Math 250A

Suggested Homework and Answers

From: Calculus, Larson & Edwards, 11th Ed.

1.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Describing Notation** Write a brief description of the meaning of the notation $\lim_{x \to 8} f(x) = 25$.
2. **Limits That Fail to Exist** Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.
3. **Formal Definition of Limit** Given the limit

   $$\lim_{x \to 2} (2x + 1) = 5$$

   use a sketch to show the meaning of the phrase "$0 < |x - 2| < 0.25$ implies $|(2x + 1) - 5| < 0.5$."
4. **Functions and Limits** Is the limit of $f(x)$ as x approaches c always equal to $f(c)$? Why or why not?

Estimating a Limit Numerically **In Exercises 5–10, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

5. $\lim_{x \to 4} \dfrac{x - 4}{x^2 - 5x + 4}$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				?			

6. $\lim_{x \to 3} \dfrac{x - 3}{x^2 - 9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

7. $\lim_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

8. $\lim_{x \to 3} \dfrac{[1/(x + 1)] - (1/4)}{x - 3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

9. $\lim_{x \to 0} \dfrac{\sin x}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

10. $\lim_{x \to 0} \dfrac{\cos x - 1}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

Estimating a Limit Numerically **In Exercises 11–18, create a table of values for the function and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

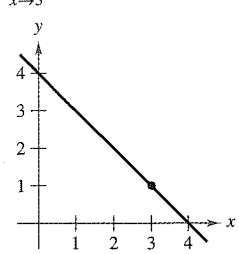
11. $\lim_{x \to 1} \dfrac{x - 2}{x^2 + x - 6}$
12. $\lim_{x \to -4} \dfrac{x + 4}{x^2 + 9x + 20}$
13. $\lim_{x \to 1} \dfrac{x^4 - 1}{x^6 - 1}$
14. $\lim_{x \to -3} \dfrac{x^3 + 27}{x + 3}$
15. $\lim_{x \to -6} \dfrac{\sqrt{10 - x} - 4}{x + 6}$
16. $\lim_{x \to 2} \dfrac{[x/(x + 1)] - (2/3)}{x - 2}$
17. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$
18. $\lim_{x \to 0} \dfrac{\tan x}{\tan 2x}$

Limits That Fail to Exist **In Exercises 19 and 20, create a table of values for the function and use the result to explain why the limit does not exist.**

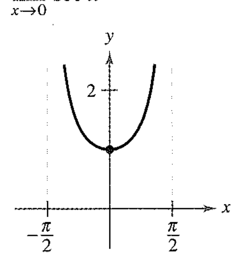
19. $\lim_{x \to 0} \dfrac{2}{x^3}$
20. $\lim_{x \to 0} \dfrac{3|x|}{x^2}$

Finding a Limit Graphically **In Exercises 21–28, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

21. $\lim_{x \to 3} (4 - x)$

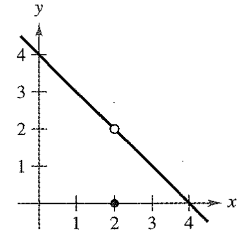


22. $\lim_{x \to 0} \sec x$



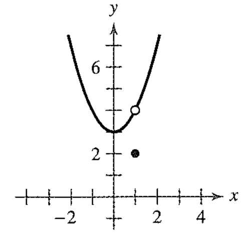
23. $\lim_{x \to 2} f(x)$

$$f(x) = \begin{cases} 4 - x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

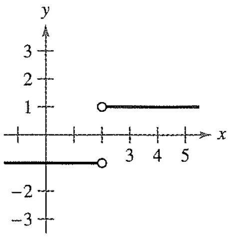


24. $\lim_{x \to 1} f(x)$

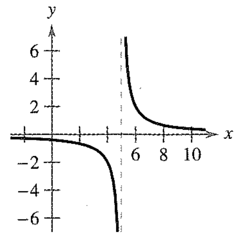
$$f(x) = \begin{cases} x^2 + 3, & x \neq 1 \\ 2, & x = 1 \end{cases}$$



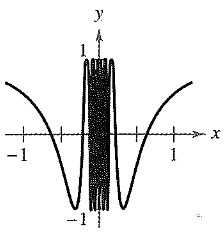
25. $\lim_{x \to 2} \dfrac{|x-2|}{x-2}$



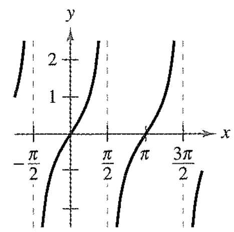
26. $\lim_{x \to 5} \dfrac{2}{x-5}$



27. $\lim_{x \to 0} \cos \dfrac{1}{x}$



28. $\lim_{x \to \pi/2} \tan x$



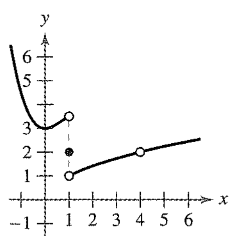
Graphical Reasoning **In Exercises 29 and 30, use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.**

29. (a) $f(1)$

(b) $\lim_{x \to 1} f(x)$

(c) $f(4)$

(d) $\lim_{x \to 4} f(x)$



30. (a) $f(-2)$

(b) $\lim_{x \to -2} f(x)$

(c) $f(0)$

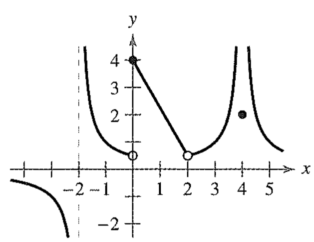
(d) $\lim_{x \to 0} f(x)$

(e) $f(2)$

(f) $\lim_{x \to 2} f(x)$

(g) $f(4)$

(h) $\lim_{x \to 4} f(x)$



Limits of a Piecewise Function **In Exercises 31 and 32, sketch the graph of f. Then identify the values of c for which $\lim_{x \to c} f(x)$ exists.**

31. $f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$

32. $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

Sketching a Graph **In Exercises 33 and 34, sketch a graph of a function f that satisfies the given values. (There are many correct answers.)**

33. $f(0)$ is undefined.

$\lim_{x \to 0} f(x) = 4$

$f(2) = 6$

$\lim_{x \to 2} f(x) = 3$

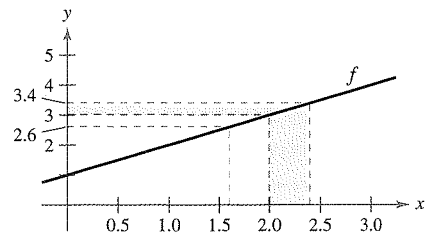
34. $f(-2) = 0$

$f(2) = 0$

$\lim_{x \to -2} f(x) = 0$

$\lim_{x \to 2} f(x)$ does not exist.

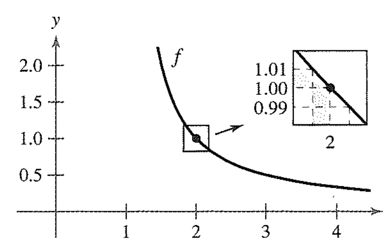
35. Finding a δ for a Given ε The graph of $f(x) = x + 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.4$.



36. Finding a δ for a Given ε The graph of

$$f(x) = \frac{1}{x-1}$$

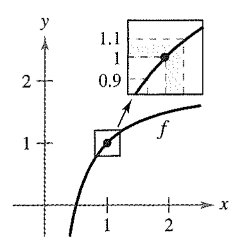
is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 1| < 0.01$.



37. Finding a δ for a Given ε The graph of

$$f(x) = 2 - \frac{1}{x}$$

is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$, then $|f(x) - 1| < 0.1$.



38. Finding a δ for a Given ε Repeat Exercise 37 for $\varepsilon = 0.05, 0.01$, and 0.005. What happens to the value of δ as the value of ε gets smaller?

1.3 Exercises

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CONCEPT CHECK

1. **Polynomial Function** Describe how to find the limit of a polynomial function $p(x)$ as x approaches c.
2. **Indeterminate Form** What is meant by an indeterminate form?
3. **Squeeze Theorem** In your own words, explain the Squeeze Theorem.
4. **Special Limits** List the two special trigonometric limits.

Finding a Limit **In Exercises 5–22, find the limit.**

5. $\lim_{x\to 2} x^3$

6. $\lim_{x\to -3} x^4$

7. $\lim_{x\to -3} (2x+5)$

8. $\lim_{x\to 9} (4x-1)$

9. $\lim_{x\to -3} (x^2+3x)$

10. $\lim_{x\to 2} (-x^3+1)$

11. $\lim_{x\to -3} (2x^2+4x+1)$

12. $\lim_{x\to 1} (2x^3-6x+5)$

13. $\lim_{x\to 3} \sqrt{x+8}$

14. $\lim_{x\to 2} \sqrt[3]{12x+3}$

15. $\lim_{x\to -4} (1-x)^3$

16. $\lim_{x\to 0} (3x-2)^4$

17. $\lim_{x\to 2} \dfrac{3}{2x+1}$

18. $\lim_{x\to -5} \dfrac{5}{x+3}$

19. $\lim_{x\to 1} \dfrac{x}{x^2+4}$

20. $\lim_{x\to 1} \dfrac{3x+5}{x+1}$

21. $\lim_{x\to 7} \dfrac{3x}{\sqrt{x+2}}$

22. $\lim_{x\to 3} \dfrac{\sqrt{x+6}}{x+2}$

Finding Limits **In Exercises 23–26, find the limits.**

23. $f(x) = 5-x,\ g(x) = x^3$

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

24. $f(x) = x+7,\ g(x) = x^2$

(a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

25. $f(x) = 4-x^2,\ g(x) = \sqrt{x+1}$

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

26. $f(x) = 2x^2-3x+1,\ g(x) = \sqrt[3]{x+6}$

(a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

Finding a Limit of a Trigonometric Function **In Exercises 27–36, find the limit of the trigonometric function.**

27. $\lim_{x\to \pi/2} \sin x$

28. $\lim_{x\to \pi} \tan x$

29. $\lim_{x\to 1} \cos \dfrac{\pi x}{3}$

30. $\lim_{x\to 2} \sin \dfrac{\pi x}{12}$

31. $\lim_{x\to 0} \sec 2x$

32. $\lim_{x\to \pi} \cos 3x$

33. $\lim_{x\to 5\pi/6} \sin x$

34. $\lim_{x\to 5\pi/3} \cos x$

35. $\lim_{x\to 3} \tan \dfrac{\pi x}{4}$

36. $\lim_{x\to 7} \sec \dfrac{\pi x}{6}$

Evaluating Limits **In Exercises 37–40, use the information to evaluate the limits.**

37. $\lim_{x\to c} f(x) = \frac{2}{5}$

$\lim_{x\to c} g(x) = 2$

(a) $\lim_{x\to c} [5g(x)]$

(b) $\lim_{x\to c} [f(x)+g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

38. $\lim_{x\to c} f(x) = 2$

$\lim_{x\to c} g(x) = \frac{3}{4}$

(a) $\lim_{x\to c} [4f(x)]$

(b) $\lim_{x\to c} [f(x)+g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 16$

(a) $\lim_{x\to c} [f(x)]^2$

(b) $\lim_{x\to c} \sqrt{f(x)}$

(c) $\lim_{x\to c} [3f(x)]$

(d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$

(a) $\lim_{x\to c} \sqrt[3]{f(x)}$

(b) $\lim_{x\to c} \dfrac{f(x)}{18}$

(c) $\lim_{x\to c} [f(x)]^2$

(d) $\lim_{x\to c} [f(x)]^{2/3}$

Finding a Limit **In Exercises 41–46, write a simpler function that agrees with the given function at all but one point. Then find the limit of the function. Use a graphing utility to confirm your result.**

41. $\lim_{x\to 0} \dfrac{x^2+3x}{x}$

42. $\lim_{x\to 0} \dfrac{x^4-5x^2}{x^2}$

43. $\lim_{x\to -1} \dfrac{x^2-1}{x+1}$

44. $\lim_{x\to -2} \dfrac{3x^2+5x-2}{x+2}$

45. $\lim_{x\to 2} \dfrac{x^3-8}{x-2}$

46. $\lim_{x\to -1} \dfrac{x^3+1}{x+1}$

Finding a Limit **In Exercises 47–62, find the limit.**

47. $\lim_{x\to 0} \dfrac{x}{x^2-x}$

48. $\lim_{x\to 0} \dfrac{7x^3-x^2}{x}$

49. $\lim_{x\to 4} \dfrac{x-4}{x^2-16}$

50. $\lim_{x\to 5} \dfrac{5-x}{x^2-25}$

51. $\lim_{x\to -3} \dfrac{x^2+x-6}{x^2-9}$

52. $\lim_{x\to 2} \dfrac{x^2+2x-8}{x^2-x-2}$

53. $\lim_{x\to 4} \dfrac{\sqrt{x+5}-3}{x-4}$

54. $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

55. $\lim_{x\to 0} \dfrac{\sqrt{x+5}-\sqrt{5}}{x}$

56. $\lim_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

1.4 Exercises

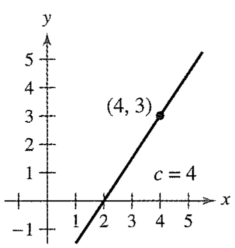
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

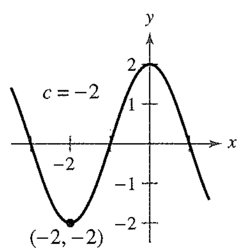
CONCEPT CHECK

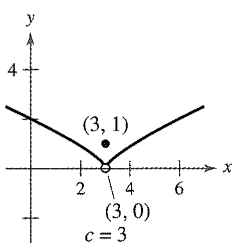
1. **Continuity** In your own words, describe what it means for a function to be continuous at a point.
2. **One-Sided Limits** What is the value of c?
$$\lim_{x\to c^+} 2\sqrt{x+1} = 0$$
3. **Existence of a Limit** Determine whether $\lim_{x\to 3} f(x)$ exists. Explain.
$$\lim_{x\to 3^-} f(x) = 1 \quad \text{and} \quad \lim_{x\to 3^+} f(x) = 1$$
4. **Intermediate Value Theorem** In your own words, explain the Intermediate Value Theorem.

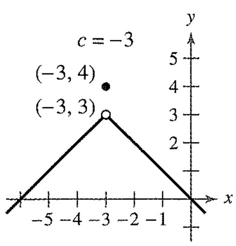
Limits and Continuity **In Exercises 5–10, use the graph to determine each limit, and discuss the continuity of the function.**

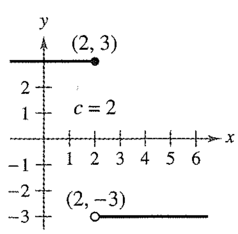
(a) $\lim_{x\to c^+} f(x)$ (b) $\lim_{x\to c^-} f(x)$ (c) $\lim_{x\to c} f(x)$

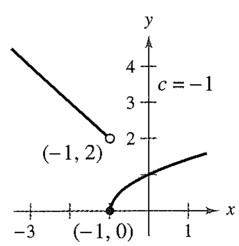
5. 

6. 

7. 

8. 

9. 

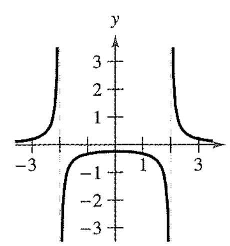
10. 

Finding a Limit **In Exercises 11–30, find the limit (if it exists). If it does not exist, explain why.**

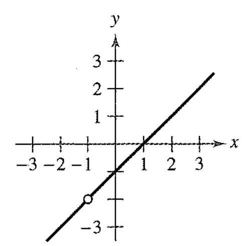
11. $\lim_{x\to 8^+} \dfrac{1}{x+8}$
12. $\lim_{x\to 3^+} \dfrac{2}{x+3}$
13. $\lim_{x\to 5^+} \dfrac{x-5}{x^2-25}$
14. $\lim_{x\to 4^+} \dfrac{4-x}{x^2-16}$
15. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$
16. $\lim_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4}$
17. $\lim_{x\to 0^-} \dfrac{|x|}{x}$
18. $\lim_{x\to 10^+} \dfrac{|x-10|}{x-10}$
19. $\lim_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x}$
20. $\lim_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2 + x + \Delta x - (x^2+x)}{\Delta x}$
21. $\lim_{x\to 3^-} f(x)$, where $f(x) = \begin{cases} \dfrac{x+2}{2}, & x < 3 \\ \dfrac{12-2x}{3}, & x > 3 \end{cases}$
22. $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 3 \\ -x^2 + 4x - 2, & x \ge 3 \end{cases}$
23. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^3 + 1, & x < 1 \\ x + 1, & x \ge 1 \end{cases}$
24. $\lim_{x\to 1^+} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$
25. $\lim_{x\to \pi} \cot x$
26. $\lim_{x\to \pi/2} \sec x$
27. $\lim_{x\to 4^-} (5[\![x]\!] - 7)$
28. $\lim_{x\to 2^+} (2x - [\![x]\!])$
29. $\lim_{x\to -1} \left(\left[\!\left[\dfrac{x}{3} \right]\!\right] + 3 \right)$
30. $\lim_{x\to 1} \left(1 - \left[\!\left[-\dfrac{x}{2} \right]\!\right] \right)$

Continuity of a Function **In Exercises 31–34, discuss the continuity of the function.**

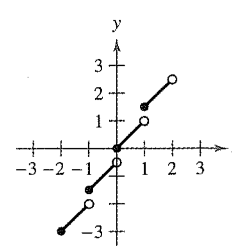
31. $f(x) = \dfrac{1}{x^2-4}$



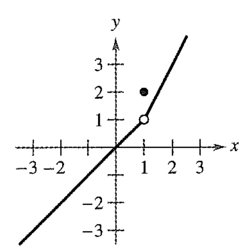
32. $f(x) = \dfrac{x^2-1}{x+1}$



33. $f(x) = \frac{1}{2}[\![x]\!] + x$



34. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



Continuity on a Closed Interval In Exercises 35–38, discuss the continuity of the function on the closed interval.

Function	Interval
35. $g(x) = \sqrt{49 - x^2}$	$[-7, 7]$
36. $f(t) = 3 - \sqrt{9 - t^2}$	$[-3, 3]$
37. $f(x) = \begin{cases} 3 - x, & x \le 0 \\ 3 + \frac{1}{2}x, & x > 0 \end{cases}$	$[-1, 4]$
38. $g(x) = \dfrac{1}{x^2 - 4}$	$[-1, 2]$

Removable and Nonremovable Discontinuities In Exercises 39–58, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

39. $f(x) = \dfrac{6}{x}$

40. $f(x) = \dfrac{4}{x - 6}$

41. $f(x) = \dfrac{1}{4 - x^2}$

42. $f(x) = \dfrac{1}{x^2 + 1}$

43. $f(x) = 3x - \cos x$

44. $f(x) = \sin x - 8x$

45. $f(x) = \dfrac{x}{x^2 - x}$

46. $f(x) = \dfrac{x}{x^2 - 4}$

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

48. $f(x) = \dfrac{x + 2}{x^2 - x - 6}$

49. $f(x) = \dfrac{|x + 7|}{x + 7}$

50. $f(x) = \dfrac{2|x - 3|}{x - 3}$

51. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

52. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

53. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

54. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

55. $f(x) = \csc 2x$

56. $f(x) = \tan \dfrac{\pi x}{2}$

57. $f(x) = [\![x - 8]\!]$

58. $f(x) = 5 - [\![x]\!]$

Making a Function Continuous In Exercises 59–64, find the constant a, or the constants a and b, such that the function is continuous on the entire real number line.

59. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

60. $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$

61. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

62. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

63. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

64. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

Continuity of a Composite Function In Exercises 65–70, discuss the continuity of the composite function $h(x) = f(g(x))$.

65. $f(x) = x^2$
$g(x) = x - 1$

66. $f(x) = 5x + 1$
$g(x) = x^3$

67. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

68. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

69. $f(x) = \tan x$
$g(x) = \dfrac{x}{2}$

70. $f(x) = \sin x$
$g(x) = x^2$

Finding Discontinuities Using Technology In Exercises 71–74, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

71. $f(x) = [\![x]\!] - x$

72. $h(x) = \dfrac{1}{x^2 + 2x - 15}$

73. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

74. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

Testing for Continuity In Exercises 75–82, describe the interval(s) on which the function is continuous.

75. $f(x) = \dfrac{x}{x^2 + x + 2}$

76. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

77. $f(x) = 3 - \sqrt{x}$

78. $f(x) = x\sqrt{x + 3}$

79. $f(x) = \sec \dfrac{\pi x}{4}$

80. $f(x) = \cos \dfrac{1}{x}$

81. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \ne 1 \\ 2, & x = 1 \end{cases}$

82. $f(x) = \begin{cases} 2x - 4, & x \ne 3 \\ 1, & x = 3 \end{cases}$

Existence of a Zero In Exercises 83–86, explain why the function has at least one zero in the given interval.

Function	Interval
83. $f(x) = \frac{1}{12}x^4 - x^3 + 4$	$[1, 2]$
84. $f(x) = x^3 + 5x - 3$	$[0, 1]$
85. $f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
86. $f(x) = -\dfrac{5}{x} + \tan \dfrac{\pi x}{10}$	$[1, 4]$

Existence of Multiple Zeros In Exercises 87 and 88, explain why the function has at least two zeros in the interval $[1, 5]$.

87. $f(x) = (x - 3)^2 - 2$

88. $f(x) = 2 \cos x$

1.5 Exercises

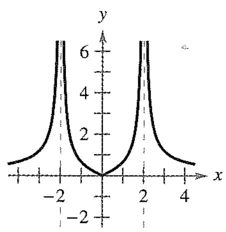
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

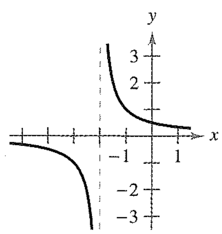
1. **Infinite Limit** In your own words, describe the meaning of an infinite limit. What does ∞ represent?
2. **Vertical Asymptote** In your own words, describe what is meant by a vertical asymptote of a graph.

Determining Infinite Limits from a Graph **In Exercises 3–6, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.**

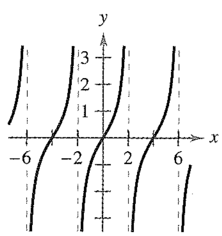
3. $f(x) = 2\left|\dfrac{x}{x^2 - 4}\right|$



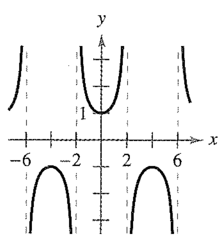
4. $f(x) = \dfrac{1}{x + 2}$



5. $f(x) = \tan \dfrac{\pi x}{4}$



6. $f(x) = \sec \dfrac{\pi x}{4}$



Determining Infinite Limits **In Exercises 7–10, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches 4 from the left and from the right.**

7. $f(x) = \dfrac{1}{x - 4}$
8. $f(x) = \dfrac{-1}{x - 4}$
9. $f(x) = \dfrac{1}{(x - 4)^2}$
10. $f(x) = \dfrac{-1}{(x - 4)^2}$

Numerical and Graphical Analysis **In Exercises 11–16, create a table of values for the function and use the result to determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right. Use a graphing utility to graph the function to confirm your answer.**

11. $f(x) = \dfrac{1}{x^2 - 9}$
12. $f(x) = \dfrac{x}{x^2 - 9}$
13. $f(x) = \dfrac{x^2}{x^2 - 9}$
14. $f(x) = -\dfrac{1}{3 + x}$
15. $f(x) = \cot \dfrac{\pi x}{3}$
16. $f(x) = \tan \dfrac{\pi x}{6}$

Finding Vertical Asymptotes **In Exercises 17–32, find the vertical asymptotes (if any) of the graph of the function.**

17. $f(x) = \dfrac{1}{x^2}$
18. $f(x) = \dfrac{2}{(x - 3)^3}$
19. $f(x) = \dfrac{x^2}{x^2 - 4}$
20. $f(x) = \dfrac{3x}{x^2 + 9}$
21. $g(t) = \dfrac{t - 1}{t^2 + 1}$
22. $h(s) = \dfrac{3s + 4}{s^2 - 16}$
23. $f(x) = \dfrac{3}{x^2 + x - 2}$
24. $g(x) = \dfrac{x^2 - 5x + 25}{x^3 + 125}$
25. $f(x) = \dfrac{4x^2 + 4x - 24}{x^4 - 2x^3 - 9x^2 + 18x}$
26. $h(x) = \dfrac{x^2 - 9}{x^3 + 3x^2 - x - 3}$
27. $f(x) = \dfrac{x^2 - 2x - 15}{x^3 - 5x^2 + x - 5}$
28. $h(t) = \dfrac{t^2 - 2t}{t^4 - 16}$
29. $f(x) = \csc \pi x$
30. $f(x) = \tan \pi x$
31. $s(t) = \dfrac{t}{\sin t}$
32. $g(\theta) = \dfrac{\tan \theta}{\theta}$

Vertical Asymptote or Removable Discontinuity **In Exercises 33–36, determine whether the graph of the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.**

33. $f(x) = \dfrac{x^2 - 1}{x + 1}$
34. $f(x) = \dfrac{x^2 - 2x - 8}{x + 1}$
35. $f(x) = \dfrac{\cos(x^2 - 1)}{x + 1}$
36. $f(x) = \dfrac{\sin(x + 1)}{x + 1}$

Finding a One-Sided Limit **In Exercises 37–50, find the one-sided limit (if it exists).**

37. $\displaystyle\lim_{x \to 2^+} \frac{x}{x - 2}$
38. $\displaystyle\lim_{x \to 2^-} \frac{x^2}{x^2 + 4}$
39. $\displaystyle\lim_{x \to -3^-} \frac{x + 3}{x^2 + x - 6}$
40. $\displaystyle\lim_{x \to (-1/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3}$
41. $\displaystyle\lim_{x \to 0^-} \left(1 + \frac{1}{x}\right)$
42. $\displaystyle\lim_{x \to 0^+} \left(6 - \frac{1}{x^3}\right)$
43. $\displaystyle\lim_{x \to -4^-} \left(x^2 + \frac{2}{x + 4}\right)$
44. $\displaystyle\lim_{x \to 0^+} \left(x - \frac{1}{x} + 3\right)$
45. $\displaystyle\lim_{x \to 0^+} \left(\sin x + \frac{1}{x}\right)$
46. $\displaystyle\lim_{x \to (\pi/2)^+} \frac{-2}{\cos x}$
47. $\displaystyle\lim_{x \to \pi^+} \frac{\sqrt{x}}{\csc x}$
48. $\displaystyle\lim_{x \to 0^-} \frac{x + 2}{\cot x}$
49. $\displaystyle\lim_{x \to (1/2)^-} x \sec \pi x$
50. $\displaystyle\lim_{x \to (1/2)^+} x^2 \tan \pi x$

Finding a One-Sided Limit Using Technology **In Exercises 51 and 52, use a graphing utility to graph the function and determine the one-sided limit.**

51. $\lim_{x \to 1^+} \dfrac{x^2 + x + 1}{x^3 - 1}$

52. $\lim_{x \to 1^-} \dfrac{x^3 - 1}{x^2 + x + 1}$

Determining Limits **In Exercises 53 and 54, use the information to determine the limits.**

53. $\lim_{x \to c} f(x) = \infty$

$\lim_{x \to c} g(x) = -2$

(a) $\lim_{x \to c} [f(x) + g(x)]$

(b) $\lim_{x \to c} [f(x)g(x)]$

(c) $\lim_{x \to c} \dfrac{g(x)}{f(x)}$

54. $\lim_{x \to c} f(x) = -\infty$

$\lim_{x \to c} g(x) = 3$

(a) $\lim_{x \to c} [f(x) + g(x)]$

(b) $\lim_{x \to c} [f(x)g(x)]$

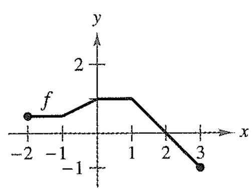
(c) $\lim_{x \to c} \dfrac{g(x)}{f(x)}$

EXPLORING CONCEPTS

55. Writing a Rational Function Write a rational function with vertical asymptotes at $x = 6$ and $x = -2$, and with a zero at $x = 3$.

56. Rational Function Does the graph of every rational function have a vertical asymptote? Explain.

57. Sketching a Graph Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$. To print an enlarged copy of the graph, go to *MathGraphs.com.*



58. Relativity According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}},$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c from the left.

59. Numerical and Graphical Reasoning Use a graphing utility to complete the table for each function and graph each function to estimate the limit. What is the value of the limit when the power of x in the denominator is greater than 3?

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$							

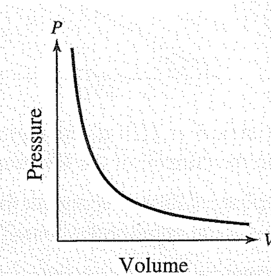
(a) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x}$

(b) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^2}$

(c) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^3}$

(d) $\lim_{x \to 0^+} \dfrac{x - \sin x}{x^4}$

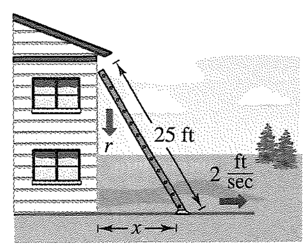
60. HOW DO YOU SEE IT? For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. What is the limit of P as V approaches 0 from the right? Explain what this means in the context of the problem.



61. Rate of Change A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, then the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house, and r is the rate in feet per second.



(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as x approaches 25 from the left.

62. Average Speed

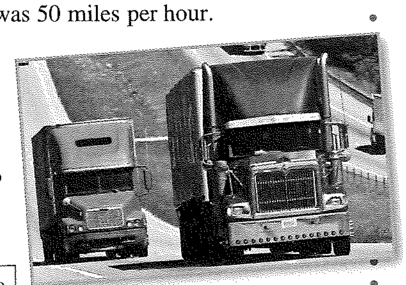
On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip, the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that

$$y = \frac{25x}{x - 25}.$$

What is the domain?

(b) Complete the table.



x	30	40	50	60
y				

Are the values of y different than you expected? Explain.

(c) Find the limit of y as x approaches 25 from the right and interpret its meaning.

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2.2 Exercises

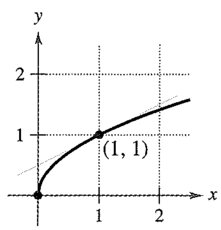
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

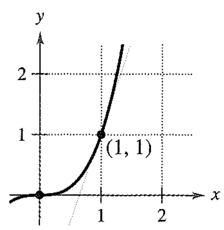
1. **Constant Rule** What is the derivative of a constant function?
2. **Finding a Derivative** Explain how to find the derivative of the function $f(x) = cx^n$.
3. **Derivatives of Trigonometric Functions** What are the derivatives of the sine and cosine functions?
4. **Average Velocity and Velocity** Describe the difference between average velocity and velocity.

Estimating Slope **In Exercises 5 and 6, use the graph to estimate the slope of the tangent line to $y = x^n$ at the point $(1, 1)$. Verify your answer analytically. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

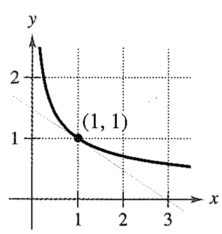
5. (a) $y = x^{1/2}$



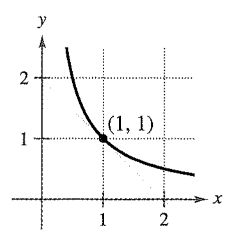
(b) $y = x^3$



6. (a) $y = x^{-1/2}$



(b) $y = x^{-1}$



Finding a Derivative **In Exercises 7–26, use the rules of differentiation to find the derivative of the function.**

7. $y = 12$
8. $f(x) = -9$
9. $y = x^7$
10. $y = x^{12}$
11. $y = \dfrac{1}{x^5}$
12. $y = \dfrac{3}{x^7}$
13. $f(x) = \sqrt[9]{x}$
14. $g(x) = \sqrt[4]{x}$
15. $f(x) = x + 11$
16. $g(x) = 6x + 3$
17. $f(t) = -3t^2 + 2t - 4$
18. $y = t^2 - 3t + 1$
19. $g(x) = x^2 + 4x^3$
20. $y = 4x - 3x^3$
21. $s(t) = t^3 + 5t^2 - 3t + 8$
22. $y = 2x^3 + 6x^2 - 1$
23. $y = \dfrac{\pi}{2}\sin\theta$
24. $g(t) = \pi\cos t$
25. $y = x^2 - \frac{1}{2}\cos x$
26. $y = 7x^4 + 2\sin x$

Rewriting a Function Before Differentiating **In Exercises 27–30, complete the table to find the derivative of the function.**

Original Function	Rewrite	Differentiate	Simplify
27. $y = \dfrac{2}{7x^4}$			
28. $y = \dfrac{8}{5x^{-5}}$			
29. $y = \dfrac{6}{(5x)^3}$			
30. $y = \dfrac{3}{(2x)^{-2}}$			

Finding the Slope of a Graph **In Exercises 31–38, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.**

Function	Point
31. $f(x) = \dfrac{8}{x^2}$	$(2, 2)$
32. $f(t) = 2 - \dfrac{4}{t}$	$(4, 1)$
33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$\left(0, -\frac{1}{2}\right)$
34. $y = 2x^4 - 3$	$(1, -1)$
35. $y = (4x + 1)^2$	$(0, 1)$
36. $f(x) = 2(x - 4)^2$	$(2, 8)$
37. $f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
38. $g(t) = -2\cos t + 5$	$(\pi, 7)$

Finding a Derivative **In Exercises 39–54, find the derivative of the function.**

39. $f(x) = x^2 + 5 - 3x^{-2}$
40. $f(x) = x^3 - 2x + 3x^{-3}$
41. $g(t) = t^2 - \dfrac{4}{t^3}$
42. $f(x) = 8x + \dfrac{3}{x^2}$
43. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$
44. $h(x) = \dfrac{4x^3 + 2x + 5}{x}$
45. $g(t) = \dfrac{3t^2 + 4t - 8}{t^{3/2}}$
46. $h(s) = \dfrac{s^5 + 2s + 6}{s^{1/3}}$
47. $y = x(x^2 + 1)$
48. $y = x^2(2x^2 - 3x)$
49. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$
50. $f(t) = t^{2/3} - t^{1/3} + 4$
51. $f(x) = 6\sqrt{x} + 5\cos x$
52. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$
53. $y = \dfrac{1}{(3x)^{-2}} - 5\cos x$
54. $y = \dfrac{3}{(2x)^3} + 2\sin x$

2.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Product Rule** Describe the Product Rule in your own words.
2. **Quotient Rule** Describe the Quotient Rule in your own words.
3. **Trigonometric Functions** What are the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$?
4. **Higher-Order Derivative** What is a higher-order derivative?

Using the Product Rule **In Exercises 5–10, use the Product Rule to find the derivative of the function.**

5. $g(x) = (2x - 3)(1 - 5x)$
6. $y = (3x - 4)(x^3 + 5)$
7. $h(t) = \sqrt{t}(1 - t^2)$
8. $g(s) = \sqrt{s}(s^2 + 8)$
9. $f(x) = x^3 \cos x$
10. $g(x) = \sqrt{x} \sin x$

Using the Quotient Rule **In Exercises 11–16, use the Quotient Rule to find the derivative of the function.**

11. $f(x) = \dfrac{x}{x - 5}$
12. $g(t) = \dfrac{3t^2 - 1}{2t + 5}$
13. $h(x) = \dfrac{\sqrt{x}}{x^3 + 1}$
14. $f(x) = \dfrac{x^2}{2\sqrt{x} + 1}$
15. $g(x) = \dfrac{\sin x}{x^2}$
16. $f(t) = \dfrac{\cos t}{t^3}$

Finding and Evaluating a Derivative **In Exercises 17–22, find $f'(x)$ and $f'(c)$.**

Function	Value of c
17. $f(x) = (x^3 + 4x)(3x^2 + 2x - 5)$	$c = 0$
18. $f(x) = (2x^2 - 3x)(9x + 4)$	$c = -1$
19. $f(x) = \dfrac{x^2 - 4}{x - 3}$	$c = 1$
20. $f(x) = \dfrac{x - 4}{x + 4}$	$c = 3$
21. $f(x) = x \cos x$	$c = \dfrac{\pi}{4}$
22. $f(x) = \dfrac{\sin x}{x}$	$c = \dfrac{\pi}{6}$

Using the Constant Multiple Rule **In Exercises 23–28, complete the table to find the derivative of the function without using the Quotient Rule.**

Function	Rewrite	Differentiate	Simplify
23. $y = \dfrac{x^3 + 6x}{3}$			
24. $y = \dfrac{5x^2 - 3}{4}$			
25. $y = \dfrac{6}{7x^2}$			
26. $y = \dfrac{10}{3x^3}$			
27. $y = \dfrac{4x^{3/2}}{x}$			
28. $y = \dfrac{2x}{x^{1/3}}$			

Finding a Derivative **In Exercises 29–40, find the derivative of the algebraic function.**

29. $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$
30. $f(x) = \dfrac{x^2 + 5x + 6}{x^2 - 4}$
31. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right)$
32. $f(x) = x^4\left(1 - \dfrac{2}{x + 1}\right)$
33. $f(x) = \dfrac{3x - 1}{\sqrt{x}}$
34. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$
35. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3}$
36. $h(x) = \dfrac{\dfrac{1}{x^2} + 5x}{x + 1}$
37. $g(s) = s^3\left(5 - \dfrac{s}{s + 2}\right)$
38. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right)$
39. $f(x) = (2x^3 + 5x)(x - 3)(x + 2)$
40. $f(x) = (x^3 - x)(x^2 + 2)(x^2 + x - 1)$

Finding a Derivative of a Trigonometric Function **In Exercises 41–56, find the derivative of the trigonometric function.**

41. $f(t) = t^2 \sin t$
42. $f(\theta) = (\theta + 1) \cos \theta$
43. $f(t) = \dfrac{\cos t}{t}$
44. $f(x) = \dfrac{\sin x}{x^3}$
45. $f(x) = -x + \tan x$
46. $y = x + \cot x$
47. $g(t) = \sqrt[4]{t} + 6 \csc t$
48. $h(x) = \dfrac{1}{x} - 12 \sec x$
49. $y = \dfrac{3(1 - \sin x)}{2 \cos x}$
50. $y = \dfrac{\sec x}{x}$
51. $y = -\csc x - \sin x$
52. $y = x \sin x + \cos x$
53. $f(x) = x^2 \tan x$
54. $f(x) = \sin x \cos x$
55. $y = 2x \sin x + x^2 \cos x$
56. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

Finding a Derivative Using Technology **In Exercises 57 and 58, use a computer algebra system to find the derivative of the function.**

57. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$
58. $f(x) = \dfrac{\cos x}{1 - \sin x}$

Finding the Slope of a Graph **In Exercises 59–62, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.**

Function	Point
59. $y = \dfrac{1 + \csc x}{1 - \csc x}$	$\left(\dfrac{\pi}{6}, -3\right)$
60. $f(x) = \tan x \cot x$	$(1, 1)$
61. $h(t) = \dfrac{\sec t}{t}$	$\left(\pi, -\dfrac{1}{\pi}\right)$
62. $f(x) = (\sin x)(\sin x + \cos x)$	$\left(\dfrac{\pi}{4}, 1\right)$

Finding an Equation of a Tangent Line **In Exercises 63–68, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *tangent* feature of a graphing utility to confirm your results.**

63. $f(x) = (x^3 + 4x - 1)(x - 2), \quad (1, -4)$

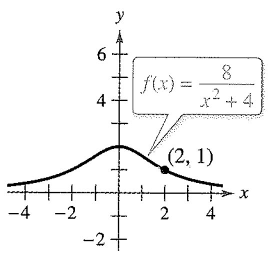
64. $f(x) = (x - 2)(x^2 + 4), \quad (1, -5)$

65. $f(x) = \dfrac{x}{x + 4}, \quad (-5, 5)$ **66.** $f(x) = \dfrac{x + 3}{x - 3}, \quad (4, 7)$

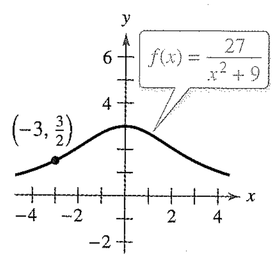
67. $f(x) = \tan x, \quad \left(\dfrac{\pi}{4}, 1\right)$ **68.** $f(x) = \sec x, \quad \left(\dfrac{\pi}{3}, 2\right)$

Famous Curves **In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *Witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentines*.)**

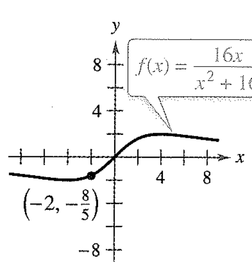
69.



70.



71.



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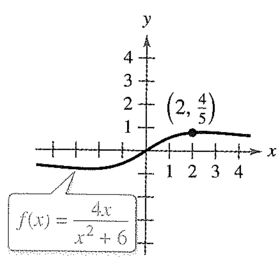


Horizontal Tangent Line **In Exercises 73–76, determine the point(s) at which the graph of the function has a horizontal tangent line.**

73. $f(x) = \dfrac{2x - 1}{x^2}$ **74.** $f(x) = \dfrac{x^2}{x^2 + 1}$

75. $f(x) = \dfrac{x^2}{x - 1}$ **76.** $f(x) = \dfrac{x - 4}{x^2 - 7}$

77. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = (x + 1)/(x - 1)$ that are parallel to the line $2y + x = 6$. Then graph the function and the tangent lines.

78. Tangent Lines Find equations of the tangent lines to the graph of $f(x) = x/(x - 1)$ that pass through the point $(-1, 5)$. Then graph the function and the tangent lines.

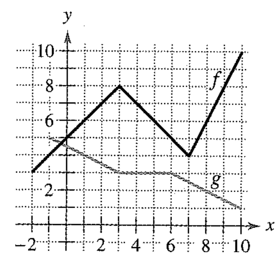
Exploring a Relationship **In Exercises 79 and 80, verify that $f'(x) = g'(x)$ and explain the relationship between f and g.**

79. $f(x) = \dfrac{3x}{x + 2}, \quad g(x) = \dfrac{5x + 4}{x + 2}$

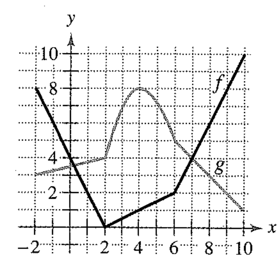
80. $f(x) = \dfrac{\sin x - 3x}{x}, \quad g(x) = \dfrac{\sin x + 2x}{x}$

Finding Derivatives **In Exercises 81 and 82, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$.**

81. (a) Find $p'(1)$.
(b) Find $q'(4)$.



82. (a) Find $p'(4)$.
(b) Find $q'(7)$.



83. Area The length of a rectangle is given by $6t + 5$ and its height is $\sqrt{t}$, where t is time in seconds and the dimensions are in centimeters. Find the rate of change of the area with respect to time.

84. Volume The radius of a right circular cylinder is given by $\sqrt{t + 2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time.

85. Inventory Replenishment The ordering and transportation cost C for the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where C is measured in thousands of dollars and x is the order size in hundreds. Find the rate of change of C with respect to x when (a) $x = 10$, (b) $x = 15$, and (c) $x = 20$. What do these rates of change imply about increasing order size?

86. Population Growth A population of 500 bacteria is introduced into a culture and grows in number according to the equation

$$P(t) = 500\left(1 + \frac{4t}{50 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when $t = 2$.

87. Proof Prove each differentiation rule.

(a) $\dfrac{d}{dx}[\sec x] = \sec x \tan x$

(b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

(c) $\dfrac{d}{dx}[\cot x] = -\csc^2 x$

88. Rate of Change Determine whether there exist any values of x in the interval $[0, 2\pi)$ such that the rate of change of $f(x) = \sec x$ and the rate of change of $g(x) = \csc x$ are equal.

89. Modeling Data The table shows the national health care expenditures h (in billions of dollars) in the United States and the population p (in millions) of the United States for the years 2008 through 2013. The year is represented by t, with $t = 8$ corresponding to 2008. *(Source: U.S. Centers for Medicare & Medicaid Services and U.S. Census Bureau)*

Year, t	8	9	10	11	12	13
h	2414	2506	2604	2705	2817	2919
p	304	307	309	311	313	315

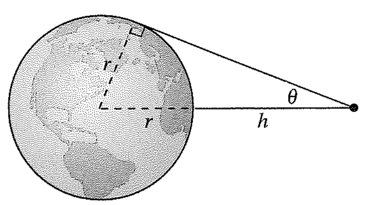
(a) Use a graphing utility to find linear models for the health care expenditures $h(t)$ and the population $p(t)$.

(b) Use a graphing utility to graph $h(t)$ and $p(t)$.

(c) Find $A = h(t)/p(t)$, then graph A using a graphing utility. What does this function represent?

(d) Find and interpret $A'(t)$ in the context of the problem.

90. Satellites When satellites observe Earth, they can scan only part of Earth's surface. Some satellites have sensors that can measure the angle θ shown in the figure. Let h represent the satellite's distance from Earth's surface, and let r represent Earth's radius.



(a) Show that $h = r(\csc \theta - 1)$.

(b) Find the rate at which h is changing with respect to θ when $\theta = 30°$. (Assume $r = 4000$ miles.)

Finding a Second Derivative In Exercises 91–100, find the second derivative of the function.

91. $f(x) = x^2 + 7x - 4$

92. $f(x) = 4x^5 - 2x^3 + 5x^2$

93. $f(x) = 4x^{3/2}$

94. $f(x) = x^2 + 3x^{-3}$

95. $f(x) = \dfrac{x}{x-1}$

96. $f(x) = \dfrac{x^2 + 3x}{x - 4}$

97. $f(x) = x \sin x$

98. $f(x) = x \cos x$

99. $f(x) = \csc x$

100. $f(x) = \sec x$

Finding a Higher-Order Derivative In Exercises 101–104, find the given higher-order derivative.

101. $f'(x) = x^3 - x^{2/5}, \quad f^{(3)}(x)$

102. $f^{(3)}(x) = \sqrt[5]{x^4}, \quad f^{(4)}(x)$

103. $f''(x) = -\sin x, \quad f^{(8)}(x)$

104. $f^{(4)}(t) = t \cos t, \quad f^{(5)}(t)$

Using Relationships In Exercises 105–108, use the given information to find $f'(2)$.

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

105. $f(x) = 2g(x) + h(x)$

106. $f(x) = 4 - h(x)$

107. $f(x) = \dfrac{g(x)}{h(x)}$

108. $f(x) = g(x)h(x)$

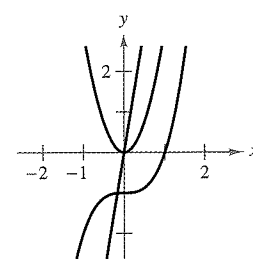
EXPLORING CONCEPTS

109. Higher-Order Derivatives Polynomials of what degree satisfy $f^{(n)} = 0$? Explain your reasoning.

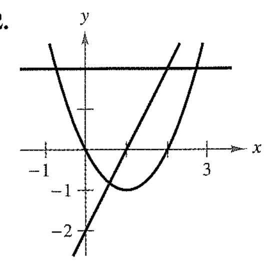
110. Differentiation of Piecewise Functions Describe how you would differentiate a piecewise function. Use your approach to find the first and second derivatives of $f(x) = x|x|$. Explain why $f''(0)$ does not exist.

Identifying Graphs In Exercises 111 and 112, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Identify each graph. Explain your reasoning. To print an enlarged copy of the graph, go to *MathGraphs.com*.

111.

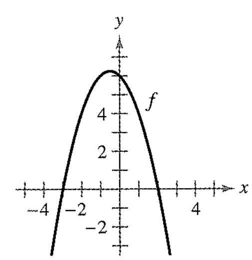


112.

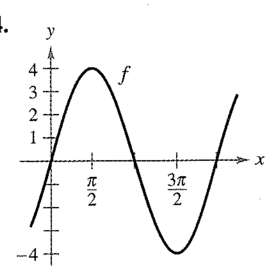


Sketching Graphs In Exercises 113 and 114, the graph of f is shown. Sketch the graphs of f' and f''. To print an enlarged copy of the graph, go to *MathGraphs.com*.

113.



114.



2.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Chain Rule** Describe the Chain Rule for the composition of two differentiable functions in your own words.
2. **General Power Rule** What is the difference between the (Simple) Power Rule and the General Power Rule?

Decomposition of a Composite Function **In Exercises 3–8, complete the table.**

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
3. $y = (6x - 5)^4$		
4. $y = \sqrt[3]{4x + 3}$		
5. $y = \dfrac{1}{3x + 5}$		
6. $y = \dfrac{2}{\sqrt{x^2 + 10}}$		
7. $y = \csc^3 x$		
8. $y = \sin \dfrac{5x}{2}$		

Finding a Derivative **In Exercises 9–34, find the derivative of the function.**

9. $y = (2x - 7)^3$
10. $y = 5(2 - x^3)^4$
11. $g(x) = 3(4 - 9x)^{5/6}$
12. $f(t) = (9t + 2)^{2/3}$
13. $h(s) = -2\sqrt{5s^2 + 3}$
14. $g(x) = \sqrt{4 - 3x^2}$
15. $y = \sqrt[3]{6x^2 + 1}$
16. $y = 2\sqrt[4]{9 - x^2}$
17. $y = \dfrac{1}{x - 2}$
18. $s(t) = \dfrac{1}{4 - 5t - t^2}$
19. $g(s) = \dfrac{6}{(s^3 - 2)^3}$
20. $y = -\dfrac{3}{(t - 2)^4}$
21. $y = \dfrac{1}{\sqrt{3x + 5}}$
22. $g(t) = \dfrac{1}{\sqrt{t^2 - 2}}$
23. $f(x) = x^2(x - 2)^7$
24. $f(x) = x(2x - 5)^3$
25. $y = x\sqrt{1 - x^2}$
26. $y = x^2\sqrt{16 - x^2}$
27. $y = \dfrac{x}{\sqrt{x^2 + 1}}$
28. $y = \dfrac{x}{\sqrt{x^4 + 4}}$
29. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$
30. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$
31. $s(t) = \left(\dfrac{1 + t}{t + 3}\right)^4$
32. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^{-2}$
33. $f(x) = ((x^2 + 3)^5 + x)^2$
34. $g(x) = (2 + (x^2 + 1)^4)^3$

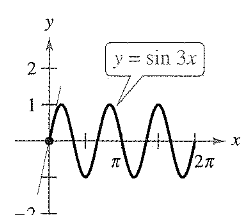
Finding a Derivative of a Trigonometric Function **In Exercises 35–54, find the derivative of the trigonometric function.**

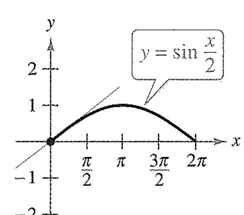
35. $y = \cos 4x$
36. $y = \sin \pi x$
37. $g(x) = 5 \tan 3x$
38. $h(x) = \sec 6x$
39. $y = \sin(\pi x)^2$
40. $y = \csc(1 - 2x)^2$
41. $h(x) = \sin 2x \cos 2x$
42. $g(\theta) = \sec(\frac{1}{2}\theta)\tan(\frac{1}{2}\theta)$
43. $f(x) = \dfrac{\cot x}{\sin x}$
44. $g(v) = \dfrac{\cos v}{\csc v}$
45. $y = 4 \sec^2 x$
46. $g(t) = 5 \cos^2 \pi t$
47. $f(\theta) = \frac{1}{4}\sin^2 2\theta$
48. $h(t) = 2 \cot^2(\pi t + 2)$
49. $f(t) = 3 \sec(\pi t - 1)^2$
50. $y = 5 \cos(\pi x)^2$
51. $y = \sin(3x^2 + \cos x)$
52. $y = \cos(5x + \csc x)$
53. $y = \sin\sqrt{\cot 3\pi x}$
54. $y = \cos\sqrt{\sin(\tan \pi x)}$

Finding a Derivative Using Technology **In Exercises 55–60, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.**

55. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$
56. $y = \sqrt{\dfrac{2x}{x + 1}}$
57. $y = \sqrt{\dfrac{x + 1}{x}}$
58. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$
59. $y = \dfrac{\cos \pi x + 1}{x}$
60. $y = x^2 \tan \dfrac{1}{x}$

Slope of a Tangent Line **In Exercises 61 and 62, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$.**

61.


62.


Finding the Slope of a Graph **In Exercises 63–70, find the slope of the graph of the function at the given point. Use the *derivative* feature of a graphing utility to confirm your results.**

63. $y = \sqrt{x^2 + 8x}$, $(1, 3)$
64. $y = \sqrt[5]{3x^3 + 4x}$, $(2, 2)$
65. $f(x) = 5(x^3 - 2)^{-1}$, $\left(-2, -\frac{1}{2}\right)$
66. $f(x) = \dfrac{1}{(x^2 - 3x)^2}$, $\left(4, \dfrac{1}{16}\right)$
67. $y = \dfrac{4}{(x + 2)^2}$, $(0, 1)$
68. $y = \dfrac{4}{(x^2 - 2x)^3}$, $(1, -4)$

69. $y = 26 - \sec^3 4x, \quad (0, 25)$ **70.** $y = \frac{1}{x} + \sqrt{\cos x}, \quad \left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

Finding an Equation of a Tangent Line **In Exercises 71–78, (a) find an equation of the tangent line to the graph of the function at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *tangent* feature of a graphing utility to confirm your results.**

71. $f(x) = \sqrt{2x^2 - 7}, \quad (4, 5)$ **72.** $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}, \quad (2, 2)$

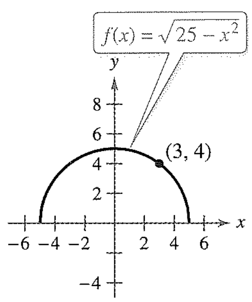
73. $y = (4x^3 + 3)^2, \quad (-1, 1)$ **74.** $f(x) = (9 - x^2)^{2/3}, \quad (1, 4)$

75. $f(x) = \sin 8x, \quad (\pi, 0)$ **76.** $y = \cos 3x, \quad \left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$

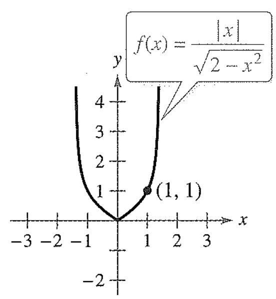
77. $f(x) = \tan^2 x, \quad \left(\frac{\pi}{4}, 1\right)$ **78.** $y = 2\tan^3 x, \quad \left(\frac{\pi}{4}, 2\right)$

Famous Curves **In Exercises 79 and 80, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line at the point in the same viewing window.**

79. Semicircle



80. Bullet-nose curve



81. Horizontal Tangent Line Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of $f(x) = 2\cos x + \sin 2x$ has a horizontal tangent.

82. Horizontal Tangent Line Determine the point(s) at which the graph of

$$f(x) = \frac{-4x}{\sqrt{2x - 1}}$$

has a horizontal tangent.

Finding a Second Derivative **In Exercises 83–88, find the second derivative of the function.**

83. $f(x) = 5(2 - 7x)^4$ **84.** $f(x) = 6(x^3 + 4)^3$

85. $f(x) = \frac{1}{11x - 6}$ **86.** $f(x) = \frac{8}{(x - 2)^2}$

87. $f(x) = \sin x^2$ **88.** $f(x) = \sec^2 \pi x$

Evaluating a Second Derivative **In Exercises 89–92, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.**

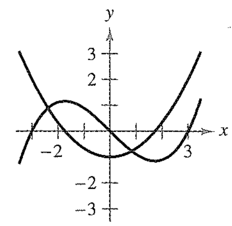
89. $h(x) = \frac{1}{9}(3x + 1)^3, \quad \left(1, \frac{64}{9}\right)$ **90.** $f(x) = \frac{1}{\sqrt{x + 4}}, \quad \left(0, \frac{1}{2}\right)$

91. $f(x) = \cos x^2, \quad (0, 1)$ **92.** $g(t) = \tan 2t, \quad \left(\frac{\pi}{6}, \sqrt{3}\right)$

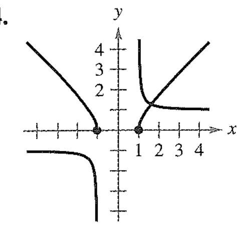
EXPLORING CONCEPTS

Identifying Graphs **In Exercises 93 and 94, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

93.



94.



95. Describing Relationships The relationship between f and g is given. Describe the relationship between f' and g'.

(a) $g(x) = f(3x)$ (b) $g(x) = f(x^2)$

96. Comparing Methods Consider the function

$$r(x) = \frac{2x - 5}{(3x + 1)^2}.$$

(a) In general, how do you find the derivative of

$$h(x) = \frac{f(x)}{g(x)}$$ using the Product Rule, where g is a composite function?

(b) Find $r'(x)$ using the Product Rule.

(c) Find $r'(x)$ using the Quotient Rule.

(d) Which method do you prefer? Explain.

97. Think About It The table shows some values of the derivative of an unknown function f. Complete the table by finding the derivative of each transformation of f, if possible.

(a) $g(x) = f(x) - 2$ (b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$ (d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

98. Using Relationships Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ for each of the following, if possible. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$ (b) $f(x) = g(h(x))$

(c) $f(x) = \frac{g(x)}{h(x)}$ (d) $f(x) = [g(x)]^3$

2.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Explicit and Implicit Functions** Describe the difference between the explicit form of a function and an implicit equation. Give an example of each.
2. **Implicit Differentiation** In your own words, state the guidelines for implicit differentiation.
3. **Implicit Differentiation** Explain when you have to use implicit differentiation to find a derivative.
4. **Chain Rule** How is the Chain Rule applied when finding dy/dx implicitly?

