Assignment 3

Haohu Shen UCID: 30063099

MATH 271 - Discrete Mathematics

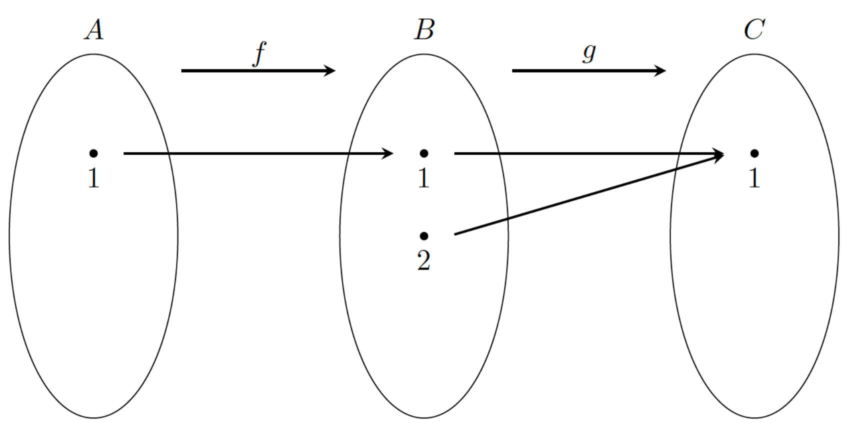
Instructor Jerrod Smith

March 15, 2019

Question 1

(a)

- The statement is false.
- Its negation is: There exists sets A, B and C and functions $f : A \to B$, $g : B \to C$ such that $g \circ f$ is onto but f is not onto. We show that the negation is true.
- **Proof** (of negation)
  For example, let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1\}$ and let function $f : A \to B$ defined by $f(1) = 1$, $g : B \to C$ defined by $g(1) = 1$, $g(2) = 1$. Since 1 is the only element in C and $(g \circ f)(1) = 1$, $1 \in A$, therefore $g \circ f$ is onto, but f is not onto since $\forall x \in A$, $f(x) \neq 2 \in B$. The arrow diagram of $g \circ f$ is given below:



(b)

- The statement is true.

- **Proof** Suppose $g \circ f : A \to C$ is onto. We show that g is onto. Suppose $b \in C$, since $g \circ f$ is onto, $\exists a \in A$ such that $(g \circ f)(a) = b$, thus $g(f(a)) = b$. Let $c = f(a) \in B$, then $g(f(a)) = g(c) = b$. Thus g is onto by its definition. Therefore, we can conclude that if $g \circ f$ is onto then g is onto.

(c)

- The statement is true.
- **Proof** Suppose $g \circ f$ is onto and g is one-to-one. We show that f is onto. Suppose $b \in B$, let $c = g(b) \in C$. Since $g \circ f$ is onto, $\exists a \in A$ such that $(g \circ f)(a) = g(f(a)) = c = g(b)$. Since g is one-to-one, we have $f(a) = b$, thus f is onto by the definition. Therefore, we can conclude that if $g \circ f$ is onto and g is one-to-one then f is onto.

Question 2

(a)

- The statement is false.
- Its negation is: $f \circ g$ is not one-to-one, that is, $\exists a, b \in \mathbb{Z}$ such that $(f \circ g)(a) = (f \circ g)(b)$ but $a \neq b$. We show its negation is true.
- **Proof** (of negation)

  For example, let $a = 0 \in \mathbb{Z}$, $b = -1 \in \mathbb{Z}$, thus

$$(f \circ g)(a) = f(g(a)) = f(\lfloor \frac{a+1}{2} \rfloor) = 3\lfloor \frac{a+1}{2} \rfloor - 1 = 3\lfloor \frac{0+1}{2} \rfloor - 1 = -1$$
$$(f \circ g)(b) = f(g(b)) = f(\lfloor \frac{b+1}{2} \rfloor) = 3\lfloor \frac{b+1}{2} \rfloor - 1 = 3\lfloor \frac{-1+1}{2} \rfloor - 1 = -1$$

  Therefore, $(f \circ g)(a) = -1 = (f \circ g)(b)$, but $a \neq b$ because $0 \neq -1$.

