

CE3102 Signal Processing Cheat sheet

Basic Formula

Trigonometric Functions

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = 1 - 2 \sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin x \sin y = \frac{1}{2} [\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2} [\cos(x-y) + \cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

$$\sin x + \sin y = 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\sin x - \sin y = 2 \cos \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$$

$$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\cos x - \cos y = -2 \sin \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$$

$$\text{sinc } x = \frac{\sin \pi t}{\pi t}$$

Euler's Formula

$$e^{jx} = \cos x + j \sin x$$

Fourier Analysis

CTFS

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}, c_n \in \mathbb{C}$$

$$c_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j2\pi n f_0 t} dt$$

$$P = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt$$

CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jw t} dw$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} S_{x_p}(f) df$$

$$S_{x_p}(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - n f_0)$$

Frequency response:

$$y(t) = \sum_{k=1}^K c'_k e^{j(\omega_k t + \theta_k)}$$

$$c'_k = c_k \mathcal{H}(\omega_k)$$

$$x(t) = \sum_{k=1}^K c_k e^{j(\omega_k t + \theta_k)}$$

$$\mathcal{H}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau) d\tau$$

Properties

$$ax_1(t) + bx_2(t) \xleftrightarrow{\mathcal{F}} aX_1(f) + bX_2(f)$$

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$X(t) \xleftrightarrow{\mathcal{F}} x(-f)$$

$$x(t - t_1) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f t_1} X(f)$$

$$e^{j2\pi f_1 t} x(t) \xleftrightarrow{\mathcal{F}} X(f - f_1)$$

$$\frac{d}{dt} x(t) \xleftrightarrow{\mathcal{F}} j2\pi f X(f)$$

$$\frac{d^n}{dt^n} x(t) \xleftrightarrow{\mathcal{F}} (j2\pi f)^n X(f)$$

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0) \delta(f)$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) d\tau \xleftrightarrow{\mathcal{F}} X_1(f) X_2(f)$$

$$x_1(t) x_2(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} X_1(\lambda) X_2(f - \lambda) d\lambda = X_1(f) * X_2(f)$$

DTFS

DTFT

Fourier Transform Pairs

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$\delta(t - t_0) f(t) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f t_0} F(f)$$

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi \delta(\omega - \omega_0)$$

$$e^{-at} u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a + j\omega}$$

$$\text{tri} \frac{t}{T} \xleftrightarrow{\mathcal{F}} T \text{sinc}^2(fT) = AT \frac{\sin(\pi f T)}{\pi f T}$$

$$\cos(2\pi f_c t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)]$$

$$\sin(2\pi f_c t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)]$$

Laplace Transform

Bilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

Unilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_0^{\infty} f(t) e^{-st} dt$$

where

$$s = \sigma + j\omega$$

Frequency Response:

$$y(t) = e^{s_0 t} H(s_0) \quad \text{if } x(t) = e^{s_0 t}$$

$$y(t) = |H(j\omega_0)| e^{j(\omega_0 t + \angle H(j\omega_0))}$$

$$y(t) = |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$$

Properties

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

$$\mathcal{L} \left[\frac{df(t)}{dt} \right] = sF(s) - f(0^+)$$

$$\mathcal{L} \left[\frac{d^n f(t)}{dt^n} \right] = s^n F(s) - s^{n-1} f(0^+) - \dots - s f^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$

$$\mathcal{L} \left[\int_0^t f(\tau) d\tau \right] = \frac{F(s)}{s}$$

$$\mathcal{L}[f(t - t_0) u(t - t_0)] = e^{-s t_0} F(s)$$

$$\mathcal{L}[e^{-at} f(t)] = F(s + a)$$

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} s F(s)$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$

$$\mathcal{L}[f(at-b)u(at-b)] = \frac{e^{-sb/a}}{a} F\left(\frac{s}{a}\right)$$

Laplace Transform Pairs

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1, \quad \text{All } s$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \quad \text{Re}(s) > 0$$

$$t \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}, \quad \text{Re}(s) > 0$$

$$t^n \xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}, \quad \text{Re}(s) > 0$$

$$e^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \text{Re}(s) > -a$$

$$te^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2}, \quad \text{Re}(s) > -a$$

$$t^n e^{-at} \xleftrightarrow{\mathcal{L}} \frac{n!}{(s+a)^{n+1}}, \quad \text{Re}(s) > -a$$

$$\sin bt \xleftrightarrow{\mathcal{L}} \frac{b}{s^2 + b^2}, \quad \text{Re}(s) > 0$$

$$\cos bt \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + b^2}, \quad \text{Re}(s) > 0$$

$$e^{-at} \sin bt \xleftrightarrow{\mathcal{L}} \frac{b}{(s+a)^2 + b^2}, \quad \text{Re}(s) > -a$$

