Solutions of Warmup Problems for Mathematical Induction

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Question 1

Proof The claim will be proved by the standard form of mathematical induction on n. The cases that $n = 0$ will be considered in the basis.

Basis (n=0) When $n = 0$,

$$n^2 - n = 0^2 - 0 = 0 = 2 \cdot 0$$

Thus n is an even number by the definition, as required.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use

Inductive Hypothesis:

For every integer k, $k^2 - k$ is an even number.

to prove

Inductive Claim:

$(k + 1)^2 - (k + 1)$ is an even number.

Since $k^2 - k$ is an even number by the inductive hypothesis, there exist an integer m such that

$$k^2 - k = 2m$$

by the definition.

Note that,

$$\begin{aligned}(k+1)^2 - (k+1) &= k^2 + 2k + 1 - k - 1 \\ &= k^2 + k \\ &= (k^2 - k) + 2k\end{aligned}$$

Since $k^2 - k = 2m$, we have

$$\begin{aligned}(k+1)^2 - (k+1) &= (k^2 - k) + 2k \\ &= 2m + 2k \\ &= 2(m+k)\end{aligned}$$

Since $m + k$ is an integer, $(k+1)^2 - (k+1)$ is an even number by the definition, as required.

Conclusion: Therefore, by standard form of mathematical induction, we can conclude that, for every integer n such that $n \geq 0$, $n^2 - n$ is an even number.

Question 2

Proof The claim will be proved by the standard form of mathematical induction on n. The cases that $n = 0$ will be considered in the basis.

Basis (n=0) When $n = 0$,

$$n^3 - n = 0^3 - 0 = 0 = 6 \cdot 0$$

Thus $n^3 - n$ is divisible by 6, as required.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use

Inductive Hypothesis:

For every integer k, $k^3 - k$ is divisible by 6.

to prove

Inductive Claim:

$(k+1)^3 - (k+1)$ is divisible by 6.

Since $k^3 - k$ is divisible by 6 by the inductive hypothesis, there exist an integer m such that

$$k^3 - k = 6m$$

by the definition.

Note that,

$$
\begin{aligned}
(k+1)^3-(k+1) &= (k+1)((k+1)^2-1) \\
&= (k+1)(k^2+2k) \\
&= k^3+k^2+2k^2+2k \\
&= k^3+3k^2+2k \\
&= k^3-k+3k^2+3k \\
&= k^3-k+3k^2-3k+6k \\
&= (k^3-k)+(3k^2-3k)+6k \\
&= (k^3-k)+3(k^2-k)+6k
\end{aligned}
$$

Notice that we have proved that k^2-k is an even number such that $k \geq 0$ in **Question 1**, thus there exist an integer p such that $k^2-k=2p$ by the definition. Meanwhile, since $k^3-k=6m$, we have

$$
\begin{aligned}
(k+1)^3-(k+1) &= (k^3-k)+3(k^2-k)+6k \\
&= 6m+3\cdot 2p+6k \\
&= 6(m+p+k)
\end{aligned}
$$

Since $m+p+k$ is an integer, by the definition, we can see that $(k+1)^3-(k+1)$ is divisible by 6, as required.

Conclusion: Therefore, by standard form of mathematical induction, we can conclude that, for every integer n such that $n \geq 0$, n^3-n is divisible by 6.

Question 3

Proof The claim will be proved by the standard form of mathematical induction on n. The cases that $n=0$ will be considered in the basis.

Basis (n=0) When $n=0$,

$$
\sum_{i=0}^{n} i^2 = \sum_{i=0}^{0} i^2 = 0^2 = 0 = \frac{0(0+1)(2\cdot 0+1)}{6} = \frac{n(n+1)(2n+1)}{6}
$$

as required.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use

Inductive Hypothesis:

For every integer $k \geq 0$,

$$
\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}
$$

to prove

Inductive Claim:

$$\sum_{i=0}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Note that,

$$\sum_{i=0}^{k+1} i^2 = (\sum_{i=0}^{k} i^2) + (k+1)^2$$

Thus, by the inductive hypothesis, we have

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(2k^2+k) + (k+1)(6k+6)}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

as required.