(b)

- The statement is false.
- Its negation is: $f \circ g$ is not onto. We prove the negation is true by contradiction.
- **Proof** (of negation by contradiction)

  Suppose $f \circ g$ is onto, that is, $\forall b \in \mathbb{Z}$, $\exists a \in \mathbb{Z}$ such that $(f \circ g)(a) = b$. Thus:

$$(f \circ g)(a) = f(g(a)) = f(\lfloor \frac{a+1}{2} \rfloor) = 3\lfloor \frac{a+1}{2} \rfloor - 1 = b$$

  Let $b = 1$, then

$$3\lfloor\frac{a+1}{2}\rfloor-1=1$$

thus

$$\lfloor\frac{a+1}{2}\rfloor=\frac{2}{3}\notin\mathbb{Z}$$

but $\frac{a+1}{2}\in\mathbb{Z}$ by the definition, thus $\lfloor\frac{a+1}{2}\rfloor\in\mathbb{Z}$ and $\lfloor\frac{a+1}{2}\rfloor\notin\mathbb{Z}$, which leads to a contradiction. Therefore, by contradiction, we can conclude that the negation is true, thus the original statement is false.

(c)

- The statement is true.
- Suppose $a, b \in \mathbb{Z}$ and $(g\circ f)(a)=(g\circ f)(b)$, we show that $a=b$.
- **Proof**

Since

$$\begin{gathered}(g\circ f)(a)=(g\circ f)(b)\\ g(f(a))=g(f(b))\\ g(3a-1)=g(3b-1)\\ \left\lfloor\frac{(3a-1)+1}{2}\right\rfloor=\left\lfloor\frac{(3b-1)+1}{2}\right\rfloor\\ \lfloor\frac{3a}{2}\rfloor=\lfloor\frac{3b}{2}\rfloor\\ a+\lfloor\frac{a}{2}\rfloor=b+\lfloor\frac{b}{2}\rfloor\end{gathered}$$

Thus, we can split the value of a into two cases by its parity.

Case 1 a is odd, thus $\exists m\in\mathbb{Z}$ such that $a=2m+1$.

- ***Subcase 1*** If b is odd, then $\exists n\in\mathbb{Z}$ such that $b=2n+1$, thus when we back-substitute to $a+\lfloor\frac{a}{2}\rfloor=b+\lfloor\frac{b}{2}\rfloor$, we have

$$\begin{gathered}(2m+1)+\left\lfloor\frac{2m+1}{2}\right\rfloor=(2n+1)+\left\lfloor\frac{2n+1}{2}\right\rfloor\\ (2m+1)+m+\lfloor\frac{1}{2}\rfloor=(2n+1)+n+\lfloor\frac{1}{2}\rfloor\\ 3m+1=3n+1\\ m=n\end{gathered}$$

Thus

$$a = 2m + 1 = 2n + 1 = b$$

- ***Subcase 2*** If b is even, then $\exists n \in \mathbb{Z}$ such that $b = 2n$, thus when we back-substitute to $a + \lfloor \frac{a}{2} \rfloor = b + \lfloor \frac{b}{2} \rfloor$, we have

$$\begin{aligned}(2m+1) + \left\lfloor \frac{2m+1}{2} \right\rfloor &= 2n + \left\lfloor \frac{2n}{2} \right\rfloor \\ (2m+1) + m + \lfloor \frac{1}{2} \rfloor &= 2n + n \\ 3m + 1 &= 3n\end{aligned}$$

Since

$$b = 2n = 2 \cdot \frac{3m+1}{3} = 2(m + \frac{1}{3}) = 2m + \frac{2}{3} \notin \mathbb{Z}$$

and

$$b = 2n \in \mathbb{Z}$$

thus $b \in \mathbb{Z}$ and $b \notin \mathbb{Z}$, which leads to a contradiction, thus it is impossible that b is even.

Case 2 a is even, thus $\exists m \in \mathbb{Z}$ such that $a = 2m$.

