CE3102 Signal Processing Cheat sheet

Basic Formula

Trigonometric Functions

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = 1 - 2\sin^2 x$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\operatorname{sinc} x = \frac{\sin \pi t}{\pi t}$$

Euler's Formula

$$e^{jx} = \cos x + j \sin x$$

Fourier Analysis

CTFS

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}, c_n \in \mathbb{C}$$
$$c_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j2\pi n f_0 t} dt$$
$$P = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt$$

CTFT

$$X(t) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f) e^{j2\pi f t} dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w) e^{jwt} dw$$
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$
$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} S_{x_p}(f) df$$
$$S_{x_p}(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

Properties

$$ax_1(t) + bx_2(t) \xleftrightarrow{\mathcal{F}} aX_1(f) + bX_2(f)$$
$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}X(\frac{f}{a})$$
$$X(t) \xleftrightarrow{\mathcal{F}} x(-f)$$
$$x(t-t_1) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f t_1} X(f)$$
$$e^{j2\pi f_1 t} x(t) \xleftrightarrow{\mathcal{F}} X(f - f_1)$$
$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{F}} j2\pi f X(f)$$
$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{F}} (j2\pi f)^n X(f)$$
$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$$
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \xleftrightarrow{\mathcal{F}} X_1(f)X_2(f)$$
$$x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} \int_{\infty}^{\infty} X_1(\lambda)X_2(f-\lambda)d\lambda = X_1(f) * X_2(f)$$

DTFS

DTFT

Fourier Transform Pairs

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$
$$\delta(t-t_0)f(t) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f t_0} f(t_0)$$
$$e^{jw_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(w - w_0)$$
$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+jw}$$
$$\operatorname{tri}\frac{t}{T} \xleftrightarrow{\mathcal{F}} T\operatorname{sinc}^2(fT) = AT\frac{\sin(\pi f t)}{\pi f t}$$
$$\cos(2\pi f_c t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)]$$

$$\sin(2\pi f_c t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j}[\delta(f-f_0) - \delta(f+f_0)]$$

Laplace Transform

Bilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

Unilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

where

$$s = \sigma + j\omega$$

Frequency Response:

$$y(t) = e^{s_0 t}H(s_0) \qquad \text{if} x(t) = e^{s_0 t}$$
$$y(t) = |H(j\omega_0)|e^{j(\omega_0 t + \angle H(j\omega_0))}$$
$$y(t) = |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$$

Properties

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$
$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$
$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}f(0^+) - \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$
$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$
$$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-}t_0 sF(s)$$
$$\mathcal{L}[e_{-at}f(t)] = F(s+a)$$
$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$
$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$
$$\mathcal{L}[f(at-b)u(at-b)] = \frac{e^{-sb/a}}{a}F(\frac{s}{a})$$

Laplace Transform Pairs

$$
\begin{aligned}
\delta(t) &\xleftrightarrow{\mathcal{L}} 1, && \text{All s}\\
u(t) &\xleftrightarrow{\mathcal{L}} \frac{1}{s}, && \mathrm{Re}(s) > 0\\
t &\xleftrightarrow{\mathcal{L}} \frac{1}{s^2}, && \mathrm{Re}(s) > 0\\
t^n &\xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}, && \mathrm{Re}(s) > 0\\
e^{-at} &\xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, && \mathrm{Re}(s) > -a\\
te^-at &\xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2}, && \mathrm{Re}(s) > -a\\
t^n e^{-at} &\xleftrightarrow{\mathcal{L}} \frac{n!}{(s+a)^{n+1}}, && \mathrm{Re}(s) > -a\\
\sin bt &\xleftrightarrow{\mathcal{L}} \frac{b}{s^2+b^2}, && \mathrm{Re}(s) > 0\\
\cos bt &\xleftrightarrow{\mathcal{L}} \frac{s}{s^2+b^2}, && \mathrm{Re}(s) > 0\\
e^{-at}\sin bt &\xleftrightarrow{\mathcal{L}} \frac{b}{(s+a)^2+b^2}, && \mathrm{Re}(s) > -a\\
e^{-at}\cos bt &\xleftrightarrow{\mathcal{L}} \frac{s+a}{(s+a)^2+b^2}, && \mathrm{Re}(s) > -a\\
t\sin bt &\xleftrightarrow{\mathcal{L}} \frac{2bs}{(s^2+b^2)^2}, && \mathrm{Re}(s) > 0\\
t\cos bt &\xleftrightarrow{\mathcal{L}} \frac{s^2-b^2}{(s^2+b^2)^2}, && \mathrm{Re}(s) > 0\\
2e^{-at}\cos(\beta t + c) &\xleftrightarrow{\mathcal{L}} \frac{e^{jc}}{s+a-j\beta} + \frac{e^{-jc}}{s+a+j\beta} && \mathrm{Re}(s) > -a
\end{aligned}
$$

Z Transform

Bilateral Z Transform:

