①

Kripke relation R
over $A \in \mathrm{obj}\,\mathcal{S}$
of arity $\sigma : \mathbb{C} \to \mathcal{S}$ where $\mathbb{C}$ is small
is
a family $\{R_c \subseteq \mathcal{S}(\sigma c, A)\}_{c \in \mathcal{S}}$
with
monotonicity:
$\forall \rho : c' \to c$ and $a : \sigma c \to A$,
if $a \in R_c$ then $a \circ \rho\sigma : \sigma(c') \to A \in R_{c'}$.

I.e. R is a unary predicate over $\mathcal{S}(\sigma -, A)$.

②

Category of Kripke relations $\underline{K}\sigma$
of arity $\sigma : \mathbb{C} \to \mathcal{S}$
has
objects: (R, A) for Kripke relation R of arity σ over $A \in \mathcal{S}$;
morphisms: $(R, A) \xrightarrow{f} (R', A')$ for $f : A \to A' \in \mathcal{S}$
with
$\forall a : \sigma c \to A$.
if $a \in R_c$ then $f \circ a : \sigma c \to A' \in R_{c'}$;
and id, $\_\circ\_$ as usual.

③

$\underline{K}\sigma$ is cartesian closed
using CC-structure of $\mathcal{S}$.

- Terminal: $(R_1, 1)$.
  $R_1(c) = \{1_{\sigma c}\}$, obviously.
- Product: $(R_1 \wedge R_2,\ A_1 \times A_2)$.
  $a : \sigma c \to A_1 \times A_2$ is in $(R_1 \wedge R_2)_c$ iff
  $\pi_1 \circ a : \sigma c \to A_1 \in R_{1c}$
  $\wedge\ \pi_2 \circ a : \sigma c \to A_2 \in R_{2c}$.
- Exponential: $(R_1 \supset R_2,\ A_1 \Rightarrow A_2)$.
  $f : \sigma c \to A_1 \Rightarrow A_2$ is in $(R_1 \supset R_2)_c$ iff
  $\forall \rho : c' \to c$ and $a : \sigma c' \to A_1$.
  if $a \in R_{1c'}$ then $ev \circ \langle f \circ \sigma\rho, a \rangle : \sigma c' \to A_2 \in R_{2c'}$

④

Fundamental Lemma.
For $I : T \to \underline{K}\sigma$, $I(\theta) = (R_\theta, I_0(\theta))$,
an interp. of base types
into Kripke relations,
the interpretation
$I_0[\![\Gamma \vdash t : \tau]\!] : I_0[\![\Gamma]\!] \to I_0[\![\tau]\!]$ in $\mathcal{S}$
(Consider I_0 an interp. of T to $\mathcal{S}$)
gives a morphism
$I[\![\Gamma]\!] \to I[\![\tau]\!]$ in $\underline{K}\sigma$,
i.e.
$I[\![\Gamma]\!] = (R_\Gamma, I_0[\![\Gamma]\!])$
and $I[\![\tau]\!] = (R_\tau, I_0[\![\tau]\!])$
gives commuting diagram

$$\begin{tikzcd}
R_\Gamma \arrow[r] \arrow[d, hook] & R_\tau \arrow[d, hook] \\
\mathcal{S}(\sigma -, I_0[\![\Gamma]\!]) \arrow[r, "I_0[\![\Gamma \vdash t : \tau]\!] \circ -"'] & \mathcal{S}(\sigma -, I_0[\![\tau]\!])
\end{tikzcd}.$$

⑤

Proof. by induction on $\Gamma \vdash t : \tau$.
E.g., case (lam): $\dfrac{\Gamma, \tau_1 \vdash t : \tau_2}{\Gamma \vdash \lambda t : \tau_1 \Rightarrow \tau_2}$.

That is, assume
$I_0[\![\Gamma, \tau_1 \vdash t : \tau_2]\!] = I_0[\![\Gamma]\!] \times I_0[\![\tau_1]\!] \xrightarrow{f} I_0[\![\tau_2]\!]$
a valid morphism in $\underline{K}\sigma$, show that
$I_0[\![\Gamma]\!] \xrightarrow{\lambda f} I_0[\![\tau_1 \Rightarrow \tau_2]\!]$
also one. I.e.
$\forall a : \sigma c \to I_0[\![\Gamma]\!]$.
$a \in R_\Gamma \Rightarrow \lambda f \circ a \in R_{\tau_1} \supset R_{\tau_2}$,
and by def., we need
$\forall \rho : c' \to c,\ a' : \sigma c' \to I_0[\![\tau_1]\!]$.
$a' \in R_{\tau_1} \Rightarrow ev \circ \langle \lambda f \circ a \circ \sigma\rho, a' \rangle \in R_{\tau_2}$
to prove it. Rewrite,

$$\begin{tikzcd}
\sigma c' \arrow[d, "\sigma\rho"'] & \times & \sigma c' \arrow[dd, "a'"] & \\
\sigma c \arrow[d, "a"'] & & & \\
\Gamma \arrow[d, "\lambda f"'] & \times & \tau_1 \arrow[d, "id"] \arrow[dr, "f"] & \\
\tau_1 \Rightarrow \tau_2 & \times & \tau_1 \arrow[r, "ev"'] & \tau_2
\end{tikzcd}$$

$ev \circ \langle \lambda f \circ a \circ \sigma\rho, a' \rangle$
$= ev \circ \langle \lambda f, id \rangle \circ \langle a \circ \sigma\rho, a' \rangle$
$= f \circ \langle a \circ \sigma\rho, a' \rangle$
and f is a morphism in $\underline{K}\sigma$ by IH. So, if
(1) $a \circ \sigma\rho \in R_\Gamma$,
(2) $a' \in R_{\tau_1}$,
then the proof is done.
Indeed, we have (1) by monotonicity of a
and (2) by assumption. $\square$

Free cocartesian category on $\mathbb{C}$
contains $c,\ c+c,\ (c+c)+c, \dots$
for all $c \in \mathbb{C}$.

