CE3102 Signal Processing Cheat sheet

Basic Formula

Trigonometric Functions

$$\sin(x+y) = \sin x\cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x\cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x \sin y$$
$$\sin(2x) = 2\sin x\cos x$$
$$\cos(2x) = 1 - 2\sin^2 x$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin x \sin y = \frac{1}{2}\big[\cos(x-y) - \cos(x+y)\big]$$
$$\cos x \cos y = \frac{1}{2}\big[\cos(x-y) + \cos(x+y)\big]$$
$$\sin x \cos y = \frac{1}{2}\big[\sin(x+y) + \sin(x-y)\big]$$
$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\operatorname{sinc} x = \frac{\sin \pi t}{\pi t}$$

Euler's Formula

$$e^{jx} = \cos x + j\sin x$$
$$\cos x = \frac{e^{jx}+e^{-jx}}{2}$$
$$\sin x = \frac{e^{jx}-e^{-jx}}{2j}$$

Fourier Analysis

CTFS

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}, c_n \in \mathbb{C}$$
$$c_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-j2\pi n f_0 t}dt$$
$$P = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt$$

Frequency response:

$$y(t) = \sum_{k=1}^{K} c'_k e^{j(\omega_k t + \theta_k)}$$
$$c'_k = c_k \mathcal{H}(\omega_k)$$
$$x(t) = \sum_{k=1}^{K} c_k e^{j(w_k t + \theta_k)}$$
$$\mathcal{H}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} h(\tau)d\tau$$

CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)e^{j2\pi f t}dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$
$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} S_{x_p}(f)df$$
$$S_{x_p}(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

Properties

$$ax_1(t) + bx_2(t) \overset{\mathcal{F}}{\longleftrightarrow} aX_1(f) + bX_2(f)$$
$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X(\frac{f}{a})$$
$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$$
$$x(t-t_1) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f t_1}X(f)$$
$$e^{j2\pi f_1 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f-f_1)$$
$$\frac{d}{dt}x(t) \overset{\mathcal{F}}{\longleftrightarrow} j2\pi f X(f)$$
$$\frac{d^n}{dt^n}x(t) \overset{\mathcal{F}}{\longleftrightarrow} (j2\pi f)^n X(f)$$
$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$$
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} X_1(f)X_2(f)$$
$$x_1(t)x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{\infty}^{\infty} X_1(\lambda)X_2(f-\lambda)d\lambda = X_1(f) * X_2(f)$$

DTFS

Analysis

$$X(\Omega) = \sum_{k=0}^{N-1} 2\pi c_k \delta(\Omega - \frac{2\pi}{N}k)$$
$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}\Omega kn}$$

Synthesis

$$\tilde{x}[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}, n \in \mathbb{Z}$$
$$\tilde{x}[n] = \frac{1}{2T}\int_0^{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega_n n}$$
$$x[n] = \frac{1}{2T}\int_0^{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$$

Fourier Transform Pairs

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$
$$\delta(t-t_0)f(t) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f t_0}f(t_0)$$
$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f-f_0)$$
$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+jw}$$
$$\operatorname{tri}\frac{t}{T} \overset{\mathcal{F}}{\longleftrightarrow} T\operatorname{sinc}^2(fT)$$
$$\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\longleftrightarrow} A\tau\frac{\sin(\pi f\tau)}{\pi f\tau}$$
$$\cos(2\pi f_c t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)]$$
$$\sin(2\pi f_c t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}[\delta(f-f_0) - \delta(f+f_0)]$$

Laplace Transform

Bilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

Unilateral Laplace Transform:

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

where

$$s = \sigma + j\omega$$

Frequency Response:

$$y(t) = e^{s_0 t}H(s_0) \qquad \text{if} x(t) = e^{s_0 t}$$
$$y(t) = |H(j\omega_0)|e^{j(\omega_0 t + \angle H(j\omega_0))}$$
$$y(t) = |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$$

Properties

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$
$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$
$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}f(0^+) - \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$
$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$
$$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-}t_0 sF(s)$$
$$\mathcal{L}[e_{-at}f(t)] = F(s+a)$$
$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$
$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$
$$\mathcal{L}[f(at-b)u(at-b)] = \frac{e^{-sb/a}}{a}F(\frac{s}{a})$$

