UNIVERSITY OF CALIFORNIA SAN DIEGO, CSE 270 WINTER 2024
GRADE 24%

Homework 1

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April 29, 2024

1 EXERCISE 1.1 — 5 PTS.

A vertex in a triangle mesh is regular if it has valence 6 (the valence of the vertex is the number of edges incident to the vertex). Show that the only closed triangulated surface where every vertex is regular is a torus or a Klein bottle

Firstly, for a triangulated surface we know that $2|E| = 3|F|$.

Further, if the vertices of this mesh are all regular, we have $2|E| = 6|V|$.

Then, using the formula of Euler characteristic of a discrete surface, we can get:

$$\chi(M) := |V| - |E| + |F| = \frac{1}{3}|E| - |E| + \frac{2}{3}|E| = 0$$

By the Euler–Poincare formula we know that this mesh has to satisfies $\chi(M) = 2 - 2g - b = 0$, where g is the number of genera and b is the number of boundaries. The only solutions are two cases: $g = 1, b = 0$ and $g = 0, b = 2$. However, the surface is asked to be closed in the preconditions, so the only case is the former one since closed surface has no boundary.

Therefore, we can conclude that the only closed triangulated surface where every vertex is regular is a torus or a Klein bottle, as they are the only two surfaces with $g = 1, b = 0$.

2 EXERCISE 1.2 — 5 PTS.

Show that the minimum possible number m of irregular vertices in a closed, orientable, triangulated surface of genus g (with a strict condition that each vertex has a valence at least 3) is

$$m(g) = \begin{cases} 4 & g = 0 \\ 0 & g = 1 \\ 1 & g \geq 2 \end{cases}$$

By the preconditions, we know that a closed, orientable, triangulated surface has Euler characteristic $\chi(g) = |V| - \frac{1}{3}|E|$.

Obviously, we have $\sum_{i \in V}$ valence$(i) = 2|E|$. Now we can split the vertices into two groups: one (R) contains all the regular vertices while the other (M) contains others. Therefore, we have $2|E| = \sum_{i \in R} 6 + \sum_{i \in M}$ valence$(i) \geq (|V| - |M|)6 + |M|3$ by given conditions.

Combining the above formulae, we have $2 - 2g = \chi(g) \leq \frac{1}{2}|M|$ for a closed orientable triangulated surface.

Tetrahedron is an example for $m(0) = 4$.

3 Exercise 1.3 — 5 pts.

Show that the mean valence $\frac{1}{|V|}\sum_{i \in V}$ valence(i) of a closed triangle surface of a fixed topology converges to 6 as $|V| \to \infty$. Conclude that the ratio of vertices to edges to faces approaches the following ratio $|V| : |E| : |F| \to 1 : 3 : 2$

From Exercises 1.1 we could know $2|E| = 3|F|$ for a trianglated surface.

Then, a closed triangle surface of a fixed topology should have a constant Euler characteristic. As $|V| \to \infty$, we can assume it is just 0.

Therefore:

$$0 \approx \chi(M) := |V| - |E| + |F| = |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|$$

Moreover, we know that $\sum_{i \in V}$ valence$(i) = 2|E|$.

Combining the above two equations we get that the mean valence converges to 6.

Then we easily conclude that he ratio of vertices to edges to faces approaches the following ratio $|V| : |E| : |F| \to 1 : 3 : 2$.

4 Exercise 1.4

Consider a triangle mesh. Place the dual vertex of a triangle at the center of its circumcircle. Note that the circumcenter of a triangle is the intersection of all three edge perpendicular bisectors, so the dual edges are already orthogonal to the primal edges

4.1 (a)—5 pts

Show that in that case the ratio of the dual and primal edge lengths is

$$\frac{|e^{\star}_{ij}|}{|e_{ij}|} = \frac{1}{2}(\cot\theta^k_{ij} + \cot\theta^l_{ji})$$

for edge e_{ij} shared by facets f_{ijk} and f_{jil}. Here, the angle $\theta^k_{ij} =_k \angle^i_j$ is the interior angle across from e_{ij} in triangle ijk, and similarly for θ^l_{ji}

Note the circumcenter of triangle ijk and jil as a and b respectively, and note the intersection of the primal and dual edges as c.

We know that $\theta^b_{jc} = \theta^l_{ji}$ and $\theta^a_{cj} = \theta^k_{ij}$ the from simple algebra.

Obviously edge ab (or dual edge $e^\star_{ij}$) is a segment bisector of primal edge e_{ij}, again through simple algebra we can get the above equation.

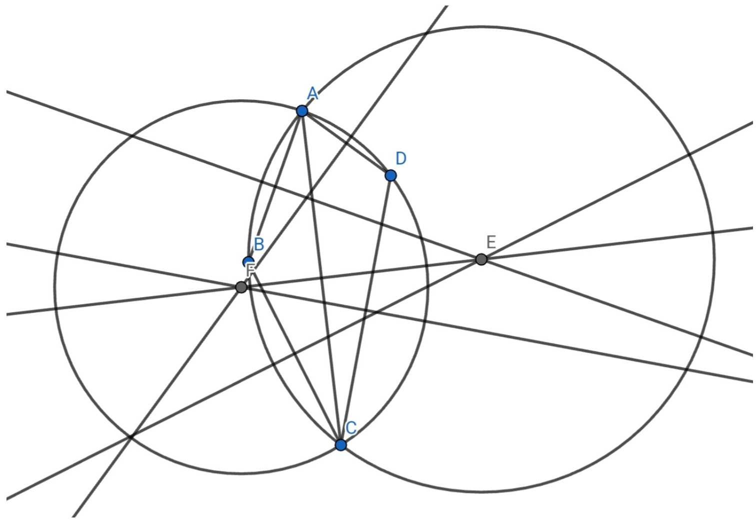
4.2 (B)—5 PTS

Show that the cotangent weight (the right-hand side of equation) is negative if and only if $\theta^k_{ij} + \theta^l_{ji} > \pi$. Illustrate a picture including both triangles, their circumcircles, primal and dual edges when this condition occurs. (The opposite condition, or that the cotan weight is non-negative, is called the Delaunay condition.)

Any expression of form $\cot\alpha + \cot\beta$ can be written as $\frac{\sin\alpha+\beta}{\sin\alpha\sin\beta}$.

Further we know that $0 < \alpha, \beta < \pi$, so the denominator is always greater than zero, and the numerator is not less than zero if and only if $0 < \alpha + \beta \le \pi$, which is just what we want to prove.

Figure 4.1: This is the picture where the circumcircles are "inverted" (by Geogebra)



5 IMPLEMENTATION (25 PTS): SURFACE LAPLACIAN

Figure 5.1: Visualization of "Armadillo Harmonic"