frac{9}{2}} + \frac{15}{2} - 1 = 6 + \frac{15}{2} - 1 = \frac{25}{2}
\end{aligned}$$

**(b) Solution** Guess for all integers $n \geq 0$,

$$b_n = \frac{n^2}{2}$$

**(c) Proof** We prove the claim by the strong form of mathematical induction on $n$. Cases that $n = 0$ and $n = 1$ will be used in the basis.

*Basis (n=0)* When $n = 0$,

$$b_n = b_0 = 0 = \frac{0^2}{2} = \frac{n^2}{2}$$

as required.

*Basis (n=1)* When $n = 1$,

$$b_n = b_1 = \frac{1}{2} = \frac{1^2}{2} = \frac{n^2}{2}$$

as required.

Thus the result holds when $n = 0$ and $n = 1$.

*Inductive Step:* Suppose for some integer $k > 1$. It is necessary and sufficient to use

**Inductive Hypothesis:**

For all integers $m$ such that $0 \leq m < k$,

$$b_m = \frac{m^2}{2}$$

to prove

**Inductive Claim:**

$$b_k = \frac{k^2}{2}$$

Since $k > 1$, $k \geq 2$, thus by the definition of $b_k$, we have

$$b_k = \sqrt{b_{k-1} b_{k-2}} + \frac{3k}{2} - 1$$

Thus, by the **inductive hypothesis** which applies since $0 \leq k-1, k-2 < k$, we have

$$\begin{aligned}
b_k &= \sqrt{\frac{(k-1)^2}{2} \frac{(k-2)^2}{2}} + \frac{3k}{2} - 1 \\
&= \frac{(k-1)(k-2)}{2} + \frac{3k}{2} - 1 \\
&= \frac{k^2 - 3k + 2}{2} + \frac{3k}{2} - \frac{2}{2} \\
&= \frac{k^2 - 3k + 2 + 3k - 2}{2} \\
&= \frac{k^2}{2}
\end{aligned}$$

as required.

**Conclusion:** Therefore, by the strong form of mathematical induction, we can conclude that, for all integers $n \geq 0$,

$$b_n = \frac{n^2}{2}$$