$\boxed{1}$

$\mathbb{F}$ is the category of finite sets with
obj: $\{0\}, \{0,1\}, \dots$
mor, id, $\_\circ\_$: as $\underline{\mathrm{Set}}$.

I.e.
$\mathbb{F}$ is the full subcategory of $\underline{\mathrm{Set}}$ with finite sets the objects;

i.e.
$\mathbb{F}$ is the free cocartesian category on one object;

i.e.
$\mathbb{F}$ is generated from an initial object with an extension $(\_)+1$ operation.

$\boxed{2}$

Free cocartesian category on $\mathbb{C}$
is also described as $\mathbb{F}\downarrow\mathbb{C}$,

i.e.
the comma category of $\mathbb{F}$ and $\mathbb{C}$;

i.e. with
obj: $\Gamma : n \to \mathbb{C}$
maps from $n \in \mathbb{F}$ to $\mathbb{C}$,

mor: $\rho : \Gamma \to \Delta$
with
$\Gamma : m \to \mathbb{C}$
$\Delta : n \to \mathbb{C}$
where
$\rho : m \to n$ and $\Delta \circ \rho = \Gamma$.

i.e.
type-preserving context renamings.

$\boxed{3}$

Category of contexts
over a set of types T
is
the free cocartesian over T,
i.e. $\mathbb{F}\downarrow T$.

Also, write $\mathbb{F}[T]$ for $(\mathbb{F}\downarrow T)^{op}$.

$\boxed{4}$

$\boxed{5}$

Semantic domain: $\underline{\mathrm{Set}}^{\mathbb{F}\downarrow T}$
i.e presheaf category
with
obj: $P : \mathbb{F}[T] \to \mathrm{Set}$
mor: $\theta : P \to Q$ as natural transformations

$\boxed{6}$

Essential Geometric Morphisms

For $f : \mathbb{C} \to \mathbb{D}$, have the following situation

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook, "y"] \arrow[d, "f"'] & \widehat{\mathbb{C}} \arrow[d, bend right=50, "f_!"'] \arrow[d, bend left=80, leftarrow, "f_*"] \arrow[d, phantom, shift right=3ex, "\dashv"] \arrow[d, phantom, shift left=3ex, "\dashv"] \\
\mathbb{D} \arrow[r, hook, "y"'] & \widehat{\mathbb{D}} \arrow[u, "f^*"]
\end{tikzcd}$$

with

- $f^*Q = \widehat{\mathbb{D}}(y \circ f-, Q) \cong Q(f-)$.
- $f_!P = \mathrm{Colim}(\int P \xrightarrow{\pi_i} \mathbb{C} \xrightarrow{f} \mathbb{D} \xrightarrow{y} \widehat{\mathbb{D}})$

$$\cong \int^{c \in \mathbb{C}} Pc \times y(fc)$$

$$\cong \int^{c \in \mathbb{C}} Pc \times \mathbb{D}(-, fc).$$

- $(f_*P)(d) = \widehat{\mathbb{C}}(\mathbb{D}(f-, d), P)$.

$\boxed{7}$

Presheaf of variables V of type τ
is
$V_\tau = y\langle\tau\rangle$
where
$y\Gamma = \mathbb{F}\downarrow T(\Gamma, \_)$;
hence,
$V_\tau(\Gamma) = \mathbb{F}\downarrow T(\langle\tau\rangle, \Gamma)$
$= \{x \mid (x:\tau) \in \Gamma\}$.

$\boxed{8}$

Consider the parameterization functor
$$\_ \times \langle\tau\rangle : \mathbb{F}[T] \to \mathbb{F}[T]$$
in EGM situation, we have

- $(\_ \times \langle\tau\rangle)^* P = P(\_ + \langle\tau\rangle)$
- $(\_ \times \langle\tau\rangle)_! P = \int^{\Gamma \in \mathbb{F}[T]} P\Gamma \times y(\Gamma \times \langle\tau\rangle)$

$$\cong \left(\int^{\Gamma \in \mathbb{F}[T]} P\Gamma \times y\Gamma\right) \times y\langle\tau\rangle \quad \text{by yoneda preserves products}$$

$$\cong P \times y\langle\tau\rangle. \quad \text{by density}$$

Therefore, the exponential of P to V_τ is
$$P^{V_\tau}(\Gamma) = \underline{\mathrm{Set}}^{\mathbb{F}\downarrow T}(y\Gamma \times V_\tau, P)$$
$$\cong \underline{\mathrm{Set}}^{\mathbb{F}\downarrow T}(y\Gamma, P(\_ + \langle\tau\rangle)) \quad \text{by adjunction}$$
$$\cong P(\Gamma + \langle\tau\rangle). \quad \text{Yoneda lemma}$$

Hence the isomorphism
$$P^{V_\tau}(\Gamma) \cong P(\Gamma + \langle\tau\rangle).$$

A typed lambda algebra over a set of base types T

is

a $\tilde{T}$-sorted algebra

with carriers $\{X_\tau\}_{\tau\in\tilde{T}}$

families of presheaves in $\underline{Set}^{\mathbb{F}\downarrow\tilde{T}}$

with operations

(var) $V_\tau \longrightarrow X_\tau$

(unit) $1 \longrightarrow X_1$

(π_1) $X_{\tau_1\times\tau_2} \longrightarrow X_{\tau_1}$

(π_2) $X_{\tau_1\times\tau_2} \longrightarrow X_{\tau_2}$

(pair) $X_{\tau_1}\times X_{\tau_2} \longrightarrow X_{\tau_1\times\tau_2}$

(abs) $V_{\tau_1}\Rightarrow X_{\tau_2} \longrightarrow X_{\tau_1\Rightarrow\tau_2}$

(app) $X_{\tau_1\Rightarrow\tau_2}\times X_{\tau_1} \longrightarrow X_{\tau_2}$.

i.