Finding a Derivative **In Exercises 5–20, find dy/dx by implicit differentiation.**

5. $x^2 + y^2 = 9$
6. $x^2 - y^2 = 25$
7. $x^5 + y^5 = 16$
8. $2x^3 + 3y^3 = 64$
9. $x^3 - xy + y^2 = 7$
10. $x^2y + y^2x = -2$
11. $x^3y^3 - y = x$
12. $\sqrt{xy} = x^2y + 1$
13. $x^3 - 3x^2y + 2xy^2 = 12$
14. $x^4y - 8xy + 3xy^2 = 9$
15. $\sin x + 2\cos 2y = 1$
16. $(\sin \pi x + \cos \pi y)^2 = 2$
17. $\csc x = x(1 + \tan y)$
18. $\cot y = x - y$
19. $y = \sin xy$
20. $x = \sec \dfrac{1}{y}$

Finding Derivatives Implicitly and Explicitly **In Exercises 21–24, (a) find two explicit functions by solving the equation for y in terms of x, (b) sketch the graph of the equation and label the parts given by the corresponding explicit functions, (c) differentiate the explicit functions, and (d) find dy/dx implicitly and show that the result is equivalent to that of part (c).**

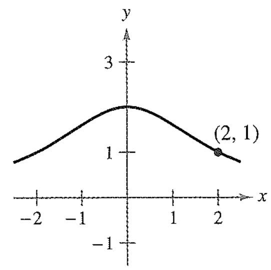
21. $x^2 + y^2 = 64$
22. $25x^2 + 36y^2 = 300$
23. $16y^2 - x^2 = 16$
24. $x^2 + y^2 - 4x + 6y + 9 = 0$

Finding the Slope of a Graph **In Exercises 25–32, find dy/dx by implicit differentiation. Then find the slope of the graph at the given point.**

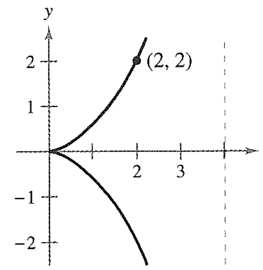
25. $xy = 6$, $(-6, -1)$
26. $3x^3y = 6$, $(1, 2)$
27. $y^2 = \dfrac{x^2 - 49}{x^2 + 49}$, $(7, 0)$
28. $4y^3 = \dfrac{x^2 - 36}{x^3 + 36}$, $(6, 0)$
29. $(x + y)^3 = x^3 + y^3$, $(-1, 1)$
30. $x^3 + y^3 = 6xy - 1$, $(2, 3)$
31. $\tan(x + y) = x$, $(0, 0)$
32. $x \cos y = 1$, $\left(2, \dfrac{\pi}{3}\right)$

Famous Curves **In Exercises 33–36, find the slope of the tangent line to the graph at the given point.**

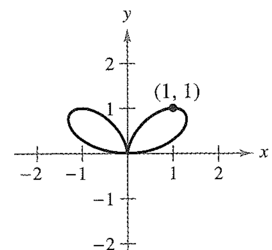
33. Witch of Agnesi: $(x^2 + 4)y = 8$



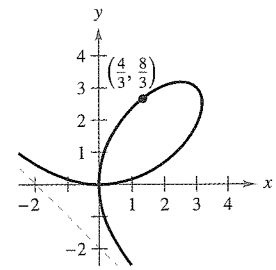
34. Cissoid: $(4 - x)y^2 = x^3$



35. Bifolium: $(x^2 + y^2)^2 = 4x^2y$

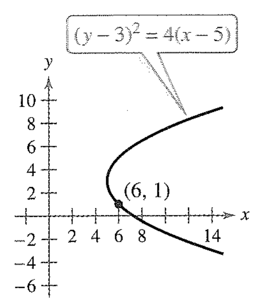


36. Folium of Descartes: $x^3 + y^3 - 6xy = 0$

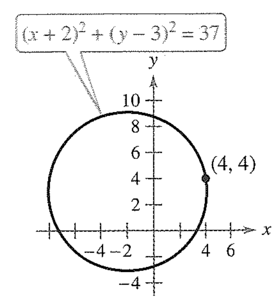


Famous Curves **In Exercises 37–42, find an equation of the tangent line to the graph at the given point. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

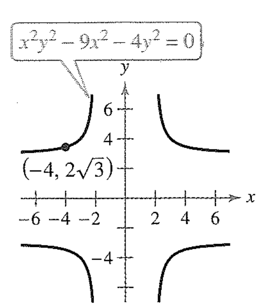
37. Parabola



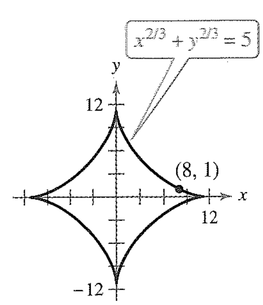
38. Circle



39. Cruciform



40. Astroid



4.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Fundamental Theorem of Calculus** Explain how to evaluate a definite integral using the Fundamental Theorem of Calculus.
2. **Mean Value Theorem** Describe the Mean Value Theorem for Integrals in your own words.
3. **Average Value of a Function** Describe the average value of a function on an interval in your own words.
4. **Accumulation Function** Why is

$$F(x) = \int_0^x f(t)\,dt$$

considered an accumulation function?

Graphical Reasoning **In Exercises 5–8, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.**

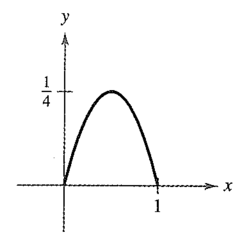
5. $\int_0^{\pi} \frac{4}{x^2+1}\,dx$
6. $\int_0^{\pi} \cos x\,dx$
7. $\int_{-2}^{2} x\sqrt{x^2+1}\,dx$
8. $\int_{-2}^{2} x\sqrt{2-x}\,dx$

Evaluating a Definite Integral **In Exercises 9–36, evaluate the definite integral. Use a graphing utility to verify your result.**

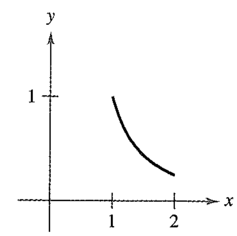
9. $\int_{-1}^{0} (2x-1)\,dx$
10. $\int_{-1}^{2} (7-3t)\,dt$
11. $\int_{-1}^{1} (t^2-5)\,dt$
12. $\int_{1}^{2} (6x^2-3x)\,dx$
13. $\int_{0}^{1} (2t-1)^2\,dt$
14. $\int_{1}^{4} (8x^3-x)\,dx$
15. $\int_{1}^{2} \left(\frac{3}{x^2}-1\right)dx$
16. $\int_{-2}^{-1} \left(u-\frac{1}{u^2}\right)du$
17. $\int_{1}^{4} \frac{u-2}{\sqrt{u}}\,du$
18. $\int_{-8}^{8} x^{1/3}\,dx$
19. $\int_{-1}^{1} (\sqrt[3]{t}-2)\,dt$
20. $\int_{1}^{8} \sqrt{\frac{2}{x}}\,dx$
21. $\int_{0}^{1} \frac{x-\sqrt{x}}{3}\,dx$
22. $\int_{0}^{2} (6-t)\sqrt{t}\,dt$
23. $\int_{-1}^{0} (t^{1/3}-t^{2/3})\,dt$
24. $\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx$
25. $\int_{0}^{5} |2x-5|\,dx$
26. $\int_{1}^{4} (3-|x-3|)\,dx$
27. $\int_{0}^{4} |x^2-9|\,dx$
28. $\int_{0}^{4} |x^2-4x+3|\,dx$
29. $\int_{0}^{\pi} (\sin x-7)\,dx$
30. $\int_{0}^{\pi} (2+\cos x)\,dx$
31. $\int_{0}^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$
32. $\int_{0}^{\pi/4} \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$
33. $\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx$
34. $\int_{\pi/4}^{\pi/2} (2-\csc^2 x)\,dx$
35. $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$
36. $\int_{-\pi/2}^{\pi/2} (2t+\cos t)\,dt$

Finding the Area of a Region **In Exercises 37–40, find the area of the given region.**

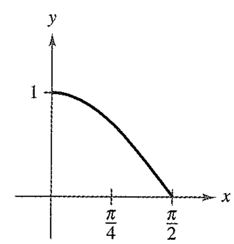
37. $y = x - x^2$



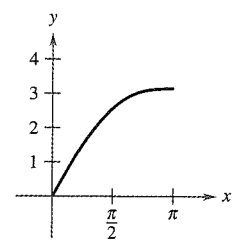
38. $y = \frac{1}{x^2}$



39. $y = \cos x$



40. $y = x + \sin x$



Finding the Area of a Region **In Exercises 41–46, find the area of the region bounded by the graphs of the equations.**

41. $y = 5x^2 + 2, \quad x = 0, \quad x = 2, \quad y = 0$
42. $y = x^3 + 6x, \quad x = 2, \quad y = 0$
43. $y = 1 + \sqrt[3]{x}, \quad x = 0, \quad x = 8, \quad y = 0$
44. $y = 2\sqrt{x} - x, \quad y = 0$
45. $y = -x^2 + 4x, \quad y = 0$
46. $y = 1 - x^4, \quad y = 0$

Using the Mean Value Theorem for Integrals **In Exercises 47–52, find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function over the given interval.**

47. $f(x) = x^3, \quad [0, 3]$
48. $f(x) = \sqrt{x}, \quad [4, 9]$
49. $y = \frac{x^2}{4}, \quad [0, 6]$
50. $f(x) = \frac{9}{x^3}, \quad [1, 3]$
51. $f(x) = 2\sec^2 x, \quad \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
52. $f(x) = \cos x, \quad \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

4.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Constant Multiple Rule** Explain how to use the Constant Multiple Rule when finding an indefinite integral.
2. **Change of Variables** In your own words, summarize the guidelines for making a change of variables when finding an indefinite integral.
3. **The General Power Rule for Integration** Describe the General Power Rule for Integration in your own words.
4. **Analyzing the Integrand** Without integrating, explain why

$$\int_{-2}^{2} x(x^2+1)^2\,dx = 0.$$

Recognizing Patterns **In Exercises 5–8, complete the table by identifying u and du for the integral.**

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
5. $\int (5x^2+1)^2(10x)\,dx$		
6. $\int x^2\sqrt{x^3+1}\,dx$		
7. $\int \tan^2 x \sec^2 x\,dx$		
8. $\int \frac{\cos x}{\sin^2 x}\,dx$		

Finding an Indefinite Integral **In Exercises 9–30, find the indefinite integral and check the result by differentiation.**

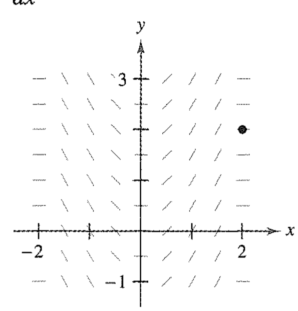
9. $\int (1+6x)^4(6)\,dx$
10. $\int (x^2-9)^3(2x)\,dx$
11. $\int \sqrt{25-x^2}\,(-2x)\,dx$
12. $\int \sqrt[3]{3-4x^2}(-8x)\,dx$
13. $\int x^3(x^4+3)^2\,dx$
14. $\int x^2(6-x^3)^5\,dx$
15. $\int x^2(2x^3-1)^4\,dx$
16. $\int x(5x^2+4)^3\,dx$
17. $\int t\sqrt{t^2+2}\,dt$
18. $\int t^3\sqrt{2t^4+3}\,dt$
19. $\int 5x\sqrt[3]{1-x^2}\,dx$
20. $\int 6u^6\sqrt{u^7+8}\,du$
21. $\int \frac{7x}{(1-x^2)^3}\,dx$
22. $\int \frac{x^3}{(1+x^4)^2}\,dx$
23. $\int \frac{x^2}{(1+x^3)^2}\,dx$
24. $\int \frac{6x^2}{(4x^3-9)^3}\,dx$
25. $\int \frac{x}{\sqrt{1-x^2}}\,dx$
26. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$
27. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$
28. $\int \left(8-\frac{1}{t^4}\right)^2\left(\frac{1}{t^5}\right)dt$
29. $\int \frac{1}{\sqrt{2x}}\,dx$
30. $\int \frac{x}{\sqrt[3]{5x^2}}\,dx$

Differential Equation **In Exercises 31–34, find the general solution of the differential equation.**

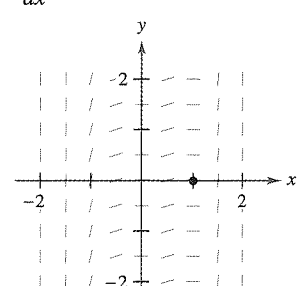
31. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$
32. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$
33. $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$
34. $\frac{dy}{dx} = \frac{18-6x^2}{\sqrt{x^3-9x+7}}$

Slope Field **In Exercises 35 and 36, a differential equation, a point, and a slope field are given. A *slope field* (or *direction field*) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration and the given point to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a) that passes through the given point.**

35. $\frac{dy}{dx} = x\sqrt{4-x^2}$, $(2, 2)$

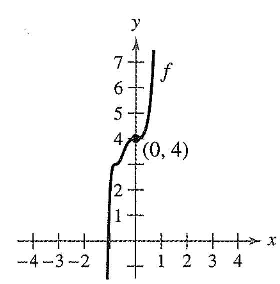


36. $\frac{dy}{dx} = x^2(x^3-1)^2$, $(1, 0)$

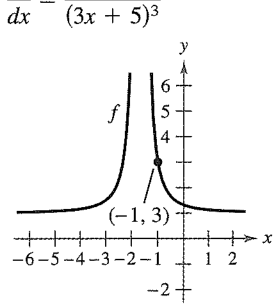


Differential Equation **In Exercises 37 and 38, the graph of a function f is shown. Use the differential equation and the given point to find an equation of the function.**

37. $\frac{dy}{dx} = 18x^2(2x^3+1)^2$



38. $\frac{dy}{dx} = \frac{-48}{(3x+5)^3}$



Finding an Indefinite Integral In Exercises 39–48, find the indefinite integral.

39. $\int \pi \sin \pi x \, dx$

40. $\int \sin 4x \, dx$

41. $\int \cos 6x \, dx$

42. $\int \csc^2\left(\frac{x}{2}\right) dx$

43. $\int \frac{1}{\theta^2} \cos \frac{1}{\theta} \, d\theta$

44. $\int x \sin x^2 \, dx$

45. $\int \sin 2x \cos 2x \, dx$

46. $\int \sqrt[3]{\tan x} \sec^2 x \, dx$

47. $\int \frac{\csc^2 x}{\cot^3 x} \, dx$

48. $\int \frac{\sin x}{\cos^3 x} \, dx$

Finding an Equation In Exercises 49–52, find an equation for the function f that has the given derivative and whose graph passes through the given point.

Derivative	Point
49. $f'(x) = -\sin \frac{x}{2}$	$(0, 6)$
50. $f'(x) = \sec^2 2x$	$\left(\frac{\pi}{2}, 2\right)$
51. $f'(x) = 2x(4x^2 - 10)^2$	$(2, 10)$
52. $f'(x) = -2x\sqrt{8 - x^2}$	$(2, 7)$

Change of Variables In Exercises 53–60, find the indefinite integral by making a change of variables.

53. $\int x\sqrt{x + 6} \, dx$

54. $\int x\sqrt{3x - 4} \, dx$

55. $\int x^2\sqrt{1 - x} \, dx$

56. $\int (x + 1)\sqrt{2 - x} \, dx$

57. $\int \frac{x^2 - 1}{\sqrt{2x - 1}} \, dx$

58. $\int \frac{2x + 1}{\sqrt{x + 4}} \, dx$

59. $\int \cos^3 2x \sin 2x \, dx$

60. $\int \sec^5 7x \tan 7x \, dx$

Evaluating a Definite Integral In Exercises 61–68, evaluate the definite integral. Use a graphing utility to verify your result.

61. $\int_{-1}^{1} x(x^2 + 1)^3 \, dx$

62. $\int_{0}^{1} x^3(2x^4 + 1)^2 \, dx$

63. $\int_{1}^{2} 2x^2\sqrt{x^3 + 1} \, dx$

64. $\int_{-1}^{0} x\sqrt{1 - x^2} \, dx$

65. $\int_{0}^{4} \frac{1}{\sqrt{2x + 1}} \, dx$

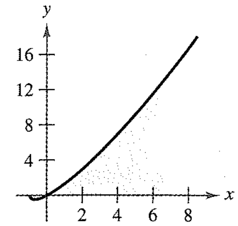
66. $\int_{0}^{2} \frac{x}{\sqrt{1 + 2x^2}} \, dx$

67. $\int_{1}^{9} \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} \, dx$

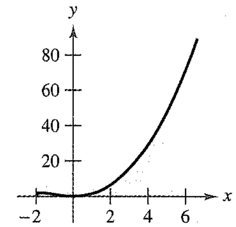
68. $\int_{4}^{5} \frac{x}{\sqrt{2x - 6}} \, dx$

Finding the Area of a Region In Exercises 69–72, find the area of the region. Use a graphing utility to verify your result.

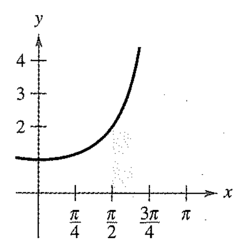
69. $\int_{0}^{7} x\sqrt[3]{x + 1} \, dx$



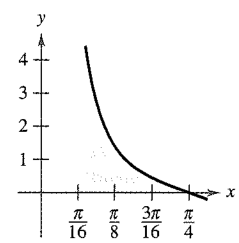
70. $\int_{-2}^{6} x^2\sqrt[3]{x + 2} \, dx$



71. $\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx$



72. $\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x \, dx$



Even and Odd Functions In Exercises 73–76, evaluate the integral using the properties of even and odd functions as an aid.

73. $\int_{-2}^{2} x^2(x^2 + 1) \, dx$

74. $\int_{-2}^{2} x(x^2 + 1)^3 \, dx$

75. $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx$

76. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx$

77. **Using an Even Function** Use $\int_0^6 x^2 \, dx = 72$ to evaluate each definite integral without using the Fundamental Theorem of Calculus.

(a) $\int_{-6}^{6} x^2 \, dx$ (b) $\int_{-6}^{0} x^2 \, dx$

(c) $\int_{0}^{6} -2x^2 \, dx$ (d) $\int_{-6}^{6} 3x^2 \, dx$

78. **Using Symmetry** Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\int_{-\pi/4}^{\pi/4} \sin x \, dx$ (b) $\int_{-\pi/4}^{\pi/4} \cos x \, dx$

(c) $\int_{-\pi/2}^{\pi/2} \cos x \, dx$ (d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx$

Even and Odd Functions In Exercises 79 and 80, write the integral as the sum of the integral of an odd function and the integral of an even function. Use this simplification to evaluate the integral.

79. $\int_{-3}^{3} (x^3 + 4x^2 - 3x - 6) \, dx$

80. $\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x) \, dx$

5.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Natural Logarithmic Function** Explain why $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$.
2. **Logarithmic Properties** What is the value of n?

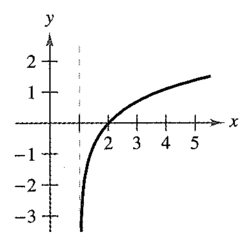
   $\ln 4 + \ln(n^{-1}) = \ln 4 - \ln 7$
3. **The Number e** How is the number e defined?
4. **Differentiation of Logarithmic Functions** State the Chain Rule version of the derivative of the natural logarithmic function in your own words.

Evaluating a Logarithm Using Technology **In Exercises 5–8, use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key and (b) using the integration capabilities to evaluate the integral $\int_1^x (1/t)\,dt$.**

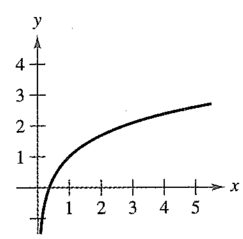
5. $\ln 45$
6. $\ln 8.3$
7. $\ln 0.8$
8. $\ln 0.6$

Matching **In Exercises 9–12, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

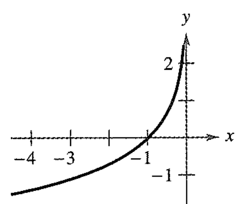
(a)



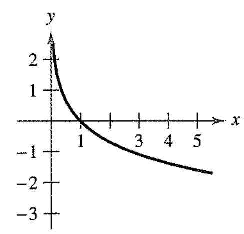
(b)



(c)



(d)



9. $f(x) = \ln x + 1$
10. $f(x) = -\ln x$
11. $f(x) = \ln(x - 1)$
12. $f(x) = -\ln(-x)$

Sketching a Graph **In Exercises 13–18, sketch the graph of the function and state its domain.**

13. $f(x) = 3 \ln x$
14. $f(x) = -2 \ln x$
15. $f(x) = \ln 2x$
16. $f(x) = \ln|x|$
17. $f(x) = \ln(x - 3)$
18. $f(x) = \ln x - 4$

Using Properties of Logarithms **In Exercises 19 and 20, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.**

19. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$
20. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

Expanding a Logarithmic Expression **In Exercises 21–30, use the properties of logarithms to expand the logarithmic expression.**

21. $\ln \dfrac{x}{4}$
22. $\ln \sqrt{x^5}$
23. $\ln \dfrac{xy}{z}$
24. $\ln(xyz)$
25. $\ln\left(x\sqrt{x^2 + 5}\right)$
26. $x \ln \sqrt{x - 4}$
27. $\ln \sqrt{\dfrac{x - 1}{x}}$
28. $\ln(3e^2)$
29. $\ln z(z - 1)^2$
30. $\ln \dfrac{z}{e}$

Condensing a Logarithmic Expression **In Exercises 31–36, write the expression as a logarithm of a single quantity.**

31. $\ln(x - 2) - \ln(x + 2)$
32. $3 \ln x + 2 \ln y - 4 \ln z$
33. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$
34. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$
35. $4 \ln 2 - \frac{1}{2} \ln(x^3 + 6x)$
36. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

Verifying Properties of Logarithms **In Exercises 37 and 38, (a) verify that $f = g$ by using a graphing utility to graph f and g in the same viewing window and (b) verify that $f = g$ algebraically.**

37. $f(x) = \ln \dfrac{x^2}{4}, \quad x > 0, \quad g(x) = 2 \ln x - \ln 4$
38. $f(x) = \ln \sqrt{x(x^2 + 1)}, \quad g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

Finding a Limit **In Exercises 39–42, find the limit.**

39. $\lim\limits_{x \to 3^+} \ln(x - 3)$
40. $\lim\limits_{x \to 6^-} \ln(6 - x)$
41. $\lim\limits_{x \to 2^-} \ln[x^2(3 - x)]$
42. $\lim\limits_{x \to 5^+} \ln \dfrac{x}{\sqrt{x - 4}}$

Finding a Derivative **In Exercises 43–66, find the derivative of the function.**

43. $f(x) = \ln 3x$
44. $f(x) = \ln(x - 1)$
45. $f(x) = \ln(x^2 + 3)$
46. $h(x) = \ln(2x^2 + 1)$
47. $y = (\ln x)^4$
48. $y = x^2 \ln x$
49. $y = \ln(t + 1)^2$
50. $y = \ln \sqrt{x^2 - 4}$

51. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

52. $y = \ln[t(t^2 + 3)^3]$

53. $f(x) = \ln \dfrac{x}{x^2 + 1}$

54. $f(x) = \ln \dfrac{2x}{x + 3}$

55. $g(t) = \dfrac{\ln t}{t^2}$

56. $h(t) = \dfrac{\ln t}{t^3 + 5}$

57. $y = \ln(\ln x^2)$

58. $y = \ln(\ln x)$

59. $y = \ln \sqrt{\dfrac{x + 1}{x - 1}}$

60. $y = \ln \sqrt[3]{\dfrac{x - 1}{x + 1}}$

61. $f(x) = \ln \dfrac{\sqrt{4 + x^2}}{x}$

62. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

63. $y = \ln|\sin x|$

64. $y = \ln|\csc x|$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

66. $y = \ln|\sec x + \tan x|$

Finding an Equation of a Tangent Line **In Exercises 67–74, (a) find an equation of the tangent line to the graph of the function at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *tangent* feature of a graphing utility to confirm your results.**

67. $y = \ln x^4, \quad (1, 0)$

68. $y = \ln x^{2/3}, \quad (-1, 0)$

69. $f(x) = 3x^2 - \ln x, \quad (1, 3)$

70. $f(x) = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

71. $f(x) = \ln\sqrt{1 + \sin^2 x}, \quad \left(\dfrac{\pi}{4}, \ln\sqrt{\dfrac{3}{2}}\right)$

72. $f(x) = \sin 2x \ln x^2, \quad (1, 0)$

73. $y = x^3 \ln x^4, \quad (-1, 0)$

74. $f(x) = \dfrac{1}{2}x \ln x^2, \quad (-1, 0)$

Logarithmic Differentiation **In Exercises 75–80, use logarithmic differentiation to find dy/dx.**

75. $y = x\sqrt{x^2 + 1}, \quad x > 0$

76. $y = \sqrt{x^2(x + 1)(x + 2)}, \quad x > 0$

77. $y = \dfrac{x^2\sqrt{3x - 2}}{(x + 1)^2}, \quad x > \dfrac{2}{3}$

78. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}, \quad x > 1$

79. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}, \quad x > 1$

80. $y = \dfrac{(x + 1)(x - 2)}{(x - 1)(x + 2)}, \quad x > 2$

Implicit Differentiation **In Exercises 81–84, use implicit differentiation to find dy/dx.**

81. $x^2 - 3\ln y + y^2 = 10$

82. $\ln xy + 5x = 30$

83. $4x^3 + \ln y^2 + 2y = 2x$

84. $4xy + \ln x^2 y = 7$

Differential Equation **In Exercises 85 and 86, verify that the function is a solution of the differential equation.**

	Function	Differential Equation
85.	$y = 2\ln x + 3$	$xy'' + y' = 0$
86.	$y = x\ln x - 4x$	$x + y - xy' = 0$

Relative Extrema and Points of Inflection **In Exercises 87–92, locate any relative extrema and points of inflection. Use a graphing utility to confirm your results.**

87. $y = \dfrac{x^2}{2} - \ln x$

88. $y = 2x - \ln 2x$

89. $y = x\ln x$

90. $y = \dfrac{\ln x}{x}$

91. $y = \dfrac{x}{\ln x}$

92. $y = x^2 \ln \dfrac{x}{4}$

Using Newton's Method **In Exercises 93 and 94, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations. Use a graphing utility to verify your result.**

93. $y = \ln x, \quad y = -x$

94. $y = \ln x, \quad y = 3 - x$

EXPLORING CONCEPTS

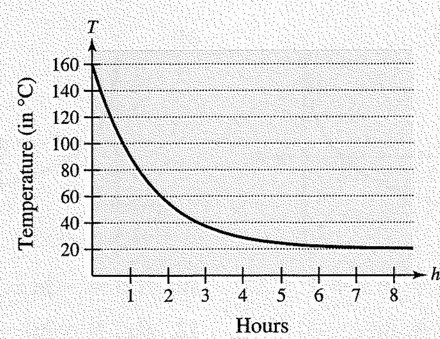
Comparing Functions **In Exercises 95 and 96, let f be a function that is positive and differentiable on the entire real number line and let $g(x) = \ln f(x)$.**

95. When g is increasing, must f be increasing? Explain.

96. When the graph of f is concave upward, must the graph of g be concave upward? Explain.

97. **Think About It** Is $\ln xy = \ln x \ln y$ a valid property of logarithms, where $x > 0$ and $y > 0$? Explain.

98. **HOW DO YOU SEE IT?** The graph shows the temperature T (in degrees Celsius) of an object h hours after it is removed from a furnace.



(a) Find $\lim_{h\to\infty} T$. What does this limit represent?

(b) When is the temperature changing most rapidly?

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. $\ln(a^{n+m}) = n\ln a + m\ln a$, where $a > 0$ and m and n are rational.

100. $\dfrac{d}{dx}[\ln(cx)] = \dfrac{d}{dx}[\ln x]$, where $c > 0$

101. If $y = \ln \pi$, then $y' = 1/\pi$.

102. If $y = \ln e$, then $y' = 1$.

5.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Log Rule for Integration** Can you use the Log Rule to find the integral below? Explain.

$$\int \frac{x}{(x^2 - 4)^3}\,dx$$

2. **Long Division** Explain when to use long division before applying the Log Rule.
3. **Guidelines for Integration** Describe two ways to alter an integrand so that it fits an integration formula.
4. **Trigonometric Functions** Integrating which trigonometric function results in $\ln|\sin x| + C$?

Finding an Indefinite Integral **In Exercises 5–28, find the indefinite integral.**

5. $\displaystyle\int \frac{5}{x}\,dx$
6. $\displaystyle\int \frac{1}{x - 5}\,dx$
7. $\displaystyle\int \frac{1}{2x + 5}\,dx$
8. $\displaystyle\int \frac{9}{5 - 4x}\,dx$
9. $\displaystyle\int \frac{x}{x^2 - 3}\,dx$
10. $\displaystyle\int \frac{x^2}{5 - x^3}\,dx$
11. $\displaystyle\int \frac{4x^3 + 3}{x^4 + 3x}\,dx$
12. $\displaystyle\int \frac{x^2 - 2x}{x^3 - 3x^2}\,dx$
13. $\displaystyle\int \frac{x^2 - 7}{7x}\,dx$
14. $\displaystyle\int \frac{x^3 - 8x}{x^2}\,dx$
15. $\displaystyle\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x}\,dx$
16. $\displaystyle\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5}\,dx$
17. $\displaystyle\int \frac{x^2 - 3x + 2}{x + 1}\,dx$
18. $\displaystyle\int \frac{2x^2 + 7x - 3}{x - 2}\,dx$
19. $\displaystyle\int \frac{x^3 - 3x^2 + 5}{x - 3}\,dx$
20. $\displaystyle\int \frac{x^3 - 6x - 20}{x + 5}\,dx$
21. $\displaystyle\int \frac{x^4 + x - 4}{x^2 + 2}\,dx$
22. $\displaystyle\int \frac{x^3 - 4x^2 - 4x + 20}{x^2 - 5}\,dx$
23. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$
24. $\displaystyle\int \frac{dx}{x(\ln x^2)^3}$
25. $\displaystyle\int \frac{1}{\sqrt{x}(1 - 3\sqrt{x})}\,dx$
26. $\displaystyle\int \frac{1}{x^{2/3}(1 + x^{1/3})}\,dx$
27. $\displaystyle\int \frac{6x}{(x - 5)^2}\,dx$
28. $\displaystyle\int \frac{x(x - 2)}{(x - 1)^3}\,dx$

Change of Variables **In Exercises 29–32, find the indefinite integral by making a change of variables (*Hint:* Let u be the denominator of the integrand.)**

29. $\displaystyle\int \frac{1}{1 + \sqrt{2x}}\,dx$
30. $\displaystyle\int \frac{4}{1 + \sqrt{5x}}\,dx$
31. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx$
32. $\displaystyle\int \frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\,dx$

Finding an Indefinite Integral of a Trigonometric Function **In Exercises 33–42, find the indefinite integral.**

33. $\displaystyle\int \cot\frac{\theta}{3}\,d\theta$
34. $\displaystyle\int \theta \tan 2\theta^2\,d\theta$
35. $\displaystyle\int \csc 2x\,dx$
36. $\displaystyle\int \sec\frac{x}{2}\,dx$
37. $\displaystyle\int (5 - \cos 3\theta)\,d\theta$
38. $\displaystyle\int \left(2 - \tan\frac{\theta}{4}\right)d\theta$
39. $\displaystyle\int \frac{\cos t}{1 + \sin t}\,dt$
40. $\displaystyle\int \frac{\csc^2 t}{\cot t}\,dt$
41. $\displaystyle\int \frac{\sec x \tan x}{\sec x - 1}\,dx$
42. $\displaystyle\int (\sec 2x + \tan 2x)\,dx$

Differential Equation **In Exercises 43–46, find the general solution of the differential equation. Use a graphing utility to graph three solutions, one of which passes through the given point.**

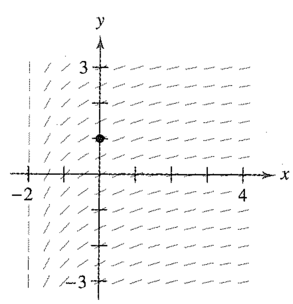
43. $\dfrac{dy}{dx} = \dfrac{3}{2 - x}$, $(1, 0)$
44. $\dfrac{dy}{dx} = \dfrac{x - 2}{x}$, $(-1, 0)$
45. $\dfrac{dy}{dx} = \dfrac{2x}{x^2 - 9}$, $(0, 4)$
46. $\dfrac{dr}{dt} = \dfrac{\sec^2 t}{\tan t + 1}$, $(\pi, 4)$

Finding a Particular Solution **In Exercises 47 and 48, find the particular solution of the differential equation that satisfies the initial conditions.**

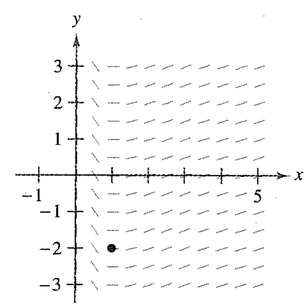
47. $f''(x) = \dfrac{2}{x^2}$, $f'(1) = 1$, $f(1) = 1$, $x > 0$
48. $f''(x) = -\dfrac{4}{(x - 1)^2} - 2$, $f'(2) = 0$, $f(2) = 3$, $x > 1$

Slope Field **In Exercises 49 and 50, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration and the given point to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a) that passes through the given point.**

49. $\dfrac{dy}{dx} = \dfrac{1}{x + 2}$, $(0, 1)$



50. $\dfrac{dy}{dx} = \dfrac{\ln x}{x}$, $(1, -2)$



Evaluating a Definite Integral **In Exercises 51–58, evaluate the definite integral. Use a graphing utility to verify your result.**

51. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx$

52. $\displaystyle\int_{-1}^1 \frac{1}{2x+3}\,dx$

53. $\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx$

54. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

55. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx$

56. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx$

57. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

58. $\displaystyle\int_{\pi/8}^{\pi/4} (\csc 2\theta - \cot 2\theta)\,d\theta$

Finding an Integral Using Technology **In Exercises 59 and 60, use a computer algebra system to find or evaluate the integral.**

59. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

60. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

Finding a Derivative **In Exercises 61–64, find $F'(x)$.**

61. $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

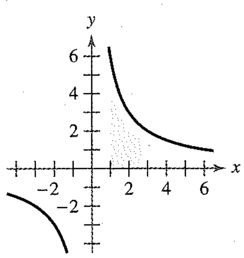
62. $\displaystyle F(x) = \int_0^x \tan t\,dt$

63. $\displaystyle F(x) = \int_1^{4x} \cot t\,dt$

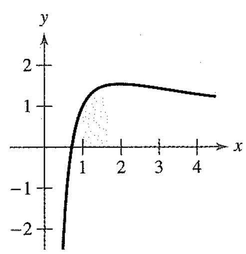
64. $\displaystyle F(x) = \int_0^{x^2} \frac{3}{t+1}\,dt$

Area **In Exercises 65–68, find the area of the given region. Use a graphing utility to verify your result.**

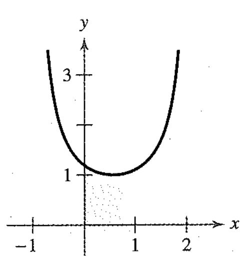
65. $y = \dfrac{6}{x}$



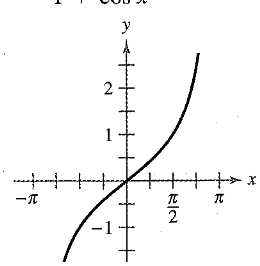
66. $y = \dfrac{1+\ln x^3}{x}$



67. $y = \csc(x+1)$



68. $y = \dfrac{\sin x}{1+\cos x}$



Area **In Exercises 69–72, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.**

69. $y = \dfrac{x^2+4}{x}$, $x = 1$, $x = 4$, $y = 0$

70. $y = \dfrac{5x}{x^2+2}$, $x = 1$, $x = 5$, $y = 0$

71. $y = 2\sec\dfrac{\pi x}{6}$, $x = 0$, $x = 2$, $y = 0$

72. $y = 2x - \tan 0.3x$, $x = 1$, $x = 4$, $y = 0$

Finding the Average Value of a Function **In Exercises 73–76, find the average value of the function over the given interval.**

73. $f(x) = \dfrac{8}{x^2}$, $[2, 4]$

74. $f(x) = \dfrac{4(x+1)}{x^2}$, $[2, 4]$

75. $f(x) = \dfrac{2\ln x}{x}$, $[1, e]$

76. $f(x) = \sec\dfrac{\pi x}{6}$, $[0, 2]$

Midpoint Rule **In Exercises 77 and 78, use the Midpoint Rule with $n = 4$ to approximate the value of the definite integral. Use a graphing utility to verify your result.**

77. $\displaystyle\int_1^3 \frac{12}{x}\,dx$

78. $\displaystyle\int_0^{\pi/4} \sec x\,dx$

EXPLORING CONCEPTS

Approximation **In Exercises 79 and 80, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. Make your selection on the basis of a sketch of the region, not by performing calculations.**

79. $f(x) = \sec x$, $[0, 1]$

(a) 6 (b) −6 (c) $\frac{1}{2}$ (d) 1.25 (e) 3

80. $f(x) = \dfrac{2x}{x^2+1}$, $[0, 4]$

(a) 3 (b) 7 (c) −2 (d) 5 (e) 1

81. **Napier's Inequality** For $0 < x < y$, use the Mean Value Theorem to show that

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}.$$

82. **Think About It** Is the function

$$F(x) = \int_x^{2x} \frac{1}{t}\,dt$$

constant, increasing, or decreasing on the interval $(0, \infty)$? Explain.

83. **Finding a Value** Find a value of x such that

$$\int_1^x \frac{3}{t}\,dt = \int_{1/4}^x \frac{1}{t}\,dt.$$

84. **Finding a Value** Find a value of x such that

$$\int_1^x \frac{1}{t}\,dt$$

is equal to (a) $\ln 5$ and (b) 1.

5.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Natural Exponential Function** Describe the graph of $f(x) = e^x$.
2. **A Function and Its Derivative** Which of the following functions are their own derivative?

   $y = e^x + 4$ $\quad y = e^x$ $\quad y = e^{4x}$ $\quad y = 4e^x$

Solving an Exponential or Logarithmic Equation **In Exercises 3–18, solve for x accurate to three decimal places.**

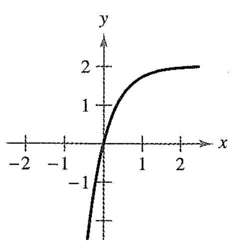
3. $e^{\ln x} = 4$
4. $e^{\ln 3x} = 24$
5. $e^x = 12$
6. $5e^x = 36$
7. $9 - 2e^x = 7$
8. $8e^x - 12 = 7$
9. $50e^{-x} = 30$
10. $100e^{-2x} = 35$
11. $\dfrac{800}{100 - e^{x/2}} = 50$
12. $\dfrac{5000}{1 + e^{2x}} = 2$
13. $\ln x = 2$
14. $\ln x^2 = -8$
15. $\ln(x - 3) = 2$
16. $\ln 4x = 1$
17. $\ln\sqrt{x + 2} = 1$
18. $\ln(x - 2)^2 = 12$

Sketching a Graph **In Exercises 19–24, sketch the graph of the function.**

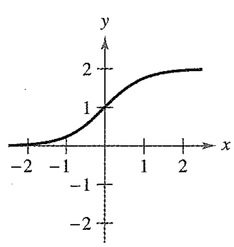
19. $y = e^{-x}$
20. $y = \frac{1}{3}e^x$
21. $y = e^x + 1$
22. $y = -e^{x-1}$
23. $y = e^{-x^2}$
24. $y = e^{-x/2}$

Matching **In Exercises 25–28, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]**

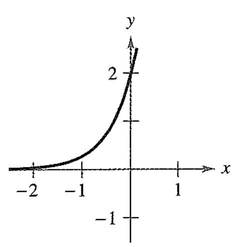
(a)



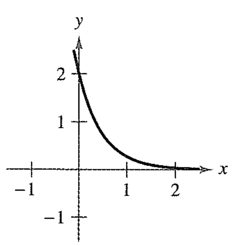
(b)



(c)



(d)



25. $y = Ce^{ax}$
26. $y = Ce^{-ax}$
27. $y = C(1 - e^{-ax})$
28. $y = \dfrac{C}{1 + e^{-ax}}$

Inverse Functions **In Exercises 29–32, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.**

29. $f(x) = e^{2x}$

    $g(x) = \ln\sqrt{x}$
30. $f(x) = e^{x/3}$

    $g(x) = \ln x^3$
31. $f(x) = e^x - 1$

    $g(x) = \ln(x + 1)$
32. $f(x) = e^{x-1}$

    $g(x) = 1 + \ln x$

Finding a Derivative **In Exercises 33–54, find the derivative of the function.**

33. $y = e^{5x}$
34. $y = e^{-8x}$
35. $y = e^{\sqrt{x}}$
36. $y = e^{-2x^3}$
37. $y = e^{x-4}$
38. $y = 5e^{x^2+5}$
39. $y = e^x \ln x$
40. $y = xe^{4x}$
41. $y = (x + 1)^2 e^x$
42. $y = x^2 e^{-x}$
43. $g(t) = (e^{-t} + e^t)^3$
44. $g(t) = e^{-3/t^2}$
45. $y = \ln(2 - e^{5x})$
46. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$
47. $y = \dfrac{2}{e^x + e^{-x}}$
48. $y = \dfrac{e^x - e^{-x}}{2}$
49. $y = \dfrac{e^x + 1}{e^x - 1}$
50. $y = \dfrac{e^{2x}}{e^{2x} + 1}$
51. $y = e^x(\sin x + \cos x)$
52. $y = e^{2x}\tan 2x$
53. $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t\, dt$
54. $F(x) = \displaystyle\int_{0}^{e^{2x}} \ln(t + 1)\, dt$

Finding an Equation of a Tangent Line **In Exercises 55–62, find an equation of the tangent line to the graph of the function at the given point.**

55. $f(x) = e^{3x}$, $(0, 1)$
56. $f(x) = e^{-x} - 6$, $(0, -5)$
57. $y = e^{3x-x^2}$, $(3, 1)$
58. $y = e^{-2x+x^2}$, $(2, 1)$
59. $f(x) = e^{-x}\ln x$, $(1, 0)$
60. $y = \ln \dfrac{e^x + e^{-x}}{2}$, $(0, 0)$
61. $y = x^2e^x - 2xe^x + 2e^x$, $(1, e)$
62. $y = xe^x - e^x$, $(1, 0)$

Implicit Differentiation **In Exercises 63 and 64, use implicit differentiation to find dy/dx.**

63. $xe^y - 10x + 3y = 0$
64. $e^{xy} + x^2 - y^2 = 10$

Finding the Equation of a Tangent Line **In Exercises 65 and 66, use implicit differentiation to find an equation of the tangent line to the graph of the equation at the given point.**

65. $xe^y + ye^x = 1$, $(0, 1)$
66. $1 + \ln xy = e^{x-y}$, $(1, 1)$

Finding a Second Derivative In Exercises 67 and 68, find the second derivative of the function.

67. $f(x) = (3 + 2x)e^{-3x}$ **68.** $g(x) = \sqrt{x} + e^x \ln x$

Differential Equation In Exercises 69 and 70, show that the function $y = f(x)$ is a solution of the differential equation.

69. $y = 4e^{-x}$
$y'' - y = 0$

70. $y = e^{3x} + e^{-3x}$
$y'' - 9y = 0$

Relative Extrema and Points of Inflection In Exercises 71–78, find the relative extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

71. $f(x) = \dfrac{e^x + e^{-x}}{2}$ **72.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

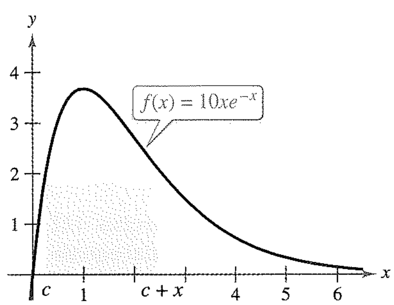
73. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$ **74.** $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

75. $f(x) = (2 - x)e^x$ **76.** $f(x) = xe^{-x}$

77. $g(t) = 1 + (2 + t)e^{-t}$ **78.** $f(x) = -2 + e^{3x}(4 - 2x)$

79. Area Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

80. Area Perform the following steps to find the maximum area of the rectangle shown in the figure.



(a) Solve for c in the equation $f(c) = f(c + x)$.

(b) Use the result in part (a) to write the area A as a function of x. [*Hint:* $A = xf(c)$]

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the maximum area.

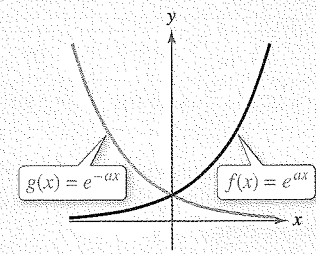
(d) Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate

$$\lim_{x\to 0^+} c \quad \text{and} \quad \lim_{x\to\infty} c.$$

Use this result to describe the changes in dimensions and position of the rectangle for $0 < x < \infty$.

81. Finding an Equation of a Tangent Line Find the point on the graph of the function $f(x) = e^{2x}$ such that the tangent line to the graph at that point passes through the origin. Use a graphing utility to graph f and the tangent line in the same viewing window.

82. HOW DO YOU SEE IT? The figure shows the graphs of f and g, where a is a positive real number. Identify the open interval(s) on which the graphs of f and g are (a) increasing or decreasing and (b) concave upward or concave downward.



83. Depreciation The value V of an item t years after it is purchased is $V = 15{,}000e^{-0.6286t}$, $0 \le t \le 10$.

(a) Use a graphing utility to graph the function.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 5$.

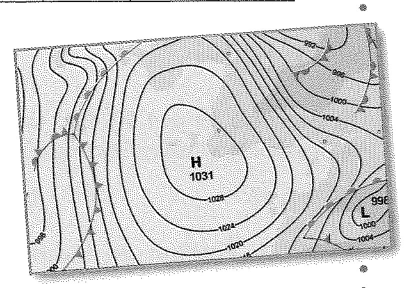
(c) Use a graphing utility to graph the tangent lines to the function when $t = 1$ and $t = 5$.

84. Harmonic Motion The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is $y = 1.56e^{-0.22t}\cos 4.9t$, where y is the displacement (in feet) and t is the time (in seconds). Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

85. Atmospheric Pressure

A meteorologist measures the atmospheric pressure P (in millibars) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	1013.2	547.5	233.0	121.6	50.7



(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form $\ln P = ah + b$. Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rates of change of the pressure when $h = 5$ and $h = 18$.

Robert Adrian Hillman/Shutterstock.com

86. Modeling Data The table lists the approximate values V of a mid-sized sedan for the years 2010 through 2016. The variable t represents the time (in years), with $t = 10$ corresponding to 2010.

t	10	11	12	13	14	15	16
V	\$23,046	\$20,596	\$18,851	\$17,001	\$15,226	\$14,101	\$12,841

(a) Use the regression capabilities of a graphing utility to fit linear and quadratic models to the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Use the exponential model to find the rates of decrease in the value of the sedan when $t = 12$ and $t = 15$.

Linear and Quadratic Approximation In Exercises 87 and 88, use a graphing utility to graph the function. Then graph

$$P_1(x) = f(0) + f'(0)(x - 0) \quad \text{and}$$

$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, P_2, and their first derivatives at $x = 0$.

87. $f(x) = e^x$ **88.** $f(x) = e^{x/2}$

Stirling's Formula For large values of n,

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n - 1) \cdot n$$

can be approximated by Stirling's Formula,

$$n! \approx \left(\frac{n}{3}\right)^n \sqrt{2\pi n}.$$

In Exercises 89 and 90, find the exact value of $n!$ and then approximate $n!$ using Stirling's Formula.

89. $n = 12$ **90.** $n = 15$

Finding an Indefinite Integral In Exercises 91–108, find the indefinite integral.

91. $\int e^{5x}(5)\,dx$ **92.** $\int e^{-x^4}(-4x^3)\,dx$

93. $\int e^{5x-3}\,dx$ **94.** $\int e^{1-3x}\,dx$

95. $\int (2x + 1)e^{x^2+x}\,dx$ **96.** $\int e^x(e^x + 1)^2\,dx$

97. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **98.** $\int \frac{e^{1/x^2}}{x^3}\,dx$

99. $\int \frac{e^{-x}}{1 + e^{-x}}\,dx$ **100.** $\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

101. $\int e^x\sqrt{1 - e^x}\,dx$ **102.** $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

103. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$ **104.** $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

105. $\int \frac{5 - e^x}{e^{2x}}\,dx$ **106.** $\int \frac{e^{-3x} + 2e^{2x} + 3}{e^x}\,dx$

107. $\int e^{-x}\tan(e^{-x})\,dx$ **108.** $\int e^{2x}\csc(e^{2x})\,dx$

Evaluating a Definite Integral In Exercises 109–118, evaluate the definite integral. Use a graphing utility to verify your result.

109. $\int_0^1 e^{-2x}\,dx$ **110.** $\int_{-1}^1 e^{1+4x}\,dx$

111. $\int_0^1 xe^{-x^2}\,dx$ **112.** $\int_{-2}^0 x^2e^{x^3/2}\,dx$

113. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$ **114.** $\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx$

115. $\int_0^2 \frac{e^{4x}}{1 + e^{4x}}\,dx$ **116.** $\int_{-2}^0 \frac{e^{x+1}}{7 - e^{x+1}}\,dx$

117. $\int_0^{\pi/2} e^{\sin \pi x}\cos \pi x\,dx$ **118.** $\int_{\pi/3}^{\pi/2} e^{\sec 2x}\sec 2x \tan 2x\,dx$

Differential Equation In Exercises 119 and 120, find the general solution of the differential equation.

119. $\frac{dy}{dx} = xe^{9x^2}$ **120.** $\frac{dy}{dx} = (e^x - e^{-x})^2$

Differential Equation In Exercises 121 and 122, find the particular solution of the differential equation that satisfies the initial conditions.

121. $f''(x) = \frac{1}{2}(e^x + e^{-x}),\ f(0) = 1,\ f'(0) = 0$

122. $f''(x) = \sin x + e^{2x},\ f(0) = \frac{1}{4},\ f'(0) = \frac{1}{2}$

Area In Exercises 123–126, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.

123. $y = e^x,\ y = 0,\ x = 0,\ x = 6$

124. $y = e^{-2x},\ y = 0,\ x = -1,\ x = 3$

125. $y = xe^{-x^2/4},\ y = 0,\ x = 0,\ x = \sqrt{6}$

126. $y = e^{-2x} + 2,\ y = 0,\ x = 0,\ x = 2$

Midpoint Rule In Exercises 127 and 128, use the Midpoint Rule with $n = 12$ to approximate the value of the definite integral. Use a graphing utility to verify your result.

127. $\int_0^4 \sqrt{x}\,e^x\,dx$ **128.** $\int_0^2 2xe^{-x}\,dx$

5.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **L'Hopital's Rule** Explain the benefit of L'Hôpital's Rule.
2. **Indeterminate Forms** For each limit, use direct substitution. Then identify the form of the limit as either indeterminate or not.

(a) $\lim_{x \to 0} \dfrac{x^2}{\sin 2x}$ (b) $\lim_{x \to \infty} (e^x + x^2)$

(c) $\lim_{x \to \infty} (\ln x - e^x)$ (d) $\lim_{x \to 0^+} \left(\ln x^2 - \dfrac{1}{x}\right)$

Numerical and Graphical Analysis **In Exercises 3–6, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.**

3. $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 3x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

4. $\lim_{x \to 0} \dfrac{1 - e^x}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

5. $\lim_{x \to \infty} x^5 e^{-x/100}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

6. $\lim_{x \to \infty} \dfrac{6x}{\sqrt{3x^2 - 2x}}$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$						

Using Two Methods **In Exercises 7–14, evaluate the limit (a) using techniques from Chapters 1 and 3 and (b) using L'Hôpital's Rule.**

7. $\lim_{x \to 4} \dfrac{3(x - 4)}{x^2 - 16}$

8. $\lim_{x \to -4} \dfrac{2x^2 + 13x + 20}{x + 4}$

9. $\lim_{x \to 6} \dfrac{\sqrt{x + 10} - 4}{x - 6}$

10. $\lim_{x \to -1} \left(\dfrac{1 - \sqrt{x + 2}}{x + 1}\right)$

11. $\lim_{x \to 0} \left(\dfrac{2 - 2\cos x}{6x}\right)$

12. $\lim_{x \to 0} \dfrac{\sin 6x}{4x}$

13. $\lim_{x \to \infty} \dfrac{5x^2 - 3x + 1}{3x^2 - 5}$

14. $\lim_{x \to \infty} \dfrac{x^3 + 2x}{4 - x}$

Evaluating a Limit **In Exercises 15–42, evaluate the limit, using L'Hôpital's Rule if necessary.**

15. $\lim_{x \to 3} \dfrac{x^2 - 2x - 3}{x - 3}$

16. $\lim_{x \to -2} \dfrac{x^2 - 3x - 10}{x + 2}$

17. $\lim_{x \to 0} \dfrac{\sqrt{25 - x^2} - 5}{x}$

18. $\lim_{x \to 5^-} \dfrac{\sqrt{25 - x^2}}{x - 5}$

19. $\lim_{x \to 0^+} \dfrac{e^x - (1 + x)}{x^3}$

20. $\lim_{x \to 1} \dfrac{\ln x^3}{x^2 - 1}$

21. $\lim_{x \to 1} \dfrac{x^{11} - 1}{x^4 - 1}$

22. $\lim_{x \to 1} \dfrac{x^a - 1}{x^b - 1}$, where $a, b \neq 0$

23. $\lim_{x \to 0} \dfrac{\sin 3x}{\sin 5x}$

24. $\lim_{x \to 0} \dfrac{\sin ax}{\sin bx}$, where $a, b \neq 0$

25. $\lim_{x \to \infty} \dfrac{7x^3 - 2x + 1}{6x^3 + 1}$

26. $\lim_{x \to \infty} \dfrac{8 - x}{x^3}$

27. $\lim_{x \to \infty} \dfrac{x^2 + 4x + 7}{x - 6}$

28. $\lim_{x \to \infty} \dfrac{x^3}{x + 2}$

29. $\lim_{x \to \infty} \dfrac{x^3}{e^{x/2}}$

30. $\lim_{x \to \infty} \dfrac{e^{x^2}}{1 - x^3}$

31. $\lim_{x \to \infty} \dfrac{x}{\sqrt{x^2 + 1}}$

32. $\lim_{x \to \infty} \dfrac{x^2}{\sqrt{x^2 + 1}}$

33. $\lim_{x \to \infty} \dfrac{\cos x}{x}$

34. $\lim_{x \to \infty} \dfrac{\sin x}{x - \pi}$

35. $\lim_{x \to \infty} \dfrac{\ln x}{x^2}$

36. $\lim_{x \to \infty} \dfrac{\ln x^4}{x^3}$

37. $\lim_{x \to \infty} \dfrac{e^x}{x^4}$

38. $\lim_{x \to \infty} \dfrac{e^{2x-9}}{3x}$

39. $\lim_{x \to 0} \dfrac{\sin 5x}{\tan 9x}$

40. $\lim_{x \to 1} \dfrac{\ln x}{\sin \pi x}$

41. $\lim_{x \to \infty} \dfrac{\int_1^x \ln(e^{4t-1})\, dt}{x}$

42. $\lim_{x \to 1^+} \dfrac{\int_1^x \cos \theta\, d\theta}{x - 1}$

Evaluating a Limit **In Exercises 43–62, (a) describe the type of indeterminate form (if any) that is obtained by direct substitution. (b) Evaluate the limit, using L'Hôpital's Rule if necessary. (c) Use a graphing utility to graph the function and verify the result in part (b).**

43. $\lim_{x \to \infty} x \ln x$

44. $\lim_{x \to 0^+} x^3 \cot x$

45. $\lim_{x \to \infty} x \sin \dfrac{1}{x}$

46. $\lim_{x \to \infty} x \tan \dfrac{1}{x}$

47. $\lim_{x \to 0^+} (e^x + x)^{2/x}$

48. $\lim_{x \to 0^+} \left(1 + \dfrac{1}{x}\right)^x$

49. $\lim_{x \to \infty} x^{1/x}$

50. $\lim_{x \to 0^+} x^{1/x}$

51. $\lim_{x \to 0^+} (1 + x)^{1/x}$

52. $\lim_{x \to \infty} (1 + x)^{1/x}$

53. $\lim_{x\to 0^+} 3x^{x/2}$

54. $\lim_{x\to 4^+} [3(x-4)]^{x-4}$

55. $\lim_{x\to 1^+} (\ln x)^{x-1}$

56. $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x$

57. $\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right)$

58. $\lim_{x\to 2^+} \left(\frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4}\right)$

59. $\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right)$

60. $\lim_{x\to 0^+} \left(\frac{10}{x} - \frac{3}{x^2}\right)$

61. $\lim_{x\to\infty} (e^x - x)$

62. $\lim_{x\to\infty} \left(x - \sqrt{x^2+1}\right)$

EXPLORING CONCEPTS

63. Finding Functions Find differentiable functions f and g that satisfy the specified condition such that

$$\lim_{x\to 5} f(x) = 0 \quad \text{and} \quad \lim_{x\to 5} g(x) = 0.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

(a) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 10$ (b) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \infty$

64. Finding Functions Find differentiable functions f and g such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \text{ and } \lim_{x\to\infty} [f(x) - g(x)] = 25.$$

Explain how you obtained your answers. (*Note:* There are many correct answers.)