$$e^{-at} \cos bt \xleftrightarrow{\mathcal{L}} \frac{s+a}{(s+a)^2 + b^2}, \quad \text{Re}(s) > -a$$

$$t \sin bt \xleftrightarrow{\mathcal{L}} \frac{2bs}{(s^2 + b^2)^2}, \quad \text{Re}(s) > 0$$

$$t \cos bt \xleftrightarrow{\mathcal{L}} \frac{s^2 - b^2}{(s^2 + b^2)^2}, \quad \text{Re}(s) > 0$$

$$2e^{-at} \cos(\beta t + c) \xleftrightarrow{\mathcal{L}} \frac{e^{jc}}{s+a-j\beta} + \frac{e^{-jc}}{s+a+j\beta} \quad \text{Re}(s) > -a$$

Z Transform

Bilateral Z Transform:

$$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$$

Unilateral Z Transform:

$$X(z) = \sum_0^{\infty} x[n]z^{-n}$$

Frequency Response:

$$y[n] = z_0^n H(z_0) \quad \text{if } x[n] = z_0^n$$

$$y[n] = |H(e^{j\omega_0})| e^{j(\omega_0 n + \angle H(e^{j\omega_0}))}$$

$$y[n] = |H(e^{j\omega_0})| \cos(\omega_0 n + \angle H(e^{j\omega_0}))$$

Stability:

$z \in \text{ROC}$ iff $\{h[n]z^{-n}\}$ is absolutely summable, i.e.,

$$S(z) = \sum_{-\infty}^{\infty} |h[n]| |z|^{-n} < \infty$$

Properties

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1[z] + bX_2[z]$$

$$x[n-m]u[n-m] \xleftrightarrow{\mathcal{Z}} z^{-m}X[z]$$

$$x[n-m]u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{z^m}X[z] + \frac{1}{z^m} \sum_{n=1}^m x[-n]z^n$$

$$x[n-1]u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{z}X[z] + x[-1]$$

$$x[n-2]u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2]$$

$$x[n+m]u[n] \xleftrightarrow{\mathcal{Z}} z^mX[z] - z^m \sum_{n=0}^{m-1} x[n]z^{-n}$$

$$a^n x[n]u[n] \xleftrightarrow{\mathcal{Z}} X\left[\frac{z}{a}\right]$$

$$nx[n]u[n] \xleftrightarrow{\mathcal{Z}} -z \frac{d}{dz} X[z]$$

$$x[-n] \xleftrightarrow{\mathcal{Z}} X\left[\frac{1}{z}\right]$$

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{Z}} X_1[z]X_2[z]$$

$$x[0] \xleftrightarrow{\mathcal{Z}} \lim_{z \rightarrow \infty} X[z]$$

$$\lim_{N \rightarrow \infty} x[N] \xleftrightarrow{\mathcal{Z}} \lim_{z \rightarrow 1} (z-1)X[z]$$

Z Transform Pairs

$$\delta[n] \xleftrightarrow{\mathcal{Z}} 1, \quad \text{All } z$$

$$u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$-a^n u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

$$na^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

$$-na^n u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| < |a|$$

$$|a|^n \cos(\beta n)u[n] \xleftrightarrow{\mathcal{Z}} \frac{z(z-|a|\cos(\beta))}{z^2-2|a|\cos(\beta)z+|a|^2}, \quad |z| > |a|$$

$$|a|^n \sin(\beta n)u[n] \xleftrightarrow{\mathcal{Z}} \frac{z|a|\sin(\beta)}{z^2-2|a|\cos(\beta)z+|a|^2}, \quad |z| > |a|$$

$$\begin{aligned} |a|^n \sin(\beta n + \theta)u[n] &\xleftrightarrow{\mathcal{Z}} \frac{z(z\cos(\theta) - |a|\cos(\beta - \theta))}{z^2 - 2|a|\cos(\beta)z + |a|^2}, \quad |z| > |a| \\ &= \frac{(0.5e^{j\theta})z}{z - |a|e^{j\beta}} + \frac{(0.5e^{-j\theta})z}{z - |a|e^{-j\beta}} \end{aligned}$$

ROC

$$\text{finity} \begin{cases} \text{right sided(causal): } \{z \in \mathbb{Z}, \sim (z \in \{0\})\} \\ \text{left sided(anti-causal): } \{z \in \mathbb{Z}, \sim (z \in \{\infty\})\} \\ \text{non-causal: } \{z \in \mathbb{Z}, \sim (z \in \{0, \infty\})\} \end{cases}$$

$$\text{infinity} \begin{cases} \text{right sided(causal): } \{z \in \mathbb{Z}, |z| > r_1\} \\ \text{left sided(anti-causal): } \{z \in \mathbb{Z}, |z| < r_2\} \\ \text{non-causal: } \{z \in \mathbb{Z}, r_1 < |z| < r_2\} \end{cases}$$