

CE3102 Signal Processing Cheat sheet

Basic Formula

Trigonometric Functions

$$\begin{aligned}\sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \\ \sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= 1 - 2 \sin^2 x \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos^2 x &= \frac{1 + \cos 2x}{2} \\ \sin x \sin y &= \frac{1}{2} [\cos(x-y) - \cos(x+y)] \\ \cos x \cos y &= \frac{1}{2} [\cos(x-y) + \cos(x+y)] \\ \sin x \cos y &= \frac{1}{2} [\sin(x+y) + \sin(x-y)] \\ \sin x + \sin y &= 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) \\ \sin x - \sin y &= 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \\ \cos x + \cos y &= 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) \\ \cos x - \cos y &= -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \\ \operatorname{sinc} x &= \frac{\sin \pi t}{\pi t}\end{aligned}$$

Euler's Formula

$$\begin{aligned}e^{jx} &= \cos x + j \sin x \\ \cos x &= \frac{e^{jx} + e^{-jx}}{2} \\ \sin x &= \frac{e^{jx} - e^{-jx}}{2j}\end{aligned}$$

Fourier Analysis

CTFS

$$\begin{aligned}x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}, c_n \in \mathbb{C} \\ c_n &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j2\pi n f_0 t} dt \\ P &= \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt\end{aligned}$$

Frequency response:

$$\begin{aligned}y(t) &= \sum_{k=1}^K c'_k e^{j(\omega_k t + \theta_k)} \\ c'_k &= c_k \mathcal{H}(\omega_k) \\ x(t) &= \sum_{k=1}^K c_k e^{j(\omega_k t + \theta_k)} \\ \mathcal{H}(\omega) &= \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau) d\tau\end{aligned}$$

CTFT

$$\begin{aligned}X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jw t} dw \\ E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \\ P &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} S_{x_p}(f) df \\ S_{x_p}(f) &= \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - n f_0)\end{aligned}$$

Properties

$$\begin{aligned}ax_1(t) + bx_2(t) &\xleftrightarrow{\mathcal{F}} aX_1(f) + bX_2(f) \\ x(at) &\xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right) \\ X(t) &\xleftrightarrow{\mathcal{F}} x(-f) \\ x(t-t_1) &\xleftrightarrow{\mathcal{F}} e^{-j2\pi f t_1} X(f) \\ e^{j2\pi f_1 t} x(t) &\xleftrightarrow{\mathcal{F}} X(f-f_1) \\ \frac{d}{dt} x(t) &\xleftrightarrow{\mathcal{F}} j2\pi f X(f) \\ \frac{d^n}{dt^n} x(t) &\xleftrightarrow{\mathcal{F}} (j2\pi f)^n X(f) \\ \int_{-\infty}^t x(\tau) d\tau &\xleftrightarrow{\mathcal{F}} \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0) \delta(f) \\ x_1(t) * x_2(t) &= \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \xleftrightarrow{\mathcal{F}} X_1(f) X_2(f) \\ x_1(t) x_2(t) &\xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} X_1(\lambda) X_2(f-\lambda) d\lambda = X_1(f) * X_2(f)\end{aligned}$$

DTFS

Analysis

$$\begin{aligned}X(\Omega) &= \sum_{k=0}^{N-1} 2\pi c_k \delta\left(\Omega - \frac{2\pi}{N} k\right) \\ c_k &= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j \frac{2\pi}{N} kn}\end{aligned}$$

Synthesis

$$\begin{aligned}\tilde{x}[n] &= \sum_{k=0}^{N-1} c_k e^{j \frac{2\pi}{N} kn}, n \in \mathbb{Z} \\ \tilde{x}[n] &= \frac{1}{2T} \int_0^{2\pi} X(\Omega) e^{j\Omega n} d\Omega\end{aligned}$$

DTFT

$$\begin{aligned}X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\ x[n] &= \frac{1}{2T} \int_0^{2\pi} X(\Omega) e^{j\Omega n} d\Omega\end{aligned}$$

DFT

$$\begin{aligned}X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} \\ x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} kn}\end{aligned}$$

Fourier Transform Pairs

$$\begin{aligned}\delta(t) &\xleftrightarrow{\mathcal{F}} 1 \\ \delta(t-t_0) f(t) &\xleftrightarrow{\mathcal{F}} e^{-j2\pi f t_0} f(t_0) \\ e^{j2\pi f_0 t} &\xleftrightarrow{\mathcal{F}} \delta(f-f_0) \\ e^{-at} u(t) &\xleftrightarrow{\mathcal{F}} \frac{1}{a+jw} \\ \operatorname{tri} \frac{t}{T} &\xleftrightarrow{\mathcal{F}} T \operatorname{sinc}^2(fT) \\ A \Pi\left(\frac{t}{\tau}\right) &\xleftrightarrow{\mathcal{F}} A \tau \frac{\sin(\pi f \tau)}{\pi f \tau} \\ \cos(2\pi f_c t) &\xleftrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f-f_0) + \delta(f+f_0)] \\ \sin(2\pi f_c t) &\xleftrightarrow{\mathcal{F}} \frac{1}{2j} [\delta(f-f_0) - \delta(f+f_0)]\end{aligned}$$