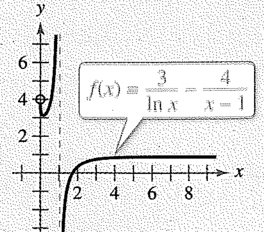
65. L'Hôpital's Rule Determine which of the following limits can be evaluated using L'Hôpital's Rule. Explain your reasoning. Do not evaluate the limit.

(a) $\lim_{x\to 2} \frac{x-2}{x^3-x-6}$ (b) $\lim_{x\to 0} \frac{x^2-4x}{2x-1}$

(c) $\lim_{x\to\infty} \frac{x^3}{e^x}$ (d) $\lim_{x\to 3} \frac{e^{x^2} - e^9}{x-3}$

(e) $\lim_{x\to 1} \frac{\cos \pi x}{\ln x}$ (f) $\lim_{x\to 1} \frac{1 + x(\ln x - 1)}{(x-1)\ln x}$

66. HOW DO YOU SEE IT? Use the graph of f to find each limit.



(a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$

67. Numerical Analysis Complete the table to show that x eventually "overpowers" $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$						

68. Numerical Analysis Complete the table to show that e^x eventually "overpowers" x^5.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

Comparing Functions **In Exercises 69–74, use L'Hôpital's Rule to determine the comparative rates of increase of the functions $f(x) = x^m$, $g(x) = e^{nx}$, and $h(x) = (\ln x)^n$, where $n > 0$, $m > 0$, and $x \to \infty$.**

69. $\lim_{x\to\infty} \frac{x^2}{e^{5x}}$

70. $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$

71. $\lim_{x\to\infty} \frac{(\ln x)^3}{x}$

72. $\lim_{x\to\infty} \frac{(\ln x)^2}{x^3}$

73. $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$

74. $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

Asymptotes and Relative Extrema **In Exercises 75–78, find any asymptotes and relative extrema that may exist and use a graphing utility to graph the function.**

75. $y = x^{1/x}, \ x > 0$

76. $y = x^x, \ x > 0$

77. $y = 2xe^{-x}$

78. $y = \frac{\ln x}{x}$

Think About It **In Exercises 79–82, L'Hôpital's Rule is used incorrectly. Describe the error.**

79. $\lim_{x\to 2} \frac{3x^2+4x+1}{x^2-x-2} = \lim_{x\to 2} \frac{6x+4}{2x-1} = \lim_{x\to 2} \frac{6}{2} = 3$

80.
$$\lim_{x\to 0} \frac{e^{2x}-1}{e^x} = \lim_{x\to 0} \frac{2e^{2x}}{e^x} = \lim_{x\to 0} 2e^x = 2$$

81.
$$\lim_{x\to\infty} \frac{e^{-x}}{1+e^{-x}} = \lim_{x\to\infty} \frac{-e^{-x}}{-e^{-x}} = \lim_{x\to\infty} 1 = 1$$

82.
$$\lim_{x\to\infty} x \cos\frac{1}{x} = \lim_{x\to\infty} \frac{\cos(1/x)}{1/x} = \lim_{x\to\infty} \frac{[-\sin(1/x)](1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sin\frac{1}{x} = 0$$

5.7 Exercises

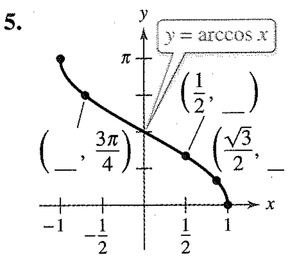
CONCEPT CHECK

1. **Inverse Trigonometric Function** Describe the meaning of $\arccos x$ in your own words.
2. **Restricted Domain** What is a restricted domain? Why are restricted domains necessary to define inverse trigonometric functions?
3. **Inverse Trigonometric Functions** Which inverse trigonometric function has a range of $0 < y < \pi$?
4. **Finding a Derivative** What is the missing value?

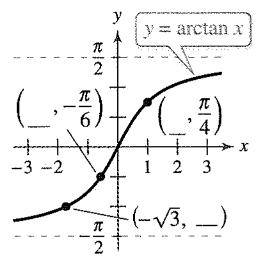
$$\frac{d}{dx}[\operatorname{arccsc} x^3] = \frac{\boxed{}}{|x^3|\sqrt{x^6 - 1}}$$

Finding Coordinates **In Exercises 5 and 6, determine the missing coordinates of the points on the graph of the function.**

5.



6.



Evaluating Inverse Trigonometric Functions **In Exercises 7–14, evaluate the expression without using a calculator.**

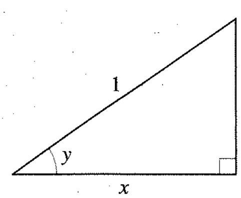
7. $\arcsin \frac{1}{2}$
8. $\arcsin 0$
9. $\arccos \frac{1}{2}$
10. $\arccos(-1)$
11. $\arctan \frac{\sqrt{3}}{3}$
12. $\operatorname{arccot}(-\sqrt{3})$
13. $\operatorname{arccsc}(-\sqrt{2})$
14. $\operatorname{arcsec} 2$

Approximating Inverse Trigonometric Functions **In Exercises 15–18, use a calculator to approximate the value. Round your answer to two decimal places.**

15. $\arccos(0.051)$
16. $\arcsin(-0.39)$
17. $\operatorname{arcsec} 1.269$
18. $\operatorname{arccsc}(-4.487)$

Using a Right Triangle **In Exercises 19–24, use the figure to write the expression in algebraic form given $y = \arccos x$, where $0 < y < \pi/2$.**

19. $\cos y$
20. $\sin y$
21. $\tan y$
22. $\cot y$
23. $\sec y$
24. $\csc y$



Evaluating an Expression **In Exercises 25–28, evaluate each expression without using a calculator. (*Hint:* Sketch a right triangle, as demonstrated in Example 3.)**

25. (a) $\sin\left(\arctan \frac{3}{4}\right)$
(b) $\sec\left(\arcsin \frac{4}{5}\right)$

26. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$
(b) $\cos\left(\arcsin \frac{5}{13}\right)$

27. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$

28. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

Simplifying an Expression Using a Right Triangle **In Exercises 29–36, write the expression in algebraic form. (*Hint:* Sketch a right triangle, as demonstrated in Example 3.)**

29. $\cos(\arcsin 2x)$
30. $\sec(\arctan 6x)$
31. $\sin(\operatorname{arcsec} x)$
32. $\cos(\operatorname{arccot} x)$
33. $\tan\left(\operatorname{arcsec} \frac{x}{3}\right)$
34. $\sec[\arcsin(x - 1)]$
35. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$
36. $\cos\left(\arcsin \frac{x - h}{r}\right)$

Solving an Equation **In Exercises 37–40, solve the equation for x.**

37. $\arcsin(3x - \pi) = \frac{1}{2}$
38. $\arctan(2x - 5) = -1$
39. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$
40. $\arccos x = \operatorname{arcsec} x$

Finding a Derivative **In Exercises 41–56, find the derivative of the function.**

41. $f(x) = \arcsin(x - 1)$
42. $f(t) = \operatorname{arccsc}(-t^2)$
43. $g(x) = 3 \arccos \frac{x}{2}$
44. $f(x) = \operatorname{arcsec} 2x$
45. $f(x) = \arctan e^x$
46. $f(x) = \operatorname{arccot} \sqrt{x}$
47. $g(x) = \frac{\arcsin 3x}{x}$
48. $h(x) = x^2 \arctan 5x$
49. $h(t) = \sin(\arccos t)$
50. $f(x) = \arcsin x + \arccos x$
51. $y = 2x \arccos x - 2\sqrt{1 - x^2}$
52. $y = x \arctan 2x - \frac{1}{4}\ln(1 + 4x^2)$
53. $y = \frac{1}{2}\left(\frac{1}{2}\ln\frac{x + 1}{x - 1} + \arctan x\right)$
54. $y = \frac{1}{2}\left[x\sqrt{4 - x^2} + 4 \arcsin \frac{x}{2}\right]$
55. $y = 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2}$
56. $y = \arctan x + \frac{x}{1 + x^2}$

5.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Integration Rules** Decide whether you can find each integral using the formulas and techniques you have studied so far. Explain.

(a) $\int \frac{2\,dx}{\sqrt{x^2+4}}$ (b) $\int \frac{dx}{x\sqrt{x^2-9}}$

2. **Completing the Square** In your own words, describe the process of completing the square of a quadratic function. Explain when completing the square is useful for finding an integral.

Finding an Indefinite Integral **In Exercises 3–22, find the indefinite integral.**

3. $\int \frac{dx}{\sqrt{9-x^2}}$

4. $\int \frac{dx}{\sqrt{1-4x^2}}$

5. $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

6. $\int \frac{12}{1+9x^2}\,dx$

7. $\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$

8. $\int \frac{7}{4+(3-x)^2}\,dx$

9. $\int \frac{t}{\sqrt{1-t^4}}\,dt$

10. $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

11. $\int \frac{t}{t^4+25}\,dt$

12. $\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

13. $\int \frac{e^{2x}}{4+e^{4x}}\,dx$

14. $\int \frac{5}{x\sqrt{9x^2-11}}\,dx$

15. $\int \frac{-\csc x \cot x}{\sqrt{25-\csc^2 x}}\,dx$

16. $\int \frac{\sin x}{7+\cos^2 x}\,dx$

17. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$

18. $\int \frac{3}{2\sqrt{x}(1+x)}\,dx$

19. $\int \frac{x-3}{x^2+1}\,dx$

20. $\int \frac{x^2+8}{x\sqrt{x^2-4}}\,dx$

21. $\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx$

22. $\int \frac{x-2}{(x+1)^2+4}\,dx$

Evaluating a Definite Integral **In Exercises 23–34, evaluate the definite integral.**

23. $\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\,dx$

24. $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx$

25. $\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$

26. $\int_{\sqrt{3}}^{3} \frac{1}{x\sqrt{4x^2-9}}\,dx$

27. $\int_1^7 \frac{1}{9+(x+2)^2}\,dx$

28. $\int_1^4 \frac{1}{x\sqrt{16x^2-5}}\,dx$

29. $\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx$

30. $\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$

31. $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

32. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

33. $\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

34. $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

Completing the Square **In Exercises 35–42, find or evaluate the integral by completing the square.**

35. $\int_0^2 \frac{dx}{x^2-2x+2}$

36. $\int_{-2}^{3} \frac{dx}{x^2+4x+8}$

37. $\int \frac{dx}{\sqrt{-2x^2+8x+4}}$

38. $\int \frac{dx}{3x^2-6x+12}$

39. $\int \frac{1}{\sqrt{-x^2-4x}}\,dx$

40. $\int \frac{2}{\sqrt{-x^2+4x}}\,dx$

41. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$

42. $\int_3^4 \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$

Integration by Substitution **In Exercises 43–46, use the specified substitution to find or evaluate the integral.**

43. $\int \sqrt{e^t-3}\,dt$

$u = \sqrt{e^t-3}$

44. $\int \frac{\sqrt{x-2}}{x+1}\,dx$

$u = \sqrt{x-2}$

45. $\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$

$u = \sqrt{x}$

46. $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$

$u = \sqrt{x+1}$

Comparing Integration Problems **In Exercises 47–50, find the indefinite integrals, if possible, using the formulas and techniques you have studied so far in the text.**

47. (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx$

(b) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

(c) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$

48. (a) $\int e^{x^2}\,dx$

(b) $\int xe^{x^2}\,dx$

(c) $\int \frac{1}{x^2}e^{1/x}\,dx$

49. (a) $\int \sqrt{x-1}\,dx$

(b) $\int x\sqrt{x-1}\,dx$

(c) $\int \frac{x}{\sqrt{x-1}}\,dx$

50. (a) $\int \frac{1}{1+x^4}\,dx$

(b) $\int \frac{x}{1+x^4}\,dx$

(c) $\int \frac{x^3}{1+x^4}\,dx$

5.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Hyperbolic Functions** Describe how the name *hyperbolic function* arose.
2. **Domains of Hyperbolic Functions** Which hyperbolic functions have domains that are not all real numbers?
3. **Hyperbolic Identities** Which hyperbolic identity corresponds to the trigonometric identity
$$\sin^2 x = \frac{1 - \cos 2x}{2}?$$
4. **Derivatives of Inverse Hyperbolic Functions** What is the missing value?
$$\frac{d}{dx}[\operatorname{sech}^{-1}(3x)] = \frac{\boxed{}}{3x\sqrt{1 - 9x^2}}$$

Evaluating a Function **In Exercises 5–10, evaluate the function. If the value is not a rational number, round your answer to three decimal places.**

5. (a) $\sinh 3$ (b) $\tanh(-2)$

6. (a) $\cosh 0$ (b) $\operatorname{sech} 1$

7. (a) $\operatorname{csch}(\ln 2)$ (b) $\coth(\ln 5)$

8. (a) $\sinh^{-1} 0$ (b) $\tanh^{-1} 0$

9. (a) $\cosh^{-1} 2$ (b) $\operatorname{sech}^{-1} \frac{2}{3}$

10. (a) $\operatorname{csch}^{-1} 2$ (b) $\coth^{-1} 3$

Verifying an Identity **In Exercises 11–18, verify the identity.**

11. $\sinh x + \cosh x = e^x$

12. $\cosh x - \sinh x = e^{-x}$

13. $\tanh^2 x + \operatorname{sech}^2 x = 1$

14. $\coth^2 x - \operatorname{csch}^2 x = 1$

15. $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

16. $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$

17. $\sinh 2x = 2 \sinh x \cosh x$

18. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

Finding Values of Hyperbolic Functions **In Exercises 19 and 20, use the value of the given hyperbolic function to find the values of the other hyperbolic functions.**

19. $\sinh x = \dfrac{3}{2}$

20. $\tanh x = \dfrac{1}{2}$

Finding a Limit **In Exercises 21–24, find the limit.**

21. $\lim\limits_{x\to\infty} \sinh x$

22. $\lim\limits_{x\to-\infty} \tanh x$

23. $\lim\limits_{x\to 0} \dfrac{\sinh x}{x}$

24. $\lim\limits_{x\to 0^-} \coth x$

Finding a Derivative **In Exercises 25–34, find the derivative of the function.**

25. $f(x) = \sinh 9x$

26. $f(x) = \cosh(8x + 1)$

27. $y = \operatorname{sech} 5x^2$

28. $f(x) = \tanh(4x^2 + 3x)$

29. $f(x) = \ln(\sinh x)$

30. $y = \ln\left(\tanh \dfrac{x}{2}\right)$

31. $h(t) = \dfrac{t}{6} \sinh(-3t)$

32. $y = (x^2 + 1) \coth \dfrac{x}{3}$

33. $f(t) = \arctan(\sinh t)$

34. $g(x) = \operatorname{sech}^2 3x$

Finding an Equation of a Tangent Line **In Exercises 35–38, find an equation of the tangent line to the graph of the function at the given point.**

35. $y = \sinh(1 - x^2)$, $(1, 0)$

36. $y = x^{\cosh x}$, $(1, 1)$

37. $y = (\cosh x - \sinh x)^2$, $(0, 1)$

38. $y = e^{\sinh x}$, $(0, 1)$

Finding Relative Extrema **In Exercises 39–42, find the relative extrema of the function. Use a graphing utility to confirm your result.**

39. $g(x) = x \operatorname{sech} x$

40. $h(x) = 2 \tanh x - x$

41. $f(x) = \sin x \sinh x - \cos x \cosh x$, $-4 \le x \le 4$

42. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

Catenary **In Exercises 43 and 44, a model for a power cable suspended between two towers is given. (a) Graph the model. (b) Find the heights of the cable at the towers and at the midpoint between the towers. (c) Find the slope of the cable at the point where the cable meets the right-hand tower.**

43. $y = 10 + 15 \cosh \dfrac{x}{15}$, $-15 \le x \le 15$

44. $y = 18 + 25 \cosh \dfrac{x}{25}$, $-25 \le x \le 25$

Finding an Indefinite Integral **In Exercises 45–54, find the indefinite integral.**

45. $\displaystyle\int \cosh 4x \, dx$

46. $\displaystyle\int \operatorname{sech}^2 3x \, dx$

47. $\displaystyle\int \sinh(1 - 2x) \, dx$

48. $\displaystyle\int \frac{\cosh \sqrt{x}}{\sqrt{x}} \, dx$

49. $\displaystyle\int \cosh^2(x - 1) \sinh(x - 1) \, dx$

50. $\displaystyle\int \frac{\sinh x}{1 + \sinh^2 x} \, dx$

8.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Integration Technique** Describe how to integrate a rational function with a numerator and denominator of the same degree.

2. **Fitting Integrands to Basic Integration Rules** What procedure should you use to fit each integrand to the basic integration rules? Do not integrate.

(a) $\displaystyle\int \frac{2+x}{x^2+9}\,dx$ (b) $\displaystyle\int \cot^2 x\,dx$

Choosing an Antiderivative **In Exercises 3 and 4, select the correct antiderivative.**

3. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$

(a) $2\sqrt{x^2+1}+C$ (b) $\sqrt{x^2+1}+C$

(c) $\frac{1}{2}\sqrt{x^2+1}+C$ (d) $\ln(x^2+1)+C$

4. $\displaystyle\int \frac{1}{x^2+1}\,dx$

(a) $\ln\sqrt{x^2+1}+C$ (b) $\dfrac{2x}{(x^2+1)^2}+C$

(c) $\arctan x + C$ (d) $\ln(x^2+1)+C$

Choosing a Formula **In Exercises 5–14, select the basic integration formula you can use to find the indefinite integral, and identify u and a when appropriate. Do not integrate.**

5. $\displaystyle\int (5x-3)^4\,dx$

6. $\displaystyle\int \frac{2t+1}{t^2+t-4}\,dt$

7. $\displaystyle\int \frac{1}{\sqrt{x}(1-2\sqrt{x})}\,dx$

8. $\displaystyle\int \frac{2}{(2t-1)^2+4}\,dt$

9. $\displaystyle\int \frac{3}{\sqrt{1-t^2}}\,dt$

10. $\displaystyle\int \frac{-2x}{\sqrt{x^2-4}}\,dx$

11. $\displaystyle\int t\sin t^2\,dt$

12. $\displaystyle\int \sec 5x\tan 5x\,dx$

13. $\displaystyle\int (\cos x)e^{\sin x}\,dx$

14. $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$

Finding an Indefinite Integral **In Exercises 15–46, find the indefinite integral.**

15. $\displaystyle\int 14(x-5)^6\,dx$

16. $\displaystyle\int \frac{5}{(t+6)^3}\,dt$

17. $\displaystyle\int \frac{7}{(z-10)^7}\,dz$

18. $\displaystyle\int t^3\sqrt{t^4+1}\,dt$

19. $\displaystyle\int \left[z^2+\frac{1}{(1-z)^6}\right]dz$

20. $\displaystyle\int \left[4x-\frac{2}{(2x+3)^2}\right]dx$

21. $\displaystyle\int \frac{t^2-3}{-t^3+9t+1}\,dt$

22. $\displaystyle\int \frac{x+1}{\sqrt{3x^2+6x}}\,dx$

23. $\displaystyle\int \frac{x^2}{x-1}\,dx$

24. $\displaystyle\int \frac{3x}{x+4}\,dx$

25. $\displaystyle\int \frac{x+2}{x+1}\,dx$

26. $\displaystyle\int \left(\frac{1}{9z-5}-\frac{1}{9z+5}\right)dz$

27. $\displaystyle\int (5+4x^2)^2\,dx$

28. $\displaystyle\int x\left(3+\frac{2}{x}\right)^2 dx$

29. $\displaystyle\int x\cos 2\pi x^2\,dx$

30. $\displaystyle\int \csc \pi x\cot \pi x\,dx$

31. $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}}\,dx$

32. $\displaystyle\int \frac{\csc^2 3t}{\cot 3t}\,dt$

33. $\displaystyle\int \frac{2}{e^{-x}+1}\,dx$

34. $\displaystyle\int \frac{4}{3-e^x}\,dx$

35. $\displaystyle\int \frac{\ln x^2}{x}\,dx$

36. $\displaystyle\int (\tan x)[\ln(\cos x)]\,dx$

37. $\displaystyle\int \frac{1+\cos\alpha}{\sin\alpha}\,d\alpha$

38. $\displaystyle\int \frac{1}{\cos\theta-1}\,d\theta$

39. $\displaystyle\int \frac{-1}{\sqrt{1-(4t+1)^2}}\,dt$

40. $\displaystyle\int \frac{1}{25+4x^2}\,dx$

41. $\displaystyle\int \frac{\tan(2/t)}{t^2}\,dt$

42. $\displaystyle\int \frac{e^{-1/t^3}}{t^4}\,dt$

43. $\displaystyle\int \frac{6}{z\sqrt{9z^2-25}}\,dz$

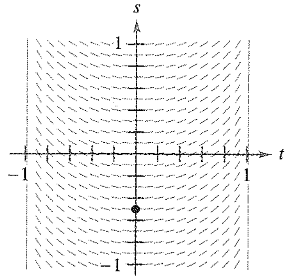
44. $\displaystyle\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}\,dx$

45. $\displaystyle\int \frac{4}{4x^2+4x+65}\,dx$

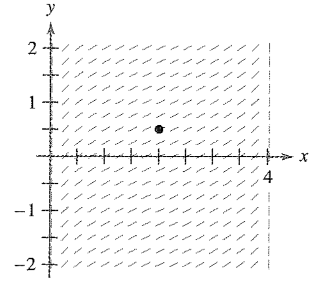
46. $\displaystyle\int \frac{1}{x^2-4x+9}\,dx$

Slope Field **In Exercises 47 and 48, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration and the given point to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a) that passes through the given point.**

47. $\dfrac{ds}{dt}=\dfrac{t}{\sqrt{1-t^4}},\ \left(0,-\dfrac{1}{2}\right)$



48. $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{4x-x^2}},\ \left(2,\dfrac{1}{2}\right)$



Slope Field **In Exercises 49 and 50, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.**

49. $\dfrac{dy}{dx} = 0.8y, \; y(0) = 4$

50. $\dfrac{dy}{dx} = 5 - y, \; y(0) = 1$

Differential Equation **In Exercises 51–56, find the general solution of the differential equation.**

51. $\dfrac{dy}{dx} = (e^x + 5)^2$

52. $\dfrac{dy}{dx} = (4 - e^{2x})^2$

53. $\dfrac{dr}{dt} = \dfrac{10e^t}{\sqrt{1 - e^{2t}}}$

54. $\dfrac{dr}{dt} = \dfrac{(1 + e^t)^2}{e^{3t}}$

55. $(4 + \tan^2 x)y' = \sec^2 x$

56. $y' = \dfrac{1}{x\sqrt{4x^2 - 9}}$

Evaluating a Definite Integral **In Exercises 57–72, evaluate the definite integral. Use a graphing utility to verify your result.**

57. $\displaystyle\int_{2/3}^{1} (2 - 3t)^4 \, dt$

58. $\displaystyle\int_{-1}^{0} \frac{5}{(t + 2)^{11}} \, dt$

59. $\displaystyle\int_{0}^{\pi/4} \cos 2x \, dx$

60. $\displaystyle\int_{0}^{\pi} \sin^2 t \cos t \, dt$

61. $\displaystyle\int_{0}^{1} xe^{-x^2} \, dx$

62. $\displaystyle\int_{1}^{e} \frac{1 - \ln x}{x} \, dx$

63. $\displaystyle\int_{2}^{3} \frac{\ln(x + 1)^3}{x + 1} \, dx$

64. $\displaystyle\int_{-3}^{1} \frac{e^x}{e^{2x} + 2e^x + 1} \, dx$

65. $\displaystyle\int_{0}^{8} \frac{2x}{\sqrt{x^2 + 36}} \, dx$

66. $\displaystyle\int_{1}^{3} \frac{2x^2 + 3x - 2}{x} \, dx$

67. $\displaystyle\int_{3}^{5} \frac{2t}{t^2 - 4t + 4} \, dt$

68. $\displaystyle\int_{2}^{4} \frac{4x^3}{x^4 - 6x^2 + 9} \, dx$

69. $\displaystyle\int_{0}^{2/\sqrt{3}} \frac{1}{4 + 9x^2} \, dx$

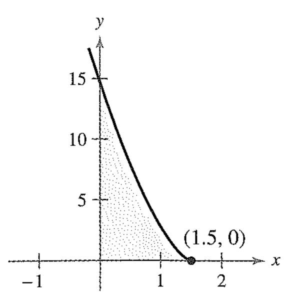
70. $\displaystyle\int_{0}^{7} \frac{1}{\sqrt{100 - x^2}} \, dx$

71. $\displaystyle\int_{-4}^{0} 3^{1-x} \, dx$

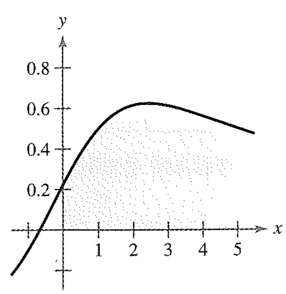
72. $\displaystyle\int_{0}^{1} 7^{x^2 + 2x}(x + 1) \, dx$

Area **In Exercises 73–76, find the area of the given region.**

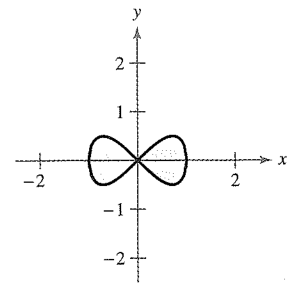
73. $y = (-4x + 6)^{3/2}$



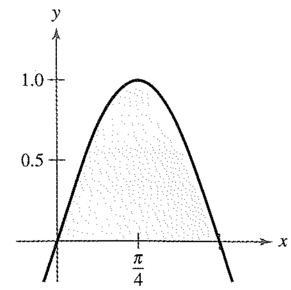
74. $y = \dfrac{3x + 2}{x^2 + 9}$



75. $y^2 = x^2(1 - x^2)$



76. $y = \sin 2x$



Finding an Integral Using Technology **In Exercises 77–80, use a computer algebra system to find the integral. Use the computer algebra system to graph two antiderivatives. Describe the relationship between the graphs of the two antiderivatives.**

77. $\displaystyle\int \frac{1}{x^2 + 4x + 13} \, dx$

78. $\displaystyle\int \frac{x - 2}{x^2 + 4x + 13} \, dx$

79. $\displaystyle\int \frac{1}{1 + \sin\theta} \, d\theta$

80. $\displaystyle\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx$

EXPLORING CONCEPTS

81. **Think About It** When evaluating

$$\int_{-1}^{1} x^2 \, dx$$

is it appropriate to substitute

$$u = x^2, \quad x = \sqrt{u}, \quad \text{and} \quad dx = \frac{du}{2\sqrt{u}}$$

to obtain

$$\frac{1}{2}\int_{1}^{1} \sqrt{u} \, du = 0?$$

Explain.

82. **Deriving a Rule** Show that

$$\sec x = \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x}.$$

Then use this identity to derive the basic integration rule

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

83. **Finding Constants** Determine the constants a and b such that

$$\sin x + \cos x = a \sin(x + b).$$

Use this result to integrate

$$\int \frac{dx}{\sin x + \cos x}.$$

84. **Area** The graphs of $f(x) = x$ and $g(x) = ax^2$ intersect at the points $(0, 0)$ and $(1/a, 1/a)$. Find a $(a > 0)$ such that the area of the region bounded by the graphs of these two functions is $\frac{2}{3}$.

8.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Integration by Parts** Integration by parts is based on what formula?
2. **Setting Up Integration by Parts** In your own words, describe how to choose u and dv when using integration by parts.
3. **Using Integration by Parts** How can you use integration by parts on an integrand with a single term that does not fit any of the basic integration rules?
4. **Using the Tabular Method** When is integrating using the tabular method useful?

Setting Up Integration by Parts **In Exercises 5–10, identify u and dv for finding the integral using integration by parts. Do not integrate.**

5. $\int xe^{9x}\,dx$
6. $\int x^2e^{2x}\,dx$
7. $\int (\ln x)^2\,dx$
8. $\int \ln 5x\,dx$
9. $\int x\sec^2 x\,dx$
10. $\int x^2\cos x\,dx$

Using Integration by Parts **In Exercises 11–14, find the indefinite integral using integration by parts with the given choices of u and dv.**

11. $\displaystyle\int x^3\ln x\,dx;\ u = \ln x,\ dv = x^3\,dx$
12. $\displaystyle\int (7 - x)e^{x/2}\,dx;\ u = 7 - x,\ dv = e^{x/2}\,dx$
13. $\displaystyle\int (2x + 1)\sin 4x\,dx;\ u = 2x + 1,\ dv = \sin 4x\,dx$
14. $\displaystyle\int x\cos 4x\,dx;\ u = x,\ dv = \cos 4x\,dx$

Finding an Indefinite Integral **In Exercises 15–34, find the indefinite integral. (*Note:* Solve by the simplest method—not all require integration by parts.)**

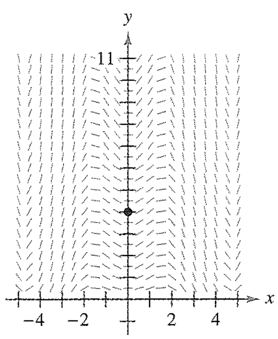
15. $\displaystyle\int xe^{4x}\,dx$
16. $\displaystyle\int \frac{5x}{e^{2x}}\,dx$
17. $\displaystyle\int x^3e^x\,dx$
18. $\displaystyle\int \frac{e^{1/t}}{t^2}\,dt$
19. $\displaystyle\int t\ln(t + 1)\,dt$
20. $\displaystyle\int x^5\ln 3x\,dx$
21. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$
22. $\displaystyle\int \frac{\ln x}{x^3}\,dx$
23. $\displaystyle\int \frac{xe^{2x}}{(2x + 1)^2}\,dx$
24. $\displaystyle\int \frac{x^3e^{x^2}}{(x^2 + 1)^2}\,dx$
25. $\displaystyle\int x\sqrt{x - 5}\,dx$
26. $\displaystyle\int \frac{2x}{\sqrt{1 - 6x}}\,dx$
27. $\displaystyle\int x\csc^2 x\,dx$
28. $\displaystyle\int t\csc t\cot t\,dt$
29. $\displaystyle\int x^3\sin x\,dx$
30. $\displaystyle\int x^2\cos x\,dx$
31. $\displaystyle\int \arctan x\,dx$
32. $\displaystyle\int 4\arccos x\,dx$
33. $\displaystyle\int e^{-3x}\sin 5x\,dx$
34. $\displaystyle\int e^{4x}\cos 2x\,dx$

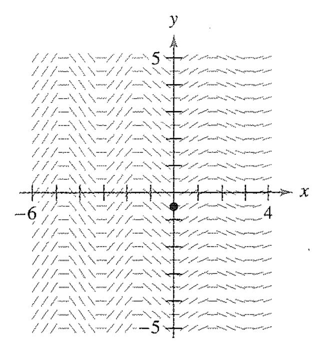
Differential Equation **In Exercises 35–38, find the general solution of the differential equation.**

35. $y' = \ln x$
36. $y' = \arctan\dfrac{x}{2}$
37. $\dfrac{dy}{dt} = \dfrac{t^2}{\sqrt{3 + 5t}}$
38. $\dfrac{dy}{dx} = x^2\sqrt{x - 3}$

Slope Field **In Exercises 39 and 40, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration and the given point to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a) that passes through the given point.**

39. $\dfrac{dy}{dx} = x\sqrt{y}\cos x,\ (0, 4)$
40. $\dfrac{dy}{dx} = e^{-x/3}\sin 2x,\ \left(0, -\frac{18}{37}\right)$





Slope Field **In Exercises 41 and 42, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.**

41. $\dfrac{dy}{dx} = \dfrac{x}{y}e^{x/8},\ y(0) = 2$
42. $\dfrac{dy}{dx} = \dfrac{x}{y}\sin x,\ y(0) = 4$

Evaluating a Definite Integral **In Exercises 43–52, evaluate the definite integral. Use a graphing utility to verify your result.**

43. $\displaystyle\int_0^3 xe^{x/2}\,dx$
44. $\displaystyle\int_0^2 x^2e^{-2x}\,dx$
45. $\displaystyle\int_0^{\pi/4} x\cos 2x\,dx$
46. $\displaystyle\int_0^{\pi} x\sin 2x\,dx$

47. $\int_0^{1/2} \arccos x\,dx$

48. $\int_0^1 x \arcsin x^2\,dx$

49. $\int_0^1 e^x \sin x\,dx$

50. $\int_0^1 \ln(4 + x^2)\,dx$

51. $\int_2^4 x \operatorname{arcsec} x\,dx$

52. $\int_0^{\pi/8} x \sec^2 2x\,dx$

Using the Tabular Method **In Exercises 53–58, use the tabular method to find the indefinite integral.**

53. $\int x^2 e^{2x}\,dx$

54. $\int (1 - x)(e^{-x} + 1)\,dx$

55. $\int (x + 2)^2 \sin x\,dx$

56. $\int x^3 \cos 2x\,dx$

57. $\int (6 + x)\sqrt{4x + 9}\,dx$

58. $\int x^2(x - 2)^{3/2}\,dx$

EXPLORING CONCEPTS

59. Integration by Parts Write an integral that requires three applications of integration by parts. Explain why three applications are needed.

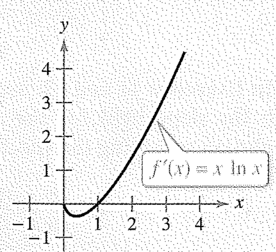
60. Integration by Parts When evaluating $\int x \sin x\,dx$, explain how letting $u = \sin x$ and $dv = x\,dx$ makes the solution more difficult to find.

61. Integration by Parts State whether you would use integration by parts to find each integral. If so, identify what you would use for u and dv. Explain your reasoning.

(a) $\int \frac{\ln x}{x}\,dx$ (b) $\int x \ln x\,dx$ (c) $\int x^2 e^{-3x}\,dx$

(d) $\int 2xe^{x^2}\,dx$ (e) $\int \frac{x}{\sqrt{x + 1}}\,dx$ (f) $\int \frac{x}{\sqrt{x^2 + 1}}\,dx$

62. HOW DO YOU SEE IT? Use the graph of f' shown in the figure to answer the following.



(a) Approximate the slope of f at $x = 2$. Explain.

(b) Approximate any open intervals on which the graph of f is increasing and any open intervals on which it is decreasing. Explain.

Using Two Methods Together **In Exercises 63–66, find the indefinite integral by using substitution followed by integration by parts.**

63. $\int \sin\sqrt{x}\,dx$

64. $\int 2x^3 \cos x^2\,dx$

65. $\int x^5 e^{x^2}\,dx$

66. $\int e^{\sqrt{2x}}\,dx$

67. Using Two Methods Integrate $\int \frac{x^3}{\sqrt{4 + x^2}}\,dx$

(a) by parts, letting $dv = \frac{x}{\sqrt{4 + x^2}}\,dx$.

(b) by substitution, letting $u = 4 + x^2$.

68. Using Two Methods Integrate $\int x\sqrt{4 - x}\,dx$

(a) by parts, letting $dv = \sqrt{4 - x}\,dx$.

(b) by substitution, letting $u = 4 - x$.

Finding a General Rule **In Exercises 69 and 70, use a computer algebra system to find the integrals for $n = 0, 1, 2,$ and 3. Use the result to obtain a general rule for the integrals for any positive integer n and test your results for $n = 4$.**

69. $\int x^n \ln x\,dx$

70. $\int x^n e^x\,dx$

Proof **In Exercises 71–76, use integration by parts to prove the formula. (For Exercises 71–74, assume that n is a positive integer.)**

71. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx$

72. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx$

73. $\int x^n \ln x\,dx = \frac{x^{n+1}}{(n + 1)^2}[-1 + (n + 1)\ln x] + C$

74. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx$

75. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C$

76. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C$

Using Formulas **In Exercises 77–82, find the indefinite integral by using the appropriate formula from Exercises 71–76.**

77. $\int x^2 \sin x\,dx$

78. $\int x^2 \cos x\,dx$

79. $\int x^5 \ln x\,dx$

80. $\int x^3 e^{2x}\,dx$

81. $\int e^{-3x} \sin 4x\,dx$

82. $\int e^{2x} \cos 3x\,dx$

8.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Analyzing Indefinite Integrals** Which integral requires more steps to find? Explain. Do not integrate.

$$\int \sin^8 x\,dx \qquad \int \sin^8 x \cos x\,dx$$

2. **Analyzing an Indefinite Integral** Describe the technique for finding $\int \sec^5 x \tan^7 x\,dx$. Do not integrate.

Finding an Indefinite Integral Involving Sine and Cosine In Exercises 3–14, find the indefinite integral.

3. $\int \cos^5 x \sin x\,dx$

4. $\int \sin^7 2x \cos 2x\,dx$

5. $\int \cos^3 x \sin^4 x\,dx$

6. $\int \sin^3 3x\,dx$

7. $\int \sin^3 x \cos^2 x\,dx$

8. $\int \cos^3 \frac{x}{3}\,dx$

9. $\int \sin^3 2\theta \sqrt{\cos 2\theta}\,d\theta$

10. $\int \frac{\cos^5 t}{\sqrt{\sin t}}\,dt$

11. $\int \cos^2 3x\,dx$

12. $\int \sin^4 6\theta\,d\theta$

13. $\int 8x \cos^2 x\,dx$

14. $\int x^2 \sin^2 x\,dx$

Using Wallis's Formulas In Exercises 15–20, use Wallis's Formulas to evaluate the integral.

15. $\int_0^{\pi/2} \cos^3 x\,dx$

16. $\int_0^{\pi/2} \cos^6 x\,dx$

17. $\int_0^{\pi/2} \sin^2 x\,dx$

18. $\int_0^{\pi/2} \sin^9 x\,dx$

19. $\int_0^{\pi/2} \sin^{10} x\,dx$

20. $\int_0^{\pi/2} \cos^{11} x\,dx$

Finding an Indefinite Integral Involving Secant and Tangent In Exercises 21–34, find the indefinite integral.

21. $\int \sec 4x\,dx$

22. $\int \sec^4 x\,dx$

23. $\int \sec^3 \pi x\,dx$

24. $\int \tan^6 3x\,dx$

25. $\int \tan^5 \frac{x}{2}\,dx$

26. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\,dx$

27. $\int \tan^3 2t \sec^3 2t\,dt$

28. $\int \tan^5 x \sec^4 x\,dx$

29. $\int \sec^6 4x \tan 4x\,dx$

30. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\,dx$

31. $\int \sec^5 x \tan^3 x\,dx$

32. $\int \tan^3 3x\,dx$

33. $\int \frac{\tan^2 x}{\sec x}\,dx$

34. $\int \frac{\tan^2 x}{\sec^5 x}\,dx$

Differential Equation In Exercises 35–38, find the general solution of the differential equation.

35. $\frac{dr}{d\theta} = \sin^4 \pi\theta$

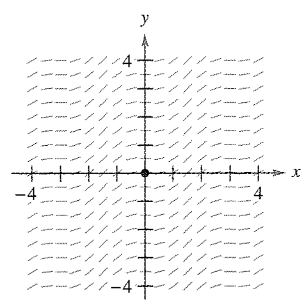
36. $\frac{ds}{d\alpha} = \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}$

37. $y' = \tan^3 3x \sec 3x$

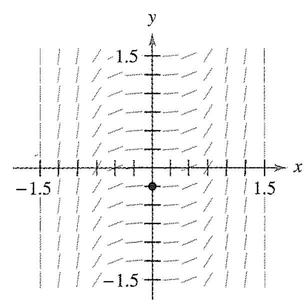
38. $y' = \sqrt{\tan x}\,\sec^4 x$

Slope Field In Exercises 39 and 40, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com.*) (b) Use integration and the given point to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a) that passes through the given point.

39. $\frac{dy}{dx} = \sin^2 x,\ (0, 0)$



40. $\frac{dy}{dx} = \sec^2 x \tan^2 x,\ \left(0, -\frac{1}{4}\right)$



Slope Field In Exercises 41 and 42, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.

41. $\frac{dy}{dx} = \frac{3 \sin x}{y},\ y(0) = 2$

42. $\frac{dy}{dx} = 3\sqrt{y}\tan^2 x,\ y(0) = 3$

Using a Product-to-Sum Formula In Exercises 43–48, find the indefinite integral.

43. $\int \cos 2x \cos 6x\,dx$

44. $\int \cos 5\theta \cos 3\theta\,d\theta$

45. $\int \sin 2t \cos 9t\,dt$

46. $\int \sin 8x \cos 7x\,dx$

47. $\int \sin \theta \sin 3\theta\,d\theta$

48. $\int \sin 5x \sin 4x\,dx$

Finding an Indefinite Integral In Exercises 49–58, find the indefinite integral. Use a computer algebra system to confirm your result.

49. $\int \cot^3 2x\,dx$

50. $\int \tan^5 \frac{x}{4} \sec^4 \frac{x}{4}\,dx$

51. $\int \csc^4 3x\,dx$

52. $\int \cot^3 \frac{x}{2} \csc^4 \frac{x}{2}\,dx$

53. $\int \frac{\cot^2 t}{\csc t}\,dt$

54. $\int \frac{\cot^3 t}{\csc t}\,dt$

55. $\int \frac{1}{\sec x \tan x}\,dx$

56. $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

57. $\int (\tan^4 t - \sec^4 t)\,dt$

58. $\int \frac{1 - \sec t}{\cos t - 1}\,dt$

Evaluating a Definite Integral In Exercises 59–66, evaluate the definite integral.

59. $\int_{-\pi}^{\pi} \sin^2 x\,dx$

60. $\int_0^{\pi/3} \tan^2 x\,dx$

61. $\int_0^{\pi/4} 6 \tan^3 x\,dx$

62. $\int_0^{\pi/3} \sec^{3/2} x \tan x\,dx$

63. $\int_0^{\pi/2} \frac{\cot t}{1 + \sin t}\,dt$

64. $\int_{\pi/6}^{\pi/3} \sin 6x \cos 4x\,dx$

65. $\int_{-\pi/2}^{\pi/2} 3 \cos^3 x\,dx$

66. $\int_0^{\pi} \sin^5 x\,dx$

EXPLORING CONCEPTS

Comparing Methods In Exercises 67 and 68, (a) find the indefinite integral in two different ways, (b) use a graphing utility to graph the antiderivative (without the constant of integration) obtained by each method to show that the results differ only by a constant, and (c) verify analytically that the results differ only by a constant.

67. $\int \sec^4 3x \tan^3 3x\,dx$

68. $\int \sec^2 x \tan x\,dx$

69. Comparing Methods Find the indefinite integral

$$\int \sin x \cos x\,dx$$

using the given method. Explain how your answers differ for each method.

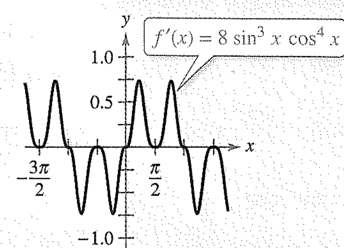
(a) Substitution where $u = \sin x$

(b) Substitution where $u = \cos x$

(c) Integration by parts

(d) Using the identity $\sin 2x = 2 \sin x \cos x$

70. HOW DO YOU SEE IT? Use the graph of f' shown in the figure to answer the following.

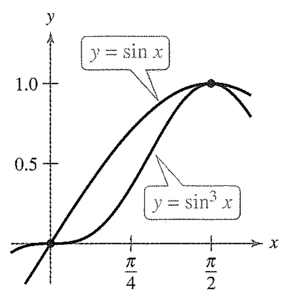


(a) Using the interval shown in the graph, approximate the value(s) of x where f is maximum. Explain.

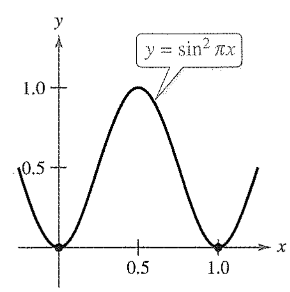
(b) Using the interval shown in the graph, approximate the value(s) of x where f is minimum. Explain.

Area In Exercises 71 and 72, find the area of the given region.

71. $y = \sin x$, $y = \sin^3 x$



72. $y = \sin^2 \pi x$



Area In Exercises 73 and 74, find the area of the region bounded by the graphs of the equations.

73. $y = \cos^2 x$, $y = \sin^2 x$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$

74. $y = \cos^2 x$, $y = \sin x \cos x$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{4}$

Volume In Exercises 75 and 76, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x-axis.

75. $y = \tan x$, $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$

76. $y = \cos \frac{x}{2}$, $y = \sin \frac{x}{2}$, $x = 0$, $x = \frac{\pi}{2}$

Volume and Centroid In Exercises 77 and 78, for the region bounded by the graphs of the equations, find (a) the volume of the solid generated by revolving the region about the x-axis and (b) the centroid of the region.

77. $y = \sin x$, $y = 0$, $x = 0$, $x = \pi$

78. $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{\pi}{2}$

8.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Trigonometric Substitution** State the trigonometric substitution you would use to find the indefinite integral. Do not integrate.

   (a) $\int (9 + x^2)^{-2}\,dx$ (b) $\int \sqrt{4 - x^2}\,dx$

   (c) $\int \frac{x^2}{\sqrt{25 - x^2}}\,dx$ (d) $\int x^2(x^2 - 25)^{3/2}\,dx$

2. **Trigonometric Substitution** Why is completing the square useful when you are considering integration by trigonometric substitution?

Using Trigonometric Substitution **In Exercises 3–6, find the indefinite integral using the substitution $x = 4 \sin \theta$.**

3. $\int \frac{1}{(16 - x^2)^{3/2}}\,dx$
4. $\int \frac{4}{x^2\sqrt{16 - x^2}}\,dx$
5. $\int \frac{\sqrt{16 - x^2}}{x}\,dx$
6. $\int \frac{x^3}{\sqrt{16 - x^2}}\,dx$

Using Trigonometric Substitution **In Exercises 7–10, find the indefinite integral using the substitution $x = 5 \sec \theta$.**

7. $\int \frac{1}{\sqrt{x^2 - 25}}\,dx$
8. $\int \frac{\sqrt{x^2 - 25}}{x}\,dx$
9. $\int x^3\sqrt{x^2 - 25}\,dx$
10. $\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx$

Using Trigonometric Substitution **In Exercises 11–14, find the indefinite integral using the substitution $x = 2 \tan \theta$.**

11. $\int \frac{x}{2}\sqrt{4 + x^2}\,dx$
12. $\int \frac{x^3}{4\sqrt{4 + x^2}}\,dx$
13. $\int \frac{4}{(4 + x^2)^2}\,dx$
14. $\int \frac{2x^2}{(4 + x^2)^2}\,dx$

Special Integration Formulas **In Exercises 15–18, use the Special Integration Formulas (Theorem 8.2) to find the indefinite integral.**

15. $\int \sqrt{49 - 16x^2}\,dx$
16. $\int \sqrt{5x^2 - 1}\,dx$
17. $\int \sqrt{36 - 5x^2}\,dx$
18. $\int \sqrt{9 + 4x^2}$

Finding an Indefinite Integral **In Exercises 19–32, find the indefinite integral.**

19. $\int \sqrt{16 - 4x^2}\,dx$
20. $\int \frac{1}{\sqrt{x^2 - 4}}\,dx$
21. $\int \frac{\sqrt{1 - x^2}}{x^4}\,dx$
22. $\int \frac{\sqrt{25x^2 + 4}}{x^4}\,dx$
23. $\int \frac{1}{x\sqrt{4x^2 + 9}}\,dx$
24. $\int \frac{1}{x\sqrt{9x^2 + 1}}\,dx$
25. $\int \frac{-3}{(x^2 + 3)^{3/2}}\,dx$
26. $\int \frac{1}{(x^2 + 5)^{3/2}}\,dx$
27. $\int e^x\sqrt{1 - e^{2x}}\,dx$
28. $\int \frac{\sqrt{1 - x}}{\sqrt{x}}\,dx$
29. $\int \frac{1}{4 + 4x^2 + x^4}\,dx$
30. $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\,dx$
31. $\int \operatorname{arcsec} 2x\,dx,\ x > \frac{1}{2}$
32. $\int x \arcsin x\,dx$

Completing the Square **In Exercises 33–36, complete the square and find the indefinite integral.**

33. $\int \frac{x}{\sqrt{4x - x^2}}\,dx$
34. $\int \frac{x^2}{\sqrt{2x - x^2}}\,dx$
35. $\int \frac{x}{\sqrt{x^2 + 6x + 12}}\,dx$
36. $\int \frac{x}{\sqrt{x^2 - 6x + 5}}\,dx$

Converting the Limits of Integration **In Exercises 37–42, evaluate the definite integral using (a) the given integration limits and (b) the limits obtained by trigonometric substitution.**

37. $\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt$
38. $\int_0^{\sqrt{3}/2} \frac{1}{(1 - t^2)^{5/2}}\,dt$
39. $\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}}\,dx$
40. $\int_0^{3/5} \sqrt{9 - 25x^2}\,dx$
41. $\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\,dx$
42. $\int_4^8 \frac{\sqrt{x^2 - 16}}{x^2}\,dx$

EXPLORING CONCEPTS

Choosing a Method **In Exercises 43 and 44, state the method of integration you would use to find each integral. Explain why you chose that method. Do not integrate.**

43. $\int x\sqrt{x^2 + 1}\,dx$
44. $\int x^2\sqrt{x^2 - 1}\,dx$

45. **Comparing Methods**

    (a) Find the integral $\int \frac{x}{\sqrt{1 - x^2}}\,dx$ using u-substitution. Then find the integral using trigonometric substitution. Discuss the results.

    (b) Find the integral $\int \frac{x^2}{x^2 + 9}\,dx$ algebraically using $x^2 = (x^2 + 9) - 9$. Then find the integral using trigonometric substitution. Discuss the results.

8.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Partial Fraction Decomposition** Write the form of the partial fraction decomposition of each rational expression. Do not solve for the constants.

(a) $\dfrac{4}{x^2 - 8x}$ (b) $\dfrac{2x^2 + 1}{(x - 3)^3}$

(c) $\dfrac{2x - 3}{x^3 + 10x}$ (d) $\dfrac{2x - 1}{x(x^2 + 1)^2}$

2. **Guidelines for Solving the Basic Equation** In your own words, explain how to solve a basic equation obtained in a partial fraction decomposition that involves quadratic factors.

Using Partial Fractions **In Exercises 3–20, use partial fractions to find the indefinite integral.**

3. $\displaystyle\int \frac{1}{x^2 - 9}\,dx$

4. $\displaystyle\int \frac{2}{9x^2 - 1}\,dx$

5. $\displaystyle\int \frac{5}{x^2 + 3x - 4}\,dx$

6. $\displaystyle\int \frac{3 - x}{3x^2 - 2x - 1}\,dx$

7. $\displaystyle\int \frac{x^2 + 12x + 12}{x^3 - 4x}\,dx$

8. $\displaystyle\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx$

9. $\displaystyle\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,dx$

10. $\displaystyle\int \frac{x + 2}{x^2 + 5x}\,dx$

11. $\displaystyle\int \frac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$

12. $\displaystyle\int \frac{5x - 2}{(x - 2)^2}\,dx$

13. $\displaystyle\int \frac{x^2 - 6x + 2}{x^3 + 2x^2 + x}\,dx$

14. $\displaystyle\int \frac{8x}{x^3 + x^2 - x - 1}\,dx$

15. $\displaystyle\int \frac{9 - x^2}{7x^3 + x}\,dx$

16. $\displaystyle\int \frac{6x}{x^3 - 8}\,dx$

17. $\displaystyle\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx$

18. $\displaystyle\int \frac{x}{16x^4 - 1}\,dx$

19. $\displaystyle\int \frac{x^2 + 5}{x^3 - x^2 + x + 3}\,dx$

20. $\displaystyle\int \frac{x^2 + 6x + 4}{x^4 + 8x^2 + 16}\,dx$

Evaluating a Definite Integral **In Exercises 21–24, use partial fractions to evaluate the definite integral. Use a graphing utility to verify your result.**

21. $\displaystyle\int_0^2 \frac{3}{4x^2 + 5x + 1}\,dx$

22. $\displaystyle\int_1^5 \frac{x - 1}{x^2(x + 1)}\,dx$

23. $\displaystyle\int_1^2 \frac{x + 1}{x(x^2 + 1)}\,dx$

24. $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx$

Finding an Indefinite Integral **In Exercises 25–32, use substitution and partial fractions to find the indefinite integral.**

25. $\displaystyle\int \frac{\sin x}{\cos x + \cos^2 x}\,dx$

26. $\displaystyle\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx$

27. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x + 5\tan x + 6}\,dx$

28. $\displaystyle\int \frac{\sec^2 x}{(\tan x)(\tan x + 1)}\,dx$

29. $\displaystyle\int \frac{e^x}{(e^x - 1)(e^x + 4)}\,dx$

30. $\displaystyle\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)}\,dx$

31. $\displaystyle\int \frac{\sqrt{x}}{x - 4}\,dx$

32. $\displaystyle\int \frac{1}{x(\sqrt{3} - \sqrt{x})}\,dx$

Verifying a Formula **In Exercises 33–36, use the method of partial fractions to verify the integration formula.**

33. $\displaystyle\int \frac{1}{x(a + bx)}\,dx = \frac{1}{a}\ln\left|\frac{x}{a + bx}\right| + C$

34. $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$

35. $\displaystyle\int \frac{x}{(a + bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a + bx} + \ln|a + bx|\right) + C$

36. $\displaystyle\int \frac{1}{x^2(a + bx)}\,dx = -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a + bx}\right| + C$

EXPLORING CONCEPTS

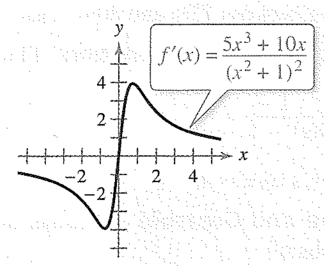
Choosing a Method **In Exercises 37–39, state the method of integration you would use to find each integral. Explain why you chose that method. Do not integrate.**

37. $\displaystyle\int \frac{x + 1}{x^2 + 2x - 8}\,dx$

38. $\displaystyle\int \frac{7x + 4}{x^2 + 2x - 8}\,dx$

39. $\displaystyle\int \frac{4}{x^2 + 2x + 5}\,dx$

40. **HOW DO YOU SEE IT?** Use the graph of f' shown in the figure to answer the following.

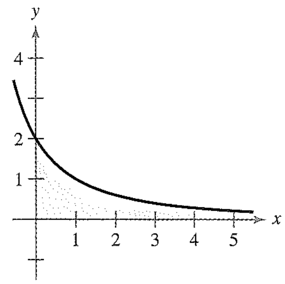


(a) Is $f(3) - f(2) > 0$? Explain.

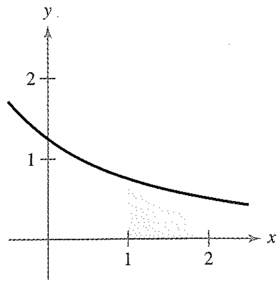
(b) Which is greater, the area under the graph of f' from 1 to 2 or the area under the graph of f' from 3 to 4?

Area **In Exercises 41–44, use partial fractions to find the area of the given region.**

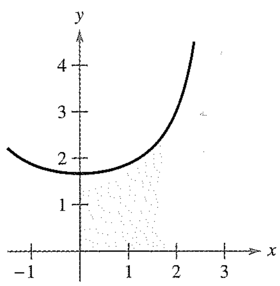
41. $y = \dfrac{12}{x^2 + 5x + 6}$



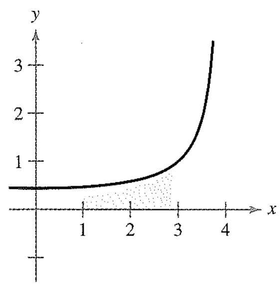
42. $y = \dfrac{15}{x^2 + 7x + 12}$



43. $y = \dfrac{15}{9 - x^2}$



44. $y = \dfrac{7}{16 - x^2}$



45. Modeling Data The predicted cost C (in hundreds of thousands of dollars) for a company to remove $p\%$ of a chemical from its waste water is shown in the table.

p	0	10	20	30	40
C	0	0.7	1.0	1.3	1.7

p	50	60	70	80	90
C	2.0	2.7	3.6	5.5	11.2

A model for the data is given by

$$C = \frac{124p}{(10 + p)(100 - p)}$$

for $0 \le p < 100$. Use the model to find the average cost of removing between 75% and 80% of the chemical.

46. Average Value of a Function Find the average value of

$$f(x) = \frac{1}{4x^2 - 1}$$

from $x = 1$ to $x = 4$.

47. Volume and Centroid Consider the region bounded by the graphs of

$$y = \frac{2x}{x^2 + 1}, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 3.$$

(a) Find the volume of the solid generated by revolving the region about the x-axis.

(b) Find the centroid of the region.

48. Volume Consider the region bounded by the graph of

$$y^2 = \frac{(2 - x)^2}{(1 + x)^2}$$

on the interval $[0, 1]$. Find the volume of the solid generated by revolving this region about the x-axis.