Laplace Transform Pairs

$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1,$	All s
$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s},$	$\mathrm{Re}(s) > 0$
$t \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2},$	$\mathrm{Re}(s) > 0$
$t^n \overset{\mathcal{L}}{\longleftrightarrow} \frac{n!}{s^{n+1}},$	$\mathrm{Re}(s) > 0$
$e^{-at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a},$	$\mathrm{Re}(s) > -a$
$te^{-}at \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^2},$	$\mathrm{Re}(s) > -a$
$t^n e^{-at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{n!}{(s+a)^{n+1}},$	$\mathrm{Re}(s) > -a$
$\sin bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{b}{s^2+b^2},$	$\mathrm{Re}(s) > 0$
$\cos bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+b^2},$	$\mathrm{Re}(s) > 0$
$e^{-at}\sin bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{b}{(s+a)^2+b^2},$	$\mathrm{Re}(s) > -a$
$e^{-at}\cos bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{s+a}{(s+a)^2+b^2},$	$\mathrm{Re}(s) > -a$
$t\sin bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{2bs}{(s^2+b^2)^2},$	$\mathrm{Re}(s) > 0$
$t\cos bt \overset{\mathcal{L}}{\longleftrightarrow} \frac{s^2-b^2}{(s^2+b^2)^2},$	$\mathrm{Re}(s) > 0$
$2e^{-at}\cos(\beta t + c) \overset{\mathcal{L}}{\longleftrightarrow} \frac{e^{jc}}{s+a-j\beta} + \frac{e^{-jc}}{s+a+j\beta}$	$\mathrm{Re}(s) > -a$

Z Transform

Bilateral Z Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Unilateral Z Transform:

$$X(z) = \sum_{0}^{\infty} x[n]z^{-n}$$

Frequency Response:

$$y[n] = z_0^n H(z_0) \qquad \text{if } x[n] = z_0^n$$
$$y[n] = |H(e^{\omega_0 j})|e^{j(\omega_0 n + \angle H(e^{\omega_0 j}))}$$
$$y[n] = |H(e^{\omega_0 j})|\cos(\omega_0 n + \angle H(e^{\omega_0 j}))$$

Stability:

$$z \in \mathrm{ROC} \text{ iff } \{h[n]z^{-n}\} \text{ is absolutely summable}, \textit{i.e.},$$
$$S(z) = \sum_{n=-\infty}^{\infty} |h[n]||z|^{-n} < \infty$$

Properties

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1[z] + bX_2[z]$$
$$x[n-m]u[n-m] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-m}X[z]$$
$$x[n-m]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^m}X[z] + \frac{1}{z^m}\sum_{n=1}^{m} x[-n]z^n$$
$$x[n-1]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z}X[z] + x[-1]$$
$$x[n-2]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2]$$
$$x[n+m]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} z^m X[z] - z^m \sum_{n=0}^{m-1} x[n]z^{-}n$$
$$a^n x[n]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X[\frac{z}{a}]$$
$$nx[n]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}X[z]$$
$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X[\frac{1}{z}]$$
$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1[z]X_2[z]$$
$$x[0] \overset{\mathcal{Z}}{\longleftrightarrow} \lim_{z\to\infty} X[z]$$
$$\lim_{N\to\infty} x[N] \overset{\mathcal{Z}}{\longleftrightarrow} \lim_{z\to 1}(z-1)X[z]$$

Z Transform Pairs

$\delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1,$	All z
$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}},$	$\lvert z\rvert > 1$
$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}},$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}},$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2},$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2},$	$\lvert z\rvert < \lvert a\rvert$
$\lvert a\rvert^n \cos(\beta n)u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z(z-\lvert a\rvert\cos(\beta))}{z^2-2\lvert a\rvert\cos(\beta)z+\lvert a\rvert^2},$	$\lvert z\rvert > \lvert a\rvert$
$\lvert a\rvert^n \sin(\beta n)u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z\lvert a\rvert\sin(\beta)}{z^2-2\lvert a\rvert\cos(\beta)z+\lvert a\rvert^2},$	$\lvert z\rvert > \lvert a\rvert$
$\lvert a\rvert^n \sin(\beta n+\theta)u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z(z\cos(\theta)-\lvert a\rvert\cos(\beta-\theta))}{z^2-2\lvert a\rvert\cos(\beta)z+\lvert a\rvert^2},$	$\lvert z\rvert > \lvert a\rvert$
$= \frac{(0.5e^{j\theta})z}{z-\lvert a\rvert e^{j\beta}} + \frac{(0.5e^{-j\theta})z}{z-\lvert a\rvert e^{-j\beta}}$	

ROC

$$\text{finity}\begin{cases} \text{right sided(causal): } \{z \in \mathbb{C}, \sim (z \in \{0\})\} \\ \text{left sided(anti-causal): } \{z \in \mathbb{C}, \sim (z \in \{\infty\})\} \\ \text{non-causal: } \{z \in \mathbb{C}, \sim (z \in \{0, \infty\})\} \end{cases}$$

$$\text{infinity}\begin{cases} \text{right sided(causal): } \{z \in \mathbb{C}, |z| > r_1\} \\ \text{left sided(anti-causal): } \{z \in \mathbb{C}, |z| < r_2\} \\ \text{non-causal: } \{z \in \mathbb{C}, r_1 < |z| < r_2\} \end{cases}$$