Conclusion: Therefore, by standard form of mathematical induction, we can conclude that, for every integer n such that $n \geq 0$,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Question 4

Proof The claim will be proved by the standard form of mathematical induction on n. The cases that $n = 0$

will be considered in the basis.

Basis (n=0) When $n = 0$,

$$\sum_{i=0}^{n} i^3 = \sum_{i=0}^{0} i^3 = 0^3 = 0 = \frac{0^2(0+1)^2}{4} = \frac{n^2(n+1)^2}{4}$$

as required.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use

Inductive Hypothesis:

For every integer $k \geq 0$,

$$\sum_{i=0}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

to prove

Inductive Claim:

$$\sum_{i=0}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

Note that,

$$\sum_{i=0}^{k+1} i^3 = (\sum_{i=0}^{k} i^3) + (k+1)^3$$

Thus, by the inductive hypothesis, we have

$$\begin{aligned}\sum_{i=0}^{k+1} i^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4}\end{aligned}$$

as required.

Conclusion: Therefore, by standard form of mathematical induction, we can conclude that, for every integer n such that $n \geq 0$,

$$\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Question 5

Proof The claim will be proved by the strong form of mathematical induction on n. The cases that $n = 0$ and $n = 1$ will be considered in the basis.

Basis (n=0) When $n = 0$, the binary tree has no edges, thus it contains exactly 1 node that is also the root of the tree. Since $1 = 0 + 1 = n + 1$, the result holds when $n = 0$.

Basis (n=1) When $n = 1$, the binary tree has only 1 edge, thus the root has only one child which is also the leaf of the tree, thus the tree has exactly 2 nodes. Since $2 = 1 + 1 = n + 1$, the result holds when $n = 1$.

Inductive Step: Let k be an integer such that $k \geq 1$. It is necessary and sufficient to use

Inductive Hypothesis: For all integers n such that $0 \leq n \leq k$, every binary tree with n edges has exactly $n + 1$ nodes.

to prove

Inductive Claim:

Every binary tree with $k + 1$ edges has exactly $(k + 1) + 1$ nodes.

Suppose a binary tree T with $k + 1$ edges, suppose the left subtree of the root of T has k_L edges and the right subtree of the root of T has k_R edges. Thus

$$k_L + k_R + 2 = k + 1$$

$$k_L + k_R = k - 1$$

Since $k_L \geq 0$, $k_R \geq 0$, $k - 1 \geq 0$,

$$k_L \leq k - 1 \leq k$$

$$k_R \leq k - 1 \leq k$$

Thus, by the **Inductive Hypothesis**, we can see the left subtree of the root of T has exactly $k_L + 1$ nodes, the right subtree of the root of T has exactly $k_R + 1$ nodes, thus the number of nodes of T is exactly

$$(k_L + 1) + (k_R + 1) + 1 = k_L + k_R + 3 = (k_L + k_R + 2) + 1 = (k + 1) + 1$$

as required.

Conclusion: Therefore, by strong form of mathematical induction, we can conclude that, every binary tree with n edges has exactly $n + 1$ nodes, for every integer $n \geq 0$.

Question 6

Proof Suppose a binary tree T contains n nodes and its depth is d_n. The claim will be proved by the strong form of mathematical induction on n. Since T is not empty, $n \geq 1$. Thus cases that $n = 1$ and $n = 2$ will be considered in the basis.

Basis (n=1) When $n = 1$, T has only 1 node which is also the root of T, thus $d_n = d_1 = 0$. Since $0 + 1 \leq 1 \leq 2^{0+1} - 1$, we have

$$d_n + 1 \leq n \leq 2^{d_n + 1} - 1$$

as required.