49. Epidemic Model A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So, $dx/dt = k(x + 1)(n - x)$ and you obtain

$$\int \frac{1}{(x + 1)(n - x)}\, dx = \int k\, dt.$$

Solve for x as a function of t.

50. Chemical Reaction

In a chemical reaction, one unit of compound Y and one unit of compound Z are converted into a single unit of compound X. Let x be the amount of compound X formed. The rate of formation of X is proportional to the product of the amounts of unconverted compounds Y and Z. So, $dx/dt = k(y_0 - x)(z_0 - x)$, where y_0 and z_0 are the initial amounts of compounds Y and Z. From this equation, you obtain

$$\int \frac{1}{(y_0 - x)(z_0 - x)}\, dx = \int k\, dt.$$

(a) Solve for x as a function of t.

(b) Use the result of part (a) to find x as $t \to \infty$ for (1) $y_0 < z_0$, (2) $y_0 > z_0$, and (3) $y_0 = z_0$.

51. Using Two Methods Evaluate

$$\int_0^1 \frac{x}{1 + x^4}\, dx$$

in two different ways, one of which is partial fractions.

PUTNAM EXAM CHALLENGE

52. Prove $\dfrac{22}{7} - \pi = \displaystyle\int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\, dx.$

53. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

8.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Improper Integrals** Describe two ways for an integral to be improper.
2. **Improper Integrals** What does it mean for an improper integral to converge?
3. **Indefinite Integration Limits** Explain how to evaluate an improper integral that has an infinite limit of integration.
4. **Finding Values** For what values of a is each integral improper? Explain.

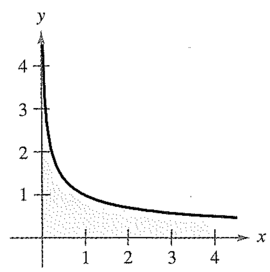
   (a) $\displaystyle\int_a^5 \frac{1}{x+2}\,dx$ (b) $\displaystyle\int_a^4 \frac{x}{3x-1}\,dx$

Determining Whether an Integral Is Improper In Exercises 5–12, decide whether the integral is improper. Explain your reasoning.

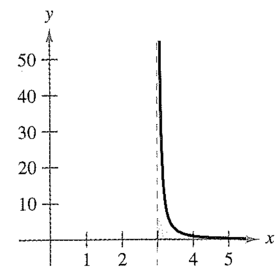
5. $\displaystyle\int_0^1 \frac{dx}{5x-3}$
6. $\displaystyle\int_1^2 \frac{dx}{x^3}$
7. $\displaystyle\int_0^1 \frac{2x-5}{x^2-5x+6}\,dx$
8. $\displaystyle\int_1^\infty \ln x^2\,dx$
9. $\displaystyle\int_0^2 e^{-x}\,dx$
10. $\displaystyle\int_0^\infty \cos x\,dx$
11. $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{4+x^2}\,dx$
12. $\displaystyle\int_0^{\pi/4} \csc x\,dx$

Evaluating an Improper Integral In Exercises 13–16, explain why the integral is improper and determine whether it diverges or converges. Evaluate the integral if it converges.

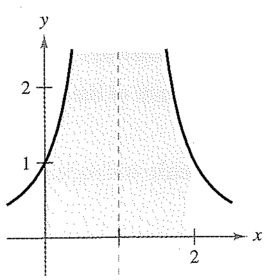
13. $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$



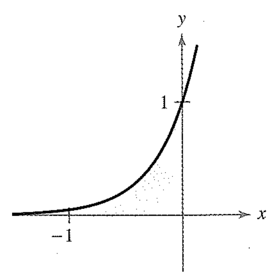
14. $\displaystyle\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx$



15. $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$



16. $\displaystyle\int_{-\infty}^0 e^{3x}\,dx$



Evaluating an Improper Integral In Exercises 17–32, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

17. $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$
18. $\displaystyle\int_3^\infty \frac{1}{(x-1)^4}\,dx$
19. $\displaystyle\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx$
20. $\displaystyle\int_1^\infty \frac{4}{\sqrt[4]{x}}\,dx$
21. $\displaystyle\int_0^\infty e^{x/3}\,dx$
22. $\displaystyle\int_{-\infty}^0 xe^{-4x}\,dx$
23. $\displaystyle\int_0^\infty x^2e^{-x}\,dx$
24. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$
25. $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3}\,dx$
26. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$
27. $\displaystyle\int_{-\infty}^\infty \frac{4}{16+x^2}\,dx$
28. $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^2}\,dx$
29. $\displaystyle\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx$
30. $\displaystyle\int_0^\infty \frac{e^x}{1+e^x}\,dx$
31. $\displaystyle\int_0^\infty \cos \pi x\,dx$
32. $\displaystyle\int_0^\infty \sin\frac{x}{2}\,dx$

Evaluating an Improper Integral In Exercises 33–48, determine whether the improper integral diverges or converges. Evaluate the integral if it converges, and check your results with the results obtained by using the integration capabilities of a graphing utility.

33. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$
34. $\displaystyle\int_0^5 \frac{10}{x}\,dx$
35. $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$
36. $\displaystyle\int_0^8 \frac{3}{\sqrt{8-x}}\,dx$
37. $\displaystyle\int_0^1 x\ln x\,dx$
38. $\displaystyle\int_0^e \ln x^2\,dx$
39. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$
40. $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta$
41. $\displaystyle\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx$
42. $\displaystyle\int_3^6 \frac{1}{\sqrt{36-x^2}}\,dx$
43. $\displaystyle\int_3^5 \frac{1}{\sqrt{x^2-9}}\,dx$
44. $\displaystyle\int_0^5 \frac{1}{25-x^2}\,dx$
45. $\displaystyle\int_3^\infty \frac{1}{x\sqrt{x^2-9}}\,dx$
46. $\displaystyle\int_4^\infty \frac{\sqrt{x^2-16}}{x^2}\,dx$
47. $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx$
48. $\displaystyle\int_1^\infty \frac{1}{x\ln x}\,dx$

Finding Values **In Exercises 49 and 50, determine all values of p for which the improper integral converges.**

49. $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$

50. $\displaystyle\int_0^{1} \frac{1}{x^p}\,dx$

51. Mathematical Induction Use mathematical induction to verify that the following integral converges for any positive integer n.

$$\int_0^{\infty} x^n e^{-x}\,dx$$

52. Comparison Test for Improper Integrals In some cases, it is impossible to find the exact value of an improper integral, but it is important to determine whether the integral converges or diverges. Suppose the functions f and g are continuous and $0 \le g(x) \le f(x)$ on the interval $[a, \infty)$. It can be shown that if $\int_a^{\infty} f(x)\,dx$ converges, then $\int_a^{\infty} g(x)\,dx$ also converges, and if $\int_a^{\infty} g(x)\,dx$ diverges, then $\int_a^{\infty} f(x)\,dx$ also diverges. This is known as the Comparison Test for improper integrals.

(a) Use the Comparison Test to determine whether $\int_1^{\infty} e^{-x^2}\,dx$ converges or diverges. (*Hint:* Use the fact that $e^{-x^2} \le e^{-x}$ for $x \ge 1$.)

(b) Use the Comparison Test to determine whether $\displaystyle\int_1^{\infty} \frac{1}{x^5+1}\,dx$ converges or diverges. (*Hint:* Use the fact that $\dfrac{1}{x^5+1} \le \dfrac{1}{x^5}$ for $x \ge 1$.)

Convergence or Divergence **In Exercises 53–60, use the results of Exercises 49–52 to determine whether the improper integral converges or diverges.**

53. $\displaystyle\int_0^{1} \frac{1}{\sqrt[6]{x}}\,dx$

54. $\displaystyle\int_0^{1} \frac{1}{x^9}\,dx$

55. $\displaystyle\int_1^{\infty} \frac{1}{x^5}\,dx$

56. $\displaystyle\int_0^{\infty} x^4 e^{-x}\,dx$

57. $\displaystyle\int_1^{\infty} \frac{1}{x^2+5}\,dx$

58. $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x-1}}\,dx$

59. $\displaystyle\int_1^{\infty} \frac{1-\sin x}{x^2}\,dx$

60. $\displaystyle\int_0^{\infty} \frac{1}{e^x+x}\,dx$

EXPLORING CONCEPTS

61. Improper Integral Explain why $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\,dx \ne 0$.

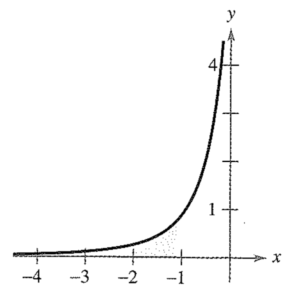
62. Improper Integral Consider the integral

$$\int_0^{3} \frac{10}{x^2-2x}\,dx.$$

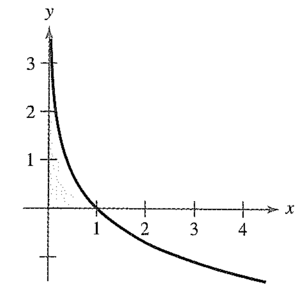
To determine the convergence or divergence of the integral, how many improper integrals must be analyzed? What must be true of each of these integrals for the given integral to converge?

Area **In Exercises 63–66, find the area of the unbounded shaded region.**

63. $y = -\dfrac{7}{(x-1)^3}$, $-\infty < x \le -1$

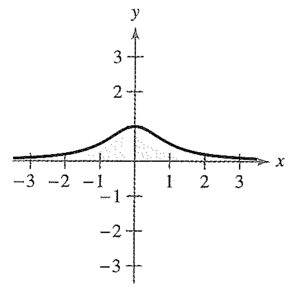


64. $y = -\ln x$



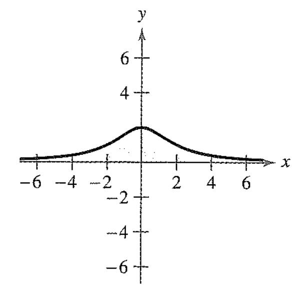
65. Witch of Agnesi:

$$y = \frac{1}{x^2+1}$$



66. Witch of Agnesi:

$$y = \frac{8}{x^2+4}$$



Area and Volume **In Exercises 67 and 68, consider the region satisfying the inequalities. (a) Find the area of the region. (b) Find the volume of the solid generated by revolving the region about the x-axis. (c) Find the volume of the solid generated by revolving the region about the y-axis.**

67. $y \le e^{-x}$, $y \ge 0$, $x \ge 0$

68. $y \le \dfrac{1}{x^2}$, $y \ge 0$, $x \ge 1$

69. Arc Length Find the arc length of the graph of $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$.

70. Surface Area Find the area of the surface formed by revolving the graph of $y = 2e^{-x}$ on the interval $[0, \infty)$ about the x-axis.

Propulsion **In Exercises 71 and 72, use the weight of the rocket to answer each question. (Use 4000 miles as the radius of Earth and do not consider the effect of air resistance.)**

(a) How much work is required to propel the rocket an unlimited distance away from Earth's surface?

(b) How far has the rocket traveled when half of the total work has occurred?

71. 5-metric-ton rocket

72. 10-metric-ton rocket

Probability A nonnegative function f is called a *probability density function* if

$$\int_{-\infty}^{\infty} f(t)\,dt = 1.$$

The probability that x lies between a and b is given by

$$P(a \le x \le b) = \int_a^b f(t)\,dt.$$

In Exercises 73 and 74, (a) show that the nonnegative function is a probability density function, and (b) find $P(0 \le x \le 6)$.

73. $f(t) = \begin{cases} \frac{1}{9}e^{-t/9}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

74. $f(t) = \begin{cases} \frac{5}{6}e^{-5t/6}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

75. Normal Probability The mean height of American men between 20 and 29 years old is 69 inches, and the standard deviation is 3 inches. A 20- to 29-year-old man is chosen at random from the population. The probability that he is 6 feet tall or taller is

$$P(72 \le x < \infty) = \int_{72}^{\infty} \frac{1}{3\sqrt{2\pi}} e^{-(x-69)^2/18}\,dx.$$

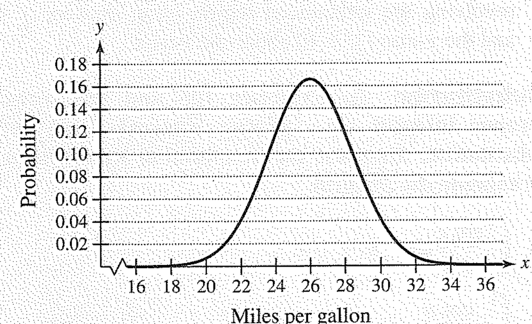
(*Source: National Center for Health Statistics*)

(a) Use a graphing utility to graph the integrand. Use the graphing utility to convince yourself that the area between the x-axis and the integrand is 1.

(b) Use a graphing utility to approximate $P(72 \le x < \infty)$.

(c) Approximate $0.5 - P(69 \le x \le 72)$ using a graphing utility. Use the graph in part (a) to explain why this result is the same as the answer in part (b).

76. HOW DO YOU SEE IT? The graph shows the probability density function for a car brand that has a mean fuel efficiency of 26 miles per gallon and a standard deviation of 2.4 miles per gallon.



(a) Which is greater, the probability of choosing a car at random that gets between 26 and 28 miles per gallon or the probability of choosing a car at random that gets between 22 and 24 miles per gallon?

(b) Which is greater, the probability of choosing a car at random that gets between 20 and 22 miles per gallon or the probability of choosing a car at random that gets at least 30 miles per gallon?

Capitalized Cost In Exercises 77 and 78, find the capitalized cost C of an asset (a) for $n = 5$ years, (b) for $n = 10$ years, and (c) forever. The capitalized cost is given by

$$C = C_0 + \int_0^n c(t)e^{-rt}\,dt$$

where C_0 is the original investment, t is the time in years, r is the annual interest rate compounded continuously, and $c(t)$ is the annual cost of maintenance.

77. $C_0 = \$700{,}000$
$c(t) = \$25{,}000$
$r = 0.06$

78. $C_0 = \$700{,}000$
$c(t) = \$25{,}000(1 + 0.08t)$
$r = 0.06$

79. Electromagnetic Theory The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k} \int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}}\,dx$$

where N, I, r, k, and c are constants. Find P.

80. Gravitational Force A "semi-infinite" uniform rod occupies the nonnegative x-axis. The rod has a linear density δ, which means that a segment of length dx has a mass of $\delta\,dx$. A particle of mass M is located at the point $(-a, 0)$. The gravitational force F that the rod exerts on the mass is given by

$$F = \int_0^{\infty} \frac{GM\delta}{(a + x)^2}\,dx$$

where G is the gravitational constant. Find F.

True or False? In Exercises 81–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

81. If f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.

82. If f is continuous on $[0, \infty)$ and $\int_0^{\infty} f(x)\,dx$ diverges, then $\lim_{x\to\infty} f(x) \ne 0$.

83. If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then

$$\int_0^{\infty} f'(x)\,dx = -f(0).$$

84. If the graph of f is symmetric with respect to the origin or the y-axis, then $\int_0^{\infty} f(x)\,dx$ converges if and only if $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

85. $\int_0^{\infty} e^{ax}\,dx$ converges for $a < 0$.

86. If $\lim_{x\to\infty} f(x) = L$, then $\int_0^{\infty} f(x)\,dx$ converges.

87. Comparing Integrals

(a) Show that $\int_{-\infty}^{\infty} \sin x\,dx$ diverges.

(b) Show that $\lim_{a\to\infty} \int_{-a}^{a} \sin x\,dx = 0$.

(c) What do parts (a) and (b) show about the definition of improper integrals?

9.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Recursively Defined Sequence** What does it mean for a sequence to be defined recursively?
2. **Properties of Limits of Sequences** What is the value of L?
$$\lim_{n\to\infty} a_n = L, \quad \lim_{n\to\infty} b_n = 8, \quad \text{and} \quad \lim_{n\to\infty} (a_n b_n) = 24$$
3. **Rate of Increase** Which function grows faster as n approaches infinity? Explain.
$$f(n) = 7^n \qquad g(n) = (n-1)!$$
4. **Bounded Monotonic Sequences** A sequence $\{a_n\}$ is bounded below and nonincreasing. Does $\{a_n\}$ converge or diverge? Use a graph to support your conclusion.

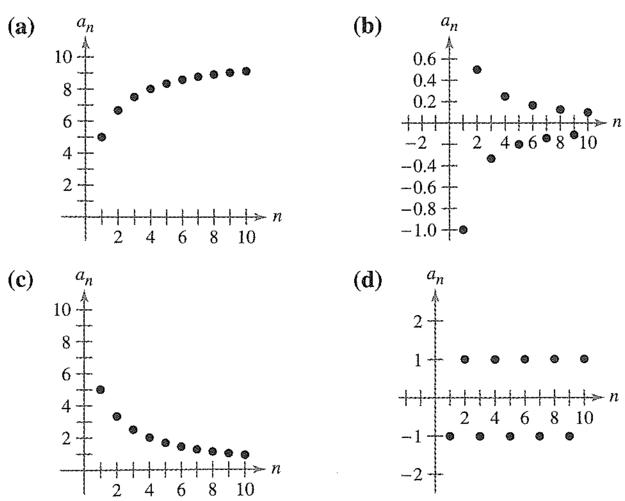
Writing the Terms of a Sequence In Exercises 5–10, write the first five terms of the sequence with the given nth term.

5. $a_n = 3^n$
6. $a_n = \left(-\frac{2}{5}\right)^n$
7. $a_n = \sin \frac{n\pi}{2}$
8. $a_n = \frac{3n}{n+4}$
9. $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$
10. $a_n = 2 + \frac{2}{n} - \frac{1}{n^2}$

Writing the Terms of a Sequence In Exercises 11 and 12, write the first five terms of the recursively defined sequence.

11. $a_1 = 3, a_{k+1} = 2(a_k - 1)$
12. $a_1 = 6, a_{k+1} = \frac{1}{3}a_k^2$

Matching In Exercises 13–16, match the sequence with the given nth term with its graph. [The graphs are labeled (a), (b), (c), and (d).]



13. $a_n = \frac{10}{n+1}$
14. $a_n = \frac{10n}{n+1}$
15. $a_n = (-1)^n$
16. $a_n = \frac{(-1)^n}{n}$

Simplifying Factorials In Exercises 17–20, simplify the ratio of factorials.

17. $\frac{(n+1)!}{(n-1)!}$
18. $\frac{(3n+1)!}{(3n)!}$
19. $\frac{n!}{(n-3)!}$
20. $\frac{(4n+1)!}{(4n+3)!}$

Finding the Limit of a Sequence In Exercises 21–24, find the limit of the sequence with the given nth term.

21. $a_n = \frac{n+1}{n}$
22. $a_n = 6 + \frac{2}{n^2}$
23. $a_n = \frac{2n}{\sqrt{n^2+1}}$
24. $a_n = \cos \frac{2}{n}$

Finding the Limit of a Sequence In Exercises 25–28, use a graphing utility to graph the first 10 terms of the sequence with the given nth term. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.

25. $a_n = \frac{4n+1}{n}$
26. $a_n = \frac{1}{n^{3/2}}$
27. $a_n = \sin \frac{n\pi}{2}$
28. $a_n = 2 - \frac{1}{4^n}$

Determining Convergence or Divergence In Exercises 29–44, determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

29. $a_n = \frac{5}{n+2}$
30. $a_n = n - \frac{1}{n!}$
31. $a_n = (-1)^n\left(\frac{n}{n+1}\right)$
32. $a_n = \frac{1+(-1)^n}{n^2}$
33. $a_n = \frac{3n+\sqrt{n}}{4n}$
34. $a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$
35. $a_n = \frac{\ln(n^3)}{2n}$
36. $a_n = \frac{e^n}{4^n}$
37. $a_n = \frac{(n+1)!}{n!}$
38. $a_n = \frac{(n-2)!}{n!}$
39. $a_n = \frac{n^p}{e^n}, \ p > 0$
40. $a_n = n \sin \frac{1}{n}$
41. $a_n = 2^{1/n}$
42. $a_n = -3^{-n}$
43. $a_n = \frac{\sin n}{n}$
44. $a_n = \frac{\cos 2n}{3^n}$

Finding the nth Term of a Sequence In Exercises 45–52, write an expression for the nth term of the sequence and then determine whether the sequence you have chosen converges or diverges. (There is more than one correct answer.)

45. $2, 8, 14, 20, \ldots$
46. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$
47. $-2, 1, 6, 13, 22, \ldots$
48. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$

49. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

50. $2, 24, 720, 40{,}320, 3{,}628{,}800, \ldots$

51. $2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

52. $\dfrac{1}{2 \cdot 3}, \dfrac{2}{3 \cdot 4}, \dfrac{3}{4 \cdot 5}, \dfrac{4}{5 \cdot 6}, \ldots$

Monotonic and Bounded Sequences **In Exercises 53–60, determine whether the sequence with the given nth term is monotonic and whether it is bounded. Use a graphing utility to confirm your results.**

53. $a_n = 4 - \dfrac{1}{n}$

54. $a_n = \dfrac{3n}{n+2}$

55. $a_n = ne^{-n/2}$

56. $a_n = \left(-\dfrac{2}{3}\right)^n$

57. $a_n = \left(\dfrac{2}{3}\right)^n$

58. $a_n = \left(\dfrac{3}{2}\right)^n$

59. $a_n = \sin \dfrac{n\pi}{6}$

60. $a_n = \dfrac{\cos n}{n}$

Using a Theorem **In Exercises 61–64, (a) use Theorem 9.5 to show that the sequence with the given nth term converges and (b) use a graphing utility to graph the first 10 terms of the sequence and find its limit.**

61. $a_n = 7 + \dfrac{1}{n}$

62. $a_n = 5 - \dfrac{2}{n}$

63. $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)$

64. $a_n = 2 + \dfrac{1}{5^n}$

65. Compound Interest Consider the sequence $\{A_n\}$ whose nth term is given by

$$A_n = P\left(1 + \frac{r}{12}\right)^n$$

where P is the principal, A_n is the account balance after n months, and r is the interest rate compounded annually.

(a) Is $\{A_n\}$ a convergent sequence? Explain.

(b) Find the first 10 terms of the sequence when $P = \$10{,}000$ and $r = 0.055$.

66. Compound Interest A deposit of \$100 is made in an account at the beginning of each month at an annual interest rate of 3% compounded monthly. The balance in the account after n months is $A_n = 100(401)[(1.0025)^n - 1]$.

(a) Compute the first six terms of the sequence $\{A_n\}$.

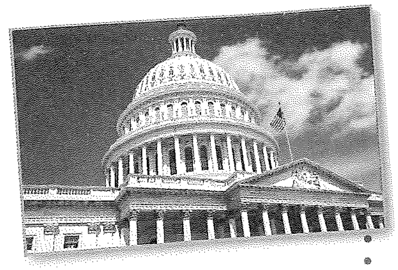
(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in the account after 20 years by computing the 240th term of the sequence.

67. Inflation When the rate of inflation is $4\frac{1}{2}\%$ per year and the average price of a car is currently \$25,000, the average price after n years is $P_n = \$25{,}000(1.045)^n$. Compute the average prices for the next 5 years.

68. Government Expenditures

A government program that currently costs taxpayers \$4.5 billion per year is cut back by 6% per year.



(a) Write an expression for the amount budgeted for this program after n years.

(b) Compute the budgets for the first 4 years.

(c) Determine the convergence or divergence of the sequence of reduced budgets. If the sequence converges, find its limit.

EXPLORING CONCEPTS

69. Writing a Sequence Give an example of a sequence satisfying the condition.

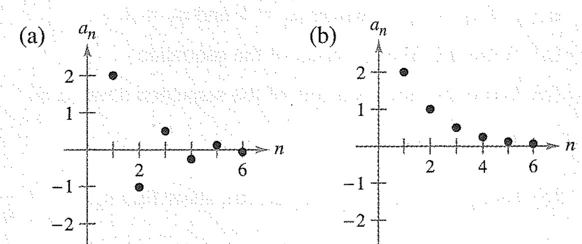
(a) A monotonically increasing sequence that converges to 10

(b) A sequence that converges to $\frac{3}{4}$

70. Writing a Sequence Give an example of a bounded sequence that has a limit and an example of a bounded sequence that does not have a limit.

71. Monotonic Sequence Let $\{a_n\}$ be a monotonic sequence such that $a_n \le 1$. Discuss the convergence of $\{a_n\}$. When $\{a_n\}$ converges, what can you conclude about its limit?

72. HOW DO YOU SEE IT? The graphs of two sequences are shown in the figures. Which graph represents the sequence with alternating signs? Explain.



73. Using a Sequence Compute the first six terms of the sequence $\{a_n\} = \{\sqrt[n]{n}\}$. If the sequence converges, find its limit.

74. Using a Sequence Compute the first six terms of the sequence

$$\{a_n\} = \left\{\sqrt{n}\ln\left(1 + \frac{1}{n}\right)\right\}.$$

If the sequence converges, find its limit.

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9.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Sequence and Series** Describe the difference between $\lim_{n\to\infty} a_n = 5$ and $\sum_{n=1}^{\infty} a_n = 5$.
2. **Geometric Series** Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.
3. **Limit of the *n*th term of a Series** The limit of the nth term of a series converges to 0. What can you conclude about the convergence or divergence of the series?
4. **Limit of the *n*th Term of a Series** The limit of the nth term of a series does not converge to 0. What can you conclude about the convergence or divergence of the series?

Finding Partial Sums **In Exercises 5–10, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5.**

5. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$
6. $\frac{1}{2\cdot 3} + \frac{2}{3\cdot 4} + \frac{3}{4\cdot 5} + \frac{4}{5\cdot 6} + \frac{5}{6\cdot 7} + \cdots$
7. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$
8. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$
9. $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$
10. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Verifying Divergence **In Exercises 11–18, verify that the infinite series diverges.**

11. $\sum_{n=0}^{\infty} 5\left(\frac{5}{2}\right)^n$
12. $\sum_{n=0}^{\infty} 4(-1.05)^n$
13. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
14. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$
15. $\sum_{n=1}^{\infty} \frac{n^3+1}{n^3+n^2}$
16. $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^2+1}}$
17. $\sum_{n=1}^{\infty} \frac{4^n+3}{4^{n+1}}$
18. $\sum_{n=1}^{\infty} \frac{(n+1)!}{5n!}$

Verifying Convergence **In Exercises 19–24, verify that the infinite series converges.**

19. $\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$
20. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$
21. $\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$
22. $\sum_{n=0}^{\infty} (-0.2)^n = 1 - 0.2 + 0.04 - 0.008 + \cdots$
23. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (*Hint:* Use partial fractions.)
24. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (*Hint:* Use partial fractions.)

Numerical, Graphical, and Analytic Analysis **In Exercises 25–28, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.**

n	5	10	20	50	100
S_n					

25. $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$
26. $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$
27. $\sum_{n=1}^{\infty} 2(0.9)^{n-1}$
28. $\sum_{n=1}^{\infty} 10\left(-\frac{1}{4}\right)^{n-1}$

Finding the Sum of a Convergent Series **In Exercises 29–38, find the sum of the convergent series.**

29. $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$
30. $\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$
31. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$
32. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$
33. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$
34. $9 - 3 + 1 - \frac{1}{3} + \cdots$
35. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$
36. $\sum_{n=0}^{\infty} [(0.3)^n + (0.8)^n]$
37. $\sum_{n=1}^{\infty} (\sin 1)^n$
38. $\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2}$

Using a Geometric Series **In Exercises 39–44, (a) write the repeating decimal as a geometric series and (b) write the sum of the series as the ratio of two integers.**

39. $0.\overline{4}$
40. $0.\overline{63}$
41. $0.\overline{12}$
42. $0.\overline{01}$
43. $0.0\overline{75}$
44. $0.2\overline{15}$

Determining Convergence or Divergence **In Exercises 45–58, determine the convergence or divergence of the series.**

45. $\sum_{n=0}^{\infty} (1.075)^n$
46. $\sum_{n=0}^{\infty} \frac{6^n}{n+1}$
47. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$
48. $\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$
49. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$
50. $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$
51. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$
52. $\sum_{n=0}^{\infty} \frac{7}{5^n}$

9.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Integral Test** What conditions have to be satisfied to use the Integral Test?
2. **p-Series** Determine whether each series is a p-series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.4}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{-2}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3}}$

Using the Integral Test **In Exercises 3–22, confirm that the Integral Test can be applied to the series. Then use the Integral Test to determine the convergence or divergence of the series.**

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+5}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$
6. $\displaystyle\sum_{n=1}^{\infty} 3^{-n}$
7. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$
8. $\displaystyle\sum_{n=1}^{\infty} ne^{-n/2}$
9. $\dfrac{\ln 2}{2} + \dfrac{\ln 3}{3} + \dfrac{\ln 4}{4} + \dfrac{\ln 5}{5} + \dfrac{\ln 6}{6} + \cdots$
10. $\dfrac{\ln 2}{\sqrt{2}} + \dfrac{\ln 3}{\sqrt{3}} + \dfrac{\ln 4}{\sqrt{4}} + \dfrac{\ln 5}{\sqrt{5}} + \dfrac{\ln 6}{\sqrt{6}} + \cdots$
11. $\dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{9} + \dfrac{1}{11} + \cdots$
12. $\dfrac{1}{4} + \dfrac{2}{7} + \dfrac{3}{12} + \dfrac{4}{19} + \dfrac{5}{28} + \cdots$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$
14. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$
15. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
16. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$
18. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$
19. $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$
20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}+9}$
21. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+2n^2+1}$

Using the Integral Test **In Exercises 23 and 24, use the Integral Test to determine the convergence or divergence of the series, where k is a positive integer.**

23. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+c}$
24. $\displaystyle\sum_{n=1}^{\infty} n^k e^{-n}$

Conditions of the Integral Test **In Exercises 25–28, explain why the Integral Test does not apply to the series.**

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
26. $\displaystyle\sum_{n=1}^{\infty} e^{-n}\cos n$
27. $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$
28. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$

Using the Integral Test **In Exercises 29–32, use the Integral Test to determine the convergence or divergence of the p-series.**

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^7}$
30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$
31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$
32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$

Using a p-Series **In Exercises 33–38, use Theorem 9.11 to determine the convergence or divergence of the p-series.**

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$
34. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$
35. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots$
36. $1 + \dfrac{1}{\sqrt[3]{4}} + \dfrac{1}{\sqrt[3]{9}} + \dfrac{1}{\sqrt[3]{16}} + \dfrac{1}{\sqrt[3]{25}} + \cdots$
37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.03}}$
38. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

39. **Numerical and Graphical Analysis** Use a graphing utility to find the indicated partial sum S_n and complete the table. Then use a graphing utility to graph the first 10 terms of the sequence of partial sums. For each series, compare the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	20	50	100
S_n					

(a) $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^{n-1} = \frac{15}{4}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

40. **Numerical Reasoning** Because the harmonic series diverges, it follows that for any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=1}^{N} \frac{1}{n} > M.$$

(a) Use a graphing utility to complete the table.

M	2	4	6	8
N				

(b) As the real number M increases in equal increments, does the number N increase in equal increments? Explain.

EXPLORING CONCEPTS

41. Think About It Without performing any calculations, determine whether the following series converges. Explain.

$$\frac{1}{10{,}000} + \frac{1}{10{,}001} + \frac{1}{10{,}002} + \cdots$$

42. Using a Function Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Use a graph to rank the following quantities in decreasing order. Explain your reasoning.

(a) $\displaystyle\sum_{n=2}^{7} a_n$ (b) $\displaystyle\int_1^7 f(x)\,dx$ (c) $\displaystyle\sum_{n=1}^{6} a_n$

43. Using a Series Use a graph to show that the inequality is true. What can you conclude about the convergence or divergence of the series? Explain.

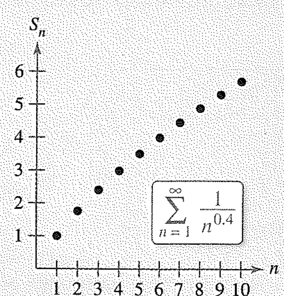
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$

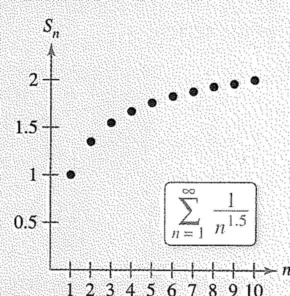
(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$

44. HOW DO YOU SEE IT? The graphs show the sequences of partial sums of the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.4}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1.5}}.$$

Using Theorem 9.11, the first series diverges and the second series converges. Explain how the graphs show this.





Finding Values **In Exercises 45–50, find the positive values of p for which the series converges.**

45. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

46. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

48. $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

49. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{p}\right)^n$

50. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^p}$

51. Proof Let f be a positive, continuous, and decreasing function for $x \geq 1$, such that $a_n = f(n)$. Prove that if the series

$$\sum_{n=1}^{\infty} a_n$$

converges to S, then the remainder $R_N = S - S_N$ is bounded by

$$0 \leq R_N \leq \int_N^{\infty} f(x)\,dx.$$

52. Using a Remainder Show that the result of Exercise 51 can be written as

$$\sum_{n=1}^{N} a_n \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{N} a_n + \int_N^{\infty} f(x)\,dx.$$

Approximating a Sum **In Exercises 53–58, use the result of Exercise 51 to approximate the sum of the convergent series using the indicated number of terms. Include an estimate of the maximum error for your approximation.**

53. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$, three terms

54. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$, six terms

55. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, eight terms

56. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3}$, ten terms

57. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$, four terms

58. $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$, five terms

Finding a Value **In Exercises 59–62, use the result of Exercise 51 to find N such that $R_N \leq 0.001$ for the convergent series.**

59. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

60. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

61. $\displaystyle\sum_{n=1}^{\infty} e^{-n/2}$

62. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

63. Comparing Series

(a) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ converges and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) Compare the first five terms of each series in part (a).

(c) Find $n > 3$ such that $\dfrac{1}{n^{1.1}} < \dfrac{1}{n \ln n}$.

64. Using a p-Series Ten terms are used to approximate a convergent p-series. Therefore, the remainder is a function of p and is

$$0 \leq R_{10}(p) \leq \int_{10}^{\infty} \frac{1}{x^p}\,dx, \quad p > 1.$$

(a) Perform the integration in the inequality.

(b) Use a graphing utility to represent the inequality graphically.

(c) Identify any asymptotes of the remainder function and interpret their meaning.

65. Euler's Constant Let

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(a) Show that $\ln(n + 1) \le S_n \le 1 + \ln n$.

(b) Show that the sequence $\{a_n\} = \{S_n - \ln n\}$ is bounded.

(c) Show that the sequence $\{a_n\}$ is decreasing.

(d) Show that the sequence $\{a_n\}$ converges to a limit γ (called Euler's constant).

(e) Approximate γ using a_{100}.

66. Finding a Sum Find the sum of the series

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right).$$

67. Using a Series Consider the series $\sum_{n=2}^{\infty} x^{\ln n}$.

(a) Determine the convergence or divergence of the series for $x = 1$.

(b) Determine the convergence or divergence of the series for $x = 1/e$.

(c) Find the positive values of x for which the series converges.

68. Riemann Zeta Function The **Riemann zeta function** for real numbers is defined for all x for which the series

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$$

converges. Find the domain of the function.

Review In Exercises 69–80, determine the convergence or divergence of the series.

69. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n - 2}$

70. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

71. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$

72. $\displaystyle 3\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

73. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

74. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{7}{5}\right)^n$

75. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2 + 3}}$

76. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right)$

77. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

78. $\displaystyle\sum_{n=4}^{\infty} \ln\frac{n}{2}$

79. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

80. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^4}$

SECTION PROJECT

The Harmonic Series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is one of the most important series in this chapter. Even though its terms tend to zero as n increases,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

the harmonic series diverges. In other words, even though the terms are getting smaller and smaller, the sum "adds up to infinity."

(a) One way to show that the harmonic series diverges is attributed to James Bernoulli. He grouped the terms of the harmonic series as follows:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> \frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{> \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{> \frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{> \frac{1}{2}} + \cdots$$

Write a short paragraph explaining how you can use this grouping to show that the harmonic series diverges.

(b) Use the proof of the Integral Test, Theorem 9.10, to show that

$$\ln(n + 1) \le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le 1 + \ln n.$$

(c) Use part (b) to determine how many terms M you would need so that

$$\sum_{n=1}^{M} \frac{1}{n} > 50.$$

(d) Show that the sum of the first million terms of the harmonic series is less than 15.

(e) Show that the following inequalities are valid.

$$\ln\frac{21}{10} \le \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{20} \le \ln\frac{20}{9}$$

$$\ln\frac{201}{100} \le \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{200} \le \ln\frac{200}{99}$$

(f) Use the inequalities in part (e) to find the limit

$$\lim_{m\to\infty} \sum_{n=m}^{2m} \frac{1}{n}.$$

9.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Direct Comparison Test** You want to compare the series $\Sigma\, a_n$ and $\Sigma\, b_n$, where $a_n > 0$, $b_n > 0$, and $\Sigma\, b_n$ converges. For $1 \le n \le 5$, $a_n > b_n$, and for $n \ge 6$, $a_n < b_n$. Explain whether the Direct Comparison Test can be used to compare the two series.

2. **Limit Comparison Test** When using the Limit Comparison Test, describe in your own words how to choose a series for comparison.

Graphical Analysis In Exercises 3 and 4, the figures show the graphs of the first 10 terms, and the graphs of the first 10 terms of the sequence of partial sums, of each series.

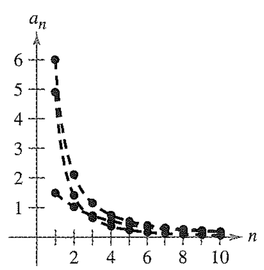
(a) Identify the series in each figure.

(b) Which series is a p-series? Does it converge or diverge?

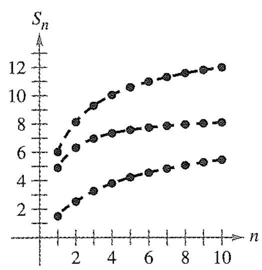
(c) For the series that are not p-series, how do the magnitudes of the terms compare with the magnitudes of the terms of the p-series? What conclusion can you draw about the convergence or divergence of the series?

(d) Explain the relationship between the magnitudes of the terms of the series and the magnitudes of the terms of the partial sums.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$, $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}}$

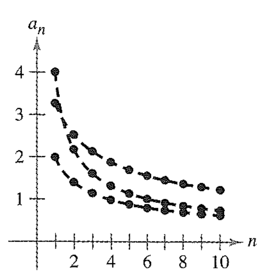


Graphs of terms

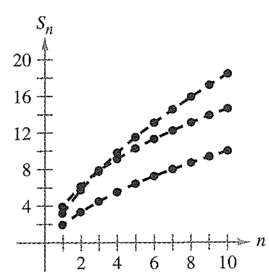


Graphs of partial sums

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$, $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt{n + 0.5}}$



Graphs of terms



Graphs of partial sums

Using the Direct Comparison Test In Exercises 5–16, use the Direct Comparison Test to determine the convergence or divergence of the series.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n - 1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n + 3}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n + 1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$

11. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n} - 1}$

13. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{6^n + n}{5^n - 1}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n + 2}{\sqrt{n}}$

Using the Limit Comparison Test In Exercises 17–26, use the Limit Comparison Test to determine the convergence or divergence of the series.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n + 1}$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 1}{5^n + 1}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 + 4)}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n + 1)2^{n-1}}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$, $k > 2$

26. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

Determining Convergence or Divergence In Exercises 27–34, test for convergence or divergence, using each test at least once. Identify which test was used.

(a) nth-Term Test
(b) Geometric Series Test
(c) p-Series Test
(d) Telescoping Series Test
(e) Integral Test
(f) Direct Comparison Test
(g) Limit Comparison Test

27. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

28. $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n + 1}$

30. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^3 - 8}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n - 2}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n + 1} - \frac{1}{n + 2}\right)$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n + 3)}$

9.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Alternating Series** An alternating series does not meet the first condition of the Alternating Series Test. What can you conclude about the convergence or divergence of the series? Explain.
2. **Alternating Series Remainder** What is the remainder of a convergent alternating series whose sum is approximated by the first N terms?
3. **Absolute and Conditional Convergence** In your own words, describe the difference between absolute and conditional convergence of an alternating series.
4. **Rearrangement of Series** Does rearranging the terms of a convergent series change the sum of the series? Explain.

Numerical and Graphical Analysis **In Exercises 5–8, explore the Alternating Series Remainder.**

(a) Use a graphing utility to find the indicated partial sum S_n and complete the table.

n	1	2	3	4	5	6	7	8	9	10
S_n										

(b) Use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum.

(c) What pattern exists between the plot of the successive points in part (b) relative to the horizontal line representing the sum of the series? Do the distances between the successive points and the horizontal line increase or decrease?

(d) Discuss the relationship between the answers in part (c) and the Alternating Series Remainder as given in Theorem 9.15.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin 1$

Determining Convergence or Divergence **In Exercises 9–30, determine the convergence or divergence of the series.**

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3n+2}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{3^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{e^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}(5n-1)}{4n+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} n}{\ln(n+1)}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

21. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1\cdot 3\cdot 5 \cdots (2n-1)}$

28. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{1\cdot 4\cdot 7 \cdots (3n-2)}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n - e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1}\operatorname{csch} n$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1}\operatorname{sech} n$

Approximating the Sum of an Alternating Series **In Exercises 31–34, approximate the sum of the series by using the first six terms. (See Example 4.)**

31. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n} 5}{n!}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{\ln(n+1)}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2}{n^3}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$

Finding the Number of Terms **In Exercises 35–40, use Theorem 9.15 to determine the number of terms required to approximate the sum of the series with an error of less than 0.001.**

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

39. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

40. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!}$

Determining Absolute and Conditional Convergence **In Exercises 41–58, determine whether the series converges absolutely or conditionally, or diverges.**

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

45. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

46. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$

48. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{5n+1}$

49. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

50. $\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$

51. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 5}$

52. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

53. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

54. $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

55. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

56. $\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$

57. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^2}$

58. $\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n}$

EXPLORING CONCEPTS

59. **Alternating Series** Determine whether S_{50} is an underestimate or an overestimate of the sum of the alternating series below. Explain.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

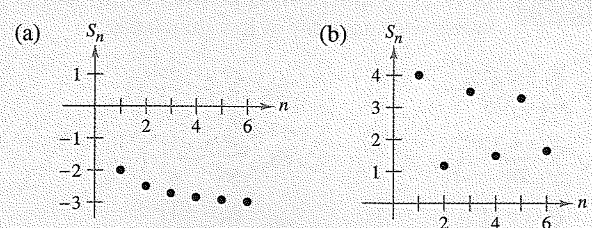
60. **Alternating Series** Give an example of convergent alternating series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma\, a_n b_n$ diverges.

61. **Think About It** Do you agree with the following statements? Why or why not?

(a) If both $\Sigma\, a_n$ and $\Sigma\,(-a_n)$ converge, then $\Sigma\, |a_n|$ converges.

(b) If $\Sigma\, a_n$ diverges, then $\Sigma\, |a_n|$ diverges.

62. **HOW DO YOU SEE IT?** The graphs of the sequences of partial sums of two series are shown in the figures. Which graph represents the partial sums of an alternating series? Explain.



Finding Values **In Exercises 63 and 64, find the values of p for which the series converges.**

63. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n^p}\right)$

64. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+p}\right)$

65. **Proof** Prove that if $\Sigma\, |a_n|$ converges, then $\Sigma\, a_n^2$ converges. Is the converse true? If not, give an example that shows it is false.

66. **Finding a Series** Use the result of Exercise 63 to give an example of an alternating p-series that converges but whose corresponding p-series diverges.

67. **Finding a Series** Give an example of a series that demonstrates the statement you proved in Exercise 65.

68. **Finding Values** Find all values of x for which the series $\Sigma\,(x^n/n)$ (a) converges absolutely and (b) converges conditionally.

Using a Series **In Exercises 69 and 70, use the given series.**

(a) Does the series meet the conditions of Theorem 9.14? Explain why or why not.

(b) Does the series converge? If so, what is the sum?

69. $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} + \cdots$

70. $\sum_{n=1}^{\infty} (-1)^{n+1} a_n,\ a_n = \begin{cases} \frac{1}{\sqrt{n}}, & \text{if } n \text{ is odd} \\ \frac{1}{n^3}, & \text{if } n \text{ is even} \end{cases}$

Review **In Exercises 71–80, determine the convergence or divergence of the series and identify the test used.**

71. $\sum_{n=1}^{\infty} \frac{8}{\sqrt[3]{n}}$

72. $\sum_{n=1}^{\infty} \frac{3n+5}{n^3+2n^2+4}$

73. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

74. $\sum_{n=1}^{\infty} \frac{1}{6^n - 5}$

75. $\sum_{n=1}^{\infty} \left(\frac{9}{8}\right)^n$

76. $\sum_{n=1}^{\infty} \frac{2n^2}{(n+1)^2}$

77. $\sum_{n=1}^{\infty} 100e^{-n/2}$

78. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4}$

79. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{3n^2 - 1}$

80. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

81. **Describing an Error** The following argument, that $0 = 1$, is *incorrect*. Describe the error.

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots \\ &= 1 \end{aligned}$$

PUTNAM EXAM CHALLENGE

82. Assume as known the (true) fact that the alternating harmonic series

(1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$

is convergent, and denote its sum by s. Rearrange the series (1) as follows:

(2) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots.$

Assume as known the (true) fact that the series (2) is also convergent, and denote its sum by S. Denote by s_k, S_k the kth partial sum of the series (1) and (2), respectively. Prove the following statements.

(i) $S_{3n} = s_{4n} + \frac{1}{2}s_{2n}$, (ii) $S \neq s$

9.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

Ratio and Root Tests In Exercises 1–6, what can you conclude about the convergence or divergence of $\Sigma\, a_n$?

1. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0$

2. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$

3. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{3}{2}$

4. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 2$

5. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1$

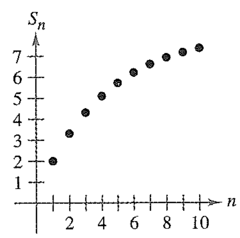
6. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = e$

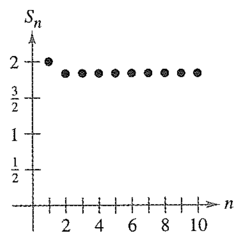
Verifying a Formula In Exercises 7 and 8, verify the formula.

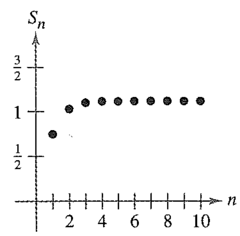
7. $\dfrac{9^{n+1}(n-1)!}{9^n(n-2)!} = 9(n-1)$

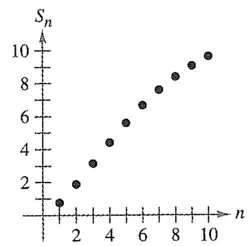
8. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{1}{(2k)(2k-1)}$

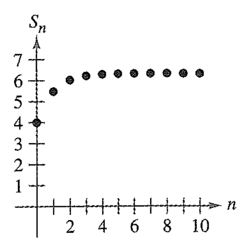
Matching In Exercises 9–14, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

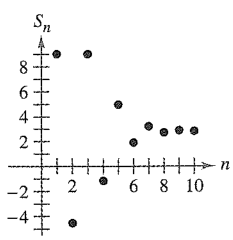
(a)


(b)


(c)


(d)


(e)


(f)


9. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

10. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n\left(\frac{1}{n!}\right)$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!}$

13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$

14. $\displaystyle\sum_{n=0}^{\infty} 4e^{-n}$

Numerical, Graphical, and Analytic Analysis In Exercises 15 and 16, (a) use the Ratio Test to verify that the series converges, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums, (d) use the table to estimate the sum of the series, and (e) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	15	20	25
S_n					

15. $\displaystyle\sum_{n=1}^{\infty} n^3\left(\frac{1}{2}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n!}$

Using the Ratio Test In Exercises 17–38, use the Ratio Test to determine the convergence or divergence of the series. If the Ratio Test is inconclusive, determine the convergence or divergence of the series using other methods.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{8^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n!}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{4^n}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(n+2)!}$

21. $\displaystyle\sum_{n=0}^{\infty} (n+2)\left(\frac{9}{7}\right)^{n+1}$

22. $\displaystyle\sum_{n=1}^{\infty} n^2\left(\frac{5}{6}\right)^n$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{n^5}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^3}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n+1)(n^2+2)}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

31. $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

34. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

35. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{2^n+1}$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

37. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1\cdot 3\cdot 5 \cdots (2n+1)}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n[2\cdot 4\cdot 6 \cdots (2n)]}{2\cdot 5\cdot 8 \cdots (3n-1)}$

Using the Root Test In Exercises 39–52, use the Root Test to determine the convergence or divergence of the series.

39. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$

40. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^n}$

41. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+2}{n+3}\right)^n$

42. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^n$

43. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$

44. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

45. $\displaystyle\sum_{n=1}^{\infty}\left(2\sqrt[n]{n}+1\right)^n$

46. $\displaystyle\sum_{n=0}^{\infty}e^{-3n}$

47. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}$

48. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^n$

49. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

50. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$

51. $\displaystyle\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$

52. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^n}{(n^n)^2}$

Review In Exercises 53–70, determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

53. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}5}{n}$

54. $\displaystyle\sum_{n=1}^{\infty}\frac{100}{n}$

55. $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n\sqrt{n}}$

56. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2\pi}{3}\right)^n$

57. $\displaystyle\sum_{n=1}^{\infty}\frac{5n}{2n-1}$

58. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2n^2+1}$

59. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3^{n-2}}{2^n}$

60. $\displaystyle\sum_{n=1}^{\infty}\frac{10}{3\sqrt{n^3}}$

61. $\displaystyle\sum_{n=1}^{\infty}\frac{10n+3}{n2^n}$

62. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{4n^2-1}$

63. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{3^n}$

64. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$

65. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n7^n}$

66. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$

67. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3^{n-1}}{n!}$

68. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n2^n}$

69. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n}{3\cdot 5\cdot 7\cdots(2n+1)}$

70. $\displaystyle\sum_{n=1}^{\infty}\frac{3\cdot 5\cdot 7\cdots(2n+1)}{18^n(2n-1)n!}$

Identifying Series In Exercises 71–74, identify the two series that are the same.

71. (a) $\displaystyle\sum_{n=1}^{\infty}\frac{n5^n}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{n5^n}{(n+1)!}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)5^{n+1}}{(n+1)!}$

72. (a) $\displaystyle\sum_{n=4}^{\infty}n\left(\frac{3}{4}\right)^n$

(b) $\displaystyle\sum_{n=0}^{\infty}(n+1)\left(\frac{3}{4}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty}n\left(\frac{3}{4}\right)^{n-1}$

73. (a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)!}$

74. (a) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)2^{n-1}}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)2^n}$

Writing an Equivalent Series In Exercises 75 and 76, write an equivalent series with the index of summation beginning at $n = 0$.

75. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{7^n}$

76. $\displaystyle\sum_{n=2}^{\infty}\frac{4^{n+1}}{(n-2)!}$

Using a Recursively Defined Series In Exercises 77–82, the terms of a series $\displaystyle\sum_{n=1}^{\infty}a_n$ are defined recursively. Determine the convergence or divergence of the series. Explain your reasoning.

77. $a_1 = \dfrac{1}{2},\ a_{n+1} = \dfrac{4n-1}{3n+2}a_n$

78. $a_1 = 2,\ a_{n+1} = \dfrac{2n+1}{5n-4}a_n$

79. $a_1 = 1,\ a_{n+1} = \dfrac{\sin n + 1}{\sqrt{n}}a_n$

80. $a_1 = \dfrac{1}{5},\ a_{n+1} = \dfrac{\cos n + 1}{n}a_n$

81. $a_1 = \dfrac{1}{3},\ a_{n+1} = \left(1+\dfrac{1}{n}\right)a_n$

82. $a_1 = \dfrac{1}{4},\ a_{n+1} = \sqrt[n]{a_n}$

9.7 Exercises

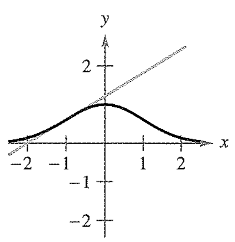
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

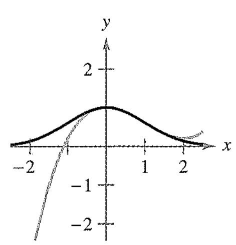
1. **Polynomial Approximation** An elementary function is approximated by a polynomial. In your own words, describe what is meant by saying that the polynomial is *expanded about c* or *centered at c*.
2. **Taylor and Maclaurin Polynomials** How are Taylor polynomials and Maclaurin polynomials related?
3. **Accuracy of a Taylor Polynomial** Describe the accuracy of the nth-degree Taylor polynomial of f centered at c as the distance between c and x increases.
4. **Accuracy of a Taylor Polynomial** In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial increases? Explain your reasoning.

Matching **In Exercises 5–8, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

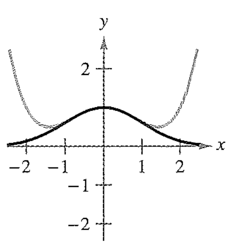
(a)



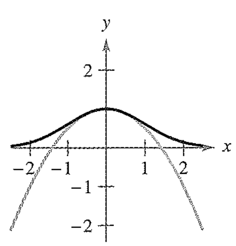
(b)



(c)



(d)



5. $g(x) = -\frac{1}{2}x^2 + 1$
6. $g(x) = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$
7. $g(x) = e^{-1/2}[(x + 1) + 1]$
8. $g(x) = e^{-1/2}\left[\frac{1}{3}(x - 1)^3 - (x - 1) + 1\right]$

Finding a First-Degree Polynomial Approximation **In Exercises 9–12, find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. Use a graphing utility to graph f and P_1.**

9. $f(x) = \dfrac{\sqrt{x}}{4}, \quad c = 4$

10. $f(x) = \dfrac{6}{\sqrt[3]{x}}, \quad c = 8$

11. $f(x) = \sec x, \quad c = \dfrac{\pi}{6}$

12. $f(x) = \tan x, \quad c = \dfrac{\pi}{4}$

Graphical and Numerical Analysis **In Exercises 13 and 14, use a graphing utility to graph f and its second-degree polynomial approximation P_2 at $x = c$. Complete the table comparing the values of f and P_2.**

13. $f(x) = \dfrac{4}{\sqrt{x}}, \quad c = 1$

$P_2(x) = 4 - 2(x - 1) + \frac{3}{2}(x - 1)^2$

x	0	0.8	0.9	1	1.1	1.2	2
$f(x)$							
$P_2(x)$							

14. $f(x) = \sec x, \quad c = \dfrac{\pi}{4}$

$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \dfrac{\pi}{4}\right) + \dfrac{3}{2}\sqrt{2}\left(x - \dfrac{\pi}{4}\right)^2$

x	-2.15	0.585	0.685	$\frac{\pi}{4}$	0.885	0.985	1.785
$f(x)$							
$P_2(x)$							

15. **Conjecture** Consider the function $f(x) = \cos x$ and its Maclaurin polynomials P_2, P_4, and P_6 (see Example 5).
(a) Use a graphing utility to graph f and the indicated polynomial approximations.
(b) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2$, 4, and 6.
(c) Use the results in part (b) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

16. **Conjecture** Consider the function $f(x) = x^2e^x$.
(a) Find the Maclaurin polynomials P_2, P_3, and P_4 for f.
(b) Use a graphing utility to graph f, P_2, P_3, and P_4.
(c) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2$, 3, and 4.
(d) Use the results in part (c) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

Finding a Maclaurin Polynomial **In Exercises 17–26, find the nth Maclaurin polynomial for the function.**

17. $f(x) = e^{4x}, \quad n = 4$
18. $f(x) = e^{-x}, \quad n = 5$
19. $f(x) = \sin x, \quad n = 5$
20. $f(x) = \cos \pi x, \quad n = 4$
21. $f(x) = xe^x, \quad n = 4$
22. $f(x) = x^2e^{-x}, \quad n = 4$
23. $f(x) = \dfrac{1}{1 - x}, \quad n = 5$
24. $f(x) = \dfrac{x}{x + 1}, \quad n = 4$
25. $f(x) = \sec x, \quad n = 2$
26. $f(x) = \tan x, \quad n = 3$

Finding a Taylor Polynomial In Exercises 27–32, find the nth Taylor polynomial for the function, centered at c.

27. $f(x) = \dfrac{2}{x}, \quad n = 3, \quad c = 1$

28. $f(x) = \dfrac{1}{x^2}, \quad n = 4, \quad c = -2$

29. $f(x) = \sqrt{x}, \quad n = 2, \quad c = 4$

30. $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

31. $f(x) = \ln x, \quad n = 4, \quad c = 2$

32. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

Finding Taylor Polynomials Using Technology In Exercises 33 and 34, use a computer algebra system to find the indicated Taylor polynomials for the function f. Graph the function and the Taylor polynomials.

33. $f(x) = \tan \pi x$
 (a) $n = 3, \quad c = 0$
 (b) $n = 3, \quad c = 1/4$

34. $f(x) = \dfrac{1}{x^2 + 1}$
 (a) $n = 4, \quad c = 0$
 (b) $n = 4, \quad c = 1$

35. **Numerical and Graphical Approximations**

(a) Use the Maclaurin polynomials $P_1(x)$, $P_3(x)$, and $P_5(x)$ for $f(x) = \sin x$ to complete the table.

x	0	0.25	0.50	0.75	1
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$					
$P_3(x)$					
$P_5(x)$					

(b) Use a graphing utility to graph $f(x) = \sin x$ and the Maclaurin polynomials in part (a).