- ***Subcase 1*** If b is even, then $\exists n \in \mathbb{Z}$ such that $b = 2n$, thus when we back-substitute to $a + \lfloor \frac{a}{2} \rfloor = b + \lfloor \frac{b}{2} \rfloor$, we have

$$\begin{aligned}2m + \left\lfloor \frac{2m}{2} \right\rfloor &= 2n + \left\lfloor \frac{2n}{2} \right\rfloor \\ 2m + m &= 2n + n \\ 3m &= 3n \\ m &= n\end{aligned}$$

Thus

$$a = 2m = 2n = b$$

- ***Subcase 2*** If b is odd, then $\exists n \in \mathbb{Z}$ such that $b = 2n + 1$, thus when we back-substitute to $a + \lfloor \frac{a}{2} \rfloor = b + \lfloor \frac{b}{2} \rfloor$, we have

$$\begin{aligned}2m + \left\lfloor \frac{2m}{2} \right\rfloor &= (2n+1) + \left\lfloor \frac{2n+1}{2} \right\rfloor \\ 2m + m &= (2n+1) + n + \lfloor \frac{1}{2} \rfloor \\ 3m &= 2n + 1 + n \\ 3m &= 3n + 1\end{aligned}$$

Since

$$a = 2m = 2 \cdot \frac{3n+1}{3} = 2(n + \frac{1}{3}) = 2n + \frac{2}{3} \notin \mathbb{Z}$$

and

$$a = 2m \in \mathbb{Z}$$

thus $a \in \mathbb{Z}$ and $a \notin \mathbb{Z}$, which leads to a contradiction, thus it is impossible that b is odd.

Conclusion Since we have $a = b$ in both cases, we can conclude that, $g \circ f$ is one-to-one by definition.

(d)

- The statement is false.
- Its negation is: $g \circ f$ is not onto. We show that the negation is true by contradiction.
- **Proof** (of negation by contradiction)
  Suppose $g \circ f$ is onto. Thus $\forall b \in \mathbb{Z}$, $\exists a \in \mathbb{Z}$ such that $(g \circ f)(a) = b$. So

$$(g \circ f)(a) = g(f(a)) = g(3a - 1) = \left\lfloor \frac{(3a-1)+1}{2} \right\rfloor = \left\lfloor \frac{3a}{2} \right\rfloor = b$$

Let $b = 5$, then

$$\left\lfloor \frac{3a}{2} \right\rfloor = 5$$

Thus, by the definition, we have

$$5 \leq \frac{3a}{2} < 6$$

therefore

$$\frac{10}{3} \leq a < 4$$

However, there is no integer a such that $a \in [\frac{10}{3}, 4)$, which contradicts that a is an integer. Hence, by contradiction, we can conclude that, the negation is true, thus the original statement is false.