The category of typed lambda algebra (in terms of categorical algebra)

is

the Σ-algebra over $(\underline{Set}^{\mathbb{F}\downarrow\tilde{T}})^{\tilde{T}}$.

with Σ induced by

$$(\Sigma X)_\theta = V_\theta + E_\theta(X)$$

$$(\Sigma X)_1 = V_1 + 1 + E_1(X)$$

$$(\Sigma X)_{\tau_1\times\tau_2} = V_{\tau_1\times\tau_2} + (\Sigma X)_{\tau_1}\times(\Sigma X)_{\tau_2} + E_{\tau_1\times\tau_2}(X)$$

$$(\Sigma X)_{\tau_1\Rightarrow\tau_2} = V_{\tau_1\Rightarrow\tau_2} + V_{\tau_1}\Rightarrow(\Sigma X)_{\tau_2} + E_{\tau_1\Rightarrow\tau_2}(X)$$

where

$$E_\tau X = \coprod_{\tau'\in\tilde{T}} X_{\tau\times\tau'} + X_{\tau'\times\tau} + X_{\tau'\Rightarrow\tau}\times X_{\tau'}.$$

ways of elimination that yields some τ

ii.

The algebra for normal and neutral terms

is

an algebra over the product category

$$(\underline{Set}^{\mathbb{F}\downarrow\tilde{T}})^{\tilde{T}} \times (\underline{Set}^{\mathbb{F}\downarrow\tilde{T}})^{\tilde{T}}$$

with signature $\langle\Sigma_1,\Sigma_2\rangle$ induced by

$$(\Sigma_1(X,\mathcal{Y}))_\tau = V_\tau + E_\tau(X,\mathcal{Y})$$

$$(\Sigma_1(X,\mathcal{Y}))_\theta = V_\theta + E_\theta(X,\mathcal{Y})$$

where

$$E_\tau(X,\mathcal{Y}) = \coprod_{\tau'\in\tilde{T}} X_{\tau'\times\tau} + X_{\tau\times\tau'} + X_{\tau'\Rightarrow\tau}\times \mathcal{Y}_{\tau'}$$

and

$$(\Sigma_2(X,\mathcal{Y}))_1 = 1$$

$$(\Sigma_2(X,\mathcal{Y}))_{\tau_1\times\tau_2} = \mathcal{Y}_{\tau_1}\times\mathcal{Y}_{\tau_2}$$

$$(\Sigma_2(X,\mathcal{Y}))_{\tau_1\Rightarrow\tau_2} = V_{\tau_1}\Rightarrow\mathcal{Y}_{\tau_2}.$$

iii.

The initial Σ-algebra $\mathcal{L}$

is

the presheaf of lambda terms

$$\mathcal{L}_\tau(\Gamma) = \{t \mid \Gamma\vdash t:\tau\}$$

with structure

$$V_\theta + E_\theta(\mathcal{L}) \cong \mathcal{L}_\theta$$

$$V_1 + 1 + E_1(\mathcal{L}) \cong \mathcal{L}_1$$

$$V_{\tau_1\times\tau_2} + \mathcal{L}_{\tau_1}\times\mathcal{L}_{\tau_2} + E_{\tau_1\times\tau_2}(\mathcal{L}) \cong \mathcal{L}_{\tau_1\times\tau_2}$$

$$V_{\tau_1\Rightarrow\tau_2} + V_{\tau_1}\Rightarrow\mathcal{L}_{\tau_2} + E_{\tau_1\Rightarrow\tau_2}(\mathcal{L}) \cong \mathcal{L}_{\tau_1\Rightarrow\tau_2}$$

and the presheaf action

is renaming on terms.

iv.

v.

The initial (Σ_1,Σ_2)-algebra $(\mathcal{M},\mathcal{N})$

are

pair of presheaves of neutral and normal terms

$$\mathcal{M}_\tau(\Gamma) = \{t \mid \Gamma\vdash_M t:\tau\}$$

$$\mathcal{N}_\tau(\Gamma) = \{t \mid \Gamma\vdash_N t:\tau\}$$

with structure

$$V_\tau + E_\tau(\mathcal{M},\mathcal{N}) \cong \mathcal{M}_\tau$$

$$V_\theta + E_\theta(\mathcal{M},\mathcal{N}) \cong \mathcal{N}_\tau$$

$$1 \cong \mathcal{N}_1$$

$$\mathcal{N}_{\tau_1}\times\mathcal{N}_{\tau_2} \cong \mathcal{N}_{\tau_1\times\tau_2}$$

$$V_{\tau_1}\Rightarrow\mathcal{N}_{\tau_2} \cong \mathcal{N}_{\tau_1\Rightarrow\tau_2}.$$

vi.

All Σ-algebra X gives a canonical (Σ_1,Σ_2)-algebra (X,X).