(c) Describe the change in accuracy of a polynomial approximation as the distance from the point where the polynomial is centered increases.

36. **Numerical and Graphical Approximations**

(a) Use the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_4(x)$ for $f(x) = e^x$, centered at $c = 1$, to complete the table.

x	1	1.25	1.50	1.75	2
e^x	e	3.4903	4.4817	5.7546	7.3891
$P_1(x)$					
$P_2(x)$					
$P_4(x)$					

(b) Use a graphing utility to graph $f(x) = e^x$ and the Taylor polynomials in part (a).

(c) Describe the change in accuracy of polynomial approximations as the degree increases.

Numerical and Graphical Approximations In Exercises 37 and 38, (a) find the Maclaurin polynomial $P_3(x)$ for $f(x)$, (b) complete the table for $f(x)$ and $P_3(x)$, and (c) sketch the graphs of $f(x)$ and $P_3(x)$, on the same set of coordinate axes.

x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$							
$P_3(x)$							

37. $f(x) = \arcsin x$

38. $f(x) = \arctan x$

Approximating a Function Value In Exercises 39–44, approximate the function at the given value of x, using the polynomial found in the indicated exercise.

39. $f(x) = e^{4x}, \quad f(\frac{1}{4}), \quad$ Exercise 17

40. $f(x) = x^2e^{-x}, \quad f(\frac{1}{5}), \quad$ Exercise 22

41. $f(x) = \dfrac{1}{x^2}, \quad f(-2.1), \quad$ Exercise 28

42. $f(x) = \sqrt[3]{x}, \quad f(8.05), \quad$ Exercise 30

43. $f(x) = \ln x, \quad f(2.1), \quad$ Exercise 31

44. $f(x) = x^2 \cos x, \quad f\left(\dfrac{7\pi}{8}\right), \quad$ Exercise 32

Using Taylor's Theorem In Exercises 45–50, use Taylor's Theorem to obtain an upper bound for the error of the approximation. Then calculate the exact value of the error.

45. $\cos(0.3) \approx 1 - \dfrac{(0.3)^2}{2!} + \dfrac{(0.3)^4}{4!}$

46. $\arccos(0.15) \approx \dfrac{\pi}{2} - 0.15$

47. $\sinh(0.2) \approx 0.2 + \dfrac{(0.2)^3}{3!} + \dfrac{(0.2)^5}{5!}$

48. $e \approx 1 + 1 + \dfrac{1^2}{2!} + \dfrac{1^3}{3!} + \dfrac{1^4}{4!} + \dfrac{1^5}{5!}$

49. $\arcsin(0.4) \approx 0.4 + \dfrac{(0.4)^3}{2 \cdot 3}$

50. $\arctan(0.4) \approx 0.4 - \dfrac{(0.4)^3}{3}$

Finding a Degree In Exercises 51–56, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.001.

51. $f(x) = \sin x, \quad$ approximate $f(0.3)$

52. $f(x) = \cos x, \quad$ approximate $f(0.4)$

53. $f(x) = e^x, \quad$ approximate $f(0.6)$

54. $f(x) = \ln(x + 1), \quad$ approximate $f(1.25)$

55. $f(x) = \dfrac{1}{x - 2}, \quad$ approximate $f(0.15)$

56. $f(x) = \dfrac{1}{x + 1}, \quad$ approximate $f(0.2)$

9.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Representing a Function** Explain how a Maclaurin polynomial and a power series centered at 0 for a function are different.
2. **Domain** What does the domain of

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

represent?

3. **Radius of Convergence** Determine the radius of convergence for the power series $\sum_{n=0}^{\infty} a_n(x - 2)^n$ given the following result of the Ratio Test, where $u_n = a_n(x - 2)^n$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x-2}{5}\right|$$

4. **Properties of Functions Defined by Power Series** In your own words, describe how a function defined by a power series behaves like a polynomial.

Finding the Center of a Power Series In Exercises 5–8, state where the power series is centered.

5. $\sum_{n=0}^{\infty} nx^n$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)}{2^n n!} x^n$

7. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3}$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-\pi)^{2n}}{(2n)!}$

Finding the Radius of Convergence In Exercises 9–14, find the radius of convergence of the power series.

9. $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

10. $\sum_{n=0}^{\infty} (3x)^n$

11. $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

12. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

13. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

14. $\sum_{n=0}^{\infty} \frac{(2n)!x^{3n}}{n!}$

Finding the Interval of Convergence In Exercises 15–38, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

15. $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

16. $\sum_{n=0}^{\infty} (2x)^n$

17. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

18. $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

19. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

20. $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

21. $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$

22. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{6^n}$

24. $\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-5)^n}{3^n}$

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$

26. $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

27. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

29. $\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

30. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

31. $\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

32. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

33. $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

34. $\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

35. $\sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)x^n}{n!}$

36. $\sum_{n=1}^{\infty} \left[\frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right] x^{2n+1}$

37. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$

38. $\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Finding the Radius of Convergence In Exercises 39 and 40, find the radius of convergence of the power series, where $c > 0$ and k is a positive integer.

39. $\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

40. $\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$

Finding the Interval of Convergence In Exercises 41–44, find the interval of convergence of the power series, where $c > 0$ and k is a positive integer. (Be sure to include a check for convergence at the endpoints of the interval.)

41. $\sum_{n=0}^{\infty} \left(\frac{x}{k}\right)^n$

42. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

43. $\sum_{n=1}^{\infty} \frac{k(k+1)(k+2)\cdots(k+n-1)x^n}{n!}$

44. $\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Writing an Equivalent Series In Exercises 45–48, write an equivalent series with the index of summation beginning at $n = 1$.

45. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

46. $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

47. $\sum_{n=2}^{\infty} \frac{x^{n-1}}{(7n-1)!}$

48. $\sum_{n=2}^{\infty} \frac{x^{3n-1}}{(2n-1)!}$

9.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Using Power Series** Explain how to use a geometric power series to represent a function of the form $f(x) = \dfrac{b}{c - x}$.

2. **Power Series Operations** Consider $f(x) = \sum_{n=0}^{\infty} 5x^{2n}$. What are the values of a and b in terms of n?

$$f\left(\frac{x^3}{5}\right) = \sum_{n=0}^{\infty} \frac{x^a}{5^b}$$

Finding a Geometric Power Series **In Exercises 3–6, find a geometric power series for the function, centered at 0, (a) by the technique shown in Examples 1 and 2 and (b) by long division.**

3. $f(x) = \dfrac{1}{4 - x}$
4. $f(x) = \dfrac{1}{2 + x}$
5. $f(x) = \dfrac{4}{3 + x}$
6. $f(x) = \dfrac{2}{5 - x}$

Finding a Power Series **In Exercises 7–18, find a power series for the function, centered at c, and determine the interval of convergence.**

7. $f(x) = \dfrac{1}{6 - x}, \quad c = 1$
8. $f(x) = \dfrac{2}{6 - x}, \quad c = -2$
9. $f(x) = \dfrac{1}{1 - 3x}, \quad c = 0$
10. $h(x) = \dfrac{1}{1 - 4x}, \quad c = 0$
11. $g(x) = \dfrac{5}{2x - 3}, \quad c = -3$
12. $f(x) = \dfrac{3}{2x - 1}, \quad c = 2$
13. $f(x) = \dfrac{2}{5x + 4}, \quad c = -1$
14. $f(x) = \dfrac{4}{3x + 2}, \quad c = 3$
15. $g(x) = \dfrac{4x}{x^2 + 2x - 3}, \quad c = 0$
16. $g(x) = \dfrac{3x - 8}{3x^2 + 5x - 2}, \quad c = 0$
17. $f(x) = \dfrac{2}{1 - x^2}, \quad c = 0$
18. $f(x) = \dfrac{5}{4 - x^2}, \quad c = 0$

Using a Power Series **In Exercises 19–28, use the power series**

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

to find a power series for the function, centered at 0, and determine the interval of convergence.

19. $h(x) = \dfrac{-2}{x^2 - 1} = \dfrac{1}{1 + x} + \dfrac{1}{1 - x}$
20. $h(x) = \dfrac{x}{x^2 - 1} = \dfrac{1}{2(1 + x)} - \dfrac{1}{2(1 - x)}$
21. $f(x) = -\dfrac{1}{(x + 1)^2} = \dfrac{d}{dx}\left[\dfrac{1}{x + 1}\right]$
22. $f(x) = \dfrac{2}{(x + 1)^3} = \dfrac{d^2}{dx^2}\left[\dfrac{1}{x + 1}\right]$
23. $f(x) = \ln(x + 1) = \displaystyle\int \frac{1}{x + 1}\,dx$
24. $f(x) = \ln(1 - x^2) = \displaystyle\int \frac{1}{1 + x}\,dx - \int \frac{1}{1 - x}\,dx$
25. $g(x) = \dfrac{1}{x^2 + 1}$
26. $f(x) = \ln(x^2 + 1)$
27. $h(x) = \dfrac{1}{4x^2 + 1}$
28. $f(x) = \arctan 2x$

Graphical and Numerical Analysis **In Exercises 29 and 30, let**

$$S_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n}.$$

Use a graphing utility to confirm the inequality graphically. Then complete the table to confirm the inequality numerically.

x	0.0	0.2	0.4	0.6	0.8	1.0
S_n						
$\ln(x + 1)$						
S_{n+1}						

29. $S_2 \le \ln(x + 1) \le S_3$
30. $S_4 \le \ln(x + 1) \le S_5$

Approximating a Sum **In Exercises 31 and 32, (a) use a graphing utility to graph several partial sums of the series, (b) find the sum of the series and its radius of convergence, (c) use a graphing utility and 50 terms of the series to approximate the sum when $x = 0.5$, and (d) determine what the approximation represents and how good the approximation is.**

31. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}$
32. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}$

Approximating a Value **In Exercises 33–36, use the power series for $f(x) = \arctan x$ to approximate the value, using $R_N \le 0.001$.**

33. $\arctan \dfrac{1}{4}$
34. $\displaystyle\int_0^{3/4} \arctan x^2\,dx$
35. $\displaystyle\int_0^{1/2} \frac{\arctan x^2}{x}\,dx$
36. $\displaystyle\int_0^{1/2} x^2 \arctan x\,dx$

9.10 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Convergence of a Taylor Series** Explain how to determine whether a Taylor series for a function f converges to f.
2. **Binomial Series** The binomial series is used to represent a function of what form? What is the radius of convergence for the binomial series?
3. **Power Series** How can you multiply and divide power series?
4. **Finding a Taylor Series** Explain how to use the series $g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to find the series for $f(x) = x^2e^{-3x}$. Do not find the series.

Finding a Taylor Series **In Exercises 5–16, use the definition of Taylor series to find the Taylor series, centered at c, for the function.**

5. $f(x) = e^{2x}, \quad c = 0$
6. $f(x) = e^{-4x}, \quad c = 0$
7. $f(x) = \cos x, \quad c = \frac{\pi}{4}$
8. $f(x) = \sin x, \quad c = \frac{\pi}{4}$
9. $f(x) = \frac{1}{x}, \quad c = 1$
10. $f(x) = \frac{1}{1-x}, \quad c = 2$
11. $f(x) = \ln x, \quad c = 1$
12. $f(x) = e^x, \quad c = 1$
13. $f(x) = \sin 3x, \quad c = 0$
14. $f(x) = \ln(x^2 + 1), \quad c = 0$
15. $f(x) = \sec x, \quad c = 0$ (first three nonzero terms)
16. $f(x) = \tan x, \quad c = 0$ (first three nonzero terms)

Proof **In Exercises 17–20, prove that the Maclaurin series for the function converges to the function for all x.**

17. $f(x) = \cos x$
18. $f(x) = e^{-2x}$
19. $f(x) = \sinh x$
20. $f(x) = \cosh x$

Using a Binomial Series **In Exercises 21–26, use the binomial series to find the Maclaurin series for the function.**

21. $f(x) = \frac{1}{\sqrt{1-x}}$
22. $f(x) = \frac{1}{(1+x)^4}$
23. $f(x) = \frac{1}{\sqrt{1-x^2}}$
24. $f(x) = \frac{1}{(2+x)^3}$
25. $f(x) = \sqrt[4]{1+x}$
26. $f(x) = \sqrt{1+x^3}$

Finding a Maclaurin Series **In Exercises 27–40, find the Maclaurin series for the function. Use the table of power series for elementary functions on page 674.**

27. $f(x) = e^{x^2/2}$
28. $g(x) = e^{-x/3}$
29. $f(x) = \ln(1 + x)$
30. $f(x) = \ln(1 + x^3)$
31. $f(x) = \cos 4x$
32. $f(x) = \sin \pi x$
33. $g(x) = \arctan 5x$
34. $f(x) = \arcsin \pi x$
35. $f(x) = \cos x^{3/2}$
36. $g(x) = 2 \sin x^3$
37. $f(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$
38. $f(x) = e^x + e^{-x} = 2 \cosh x$
39. $f(x) = \cos^2 x$
40. $f(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

    $\left(\textit{Hint:} \text{ Integrate the series for } \frac{1}{\sqrt{x^2+1}}.\right)$

Verifying a Formula **In Exercises 41 and 42, use a power series and the fact that $i^2 = -1$ to verify the formula.**

41. $g(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) = \sin x$
42. $g(x) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$

Finding a Maclaurin Series **In Exercises 43–46, find the Maclaurin series for the function.**

43. $f(x) = x \sin x$
44. $h(x) = x \cos x$
45. $g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$
46. $f(x) = \begin{cases} \frac{\arcsin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Finding Terms of a Maclaurin Series **In Exercises 47–52, find the first four nonzero terms of the Maclaurin series for the function by multiplying or dividing the appropriate power series. Use the table of power series for elementary functions on page 674. Use a graphing utility to graph the function and its corresponding polynomial approximation.**

47. $f(x) = e^x \sin x$
48. $g(x) = e^x \cos x$
49. $h(x) = (\cos x) \ln(1 + x)$
50. $f(x) = e^x \ln(1 + x)$
51. $g(x) = \frac{\sin x}{1 + x}$
52. $f(x) = \frac{e^x}{1 + x}$

Finding a Maclaurin Series **In Exercises 53 and 54, find a Maclaurin series for $f(x)$.**

53. $f(x) = \int_0^x (e^{-t^2} - 1)\, dt$
54. $f(x) = \int_0^x \sqrt{1 + t^3}\, dt$

Verifying a Sum In Exercises 55–58, verify the sum. Then use a graphing utility to approximate the sum with an error of less than 0.0001.

55. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = \ln 2$

56. $\sum_{n=0}^{\infty} (-1)^{n}\left[\frac{1}{(2n+1)!}\right] = \sin 1$

57. $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

58. $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n!}\right) = \frac{e-1}{e}$

Finding a Limit In Exercises 59–62, use the series representation of the function f to find $\lim_{x\to 0} f(x)$, if it exists.

59. $f(x) = \frac{1-\cos x}{x}$

60. $f(x) = \frac{\sin x}{x}$

61. $f(x) = \frac{e^x - 1}{x}$

62. $f(x) = \frac{\ln(x+1)}{x}$

Approximating an Integral In Exercises 63–70, use a power series to approximate the value of the definite integral with an error of less than 0.0001. (In Exercises 65 and 67, assume that the integrand is defined as 1 when $x = 0$.)

63. $\int_0^1 e^{-x^3}\,dx$

64. $\int_0^{1/4} x\ln(x+1)\,dx$

65. $\int_0^1 \frac{\sin x}{x}\,dx$

66. $\int_0^1 \cos x^2\,dx$

67. $\int_0^{1/2} \frac{\arctan x}{x}\,dx$

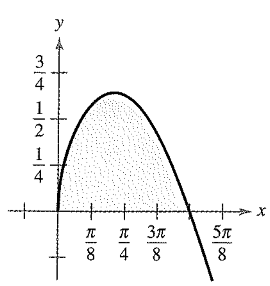
68. $\int_0^{1/2} \arctan x^2\,dx$

69. $\int_{0.1}^{0.3} \sqrt{1+x^3}\,dx$

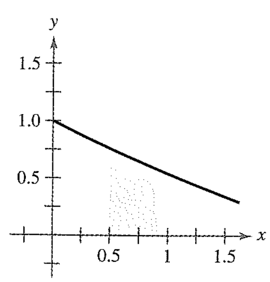
70. $\int_0^{0.2} \sqrt{1+x^2}\,dx$

Area In Exercises 71 and 72, use a power series to approximate the area of the region with an error of less than 0.0001. Use a graphing utility to verify the result.

71. $\int_0^{\pi/2} \sqrt{x}\cos x\,dx$

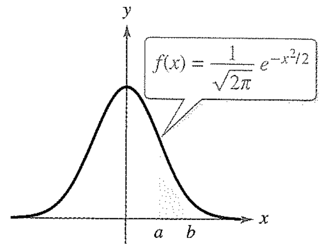


72. $\int_{0.5}^1 \cos\sqrt{x}\,dx$



Probability In Exercises 73 and 74, approximate the probability with an error of less than 0.0001, where the probability is given by

$$P(a < x < b) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx.$$



73. $P(0 < x < 1)$

74. $P(1 < x < 2)$

EXPLORING CONCEPTS

75. **Comparing Methods** Describe three ways to find the Maclaurin series for $\cos^2 x$. Show that each method produces the same first three terms.

76. **Maclaurin Series** Explain how to use the power series for $f(x) = \arctan x$ to find the Maclaurin series for

$$g(x) = \frac{1}{1+x^2}.$$

What is another way to find the Maclaurin series for g using a power series for an elementary function?

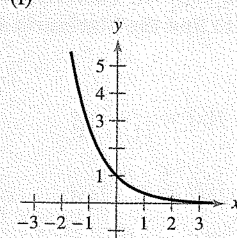
77. **Finding a Function** Which function has the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{(-1)^n(x+3)^{2n+1}}{2^2(2n+1)!}?$$

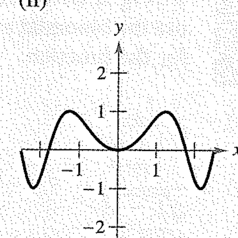
Explain your reasoning.

78. **HOW DO YOU SEE IT?** Identify the function represented by each power series and match the function with its graph. [The graphs are labeled (i) and (ii).]

(i)



(ii)



(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

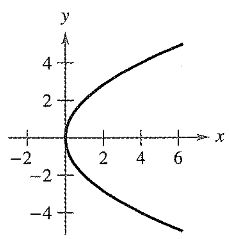
10.1 Exercises

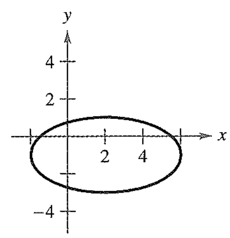
See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

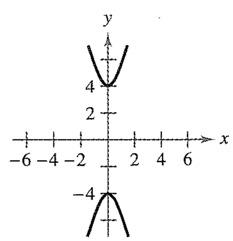
CONCEPT CHECK

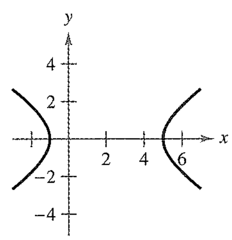
1. **Conic Sections** State the definitions of parabola, ellipse, and hyperbola in your own words.
2. **Reflective Property** Use a sketch to illustrate the reflective property of an ellipse.
3. **Eccentricity** Consider an ellipse with eccentricity e.
   (a) What are the possible values of e?
   (b) What happens to the graph of the ellipse as e increases?
4. **Hyperbola** Explain how to sketch a hyperbola with a vertical transverse axis.

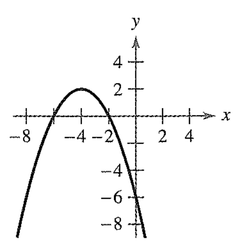
Matching In Exercises 5–10, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

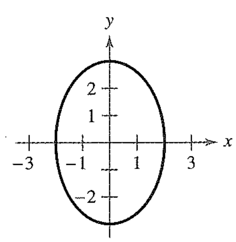
(a)


(b)


(c)


(d)


(e)


(f)


5. $y^2 = 4x$
6. $(x + 4)^2 = -2(y - 2)$
7. $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$
8. $\dfrac{(x - 2)^2}{16} + \dfrac{(y + 1)^2}{4} = 1$
9. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
10. $\dfrac{(x - 2)^2}{9} - \dfrac{y^2}{4} = 1$

Sketching a Parabola In Exercises 11–16, find the vertex, focus, and directrix of the parabola, and sketch its graph.

11. $(x + 5) + (y - 3)^2 = 0$
12. $(x - 6)^2 - 2(y + 7) = 0$
13. $y^2 - 4y - 4x = 0$
14. $y^2 + 6y + 8x + 25 = 0$
15. $x^2 + 4x + 4y - 4 = 0$
16. $x^2 - 2x - 4y - 7 = 0$

Finding the Standard Equation of a Parabola In Exercises 17–24, find the standard form of the equation of the parabola with the given characteristics.

17. Vertex: $(5, 4)$
    Focus: $(3, 4)$
18. Vertex: $(-3, -1)$
    Focus: $(-3, 1)$
19. Vertex: $(0, 5)$
    Directrix: $y = -3$
20. Focus: $(2, 2)$
    Directrix: $x = -2$
21. Vertex: $(1, -1)$
    Points on the parabola: $(-1, -4), (3, -4)$
22. Vertex: $(2, 4)$
    Points on the parabola: $(0, 0), (4, 0)$
23. Axis is parallel to y-axis; graph passes through $(0, 3)$, $(3, 4)$, and $(4, 11)$.
24. Directrix: $y = -2$; endpoints of latus rectum are $(0, 2)$ and $(8, 2)$.

Sketching an Ellipse In Exercises 25–30, find the center, foci, vertices, and eccentricity of the ellipse, and sketch its graph.

25. $16x^2 + y^2 = 16$
26. $3x^2 + 7y^2 = 63$
27. $\dfrac{(x - 3)^2}{16} + \dfrac{(y - 1)^2}{25} = 1$
28. $(x + 4)^2 + \dfrac{(y + 6)^2}{1/4} = 1$
29. $9x^2 + 4y^2 + 36x - 24y - 36 = 0$
30. $x^2 + 10y^2 - 6x + 20y + 18 = 0$

Finding the Standard Equation of an Ellipse In Exercises 31–36, find the standard form of the equation of the ellipse with the given characteristics.

31. Center: $(0, 0)$
    Focus: $(5, 0)$
    Vertex: $(6, 0)$
32. Vertices: $(0, 3), (8, 3)$
    Eccentricity: $\frac{3}{4}$
33. Vertices: $(3, 1), (3, 9)$
    Minor axis length: 6
34. Foci: $(0, \pm 9)$
    Major axis length: 22
35. Center: $(0, 0)$
    Major axis: horizontal
    Points on the ellipse: $(3, 1), (4, 0)$
36. Center: $(1, 2)$
    Major axis: vertical
    Points on the ellipse: $(1, 6), (3, 2)$

Sketching a Hyperbola In Exercises 37–40, find the center, foci, vertices, and eccentricity of the hyperbola, and sketch its graph using asymptotes as an aid.

37. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$
38. $\dfrac{(y + 3)^2}{225} - \dfrac{(x - 5)^2}{64} = 1$
39. $9x^2 - y^2 - 36x - 6y + 18 = 0$
40. $y^2 - 16x^2 + 64x - 208 = 0$

Finding the Standard Equation of a Hyperbola In Exercises 41–48, find the standard form of the equation of the hyperbola with the given characteristics.

41. Vertices: $(\pm 1, 0)$
Asymptotes: $y = \pm 5x$

42. Vertices: $(0, \pm 4)$
Asymptotes: $y = \pm 2x$

43. Vertices: $(2, \pm 3)$
Point on graph: $(0, 5)$

44. Vertices: $(2, \pm 3)$
Foci: $(2, \pm 5)$

45. Center: $(0, 0)$
Vertex: $(0, 2)$
Focus: $(0, 4)$

46. Center: $(0, 0)$
Vertex: $(6, 0)$
Focus: $(10, 0)$

47. Vertices: $(0, 2), (6, 2)$
Asymptotes: $y = \frac{2}{3}x$
$y = 4 - \frac{2}{3}x$

48. Focus: $(20, 0)$
Asymptotes: $y = \pm\frac{3}{4}x$

Finding Equations of Tangent Lines and Normal Lines In Exercises 49 and 50, find equations for (a) the tangent lines and (b) the normal lines to the hyperbola for the given value of x. (The *normal line* at a point is perpendicular to the tangent line at the point.)

49. $\dfrac{x^2}{9} - y^2 = 1, \quad x = 6$

50. $\dfrac{y^2}{4} - \dfrac{x^2}{2} = 1, \quad x = 4$

Classifying the Graph of an Equation In Exercises 51–56, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

51. $25x^2 - 10x - 200y - 119 = 0$

52. $4x^2 - y^2 - 4x - 3 = 0$

53. $3(x - 1)^2 = 6 + 2(y + 1)^2$

54. $9(x + 3)^2 = 36 - 4(y - 2)^2$

55. $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

56. $y^2 - 4y = x + 5$

EXPLORING CONCEPTS

57. Using an Equation Consider the equation $9x^2 + 4y^2 - 36x - 24y - 36 = 0$.

(a) Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

(b) Change the $4y^2$-term in the equation to $-4y^2$. Classify the graph of the new equation.

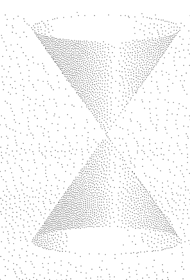
(c) Change the $9x^2$-term in the original equation to $4x^2$. Classify the graph of the new equation.

(d) Describe one way you could change the original equation so that its graph is a parabola.

58. Investigation Sketch the graphs of $x^2 = 4py$ for $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}$, and 2 on the same coordinate axes. Discuss the change in the graphs as p increases.

59. Ellipse Let C be the circumference of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $b < a$. Explain why $2\pi b < C < 2\pi a$. Use a graph to support your explanation.

60. HOW DO YOU SEE IT? Describe in words how a plane could intersect with the double-napped cone to form each conic section (see figure).



(a) Circle (b) Ellipse
(c) Parabola (d) Hyperbola

61. Solar Collector A solar collector for heating water is constructed with a sheet of stainless steel that is formed into the shape of a parabola (see figure). The water will flow through a pipe that is located at the focus of the parabola. At what distance from the vertex is the pipe?

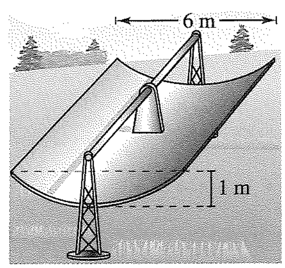


Figure for 61

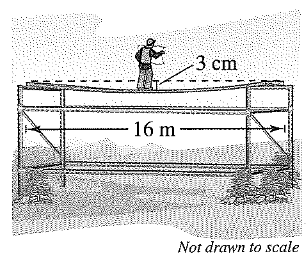


Figure for 62

62. Beam Deflection A simply supported beam that is 16 meters long has a load concentrated at the center (see figure). The deflection of the beam at its center is 3 centimeters. Assume that the shape of the deflected beam is parabolic.

(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)

(b) How far from the center of the beam is the deflection 1 centimeter?

63. Proof

(a) Prove that any two distinct tangent lines to a parabola intersect.

(b) Demonstrate the result of part (a) by finding the point of intersection of the tangent lines to the parabola $x^2 - 4x - 4y = 0$ at the points $(0, 0)$ and $(6, 3)$.

64. Proof

(a) Prove that if any two tangent lines to a parabola intersect at right angles, then their point of intersection must lie on the directrix.

(b) Demonstrate the result of part (a) by showing that the tangent lines to the parabola $x^2 - 4x - 4y + 8 = 0$ at the points $(-2, 5)$ and $\left(3, \frac{5}{4}\right)$ intersect at right angles and that their point of intersection lies on the directrix.

10.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Parametric Equations** What information does a set of parametric equations provide that is lacking in a rectangular equation for describing the motion of an object?
2. **Plane Curve** Explain the process of sketching a plane curve given by parametric equations. What is meant by the orientation of the curve?
3. **Think About It** How can two sets of parametric equations represent the same graph but different curves?
4. **Adjusting a Domain** Consider the parametric equations

   $x = \sqrt{t-2}$ and $y = \frac{1}{2}t + 1, \quad t \geq 2.$

   What is implied about the domain of the resulting rectangular equation?

Using Parametric Equations In Exercises 5–22, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

5. $x = 2t - 3, \quad y = 3t + 1$
6. $x = 5 - 4t, \quad y = 2 + 5t$
7. $x = t + 1, \quad y = t^2$
8. $x = 2t^2, \quad y = t^4 + 1$
9. $x = t^3, \quad y = \dfrac{t^2}{2}$
10. $x = t^2 + t, \quad y = t^2 - t$
11. $x = \sqrt{t}, \quad y = t - 5$
12. $x = \sqrt[4]{t}, \quad y = 8 - t$
13. $x = t - 3, \quad y = \dfrac{t}{t-3}$
14. $x = 1 + \dfrac{1}{t}, \quad y = t - 1$
15. $x = 2t, \quad y = |t - 2|$
16. $x = |t - 1|, \quad y = t + 2$
17. $x = e^t, \quad y = e^{3t} + 1$
18. $x = e^{-t}, \quad y = e^{2t} - 1$
19. $x = 8\cos\theta, \quad y = 8\sin\theta$
20. $x = 3\cos\theta, \quad y = 7\sin\theta$
21. $x = \sec\theta, \quad y = \cos\theta, \quad 0 \leq \theta < \pi/2, \quad \pi/2 < \theta \leq \pi$
22. $x = \tan^2\theta, \quad y = \sec^2\theta$

Using Parametric Equations In Exercises 23–34, use a graphing utility to graph the curve represented by the parametric equations (indicate the orientation of the curve). Eliminate the parameter and write the corresponding rectangular equation.

23. $x = 6\sin 2\theta$, $y = 4\cos 2\theta$
24. $x = \cos\theta$, $y = 2\sin 2\theta$
25. $x = 4 + 2\cos\theta$, $y = -1 + \sin\theta$
26. $x = -2 + 3\cos\theta$, $y = -5 + 3\sin\theta$
27. $x = -3 + 4\cos\theta$, $y = 2 + 5\sin\theta$
28. $x = \sec\theta$, $y = \tan\theta$
29. $x = 4\sec\theta$, $y = 3\tan\theta$
30. $x = \cos^3\theta$, $y = \sin^3\theta$
31. $x = t^3, \quad y = 3\ln t$
32. $x = \ln 2t, \quad y = t^2$
33. $x = e^{-t}, \quad y = e^{3t}$
34. $x = e^{2t}, \quad y = e^t$

Comparing Plane Curves In Exercises 35–38, determine any differences between the curves of the parametric equations. Are the graphs the same? Are the orientations the same? Are the curves smooth? Explain.

35. (a) $x = t, y = t^2$ (b) $x = -t, y = t^2$
36. (a) $x = t + 1, y = t^3$ (b) $x = -t + 1, y = (-t)^3$
37. (a) $x = t$, $y = 2t + 1$ (b) $x = \cos\theta$, $y = 2\cos\theta + 1$
    (c) $x = e^{-t}$, $y = 2e^{-t} + 1$ (d) $x = e^t$, $y = 2e^t + 1$
38. (a) $x = 2\cos\theta$, $y = 2\sin\theta$ (b) $x = \sqrt{4t^2 - 1}/|t|$, $y = 1/t$
    (c) $x = \sqrt{t}$, $y = \sqrt{4 - t}$ (d) $x = -\sqrt{4 - e^{2t}}$, $y = e^t$

Eliminating a Parameter In Exercises 39–42, eliminate the parameter and obtain the standard form of the rectangular equation.

39. Line through (x_1, y_1) and (x_2, y_2):
    $x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1)$
40. Circle: $x = h + r\cos\theta, \quad y = k + r\sin\theta$
41. Ellipse: $x = h + a\cos\theta, \quad y = k + b\sin\theta$
42. Hyperbola: $x = h + a\sec\theta, \quad y = k + b\tan\theta$

Writing a Set of Parametric Equations In Exercises 43–50, use the results of Exercises 39–42 to find a set of parametric equations for the line or conic.

43. Line: passes through $(0, 0)$ and $(4, -7)$
44. Line: passes through $(-3, 1)$ and $(1, 9)$
45. Circle: center: $(1, 1)$; radius: 2
46. Circle: center: $\left(-\frac{1}{2}, -4\right)$; radius: $\frac{1}{2}$
47. Ellipse: vertices: $(-3, 0), (7, 0)$; foci: $(-1, 0), (5, 0)$
48. Ellipse: vertices: $(-1, 8), (-1, -12)$; foci: $(-1, 4), (-1, -8)$
49. Hyperbola: vertices: $(0, \pm 1)$; foci: $\left(0, \pm\sqrt{5}\right)$
50. Hyperbola: vertices: $(-2, 1), (0, 1)$; foci: $(-3, 1), (1, 1)$

Finding Parametric Equations In Exercises 51–54, find two different sets of parametric equations for the rectangular equation.

51. $y = 6x - 5$
52. $y = 4/(x - 1)$
53. $y = x^3$
54. $y = x^2$

Finding Parametric Equations **In Exercises 55–58, find a set of parametric equations for the rectangular equation that satisfies the given condition.**

55. $y = 2x - 5$, $t = 0$ at the point $(3, 1)$

56. $y = 4x + 1$, $t = -1$ at the point $(-2, -7)$

57. $y = x^2$, $t = 4$ at the point $(4, 16)$

58. $y = 4 - x^2$, $t = 1$ at the point $(1, 3)$

Graphing a Plane Curve **In Exercises 59–66, use a graphing utility to graph the curve represented by the parametric equations. Indicate the orientation of the curve. Identify any points at which the curve is not smooth.**

59. Cycloid: $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$

60. Cycloid: $x = \theta + \sin\theta$, $y = 1 - \cos\theta$

61. Prolate cycloid: $x = \theta - \frac{3}{2}\sin\theta$, $y = 1 - \frac{3}{2}\cos\theta$

62. Prolate cycloid: $x = 2\theta - 4\sin\theta$, $y = 2 - 4\cos\theta$

63. Hypocycloid: $x = 3\cos^3\theta$, $y = 3\sin^3\theta$

64. Curtate cycloid: $x = 2\theta - \sin\theta$, $y = 2 - \cos\theta$

65. Witch of Agnesi: $x = 2\cot\theta$, $y = 2\sin^2\theta$

66. Folium of Descartes: $x = 3t/(1 + t^3)$, $y = 3t^2/(1 + t^3)$

EXPLORING CONCEPTS

67. **Orientation** Describe the orientation of the parametric equations $x = t^2$ and $y = t^4$ for $-1 \le t \le 1$.

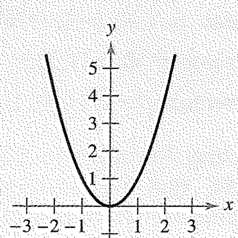
68. **Conjecture** Make a conjecture about the change in the graph of parametric equations when the sign of the parameter is changed. Explain your reasoning using examples to support your conjecture.

69. **Think About It** The following sets of parametric equations have the same graph. Does this contradict your conjecture from Exercise 68? Explain.

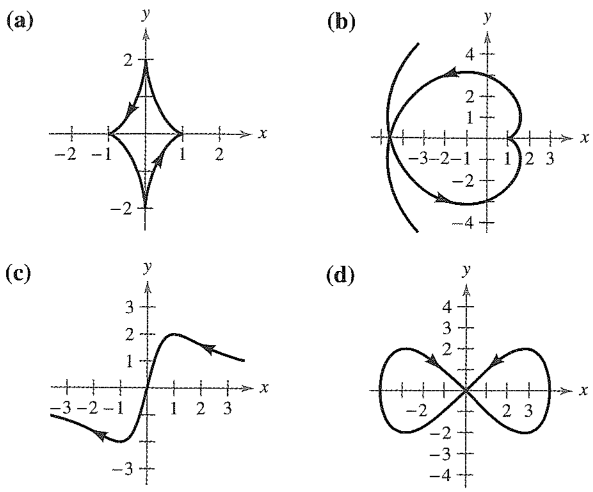
$$x = \cos\theta,\ y = \sin^2\theta,\ 0 < \theta < \pi$$
$$x = \cos(-\theta),\ y = \sin^2(-\theta),\ 0 < \theta < \pi$$

70. **HOW DO YOU SEE IT?** Which set of parametric equations is shown in the graph below? Explain your reasoning.

(a) $x = t$, $y = t^2$ (b) $x = t^2$, $y = t$



Matching **In Exercises 71–74, match the set of parametric equations with its graph. [The graphs are labeled (a), (b), (c), and (d).] Explain your reasoning.**



71. Lissajous curve: $x = 4\cos\theta$, $y = 2\sin 2\theta$

72. Evolute of ellipse: $x = \cos^3\theta$, $y = 2\sin^3\theta$

73. Involute of circle: $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

74. Serpentine curve: $x = \cot\theta$, $y = 4\sin\theta\cos\theta$

75. **Curtate Cycloid** A wheel of radius a rolls along a line without slipping. The curve traced by a point P that is b units from the center ($b < a$) is called a **curtate cycloid** (see figure). Use the angle θ to find a set of parametric equations for this curve.

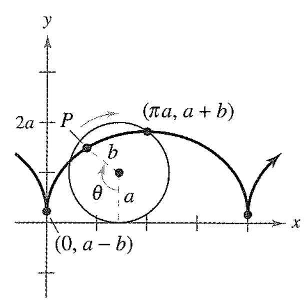


Figure for 75

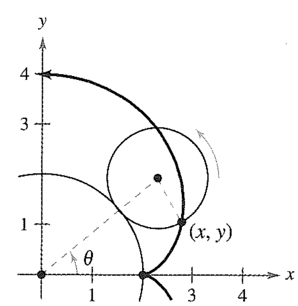


Figure for 76

76. **Epicycloid** A circle of radius 1 rolls around the outside of a circle of radius 2 without slipping. The curve traced by a point on the circumference of the smaller circle is called an **epicycloid** (see figure). Use the angle θ to find a set of parametric equations for this curve.

True or False? **In Exercises 77–79, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. The graph of the parametric equations $x = t^2$ and $y = t^2$ is the line $y = x$.

78. If y is a function of t and x is a function of t, then y is a function of x.

79. The curve represented by the parametric equations $x = t$ and $y = \cos t$ can be written as an equation of the form $y = f(x)$.

10.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Parametric Form of the Derivative** What does the parametric form of the derivative represent?
2. **Tangent Lines** Under what circumstances can a graph that represents a set of parametric equations have more than one tangent line at a given point?
3. **Tangent Lines** Consider a curve represented by the parametric equations $x = f(t)$ and $y = g(t)$. When does the graph have horizontal tangent lines? Vertical tangent lines?
4. **Arc Length** Why does the arc length formula require that the curve not intersect itself on an interval, except possibly at the endpoints?

Finding a Derivative **In Exercises 5–8, find dy/dx.**

5. $x = t^2, \quad y = 7 - 6t$
6. $x = \sqrt[3]{t}, \quad y = 4 - t$
7. $x = \sin^2 \theta, \quad y = \cos^2 \theta$
8. $x = 2e^{\theta}, \quad y = e^{-\theta/2}$

Finding Slope and Concavity **In Exercises 9–18, find dy/dx and d^2y/dx^2, and find the slope and concavity (if possible) at the given value of the parameter.**

Parametric Equations	Parameter
9. $x = 4t, \quad y = 3t - 2$	$t = 3$
10. $x = \sqrt{t}, \quad y = 3t - 1$	$t = 1$
11. $x = t + 1, \quad y = t^2 + 3t$	$t = -2$
12. $x = t^2 + 5t + 4, \quad y = 4t$	$t = 0$
13. $x = 4 \cos \theta, \quad y = 4 \sin \theta$	$\theta = \frac{\pi}{4}$
14. $x = \cos \theta, \quad y = 3 \sin \theta$	$\theta = 0$
15. $x = 2 + \sec \theta, \quad y = 1 + 2 \tan \theta$	$\theta = -\frac{\pi}{3}$
16. $x = \sqrt{t}, \quad y = \sqrt{t - 1}$	$t = 5$
17. $x = \cos^3 \theta, \quad y = \sin^3 \theta$	$\theta = \frac{\pi}{4}$
18. $x = \theta - \sin \theta, \quad y = 1 - \cos \theta$	$\theta = \pi$

Finding Equations of Tangent Lines **In Exercises 19–22, find an equation of the tangent line to the curve at each given point.**

19. $x = 2 \cot \theta, \quad y = 2 \sin^2 \theta, \quad \left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right), (0, 2), \left(2\sqrt{3}, \frac{1}{2}\right)$
20. $x = 2 - 3 \cos \theta, \quad y = 3 + 2 \sin \theta,$
$(-1, 3), (2, 5), \left(\frac{4 + 3\sqrt{3}}{2}, 2\right)$
21. $x = t^2 - 4, \quad y = t^2 - 2t, \quad (0, 0), (-3, -1), (-3, 3)$
22. $x = t^4 + 2, \quad y = t^3 + t, \quad (2, 0), (3, -2), (18, 10)$

Finding an Equation of a Tangent Line **In Exercises 23–26, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find dx/dt, dy/dt, and dy/dx at the given value of the parameter, (c) find an equation of the tangent line to the curve at the given value of the parameter, and (d) use a graphing utility to graph the curve and the tangent line from part (c).**

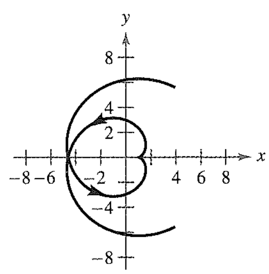
Parametric Equations	Parameter
23. $x = 6t, \quad y = 1 - 4t^2$	$t = -\frac{1}{2}$
24. $x = t - 2, \quad y = \frac{1}{t} + 3$	$t = 1$
25. $x = t^2 - t + 2, \quad y = t^3 - 3t$	$t = -1$
26. $x = 3t - t^2, \quad y = 2t^{3/2}$	$t = \frac{1}{4}$

Finding Equations of Tangent Lines **In Exercises 27–30, find the equations of the tangent lines at the point where the curve crosses itself.**

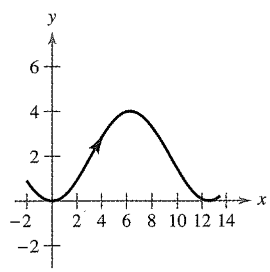
27. $x = 2 \sin 2t, \quad y = 3 \sin t$
28. $x = 2 - \pi \cos t, \quad y = 2t - \pi \sin t$
29. $x = t^2 - t, \quad y = t^3 - 3t - 1$
30. $x = t^3 - 6t, \quad y = t^2$

Horizontal and Vertical Tangency **In Exercises 31 and 32, find all points (if any) of horizontal and vertical tangency to the curve on the given interval.**

31. $x = \cos \theta + \theta \sin \theta$
$y = \sin \theta - \theta \cos \theta$
$-2\pi \le \theta \le 2\pi$



32. $x = 2\theta$
$y = 2(1 - \cos \theta)$
$0 \le \theta \le 2\pi$



Horizontal and Vertical Tangency **In Exercises 33–42, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.**

33. $x = 9 - t, \quad y = -t^2$
34. $x = t + 1, \quad y = t^2 + 3t$
35. $x = t + 4, \quad y = t^3 - 12t + 6$
36. $x = t^2 - t + 2, \quad y = t^3 - 3t$
37. $x = 7 \cos \theta, \quad y = 7 \sin \theta$
38. $x = \cos \theta, \quad y = 2 \sin 2\theta$

39. $x = 5 + 3\cos\theta, \quad y = -2 + \sin\theta$

40. $x = \sec\theta, \quad y = \tan\theta$

41. $x = 4\cos^2\theta, \quad y = 2\sin\theta$ **42.** $x = \cos^2\theta, \quad y = \cos\theta$

Determining Concavity **In Exercises 43–48, determine the open t-intervals on which the curve is concave downward or concave upward.**

43. $x = 3t^2, \quad y = t^3 - t$ **44.** $x = 2 + t^2, \quad y = t^2 + t^3$

45. $x = 2t + \ln t, \quad y = 2t - \ln t$

46. $x = t^2, \quad y = \ln t$

47. $x = \sin t, \quad y = \cos t, \quad 0 < t < \pi$

48. $x = 4\cos t, \quad y = 2\sin t, \quad 0 < t < 2\pi$

Arc Length **In Exercises 49–54, find the arc length of the curve on the given interval.**

Parametric Equations	**Interval**
49. $x = 3t + 5, \quad y = 7 - 2t$	$-1 \le t \le 3$
50. $x = 6t^2, \quad y = 2t^3$	$1 \le t \le 4$
51. $x = e^{-t}\cos t, \quad y = e^{-t}\sin t$	$0 \le t \le \frac{\pi}{2}$
52. $x = \arcsin t, \quad y = \ln\sqrt{1 - t^2}$	$0 \le t \le \frac{1}{2}$
53. $x = \sqrt{t}, \quad y = 3t - 1$	$0 \le t \le 1$
54. $x = t, \quad y = \frac{t^5}{10} + \frac{1}{6t^3}$	$1 \le t \le 2$

Arc Length **In Exercises 55–58, find the arc length of the curve on the interval $[0, 2\pi]$.**

55. Hypocycloid perimeter: $x = a\cos^3\theta, y = a\sin^3\theta$

56. Involute of a circle: $x = \cos\theta + \theta\sin\theta$

$y = \sin\theta - \theta\cos\theta$

57. Cycloid arch: $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

58. Nephroid perimeter: $x = a(3\cos t - \cos 3t)$

$y = a(3\sin t - \sin 3t)$

59. Path of a Projectile The path of a projectile is modeled by the parametric equations

$$x = (90\cos 30°)t \quad \text{and} \quad y = (90\sin 30°)t - 16t^2$$

where x and y are measured in feet.

(a) Use a graphing utility to graph the path of the projectile.

(b) Use a graphing utility to approximate the range of the projectile.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the path. Compare this result with the range of the projectile.

60. Path of a Projectile When the projectile in Exercise 59 is launched at an angle θ with the horizontal, its parametric equations are $x = (90\cos\theta)t$ and $y = (90\sin\theta)t - 16t^2$. Find the angle that maximizes the range of the projectile. Use a graphing utility to find the angle that maximizes the arc length of the trajectory.

61. Folium of Descartes Consider the parametric equations

$$x = \frac{4t}{1 + t^3} \quad \text{and} \quad y = \frac{4t^2}{1 + t^3}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the closed loop. (*Hint:* Use symmetry and integrate over the interval $0 \le t \le 1$.)

62. Witch of Agnesi Consider the parametric equations

$$x = 4\cot\theta \quad \text{and} \quad y = 4\sin^2\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length over the interval $\pi/4 \le \theta \le \pi/2$.

Surface Area **In Exercises 63–68, find the area of the surface generated by revolving the curve about each given axis.**

63. $x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

(a) x-axis

(b) y-axis

64. $x = t, \quad y = 4 - 2t, \quad 0 \le t \le 2$

(a) x-axis

(b) y-axis

65. $x = 5\cos\theta, \quad y = 5\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad y$-axis

66. $x = \frac{1}{3}t^3, \quad y = t + 1, \quad 1 \le t \le 2, \quad y$-axis

67. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \pi, \quad x$-axis

68. $x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$

(a) x-axis

(b) y-axis

Surface Area **In Exercises 69–72, write an integral that represents the area of the surface generated by revolving the curve about the x-axis. Use a graphing utility to approximate the integral.**

Parametric Equations	**Interval**
69. $x = t^3, \quad y = t + 2$	$0 \le t \le 2$
70. $x = t^2, \quad y = \sqrt{t}$	$1 \le t \le 3$
71. $x = \cos^2\theta, \quad y = \cos\theta$	$0 \le \theta \le \frac{\pi}{2}$
72. $x = \theta + \sin\theta, \quad y = \theta + \cos\theta$	$0 \le \theta \le \frac{\pi}{2}$

10.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Polar Coordinates** Consider the polar coordinates (r, θ). What does r represent? What does θ represent?
2. **Plotting Points** Plot the points below on the same set of coordinate axes.

$$(r, \theta) = \left(2, \frac{\pi}{2}\right) \quad \text{and} \quad (x, y) = \left(2, \frac{\pi}{2}\right)$$

3. **Comparing Coordinate Systems** Describe the differences between the rectangular coordinate system and the polar coordinate system.
4. **Parametric Form of a Polar Equation** Explain how to write a polar equation in parametric form.

Polar-to-Rectangular Conversion **In Exercises 5–14, the polar coordinates of a point are given. Plot the point and find the corresponding rectangular coordinates for the point.**

5. $\left(8, \frac{\pi}{2}\right)$ **6.** $\left(-2, \frac{5\pi}{3}\right)$

7. $\left(-4, -\frac{3\pi}{4}\right)$ **8.** $\left(0, -\frac{7\pi}{6}\right)$

9. $\left(7, \frac{5\pi}{4}\right)$ **10.** $\left(-2, \frac{11\pi}{6}\right)$

11. $(\sqrt{2}, 2.36)$ **12.** $(-3, -1.57)$

13. $(-8, 0.75)$ **14.** $(1.25, -5)$

Rectangular-to-Polar Conversion **In Exercises 15–24, the rectangular coordinates of a point are given. Plot the point and find *two* sets of polar coordinates for the point for $0 \le \theta < 2\pi$.**

15. $(1, 0)$ **16.** $(0, -9)$

17. $(-3, 4)$ **18.** $(6, -2)$

19. $(-5, -5\sqrt{3})$ **20.** $(3, -\sqrt{3})$

21. $(\sqrt{7}, -\sqrt{7})$ **22.** $(-2\sqrt{2}, -2\sqrt{2})$

23. $(4, 5)$ **24.** $(1, 8)$

Rectangular-to-Polar Conversion **In Exercises 25–34, convert the rectangular equation to polar form and sketch its graph.**

25. $x^2 + y^2 = 9$ **26.** $x^2 - y^2 = 9$

27. $x^2 + y^2 = a^2$ **28.** $x^2 + y^2 - 2ax = 0$

29. $y = 8$ **30.** $x = 12$

31. $3x - y + 2 = 0$ **32.** $xy = 4$

33. $y^2 = 9x$

34. $(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$

Polar-to-Rectangular Conversion **In Exercises 35–44, convert the polar equation to rectangular form and sketch its graph.**

35. $r = 4$ **36.** $r = -1$

37. $r = 3 \sin \theta$ **38.** $r = 5 \cos \theta$

39. $r = \theta$ **40.** $\theta = \frac{5\pi}{6}$

41. $r = 3 \sec \theta$ **42.** $r = -6 \csc \theta$

43. $r = \sec \theta \tan \theta$ **44.** $r = \cot \theta \csc \theta$

Graphing a Polar Equation **In Exercises 45–54, use a graphing utility to graph the polar equation. Find an interval for θ over which the graph is traced *only once*.**

45. $r = 2 - 5 \cos \theta$ **46.** $r = 3(1 - 4 \cos \theta)$

47. $r = -1 + \sin \theta$ **48.** $r = 4 + 3 \cos \theta$

49. $r = \frac{2}{1 + \cos \theta}$ **50.** $r = \frac{1}{4 - 3 \sin \theta}$

51. $r = 5 \cos \frac{3\theta}{2}$ **52.** $r = 3 \sin \frac{5\theta}{2}$

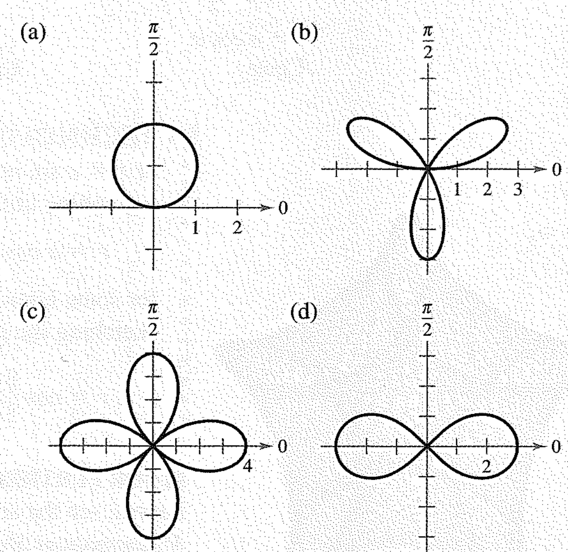
53. $r^2 = 4 \sin 2\theta$ **54.** $r^2 = \frac{1}{\theta}$

55. Verifying a Polar Equation Convert the equation

$$r = 2(h \cos \theta + k \sin \theta)$$

to rectangular form and verify that it is the equation of a circle. Find the radius and the rectangular coordinates of the center of the circle.

56. **HOW DO YOU SEE IT?** Identify each special polar graph and write its equation.



57. Sketching a Graph Sketch the graph of $r = 4 \sin \theta$ over each interval.

(a) $0 \le \theta \le \frac{\pi}{2}$ (b) $\frac{\pi}{2} \le \theta \le \pi$ (c) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

58. Distance Formula

(a) Verify that the Distance Formula for the distance between the two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

(b) Describe the positions of the points relative to each other for $\theta_1 = \theta_2$. Simplify the Distance Formula for this case. Is the simplification what you expected? Explain.

(c) Simplify the Distance Formula for $\theta_1 - \theta_2 = 90°$. Is the simplification what you expected? Explain.

(d) Choose two points on the polar coordinate system and find the distance between them. Then choose different polar representations of the same two points and apply the Distance Formula again. Discuss the result.

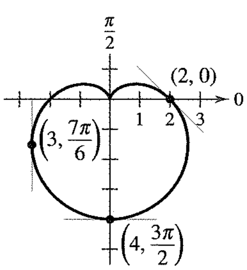
Distance Formula In Exercises 59–62, use the result of Exercise 58 to find the distance between the two points in polar coordinates.

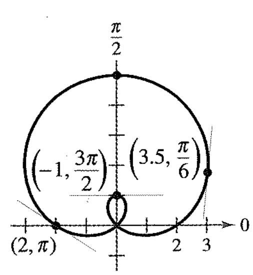
59. $\left(1, \frac{5\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$ **60.** $\left(8, \frac{7\pi}{4}\right), (5, \pi)$

61. $(2, 0.5), (7, 1.2)$ **62.** $(4, 2.5), (12, 1)$

Finding Slopes of Tangent Lines In Exercises 63 and 64, find dy/dx and the slopes of the tangent lines shown on the graph of the polar equation.

63. $r = 2(1 - \sin \theta)$ **64.** $r = 2 + 3 \sin \theta$





Finding Slopes of Tangent Lines Using Technology In Exercises 65–68, use a graphing utility to (a) graph the polar equation, (b) draw the tangent line at the given value of θ, and (c) find dy/dx at the given value of θ. (*Hint:* Let the increment between the values of θ equal $\pi/24$.)

65. $r = 3(1 - \cos \theta), \theta = \frac{\pi}{2}$ **66.** $r = 3 - 2\cos \theta, \theta = 0$

67. $r = 3 \sin \theta, \theta = \frac{\pi}{3}$ **68.** $r = 4, \theta = \frac{\pi}{4}$

Horizontal and Vertical Tangency In Exercises 69 and 70, find the points of horizontal and vertical tangency to the polar curve.

69. $r = 1 - \sin \theta$ **70.** $r = a \sin \theta$

Horizontal Tangency In Exercises 71 and 72, find the points of horizontal tangency to the polar curve.

71. $r = 2 \csc \theta + 3$ **72.** $r = a \sin \theta \cos^2 \theta$

Tangent Lines at the Pole In Exercises 73–80, sketch a graph of the polar equation and find the tangent line(s) at the pole (if any).

73. $r = 5 \sin \theta$ **74.** $r = 5 \cos \theta$

75. $r = 4(1 - \sin \theta)$ **76.** $r = 2(1 - \cos \theta)$

77. $r = 4 \cos 3\theta$ **78.** $r = -\sin 5\theta$

79. $r = 3 \sin 2\theta$ **80.** $r = 3 \cos 2\theta$

Sketching a Polar Graph In Exercises 81–92, sketch a graph of the polar equation.

81. $r = 8$ **82.** $r = 1$

83. $r = 4(1 + \cos \theta)$ **84.** $r = 1 + \sin \theta$

85. $r = 3 - 2\cos \theta$ **86.** $r = 5 - 4 \sin \theta$

87. $r = -7 \csc \theta$ **88.** $r = \dfrac{6}{2\sin \theta - 3\cos \theta}$

89. $r = 3\theta$ **90.** $r = \dfrac{1}{\theta}$

91. $r^2 = 4 \cos 2\theta$ **92.** $r^2 = 4 \sin \theta$

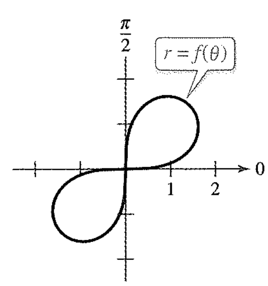
Asymptote In Exercises 93–96, use a graphing utility to graph the equation and show that the given line is an asymptote of the graph.

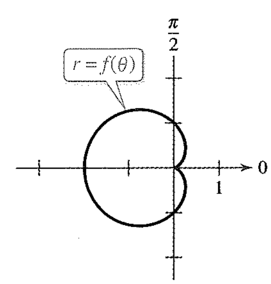
Name of Graph	Polar Equation	Asymptote
93. Conchoid	$r = 2 - \sec \theta$	$x = -1$
94. Conchoid	$r = 2 + \csc \theta$	$y = 1$
95. Hyperbolic spiral	$r = 2/\theta$	$y = 2$
96. Strophoid	$r = 2 \cos 2\theta \sec \theta$	$x = -2$

EXPLORING CONCEPTS

Transformations of Polar Graphs In Exercises 97 and 98, use the graph of $r = f(\theta)$ to sketch a graph of the transformation.