Question 3

(a) Solution Since $I_A : A \to A$ and $I_A = x$ for each $x \in A$, we have

$$I_A = \{(1,1), (2,2), (3,3), (4,4)\}$$

Let $f : A \to A$ be defined by

$$f = \{(1,1),(2,2),(3,3),(4,4)\}$$

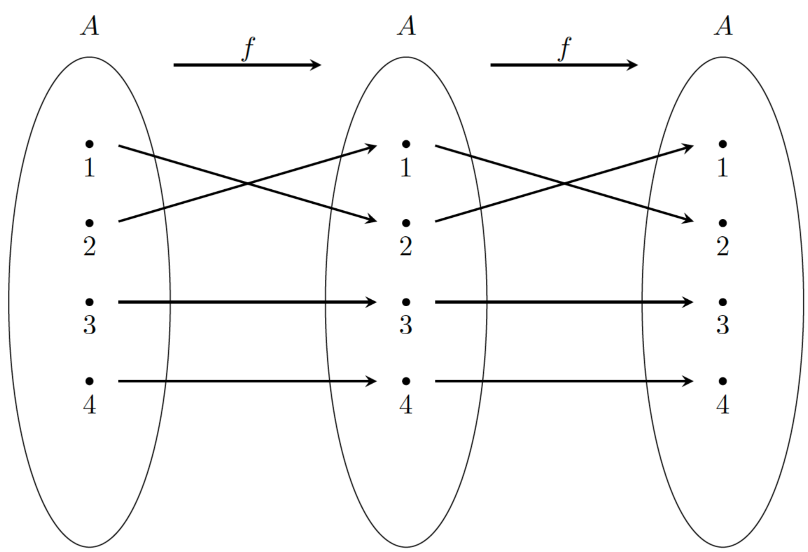
Then we have

$$f(1) = 2, f(2) = 1, f(3) = 3, f(4) = 4$$

- Since $(1,2) \in f$ and $(1,2) \notin I_A$, $f \neq I_A$.
- Since

$$\begin{aligned}
(f \circ f)(1) &= f(f(1)) = f(2) = 1 = I_A(1) \\
(f \circ f)(2) &= f(f(2)) = f(1) = 2 = I_A(2) \\
(f \circ f)(3) &= f(f(3)) = f(3) = 3 = I_A(3) \\
(f \circ f)(4) &= f(f(4)) = f(4) = 4 = I_A(4)
\end{aligned}$$

We have $f \circ f = I_A$ from A to A.
The arrow diagram of $f \circ f$ is shown below:



(b) We introduce two lemmas and prove them are both true at first.

- **Lemma 1** I_A is one-to-one.
- **Proof** (of Lemma 1) Suppose $x, y \in A$ such that $I_A(x) = I_A(y)$. We prove I_A is one-to-one by showing $x = y$. Since $I_A(x) = I_A(y)$, we have $x = y$ by the definition of I_A. Thus I_A is one-to-one.
- **Lemma 2** I_A is onto.
- **Proof** (of Lemma 2) Suppose $x \in A$, since $I_A(x) = x$, we have I_A is onto by the definition.

Now we claim the statement is true and we prove that f is one-to-one and onto separately.

- **Proof** We suppose $f \in F$ and $f \circ f = I_A$. We also suppose $x, y \in A$ and $f(x) = f(y)$. Let $c = f(x) = f(y)$, then

$$f(c) = f(f(x)) = f(f(y))$$

Since $f \circ f = I_A$, we have $I_A(x) = I_A(y)$, from **Lemma 1** we know I_A is one-to-one, thus

$$x = y$$

Since $f(f(x)) = f(f(y))$ and $x = y$ such that $x, y \in A$, we have f is one-to-one by the definition.

From **Lemma 2** we know I_A is onto, thus $\forall b \in A$, $\exists a \in A$ such that $I_A(a) = b$. Therefore

$$I_A(a) = (f \circ f)(a) = f(f(a)) = b$$

Let $c = f(a)$, thus $c \in A$ and $f(c) = b$. So we have f is onto by the definition.

Conclusion Thus we can conclude that, for all $f \in F$, if $f \circ f = I_A$ then f is one-to-one and onto.

(c)

- The statement is false.
- Its negation is: $\exists f, g \in F$ such that $f \circ f = g \circ g$ but $f \neq g$. We prove that the negation is true.
- **Proof** (of negation)

