1 MILP

1.1 Index Sets and Variables

Sets:
Let $\mathcal{A}$ be the set of possible aperture stops,

$$\mathcal{A} = \{a_{\min}, ..., a_{\max}\} \tag{1}$$

indexed by the set,

$$[n_A] = \{1, 2, ..., n_A\}, \quad n_A = |\mathcal{A}| \tag{2}$$

Let $\mathcal{T}$ be the set of possible exposure time stops,

$$\mathcal{T} = \{t_{\min}, ..., t_{\max}\} \tag{3}$$

indexed by the set,

$$[n_T] = \{1, 2, ..., n_T\}, \quad n_T = |\mathcal{T}| \tag{4}$$

Let $\mathcal{S}$ be the set of possible ISO stops,

$$\mathcal{S} = \{s_{\min}, ..., s_{\max}\} \tag{5}$$

indexed by the set,

$$[n_S] = \{1, 2, ..., n_S\}, \quad n_S = |\mathcal{S}| \tag{6}$$

Decision variables:

$$h_{flt} \in \mathbb{R} \quad \text{(Flight height above launch)} \tag{7}$$

$$v_{vel} \in \mathbb{R} \quad \text{(UAV velocity)} \tag{8}$$

$$y \in \mathbb{R} \quad \text{(Epigraph variable for objective)} \tag{9}$$

$$x \in \mathbb{R} \quad \text{(Auxiliary variable: } x = yv\text{)} \tag{10}$$

$$b_i^{(A)} \in \{0, 1\}, \quad i \in [n_A] \quad \text{(Aperture binary choice)} \tag{11}$$

$$b_j^{(T)} \in \{0, 1\}, \quad j \in [n_T] \quad \text{(Integration-time binary choice)} \tag{12}$$

$$b_k^{(S)} \in \{0, 1\}, \quad k \in [n_S] \quad \text{(ISO binary choice)} \tag{13}$$

$$b_p^{(L)} \in \{0, 1\}, \quad p \in [n_L] \quad \text{(ceiling function binary choice)} \tag{14}$$

1.2 Constraints

One hot encoding:

$$\sum_{i \in [n_A]} b_i^{(A)} = 1 \tag{15}$$

$$\sum_{j \in [n_T]} b_j^{(T)} = 1 \tag{16}$$

$$\sum_{k \in [n_S]} b_k^{(S)} = 1 \tag{17}$$

1.2.1 Sensor Resolution

$$h_l \leq \frac{GSD_{req} \cdot f}{S_\delta} + z_{s,min} - z_l \tag{18}$$

1.2.2 Diffraction Resolution

Given,

$$h_l \leq \frac{GSD_{req} \cdot f}{2.44 \cdot N \cdot \lambda} + z_{s,min} - z_l \tag{19}$$

The linear constraint becomes,

$$h_l \leq \alpha_i + z_{s,\min} - z_l + M_i^{\alpha}(1 - b_i^{(A)}), \quad i \in [n_A] \tag{20}$$

Where,

$$\alpha_i = \frac{GSD_{req} \cdot f}{a_i \cdot 2.44 \cdot \lambda}, \quad i \in [n_A] \tag{21}$$

$$M_i^{\alpha} = \max\{0, l, -\alpha_i - z_{s,\min} + z_l\}, \quad i \in [n_A] \tag{22}$$

1.2.3 DOF Near Limit

Given,

$$h_l \geq \frac{N \cdot c \cdot f \cdot s + s \cdot f^2}{N \cdot c \cdot (f + s) + f^2} + z_{s,\max} - z_l \tag{23}$$

the linear constraint becomes,

$$h_l \geq \beta_i + z_{s,\max} - z_l + M_i^{\beta}(1 - b_i^{(A)}), \quad i \in [n_A] \tag{24}$$

where,

$$\beta_i = \frac{a_i \cdot c \cdot f \cdot s + s \cdot f^2}{a_i \cdot c \cdot (f + s) + f^2}, \quad i \in [n_A] \tag{25}$$

$$M_i^{\beta} = \max\{0, \beta_i + z_{s,\max} - z_l - l, \}, \quad i \in [n_A] \tag{26}$$