97. $r = f(-\theta)$ **98.** $r = -f(\theta)$





99. Symmetry of Polar Graphs Describe how to test whether a polar graph is symmetric about (a) the x-axis and (b) the y-axis.

10.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

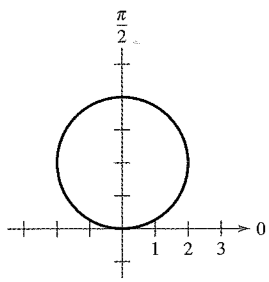
CONCEPT CHECK

1. **Area of a Polar Region** What should you check before applying Theorem 10.13 to find the area of the region bounded by the graph of $r = f(\theta)$?

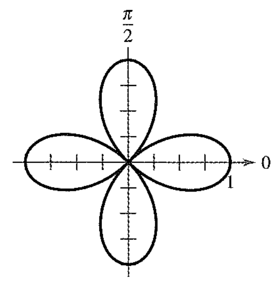
2. **Points of Intersection** Explain why finding points of intersection of polar graphs may require further analysis beyond solving two equations simultaneously.

Area of a Polar Region **In Exercises 3–6, write an integral that represents the area of the shaded region of the figure. Do not evaluate the integral.**

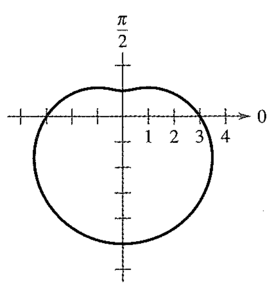
3. $r = 4 \sin \theta$



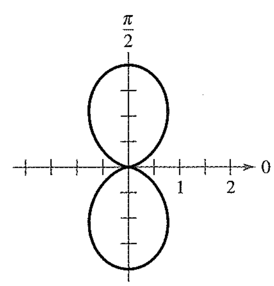
4. $r = \cos 2\theta$



5. $r = 3 - 2 \sin \theta$



6. $r = 1 - \cos 2\theta$



Finding the Area of a Polar Region **In Exercises 7–18, find the area of the region.**

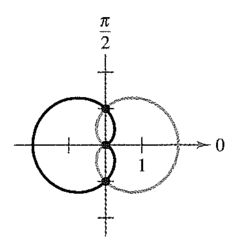
7. Interior of $r = 6 \sin \theta$
8. Interior of $r = 3 \cos \theta$
9. One petal of $r = 2 \cos 3\theta$
10. Two petals of $r = 4 \sin 3\theta$
11. Two petals of $r = \sin 8\theta$
12. Three petals of $r = \cos 5\theta$
13. Interior of $r = 6 + 5 \sin \theta$ (below the polar axis)
14. Interior of $r = 9 - \sin \theta$ (above the polar axis)
15. Interior of $r = 4 + \sin \theta$
16. Interior of $r = 1 - \cos \theta$
17. Interior of $r^2 = 4 \cos 2\theta$
18. Interior of $r^2 = 6 \sin 2\theta$

Finding the Area of a Polar Region **In Exercises 19–26, use a graphing utility to graph the polar equation. Find the area of the given region analytically.**

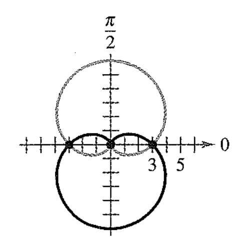
19. Inner loop of $r = 1 + 2 \cos \theta$
20. Inner loop of $r = 2 - 4 \cos \theta$
21. Inner loop of $r = 1 + 2 \sin \theta$
22. Inner loop of $r = 4 - 6 \sin \theta$
23. Between the loops of $r = 1 + 2 \cos \theta$
24. Between the loops of $r = 2(1 + 2 \sin \theta)$
25. Between the loops of $r = 3 - 6 \sin \theta$
26. Between the loops of $r = \frac{1}{2} + \cos \theta$

Finding Points of Intersection **In Exercises 27–34, find the points of intersection of the graphs of the equations.**

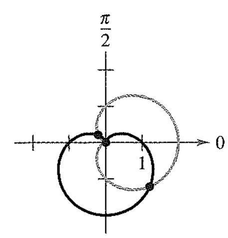
27. $r = 1 + \cos \theta$
$r = 1 - \cos \theta$



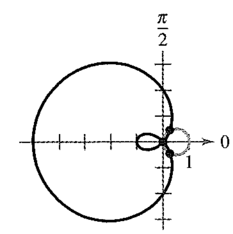
28. $r = 3(1 + \sin \theta)$
$r = 3(1 - \sin \theta)$



29. $r = 1 + \cos \theta$
$r = 1 - \sin \theta$



30. $r = 2 - 3 \cos \theta$
$r = \cos \theta$



31. $r = 4 - 5 \sin \theta$
$r = 3 \sin \theta$

32. $r = 3 + \sin \theta$
$r = 2 \csc \theta$

33. $r = \dfrac{\theta}{2}$
$r = 2$

34. $\theta = \dfrac{\pi}{4}$
$r = 2$

Writing **In Exercises 35 and 36, use a graphing utility to graph the polar equations and approximate the points of intersection of the graphs. Watch the graphs as they are traced in the viewing window. Explain why the pole is not a point of intersection obtained by solving the equations simultaneously.**

35. $r = \cos \theta$
$r = 2 - 3 \sin \theta$

36. $r = 4 \sin \theta$
$r = 2(1 + \sin \theta)$

Finding the Area of a Polar Region Between Two Curves **In Exercises 37–44, use a graphing utility to graph the polar equations. Find the area of the given region analytically.**

37. Common interior of $r = 4 \sin 2\theta$ and $r = 2$

38. Common interior of $r = 2(1 + \cos \theta)$ and $r = 2(1 - \cos \theta)$

39. Common interior of $r = 3 - 2 \sin \theta$ and $r = -3 + 2 \sin \theta$

40. Common interior of $r = 5 - 3 \sin \theta$ and $r = 5 - 3 \cos \theta$

41. Common interior of $r = 4 \sin \theta$ and $r = 2$

42. Common interior of $r = 2 \cos \theta$ and $r = 2 \sin \theta$

43. Inside $r = 2 \cos \theta$ and outside $r = 1$

44. Inside $r = 3 \sin \theta$ and outside $r = 1 + \sin \theta$

Finding the Area of a Polar Region Between Two Curves **In Exercises 45–48, find the area of the region.**

45. Inside $r = a(1 + \cos \theta)$ and outside $r = a \cos \theta$

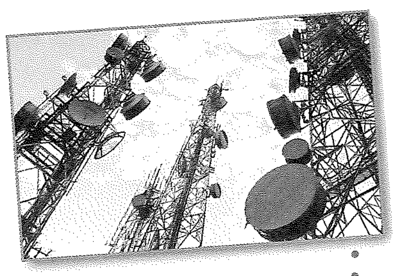
46. Inside $r = 2a \cos \theta$ and outside $r = a$

47. Common interior of $r = a(1 + \cos \theta)$ and $r = a \sin \theta$

48. Common interior of $r = a \cos \theta$ and $r = a \sin \theta$, where $a > 0$

49. Antenna Radiation

The radiation from a transmitting antenna is not uniform in all directions. The intensity from a particular antenna is modeled by $r = a \cos^2 \theta$.



(a) Convert the polar equation to rectangular form.

(b) Use a graphing utility to graph the model for $a = 4$ and $a = 6$.

(c) Find the area of the geographical region between the two curves in part (b).

50. **Area** The area inside one or more of the three interlocking circles $r = 2a \cos \theta$, $r = 2a \sin \theta$, and $r = a$ is divided into seven regions. Find the area of each region.

51. **Conjecture** Find the area of the region enclosed by

$$r = a \cos(n\theta)$$

for $n = 1, 2, 3, \ldots$. Use the results to make a conjecture about the area enclosed by the function when n is even and when n is odd.

52. **Area** Sketch the strophoid

$$r = \sec \theta - 2 \cos \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Convert this equation to rectangular coordinates. Find the area enclosed by the loop.

Finding the Arc Length of a Polar Curve **In Exercises 53–58, find the length of the curve over the given interval.**

53. $r = 8, \quad \left[0, \frac{\pi}{6}\right]$

54. $r = a, \quad [0, 2\pi]$

55. $r = 4 \sin \theta, \quad [0, \pi]$

56. $r = 2a \cos \theta, \quad \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

57. $r = 1 + \sin \theta, \quad [0, 2\pi]$

58. $r = 8(1 + \cos \theta), \quad \left[0, \frac{\pi}{3}\right]$

Finding the Arc Length of a Polar Curve **In Exercises 59–64, use a graphing utility to graph the polar equation over the given interval. Use the integration capabilities of the graphing utility to approximate the length of the curve.**

59. $r = 2\theta, \quad \left[0, \frac{\pi}{2}\right]$

60. $r = \sec \theta, \quad \left[0, \frac{\pi}{3}\right]$

61. $r = \frac{1}{\theta}, \quad [\pi, 2\pi]$

62. $r = e^{\theta}, \quad [0, \pi]$

63. $r = \sin(3 \cos \theta), \quad [0, \pi]$

64. $r = 2 \sin(2 \cos \theta), \quad [0, \pi]$

Finding the Area of a Surface of Revolution **In Exercises 65–68, find the area of the surface formed by revolving the polar equation over the given interval about the given line.**

Polar Equation	Interval	Axis of Revolution
65. $r = 6 \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$	Polar axis
66. $r = a \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
67. $r = e^{a\theta}$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
68. $r = a(1 + \cos \theta)$	$0 \le \theta \le \pi$	Polar axis

Finding the Area of a Surface of Revolution **In Exercises 69 and 70, use the integration capabilities of a graphing utility to approximate the area of the surface formed by revolving the polar equation over the given interval about the polar axis.**

69. $r = 4 \cos 2\theta, \quad \left[0, \frac{\pi}{4}\right]$

70. $r = \theta, \quad [0, \pi]$

EXPLORING CONCEPTS

Using Different Methods **In Exercises 71 and 72, (a) sketch the graph of the polar equation, (b) determine the interval that traces the graph only once, (c) find the area of the region bounded by the graph using a geometric formula, and (d) find the area of the region bounded by the graph using integration.**

71. $r = 10 \cos \theta$

72. $r = 5 \sin \theta$

73. **Think About It** Let $f(\theta) > 0$ for all θ and let $g(\theta) < 0$ for all θ. Find polar equations $r = f(\theta)$ and $r = g(\theta)$ such that their graphs intersect.

12.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Vector-Valued Function** Describe how you can use a vector-valued function to represent a curve.
2. **Continuity of a Vector-Valued Function** Describe what it means for a vector-valued function $\mathbf{r}(t)$ to be continuous at a point.

Finding the Domain **In Exercises 3–10, find the domain of the vector-valued function.**

3. $\mathbf{r}(t) = \dfrac{1}{t+1}\mathbf{i} + \dfrac{t}{2}\mathbf{j} - 3t\mathbf{k}$
4. $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$
5. $\mathbf{r}(t) = \ln t\,\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$
6. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + t\mathbf{k}$
7. $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t)$, where
   $\mathbf{F}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $\mathbf{G}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$
8. $\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t)$, where
   $\mathbf{F}(t) = \ln t\,\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}$, $\mathbf{G}(t) = \mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k}$
9. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$, where
   $\mathbf{F}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$, $\mathbf{G}(t) = \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$
10. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$, where
   $\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j} + t\mathbf{k}$, $\mathbf{G}(t) = \sqrt[3]{t}\,\mathbf{i} + \dfrac{1}{t+1}\mathbf{j} + (t+2)\mathbf{k}$

Evaluating a Function **In Exercises 11 and 12, evaluate the vector-valued function at each given value of t.**

11. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$
   (a) $\mathbf{r}(1)$ (b) $\mathbf{r}(0)$ (c) $\mathbf{r}(s+1)$
   (d) $\mathbf{r}(2+\Delta t) - \mathbf{r}(2)$
12. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$
   (a) $\mathbf{r}(0)$ (b) $\mathbf{r}(\pi/4)$ (c) $\mathbf{r}(\theta - \pi)$
   (d) $\mathbf{r}(\pi/6 + \Delta t) - \mathbf{r}(\pi/6)$

Writing a Vector-Valued Function **In Exercises 13–16, represent the line segment from P to Q by a vector-valued function and by a set of parametric equations.**

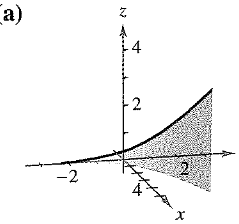
13. $P(0, 0, 0)$, $Q(5, 2, 2)$
14. $P(0, 2, -1)$, $Q(4, 7, 2)$
15. $P(-3, -6, -1)$, $Q(-1, -9, -8)$
16. $P(1, -6, 8)$, $Q(-3, -2, 5)$

Think About It **In Exercises 17 and 18, find $\mathbf{r}(t) \cdot \mathbf{u}(t)$. Is the result a vector-valued function? Explain.**

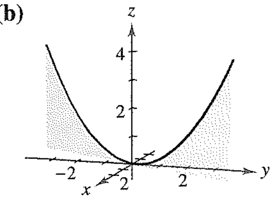
17. $\mathbf{r}(t) = (3t-1)\mathbf{i} + \frac{1}{4}t^3\mathbf{j} + 4\mathbf{k}$, $\mathbf{u}(t) = t^2\mathbf{i} - 8\mathbf{j} + t^3\mathbf{k}$
18. $\mathbf{r}(t) = \langle 3\cos t, 2\sin t, t-2\rangle$, $\mathbf{u}(t) = \langle 4\sin t, -6\cos t, t^2\rangle$

Matching **In Exercises 19–22, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

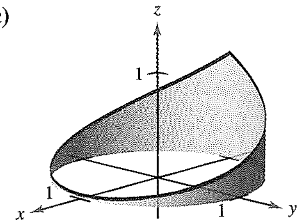
(a)



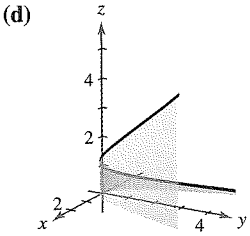
(b)



(c)



(d)



19. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$, $-2 \le t \le 2$
20. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}$, $-1 \le t \le 1$
21. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}$, $-2 \le t \le 2$
22. $\mathbf{r}(t) = t\mathbf{i} + \ln t\,\mathbf{j} + \dfrac{2t}{3}\mathbf{k}$, $0.1 \le t \le 5$

Sketching a Plane Curve **In Exercises 23–30, sketch the plane curve represented by the vector-valued function and give the orientation of the curve.**

23. $\mathbf{r}(t) = \dfrac{t}{4}\mathbf{i} + (t-1)\mathbf{j}$
24. $\mathbf{r}(t) = (5-t)\mathbf{i} + \sqrt{t}\,\mathbf{j}$
25. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$
26. $\mathbf{r}(t) = (t^2+t)\mathbf{i} + (t^2-t)\mathbf{j}$
27. $\mathbf{r}(\theta) = \cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j}$
28. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$
29. $\mathbf{r}(\theta) = 3\sec\theta\,\mathbf{i} + 2\tan\theta\,\mathbf{j}$
30. $\mathbf{r}(t) = 2\cos^3 t\,\mathbf{i} + 2\sin^3 t\,\mathbf{j}$

Sketching a Space Curve **In Exercises 31–38, sketch the space curve represented by the vector-valued function and give the orientation of the curve.**

31. $\mathbf{r}(t) = (-t+1)\mathbf{i} + (4t+2)\mathbf{j} + (2t+3)\mathbf{k}$
32. $\mathbf{r}(t) = t\mathbf{i} + (2t-5)\mathbf{j} + 3t\mathbf{k}$
33. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$
34. $\mathbf{r}(t) = t\mathbf{i} + 3\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}$
35. $\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + e^{-t}\mathbf{k}$
36. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \frac{3}{2}t\mathbf{k}$
37. $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3\right\rangle$
38. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

12.2 Exercises

CONCEPT CHECK

1. **Derivative** Describe the relationship between the graph of $\mathbf{r}'(t_0)$ and the curve represented by $\mathbf{r}(t)$.
2. **Integration** Explain why the family of vector-valued functions that are the antiderivatives of a vector-valued function differ by a constant vector.

Differentiation of Vector-Valued Functions In Exercises 3–10, find $\mathbf{r}'(t)$, $\mathbf{r}(t_0)$, and $\mathbf{r}'(t_0)$ for the given value of t_0. Then sketch the curve represented by the vector-valued function and sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$.

3. $\mathbf{r}(t) = (1 - t^2)\mathbf{i} + t\mathbf{j}, \quad t_0 = 3$
4. $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j}, \quad t_0 = 1$
5. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad t_0 = \dfrac{\pi}{2}$
6. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}, \quad t_0 = \dfrac{\pi}{2}$
7. $\mathbf{r}(t) = \langle e^t, e^{2t}\rangle, \quad t_0 = 0$
8. $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle, \quad t_0 = 0$
9. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad t_0 = \dfrac{3\pi}{2}$
10. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, \quad t_0 = 2$

Finding a Derivative In Exercises 11–18, find $\mathbf{r}'(t)$.

11. $\mathbf{r}(t) = t^4\mathbf{i} - 5t\mathbf{j}$
12. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (1 - t^3)\mathbf{j}$
13. $\mathbf{r}(t) = 3\cos^3 t\mathbf{i} + 2\sin^3 t\mathbf{j} + \mathbf{k}$
14. $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$
15. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j} + 5te^t\mathbf{k}$
16. $\mathbf{r}(t) = \langle t^3, \cos 3t, \sin 3t\rangle$
17. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$
18. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

Higher-Order Differentiation In Exercises 19–22, find (a) $\mathbf{r}'(t)$, (b) $\mathbf{r}''(t)$, and (c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

19. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$
20. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$
21. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$
22. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

Higher-Order Differentiation In Exercises 23–26, find (a) $\mathbf{r}'(t)$, (b) $\mathbf{r}''(t)$, (c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$, and (d) $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

23. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$
24. $\mathbf{r}(t) = t^3\mathbf{i} + (2t^2 + 3)\mathbf{j} + (3t - 5)\mathbf{k}$
25. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$
26. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

Finding Intervals on Which a Curve Is Smooth In Exercises 27–34, find the open interval(s) on which the curve given by the vector-valued function is smooth.

27. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$
28. $\mathbf{r}(t) = 5t^5\mathbf{i} - t^4\mathbf{j}$
29. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}, \quad 0 \le \theta \le 2\pi$
30. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}, \quad 0 \le \theta \le 2\pi$
31. $\mathbf{r}(t) = \dfrac{2t}{8 + t^3}\mathbf{i} + \dfrac{2t^2}{8 + t^3}\mathbf{j}$
32. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$
33. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$
34. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

Using Properties of the Derivative In Exercises 35 and 36, use the properties of the derivative to find the following.

(a) $\mathbf{r}'(t)$ (b) $\dfrac{d}{dt}[3\mathbf{r}(t) - \mathbf{u}(t)]$ (c) $\dfrac{d}{dt}[(5t)\mathbf{u}(t)]$

(d) $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ (e) $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)]$ (f) $\dfrac{d}{dt}[\mathbf{r}(2t)]$

35. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}, \quad \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$
36. $\mathbf{r}(t) = \langle t, 2\sin t, 2\cos t\rangle, \mathbf{u}(t) = \left\langle \dfrac{1}{t}, 2\sin t, 2\cos t\right\rangle$

Using Two Methods In Exercises 37 and 38, find (a) $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and (b) $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)]$ in two different ways.

(i) Find the product first, then differentiate.
(ii) Apply the properties of Theorem 12.2.

37. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{u}(t) = t^4\mathbf{k}$
38. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

Finding an Indefinite Integral In Exercises 39–46, find the indefinite integral.

39. $\displaystyle\int (2t\mathbf{i} + \mathbf{j} + 9\mathbf{k})\,dt$
40. $\displaystyle\int \left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right)dt$
41. $\displaystyle\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right)dt$
42. $\displaystyle\int \left(\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right)dt$
43. $\displaystyle\int (\mathbf{i} + 4t^3\mathbf{j} + 5^t\mathbf{k})\,dt$
44. $\displaystyle\int \left(\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right)dt$
45. $\displaystyle\int (e^t\mathbf{i} + \mathbf{j} + t\cos t\mathbf{k})\,dt$
46. $\displaystyle\int (e^{-t}\sin t\mathbf{i} + \cot t\mathbf{j})\,dt$

Evaluating a Definite Integral **In Exercises 47–52, evaluate the definite integral.**

47. $\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt$

48. $\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}\right)dt$

49. $\int_0^{\pi/2} [(5\cos t)\mathbf{i} + (6\sin t)\mathbf{j} + \mathbf{k}]\,dt$

50. $\int_0^{\pi/4} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt$

51. $\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k})\,dt$

52. $\int_0^3 \|t\mathbf{i} + t^2\mathbf{j}\|\,dt$

Finding an Antiderivative **In Exercises 53–58, find r(t) that satisfies the initial condition(s).**

53. $\mathbf{r}'(t) = 4e^{2t}\mathbf{i} + 3e^t\mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i}$

54. $\mathbf{r}'(t) = 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

55. $\mathbf{r}''(t) = -32\mathbf{j}, \quad \mathbf{r}'(0) = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

56. $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}, \quad \mathbf{r}'(0) = 3\mathbf{k}, \quad \mathbf{r}(0) = 4\mathbf{j}$

57. $\mathbf{r}'(t) = te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}$

58. $\mathbf{r}'(t) = \frac{1}{1+t^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}, \quad \mathbf{r}(1) = 2\mathbf{i}$

EXPLORING CONCEPTS

59. **Using a Derivative** The three components of the derivative of the vector-valued function **u** are positive at $t = t_0$. Describe the behavior of **u** at $t = t_0$.

60. **Think About It** Find two vector-valued functions **f**(t) and **g**(t) such that

$$\int_a^b [\mathbf{f}(t)\cdot\mathbf{g}(t)]\,dt \neq \left[\int_a^b \mathbf{f}(t)\,dt\right]\cdot\left[\int_a^b \mathbf{g}(t)\,dt\right].$$

Proof **In Exercises 61–68, prove the property. In each case, assume r, u, and v are differentiable vector-valued functions of t in space, w is a differentiable real-valued function of t, and c is a scalar.**

61. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

62. $\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

63. $\frac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$

64. $\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)] = \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t)$

65. $\frac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$

66. $\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t)$

67. $\frac{d}{dt}\{\mathbf{r}(t)\cdot[\mathbf{u}(t)\times\mathbf{v}(t)]\} = \mathbf{r}'(t)\cdot[\mathbf{u}(t)\times\mathbf{v}(t)] + \mathbf{r}(t)\cdot[\mathbf{u}'(t)\times\mathbf{v}(t)] + \mathbf{r}(t)\cdot[\mathbf{u}(t)\times\mathbf{v}'(t)]$

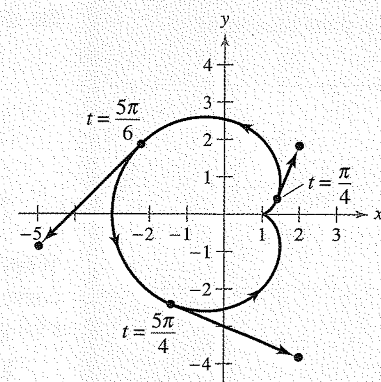
68. If $\mathbf{r}(t)\cdot\mathbf{r}(t)$ is a constant, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.

69. **Particle Motion** A particle moves in the xy-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$.
    (a) Use a graphing utility to graph **r**. Describe the curve.
    (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

70. **Particle Motion** A particle moves in the yz-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (2\cos t)\mathbf{j} + (3\sin t)\mathbf{k}$.
    (a) Describe the curve.
    (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

71. **Perpendicular Vectors** Consider the vector-valued function $\mathbf{r}(t) = (e^t\sin t)\mathbf{i} + (e^t\cos t)\mathbf{j}$. Show that $\mathbf{r}(t)$ and $\mathbf{r}''(t)$ are always perpendicular to each other.

72. **HOW DO YOU SEE IT?** The graph shows a vector-valued function $\mathbf{r}(t)$ for $0 \le t \le 2\pi$ and its derivative $\mathbf{r}'(t)$ for several values of t.



(a) For each derivative shown in the graph, determine whether each component is positive or negative.
(b) Is the curve smooth on the interval $[0, 2\pi]$? Explain.

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.

74. The definite integral of a vector-valued function is a real number.

75. $\frac{d}{dt}[\|\mathbf{r}(t)\|] = \|\mathbf{r}'(t)\|$

76. If **r** and **u** are differentiable vector-valued functions of t, then

$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] = \mathbf{r}'(t)\cdot\mathbf{u}'(t)$.

12.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Velocity Vector** An object moves along a curve in the plane. What information do you gain about the motion of the object from the velocity vector to the curve at time t?
2. **Acceleration Vectors** For each scenario, describe the direction of the acceleration vectors. Explain your reasoning.
   (a) A comet traveling through our solar system in a parabolic path
   (b) An object thrown on Earth's surface

Finding Velocity and Acceleration Along a Plane Curve **In Exercises 3–10, the position vector r describes the path of an object moving in the xy-plane.**

(a) Find the velocity vector, speed, and acceleration vector of the object.

(b) Evaluate the velocity vector and acceleration vector of the object at the given point.

(c) Sketch a graph of the path and sketch the velocity and acceleration vectors at the given point.

Position Vector	Point
3. $\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}$	$(3, 0)$
4. $\mathbf{r}(t) = t\mathbf{i} + (-t^2+4)\mathbf{j}$	$(1, 3)$
5. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$	$(4, 2)$
6. $\mathbf{r}(t) = \left(\frac{1}{4}t^3 + 1\right)\mathbf{i} + t\mathbf{j}$	$(3, 2)$
7. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$	$\left(\sqrt{2}, \sqrt{2}\right)$
8. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$	$(3, 0)$
9. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$	$(\pi, 2)$
10. $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle$	$(1, 1)$

Finding Velocity and Acceleration Vectors in Space **In Exercises 11–20, the position vector r describes the path of an object moving in space.**

(a) Find the velocity vector, speed, and acceleration vector of the object.

(b) Evaluate the velocity vector and acceleration vector of the object at the given value of t.

Position Vector	Time
11. $\mathbf{r}(t) = t\mathbf{i} + 5t\mathbf{j} + 3t\mathbf{k}$	$t = 1$
12. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} - 2t\mathbf{k}$	$t = 3$
13. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$	$t = 4$
14. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$	$t = 2$
15. $\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + \sqrt{9-t^2}\,\mathbf{k}$	$t = 0$
16. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$	$t = 4$
17. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$	$t = \pi$
18. $\mathbf{r}(t) = \langle 2\cos t, \sin 3t, t^2\rangle$	$t = \dfrac{\pi}{4}$
19. $\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$	$t = 0$
20. $\mathbf{r}(t) = \left\langle \ln t, \dfrac{1}{t^2}, t^4\right\rangle$	$t = \sqrt{3}$

Finding a Position Vector by Integration **In Exercises 21–26, use the given acceleration vector and initial conditions to find the velocity and position vectors. Then find the position at time $t = 2$.**

21. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v}(0) = \mathbf{0}$, $\mathbf{r}(0) = \mathbf{0}$
22. $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}$, $\mathbf{v}(0) = 4\mathbf{j}$, $\mathbf{r}(0) = \mathbf{0}$
23. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}$, $\mathbf{v}(1) = 5\mathbf{j}$, $\mathbf{r}(1) = \mathbf{0}$
24. $\mathbf{a}(t) = -32\mathbf{k}$, $\mathbf{v}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{r}(0) = 5\mathbf{j} + 2\mathbf{k}$
25. $\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$, $\mathbf{v}(0) = \mathbf{j} + \mathbf{k}$, $\mathbf{r}(0) = \mathbf{i}$
26. $\mathbf{a}(t) = e^t\mathbf{i} - 8\mathbf{k}$, $\mathbf{v}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{r}(0) = \mathbf{0}$

Projectile Motion **In Exercises 27–40, use the model for projectile motion, assuming there is no air resistance and $g = 32$ feet per second per second.**

27. A baseball is hit from a height of 2.5 feet above the ground with an initial speed of 140 feet per second and at an angle of 22° above the horizontal. Find the maximum height reached by the baseball. Determine whether it will clear a 10-foot-high fence located 375 feet from home plate.
28. Determine the maximum height and range of a projectile fired at a height of 3 feet above the ground with an initial speed of 900 feet per second and at an angle of 45° above the horizontal.
29. A baseball, hit 3 feet above the ground, leaves the bat at an angle of 45° and is caught by an outfielder 3 feet above the ground and 300 feet from home plate. What is the initial speed of the ball, and how high does it rise?
30. A baseball player at second base throws a ball 90 feet to the player at first base. The ball is released at a point 5 feet above the ground with an initial speed of 50 miles per hour and at an angle of 15° above the horizontal. At what height does the player at first base catch the ball?
31. Eliminate the parameter t from the position vector for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{g\sec^2\theta}{2v_0^2}x^2 + (\tan\theta)x + h.$$

32. The path of a ball is given by the rectangular equation $y = x - 0.005x^2$. Use the result of Exercise 31 to find the position vector. Then find the speed and direction of the ball at the point at which it has traveled 60 feet horizontally.

33. The Rogers Centre in Toronto, Ontario, has a center field fence that is 10 feet high and 400 feet from home plate. A ball is hit 3 feet above the ground and leaves the bat at a speed of 100 miles per hour.

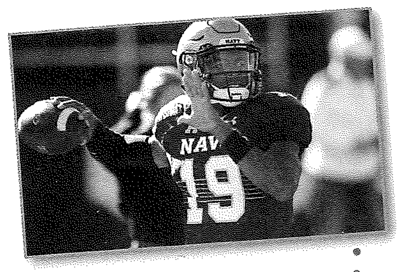
(a) The ball leaves the bat at an angle of $\theta = \theta_0$ with the horizontal. Write the vector-valued function for the path of the ball.

(b) Use a graphing utility to graph the vector-valued function for $\theta_0 = 10°$, $\theta_0 = 15°$, $\theta_0 = 20°$, and $\theta_0 = 25°$. Use the graphs to approximate the minimum angle required for the hit to be a home run.

(c) Determine analytically the minimum angle required for the hit to be a home run.

34. Football

The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver 30 yards directly downfield at a height of 4 feet. The pass is released at an angle of 35° with the horizontal.



(a) Find the speed of the football when it is released.

(b) Find the maximum height of the football.

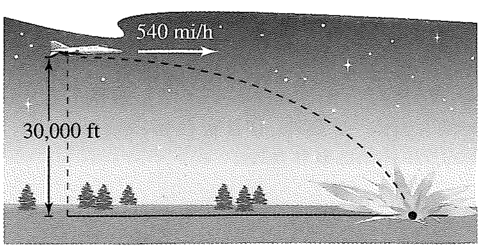
(c) Find the time the receiver has to reach the proper position after the quarterback releases the football.

35. A bale ejector consists of two variable-speed belts at the end of a baler. Its purpose is to toss bales into a trailing wagon. In loading the back of a wagon, a bale must be thrown to a position 8 feet above and 16 feet behind the ejector.

(a) Find the minimum initial speed of the bale and the corresponding angle at which it must be ejected from the baler.

(b) The ejector has a fixed angle of 45°. Find the initial speed required.

36. A bomber is flying horizontally at an altitude of 30,000 feet with a speed of 540 miles per hour (see figure). When should the bomb be released for it to hit the target? (Give your answer in terms of the angle of depression from the plane to the target.) What is the speed of the bomb at the time of impact?



37. A shot fired from a gun with a muzzle speed of 1200 feet per second is to hit a target 3000 feet away. Determine the minimum angle of elevation of the gun.

38. A projectile is fired from ground level at an angle of 12° with the horizontal. The projectile is to have a range of 200 feet. Find the minimum initial speed necessary.

39. Use a graphing utility to graph the paths of a projectile for the given values of θ and v_0. For each case, use the graph to approximate the maximum height and range of the projectile. (Assume that the projectile is launched from ground level.)

(a) $\theta = 10°$, $v_0 = 66$ ft/sec

(b) $\theta = 10°$, $v_0 = 146$ ft/sec

(c) $\theta = 45°$, $v_0 = 66$ ft/sec

(d) $\theta = 45°$, $v_0 = 146$ ft/sec

(e) $\theta = 60°$, $v_0 = 66$ ft/sec

(f) $\theta = 60°$, $v_0 = 146$ ft/sec

40. Find the angles at which an object must be thrown to obtain (a) the maximum range and (b) the maximum height.

Projectile Motion **In Exercises 41 and 42, use the model for projectile motion, assuming there is no air resistance and $g = 9.8$ meters per second per second.**

41. Determine the maximum height and range of a projectile fired at a height of 1.5 meters above the ground with an initial speed of 100 meters per second and at an angle of 30° above the horizontal.

42. A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 meters. Find the minimum initial speed necessary.

43. Shot-Put Throw The path of a shot thrown at an angle θ is

$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where v_0 is the initial speed, h is the initial height, t is the time in seconds, and g is the acceleration due to gravity. Verify that the shot will remain in the air for a total of

$$t = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \text{ seconds}$$

and will travel a horizontal distance of

$$\frac{v_0^2 \cos \theta}{g}\left(\sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}}\right) \text{ feet.}$$

44. Shot-Put Throw

A shot is thrown from a height of $h = 5.75$ feet with an initial speed of $v_0 = 41$ feet per second and at an angle of $\theta = 42.5°$ with the horizontal. Use the result of Exercise 43 to find the total time of travel and the total horizontal distance traveled.

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12.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Unit Tangent Vector** How is the unit tangent vector related to the orientation of a curve? Explain.
2. **Principal Unit Normal Vector** In what direction does the principal unit normal vector point?

Finding the Unit Tangent Vector **In Exercises 3–8, find the unit tangent vector to the curve at the specified value of the parameter.**

3. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad t = 1$
4. $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}, \quad t = 1$
5. $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}, \quad t = \dfrac{\pi}{3}$
6. $\mathbf{r}(t) = 6\sin t\mathbf{i} - 2\cos t\mathbf{j}, \quad t = \dfrac{\pi}{6}$
7. $\mathbf{r}(t) = 3t\mathbf{i} - \ln t\mathbf{j}, \quad t = e$
8. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\mathbf{j}, \quad t = 0$

Finding a Tangent Line **In Exercises 9–14, find the unit tangent vector $\mathbf{T}(t)$ and a set of parametric equations for the tangent line to the space curve at point P.**

9. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, \quad P(0, 0, 0)$
10. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k}, \quad P\left(1, 1, \frac{4}{3}\right)$
11. $\mathbf{r}(t) = \cos t\mathbf{i} + 3\sin t\mathbf{j} + (3t - 4)\mathbf{k}, \quad P(1, 0, -4)$
12. $\mathbf{r}(t) = \left\langle t, t, \sqrt{4 - t^2}\right\rangle, \quad P\left(1, 1, \sqrt{3}\right)$
13. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle, \quad P\left(\sqrt{2}, \sqrt{2}, 4\right)$
14. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle, \quad P\left(1, \sqrt{3}, 1\right)$

Finding the Principal Unit Normal Vector **In Exercises 15–20, find the principal unit normal vector to the curve at the specified value of the parameter.**

15. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}, \quad t = 2$
16. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{6}{t}\mathbf{j}, \quad t = 3$
17. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, \quad t = 1$
18. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, \quad t = 0$
19. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \mathbf{k}, \quad t = \dfrac{3\pi}{4}$
20. $\mathbf{r}(t) = \cos 3t\mathbf{i} + 2\sin 3t\mathbf{j} + \mathbf{k}, \quad t = \pi$

Sketching a Graph and Vectors **In Exercises 21–24, sketch the graph of the plane curve $\mathbf{r}(t)$ and sketch the vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ at the given value of t.**

21. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad t = 2$
22. $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}, \quad t = 1$
23. $\mathbf{r}(t) = (2t + 1)\mathbf{i} - t^2\mathbf{j}, \quad t = 2$
24. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad t = \dfrac{7\pi}{6}$

Finding Tangential and Normal Components of Acceleration **In Exercises 25–30, find the tangential and normal components of acceleration at the given time t for the plane curve $\mathbf{r}(t)$.**

25. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad t = 1$
26. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad t = 1$
27. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j}, \quad t = 0$
28. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}, \quad t = 0$
29. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j}, \quad t = \dfrac{\pi}{2}$
30. $\mathbf{r}(t) = 4\cos 3t\mathbf{i} + 4\sin 3t\mathbf{j}, \quad t = \pi$

Circular Motion **In Exercises 31–34, consider an object moving according to the position vector**

$$\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}.$$

31. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$.
32. Determine the directions of $\mathbf{T}$ and $\mathbf{N}$ relative to the position vector $\mathbf{r}$.
33. Determine the speed of the object at any time t and explain its value relative to the value of $a_{\mathbf{T}}$.
34. When the angular speed ω is halved, by what factor is $a_{\mathbf{N}}$ changed?

Finding Tangential and Normal Components of Acceleration **In Exercises 35–40, find the tangential and normal components of acceleration at the given time t for the space curve $\mathbf{r}(t)$.**

35. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}, \quad t = 1$
36. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}, \quad t = \dfrac{\pi}{3}$
37. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}, \quad t = 1$
38. $\mathbf{r}(t) = (2t - 1)\mathbf{i} + t^2\mathbf{j} - 4t\mathbf{k}, \quad t = 2$
39. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j} + e^t\mathbf{k}, \quad t = 0$
40. $\mathbf{r}(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k}, \quad t = 0$

EXPLORING CONCEPTS

41. **Acceleration** Describe the motion of a particle when the normal component of acceleration is 0.
42. **Acceleration** Describe the motion of a particle when the tangential component of acceleration is 0.

12.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

CONCEPT CHECK

1. **Curvature** Consider points P and Q on a curve. What does it mean for the curvature at P to be less than the curvature at Q?

2. **Arc Length Parameter** Let $\mathbf{r}(t)$ be a space curve. How can you determine whether t is the arc length parameter?

Finding the Arc Length of a Plane Curve **In Exercises 3–8, sketch the plane curve and find its length over the given interval.**

3. $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j}$, $[0, 3]$

4. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $[0, 4]$

5. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, $[0, 1]$

6. $\mathbf{r}(t) = t^2\mathbf{i} - 4t\mathbf{j}$, $[0, 5]$

7. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}$, $[0, 2\pi]$

8. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $[0, 2\pi]$

9. **Projectile Motion** The position of a baseball is represented by $\mathbf{r}(t) = 50\sqrt{2}t\mathbf{i} + \left(3 + 50\sqrt{2}t - 16t^2\right)\mathbf{j}$. Find the arc length of the trajectory of the baseball.

10. **Projectile Motion** The position of a baseball is represented by $\mathbf{r}(t) = 40\sqrt{3}t\mathbf{i} + (4 + 40t - 16t^2)\mathbf{j}$. Find the arc length of the trajectory of the baseball.

Finding the Arc Length of a Curve in Space **In Exercises 11–16, sketch the space curve and find its length over the given interval.**

11. $\mathbf{r}(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}$, $[0, 1]$

12. $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $[0, 2]$

13. $\mathbf{r}(t) = \langle 4t, -\cos t, \sin t\rangle$, $\left[0, \dfrac{3\pi}{2}\right]$

14. $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$, $[0, \pi]$

15. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$, $[0, 2\pi]$

16. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle$, $\left[0, \dfrac{\pi}{2}\right]$

17. **Investigation** Consider the graph of the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k}$ on the interval $[0, 2]$.

(a) Approximate the length of the curve by finding the length of the line segment connecting its endpoints.

(b) Approximate the length of the curve by summing the lengths of the line segments connecting the terminal points of the vectors $\mathbf{r}(0)$, $\mathbf{r}(0.5)$, $\mathbf{r}(1)$, $\mathbf{r}(1.5)$, and $\mathbf{r}(2)$.

(c) Describe how you could obtain a more accurate approximation by continuing the processes in parts (a) and (b).

(d) Use the integration capabilities of a graphing utility to approximate the length of the curve. Compare this result with the answers in parts (a) and (b).

18. **Investigation** Consider the helix represented by the vector-valued function $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$.

(a) Write the length of the arc s on the helix as a function of t by evaluating the integral

$$s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du.$$

(b) Solve for t in the relationship derived in part (a), and substitute the result into the original vector-valued function. This yields a parametrization of the curve in terms of the arc length parameter s.

(c) Find the coordinates of the point on the helix for arc lengths $s = \sqrt{5}$ and $s = 4$.

(d) Verify that $\|\mathbf{r}'(s)\| = 1$.

Finding Curvature **In Exercises 19–22, find the curvature of the curve, where s is the arc length parameter.**

19. $\mathbf{r}(s) = \left(1 + \dfrac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \dfrac{\sqrt{2}}{2}s\right)\mathbf{j}$

20. $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$

21. $\mathbf{r}(s) = \cos\dfrac{1}{2}s\mathbf{i} + \dfrac{\sqrt{3}}{2}s\mathbf{j} + \sin\dfrac{1}{2}s\mathbf{k}$

22. $\mathbf{r}(s) = \cos s\mathbf{i} + \sin s\mathbf{j} + 5\mathbf{k}$

Finding Curvature **In Exercises 23–28, find the curvature of the plane curve at the given value of the parameter.**

23. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$, $t = 1$

24. $\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j}$, $t = 2$

25. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}$, $t = 1$

26. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{9}t^3\mathbf{j}$, $t = 2$

27. $\mathbf{r}(t) = \langle t, \sin t\rangle$, $t = \dfrac{\pi}{2}$

28. $\mathbf{r}(t) = \langle 5\cos t, 4\sin t\rangle$, $t = \dfrac{\pi}{3}$

Finding Curvature **In Exercises 29–36, find the curvature of the curve.**

29. $\mathbf{r}(t) = 4\cos 2\pi t\mathbf{i} + 4\sin 2\pi t\mathbf{j}$

30. $\mathbf{r}(t) = 2\cos \pi t\mathbf{i} + \sin \pi t\mathbf{j}$

31. $\mathbf{r}(t) = a\cos \omega t\mathbf{i} + a\sin \omega t\mathbf{j}$

32. $\mathbf{r}(t) = a\cos \omega t\mathbf{i} + b\sin \omega t\mathbf{j}$

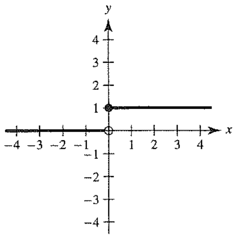
33. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

34. $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}$

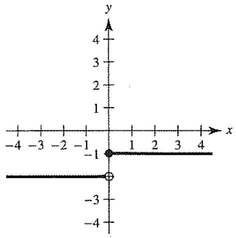
35. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

36. $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{2t}\cos t\mathbf{j} + e^{2t}\sin t\mathbf{k}$

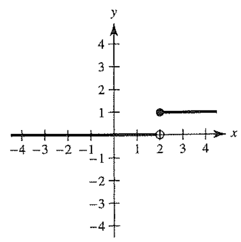
3.



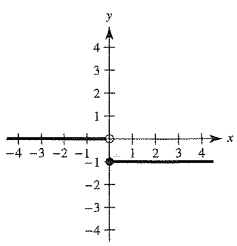
(a)



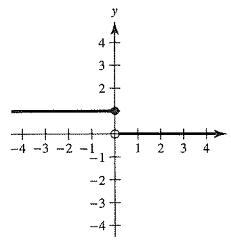
(b)



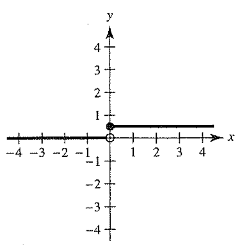
(c)



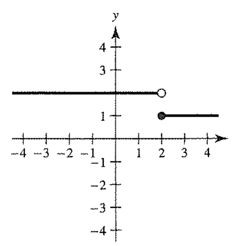
(d)



(e)

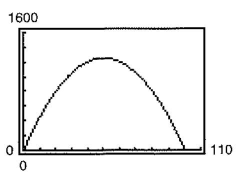


(f)



5. (a) $A(x) = x\left(\dfrac{100 - x}{2}\right)$; Domain: $(0, 100)$

(b)



Dimensions 50 m × 25 m yield maximum area of 1250 m^2.

(c) 50 m × 25 m; Area = 1250 m^2

7. $T(x) = \dfrac{2\sqrt{4 + x^2} + \sqrt{(3 - x)^2 + 1}}{4}$

9. (a) 5, less (b) 3, greater (c) 4.1, less
(d) $4 + h$ (e) 4; Answers will vary.

11. (a) Domain: $(-\infty, 1) \cup (1, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$

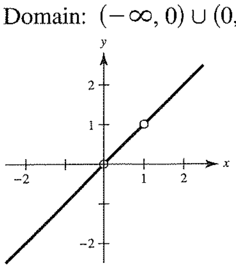
(b) $f(f(x)) = \dfrac{x - 1}{x}$

Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

(c) $f(f(f(x))) = x$

Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

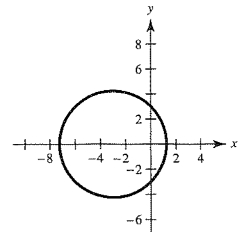
(d)



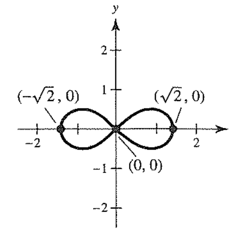
The graph is not a line because there are holes at $x = 0$ and $x = 1$.

13. (a) $x \approx 1.2426, -7.2426$

(b) $(x + 3)^2 + y^2 = 18$



15. Proof



Chapter 1

Section 1.1 *(page 51)*

1. Calculus is the mathematics of change. Precalculus mathematics is more static.

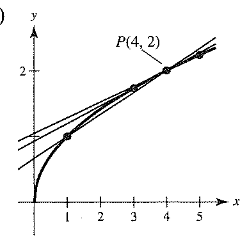
Answers will vary. Sample answer:

Precalculus	**Calculus**
Area of a rectangle	Area under a curve
Work done by a constant force	Work done by a variable force
Center of a rectangle	Centroid of a region

3. Precalculus: 300 ft

5. Calculus: Slope of the tangent line at $x = 2$ is 0.16.

7. (a)



(b) $x = 1$: $m = \frac{1}{3}$

$x = 3$: $m = \dfrac{1}{\sqrt{3} + 2} \approx 0.2679$

$x = 5$: $m = \dfrac{1}{\sqrt{5} + 2} \approx 0.2361$

(c) $\frac{1}{4}$; You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

9. Area ≈ 10.417; Area ≈ 9.145; Use more rectangles.

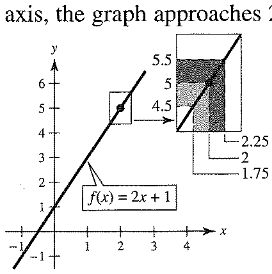
11. (a) About 5.66 (b) About 6.11

(c) Increase the number of line segments.

Section 1.2 *(page 59)*

1. As the graph of the function approaches 8 on the horizontal axis, the graph approaches 25 on the vertical axis.

3.



5.

x	3.9	3.99	3.999	4
$f(x)$	0.3448	0.3344	0.3334	?

x	4.001	4.01	4.1
$f(x)$	0.3332	0.3322	0.3226

$\lim_{x\to 4} \frac{x-4}{x^2-5x+4} \approx 0.3333$ $\left(\text{Actual limit is } \frac{1}{3}.\right)$

7.

x	-0.1	-0.01	-0.001	0
$f(x)$	0.5132	0.5013	0.5001	?

x	0.001	0.01	0.1
$f(x)$	0.4999	0.4988	0.4881

$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

9.

x	-0.1	-0.01	-0.001	0
$f(x)$	0.9983	0.99998	1.0000	?

x	0.001	0.01	0.1
$f(x)$	1.0000	0.99998	0.9983

$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.)

11.

x	0.9	0.99	0.999	1
$f(x)$	0.2564	0.2506	0.2501	?

x	1.001	1.01	1.1
$f(x)$	0.2499	0.2494	0.2439

$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

13.

x	0.9	0.99	0.999	1
$f(x)$	0.7340	0.6733	0.6673	?

x	1.001	1.01	1.1
$f(x)$	0.6660	0.6600	0.6015

$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666$ $\left(\text{Actual limit is } \frac{2}{3}.\right)$

15.

x	-6.1	-6.01	-6.001	-6
$f(x)$	-0.1248	-0.1250	-0.1250	?

x	-5.999	-5.99	-5.9
$f(x)$	-0.1250	-0.1250	-0.1252

$\lim_{x\to -6} \frac{\sqrt{10-x}-4}{x+6} \approx -0.1250$ $\left(\text{Actual limit is } -\frac{1}{8}.\right)$

17.

x	-0.1	-0.01	-0.001	0
$f(x)$	1.9867	1.9999	2.0000	?

x	0.001	0.01	0.1
$f(x)$	2.0000	1.9999	1.9867

$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000$ (Actual limit is 2.)

19.

x	-0.1	-0.01	-0.001	0
$f(x)$	-2000	-2×10^6	-2×10^9	?

x	0.001	0.01	0.1
$f(x)$	2×10^9	2×10^6	2000

As x approaches 0 from the left, the function decreases without bound. As x approaches 0 from the right, the function increases without bound.

21. 1 **23.** 2

25. Limit does not exist. The function approaches 1 from the right side of 2, but it approaches -1 from the left side of 2.

27. Limit does not exist. The function oscillates between 1 and -1 as x approaches 0.

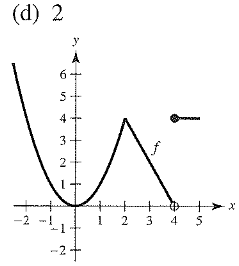
29. (a) 2

(b) Limit does not exist. The function approaches 1 from the right side of 1, but it approaches 3.5 from the left side of 1.

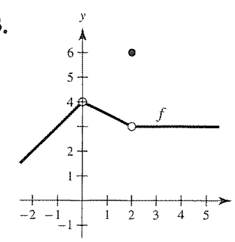
(c) Value does not exist. The function is undefined at $x = 4$.

(d) 2

31.



33.



$\lim_{x\to c} f(x)$ exists for all points on the graph except where $c = 4$.

35. $\delta = 0.4$ **37.** $\delta = \frac{1}{11} \approx 0.091$

39. $L = 8$

Answers will vary. Sample answers:

(a) $\delta \approx 0.0033$ (b) $\delta \approx 0.00167$

41. $L = 1$

Answers will vary.
Sample answers:
(a) $\delta = 0.002$
(b) $\delta = 0.001$

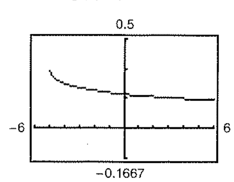
43. $L = 12$

Answers will vary.
Sample answers:
(a) $\delta = 0.00125$
(b) $\delta = 0.000625$

45. 6 **47.** -3 **49.** 3 **51.** 0 **53.** 10

55. 2 **57.** 4

59.

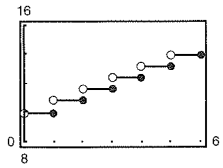


$\lim_{x\to 4} f(x) = \frac{1}{6}$

Domain: $[-5, 4) \cup (4, \infty)$

The graph has a hole at $x = 4$.

61. (a) \$17.89; the cost of a 10-minute, 45-second phone call

(b) 

The limit does not exist because the limits from the right and left are not equal.

63. Choosing a smaller positive value of δ will still satisfy the inequality $|f(x) - L| < \varepsilon$.

65. No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

67. (a) $r = \dfrac{3}{\pi} \approx 0.9549$ cm

(b) $\dfrac{5.5}{2\pi} \le r \le \dfrac{6.5}{2\pi}$, or approximately $0.8754 < r < 1.0345$

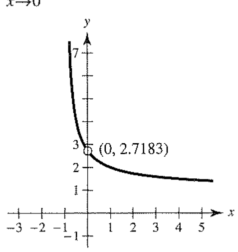
(c) $\lim\limits_{r\to 3/\pi} 2\pi r = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

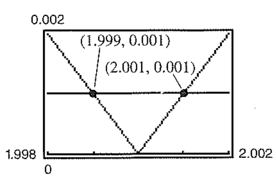
69.

x	-0.001	-0.0001	-0.00001
$f(x)$	2.7196	2.7184	2.7183

x	0.00001	0.0001	0.001
$f(x)$	2.7183	2.7181	2.7169

$\lim\limits_{x\to 0} f(x) \approx 2.7183$



71. 

$\delta = 0.001$, $(1.999, 2.001)$

73. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

75. False. See Exercise 23.

77. Yes. As x approaches 0.25 from either side, $\sqrt{x}$ becomes arbitrarily close to 0.5.

79. $\lim\limits_{x\to 0} \dfrac{\sin nx}{x} = n$ **81–83.** Proofs

85. Putnam Problem B1, 1986

Section 1.3 *(page 71)*

1. Substitute c for x and simplify.

3. If a function f is squeezed between two functions h and g, $h(x) \le f(x) \le g(x)$, and h and g have the same limit L as $x \to c$, then $\lim\limits_{x\to c} f(x)$ exists and equals L.

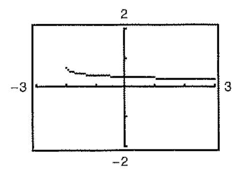
5. 8 **7.** -1 **9.** 0 **11.** 7 **13.** $\sqrt{11}$ **15.** 125
17. $\frac{3}{5}$ **19.** $\frac{1}{5}$ **21.** 7 **23.** (a) 4 (b) 64 (c) 64
25. (a) 3 (b) 2 (c) 2 **27.** 1 **29.** $\frac{1}{2}$ **31.** 1
33. $\frac{1}{2}$ **35.** -1 **37.** (a) 10 (b) $\frac{12}{5}$ (c) $\frac{4}{5}$ (d) $\frac{1}{5}$
39. (a) 256 (b) 4 (c) 48 (d) 64

41. $f(x) = \dfrac{x^2 + 3x}{x}$ and $g(x) = x + 3$ agree except at $x = 0$.
$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 3$

43. $f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.
$\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} g(x) = -2$

45. $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.
$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} g(x) = 12$

47. -1 **49.** $\dfrac{1}{8}$ **51.** $\dfrac{5}{6}$ **53.** $\dfrac{1}{6}$ **55.** $\dfrac{\sqrt{5}}{10}$
57. $-\frac{1}{9}$ **59.** 2 **61.** $2x - 2$ **63.** $\frac{1}{5}$ **65.** 0
67. 0 **69.** 0 **71.** 0 **73.** $\frac{3}{2}$

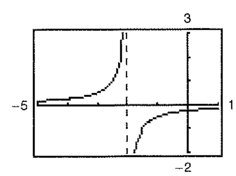
75. 

The graph has a hole at $x = 0$.

Answers will vary. Sample answer:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$\lim\limits_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354$; Actual limit is $\dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$.

77. 

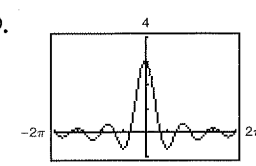
The graph has a hole at $x = 0$.

Answers will vary. Sample answer:

x	-0.1	-0.01	-0.001
$f(x)$	-0.263	-0.251	-0.250

x	0.001	0.01	0.1
$f(x)$	-0.250	-0.249	-0.238

$\lim\limits_{x\to 0} \dfrac{[1/(2 + x)] - (1/2)}{x} \approx -0.250$; Actual limit is $-\dfrac{1}{4}$.

79. 

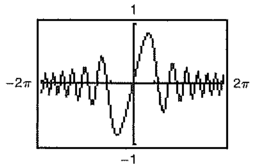
The graph has a hole at $t = 0$.

Answers will vary. Sample answer:

t	-0.1	-0.01	0	0.01	0.1
$f(t)$	2.96	2.9996	?	2.9996	2.96

$\lim\limits_{t\to 0} \dfrac{\sin 3t}{t} \approx 3.0000$; Actual limit is 3.

81.



The graph has a hole at $x = 0$.

Answers will vary. Sample answer:

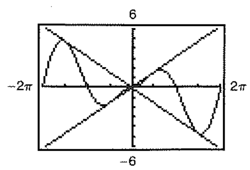
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.1	-0.01	-0.001	?	0.001	0.01	0.1

$\lim_{x \to 0} \dfrac{\sin x^2}{x} = 0$; Actual limit is 0.

83. 3 **85.** $2x - 4$ **87.** $x^{-1/2}$

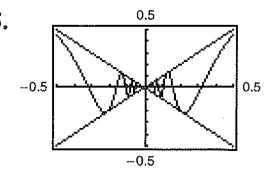
89. $-1/(x + 3)^2$ **91.** 4

93.



0

95.



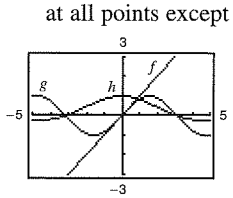
0

The graph has a hole at $x = 0$.

97. (a) f and g agree at all but one point if c is a real number such that $f(x) = g(x)$ for all $x \neq c$.

(b) Sample answer: $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

99.



The magnitudes of $f(x)$ and $g(x)$ are approximately equal when x is close to 0. Therefore, their ratio is approximately 1.