So, $\mathcal{L}$ gives $(\mathcal{L},\mathcal{L})$ and by initiality of $(\mathcal{M},\mathcal{N})$,

we have embeddings of neutral and normal into terms

$\mathcal{M}\hookrightarrow\mathcal{L}$ and $\mathcal{N}\hookrightarrow\mathcal{L}$.

Later, we give the relative-hom semantic typed lambda algebra $\mathcal{C}$,

and induce the following situation

$$\begin{tikzcd}
\mathcal{M} \arrow[r, hook] \arrow[dr, "m"'] & \mathcal{L} \arrow[d, "l"] & \mathcal{N} \arrow[l, hook'] \arrow[dl, "n"] \\
& \mathcal{C} &
\end{tikzcd}$$

where l, m, n are obtained by initiality.

Explicitly, $l_\tau(\Gamma) : \mathcal{L}_\tau\Gamma \longrightarrow \mathcal{C}_\tau\Gamma = \mathcal{S}(S[\![\Gamma]\!], S[\![\tau]\!])$

$$t \longmapsto S[\![\Gamma\vdash t:\tau]\!]$$

and similarly for m and n.

The relative hom functor $\langle\sigma\rangle : \mathcal{S} \to \widehat{\mathbb{C}}$
for $\sigma : \mathbb{C} \to \mathcal{S}$
is defined as

$$\langle\sigma\rangle(s) = \mathcal{S}(\sigma -, s)$$
$$\langle\sigma\rangle(f : s \to s')_m = f \circ - : \mathcal{S}(\sigma m, s) \to \mathcal{S}(\sigma m, s').$$

If $\mathbb{C}$ and $\mathcal{S}$ are cartesian closed, then

- $\langle\sigma\rangle$ is a right adjoint, hence preserving limits;
- $\langle\sigma\rangle$ commutes with exponentiation by representables, i.e. has the following isomorphism.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\sigma\Gamma \Rightarrow -"] \arrow[d, "\langle\sigma\rangle"'] \arrow[dr, phantom, "\cong"] & \mathcal{S} \arrow[d, "\langle\sigma\rangle"] \\
\widehat{\mathbb{C}} \arrow[r, "y\Gamma \Rightarrow -"'] & \widehat{\mathbb{C}}
\end{tikzcd}$$

Proof: $\forall s \in \mathcal{S}$,

$$\begin{aligned}
\langle\sigma\rangle(\sigma\Gamma \Rightarrow s) &= \mathcal{S}(\sigma -, \sigma\Gamma \Rightarrow s) \\
&\cong \mathcal{S}(\sigma - \times \sigma\Gamma, s) && \text{uncurrying.} \\
&\cong \mathcal{S}(\sigma(- \times \Gamma), s) && \sigma \text{ preserve limits.} \\
&\cong \langle\sigma\rangle(s)(\Gamma \times -) && \text{by def.} \\
&\cong \widehat{\mathbb{C}}(y\Gamma \times y -, \langle\sigma\rangle(s)) && \text{Yoneda lemma.} \\
&\cong y\Gamma \Rightarrow \langle\sigma\rangle(s). \ \square && \text{by def.}
\end{aligned}$$

Initial algebra semantics.

Given an interpretation of base types in $\mathcal{S}$ (which is ccc)
$$s : T \to \mathcal{S},$$
it induces a cartesian extension
$$s[\![-]\!] : \mathbb{FL}[\tilde{T}] \to \mathcal{S},$$
and the relative hom-functor of $s[\![-]\!]$
$$\langle s\rangle : \mathcal{S} \to \mathbf{Set}^{\mathbb{FL}\tilde{T}}$$
can be used to define a typed lambda algebra on
$$\mathcal{C} = \{\mathcal{S}(s[\![-]\!], s[\![\tau]\!])\}_{\tau \in \tilde{T}} = \{\langle s\rangle(s[\![\tau]\!])\}_{\tau\in\tilde{T}}$$
with operations given by

$$V_\tau \xrightarrow{s[\![-]\!]\circ -} \langle s\rangle(s[\![\tau]\!]) = \mathcal{S}(s[\![-]\!], s[\![\tau]\!])$$
$$1 \xrightarrow{\cong} \langle s\rangle(s[\![1]\!]) = \mathcal{S}(s[\![-]\!], 1) \quad \text{terminal}$$
$$\langle s\rangle(s[\![\tau_1 \times \tau_2]\!]) \xrightarrow{\langle s\rangle(\pi_1)} \langle s\rangle(s[\![\tau_1]\!]) \quad \langle s\rangle \text{ preserve limits}$$
$$\langle s\rangle(s[\![\tau_1 \times \tau_2]\!]) \xrightarrow{\langle s\rangle(\pi_2)} \langle s\rangle(s[\![\tau_2]\!])$$
$$\langle s\rangle(s[\![\tau_1]\!]) \times \langle s\rangle(s[\![\tau_2]\!]) \xrightarrow{\cong} \langle s\rangle(s[\![\tau_1 \times \tau_2]\!])$$
$$\begin{aligned}
&\langle s\rangle(s[\![\tau_1 \Rightarrow \tau_2]\!]) \times \langle s\rangle(s[\![\tau_1]\!]) \\
&\cong \\
&\langle s\rangle((s[\![\tau_1]\!] \Rightarrow s[\![\tau_2]\!]) \times s[\![\tau_1]\!]) \xrightarrow{\langle s\rangle(ev)} \langle s\rangle(s[\![\tau_2]\!])
\end{aligned}$$
$$V_{\tau_1} \Rightarrow \langle s\rangle(s[\![\tau_2]\!]) \xrightarrow{\cong} \langle s\rangle(s[\![\tau_1]\!] \Rightarrow s[\![\tau_2]\!]).$$

The yoneda embedding extends to the glued category $\widehat{\mathbb{C}}\downarrow\langle\sigma\rangle$.