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

Unilateral Z Transform:

$$X(z) = \sum_{0}^{\infty} x[n] z^{-n}$$

Frequency Response:

$$
\begin{aligned}
y[n] &= z_0^n H(z_0) \qquad \text{if } x[n] = z_0^n\\
y[n] &= |H(e^{\omega_0 j})| e^{j(\omega_0 n + \angle H(e^{\omega_0 j}))}\\
y[n] &= |H(e^{\omega_0 j})| \cos(\omega_0 n + \angle H(e^{\omega_0 j}))
\end{aligned}
$$

Stability:

$$z \in \text{ROC iff } \{h[n]z^{-n}\} \text{ is absolutely summable}, \textit{i.e.},$$

$$S(z) = \sum_{-\infty}^{\infty} |h[n]||z|^{-n} < \infty$$

Properties

$$
\begin{aligned}
ax_1[n] + bx_2[n] &\xleftrightarrow{\mathcal{Z}} aX_1[z] + bX_2[z]\\
x[n-m]u[n-m] &\xleftrightarrow{\mathcal{Z}} z^{-m} X[z]\\
x[n-m]u[n] &\xleftrightarrow{\mathcal{Z}} \frac{1}{z^m} X[z] + \frac{1}{z^m} \sum_{n=1}^{m} x[-n] z^n\\
x[n-1]u[n] &\xleftrightarrow{\mathcal{Z}} \frac{1}{z} X[z] + x[-1]\\
x[n-2]u[n] &\xleftrightarrow{\mathcal{Z}} \frac{1}{z^2} X[z] + \frac{1}{z} x[-1] + x[-2]\\
x[n+m]u[n] &\xleftrightarrow{\mathcal{Z}} z^m X[z] - z^m \sum_{n=0}^{m-1} x[n] z^- n\\
a^n x[n] u[n] &\xleftrightarrow{\mathcal{Z}} X[\frac{z}{a}]\\
nx[n]u[n] &\xleftrightarrow{\mathcal{Z}} -z \frac{d}{dz} X[z]\\
x[-n] &\xleftrightarrow{\mathcal{Z}} X[\frac{1}{z}]\\
x_1[n] * x_2[n] &\xleftrightarrow{\mathcal{Z}} X_1[z] X_2[z]
\end{aligned}
$$

$$
\begin{aligned}
x[0] &\xleftrightarrow{\mathcal{Z}} \lim_{z\to\infty} X[z]\\
\lim_{N\to\infty} x[N] &\xleftrightarrow{\mathcal{Z}} \lim_{z\to 1} (z-1) X[z]
\end{aligned}
$$

Z Transform Pairs

$$
\begin{aligned}
\delta[n] &\xleftrightarrow{\mathcal{Z}} 1, && \text{All z}\\
u[n] &\xleftrightarrow{\mathcal{Z}} \frac{1}{1-z^{-1}}, && |z| > 1\\
a^n u[n] &\xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}, && |z| > |a|\\
-a^n u[-n-1] &\xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}, && |z| < |a|\\
na^n u[n] &\xleftrightarrow{\mathcal{Z}} \frac{az^{-1}}{(1-az^{-1})^2}, && |z| > |a|\\
-na^n u[-n-1] &\xleftrightarrow{\mathcal{Z}} \frac{az^{-1}}{(1-az^{-1})^2}, && |z| < |a|\\
|a|^n \cos(\beta n) u[n] &\xleftrightarrow{\mathcal{Z}} \frac{z(z-|a|\cos(\beta))}{z^2 - 2|a|\cos(\beta)z + |a|^2}, && |z| > |a|\\
|a|^n \sin(\beta n) u[n] &\xleftrightarrow{\mathcal{Z}} \frac{z|a|\sin(\beta)}{z^2 - 2|a|\cos(\beta)z + |a|^2}, && |z| > |a|\\
|a|^n \sin(\beta n + \theta) u[n] &\xleftrightarrow{\mathcal{Z}} \frac{z(z\cos(\theta) - |a|\cos(\beta-\theta))}{z^2 - 2|a|\cos(\beta)z + |a|^2}, && |z| > |a|\\
&= \frac{(0.5e^{j\theta})z}{z-|a|e^{j\beta}} + \frac{(0.5e^{-j\theta})z}{z-|a|e^{-j\beta}}
\end{aligned}
$$

ROC

$$
\text{finity}\begin{cases}
\text{right sided(causal): } \{z \in \mathbb{Z}, \sim (z \in \{0\})\}\\
\text{left sided(anti-causal): } \{z \in \mathbb{Z}, \sim (z \in \{\infty\})\}\\
\text{non-causal: } \{z \in \mathbb{Z}, \sim (z \in \{0, \infty\})\}
\end{cases}
$$

$$
\text{infinity}\begin{cases}
\text{right sided(causal): } \{z \in \mathbb{Z}, |z| > r_1\}\\
\text{left sided(anti-causal): } \{z \in \mathbb{Z}, |z| < r_2\}\\
\text{non-causal: } \{z \in \mathbb{Z}, r_1 < |z| < r_2\}
\end{cases}
$$