101. -64 ft/sec (speed $= 64$ ft/sec) **103.** -29.4 m/sec

105. Let $f(x) = 1/x$ and $g(x) = -1/x$.

$\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist. However,

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x \to 0} 0 = 0$$

and therefore does exist.

107–111. Proofs

113. Let $f(x) = \begin{cases} 4, & x \geq 0 \\ -4, & x < 0 \end{cases}$.

$\lim_{x \to 0} |f(x)| = \lim_{x \to 0} 4 = 4$

$\lim_{x \to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

115. False. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0.

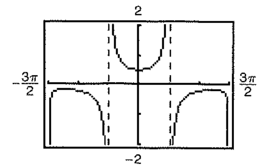
117. True.

119. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2.

121. Proof

123. (a) All $x \neq 0, \dfrac{\pi}{2} + n\pi$

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent from the graph.

(c) $\frac{1}{2}$ (d) $\frac{1}{2}$

Section 1.4 *(page 83)*

1. A function is continuous at a point c if there is no interruption of the graph at c.

3. The limit exists because the limit from the left and the limit from the right are equivalent.

5. (a) 3 (b) 3 (c) 3; $f(x)$ is continuous on $(-\infty, \infty)$.

7. (a) 0 (b) 0 (c) 0; Discontinuity at $x = 3$

9. (a) -3 (b) 3 (c) Limit does not exist. Discontinuity at $x = 2$

11. $\frac{1}{16}$ **13.** $\frac{1}{10}$

15. Limit does not exist. The function decreases without bound as x approaches -3 from the left.

17. -1 **19.** $-\dfrac{1}{x^2}$ **21.** $\dfrac{5}{2}$ **23.** 2

25. Limit does not exist. The function decreases without bound as x approaches π from the left and increases without bound as x approaches π from the right.

27. 8 **29.** 2

31. Discontinuities at $x = -2$ and $x = 2$

33. Discontinuities at every integer

35. Continuous on $[-7, 7]$ **37.** Continuous on $[-1, 4]$

39. Nonremovable discontinuity at $x = 0$

41. Nonremovable discontinuities at $x = -2$ and $x = 2$

43. Continuous for all real x

45. Nonremovable discontinuity at $x = 1$
Removable discontinuity at $x = 0$

47. Removable discontinuity at $x = -2$
Nonremovable discontinuity at $x = 5$

49. Nonremovable discontinuity at $x = -7$

51. Nonremovable discontinuity at $x = 2$

53. Continuous for all real x

55. Nonremovable discontinuities at integer multiples of $\dfrac{\pi}{2}$

57. Nonremovable discontinuities at each integer

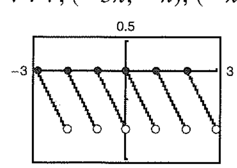
59. $a = 7$ **61.** $a = 2$ **63.** $a = -1, b = 1$

65. Continuous for all real x

67. Nonremovable discontinuities at $x = 1$ and $x = -1$

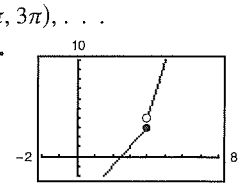
69. Continuous on the open intervals
$\ldots, (-3\pi, -\pi), (-\pi, \pi), (\pi, 3\pi), \ldots$

71.



Nonremovable discontinuity at each integer

73.



Nonremovable discontinuity at $x = 4$

75. Continuous on $(-\infty, \infty)$ **77.** Continuous on $[0, \infty)$

79. Continuous on the open intervals . . . , $(-6, -2)$, $(-2, 2)$, $(2, 6)$, . . .

81. Continuous on $(-\infty, \infty)$

83. Because $f(x)$ is continuous on the interval $[1, 2]$ and $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$, by the Intermediate Value Theorem there exists a real number c in $[1, 2]$ such that $f(c) = 0$.

85. Because $f(x)$ is continuous on the interval $[0, \pi]$ and $f(0) = -3$ and $f(\pi) \approx 8.87$, by the Intermediate Value Theorem there exists a real number c in $[0, \pi]$ such that $f(c) = 0$.

87. Consider the intervals $[1, 3]$ and $[3, 5]$.
$f(1) = 2 > 0$ and $f(3) = -2 < 0$. So, there is at least one zero in the interval $[1, 3]$.
$f(3) = -2 < 0$ and $f(5) = 2 > 0$. So, there is at least one zero in the interval $[3, 5]$.

89. 0.68, 0.6823 **91.** 0.95, 0.9472 **93.** 0.56, 0.5636

95. $f(3) = 11$; $c = 3$

97. $f(0) \approx 0.6458$, $f(5) \approx 1.464$; $c = 2$

99. $f(1) = 0$, $f(3) = 24$; $c = 2$

101. Answers will vary. Sample answer:
$$f(x) = \frac{-1}{(x - a)(x - b)}$$

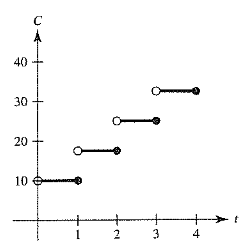
103. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, $\frac{f}{g}$ might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but $\frac{f}{g}$ is not continuous at $x = \pm 1$.

105. True

107. False. $f(x) = \cos x$ has two zeros in $[0, 2\pi]$. However, $f(0)$ and $f(2\pi)$ have the same sign.

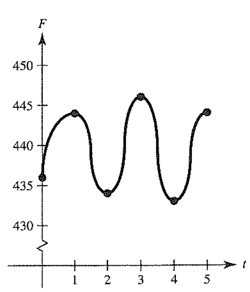
109. False. A rational function can be written as $\frac{P(x)}{Q(x)}$, where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

111. The functions differ by 1 for non-integer values of x.

113.


There is a jump discontinuity every gigabyte.

115–117. Proofs **119.** Answers will vary.

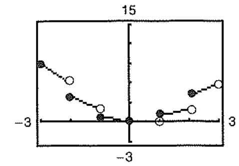
121. (a)


(b) No. The frequency is oscillating.

123. $c = \dfrac{-1 \pm \sqrt{5}}{2}$

125. Domain: $[-c^2, 0) \cup (0, \infty)$; Let $f(0) = \dfrac{1}{2c}$.

127. $h(x)$ has a nonremovable discontinuity at every integer except 0.



129. Putnam Problem B2, 1988

Section 1.5 *(page 92)*

1. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol $\lim_{x \to c} f(x) = \infty$ says how the limit fails to exist.

3. $\lim_{x \to -2^+} 2\left|\frac{x}{x^2 - 4}\right| = \infty$, $\lim_{x \to -2^-} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

5. $\lim_{x \to -2^+} \tan \frac{\pi x}{4} = -\infty$, $\lim_{x \to -2^-} \tan \frac{\pi x}{4} = \infty$

7. $\lim_{x \to 4^+} \frac{1}{x - 4} = \infty$, $\lim_{x \to 4^-} \frac{1}{x - 4} = -\infty$

9. $\lim_{x \to 4^+} \frac{1}{(x - 4)^2} = \infty$, $\lim_{x \to 4^-} \frac{1}{(x - 4)^2} = \infty$

11.

x	-3.5	-3.1	-3.01	-3.001	-3
$f(x)$	0.31	1.64	16.6	167	?

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-167	-16.7	-1.69	-0.36

$\lim_{x \to -3^+} f(x) = -\infty$; $\lim_{x \to -3^-} f(x) = \infty$

13.

x	-3.5	-3.1	-3.01	-3.001	-3
$f(x)$	3.8	16	151	1501	?

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1499	-149	-14	-2.3

$\lim_{x \to -3^+} f(x) = -\infty$; $\lim_{x \to -3^-} f(x) = \infty$

15.

x	-3.5	-3.1	-3.01	-3.001	-3
$f(x)$	-1.7321	-9.514	-95.49	-954.9	?

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	954.9	95.49	9.514	1.7321

$\lim_{x \to -3^-} f(x) = -\infty$; $\lim_{x \to -3^+} f(x) = \infty$

17. $x = 0$ **19.** $x = \pm 2$ **21.** No vertical asymptote

23. $x = -2$, $x = 1$ **25.** $x = 0$, $x = 3$

27. No vertical asymptote **29.** $x = n$, n is an integer

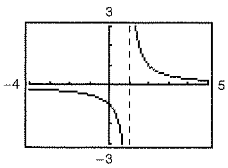
31. $t = n\pi$, n is a nonzero integer

33. Removable discontinuity at $x = -1$

35. Vertical asymptote at $x = -1$ **37.** ∞ **39.** $-\frac{1}{5}$

41. $-\infty$ **43.** $-\infty$ **45.** ∞ **47.** 0 **49.** ∞

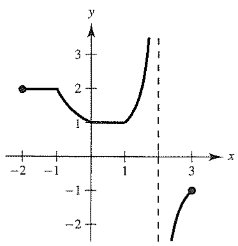
51. $\lim_{x\to 1^+} f(x) = \infty$



53. (a) ∞ (b) $-\infty$ (c) 0

55. Answers will vary. Sample answer: $f(x) = \dfrac{x-3}{x^2-4x-12}$

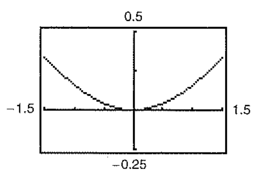
57.



59. (a)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.0411	0.0067	0.0017

x	0.01	0.001	0.0001
$f(x)$	≈ 0	≈ 0	≈ 0

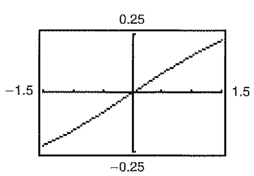


$\lim_{x\to 0^+} \dfrac{x - \sin x}{x} = 0$

(b)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.0823	0.0333	0.0167

x	0.01	0.001	0.0001
$f(x)$	0.0017	≈ 0	≈ 0

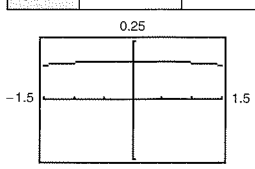


$\lim_{x\to 0^+} \dfrac{x - \sin x}{x^2} = 0$

(c)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.1646	0.1663	0.1666

x	0.01	0.001	0.0001
$f(x)$	0.1667	0.1667	0.1667

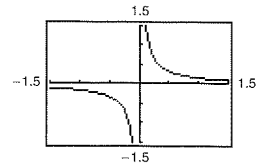


$\lim_{x\to 0^+} \dfrac{x - \sin x}{x^3} = 0.1\overline{6} \text{ or } \dfrac{1}{6}$

(d)

x	1	0.5	0.2	0.1
$f(x)$	0.1585	0.3292	0.8317	1.6658

x	0.01	0.001	0.0001
$f(x)$	16.67	166.7	1667.0



$\lim_{x\to 0^+} \dfrac{x - \sin x}{x^4} = \infty$

For $n > 3$, $\lim_{x\to 0^+} \dfrac{x - \sin x}{x^n} = \infty$.

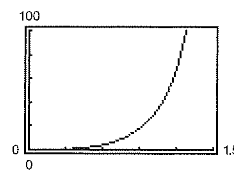
61. (a) $\frac{7}{12}$ ft/sec (b) $\frac{3}{2}$ ft/sec

(c) $\lim_{x\to 25^-} \dfrac{2x}{\sqrt{625 - x^2}} = \infty$

63. (a) $A = 50 \tan\theta - 50\theta$; Domain: $\left(0, \dfrac{\pi}{2}\right)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1



(c) $\lim_{\theta\to\pi/2} A = \infty$

65. True. **67.** False; let $f(x) = \tan x$

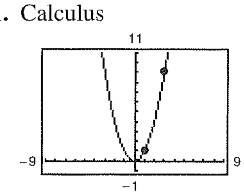
69. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and let $c = 0$. $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$ and $\lim_{x\to 0} \dfrac{1}{x^4} = \infty$, but $\lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right) = \lim_{x\to 0}\left(\dfrac{x^2-1}{x^4}\right) = -\infty \neq 0$.

71. Given $\lim_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

73–75. Proofs

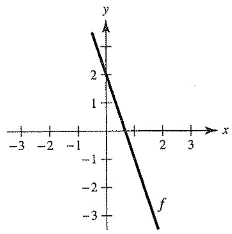
Review Exercises for Chapter 1 *(page 95)*

1. Calculus



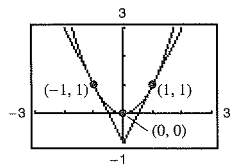
Estimate: 8.3

59. $f(x) = -3x + 2$



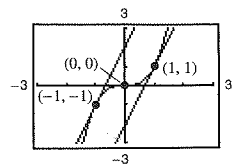
61. $y = 2x + 1,\ y = -2x + 9$

63. (a)



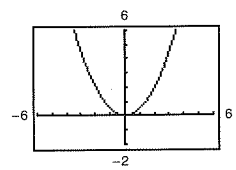
For this function, the slopes of the tangent lines are always distinct for different values of x.

(b)



For this function, the slopes of the tangent lines are sometimes the same.

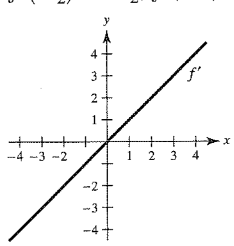
65. (a)



$f'(0) = 0,\ f'\left(\frac{1}{2}\right) = \frac{1}{2},\ f'(1) = 1,\ f'(2) = 2$

(b) $f'\left(-\frac{1}{2}\right) = -\frac{1}{2},\ f'(-1) = -1,\ f'(-2) = -2$

(c)



(d) $f'(x) = x$

67. $f(2) = 4,\ f(2.1) = 3.99,\ f'(2) \approx -0.1$ **69.** 4

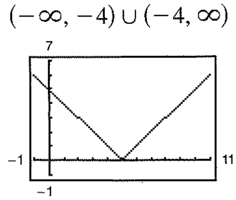
71. $g(x)$ is not differentiable at $x = 0$.

73. $f(x)$ is not differentiable at $x = 6$.

75. $h(x)$ is not differentiable at $x = -7$.

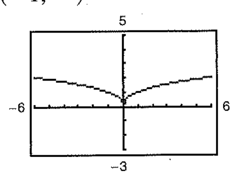
77. $(-\infty, -4) \cup (-4, \infty)$ **79.** $(-1, \infty)$

81.



$(-\infty, 5) \cup (5, \infty)$

83.



$(-\infty, 0) \cup (0, \infty)$

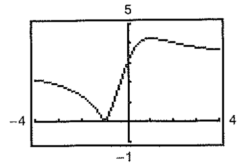
85. The derivative from the left is -1 and the derivative from the right is 1, so f is not differentiable at $x = 1$.

87. The derivatives from both the right and the left are 0, so $f'(1) = 0$.

89. f is differentiable at $x = 2$.

91. (a) $d = \dfrac{3|m + 1|}{\sqrt{m^2 + 1}}$

(b)



Not differentiable at $m = -1$

93. False. The slope is $\lim_{\Delta x \to 0} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

95. False. For example, $f(x) = |x|$. The derivative from the left and the derivative from the right both exist but are not equal.

97. Proof

Section 2.2 *(page 118)*

1. 0

3. The derivative of the sine function is the cosine function. The derivative of the cosine function is the negative of the sine function.

5. (a) $\frac{1}{2}$ (b) 3 **7.** 0 **9.** $7x^6$ **11.** $-5/x^6$

13. $1/(9x^{8/9})$ **15.** 1 **17.** $-6t + 2$ **19.** $2x + 12x^2$

21. $3t^2 + 10t - 3$ **23.** $\dfrac{\pi}{2}\cos\theta$ **25.** $2x + \dfrac{1}{2}\sin x$

	Function	Rewrite	Differentiate	Simplify
27.	$y = \dfrac{2}{7x^4}$	$y = \dfrac{2}{7}x^{-4}$	$y' = -\dfrac{8}{7}x^{-5}$	$y' = -\dfrac{8}{7x^5}$
29.	$y = \dfrac{6}{(5x)^3}$	$y = \dfrac{6}{125}x^{-3}$	$y' = -\dfrac{18}{125}x^{-4}$	$y' = -\dfrac{18}{125x^4}$

31. -2 **33.** 0 **35.** 8 **37.** 3 **39.** $\dfrac{2x + 6}{x^3}$

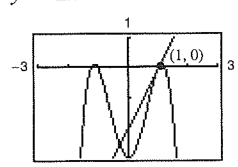
41. $\dfrac{2t + 12}{t^4}$ **43.** $\dfrac{x^3 - 8}{x^3}$ **45.** $\dfrac{3t^2 - 4t + 24}{2t^{5/2}}$

47. $3x^2 + 1$ **49.** $\dfrac{1}{2\sqrt{x}} - \dfrac{2}{x^{2/3}}$ **51.** $\dfrac{3}{\sqrt{x}} - 5\sin x$

53. $18x + 5\sin x$

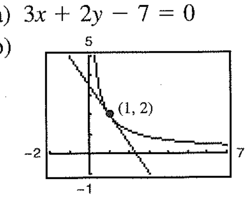
55. (a) $y = 2x - 2$

(b)



57. (a) $3x + 2y - 7 = 0$

(b)

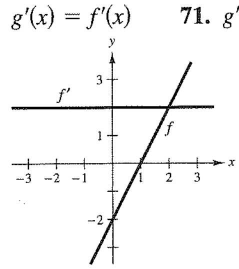


59. $(-1, 2), (0, 3), (1, 2)$ **61.** No horizontal tangents

63. (π, π) **65.** $k = -8$ **67.** $k = 3$

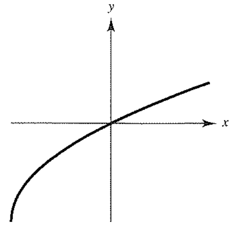
69. $g'(x) = f'(x)$ **71.** $g'(x) = -5f'(x)$

73.

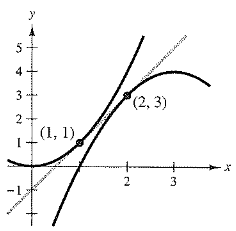


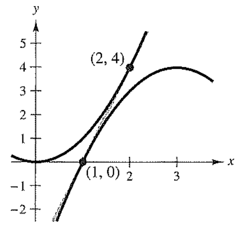
The rate of change of f is constant, and therefore f' is a constant function.

75.



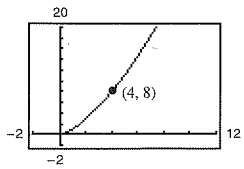
77. $y = 2x - 1$ $\qquad$ $y = 4x - 4$





79. $f'(x) = 3 + \cos x \neq 0$ for all x. **81.** $x - 4y + 4 = 0$

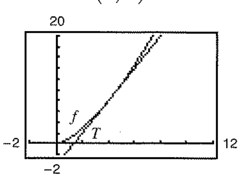
83. (a)



$(3.9, 7.7019)$,
$S(x) = 2.981x - 3.924$

(b) $T(x) = 3(x - 4) + 8 = 3x - 4$
The slope (and equation) of the secant line approaches that of the tangent line at $(4, 8)$ as you choose points closer and closer to $(4, 8)$.

(c)



The approximation becomes less accurate.

(d)

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	8.3	9.5	11	14	17

85. False. Let $f(x) = x$ and $g(x) = x + 1$.

87. False. $\dfrac{dy}{dx} = 0$ **89.** False. $f'(x) = 0$

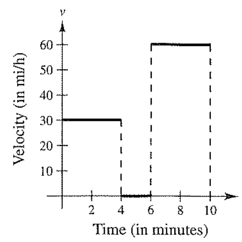
91. Average rate: 3
Instantaneous rates:
$f'(1) = 3,\ f'(2) = 3$

93. Average rate: $\frac{1}{2}$
Instantaneous rates:
$f'(1) = 1,\ f'(2) = \frac{1}{4}$

95. (a) $s(t) = -16t^2 + 1362$, $v(t) = -32t$ (b) -48 ft/sec
(c) $s'(1) = -32$ ft/sec, $s'(2) = -64$ ft/sec
(d) $t = \dfrac{\sqrt{1362}}{4} \approx 9.226$ sec (e) -295.242 ft/sec

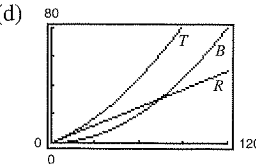
97. $v(5) = 71$ m/sec; $v(10) = 22$ m/sec

99.



101. $V'(6) = 108\ \text{cm}^3/\text{cm}$

103. (a) $R(v) = 0.417v - 0.02$
(b) $B(v) = 0.0056v^2 + 0.001v + 0.04$
(c) $T(v) = 0.0056v^2 + 0.418v + 0.02$
(d)



(e) $T'(v) = 0.0112v + 0.418$
$T'(40) = 0.866$
$T'(80) = 1.314$
$T'(100) = 1.538$

(f) Stopping distance increases at an increasing rate.

105. Proof **107.** $y = 2x^2 - 3x + 1$

109. $9x + y = 0,\ 9x + 4y + 27 = 0$ **111.** $a = \frac{1}{3},\ b = -\frac{4}{3}$

113. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.
$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

115. Putnam Problem A2, 2010

Section 2.3 *(page 129)*

1. To find the derivative of the product of two differentiable functions f and g, multiply the first function f by the derivative of the second function g, and then add the second function g times the derivative of the first function f.

3. $\dfrac{d}{dx}\tan x = \sec^2 x$

$\dfrac{d}{dx}\cot x = -\csc^2 x$

$\dfrac{d}{dx}\sec x = \sec x \tan x$

$\dfrac{d}{dx}\csc x = -\csc x \cot x$

5. $-20x + 17$ **7.** $\dfrac{1 - 5t^2}{2\sqrt{t}}$ **9.** $x^2(3\cos x - x\sin x)$

11. $-\dfrac{5}{(x - 5)^2}$ **13.** $\dfrac{1 - 5x^3}{2\sqrt{x}(x^3 + 1)^2}$ **15.** $\dfrac{x\cos x - 2\sin x}{x^3}$

17. $f'(x) = (x^3 + 4x)(6x + 2) + (3x^2 + 2x - 5)(3x^2 + 4)$
$= 15x^4 + 8x^3 + 21x^2 + 16x - 20$
$f'(0) = -20$

19. $f'(x) = \dfrac{x^2 - 6x + 4}{(x - 3)^2}$
$f'(1) = -\dfrac{1}{4}$

21. $f'(x) = \cos x - x\sin x$
$f'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{8}(4 - \pi)$

	Function	Rewrite	Differentiate	Simplify
23.	$y = \dfrac{x^3 + 6x}{3}$	$y = \frac{1}{3}x^3 + 2x$	$y' = \frac{1}{3}(3x^2) + 2$	$y' = x^2 + 2$
25.	$y = \dfrac{6}{7x^2}$	$y = \frac{6}{7}x^{-2}$	$y' = -\frac{12}{7}x^{-3}$	$y' = -\dfrac{12}{7x^3}$
27.	$y = \dfrac{4x^{3/2}}{x}$	$y = 4x^{1/2}$, $x > 0$	$y' = 2x^{-1/2}$	$y' = \dfrac{2}{\sqrt{x}}$, $x > 0$

29. $\dfrac{3}{(x + 1)^2},\ x \neq 1$ **31.** $\dfrac{x^2 + 6x - 3}{(x + 3)^2}$ **33.** $\dfrac{3x + 1}{2x^{3/2}}$

35. $-\dfrac{2x^2-2x+3}{x^2(x-3)^2}$ **37.** $\dfrac{4s^2(3s^2+13s+15)}{(s+2)^2}$

39. $10x^4-8x^3-21x^2-10x-30$ **41.** $t(t\cos t+2\sin t)$

43. $\dfrac{-(t\sin t+\cos t)}{t^2}$ **45.** $-1+\sec^2 x$, or $\tan^2 x$

47. $\dfrac{1}{4t^{3/4}}-6\csc t\cot t$ **49.** $\dfrac{3}{2}\sec x(\tan x-\sec x)$

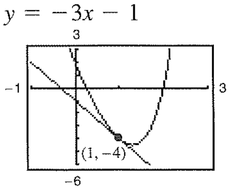
51. $\cos x\cot^2 x$ **53.** $x(x\sec^2 x+2\tan x)$

55. $4x\cos x+(2-x^2)\sin x$ **57.** $\dfrac{2x^2+8x-1}{(x+2)^2}$

59. $-4\sqrt{3}$ **61.** $\dfrac{1}{\pi^2}$

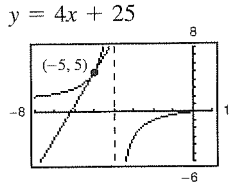
63. (a) $y=-3x-1$

(b)



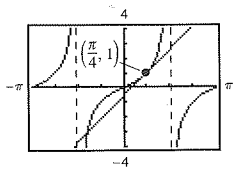
65. (a) $y=4x+25$

(b)



67. (a) $4x-2y-\pi+2=0$

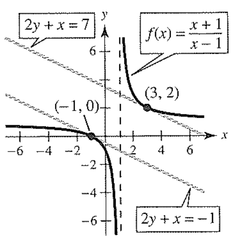
(b)



69. $2y+x-4=0$

71. $25y-12x+16=0$ **73.** $(1, 1)$ **75.** $(0, 0)$, $(2, 4)$

77. Tangent lines: $2y+x=7$, $2y+x=-1$



79. $f(x)+2=g(x)$ **81.** (a) $p'(1)=1$ (b) $q'(4)=-\frac{1}{3}$

83. $\dfrac{18t+5}{2\sqrt{t}}$ cm²/sec

85. (a) −\$38.13 thousand/100 components

(b) −\$10.37 thousand/100 components

(c) −\$3.80 thousand/100 components

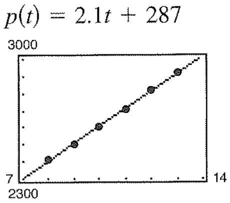
The cost decreases with increasing order size.

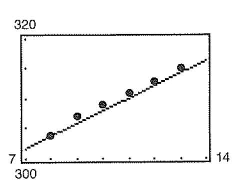
87. Proof

89. (a) $h(t)=101.7t+1593$

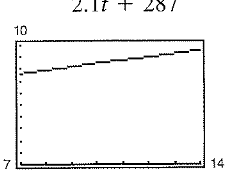
$p(t)=2.1t+287$

(b)





(c) $A=\dfrac{101.7t+1593}{2.1t+287}$



A represents the average health care expenditures per person (in thousands of dollars).

(d) $A'(t)=\dfrac{25{,}842.6}{4.41t^2+1205.4t+82{,}369}$

$A'(t)$ represents the rate of change of the average health care expenditures per person for the given year t.

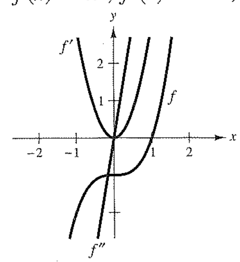
91. 2 **93.** $\dfrac{3}{\sqrt{x}}$ **95.** $\dfrac{2}{(x-1)^3}$ **97.** $2\cos x-x\sin x$

99. $\csc^3 x+\csc x\cot^2 x$ **101.** $6x+\dfrac{6}{25x^{8/5}}$ **103.** $\sin x$

105. 0 **107.** −10

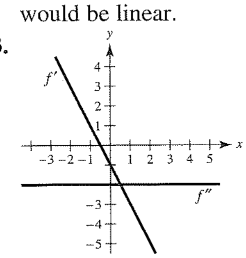
109. $n-1$ or lower; Answers will vary. Sample answer: $f(x)=x^3$, $f'(x)=3x^2$, $f''(x)=6x$, $f'''(x)=6$, $f^{(4)}(x)=0$

111.



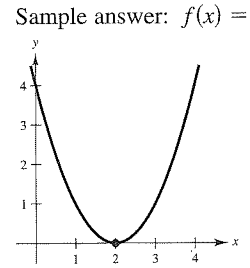
It appears that f is cubic, so f' would be quadratic and f'' would be linear.

113.



115. Answers will vary.

Sample answer: $f(x)=(x-2)^2$



117. $v(3)=27$ m/sec

$a(3)=-6$ m/sec²

The speed of the object is decreasing.

119.

t	0	1	2	3	4
$s(t)$	0	57.75	99	123.75	132
$v(t)$	66	49.5	33	16.5	0
$a(t)$	−16.5	−16.5	−16.5	−16.5	−16.5

The average velocity on [0, 1] is 57.75, on [1, 2] is 41.25, on [2, 3] is 24.75, and on [3, 4] is 8.25.

121. $f^{(n)}(x)=n(n-1)(n-2)\cdot\cdot\cdot(2)(1)=n!$

123. (a) $f''(x) = g(x)h''(x) + 2g'(x)h'(x) + g''(x)h(x)$
$f'''(x) = g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$
$f^{(4)}(x) = g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$

(b) $f^{(n)}(x) = g(x)h^{(n)}(x) + \dfrac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \dfrac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots + \dfrac{n!}{(n-2)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$

125. $n = 1$: $f'(x) = x\cos x + \sin x$
$n = 2$: $f'(x) = x^2\cos x + 2x\sin x$
$n = 3$: $f'(x) = x^3\cos x + 3x^2\sin x$
$n = 4$: $f'(x) = x^4\cos x + 4x^3\sin x$
General rule: $f'(x) = x^n\cos x + nx^{(n-1)}\sin x$

127. $y' = -\dfrac{1}{x^2}$, $y'' = \dfrac{2}{x^3}$,
$$x^3y'' + 2x^2y' = x^3\left(\frac{2}{x^3}\right) + 2x^2\left(\frac{-1}{x^2}\right) = 2 - 2 = 0$$

129. $y' = 2\cos x$, $y'' = -2\sin x$,
$y'' + y = -2\sin x + 2\sin x + 3 = 3$

131. False. $\dfrac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$ **133.** True

135. True **137.** Proof

Section 2.4 *(page 140)*

1. To find the derivative of the composition of two differentiable functions, take the derivative of the outer function and keep the inner function the same. Then multiply by the derivative of the inner function.

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
3.	$y = (6x-5)^4$	$u = 6x - 5$	$y = u^4$
5.	$y = \dfrac{1}{3x+5}$	$u = 3x + 5$	$y = \dfrac{1}{u}$
7.	$y = \csc^3 x$	$u = \csc x$	$y = u^3$

9. $6(2x-7)^2$ **11.** $-\dfrac{45}{2(4-9x)^{1/6}}$ **13.** $-\dfrac{10s}{\sqrt{5s^2+3}}$

15. $\dfrac{4x}{\sqrt[3]{(6x^2+1)^2}}$ **17.** $-\dfrac{1}{(x-2)^2}$ **19.** $-\dfrac{54s^2}{(s^3-2)^4}$

21. $-\dfrac{3}{2\sqrt{(3x+5)^3}}$ **23.** $x(x-2)^6(9x-4)$

25. $\dfrac{1-2x^2}{\sqrt{1-x^2}}$ **27.** $\dfrac{1}{\sqrt{(x^2+1)^3}}$

29. $\dfrac{-2(x+5)(x^2+10x-2)}{(x^2+2)^3}$ **31.** $\dfrac{8(t+1)^3}{(t+3)^5}$

33. $20x(x^2+3)^9 + 2(x^2+3)^5 + 20x^2(x^2+3)^4 + 2x$

35. $-4\sin 4x$ **37.** $15\sec^2 3x$ **39.** $2\pi^2 x\cos(\pi x)^2$

41. $2\cos 4x$ **43.** $\dfrac{-1-\cos^2 x}{\sin^3 x}$ **45.** $8\sec^2 x\tan x$

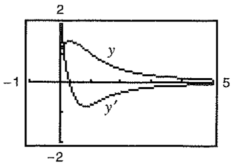
47. $\sin 2\theta\cos 2\theta$, or $\dfrac{1}{2}\sin 4\theta$

49. $6\pi(\pi t - 1)\sec(\pi t - 1)^2\tan(\pi t - 1)^2$

51. $(6x - \sin x)\cos(3x^2 + \cos x)$

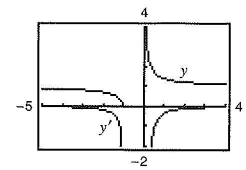
53. $-\dfrac{3\pi\cos\sqrt{\cot 3\pi x}\csc^2(3\pi x)}{2\sqrt{\cot 3\pi x}}$

55. $\dfrac{1-3x^2-4x^{3/2}}{2\sqrt{x}(x^2+1)^2}$



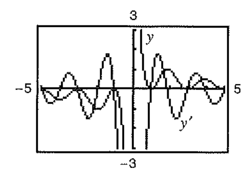
The zero of y' corresponds to the point on the graph of the function where the tangent line is horizontal.

57. $-\dfrac{\sqrt{\dfrac{x+1}{x}}}{2x(x+1)}$



y' has no zeros.

59. $-\dfrac{\pi x\sin(\pi x) + \cos(\pi x) + 1}{x^2}$

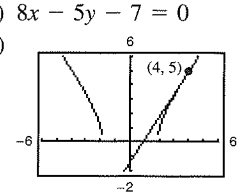


The zeros of y' correspond to the points on the graph of the function where the tangent lines are horizontal.

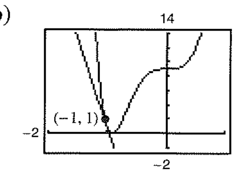
61. 3; 3 cycles in $[0, 2\pi]$ **63.** $\frac{5}{3}$

65. $-\frac{3}{5}$ **67.** -1 **69.** 0

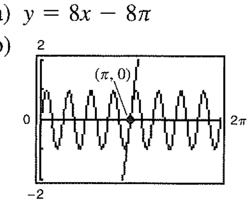
71. (a) $8x - 5y - 7 = 0$
(b)



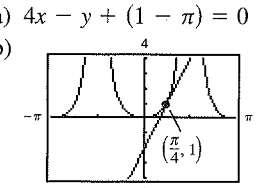
73. (a) $24x + y + 23 = 0$
(b)



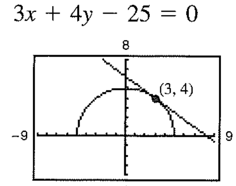
75. (a) $y = 8x - 8\pi$
(b)



77. (a) $4x - y + (1 - \pi) = 0$
(b)



79. $3x + 4y - 25 = 0$

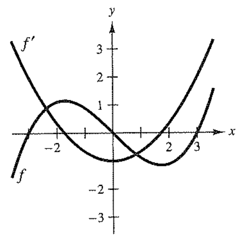


81. $\left(\dfrac{\pi}{6}, \dfrac{3\sqrt{3}}{2}\right), \left(\dfrac{5\pi}{6}, -\dfrac{3\sqrt{3}}{2}\right), \left(\dfrac{3\pi}{2}, 0\right)$ **83.** $2940(2-7x)^2$

85. $\dfrac{242}{(11x-6)^3}$ **87.** $2(\cos x^2 - 2x^2\sin x^2)$

89. $h''(x) = 18x + 6$, 24

91. $f''(x) = -4x^2\cos x^2 - 2\sin x^2$, 0

93. 

The zeros of f' correspond to the points where the graph of f has horizontal tangents.

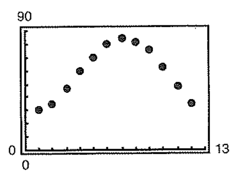
95. (a) The rate of change of g is three times as fast as the rate of change of f.
(b) The rate of change of g is $2x$ times as fast as the rate of change of f.

97. (a) $g'(x) = f'(x)$ (b) $h'(x) = 2f'(x)$
(c) $r'(x) = -3f'(-3x)$ (d) $s'(x) = f'(x + 2)$

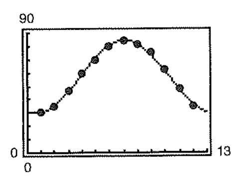
x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

99. (a) $\frac{1}{2}$
(b) $s'(5)$ does not exist because g is not differentiable at 6.

101. (a) 1.461 (b) -1.016 **103.** 0.2 rad, 1.45 rad/sec

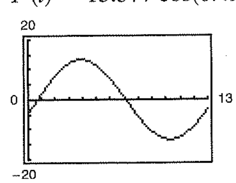
105. (a) 

$T(t) = 27.3 \sin(0.49t - 1.90) + 57.1$

(b) 

The model is a good fit.

(c) $T'(t) = 13.377 \cos(0.49t - 1.90)$



(d) The temperature changes most rapidly around spring (March–May) and fall (Oct.–Nov.)
The temperature changes most slowly around winter (Dec.–Feb.) and summer (Jun.–Aug.)
Yes. Explanations will vary.

107. (a) 0 bacteria per day (b) 177.8 bacteria per day
(c) 44.4 bacteria per day (d) 10.8 bacteria per day
(e) 3.3 bacteria per day
(f) The rate of change of the population is decreasing as time passes.

109. (a) $f'(x) = \beta \cos \beta x$
$f''(x) = -\beta^2 \sin \beta x$
$f'''(x) = -\beta^3 \cos \beta x$
$f^{(4)}(x) = \beta^4 \sin \beta x$
(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$
(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$
$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$

111. (a) $r'(1) = 0$ (b) $s'(4) = \frac{5}{8}$

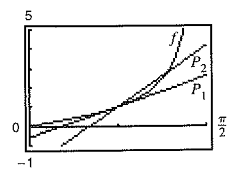
113. (a) and (b) Proofs

115. $g'(x) = 3\left(\dfrac{3x - 5}{|3x - 5|}\right), \quad x \neq \dfrac{5}{3}$

117. $h'(x) = -|x|\sin x + \dfrac{x}{|x|}\cos x, \quad x \neq 0$

119. (a) $P_1(x) = 2\left(x - \dfrac{\pi}{4}\right) + 1$

$P_2(x) = 2\left(x - \dfrac{\pi}{4}\right)^2 + 2\left(x - \dfrac{\pi}{4}\right) + 1$

(b) 

(c) P_2

(d) The accuracy worsens as you move away from $x = \dfrac{\pi}{4}$.

121. True. **123.** True **125.** Putnam Problem A1, 1967

Section 2.5 *(page 149)*

1. Answers will vary. Sample answer: In the explicit form of a function, the dependent variable y is explicitly written as a function of the independent variable x $[y = f(x)]$. In an implicit equation, the dependent variable y is not necessarily written in the form $y = f(x)$. An example of an implicit function is $x^2 + xy = 5$. In explicit form, it would be

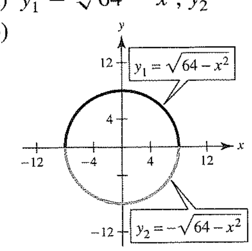
$$y = \frac{5 - x^2}{x}.$$

3. You use implicit differentiation to find the derivative in cases where it is difficult to express y as a function of x explicitly.

5. $-\dfrac{x}{y}$ **7.** $-\dfrac{x^4}{y^4}$ **9.** $\dfrac{y - 3x^2}{2y - x}$

11. $\dfrac{1 - 3x^2y^3}{3x^3y^2 - 1}$ **13.** $\dfrac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$ **15.** $\dfrac{\cos x}{4 \sin 2y}$

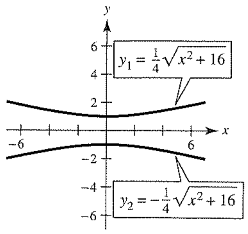
17. $-\dfrac{\cot x \csc x + \tan y + 1}{x \sec^2 y}$ **19.** $\dfrac{y \cos xy}{1 - x \cos xy}$

21. (a) $y_1 = \sqrt{64 - x^2}$, $y_2 = -\sqrt{64 - x^2}$
(b) 

(c) $y' = \mp \dfrac{x}{\sqrt{64 - x^2}} = -\dfrac{x}{y}$ (d) $y' = -\dfrac{x}{y}$

23. (a) $y_1 = \dfrac{\sqrt{x^2+16}}{4},\ y_2 = \dfrac{-\sqrt{x^2+16}}{4}$

(b)



(c) $y' = \dfrac{\pm x}{4\sqrt{x^2+16}} = \dfrac{x}{16y}$ (d) $y' = \dfrac{x}{16y}$

25. $-\dfrac{y}{x};\ -\dfrac{1}{6}$ **27.** $\dfrac{98x}{y(x^2+49)^2}$; Undefined

29. $-\dfrac{y(y+2x)}{x(x+2y)};\ -1$ **31.** $-\sin^2(x+y)$ or $-\dfrac{x^2}{x^2+1};\ 0$

33. $-\frac{1}{2}$ **35.** 0 **37.** $y = -x + 7$

39. $y = \dfrac{\sqrt{3}x}{6} + \dfrac{8\sqrt{3}}{3}$ **41.** $y = -\dfrac{2}{11}x + \dfrac{30}{11}$

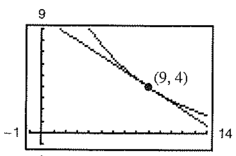
43. Answers will vary. Sample answers:
$xy = 2,\ yx^2 + x = 2;\ x^2 + y^2 + y = 4,\ xy + y^2 = 2$

45. (a) $y = -2x + 4$ (b) Answers will vary.

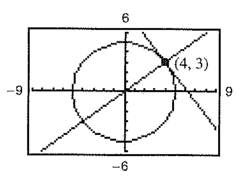
47. $\cos^2 y,\ -\dfrac{\pi}{2} < y < \dfrac{\pi}{2},\ \dfrac{1}{1+x^2}$ **49.** $-\dfrac{4}{y^3}$

51. $\dfrac{6x^2y + 2y - 20x}{(x^2-1)^2}$ **53.** $\dfrac{x\sin x + 2\cos x + 14y}{7x^2}$

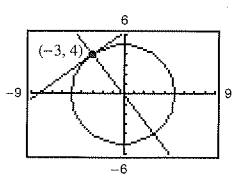
55. $2x + 3y - 30 = 0$



57. At $(4, 3)$:
Tangent line: $4x + 3y - 25 = 0$
Normal line: $3x - 4y = 0$



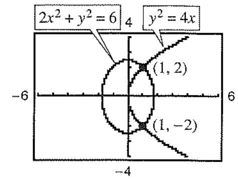
At $(-3, 4)$
Tangent line: $3x - 4y + 25 = 0$
Normal line: $4x + 3y = 0$



59. $x^2 + y^2 = r^2 \Rightarrow y' = -\dfrac{x}{y} \Rightarrow \dfrac{y}{x} =$ slope of normal line.
Then for (x_0, y_0) on the circle, $x_0 \neq 0$, an equation of the normal line is $y = \left(\dfrac{y_0}{x_0}\right)x$, which passes through the origin. If $x_0 = 0$, the normal line is vertical and passes through the origin.

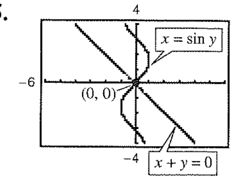
61. Horizontal tangents: $(-4, 0), (-4, 10)$
Vertical tangents: $(0, 5), (-8, 5)$

63.



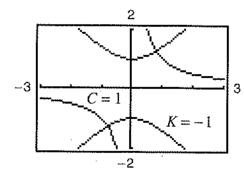
At $(1, 2)$:
Slope of ellipse: -1
Slope of parabola: 1
At $(1, -2)$:
Slope of ellipse: 1
Slope of parabola: -1

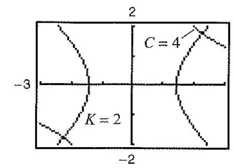
65.



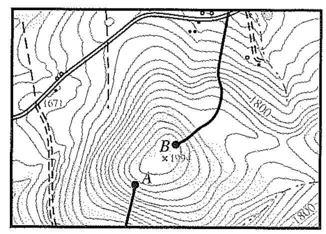
At $(0, 0)$:
Slope of line: -1
Slope of sine curve: 1

67. Derivatives: $\dfrac{dy}{dx} = -\dfrac{y}{x},\ \dfrac{dy}{dx} = \dfrac{x}{y}$



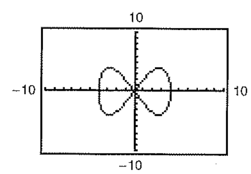


69.

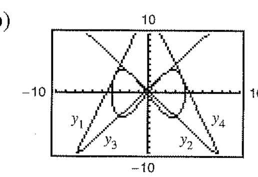


Use starting point B.

71. (a)



(b)

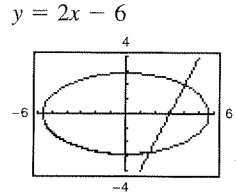


$y_1 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + \left(8\sqrt{7} + 23\right)\right]$
$y_2 = -\frac{1}{3}\left[\left(-\sqrt{7} + 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$
$y_3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$
$y_4 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right]$

(c) $\left(\dfrac{8\sqrt{7}}{7}, 5\right)$

73. Proof **75.** $y = -\dfrac{\sqrt{3}}{2}x + 2\sqrt{3},\ y = \dfrac{\sqrt{3}}{2}x - 2\sqrt{3}$

77. (a) $y = 2x - 6$
(b)



(c) $\left(\frac{28}{17}, -\frac{46}{17}\right)$

Section 2.6 *(page 157)*

1. A related-rate equation is an equation that relates the rates of change of various quantities.

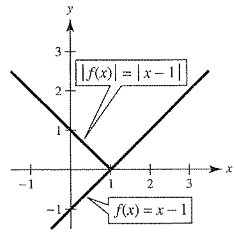
3. (a) $\frac{3}{4}$ (b) 20 **5.** (a) $-\frac{5}{8}$ (b) $\frac{3}{2}$

7. (a) -8 cm/sec (b) 0 cm/sec (c) 8 cm/sec

9. (a) 12 ft/sec (b) 6 ft/sec (c) 3 ft/sec

11. $296\pi\ \text{cm}^2/\text{min}$

77. Answers will vary. Sample answer:



The integrals are equal when f is always greater than or equal to 0 on $[a, b]$.

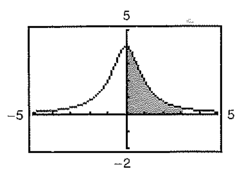
79. $\frac{1}{3}$

Section 4.4 *(page 292)*

1. Find an antiderivative of the function and evaluate the difference of the antiderivative at the upper limit of integration and the lower limit of integration.

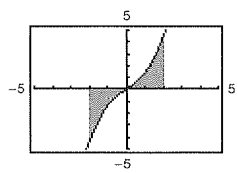
3. The average value of a function on an interval is the integral of the function on $[a, b]$ times $\dfrac{1}{b-a}$.

5.



Positive

7.



Zero

9. -2 **11.** $-\frac{28}{3}$ **13.** $\frac{1}{3}$ **15.** $\frac{1}{2}$ **17.** $\frac{2}{3}$

19. -4 **21.** $-\frac{1}{18}$ **23.** $-\frac{27}{20}$ **25.** $\frac{25}{2}$ **27.** $\frac{64}{3}$

29. $2 - 7\pi$ **31.** $\dfrac{\pi}{4}$ **33.** $\dfrac{2\sqrt{3}}{3}$ **35.** 0 **37.** $\dfrac{1}{6}$

39. 1 **41.** $\dfrac{52}{3}$ **43.** 20 **45.** $\dfrac{32}{3}$ **47.** $\dfrac{3\sqrt[3]{2}}{2} \approx 1.8899$

49. $2\sqrt{3} \approx 3.4641$ **51.** $\pm\arccos\dfrac{\sqrt{\pi}}{2} \approx \pm 0.4817$

53. Average value $= \frac{8}{3}$

$x = \pm\dfrac{2\sqrt{3}}{3}$

55. Average value $= 10.2$

$x \approx 1.3375$

57. Average value $= \dfrac{2}{\pi}$

$x \approx 0.690,\ x \approx 2.451$

59. (a) $F(x) = 500 \sec^2 x$

(b) $\dfrac{1500\sqrt{3}}{\pi} \approx 827$ N

61. $\dfrac{2}{\pi} \approx 63.7\%$

63. $F(x) = -\dfrac{20}{x} + 20$

$F(2) = 10$

$F(5) = 16$

$F(8) = \dfrac{35}{2}$

65. $F(x) = \sin x$

$F(0) = 0$

$F\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

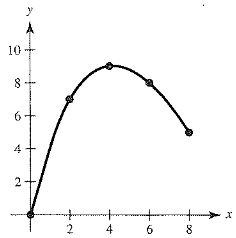
$F\left(\dfrac{\pi}{2}\right) = 1$

67. (a) $g(0) = 0,\ g(2) \approx 7,\ g(4) \approx 9,\ g(6) \approx 8,\ g(8) \approx 5$

(b) Increasing: $(0, 4)$; Decreasing: $(4, 8)$

(c) A maximum occurs at $x = 4$.

(d)

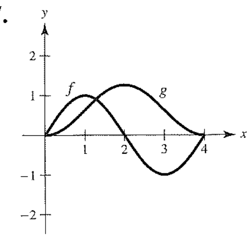


69. $\frac{1}{2}x^2 + 2x$ **71.** $\frac{3}{4}x^{4/3} - 12$ **73.** $\tan x - 1$

75. $x^2 - 2x$ **77.** $\sqrt{x^4 + 1}$ **79.** $\sqrt{x} \csc x$ **81.** 8

83. $\cos x \sqrt{\sin x}$ **85.** $3x^2 \sin x^6$

87.



An extremum of g occurs at $x = 2$.

89. 8190 L

91. About 540 ft **93.** (a) $\frac{3}{2}$ ft to the right (b) $\frac{113}{10}$ ft

95. (a) 0 ft (b) $\frac{63}{2}$ ft **97.** (a) 2 ft to the right (b) 2 ft

99. The displacement and total distance traveled are equal when the particle is always moving in the same direction on an interval.

101. The Fundamental Theorem of Calculus requires that f be continuous on $[a, b]$ and that F be an antiderivative for f on the entire interval. On an interval containing c, the function $f(x) = \dfrac{1}{x - c}$ is not continuous at c.

103. 28 units

105. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

107. $f(x) = \sec^2 x$ has a nonremovable discontinuity at $x = \dfrac{\pi}{2}$.

109. True

111. $f'(x) = \dfrac{1}{(1/x)^2 + 1}\left(-\dfrac{1}{x^2}\right) + \dfrac{1}{x^2 + 1} = 0$

Because $f'(x) = 0$, $f(x)$ is constant.

113. (a) 0 (b) 0 (c) $xf(x) + \int_0^x f(t)\,dt$ (d) 0

115. Putnam Problem B5, 2006

Section 4.5 *(page 305)*

1. You can move constant multiples outside the integral sign.

$$\int kf(x)\,dx = k\int f(x)\,dx$$

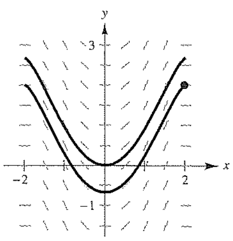
3. The integral of $[g(x)]^n g'(x)$ is $\dfrac{[g(x)]^{n+1}}{n+1} + C,\ n \neq -1$.

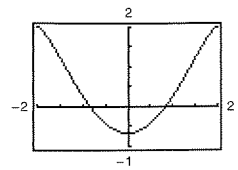
Recall the power rule for polynomials.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
5. $\int (5x^2 + 1)^2(10x)\,dx$	$5x^2 + 1$	$10x\,dx$
7. $\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$

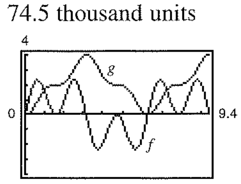
9. $\frac{1}{5}(1 + 6x)^5 + C$ **11.** $\frac{2}{3}(25 - x^2)^{3/2} + C$

13. $\frac{1}{12}(x^4+3)^3+C$ **15.** $\frac{1}{30}(2x^3-1)^5+C$
17. $\frac{1}{3}(t^2+2)^{3/2}+C$ **19.** $-\frac{15}{8}(1-x^2)^{4/3}+C$
21. $\dfrac{7}{4(1-x^2)^2}+C$ **23.** $-\dfrac{1}{3(1+x^3)}+C$
25. $-\sqrt{1-x^2}+C$ **27.** $-\dfrac{1}{4}\left(1+\dfrac{1}{t}\right)^4+C$
29. $\sqrt{2x}+C$ **31.** $2x^2-4\sqrt{16-x^2}+C$
33. $-\dfrac{1}{2(x^2+2x-3)}+C$
35. (a) Answers will vary. (b) $y=-\frac{1}{3}(4-x^2)^{3/2}+2$
Sample answer:

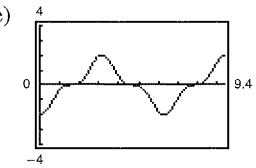




37. $f(x)=(2x^3+1)^3+3$ **39.** $-\cos\pi x+C$
41. $\dfrac{1}{6}\sin 6x+C$ **43.** $-\sin\dfrac{1}{\theta}+C$
45. $\frac{1}{4}\sin^2 2x+C$ or $-\frac{1}{4}\cos^2 2x+C_1$ or $-\frac{1}{8}\cos 4x+C_2$
47. $\dfrac{1}{2}\tan^2 x+C$ or $\dfrac{1}{2}\sec^2 x+C_1$ **49.** $f(x)=2\cos\dfrac{x}{2}+4$
51. $f(x)=\frac{1}{12}(4x^2-10)^3-8$
53. $\frac{2}{5}(x+6)^{5/2}-4(x+6)^{3/2}+C=\frac{2}{5}(x+6)^{3/2}(x-4)+C$
55. $-\left[\frac{2}{3}(1-x)^{3/2}-\frac{4}{5}(1-x)^{5/2}+\frac{2}{7}(1-x)^{7/2}\right]+C=$
$-\frac{2}{105}(1-x)^{3/2}(15x^2+12x+8)+C$
57. $\frac{1}{8}\left[\frac{2}{5}(2x-1)^{5/2}+\frac{4}{3}(2x-1)^{3/2}-6(2x-1)^{1/2}\right]+C=$
$\dfrac{\sqrt{2x-1}}{15}(3x^2+2x-13)+C$
59. $-\frac{1}{8}\cos^4 2x+C$ **61.** 0 **63.** $12-\frac{8}{9}\sqrt{2}$
65. 2 **67.** $\frac{1}{2}$ **69.** $\frac{1209}{28}$ **71.** $2(\sqrt{3}-1)$ **73.** $\frac{272}{15}$
75. 0 **77.** (a) 144 (b) 72 (c) -144 (d) 432
79. $2\displaystyle\int_0^3 (4x^2-6)\,dx=36$
81. (a) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$; Use substitution with $u=x^3+1$.
(b) $\displaystyle\int \cot^3(2x)\csc^2(2x)\,dx$; Use substitution with $u=\cot 2x$.
83. \$340,000
85. (a) 102.532 thousand units (b) 102.352 thousand units
(c) 74.5 thousand units
87. (a)



(b) g is nonnegative, because the graph of f is positive at the beginning and generally has more positive sections than negative ones.
(c) The points on g that correspond to the extrema of f are points of inflection of g.
(d) No, some zeros of f, such as $x=\dfrac{\pi}{2}$, do not correspond to extrema of g. The graph of g continues to increase after $x=\dfrac{\pi}{2}$, because f remains above the x-axis.
(e)

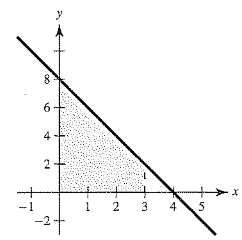


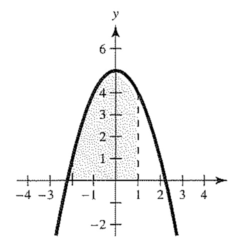
The graph of h is that of g shifted 2 units downward.

89. (a) and (b) Proofs
91. (a) $P_{0.50,\,0.75}\approx 35.3\%$ (b) $b\approx 58.6\%$
93. True **95.** True **97.** True **99–101.** Proofs
103. Putnam Problem A1, 1958

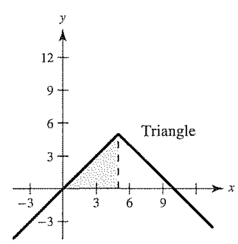
Review Exercises for Chapter 4 *(page 309)*

1. $\dfrac{x^4}{4}+4x+C$ **3.** $\dfrac{4}{3}x^3+\dfrac{1}{2}x^2+3x+C$
5. $\dfrac{x^2}{2}-\dfrac{4}{x^2}+C$ **7.** $9\cos x-2\cot x+C$
9. $y=1-3x^2$ **11.** $f(x)=4x^3-5x-3$
13. (a) 3 sec; 144 ft (b) $\frac{3}{2}$ sec (c) 108 ft
15. 60 **17.** $\displaystyle\sum_{i=1}^{10}\frac{i}{5(i+2)}$ **19.** 192 **21.** 420
23. 3310 **25.** $9.038<$ (Area of region) <13.038
27. $s(n)=11-\dfrac{2}{n}$, $S(n)=11+\dfrac{2}{n}$
29. $A=15$ **31.** $A=12$





33. 43 **35.** 48
37.