- Extended yoneda embedding

$$\bar{y} : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}\downarrow\langle\sigma\rangle$$
$$\bar{y}(\Gamma) = (y\Gamma, \sigma\circ -, \sigma\Gamma)$$

Note: $\sigma\circ - : y\Gamma \to \langle\sigma\rangle(\sigma\Gamma)$, where $y\Gamma = \mathcal{S}(-,\Gamma)$ and $\langle\sigma\rangle(\sigma\Gamma) = \mathcal{S}(\sigma -, \sigma\Gamma)$

gives the situation

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dl, "y"'] \arrow[d, "\bar{y}"] \arrow[dr, "\sigma"] & \\
\widehat{\mathbb{C}} & \widehat{\mathbb{C}}\downarrow\langle\sigma\rangle \arrow[l, "w"'] \arrow[r, "\pi"] & \mathcal{S}
\end{tikzcd}$$

where (w, π) are forgetful functors.

The extended yoneda lemma states that

$$\begin{tikzcd}
\widehat{\mathbb{C}}\downarrow\langle\sigma\rangle[\bar{y}-, (P,p,A)] \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & P \arrow[dl, "p"] \\
 & \mathcal{S}(\sigma -, A) &
\end{tikzcd}$$

where the isomorphism is $(\varphi, f) \mapsto \varphi(id)$, like in Yoneda lemma.

If $\mathbb{C}, \sigma$ are cartesian and $\mathcal{S}$ is cartesian closed, then

- $\bar{y}$ preserves products;
- Exponential $\bar{y}\Gamma \Rightarrow (P, p, A)$ can be written as
$$(y\Gamma \Rightarrow P, p', \sigma\Gamma \Rightarrow A)$$
i.e. $y\Gamma \Rightarrow P$ is the pullback in situation (easy to show!)

$$\begin{tikzcd}
y\Gamma\Rightarrow P \arrow[r, "id"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & y\Gamma \Rightarrow P \arrow[d, "y\Gamma\Rightarrow p"] \\
\langle\sigma\rangle(\sigma\Gamma\Rightarrow A) \arrow[r, "\cong"] & y\Gamma \Rightarrow \langle\sigma\rangle A
\end{tikzcd}$$

where p is $y\Gamma \Rightarrow P \xrightarrow{y\Gamma\Rightarrow p} y\Gamma \Rightarrow \langle\sigma\rangle A \xrightarrow{\cong} \langle\sigma\rangle(\sigma\Gamma \Rightarrow A)$

and the isomorphism is

$$y\Gamma \Rightarrow \langle\sigma\rangle A = \widehat{\mathbb{C}}(y\Gamma \times y -, \langle\sigma\rangle A) \cong \langle\sigma\rangle A(\Gamma \times -) = \mathcal{S}(\sigma(\Gamma\times -), A)$$
$$\cong \mathcal{S}(\sigma -, \sigma\Gamma \Rightarrow A).$$

Category of intensional Kripke relations of arity $\sigma : \mathcal{C} \to \mathcal{S}$

is

the glueing of $\widehat{\mathcal{C}}$

along the relative hom functor $\langle\sigma\rangle : \mathcal{S} \to \widehat{\mathcal{C}}$,

i.e.

the comma category $\widehat{\mathcal{C}} \downarrow \langle\sigma\rangle$

with

objects (P, p, A)

where $P \in \widehat{\mathcal{C}}$, $A \in \mathcal{S}$, $p : P \to \langle\sigma\rangle A$,

morphisms $(P, p, A) \longrightarrow (P', p', A')$

are pairs $\varphi : P \to P'$, $f : A \to A'$

with

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "p"'] & P' \arrow[d, "p'"] \\ \langle\sigma\rangle A \arrow[r, "\langle\sigma\rangle f"'] & \langle\sigma\rangle A' \end{tikzcd}$$

commutes.

- $\underline{K\langle\sigma\rangle}$ is a full subcategory of $\widehat{\mathcal{C}} \downarrow \langle\sigma\rangle$

  via the embedding

  $$R \longmapsto (R,\ R \hookrightarrow \langle\sigma\rangle A,\ A)$$

  for $\underline{K\langle\sigma\rangle}$ relation R over $A \in \mathcal{S}$

  Since R is a presheaf

  and $R c$ is a subset of $\mathcal{S}(\sigma c, A) = (\langle\sigma\rangle A)(c)$.