Basis (n=2) When $n = 2$, T has only 2 nodes, thus the root of T has only 1 child which is also the leaf of T, thus $d_2 = 1$. Since $1 + 1 \leq 2 \leq 2^{1+1} - 1$, we have

$$d_n + 1 \leq n \leq 2^{d_n + 1} - 1$$

as required.

Inductive Step: Let k be an integer such that $k \geq 2$. It is necessary and sufficient to use

Inductive Hypothesis: For all integers n such that $0 \leq n \leq k$, if a binary tree T whose depth is d_n and it has n_d nodes,

$$d_n + 1 \leq n \leq 2^{d_n + 1} - 1$$

to prove

Inductive Claim:

If a binary tree T whose depth is d_{k+1} and it has $k + 1$ nodes,

$$d_{k+1} + 1 \leq k + 1 \leq 2^{d_{k+1} + 1} - 1$$

Suppose a binary tree T whose depth is d_{k+1} and it has $k + 1$ nodes, also suppose the left subtree of the root of T has n_L nodes and its depth is d_{n_L}, the right subtree of the root of T has n_R nodes and its depth is d_{n_R}. Thus

$$n_L + n_R + 1 = k + 1$$

$$n_L + n_R = k$$

Since $n_L \geq 0$, $n_R \geq 0$ and $k \geq 2 \geq 0$, we have

$$n_L \leq k$$

$$n_R \leq k$$

Thus, by the **Inductive Hypothesis**, we have

$$d_{n_L} + 1 \leq n_L \leq 2^{d_{n_L}+1} - 1$$

$$d_{n_R} + 1 \leq n_R \leq 2^{d_{n_R}+1} - 1$$

Thus

$$(d_{n_L} + 1) + (d_{n_R} + 1) + 1 \leq n_L + n_R + 1 = k + 1 \leq (2^{d_{n_L}+1} - 1) + (2^{d_{n_R}+1} - 1) + 1$$

$$d_{n_L} + d_{n_R} + 3 \leq k + 1 \leq 2^{d_{n_L}+1} + 2^{d_{n_R}+1} - 1$$

By the definition, we have

$$\begin{aligned}
d_{k+1} &= \max(d_{n_L}, d_{n_R}) + 1 \\
d_{k+1} + 1 &= \max(d_{n_L}, d_{n_R}) + 2 \\
&\leq \max(d_{n_L}, d_{n_R}) + \min(d_{n_L}, d_{n_R}) + 2 \\
&\leq \max(d_{n_L}, d_{n_R}) + \min(d_{n_L}, d_{n_R}) + 3 \\
&\leq d_{n_L} + d_{n_R} + 3
\end{aligned}$$

Thus

$$d_{k+1} + 1 \leq k + 1 \leq 2^{d_{n_L}+1} + 2^{d_{n_R}+1} - 1$$

On the other hand, since

$$d_{k+1} + 1 = \max(d_{n_L}, d_{n_R}) + 2$$

We have

$$2^{d_{k+1}+1} = 2^{\max(d_{n_L}, d_{n_R})+2}$$

Thus

$$
\begin{aligned}
2^{d_{k+1}+1} - 1 &= 2^{\max(d_{n_L}, d_{n_R})+2} - 1 \\
&= 2 \cdot 2^{\max(d_{n_L}, d_{n_R})+1} - 1 \\
&= 2^{\max(d_{n_L}, d_{n_R})+1} + 2^{\max(d_{n_L}, d_{n_R})+1} - 1 \\
&\geq 2^{d_{n_L}+1} + 2^{d_{n_R}+1} - 1
\end{aligned}
$$

Therefore

$$d_{k+1} + 1 \leq k + 1 \leq 2^{d_{k+1}+1} - 1$$

as required.

Conclusion: Therefore, by strong form of mathematical induction, we can conclude that, a nonempty binary tree with depth d has at least $d+1$ nodes and at most $2^{d+1}-1$ nodes, for every integer $d \geq 0$.