UNIVERSITY OF CALIFORNIA SAN DIEGO, CSE 270 WINTER 2024
GRADE 24%

# Homework 2

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## EXTERIOR PRODUCT: EXERCISE 2.1 — 5 PTS.

**Let $dx, dy, dz, dt$ denote a basis for $(R^4)^\star$. Define the 2-form $\alpha = u_{12}dx\wedge dy + u_{24}dy\wedge dt + u_{34}dz\wedge dt$ and the 1-form $\beta = w_2 dy + w_3 dz$. Compute $\alpha\wedge\beta$ and $\alpha\wedge\alpha$**

$$\begin{aligned}\alpha\wedge\beta &= u_{12}w_3 dx\wedge dy\wedge dz + u_{24}w_3 dy\wedge dt\wedge dz + u_{34}w_2 dz\wedge dt\wedge dy\\ &= u_{12}w_3 dx\wedge dy\wedge dz - (u_{24}w_3 + u_{34}w_2) dy\wedge dz\wedge dt\end{aligned}$$

$$\begin{aligned}\alpha\wedge\alpha &= u_{12}u_{34}dx\wedge dy\wedge dz\wedge dt + u_{34}u_{12}dz\wedge dt\wedge dx\wedge dy\\ &= 2u_{12}u_{34}dx\wedge dy\wedge dz\wedge dt\end{aligned}$$

## VECTOR IDENTITIES IN 3D: EXERCISE 2.2 — 10 PTS.

**Use the Leibniz rule, and the vector-form correspondence in 3D, to show the BAC-CAB formula (3) and the Binet–Cauchy identity (4)**

Prove the BAC-CAB formula (assume that $\alpha = \mathbf{a}^\flat$, $\beta = \mathbf{b}^\flat$ and $\gamma = \mathbf{c}^\flat$):

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} \\
&= (i_{\mathbf{a}} \star (\mathbf{c} \times \mathbf{b})^\flat)^\sharp && \text{(Interior product applied to a 2-form corresponds to a cross product)} \\
&= (i_{\mathbf{a}}(\gamma \wedge \beta))^\sharp && \text{(Wedge between two 1-forms corresponds to a cross product)} \\
&= ((i_{\mathbf{a}}\gamma) \wedge \beta - \gamma \wedge (i_{\mathbf{a}}\beta))^\sharp && \text{(Leibniz rule)} \\
&= (\beta(i_{\mathbf{a}}\gamma) - \gamma(i_{\mathbf{a}}\beta))^\sharp && \text{(Wedge between a 0-form and a 1-form equals their "scalar product")} \\
&= \mathbf{b}(i_{\mathbf{a}}\gamma) - \mathbf{c}(i_{\mathbf{a}}\beta) && \text{(The constant part does not effect the sharp of a 1-form)} \\
&= \mathbf{b}\langle \mathbf{a}, \mathbf{c}\rangle - \mathbf{c}\langle \mathbf{a}, \mathbf{b}\rangle && \text{(Interior product applied to a 1-form corresponds to a dot product)}
\end{aligned}$$

Prove the Binet–Cauchy identity (assume that $\alpha = \mathbf{a}^\flat$ and $\beta = \mathbf{b}^\flat$):

$$\begin{aligned}
\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d}\rangle &= i_{\mathbf{c}\times\mathbf{d}}(\star(\alpha \wedge \beta)) \\
&= i_{\mathbf{d}} i_{\mathbf{c}}(\alpha \wedge \beta) \\
&= (\alpha \wedge \beta)(\mathbf{c}, \mathbf{d}) \\
&= \alpha(\mathbf{c})\beta(\mathbf{d}) - \alpha(\mathbf{d})\beta(\mathbf{c}) \\
&= \langle \mathbf{a}, \mathbf{c}\rangle\langle \mathbf{b}, \mathbf{d}\rangle - \langle \mathbf{b}, \mathbf{c}\rangle\langle \mathbf{a}, \mathbf{d}\rangle
\end{aligned}$$