1.2.4 DOF Far Limit

Given,

$$h_l \leq \frac{N \cdot c \cdot f \cdot s + s \cdot f^2}{N \cdot c \cdot (f - s) + f^2} + z_{s,\min} - z_l \tag{27}$$

the linear constraint becomes,

$$h_l \leq \gamma_i + z_{s,\min} - z_l + M_i^{\gamma}(1 - b_i^{(A)}), \quad i \in [n_A] \tag{28}$$

where,

$$\gamma_i = \frac{a_i \cdot c \cdot f \cdot s + s \cdot f^2}{a_i \cdot c \cdot (f - s) + f^2}, \quad i \in [n_A] \tag{29}$$

$$M_i^{\gamma} = \max\{0, h_{l,\max} - \gamma_i - z_{s,\min} + z_l\}, \quad i \in [n_A] \tag{30}$$

$$\tag{31}$$

1.2.5 Motion Blur

Given,

$$v \leq \frac{S_\delta \cdot (h_l + z_l - z_{s,\max}) \cdot \delta_{\max}}{f \cdot I_t} \tag{32}$$

the linear constraint becomes,

$$v \leq \zeta_j (h_l + z_l - z_{s,\max}) + M_j^\zeta (1 - b_j^{(T)}), \quad j \in [n_T] \tag{33}$$

$$\tag{34}$$

where,

$$\zeta_j = \frac{S_\delta \cdot \delta_{\max}}{t_j \cdot f}, \quad j \in [n_T] \tag{35}$$

$$M_j^\zeta = \max\{0, V_{\max} - \zeta_j (h_{l,\min} + z_l - z_{s,\max})\}, \quad j \in [n_T] \tag{36}$$

1.2.6 Endlap

$$v \leq \frac{I_{f,\max} \cdot S_y \cdot (1 - \frac{EO_{req}}{100})}{f} \cdot (h_l + z_l - z_{s,max}) \tag{37}$$

1.2.7 Exposure

Given,

$$\log_2(\frac{N^2}{I_t}) + \log_2(\frac{100}{ISO}) = EV \tag{38}$$

the linear constraint becomes,

$$\sum_{i \in [n_A]} \eta_i b_i^{(A)} - \sum_{j \in [n_T]} \theta_j b_j^{(T)} - \sum_{k \in [n_S]} \mu_k b_k^{(S)} = \xi + s^+ - s^- \tag{39}$$

where,

$$\eta_i = 2 \log_2(a_i), \quad i \in [n_A] \tag{40}$$

$$\theta_j = \log_2(t_j), \quad j \in [n_T] \tag{41}$$

$$\mu_k = \log_2(s_k), \quad k \in [n_S] \tag{42}$$

$$\xi = EV - \log_2(100) \tag{43}$$

$$s^- \geq 0 \tag{44}$$

$$s^+ \leq \epsilon \tag{45}$$

1.3 Objective

$$\text{Flight Time} = \frac{S_n \cdot (A_y + G_S) + A_x + G_E}{F_v} \tag{46}$$

$$= \frac{\lceil \frac{A_x + \frac{S_x \cdot h}{f}}{\frac{S_x \cdot h}{f} \cdot (1 - \frac{SO}{100})} \rceil \cdot (A_y + \frac{S_x \cdot h}{f}) + A_x + \frac{S_y \cdot h}{f}}{v} \tag{47}$$

$$f(h, v) = \frac{\lceil \frac{\sigma_1 + \sigma_2 h}{\sigma_3 h} \rceil \cdot (\sigma_4 + \sigma_2 h) + \sigma_1 + \sigma_5 h}{v} \tag{48}$$

$$\tag{49}$$

where,

$$\sigma_1 = A_x \tag{50}$$

$$\sigma_2 = \frac{S_x}{f} \tag{51}$$

$$\sigma_3 = \frac{S_x}{f}(1 - \frac{SO}{100}) \tag{52}$$

$$\sigma_4 = A_y \tag{53}$$

$$\sigma_5 = \frac{S_y}{f} \tag{54}$$

1.4 Flight Lines

To linearise the ceiling function

$$\lceil \frac{\sigma_1 + \sigma_2 h_l}{\sigma_3 h_l} \rceil \tag{55}$$

we introduce the variable,

$$l \in \mathbb{Z} \tag{56}$$

$$l - 1 \leq \frac{\sigma_1 + \sigma_2 h_l}{\sigma_3 h_l} < l \tag{57}$$

$$l_{\min} \leq l \leq l_{\max} \tag{58}$$

This set of inequalities ensures that $l = \lceil \frac{\sigma_1 + \sigma_2 h}{\sigma_3 h} \rceil$. To determine the bounds, $l_{\min}$ and $l_{\max}$, notice,

$$\frac{\sigma_1 + \sigma_2 h_l}{\sigma_3 h_l} = \frac{\sigma_1}{\sigma_3 \, h_l} + \frac{\sigma_2}{\sigma_3} \tag{59}$$

which is strictly decreasing for $\sigma_1, \sigma_2, \sigma_3, h_l > 0$. Therefore,

$$\min_{h_l \in [h_{l,\min}, h_{l,\max}]} \{ \frac{\sigma_1 + \sigma_2 h_l}{\sigma_3 h_l} \} = \frac{\sigma_1 + \sigma_2 h_{l,\max}}{\sigma_3 h_{l,\max}} \tag{60}$$

$$\max_{h_l \in [h_{l,\min}, h_{l,\max}]} \{ \frac{\sigma_1 + \sigma_2 h_l}{\sigma_3 h_l} \} = \frac{\sigma_1 + \sigma_2 h_{l,\min}}{\sigma_3 h_{l,\min}} \tag{61}$$

and,

$$l_{\min} = \lceil \frac{\sigma_1 + \sigma_2 h_{l,\max}}{\sigma_3 h_{l,\max}} \rceil \tag{62}$$

$$l_{\max} = \lceil \frac{\sigma_1 + \sigma_2 h_{l,\min}}{\sigma_3 h_{l,\min}} \rceil \tag{63}$$

Whilst the ceiling function has been handled, Equation 57 is is bilinear. Thus we define the set of all possible values of l,

$$\mathcal{L} = \{l_{\min}, l_{\min} + 1, ..., l_{\max}\} \tag{64}$$

indexed by,

$$[n_L] = \{1, 2, ..., n_L\}, \quad n_L = |\mathcal{L}| \tag{65}$$

and the binary decision variable,

$$b_p^{(L)} \in \{0, 1\}, \quad p \in [n_L] \tag{66}$$

$$\sum_{p \in [n_L]} b_p^{(L)} = 1 \tag{67}$$

$$\tag{68}$$

Now Equation 57 can be expressed using big M notation as,

$$\sigma_3 h_l (l_p - 1) \leq \sigma_1 + \sigma_2 h_l + M_p^{(lower)} (1 - b_p^{(L)}), \quad \forall \quad p \in [n_L] \tag{69}$$

where,

$$M_p^{(lower)} = \max_{h_l \in [h_{l,\min}, h_{l,\max}]} \{\sigma_3 h_l (l_p - 1) - \sigma_1 - \sigma_2 h_l\}, \quad \forall \quad p \in [n_L] \tag{70}$$

and

$$\sigma_1 + \sigma_2 h_l \leq \sigma_3 h_l l_p - \epsilon + M_p^{(upper)} (1 - b_p^{(L)}), \quad \forall \quad p \in [n_L] \tag{71}$$

where $\epsilon > 0$ is a small constant used to allow the soft inequality and,

$$M_p^{(upper)} = \max_{h_l \in [h_{l,\min}, h_{l,\max}]} \{\sigma_1 + \sigma_2 h_l - \sigma_3 h_l l_p + \epsilon\}, \quad \forall \quad p \in [n_L] \tag{72}$$

1.5 Binary Continuous Product Linearisation

This transforms the objective function to,

$$f(h, v) = \frac{\sum_{p \in [n_L]} l_p b_p (\sigma_4 + \sigma_2 h_l) + \sigma_1 + \sigma_5 h_l}{v} \tag{73}$$

Let q reprent the continuous linear function of h_l,

$$q = \sigma_4 + \sigma_2 h_l \tag{74}$$

$$q_{\max} = \sigma_4 + \sigma_2 h_{l,\max} \tag{75}$$

$$q_{\min} = \sigma_4 + \sigma_2 h_{l,\min} \tag{76}$$

and r_p be the product of the binary variable $b_p^{(L)}$ and q,

$$r_p = b_p q, \quad \forall \quad p \in [n_L] \tag{77}$$

The following linear constraints enforce Equation 77,

$$r_p \leq q_{\max} b_p^{(L)}, \quad \forall p \in [n_L] \tag{78}$$

$$r_p \geq q_{\min} b_p^{(L)}, \quad \forall p \in [n_L] \tag{79}$$

$$r_p \leq q - q_{\min} (1 - b_p^{(L)}), \quad \forall p \in [n_L] \tag{80}$$

$$r_p \geq q - q_{\max} (1 - b_p^{(L)}), \quad \forall p \in [n_L] \tag{81}$$

When $b_p = 1$, these constraints force r_p to equal q. When $b_p = 0$, they force $r_p = 0$, regardless of the value of q. Now,

$$f(h, v) = \frac{w}{v} \tag{82}$$

where w is the linear function of h_l,

$$w = \sum_{p \in [n_L]} l_p r_p + \sigma_1 + \sigma_5 h_l \tag{83}$$

Introducing the epigraph variable y allows a linear objective,

$$\min y \tag{84}$$

but introduces the bilinear constraint,

$$yv \geq w \tag{85}$$

however, using McCormick envelopes we can linearise Equation 85 given we can calculate the bounds on y and v.

1.5.1 Epigraph linearisation via McCormick envelopes

Let x represent the bilinear term,

$$x = yv \tag{86}$$

and enforce

$$w \leq x \tag{87}$$

The McCormick envelopes provide the tightest convex relaxation of the bilinear constraint 86:

$$x \geq y_{\min} v + v_{\min} y - y_{\min} v_{\min} \tag{88}$$
$$x \geq y_{\max} v + v_{\max} y - y_{\max} v_{\max} \tag{89}$$
$$x \leq y_{\min} v + v_{\max} y - y_{\min} v_{\max} \tag{90}$$
$$x \leq y_{\max} v + v_{\min} y - y_{\max} v_{\min} \tag{91}$$

where,

$$v_{\min} = v_{\min} \tag{92}$$
$$v_{\max} = v_{\max} \tag{93}$$
$$y_{\min} = \frac{l_{\min} q_{\min} + \sigma_1 + \sigma_5 h_{l,min}}{v_{\max}} \tag{94}$$
$$y_{\max} = \frac{l_{\max} q_{\max} + \sigma_1 + \sigma_5 h_{l,max}}{v_{\min}} \tag{95}$$

These four linear inequalities, together with the auxiliary variable x, replace the bilinear constraint $yv \geq w$.

2 Program Summary

$$
\begin{aligned}
\min_{v,h_l,q,r,y,x,w,b^{(A)},b^{(T)},b^{(S)},b^{(L)},s^+,s^-} \quad & y \\
\text{s.t.} \quad & w = \sum_{p\in[n_L]} l_p r_p + \sigma_1 + \sigma_5 h_l \\
& w \le x \\
\\
& x \ge y_{\min} v + v_{\min} y - y_{\min} v_{\min} \\
& x \ge y_{\max} v + v_{\max} y - y_{\max} v_{\max} \\
& x \le y_{\min} v + v_{\max} y - y_{\min} v_{\max} \\
& x \le y_{\max} v + v_{\min} y - y_{\max} v_{\min} \\
\\
& q = \sigma_4 + \sigma_2 h_l \\
& r_p \le q_{\max}\, b_p^{(L)}, \quad \forall\, p \in [n_L] \\
& r_p \ge q_{\min}\, b_p^{(L)}, \quad \forall\, p \in [n_L] \\
& r_p \le q - q_{\min}(1 - b_p^{(L)}), \quad \forall\, p \in [n_L] \\
& r_p \ge q - q_{\max}(1 - b_p^{(L)}), \quad \forall\, p \in [n_L] \\
\\
\sigma_3 h_l (l_p - 1) \le{} & \sigma_1 + \sigma_2 h_l + M_p^{(lower)}(1 - b_p^{(L)}), \quad \forall \quad p \in [n_L] \\
\sigma_1 + \sigma_2 h_l \le{} & \sigma_3 h_l l_p - \epsilon + M_p^{(upper)}(1 - b_p^{(L)}), \quad \forall \quad p \in [n_L] \\
\\
& h_l \le \frac{GSD_{req} \cdot f}{S_\delta} + z_{s,min} - z_l \\
& h_l \le \alpha_i + z_{s,\min} - z_l + M_i^{\alpha}(1 - b_i^{(A)}), \quad i \in [n_A] \\
& h_l \ge \beta_i + z_{s,\max} - z_l + M_i^{\beta}(1 - b_i^{(A)}), \quad i \in [n_A] \\
& h_l \le \gamma_i + z_{s,\min} - z_l + M_i^{\gamma}(1 - b_i^{(A)}), \quad i \in [n_A] \\
\\
& v \le \zeta_j (h_l + z_l - z_{s,\max}) + M_j^{\zeta}(1 - b_j^{(T)}), \quad j \in [n_T] \\
& v \le \frac{I_{f,\max} \cdot S_y \cdot (1 - \frac{EO_{req}}{100})}{f} \cdot (h_l + z_l - z_{s,max}) \\
\\
\xi + s^+ - s^- ={} & \sum_{i\in[n_A]} \eta_i b_i^{(A)} - \sum_{j\in[n_T]} \theta_j b_j^{(T)} - \sum_{k\in[n_S]} \mu_k b_k^{(S)} \\
\\
& \sum_{i\in[n_A]} b_i^{(A)} = 1 \\
& \sum_{j\in[n_T]} b_j^{(T)} = 1 \\
& \sum_{k\in[n_S]} b_k^{(S)} = 1 \\
& \sum_{p\in[n_L]} b_p^{(L)} = 1
\end{aligned}
$$