Laplace Transform

Bilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$$

Unilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_0^{\infty} f(t)e^{-st} dt$$

where

$$s = \sigma + j\omega$$

Frequency Response:

$$y(t) = e^{s_0 t} H(s_0) \quad \text{if } x(t) = e^{s_0 t}$$

$$y(t) = |H(j\omega_0)| e^{j(\omega_0 t + \angle H(j\omega_0))}$$

$$y(t) = |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$$

Properties

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^+) - \dots - s f^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$$

$$\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0 s} F(s)$$

$$\mathcal{L}[e^{-at} f(t)] = F(s + a)$$

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$

$$\mathcal{L}[f(at - b)u(at - b)] = \frac{e^{-sb/a}}{a} F\left(\frac{s}{a}\right)$$

Laplace Transform Pairs

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1, \quad \text{All } s$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \quad \text{Re}(s) > 0$$

$$t \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}, \quad \text{Re}(s) > 0$$

$$t^n \xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}, \quad \text{Re}(s) > 0$$

$$e^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \text{Re}(s) > -a$$

$$te^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2}, \quad \text{Re}(s) > -a$$

$$t^n e^{-at} \xleftrightarrow{\mathcal{L}} \frac{n!}{(s+a)^{n+1}}, \quad \text{Re}(s) > -a$$

$$\sin bt \xleftrightarrow{\mathcal{L}} \frac{b}{s^2 + b^2}, \quad \text{Re}(s) > 0$$

$$\cos bt \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + b^2}, \quad \text{Re}(s) > 0$$

$$e^{-at} \sin bt \xleftrightarrow{\mathcal{L}} \frac{b}{(s+a)^2 + b^2}, \quad \text{Re}(s) > -a$$

$$e^{-at} \cos bt \xleftrightarrow{\mathcal{L}} \frac{s+a}{(s+a)^2 + b^2}, \quad \text{Re}(s) > -a$$

$$t \sin bt \xleftrightarrow{\mathcal{L}} \frac{2bs}{(s^2 + b^2)^2}, \quad \text{Re}(s) > 0$$

$$t \cos bt \xleftrightarrow{\mathcal{L}} \frac{s^2 - b^2}{(s^2 + b^2)^2}, \quad \text{Re}(s) > 0$$

$$2e^{-at} \cos(\beta t + c) \xleftrightarrow{\mathcal{L}} \frac{e^{jc}}{s+a-j\beta} + \frac{e^{-jc}}{s+a+j\beta} \quad \text{Re}(s) > -a$$

Z Transform

Bilateral Z Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Unilateral Z Transform:

$$X(z) = \sum_0^{\infty} x[n]z^{-n}$$

Frequency Response:

$$y[n] = z_0^n H(z_0) \quad \text{if } x[n] = z_0^n$$

$$y[n] = |H(e^{j\omega_0})| e^{j(\omega_0 n + \angle H(e^{j\omega_0}))}$$

$$y[n] = |H(e^{j\omega_0})| \cos(\omega_0 n + \angle H(e^{j\omega_0}))$$

Stability:

$$z \in \text{ROC} \text{ iff } \{h[n]z^{-n}\} \text{ is absolutely summable, i.e.,}$$

$$S(z) = \sum_{n=-\infty}^{\infty} |h[n]| |z|^{-n} < \infty$$

Properties

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{Z}} aX_1[z] + bX_2[z]$$

$$x[n - m]u[n - m] \xleftrightarrow{\mathcal{Z}} z^{-m} X[z]$$

Z Transform Pairs

$$\delta[n] \xleftrightarrow{\mathcal{Z}} 1, \quad \text{All } z$$

$$u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

$$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$-a^n u[-n - 1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

$$na^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| > |a|$$

$$-na^n u[-n - 1] \xleftrightarrow{\mathcal{Z}} \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| < |a|$$

$$|a|^n \cos(\beta n) u[n] \xleftrightarrow{\mathcal{Z}} \frac{z(z - |a| \cos(\beta))}{z^2 - 2|a| \cos(\beta)z + |a|^2}, \quad |z| > |a|$$

$$|a|^n \sin(\beta n) u[n] \xleftrightarrow{\mathcal{Z}} \frac{z|a| \sin(\beta)}{z^2 - 2|a| \cos(\beta)z + |a|^2}, \quad |z| > |a|$$

$$|a|^n \sin(\beta n + \theta) u[n] \xleftrightarrow{\mathcal{Z}} \frac{z(z \cos(\theta) - |a| \cos(\beta - \theta))}{z^2 - 2|a| \cos(\beta)z + |a|^2}, \quad |z| > |a|$$

$$= \frac{(0.5e^{j\theta})z}{z - |a|e^{j\beta}} + \frac{(0.5e^{-j\theta})z}{z - |a|e^{-j\beta}}$$

ROC

$$\text{finity} \left\{ \begin{array}{l} \text{right sided(causal): } \{z \in \mathbb{C}, \sim (z \in \{0\})\} \\ \text{left sided(anti-causal): } \{z \in \mathbb{C}, \sim (z \in \{\infty\})\} \\ \text{non-causal: } \{z \in \mathbb{C}, \sim (z \in \{0, \infty\})\} \end{array} \right.$$

$$\text{infinity} \left\{ \begin{array}{l} \text{right sided(causal): } \{z \in \mathbb{C}, |z| > r_1\} \\ \text{left sided(anti-causal): } \{z \in \mathbb{C}, |z| < r_2\} \\ \text{non-causal: } \{z \in \mathbb{C}, r_1 < |z| < r_2\} \end{array} \right.$$