## Vector Calculus Operators: Exercise 2.3 — 20 pts.

**Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and $\mathbf{a}, \mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields in 3D. Use Leibniz rules (for $d$) and the text before Exercise 2.2 to show formula (a) (b) (c) (d)**

(a)

$$\begin{aligned}
\nabla \cdot (\mathbf{a} \times \mathbf{b}) \det &= d \star (\mathbf{a} \times \mathbf{b})^\flat \\
&= d(\alpha \wedge \beta) \\
&= d\alpha \wedge \beta - \alpha \wedge d\beta \\
&= i_{\nabla\times\mathbf{a}} \det \wedge \beta - \alpha \wedge i_{\nabla\times\mathbf{b}} \det \\
&= i_{\nabla\times\mathbf{a}}\beta \det - i_{\nabla\times\mathbf{b}}\alpha \det \\
&= ((\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})) \det
\end{aligned}$$

Therefore we have $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ by removing det.

(b)

$$\begin{aligned}
\nabla\cdot(f\mathbf{a})\det &= d\star(f\mathbf{a})^\flat\\
&= d\star(f\mathbf{a}^\flat)\\
&= d(f\star\mathbf{a}^\flat)\\
&= d(f i_{\mathbf{a}}\det)\\
&= d(f\wedge i_{\mathbf{a}}\det)\\
&= df\wedge i_{\mathbf{a}}\det + f\wedge d i_{\mathbf{a}}\det\\
&= df\wedge i_{\mathbf{a}}\det + f(\nabla\cdot\mathbf{a})\det\\
&= (i_{\mathbf{a}}df\wedge\det - i_{\mathbf{a}}(df\wedge\det)) + f(\nabla\cdot\mathbf{a})\det\\
&= (i_{\mathbf{a}}df\wedge\det - 0) + f(\nabla\cdot\mathbf{a})\det\\
&= df(\mathbf{a})\det + f(\nabla\cdot\mathbf{a})\det\\
&= (df)^\sharp\cdot\mathbf{a}\det + f(\nabla\cdot\mathbf{a})\det\\
&= (\nabla f)\cdot\mathbf{a}\det + f(\nabla\cdot\mathbf{a})\det\\
&= ((\nabla f)\cdot\mathbf{a} + f(\nabla\cdot\mathbf{a}))\det
\end{aligned}$$

Therefore we have $\nabla\cdot(f\mathbf{a}) = ((\nabla f)\cdot\mathbf{a} + f(\nabla\cdot\mathbf{a}))$ by removing det.

(c)

$$\begin{aligned}
\nabla\times(f\mathbf{a}) &= (\star d(f\mathbf{a})^\flat)^\sharp\\
&= (\star d(f\wedge\mathbf{a}^\flat))^\sharp\\
&= (\star(df\wedge\mathbf{a}^\flat f\wedge d\mathbf{a}^\flat))^\sharp\\
&= (\star(df\wedge\mathbf{a}^\flat) + f\star d\mathbf{a}^\flat)^\sharp\\
&= (((df)^\sharp\times\mathbf{a})^\flat + f(\nabla\times\mathbf{a})^\flat)^\sharp\\
&= \nabla f\times\mathbf{a} + f\nabla\times\mathbf{a}
\end{aligned}$$

Thus $\nabla\times(f\mathbf{a}) = \nabla f\times\mathbf{a} + f\nabla\times\mathbf{a}$.

(d)

To prove this formula, we just need to replace $\mathbf{a}$ in (c) with $\nabla g$, then use the rule curl$\circ$grad $= 0$ to cancel the second term.