Let $f : A \to A$ be defined by

$$f = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$$

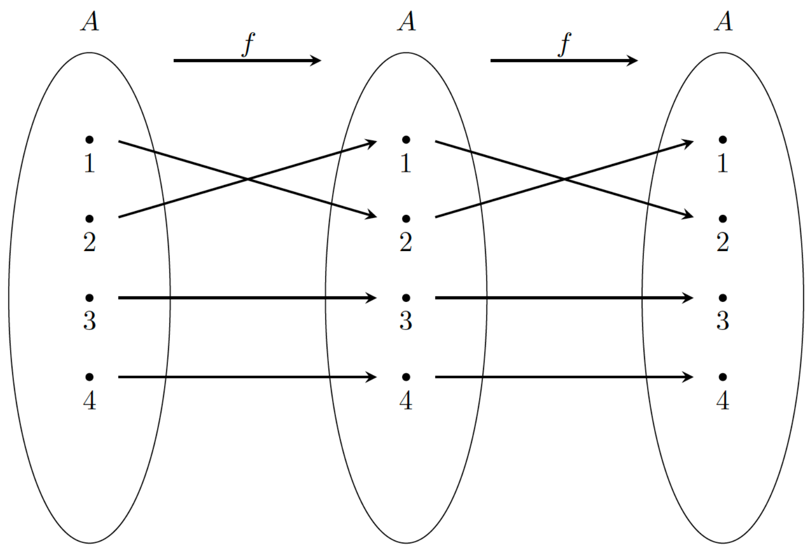
also let $g : A \to A$ be defined by

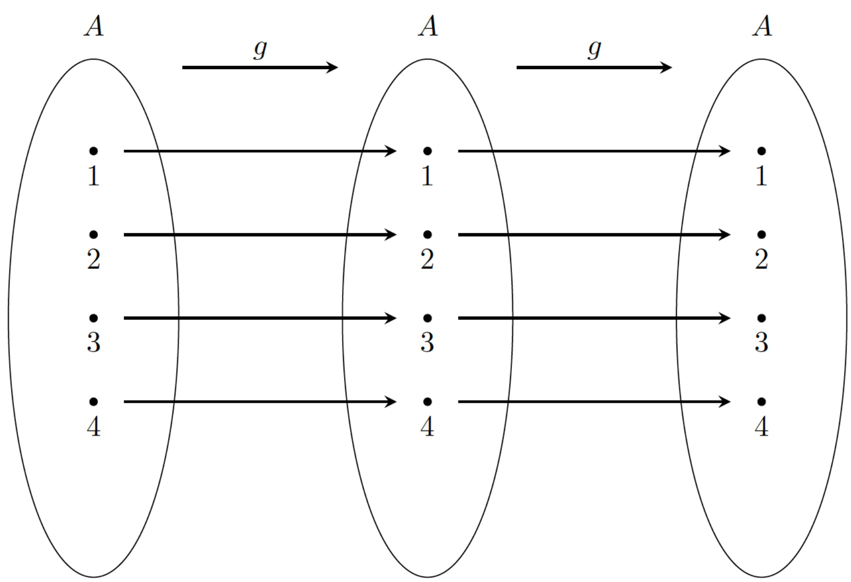
$$g = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

Since

$$\begin{aligned}
(f \circ f)(1) = f(f(1)) = f(2) = 1 = g(1) = g(g(1)) = (g \circ g)(1) \\
(f \circ f)(2) = f(f(2)) = f(1) = 2 = g(2) = g(g(2)) = (g \circ g)(2) \\
(f \circ f)(3) = f(f(3)) = f(3) = 3 = g(3) = g(g(3)) = (g \circ g)(3) \\
(f \circ f)(4) = f(f(4)) = f(4) = 4 = g(4) = g(g(4)) = (g \circ g)(4)
\end{aligned}$$

We have $\forall x \in A$, $(f \circ f)(x) = I_A(x)$, thus $f \circ f = I_A$, but since $(1, 2) \in f$ and $(1, 2) \notin g$, $f \neq g$. Thus the negation is true and the arrow diagrams of $f \circ f$ and $g \circ g$ are shown below:





(d)

- The statement is false.
- Its negation is: $\exists f, g \in F$ such that $f \circ g = g \circ f$ but $f \neq g$. We prove that the negation is true.
- **Proof** (of negation)

  Let $f : A \to A$ be defined by

  $$f = \{(1,2),(2,2),(3,3),(4,3)\}$$

  also let $g : A \to A$ be defined by

  $$g = \{(1,1),(2,2),(3,3),(4,4)\}$$

Since

$$
\begin{aligned}
(f \circ g)(1) = f(g(1)) = f(1) = 1 = g(1) = g(f(1)) = (g \circ f)(1) \\
(f \circ g)(2) = f(g(2)) = f(2) = 2 = g(2) = g(f(2)) = (g \circ f)(2) \\
(f \circ g)(3) = f(g(3)) = f(3) = 3 = g(3) = g(f(3)) = (g \circ f)(3) \\
(f \circ g)(4) = f(g(4)) = f(4) = 3 = g(3) = g(f(4)) = (g \circ f)(4)
\end{aligned}
$$

We have $\forall x \in A$, $(f \circ g)(x) = (g \circ f)(x)$, thus $f \circ g = g \circ f$. But since $(4, 3) \in f$ and $(4, 3) \notin f$, $f \neq g$. Thus the negation is true and the arrow diagrams of $g \circ f$ and $f \circ g$ are shown below: