Paper Summary: A Practical Walk-on-Boundary Method for Boundary Value Problems

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1 Contribution

Introduce walk-on-boundary (WoB) method to solve boundary value problems. WoB is similar to walk-on-spheres (WoS) as a Monte Carlo solver, but it has some advantages over WoS since it is built upon boundary integral equations (BIEs):

- **Versatility**: WoB supports
  - various **boundary conditions** like Dirichlet, Neumann, Robin, and mixed
  - both interior and **exterior** domains
- **Easy to implement**: WoB can reuse many rendering techniques thanks to its similarity to Monte Carlo ray tracing algorithms.
- **Accuracy on boundary**: WoB samples points directly on the boundary rather than inside the domain (compared to WoS which has an ϵ-shell error).

Primary contributions that distinguish this paper from prior work on WoB:

- Apply WoB to **direct BIE formulations**
- Apply WoB on **mixed boundary conditions**

2 Method

The key of WoB is similar to WoS and MC ray tracing (recursively), but while WoS terminates when the distance between the current sample point and the boundary is less than the prescribed ϵ value, WoB terminates when it reaches the maximum path length like ray tracing.

Boundary Integral Equations define the solution of some kinds of PDEs based on **integrals only of boundary values**. We need the **fundamental solutions** in both types of formulation, but:

- **Direct** BIEs use values on boundary directly, and

- **Indirect** BIEs introduce an **unknown source density function** on the boundary

To solve **Dirichlet problems with double layer BIE**, we:

1. find the value of the unknown source density function on a sample point by recursively using the Fredholm equation of the second kind defining the relationship between the unknown source density function on the boundary and the given boundary conditions

2. assemble multiple sample points' source density function value to get the function value we want at the queried interior point using the Monte Carlo method

The Fredholm equation of the second kind is derived by taking the limit to the boundary in double layer potential.

Similar procedures can be used for **Neumann problems with direct BIE** and **mixed boundary problems with single layer BIE**.

For **direct BIE** we don't need to estimate the unknown source density function, so we just directly estimate the value recursively.

For **Dirichlet boundary conditions in the mixed case**, we get the Fredholm equation of the first kind that cannot be directly estimated elegantly by WoB. However, we can transform it into a similar equation to the second kind. Then, we can use a similar procedure above.

To estimate the gradient of the function, we just need to take the derivative of both sides in BIEs and use the same procedure.

2.1 Path truncation

The rendering equation is also regarded as a Fredholm equation of the second kind.

We can truncate the path easily in ray tracing due to the decreasing nature of light transportation, but we cannot do the same thing in WoB for BVPs since the integral kernel of the BIE will not "attenuate" its contribution per recursion. That is, the Neumann series expansion in the BIEs may not converge.

This paper proposes to rearrange the series so that the sequence converges to zero. Then, we can truncate the path with a modification to the last term.

3 Question?

I have a tiny question in section 3.1 of this paper: it says "where Ω is a closed domain" but " Let us denote the boundary of Ω by $\Gamma = \partial\Omega$; note that Γ is not included in Ω."

In my understanding, a closed set/domain contains its boundary, so it seems contradictory to say that Γ is not included in Ω.