## Non-Cartesian Coordinate: Exercise 2.4 — 5 pts.

**Use (10) and (11), compute $\star dr$ and $\star d\theta$. (Write them in the form of $g(r,\theta)dr + h(r,\theta)d\theta$.)**

Let $\star dr = g_r(r,\theta)dr + h_r(r,\theta)d\theta$ and $\star d\theta = g_\theta(r,\theta)dr + h_\theta(r,\theta)d\theta$, then we could use the formula $\det_{\mathbb{R}^2} = r dr\wedge d\theta$ and the formulae of inner products to get the answer:

$$\begin{aligned}
\star dr &= r d\theta\\
\star d\theta &= -\frac{1}{r}dr
\end{aligned}$$

**Show that the Laplacian $\Delta f = \star d \star df$ in the polar coordinate is given by the formula**

$$\begin{aligned}
\star d \star df &= \star d \star (\frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta) \\
&= \star d (\frac{\partial f}{\partial r} r d\theta - \frac{\partial f}{\partial \theta}\frac{1}{r} dr) \\
&= \star ((\frac{\partial^2 f}{\partial r^2} r + \frac{\partial f}{\partial r}) dr \wedge d\theta - \frac{\partial^2 f}{\partial \theta^2}\frac{1}{r} d\theta \wedge dr) \\
&= \star ((\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r}\frac{1}{r} + \frac{\partial^2 f}{\partial \theta^2}\frac{1}{r^2}) r dr \wedge d\theta) \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r}\frac{1}{r} + \frac{\partial^2 f}{\partial \theta^2}\frac{1}{r^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}
\end{aligned}$$

## PIECEWISE LINEAR FINITE ELEMENT: EXERCISE 2.5 — 10 PTS.

**(a) Show that the aspect ratio of a triangle can be expressed as the sum of the cotangents of the interior angles at its base**

We have $\cot\alpha = \frac{w_1}{h}$, $\cot\beta = \frac{w_2}{h}$ and $w_1 + w_2 = h$, sum up cots to get $\frac{w}{h} = \cot\alpha + \cot\beta$.

**(b) Show that the gradient of the hat function on triangle $t_{ijk}$ is given by**

We can know that the gradient of the hat function at vertex $i$ on triangle $t_{ijk}$ is the vector of the shortest path pointing from $e_{jk}$ to vertex $i$ based on two facts: (1) the hat function $phi_i$ is linear on triangle $t_{ijk}$; (2) the value at vertex $i$ is 1 and the value on edge $e_{jk}$ is 0.

Then obviously the length of this shortest path equals the height of triangle $t_{ijk}$, which is $h = \frac{2A_{ijk}}{|\vec{e_{ik}}|}$. Combining the direction we get the answer $\text{grad}\phi_i = \frac{\vec{e_{ik}}^\perp}{2A_{ijk}}$

**(c) Show that for any hat function $\phi_i$ associated with vertex $p_i$ of triangle $t_{ijk}$**

From (b) and (a) we have:

$$\begin{aligned}
\langle \text{grad}\phi_i, \text{grad}\phi_i \rangle A_{ijk} &= \langle \vec{e_{ik}}^\perp, \vec{e_{ik}}^\perp \rangle \frac{1}{4A_{ijk}} \\
&= \frac{h^2}{4\frac{1}{2}hw} \\
&= \frac{h}{2w} \\
&= \frac{1}{2}(\cot\alpha + \cot\beta)
\end{aligned}$$

**(d) Show that for the hat function $\phi_i$ and $\phi_j$ associated with vertex $p_i$ and $p_j$ of triangle $t_{ijk}$, we have**

From (b) and (a) we have:

$$
\begin{aligned}
\langle \text{grad}\phi_i, \text{grad}\phi_j \rangle A_{ijk} &= \langle \vec{e_{ik}}^{\perp}, \vec{e_{jk}}^{\perp} \rangle \frac{1}{4A_{ijk}} \\
&= -\langle \vec{e_{ik}}, \vec{e_{jk}} \rangle \frac{1}{4A_{ijk}} \\
&= -|\vec{e_{ik}}||\vec{e_{jk}}| \cos\theta \frac{1}{4A_{ijk}} \\
&= -\frac{2A_{ijk}}{\sin\theta} \cos\theta \frac{1}{4A_{ijk}} \\
&= -\frac{1}{2} \cot\theta
\end{aligned}
$$

## Implementation (25 pts): Geodesic Distance using Heat Equation

Figure 0.1: Visualization of geodesic distance using Heat method