- Every glued object (P, p, A)

  has an associated relation

  given by the extension of p as below

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p"'] & \mathrm{im}(p) \arrow[dl, hook] \\ \langle\sigma\rangle A & \end{tikzcd}$$

where $\mathrm{im}(p)$ is the image of p

i.e.

$$\mathrm{im}(p)(c) = \{ f \in \mathcal{S}(\sigma c, A) \mid \exists z \in Pc.\ p_c(z) = f \}.$$

Monotonicity:

$\forall \rho : c' \to c.$

if $f : \sigma c \to A \in \mathrm{im}(p)(c)$,

then $\exists z \in Pc.\ p_c(z) = f$,

then $f \circ \sigma\rho = p_c(z) \circ \sigma\rho$

$= p_{c'}(P\rho(z))$,

by naturality of p.

$$\begin{tikzcd} Pc \arrow[r, "P\rho"] \arrow[d, "p_c"'] & Pc' \arrow[d, "p_c"] \\ \mathcal{S}(\sigma c, A) \arrow[r, "- \circ \sigma\rho"'] & \mathcal{S}(\sigma c', A) \end{tikzcd}$$

Hence $f \circ \sigma\rho \in \mathrm{im}(p)(c')$. □

- If $\mathcal{S}$ is cartesian closed, then also is $\widehat{\mathcal{C}} \downarrow \langle\sigma\rangle$.
  - Terminal: $(1, t, 1)$

    where $t : 1 \xrightarrow{\cong} \mathcal{S}(\sigma -, 1)$.
  - Product: $(P, p, A) \times (Q, q, B)$

    is $(P \times Q,\ r,\ A \times B)$

    where r is

    $$P \times Q \xrightarrow{\langle p, q\rangle} \langle\sigma\rangle A \times \langle\sigma\rangle B \xrightarrow{\cong} \langle\sigma\rangle(A \times B).$$
  - Exponential: $(P, p, A) \Rightarrow (Q, q, B)$

    is $(R, r, A \Rightarrow B)$

    in the pullback

$$\begin{tikzcd} R \arrow[rr, "\pi"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & & P \Rightarrow Q \arrow[d, "P \Rightarrow q"] \\ \langle\sigma\rangle(A \Rightarrow B) \arrow[r, "f"'] & \langle\sigma\rangle A \Rightarrow \langle\sigma\rangle B \arrow[r, "p \Rightarrow \langle\sigma\rangle B"'] & P \Rightarrow \langle\sigma\rangle B \end{tikzcd}$$

where f is the currying of

$$\langle\sigma\rangle(A \Rightarrow B) \times \langle\sigma\rangle A \xrightarrow{\cong} \langle\sigma\rangle(A \Rightarrow B \times A) \xrightarrow{\langle\sigma\rangle ev} \langle\sigma\rangle B,$$

and $ev : (R, r, A \Rightarrow B) \times (P, p, A) \longrightarrow (Q, q, B)$

is the pair

$ev_R : R \times P \longrightarrow Q$ the uncurrying of π

$ev : A \Rightarrow B \times B \longrightarrow B$.

Proof:

- Morphisms ev_R and ev should satisfy

$$\begin{tikzcd} R \times P \arrow[r, "ev_R"] \arrow[d, "r \times p"'] & Q \arrow[d, "q"] \\ \langle\sigma\rangle(A \Rightarrow B) \times \langle\sigma\rangle A \arrow[r, "\langle\sigma\rangle ev"'] & \langle\sigma\rangle B \end{tikzcd}$$

  which is exactly the pullback square applied with the currying adjunction.
- Given $(P, p, A) \times (Q, q, B) \xrightarrow{(\theta, f)} (S, s, C)$

  we have $(\theta', f') : (P, p, A) \longrightarrow (R, r, B \Rightarrow C)$ with

  $f' = \mathrm{cur}\, f : A \to B \Rightarrow C$

  and θ' given by the following situation (holds via currying)

$$\begin{tikzcd} P \arrow[drr, bend left, "\mathrm{cur}\,\theta"] \arrow[dr, dashed, "\theta'"] \arrow[dd, "p"'] & & \\ & R \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & Q \Rightarrow S \arrow[dd, "Q \Rightarrow s"] \\ \langle\sigma\rangle A \arrow[dr, "\langle\sigma\rangle(\mathrm{cur} f)"'] & & \\ & \langle\sigma\rangle(B \Rightarrow C) \arrow[r] & Q \Rightarrow \langle\sigma\rangle C \end{tikzcd}$$

i.e. P is a non-terminal pullback,

and all desired properties are given by the situation. □

Glueing syntax and semantics.

- Consider the glued category $\mathbf{Set}^{\mathbb{F}\downarrow\tilde{T}}\downarrow\langle s\rangle$
  for a set of base types T
  and interpretation $s: T \to \mathcal{S}$.

- Have $\mathcal{V}_\tau = \bar{y}\langle\tau\rangle = (V_\tau,\ S[\![-]\!]: V_\tau \to \mathcal{C}_\tau,\ S[\![\tau]\!])$
  Recall: $\mathcal{C}_\tau = \langle s\rangle(S[\![\tau]\!])$
  an object glueing syntax and semantics.

- Similarly, have
  $\mathcal{M}_\tau = (M_\tau,\ m_\tau: M_\tau \to \mathcal{C}_\tau,\ S[\![\tau]\!])$
  and
  $\mathcal{N}_\tau = (N_\tau,\ n_\tau: N_\tau \to \mathcal{C}_\tau,\ S[\![\tau]\!])$.
  Recall: we have l, m, n by initiality.

Consider the interpretation into the glued category

$$\bar{S}: T \longrightarrow \mathbf{Set}^{\mathbb{F}\downarrow T}\downarrow\langle s\rangle$$

with $\bar{S}(\theta) = \mathcal{M}_\theta$,
since glueing preserves cartesian closed structures,
we have
an extension of semantics $\bar{S}[\![-]\!]$
and
$\bar{S}[\![\Gamma\vdash t:\tau]\!]$ is a pair
$(S'[\![\Gamma\vdash t:\tau]\!],\ S[\![\Gamma\vdash t:\tau]\!])$,
i.e. a syntactic and a semantic interpretation.

Let $(\mathcal{G}_\tau, \sigma_\tau, S[\![\tau]\!])$ be $\bar{S}[\![\tau]\!]$, we wish to have

$$\begin{tikzcd}
M_\tau \arrow[r, "?"] \arrow[dr, "m_\tau"'] & \mathcal{G}_\tau \arrow[r, "?"] \arrow[d, "\sigma_\tau"] & N_\tau \arrow[dl, "n_\tau"] \\
& \mathcal{S}(S[\![-]\!], S[\![\tau]\!]) &
\end{tikzcd}$$

such that the following commutes
for all $\Gamma\vdash t:\tau$
and write $\Gamma = x_1:\tau_1, \dots, x_n:\tau_n$:

$$\begin{tikzcd}
\prod_i M_{\tau_i} \arrow[r, "?"] \arrow[dr, "\prod_i m_{\tau_i}"'] & \prod_i \mathcal{G}_{\tau_i} \arrow[r, "S'[\![\Gamma\vdash t:\tau]\!]"] \arrow[d, "\prod_i \sigma_{\tau_i}"] & \mathcal{G}_\tau \arrow[r, "?"] \arrow[dd, "\sigma_\tau"] & N_\tau \arrow[ddl, "n_\tau"] \\
& \prod_i \mathcal{S}(S[\![-]\!], S[\![\tau_i]\!]) \arrow[d, phantom, "\cong"] & & \\
& \mathcal{S}(S[\![-]\!], S[\![\Gamma]\!]) \arrow[r, "S[\![\Gamma\vdash t:\tau]\!]\circ -"'] & \mathcal{S}(S[\![\Gamma]\!], S[\![\tau]\!]) &
\end{tikzcd}$$

(Σ_1, Σ_2)-structure on $\mathbf{Set}^{\mathbb{F}\downarrow T}\downarrow\langle s\rangle$.

- Variable: $\mathcal{V}_\tau \longrightarrow \mathcal{M}_\tau$ is the pair
  $\mathrm{Var}_\tau: V_\tau \to M_\tau$, $id_{S[\![\tau]\!]}$.

- Eliminators
  - $\pi_1: \mathcal{M}_{\tau\times\tau'} \longrightarrow \mathcal{M}_\tau$ is the pair
    $fst_\tau^{\tau'}: M_{\tau\times\tau'} \to M_\tau$, $\pi_1: S[\![\tau_1]\!]\times S[\![\tau_2]\!] \to S[\![\tau_1]\!]$
  - π_2: similarly, $snd_\tau^{\tau'}$ and π_2.
  - app: $\mathcal{M}_{\tau'\Rightarrow\tau}\times\mathcal{N}_{\tau'} \longrightarrow \mathcal{M}_\tau$ is the pair
    $app_\tau^{\tau'}: M_{\tau'\Rightarrow\tau}\times N_{\tau'} \longrightarrow N_\tau$,
    $ev: S[\![\tau'\Rightarrow\tau]\!]\times S[\![\tau']\!] \longrightarrow S[\![\tau]\!]$.

- Base types
  Isomorphism $\mathcal{M}_\theta \cong \mathcal{N}_\theta$ is witnessed by the pair
  $M_\theta \cong N_\theta \cong V_\theta + E_\theta(M, N)$,
  $id_{S(\theta)}$.

- Constructors
  - unit: $1 \xrightarrow{\cong} \mathcal{N}_1$ is witnessed by the pair
    $1 \cong N_1$, id_1.
  - pair: $\mathcal{N}_{\tau_1}\times\mathcal{N}_{\tau_2} \xrightarrow{\cong} \mathcal{N}_{\tau_1\times\tau_2}$ is witnessed by
    $N_{\tau_1}\times N_{\tau_2} \cong N_{\tau_1\times\tau_2}$, $S[\![\tau_1]\!]\times S[\![\tau_2]\!] \cong S[\![\tau_1\times\tau_2]\!]$.
  - abs: $\mathcal{V}_\tau\Rightarrow\mathcal{N}_{\tau'} \xrightarrow{\cong} \mathcal{N}_{\tau\Rightarrow\tau'}$ is witnessed by
    $V_\tau\Rightarrow N_{\tau'} \cong N_{\tau\Rightarrow\tau'}$, $S[\![\tau]\!]\Rightarrow S[\![\tau']\!] \cong S[\![\tau\Rightarrow\tau']\!]$.

I.e.
the operations on glued objects
are
pairs of syntactic operations
and semantic interpretations.

- If so, it implies that
  interpreting the term t
  in the context of (reflected) neutrals
  gives a normal term
  that has the same semantics
  as
  interpreting t in the semantic context.

- Also, since $\beta\eta$-equivalent terms have equal semantics,
  they will have equal normal forms.

Normalization by Evaluation.

- We define two maps

  ↑/unquote/reflect : $\mathcal{M}_\tau \xrightarrow{u_\tau} \bar{S}[\![\tau]\!]$

  and

  ↓/quote/reify : $\bar{S}[\![\tau]\!] \xrightarrow{q_\tau} \mathcal{N}_\tau$

  that are identities at the second component,

  i.e. $u_\tau = (u_\tau : M_\tau \to G_\tau,\ id_{S[\![\tau]\!]})$,

  $q_\tau = (q_\tau : G_\tau \to N_\tau,\ id_{S[\![\tau]\!]})$.

- The definition is defined inductively on types.
  - Base types:

    $u_\theta = id_{\mathcal{M}\theta}$, $q_\theta = \mathcal{M}\theta \xrightarrow{\cong} \mathcal{N}\theta$.

    Recall that $\bar{S}[\![\theta]\!] = \bar{S}(\theta) = \mathcal{M}\theta$ by def.
  - Unit type:

    $u_1 = \mathcal{M}_1 \to 1$, $q_1 = 1 \xrightarrow{\cong} \mathcal{N}_1$.

    i.e. u_1 is the unique morphism to terminal object.
  - Product types:

    $u_{\tau\times\tau'} = p_1 \times p_2$

    where $p_1 = \mathcal{M}_{\tau\times\tau'} \xrightarrow{fst} \mathcal{M}_\tau \xrightarrow{u_\tau} \bar{S}[\![\tau']\!]$

    $p_2 = \mathcal{M}_{\tau\times\tau'} \xrightarrow{snd} \mathcal{M}_{\tau'} \xrightarrow{u_{\tau'}} \bar{S}[\![\tau']\!]$

    and

    $q_{\tau\times\tau'} = \bar{S}[\![\tau]\!] \times \bar{S}[\![\tau']\!] \xrightarrow{q_\tau \times q_{\tau'}} \mathcal{N}_\tau \times \mathcal{N}_{\tau'} \cong \mathcal{N}_{\tau\times\tau'}$.
  - Arrow types:

    $u_{\tau\Rightarrow\tau'} = \mathcal{M}_{\tau\Rightarrow\tau'} \xrightarrow{cur\ app} \mathcal{N}_\tau \Rightarrow \mathcal{M}_{\tau'} \xrightarrow{q_\tau \Rightarrow u_{\tau'}} \bar{S}[\![\tau]\!] \Rightarrow \bar{S}[\![\tau']\!]$,

    $q_{\tau\times\tau'} = \bar{S}[\![\tau]\!] \Rightarrow \bar{S}[\![\tau']\!] \xrightarrow{f_\tau \Rightarrow q_{\tau'}} V_\tau \Rightarrow \mathcal{N}_{\tau'} \cong \mathcal{N}_{\tau\Rightarrow\tau'}$

    where $f_\tau = V_\tau \xrightarrow{var} M_\tau \xrightarrow{u_\tau} \bar{S}[\![\tau]\!]$.

Normalization.

For any interpretation $s : T \to S$,
it induces a normalization function

$$s\text{-}nf_\tau : \mathcal{L}_\tau \to \mathcal{N}_\tau$$

defined as the composite

$$\mathcal{L}_\tau \xrightarrow{\bar{l}_\tau} [\bar{S}[\![-]\!], \bar{S}[\![\tau]\!]] \xrightarrow{[uv,\ q_\tau]} [\bar{y}(-), \mathcal{N}_\tau] \cong \mathcal{N}_\tau$$

where

$(uv)_\Gamma = \bar{y}\Gamma \xrightarrow{v_\Gamma} \mathcal{M}[\![\Gamma]\!] \xrightarrow{u_\Gamma} \bar{S}[\![\Gamma]\!]$

where

$\mathcal{M}[\![\Gamma]\!] = \prod_i \mathcal{M}_{\tau_i}$ for $\tau_i \in \Gamma$

$u_\Gamma = \prod_i u_{\tau_i}$

$v = \bar{y}\Gamma \cong \prod_i V_{\tau_i} \xrightarrow{\prod_i var_{\tau_i}} \mathcal{M}[\![\Gamma]\!]$,

i.e.

$$s\text{-}nf_{\tau,\Gamma}(t) = (q_\tau,\ \bar{S}[\![\Gamma \vdash t : \tau]\!](uv)_\Gamma)(id_\Gamma)$$

for all $t \in \mathcal{L}_\tau(\Gamma)$.

Theorems.

- $\pi(u_\tau) = id_{S[\![\tau]\!]} = \pi(q_\tau)$,
  clearly by definition.

- If $t =_{\beta\eta} t' \in \mathcal{L}_\tau\Gamma$, then

  $s\text{-}nf(t) = s\text{-}nf(t')$,

  since $\beta\eta$-equivalent terms have same interpretations, i.e.

  $\bar{S}[\![\Gamma \vdash t : \tau]\!] = \bar{S}[\![\Gamma \vdash t' : \tau]\!]$

  and hence the theorem.

- Since the second component of u_τ and q_τ are identities, we know that terms has same semantics as their normal forms,

  i.e.

$$\begin{tikzcd}
\mathcal{L}_\tau \arrow[rr, "s\text{-}nf_\tau"] \arrow[dr, "l_\tau"'] & & \mathcal{N}_\tau \arrow[dl, "n_\tau"] \\
& \mathcal{S}(S[\![-]\!], S[\![\tau]\!]) &
\end{tikzcd}$$

  commutes,

  i.e. $\forall t \in \mathcal{L}_\tau(\Gamma).\ t =_{\beta\eta} s\text{-}nf(t)$.

Idempotent:

for all $N \in \mathcal{N}_\tau(\Gamma)$, $s\text{-}nf(N) = N$.

From here, we can deduce:

- if $N =_{\beta\eta} N'$ then $s\text{-}nf(N) = s\text{-}nf(N')$, hence $N = N'$.
- for any interpretation $f : T \to \mathcal{S}'$,

  we have $s\text{-}nf(t) = s\text{-}nf(f\text{-}nf(t))$

  since $t =_{\beta\eta} f\text{-}nf(t)$

  and hence $s\text{-}nf(t) = f\text{-}nf(t)$.
- let $f : T \to \mathbb{F}[T]$ be the interpretation into the free cartesian closed category over T, all normalization functions $s\text{-}nf$ are equal to $f\text{-}nf$.