$A=\frac{25}{2}$
39. (a) 17 (b) 7 (c) 9 (d) 84
41. 12 **43.** $\dfrac{422}{5}$ **45.** $\dfrac{\sqrt{2}+2}{2}$ **47.** 1 **49.** 30
51. $\frac{1}{4}$ **53.** $\sqrt{\frac{13}{3}}$
55. Average value $=\frac{2}{5}$
$x=\frac{25}{4}$
57. $x^2\sqrt{1+x^3}$ **59.** $-\frac{1}{30}(1-3x^2)^5+C=\frac{1}{30}(3x^2-1)^5+C$
61. $\frac{1}{4}\sin^4 x+C$ **63.** $-2\sqrt{1-\sin\theta}+C$
65. $\frac{2}{5}(8-x)^{5/2}-\frac{16}{3}(8-x)^{3/2}+C$

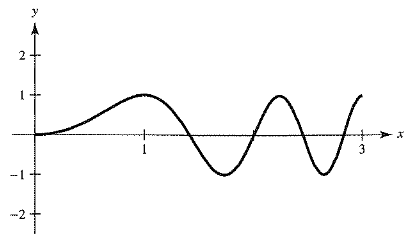
67. $\frac{455}{2}$ **69.** 2 **71.** $\frac{28\pi}{15}$ **73.** $\frac{468}{7}$ **75.** 0

P.S. Problem Solving *(page 311)*

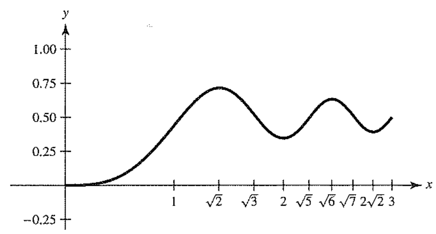
1. (a) $L(1) = 0$ (b) $L'(x) - = \frac{1}{x}, L'(1) = 1$
(c) $x \approx 2.718$ (d) Proof

3. (a) $\lim_{n\to\infty} \left[\frac{32}{n^5}\sum_{i=1}^{n} i^4 - \frac{64}{n^4}\sum_{i=1}^{n} i^3 + \frac{32}{n^3}\sum_{i=1}^{n} i^2\right]$
(b) $\frac{16n^4 - 16}{15n^4}$ (c) $\frac{16}{15}$

5. (a)

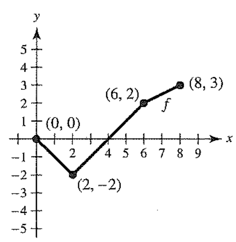


(b)



(c) Relative maxima at $x = \sqrt{2}, \sqrt{6}$
Relative minima at $x = 2, 2\sqrt{2}$
(d) Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}$

7. (a)



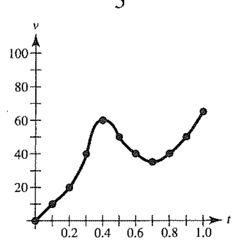
(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $x = 4, 8$ (d) $x = 2$

9. Proof **11.** $\frac{2}{3}$ **13.** Proof; $1 \le \int_0^1 \sqrt{1 + x^4}\, dx \le \sqrt{2}$

15. (a)

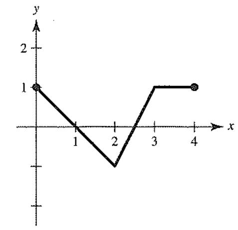


(b) (0, 0.4) and (0.7, 1.0) (c) 150 mi/h^2
(d) Total distance traveled in miles; 37 mi
(e) Answers will vary. Sample answer: 100 mi/h^2

17. (a)–(c) Proofs

19. (a) $S = \frac{5mb^2}{8}, s = \frac{3mb^2}{8}$
(b) $S(n) = \frac{mb^2(n + 1)}{2n}, s(n) = \frac{mb^2(n - 1)}{2n}$
(c) Area $= \frac{1}{2}(b)(mb) = \frac{1}{2}$(base)(height)

21. $f(x) = \begin{cases} -x + 1, & 0 \le x < 2 \\ 2x - 5, & 2 \le x < 3 \\ 1, & 3 \le x \le 4 \end{cases}$



Chapter 5

Section 5.1 *(page 321)*

1. For $x > 1$, $\ln x = \int_1^x \frac{1}{t}\, dt > 0$. For $0 < x < 1$,
$\ln x = \int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt.$

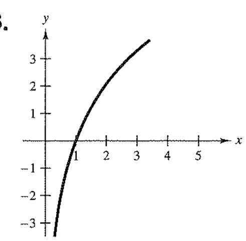
3. The number e is the base for the natural logarithm:
$\ln e = \int_1^e \frac{1}{t}\, dt = 1.$

5. (a) 3.8067 (b) $\ln 45 = \int_1^{45} \frac{1}{t}\, dt \approx 3.8067$

7. (a) -0.2231 (b) $\ln 0.8 = \int_1^{0.8} \frac{1}{t}\, dt \approx -0.2231$

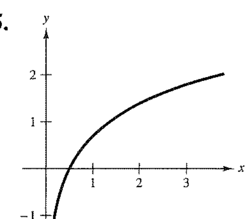
9. b **10.** d **11.** a **12.** c

13.



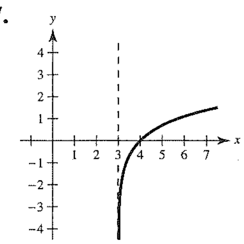
Domain: $x > 0$

15.



Domain: $x > 0$

17.



Domain: $x > 3$

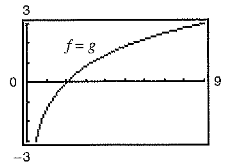
19. (a) 1.7917 (b) -0.4055 (c) 4.3944 (d) 0.5493

21. $\ln x - \ln 4$ **23.** $\ln x + \ln y - \ln z$

25. $\ln x + \frac{1}{2}\ln(x^2 + 5)$ **27.** $\frac{1}{2}[\ln(x - 1) - \ln x]$

29. $\ln z + 2\ln(z - 1)$ **31.** $\ln \frac{x - 2}{x + 2}$

33. $\ln \sqrt[3]{\frac{x(x + 3)^2}{x^2 - 1}}$ **35.** $\ln \frac{16}{\sqrt{x^3 + 6x}}$

37. (a)


(b) $f(x) = \ln \dfrac{x^2}{4} = \ln x^2 - \ln 4$
$= 2 \ln x - \ln 4$
$= g(x)$

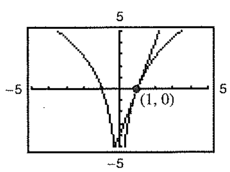
39. $-\infty$ **41.** $\ln 4 \approx 1.3863$ **43.** $\dfrac{1}{x}$ **45.** $\dfrac{2x}{x^2 + 3}$

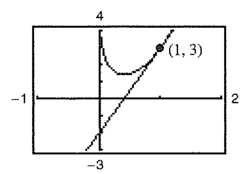
47. $\dfrac{4(\ln x)^3}{x}$ **49.** $\dfrac{2}{t + 1}$ **51.** $\dfrac{2x^2 - 1}{x(x^2 - 1)}$

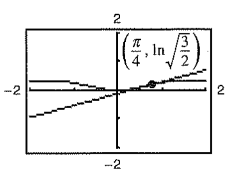
53. $\dfrac{1 - x^2}{x(x^2 + 1)}$ **55.** $\dfrac{1 - 2 \ln t}{t^3}$ **57.** $\dfrac{2}{x \ln x^2} = \dfrac{1}{x \ln x}$

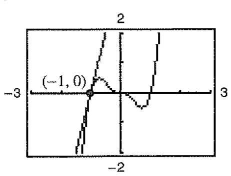
59. $\dfrac{1}{1 - x^2}$ **61.** $\dfrac{-4}{x(x^2 + 4)}$ **63.** $\cot x$

65. $-\tan x + \dfrac{\sin x}{\cos x - 1}$

67. (a) $y = 4x - 4$
(b)


69. (a) $5x - y - 2 = 0$
(b)


71. (a) $y = \frac{1}{3}x - \frac{1}{12}\pi + \frac{1}{2}\ln \frac{3}{2}$
(b)


73. (a) $y = 4x + 4$
(b)


75. $\dfrac{2x^2 + 1}{\sqrt{x^2 + 1}}$ **77.** $\dfrac{3x^3 + 15x^2 - 8x}{2(x + 1)^3\sqrt{3x - 2}}$

79. $\dfrac{(2x^2 + 2x - 1)\sqrt{x - 1}}{(x + 1)^{3/2}}$ **81.** $\dfrac{2xy}{3 - 2y^2}$

83. $\dfrac{y(1 - 6x^2)}{1 + y}$ **85.** $xy'' + y' = x\left(-\dfrac{2}{x^2}\right) + \dfrac{2}{x} = 0$

87. Relative minimum: $\left(1, \frac{1}{2}\right)$

89. Relative minimum: $(e^{-1}, -e^{-1})$

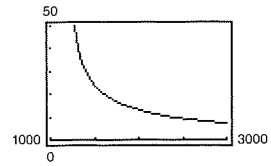
91. Relative minimum: (e, e); Point of inflection: $\left(e^2, \dfrac{e^2}{2}\right)$

93. $x \approx 0.567$

95. Yes. If the graph of g is increasing, then $g'(x) > 0$. Because $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus $f'(x) > 0$. Therefore, the graph of f is increasing.

97. No. For example,
$(\ln 2)(\ln 3) \approx 0.76 \neq 1.79 \approx \ln(2 \cdot 3) = \ln 6$.

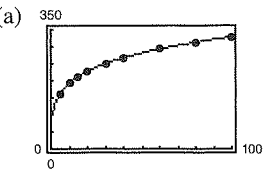
99. True. **101.** False. π is a constant, so $\dfrac{d}{dx}[\ln \pi] = 0$.

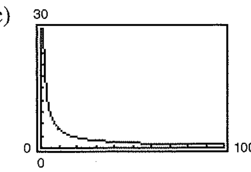
103. (a)


(b) 30 yr; \$503,434.80
(c) 20 yr; \$386,685.60

(d) When $x = 1398.43$, $\dfrac{dt}{dx} \approx -0.0805$. When $x = 1611.19$, $\dfrac{dt}{dx} \approx -0.0287$.

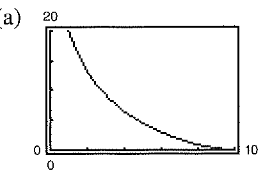
(e) Two benefits of a higher monthly payment are a shorter term and a lower total amount paid.

105. (a)


(c)


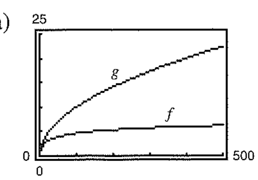
(b) $T'(10) \approx 4.75°/\text{lb/in.}^2$
$T'(70) \approx 0.97°/\text{lb/in.}^2$

$\lim_{p \to \infty} T'(p) = 0$
Answers will vary.

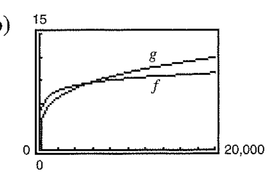
107. (a)


(b) When $x = 5$,
$\dfrac{dy}{dx} = -\sqrt{3}$.
When $x = 9$,
$\dfrac{dy}{dx} = -\dfrac{\sqrt{19}}{9}$.

(c) $\displaystyle\lim_{x \to 10^-} \frac{dy}{dx} = 0$

109. (a)


For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for large values of x.

(b)


For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for large values of x.

$f(x) = \ln x$ increases very slowly for large values of x.

Section 5.2 *(page 330)*

1. No. To use the Log Rule, look for quotients in which the numerator is the derivative of the denominator, with rewriting in mind.

3. Ways to alter an integrand are to rewrite using a trigonometric identity, multiply and divide by the same quantity, add and subtract the same quantity, or use long division.

5. $5 \ln|x| + C$ **7.** $\frac{1}{2}\ln|2x + 5| + C$

9. $\frac{1}{2}\ln|x^2 - 3| + C$ **11.** $\ln|x^4 + 3x| + C$

13. $\dfrac{x^2}{14} - \ln|x| + C$ **15.** $\dfrac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$

17. $\frac{1}{2}x^2 - 4x + 6 \ln|x + 1| + C$

19. $\frac{1}{3}x^3 + 5 \ln|x - 3| + C$

21. $\frac{1}{3}x^3 - 2x + \ln\sqrt{x^2 + 2} + C$ **23.** $\frac{1}{3}(\ln x)^3 + C$

25. $-\frac{2}{3}\ln\left|1 - 3\sqrt{x}\right| + C$ **27.** $6\ln|x - 5| - \dfrac{30}{x - 5} + C$

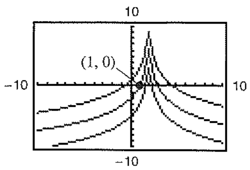
29. $\sqrt{2x} - \ln\left|1 + \sqrt{2x}\right| + C$

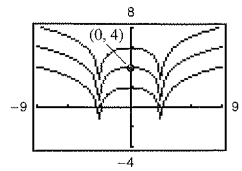
31. $x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C$ **33.** $3\ln\left|\sin\dfrac{\theta}{3}\right| + C$

35. $-\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$ **37.** $5\theta - \frac{1}{3}\sin 3\theta + C$

39. $\ln|1 + \sin t| + C$ **41.** $\ln|\sec x - 1| + C$

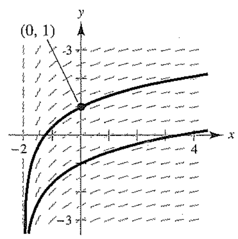
43. $y = -3\ln|2 - x| + C$ **45.** $y = \ln|x^2 - 9| + C$



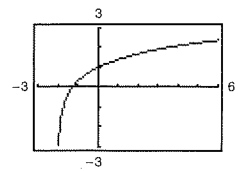


47. $f(x) = -2\ln x + 3x - 2$

49. (a)



(b) $y = \ln\left(\dfrac{x + 2}{2}\right) + 1$



51. $\frac{5}{3}\ln 13 \approx 4.275$ **53.** $\frac{7}{3}$ **55.** $-\ln 3 \approx 1.099$

57. $\ln\left|\dfrac{2 - \sin 2}{1 - \sin 1}\right| \approx 1.929$

59. $4\sqrt{x} - x - 4\ln\left(1 + \sqrt{x}\right) + C$ **61.** $\dfrac{1}{x}$

63. $4\cot 4x$ **65.** $6\ln 3 \approx 6.592$

67. $\ln|\csc 1 + \cot 1| - \ln|\csc 2 + \cot 2| \approx 1.048$

69. $\dfrac{15}{2} + 8\ln 2 \approx 13.045$ **71.** $\dfrac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.03$

73. 1 **75.** $\dfrac{1}{e - 1} \approx 0.582$ **77.** About 13.077

79. d **81.** Proof **83.** $x = 2$ **85.** Proof

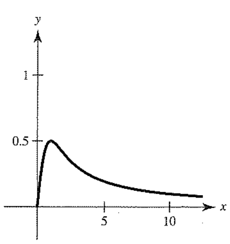
87. $-\ln|\cos x| + C = \ln\left|\dfrac{1}{\cos x}\right| + C = \ln|\sec x| + C$

89. $$\ln|\sec x + \tan x| + C = \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C$$

91. (a) $P(t) = 1000(12\ln|1 + 0.25t| + 1)$ (b) $P(3) \approx 7715$

93. About 4.15 min

95.



(a) $A = \frac{1}{2}\ln 2 - \frac{1}{4}$
(b) $0 < m < 1$
(c) $A = \frac{1}{2}(m - \ln m - 1)$

97. True **99.** True **101.** Putnam Problem B2, 2014

Section 5.3 *(page 339)*

1. The functions f and g have the effect of "undoing" each other.

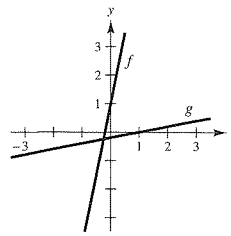
3. No. The domain of f^{-1} is the range of f.

5. c **6.** b **7.** a **8.** d

9. (a) $f(g(x)) = 5\left(\dfrac{x - 1}{5}\right) + 1 = x$

$g(f(x)) = \dfrac{(5x + 1) - 1}{5} = x$

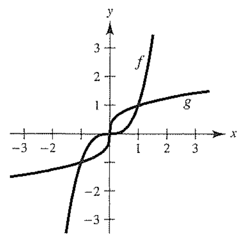
(b)



11. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$

$g(f(x)) = \sqrt[3]{x^3} = x$

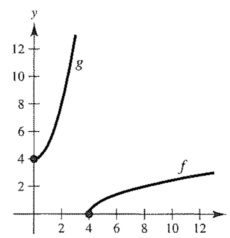
(b)



13. (a) $f(g(x)) = \sqrt{x^2 + 4 - 4} = x$

$g(f(x)) = \left(\sqrt{x - 4}\right)^2 + 4 = x$

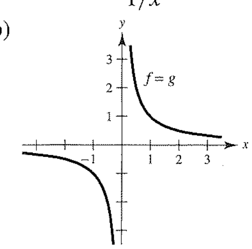
(b)



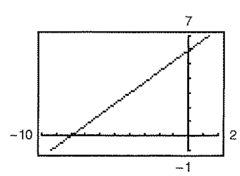
15. (a) $f(g(x)) = \dfrac{1}{1/x} = x$

$g(f(x)) = \dfrac{1}{1/x} = x$

(b)

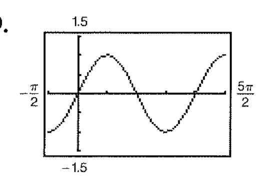


17.



One-to-one, inverse exists

19.



Not one-to-one, inverse does not exist

75. 32 **77.** 88 **79.** $(g^{-1} \circ f^{-1})(x) = \dfrac{x+1}{2}$

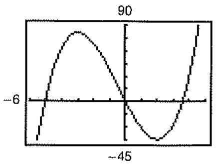
81. $(f \circ g)^{-1}(x) = \dfrac{x+1}{2}$

83. Yes. Functions of the form $f(x) = x^n$, n is odd, are always increasing or always decreasing. So, it is one-to-one and therefore has an inverse function.

85. Many x-values yield the same y-value. For example, $f(\pi) = 0 = f(0)$. The graph is not continuous at $\dfrac{(2n-1)\pi}{2}$, where n is an integer.

87. $k = \frac{1}{4}$ **89.** False. Let $f(x) = x^2$.

91. (a)



(b) $c = 2$

f does not pass the Horizontal Line Test.

93–95. Proofs **97.** Proof; $\dfrac{\sqrt{5}}{5}$

99. Proof; The graph of f is symmetric about the line $y = x$.

101. Proof; concave upward

Section 5.4 *(page 348)*

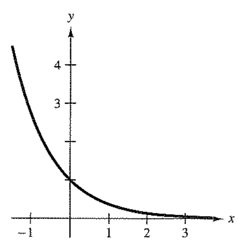
1. The graph of $f(x) = e^x$ is concave upward and increasing on the entire domain.

3. $x = 4$ **5.** $x \approx 2.485$ **7.** $x = 0$ **9.** $x \approx 0.511$

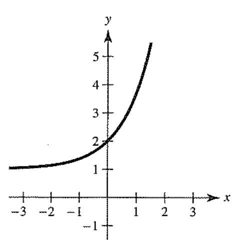
11. $x \approx 8.862$ **13.** $x \approx 7.389$ **15.** $x \approx 10.389$

17. $x \approx 5.389$

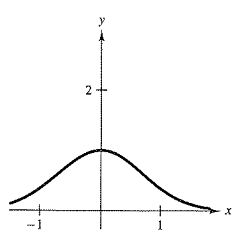
19.



21.

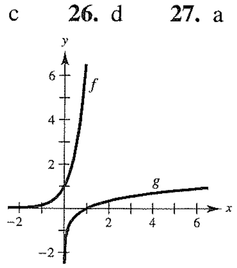


23.

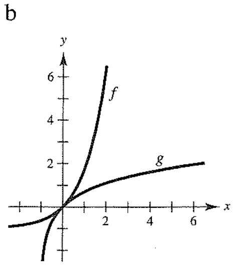


25. c **26.** d **27.** a **28.** b

29.



31.



33. $5e^{5x}$ **35.** $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ **37.** e^{x-4} **39.** $e^x\left(\dfrac{1}{x} + \ln x\right)$

41. $e^x(x+1)(x+3)$ **43.** $3(e^{-t} + e^t)^2(e^t - e^{-t})$

45. $-\dfrac{5e^{5x}}{2 - e^{5x}}$ **47.** $\dfrac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$ **49.** $-\dfrac{2e^x}{(e^x - 1)^2}$

51. $2e^x \cos x$ **53.** $\dfrac{\cos x}{x}$ **55.** $y = 3x + 1$

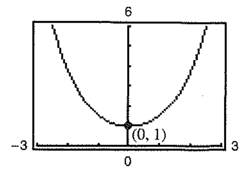
57. $y = -3x + 10$ **59.** $y = \left(\dfrac{1}{e}\right)x - \dfrac{1}{e}$

61. $y = ex$ **63.** $\dfrac{10 - e^y}{xe^y + 3}$ **65.** $y = (-e - 1)x + 1$

67. $3(6x + 5)e^{-3x}$

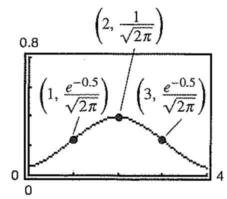
69. $y'' - y = 0$
$4e^{-x} - 4e^{-x} = 0$

71. Relative minimum: $(0, 1)$

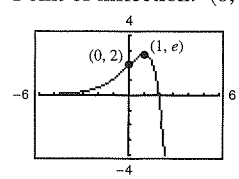


73. Relative maximum: $\left(2, \dfrac{1}{\sqrt{2\pi}}\right)$

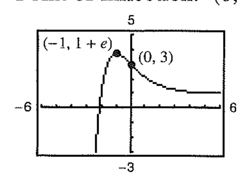
Points of inflection: $\left(1, \dfrac{e^{-0.5}}{\sqrt{2\pi}}\right), \left(3, \dfrac{e^{-0.5}}{\sqrt{2\pi}}\right)$



75. Relative maximum: $(1, e)$
Point of inflection: $(0, 2)$

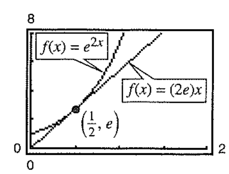


77. Relative maximum: $(-1, 1 + e)$
Point of inflection: $(0, 3)$

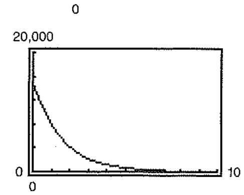


79. $A = \sqrt{2}e^{-1/2}$

81. $\left(\frac{1}{2}, e\right)$

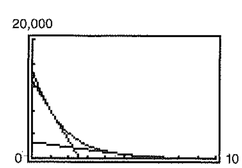


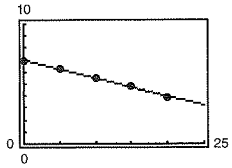
83. (a)



(b) When $t = 1$, $\dfrac{dV}{dt} \approx -5028.84$.
When $t = 5$, $\dfrac{dV}{dt} \approx -406.89$.

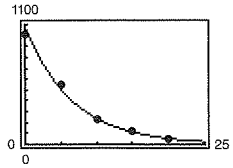
(c)



85. (a) 

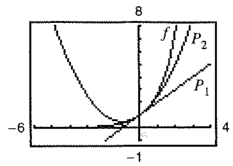
$\ln P = -0.1499h + 6.9797$

(b) $P = 1074.6e^{-0.1499h}$

(c) 

(d) $h = 5$: -76.13 millibars/km
$h = 18$: -10.84 millibars/km

87. $P_1 = 1 + x$; $P_2 = 1 + x + \frac{1}{2}x^2$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 0$.

89. $12! = 479{,}001{,}600$
Stirling's Formula: $12! \approx 475{,}687{,}487$

91. $e^{5x} + C$ **93.** $\frac{1}{5}e^{5x-3} + C$ **95.** $e^{x^2+x} + C$

97. $2e^{\sqrt{x}} + C$

99. $x - \ln(e^x + 1) + C_1$ or $-\ln(1 + e^{-x}) + C_2$

101. $-\frac{2}{3}(1 - e^x)^{3/2} + C$ **103.** $\ln|e^x - e^{-x}| + C$

105. $-\frac{5}{2}e^{-2x} + e^{-x} + C$ **107.** $\ln|\cos e^{-x}| + C$

109. $\dfrac{e^2 - 1}{2e^2}$ **111.** $\dfrac{e - 1}{2e}$ **113.** $\dfrac{e}{3}(e^2 - 1)$

115. $\dfrac{1}{4}\ln\dfrac{1 + e^8}{2}$ **117.** $\dfrac{1}{\pi}[e^{\sin(\pi^2/2)} - 1]$

119. $y = \frac{1}{18}e^{9x^2} + C$ **121.** $f(x) = \frac{1}{2}(e^x + e^{-x})$

123. $e^6 - 1 \approx 402.4$ **125.** $2(1 - e^{-3/2}) \approx 1.554$

127. 92.190

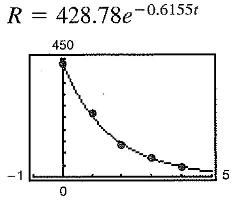
129. The natural exponential function has a horizontal asymptote $y = 0$ to the left and the natural logarithmic function has a vertical asymptote $x = 0$ from the right.

131. False. The derivative is $e^x(g'(x) + g(x))$.

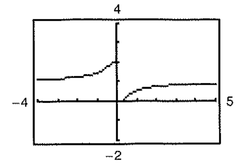
133. True

135. The probability that a given battery will last between 48 months and 60 months is approximately 47.72%.

137. (a) $R = 428.78e^{-0.6155t}$

(b) 

(c) About 637.2 L

139. (a) 

(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. So, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞, and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist because the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

141. $\displaystyle\int_0^x e^t\,dt \geq \int_0^x 1\,dt$; $e^x - 1 \geq x$; $e^x \geq x + 1$ for $x \geq 0$

143. Relative maximum: $\left(\dfrac{1}{k}, \dfrac{1}{ke}\right)$

Point of inflection: $\left(\dfrac{2}{k}, \dfrac{2}{ke^2}\right)$

145. Putnam Problem B1, 2012

Section 5.5 *(page 358)*

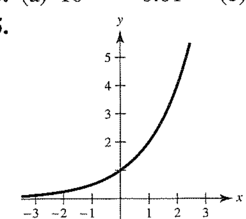
1. $a = 4, b = 6$

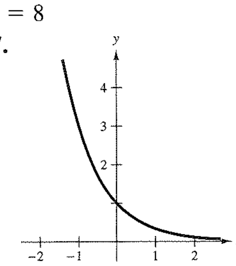
3. It is necessary when you have a function of the form $y = u(x)^{v(x)}$

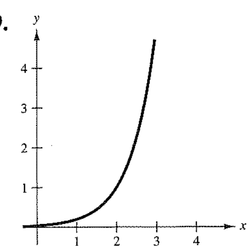
5. -3 **7.** 0 **9.** $\frac{5}{6}$

11. (a) $\log_2 8 = 3$ (b) $\log_3\left(\frac{1}{3}\right) = -1$

13. (a) $10^{-2} = 0.01$ (b) $\left(\frac{1}{2}\right)^{-3} = 8$

15. 

17. 

19. 

21. (a) $x = 3$ (b) $x = -1$ **23.** (a) $x = \frac{1}{3}$ (b) $x = \frac{1}{16}$

25. (a) $x = -1, 2$ (b) $x = \frac{1}{3}$ **27.** 1.965 **29.** -6.288

31. 12.253 **33.** 33.000 **35.** 3.429

Section 5.6 *(page 369)*

1. L'Hôpital's Rule allows you to address limits of the form $0/0$ and ∞/∞.

3.

x	-0.1	-0.01	-0.001	0
$f(x)$	1.3177	1.3332	1.3333	?

x	0.001	0.01	0.1
$f(x)$	1.3333	1.3332	1.3177

$\frac{4}{3}$

5.

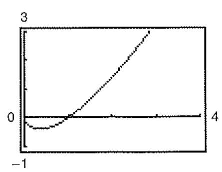
x	1	10	10^2
$f(x)$	0.9900	90,483.7	3.7×10^9

x	10^3	10^4	10^5
$f(x)$	4.5×10^{10}	0	0

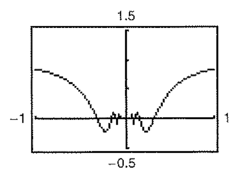
0

7. $\frac{3}{8}$ **9.** $\frac{1}{8}$ **11.** 0 **13.** $\frac{5}{3}$ **15.** 4 **17.** 0
19. ∞ **21.** $\frac{11}{4}$ **23.** $\frac{3}{5}$ **25.** $\frac{7}{6}$ **27.** ∞
29. 0 **31.** 1 **33.** 0 **35.** 0 **37.** ∞
39. $\frac{5}{9}$ **41.** ∞

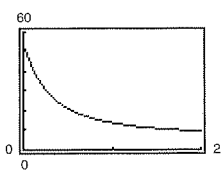
43. (a) Not indeterminate
(b) ∞
(c)



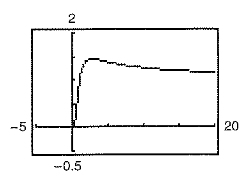
45. (a) $0 \cdot \infty$
(b) 1
(c)



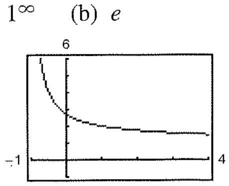
47. (a) 1^∞
(b) e^4
(c)



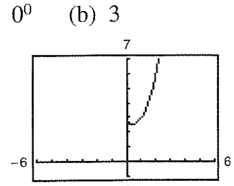
49. (a) ∞^0
(b) 1
(c)



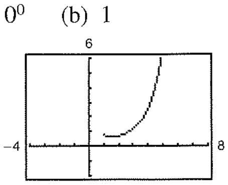
51. (a) 1^∞ (b) e
(c)



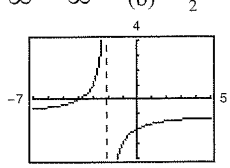
53. (a) 0^0 (b) 3
(c)



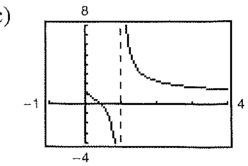
55. (a) 0^0 (b) 1
(c)



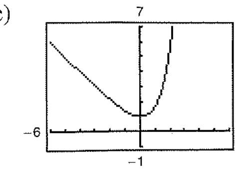
57. (a) $\infty - \infty$ (b) $-\frac{3}{2}$
(c)



59. (a) $\infty - \infty$ (b) ∞
(c)



61. (a) $\infty - \infty$ (b) ∞
(c)



63. Answers will vary. Sample answers:
(a) $f(x) = x^2 - 25$, $g(x) = x - 5$
(b) $f(x) = (x - 5)^2$, $g(x) = x^2 - 25$
(c) $f(x) = x^2 - 25$, $g(x) = (x - 5)^3$

65. (a) Yes; $\frac{0}{0}$ (b) No; $\frac{0}{-1}$ (c) Yes; $\frac{\infty}{\infty}$ (d) Yes; $\frac{0}{0}$
(e) No; $\frac{-1}{0}$ (f) Yes; $\frac{0}{0}$

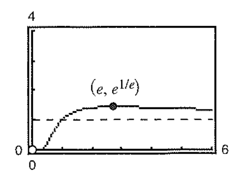
67.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

69. 0 **71.** 0 **73.** 0

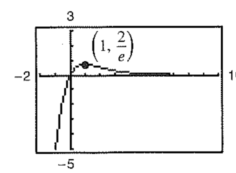
75. Horizontal asymptote: $y = 1$

Relative maximum: $(e, e^{1/e})$



77. Horizontal asymptote: $y = 0$

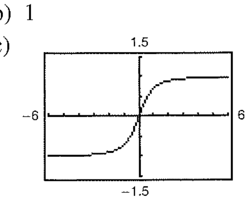
Relative maximum: $\left(1, \frac{2}{e}\right)$



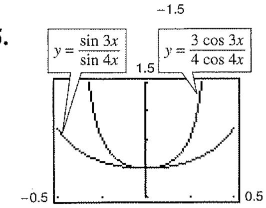
79. Limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

81. Limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

83. (a) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$
Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails.
(b) 1
(c)



85.



As $x \to 0$, the graphs get closer together (they both approach 0.75). By L'Hôpital's Rule,
$\lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{3 \cos 3x}{4 \cos 4x} = \frac{3}{4}$.

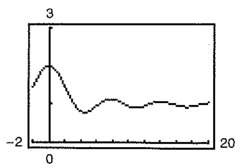
87. $\dfrac{Vt}{L}$ **89.** Proof **91.** $c = \dfrac{2}{3}$ **93.** $c = \dfrac{\pi}{4}$

95. False. $\dfrac{\infty}{0} = \pm\infty$ **97.** True **99.** True **101.** $\frac{3}{4}$

103. $c = \frac{4}{3}$ **105.** $a = 1, b = \pm 2$ **107.** Proof

109. (a) $0 \cdot \infty$ (b) 0 **111.** Proof **113.** (a)–(c) 2

115. (a)



(b) $\lim_{x\to\infty} h(x) = 1$ (c) No

117. Putnam Problem A1, 1956

Section 5.7 *(page 379)*

1. arccos x is the angle, $0 \le \theta \le \pi$, whose cosine is x.

3. arccot **5.** $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{3\pi}{4}\right), \left(\dfrac{1}{2}, \dfrac{\pi}{3}\right), \left(\dfrac{\sqrt{3}}{2}, \dfrac{\pi}{6}\right)$

7. $\dfrac{\pi}{6}$ **9.** $\dfrac{\pi}{3}$ **11.** $\dfrac{\pi}{6}$ **13.** $-\dfrac{\pi}{4}$ **15.** 1.52

17. $\arccos \dfrac{1}{1.269} \approx 0.66$ **19.** x **21.** $\dfrac{\sqrt{1-x^2}}{x}$

23. $\dfrac{1}{x}$ **25.** (a) $\dfrac{3}{5}$ (b) $\dfrac{5}{3}$ **27.** (a) $-\sqrt{3}$ (b) $-\frac{13}{5}$

29. $\sqrt{1-4x^2}$ **31.** $\dfrac{\sqrt{x^2-1}}{|x|}$ **33.** $\dfrac{\sqrt{x^2-9}}{3}$

35. $\dfrac{\sqrt{x^2+2}}{x}$ **37.** $x = \frac{1}{3}\left(\sin \frac{1}{2} + \pi\right) \approx 1.207$ **39.** $x = \frac{1}{3}$

41. $\dfrac{1}{\sqrt{2x-x^2}}$ **43.** $-\dfrac{3}{\sqrt{4-x^2}}$ **45.** $\dfrac{e^x}{1+e^{2x}}$

47. $\dfrac{3x - \sqrt{1-9x^2}\arcsin 3x}{x^2\sqrt{1-9x^2}}$ **49.** $-\dfrac{t}{\sqrt{1-t^2}}$

51. $2 \arccos x$ **53.** $\dfrac{1}{1-x^4}$ **55.** $\dfrac{x^2}{\sqrt{16-x^2}}$

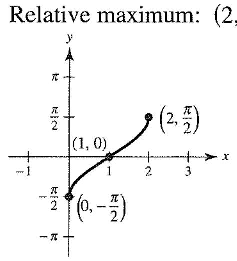
57. $y = \dfrac{1}{3}\left(4\sqrt{3}x - 2\sqrt{3} + \pi\right)$ **59.** $y = \dfrac{1}{4}x + \dfrac{\pi - 2}{4}$

61. $y = (2\pi - 4)x + 4$

63. Relative maximum: $(1.272, -0.606)$
Relative minimum: $(-1.272, 3.747)$

65. Relative maximum: $(2, 2.214)$

67.

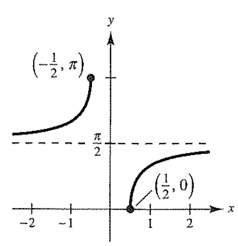


Maximum: $\left(2, \dfrac{\pi}{2}\right)$

Minimum: $\left(0, -\dfrac{\pi}{2}\right)$

Point of inflection: $(1, 0)$

69.



Maximum: $\left(-\dfrac{1}{2}, \pi\right)$

Minimum: $\left(\dfrac{1}{2}, 0\right)$

Asymptote: $y = \dfrac{\pi}{2}$

71. $y = -\dfrac{2\pi x}{\pi + 8} + 1 - \dfrac{\pi^2}{2\pi + 16}$ **73.** $y = -x + \sqrt{2}$

75. (a) $\arcsin(\arcsin 0.5) \approx 0.551$
$\arcsin(\arcsin 1)$ does not exist.
(b) $\sin(-1) \le x \le \sin 1$

77. No

79. In order to have a true inverse function, the domain of sine must be restricted. As a result, 2π is not in the range of the arcsine function.

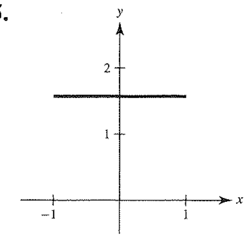
81. (a) and (b) Proofs **83.** True **85.** True

87. (a) $\theta = \operatorname{arccot} \dfrac{x}{5}$
(b) $x = 10$: 16 rad/h
$x = 3$: 58.824 rad/h

89. (a) $h(t) = -16t^2 + 256$; $t = 4$ sec
(b) $t = 1$: -0.0520 rad/sec
$t = 2$: -0.1116 rad/sec

91. $50\sqrt{2} \approx 70.71$ ft **93.** (a) and (b) Proofs

95.



(a) The graph is a horizontal line at $\dfrac{\pi}{2}$.
(b) Proof

97. $c = 2$ **99.** Proof

Section 5.8 *(page 387)*

1. (a) No
(b) Yes. Use the rule involving the arcsecant function.

3. $\arcsin \dfrac{x}{3} + C$ **5.** $\operatorname{arcsec}|2x| + C$

7. $\arcsin(x+1) + C$ **9.** $\frac{1}{2}\arcsin t^2 + C$

11. $\dfrac{1}{10}\arctan \dfrac{t^2}{5} + C$ **13.** $\dfrac{1}{4}\arctan \dfrac{e^{2x}}{2} + C$

15. $\arcsin \dfrac{\csc x}{5} + C$ **17.** $2\arcsin\sqrt{x} + C$

19. $\frac{1}{2}\ln(x^2+1) - 3\arctan x + C$

21. $8\arcsin \dfrac{x-3}{3} - \sqrt{6x - x^2} + C$ **23.** $\dfrac{\pi}{6}$ **25.** $\dfrac{\pi}{6}$

27. $\dfrac{1}{3}\left(\arctan 3 - \dfrac{\pi}{4}\right) \approx 0.155$ **29.** $\arctan 5 - \dfrac{\pi}{4} \approx 0.588$

31. $\dfrac{\pi}{4}$ **33.** $\dfrac{1}{32}\pi^2 \approx 0.308$ **35.** $\dfrac{\pi}{2}$

37. $\dfrac{\sqrt{2}}{2}\arcsin\left[\dfrac{\sqrt{6}}{6}(x-2)\right] + C$ **39.** $\arcsin \dfrac{x+2}{2} + C$

41. $4 - 2\sqrt{3} + \dfrac{1}{6}\pi \approx 1.059$

43. $2\sqrt{e^t - 3} - 2\sqrt{3}\arctan \dfrac{\sqrt{e^t-3}}{\sqrt{3}} + C$ **45.** $\dfrac{\pi}{6}$

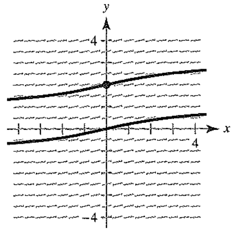
47. (a) $\arcsin x + C$ (b) $-\sqrt{1-x^2} + C$ (c) Not possible

49. (a) $\frac{2}{3}(x-1)^{3/2} + C$ (b) $\frac{2}{15}(x-1)^{3/2}(3x+2) + C$
(c) $\frac{2}{3}\sqrt{x-1}(x+2) + C$

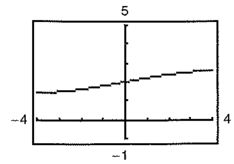
51. Proof

53. No. Graphing $f(x) = \arcsin x$ and $g(x) = -\arccos x$, you can see that the graph of f is the graph of g shifted vertically.

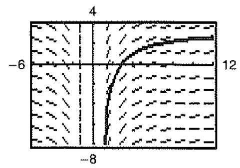
55. (a)



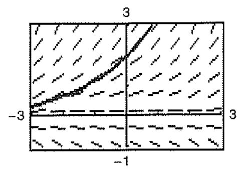
(b) $y = \frac{2}{3}\arctan\frac{x}{3} + 2$



57.

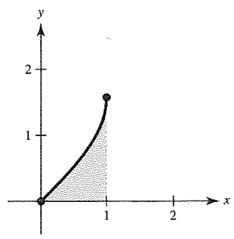


59.



61. $y = \arcsin\frac{x}{2} + \pi$ **63.** $\frac{\pi}{3}$ **65.** $\frac{3\pi}{2}$

67. (a)



(b) 0.5708 (c) $\frac{\pi - 2}{2}$

69. (a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x + 2]$; Maximum at $x = -1$
(b) Maximum at $x = -1$

71. False. $\int \frac{dx}{3x\sqrt{9x^2 - 16}} = \frac{1}{12}\operatorname{arcsec}\frac{|3x|}{4} + C$

73–75. Proofs

77. (a) $\int_0^1 \frac{1}{1 + x^2}\,dx$ (b) About 0.7857
(c) Because $\int_0^1 \frac{1}{1 + x^2}\,dx = \frac{\pi}{4}$, you can use the Midpoint Rule to approximate $\frac{\pi}{4}$. Multiplying the result by 4 gives an estimation of π.

Section 5.9 *(page 397)*

1. Hyperbolic function came from the comparison of the area of a semicircular region with the area of a region under a hyperbola.

3. $\sinh^2 x = \frac{-1 + \cosh 2x}{2}$ **5.** (a) 10.018 (b) -0.964

7. (a) $\frac{4}{3}$ (b) $\frac{13}{12}$ **9.** (a) 1.317 (b) 0.962

11–17. Proofs

19. $\cosh x = \frac{\sqrt{13}}{2}$, $\tanh x = \frac{3\sqrt{13}}{13}$, $\operatorname{csch} x = \frac{2}{3}$,
$\operatorname{sech} x = \frac{2\sqrt{13}}{13}$, $\coth x = \frac{\sqrt{13}}{3}$

21. ∞ **23.** 1 **25.** $9\cosh 9x$

27. $-10x(\operatorname{sech} 5x^2 \tanh 5x^2)$ **29.** $\coth x$

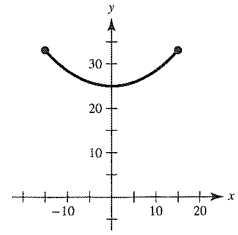
31. $-\frac{t}{2}\cosh(-3t) + \frac{\sinh(-3t)}{6}$ **33.** $\operatorname{sech} t$

35. $y = -2x + 2$ **37.** $y = 1 - 2x$

39. Relative maximum: $(1.20, 0.66)$
Relative minimum: $(-1.20, -0.66)$

41. Relative maxima: $(\pm\pi, \cosh\pi)$, Relative minimum: $(0, -1)$

43. (a)



(b) 33.146 units, 25 units
(c) $m = \sinh 1 \approx 1.175$

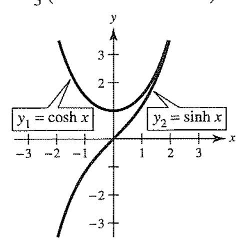
45. $\frac{1}{4}\sinh 4x + C$ **47.** $-\frac{1}{2}\cosh(1 - 2x) + C$

49. $\frac{1}{3}\cosh^3(x - 1) + C$ **51.** $\ln|\sinh x| + C$

53. $-\coth\frac{x^2}{2} + C$ **55.** $\ln\frac{5}{4}$ **57.** $\coth 1 - \coth 2$

59. $-\frac{1}{3}(\operatorname{csch} 2 - \operatorname{csch} 1)$

61.



The graphs do not intersect.

63. Proof **65.** $\frac{3}{\sqrt{9x^2 - 1}}$ **67.** $\frac{1}{2\sqrt{x}(1 - x)}$

69. $|\sec x|$ **71.** $-\csc x$ **73.** $2\sinh^{-1}(2x)$

75. $\frac{\sqrt{3}}{18}\ln\left|\frac{1 + \sqrt{3}x}{1 - \sqrt{3}x}\right| + C$ **77.** $\ln\left(\sqrt{e^{2x} + 1} - 1\right) - x + C$

79. $2\sinh^{-1}\sqrt{x} + C = 2\ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$

81. $\frac{1}{4}\ln\left|\frac{x - 4}{4}\right| + C$ **83.** $\ln\left(\frac{3 + \sqrt{5}}{2}\right)$ **85.** $\frac{\ln 7}{12}$

87. $-\frac{x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{x - 5}{x + 1}\right| + C$

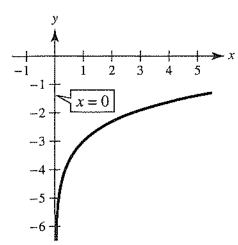
89. $8\arctan e^2 - 2\pi \approx 5.207$ **91.** $\frac{5}{2}\ln\left(\sqrt{17} + 4\right) \approx 5.237$

93. (a) $-\frac{\sqrt{a^2 - x^2}}{x}$ (b) Proof

95–103. Proofs **105.** Putnam Problem 8, 1939

Review Exercises for Chapter 5 *(page 400)*

1.



Domain: $x > 0$

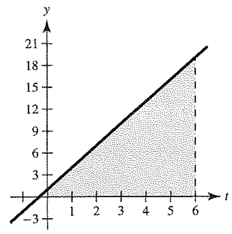
3. (a) 2.9957 (b) -0.2231 (c) 6.4376 (d) 0.8047

5. $\frac{1}{5}[\ln(2x + 1) + \ln(2x - 1) - \ln(4x^2 + 1)]$

7. $\ln\frac{3\sqrt[3]{4 - x^2}}{x}$ **9.** $\frac{1}{2x}$ **11.** $\frac{1 + 2\ln x}{2\sqrt{\ln x}}$

13. $-\frac{8x}{x^4 - 16}$ **15.** $\frac{7}{(1 - 7x)[\ln(1 - 7x)]^2}$

(c) $F(6) = 60$



17. R_1; \$1.125 million **19.** $\dfrac{\pi^2}{2}$ **21.** $\dfrac{\pi^2}{4}$

23. (a) 9π (b) 18π (c) 9π (d) 36π

25. 3 ft^3 **27.** $\frac{8}{15}(1 + 6\sqrt{3}) \approx 6.076$

29. $2\pi \displaystyle\int_3^6 \frac{x^3}{18}\sqrt{1 + \frac{x^4}{36}}\, dx \approx 459.098$

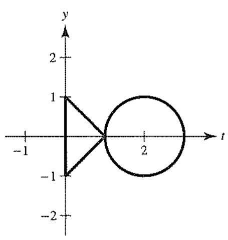
31. $2\pi \displaystyle\int_0^2 x\sqrt{1 + x^2}\, dx \approx 21.322$ **33.** 5.208 ft-lb

35. 952.4 mile-tons $\approx 1.109 \times 10^{10}$ ft-lb **37.** 200 ft-lb

39. 693.15 ft-lb

41. 3.6 **43.** $M_x = \dfrac{544\rho}{15}$, $M_y = \dfrac{32\rho}{3}$, $(\bar{x}, \bar{y}) = \left(1, \dfrac{17}{5}\right)$

45. Answers will vary. Sample answer:



$(\bar{x}, \bar{y}) = (1.596, 0)$

47. 374.4 lb **49.** 3072 lb **51.** 723,822.95 lb

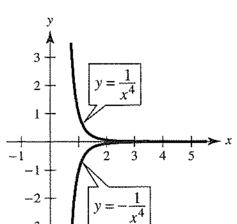
P.S. Problem Solving *(page 513)*

1. 3 **3.** $y = 0.2063x$ **5.** $(\bar{x}, \bar{y}) = \left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$

7. $V = 2\pi\left(d + \frac{1}{2}\sqrt{w^2 + l^2}\right)lw$

9. $f(x) = 2e^{x/2} - 2$ **11.** 89.3%

13.



(a) $(\bar{x}, \bar{y}) = \left(\dfrac{63}{43}, 0\right)$

(b) $(\bar{x}, \bar{y}) = \left(\dfrac{3b(b + 1)}{2(b^2 + b + 1)}, 0\right)$

(c) $\left(\dfrac{3}{2}, 0\right)$

15. Consumer surplus: 1600, Producer surplus: 400

17. Wall at shallow end: 9984 lb
Wall at deep end: 39,936 lb
Side wall: 19,968 + 26,624 = 46,592 lb

Chapter 8

Section 8.1 *(page 520)*

1. Use long division to rewrite the function as the sum of a polynomial and a proper rational function.

3. b

5. $\displaystyle\int u^n\, du$, $u = 5x - 3$, $n = 4$ **7.** $\displaystyle\int \frac{du}{u}$, $u = 1 - 2\sqrt{x}$ **9.** $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}}$, $u = t$, $a = 1$

11. $\displaystyle\int \sin u\, du$, $u = t^2$ **13.** $\displaystyle\int e^u\, du$, $u = \sin x$ **15.** $2(x - 5)^7 + C$

17. $-\dfrac{7}{6(z - 10)^6} + C$ **19.** $\dfrac{z^3}{3} - \dfrac{1}{5(z - 1)^5} + C$

21. $-\frac{1}{3}\ln|-t^3 + 9t + 1| + C$ **23.** $\frac{1}{2}x^2 + x + \ln|x - 1| + C$

25. $x + \ln|x + 1| + C$ **27.** $\dfrac{x}{15}(48x^4 + 200x^2 + 375) + C$

29. $\dfrac{\sin 2\pi x^2}{4\pi} + C$ **31.** $-2\sqrt{\cos x} + C$

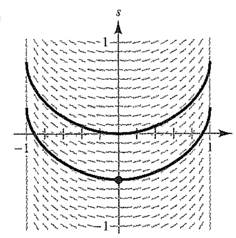
33. $2\ln(1 + e^x) + C$ **35.** $(\ln x)^2 + C$

37. $-\ln|\csc\alpha + \cot\alpha| + \ln|\sin\alpha| + C$

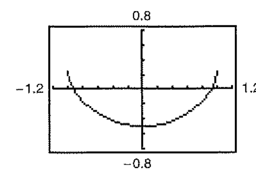
39. $-\dfrac{1}{4}\arcsin(4t + 1) + C$ **41.** $\dfrac{1}{2}\ln\left|\cos\dfrac{2}{t}\right| + C$

43. $\dfrac{6}{5}\operatorname{arcsec}\dfrac{|3z|}{5} + C$ **45.** $\dfrac{1}{4}\arctan\dfrac{2x + 1}{8} + C$

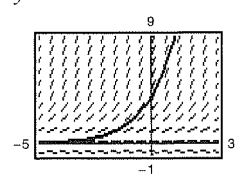
47. (a)



(b) $\frac{1}{2}\arcsin t^2 - \frac{1}{2}$



49. $y = 4e^{0.8x}$



51. $y = \frac{1}{2}e^{2x} + 10e^x + 25x + C$ **53.** $r = 10\arcsin e^t + C$

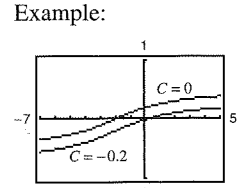
55. $y = \dfrac{1}{2}\arctan\dfrac{\tan x}{2} + C$ **57.** $\dfrac{1}{15}$ **59.** $\dfrac{1}{2}$

61. $\dfrac{1}{2}(1 - e^{-1}) \approx 0.316$ **63.** $\dfrac{3}{2}[(\ln 4)^2 - (\ln 3)^2] \approx 1.072$

65. 8 **67.** $\ln 9 + \dfrac{8}{3} \approx 4.864$ **69.** $\dfrac{\pi}{18}$

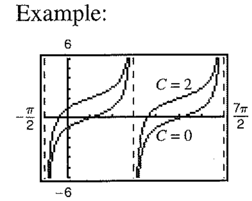
71. $\dfrac{240}{\ln 3} \approx 218.457$ **73.** $\dfrac{18\sqrt{6}}{5} \approx 8.82$ **75.** $\dfrac{4}{3} \approx 1.333$

77. $\frac{1}{3}\arctan\left[\frac{1}{3}(x + 2)\right] + C$
Graphs will vary.
Example:



One graph is a vertical translation of the other.

79. $\tan\theta - \sec\theta + C$
Graphs will vary.
Example:



One graph is a vertical translation of the other.

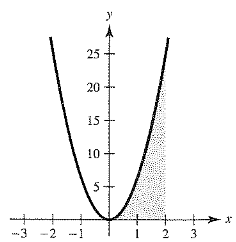
81. No. When $u = x^2$, it does not follow that $x = \sqrt{u}$ because x is negative on $[-1, 0)$.

83. $a = \sqrt{2},\ b = \dfrac{\pi}{4};\ -\dfrac{1}{\sqrt{2}}\ln\left|\csc\left(x + \dfrac{\pi}{4}\right) + \cot\left(x + \dfrac{\pi}{4}\right)\right| + C$

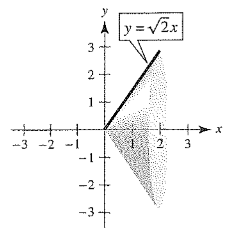
85. (a) They are equivalent because $e^{x+C_1} = e^x \cdot e^{C_1} = Ce^x,\ C = e^{C_1}$.
(b) They differ by a constant. $\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C$

87. a

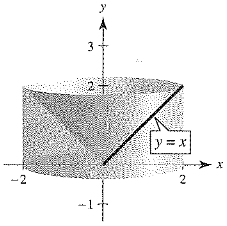
89. (a)



(b)



(c)



91. (a) $\pi(1 - e^{-1}) \approx 1.986$
(b) $b = \sqrt{\ln\dfrac{3\pi}{3\pi - 4}} \approx 0.743$

93. $\ln(\sqrt{2} + 1) \approx 0.8814$

95. $\dfrac{8\pi}{3}(10\sqrt{10} - 1) \approx 256.545$ **97.** $\dfrac{1}{3}\arctan 3 \approx 0.416$

99. About 1.0320

101. (a) $\frac{1}{3}\sin x(\cos^2 x + 2)$
(b) $\frac{1}{15}\sin x(3\cos^4 x + 4\cos^2 x + 8)$
(c) $\frac{1}{35}\sin x(5\cos^6 x + 6\cos^4 x + 8\cos^2 x + 16)$
(d) $\displaystyle\int \cos^{15} x\,dx = \int (1 - \sin^2 x)^7 \cos x\,dx$
You would expand $(1 - \sin^2 x)^7$.

103. Proof

Section 8.2 *(page 529)*

1. The formula for the derivative of a product

3. Let $dv = dx$. **5.** $u = x,\ dv = e^{9x}\,dx$

7. $u = (\ln x)^2,\ dv = dx$ **9.** $u = x,\ dv = \sec^2 x\,dx$

11. $\frac{1}{16}x^4(4\ln x - 1) + C$

13. $-\frac{1}{4}(2x + 1)\cos 4x + \frac{1}{8}\sin 4x + C$

15. $\dfrac{e^{4x}}{16}(4x - 1) + C$ **17.** $e^x(x^3 - 3x^2 + 6x - 6) + C$

19. $\frac{1}{4}[2(t^2 - 1)\ln|t + 1| - t^2 + 2t] + C$ **21.** $\frac{1}{3}(\ln x)^3 + C$

23. $\dfrac{e^{2x}}{4(2x + 1)} + C$ **25.** $\dfrac{2}{15}(x - 5)^{3/2}(3x + 10) + C$

27. $-x\cot x + \ln|\sin x| + C$

29. $(6x - x^3)\cos x + (3x^2 - 6)\sin x + C$

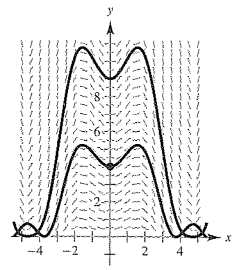
31. $x\arctan x - \frac{1}{2}\ln(1 + x^2) + C$

33. $-\frac{3}{34}e^{-3x}\sin 5x - \frac{5}{34}e^{-3x}\cos 5x + C$

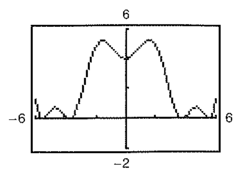
35. $x\ln x - x + C$

37. $y = \dfrac{2}{5}t^2\sqrt{3 + 5t} - \dfrac{8t}{75}(3 + 5t)^{3/2} + \dfrac{16}{1875}(3 + 5t)^{5/2} + C$
$= \dfrac{2}{625}\sqrt{3 + 5t}(25t^2 - 20t + 24) + C$

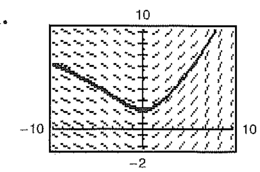
39. (a)



(b) $2\sqrt{y} - \cos x - x\sin x = 3$



41.



43. $2e^{3/2} + 4 \approx 12.963$ **45.** $\dfrac{\pi}{8} - \dfrac{1}{4} \approx 0.143$

47. $\dfrac{\pi - 3\sqrt{3} + 6}{6} \approx 0.658$

49. $\frac{1}{2}[e(\sin 1 - \cos 1) + 1] \approx 0.909$

51. $8\operatorname{arcsec} 4 + \dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{15}}{2} - \dfrac{2\pi}{3} \approx 7.380$

53. $\dfrac{e^{2x}}{4}(2x^2 - 2x + 1) + C$

55. $-\cos x(x + 2)^2 + 2\sin x(x + 2) + 2\cos x + C$

57. $\frac{1}{20}(4x + 9)^{3/2}(2x + 17) + C$

59. Answers will vary. Sample answer: $\int x^3 \sin x\,dx$
It takes three applications until the algebraic factor becomes a constant.

61. (a) No, substitution (b) Yes, $u = \ln x,\ dv = x\,dx$
(c) Yes, $u = x^2,\ dv = e^{-3x}\,dx$ (d) No, substitution
(e) Yes, $u = x$ and $dv = \dfrac{1}{\sqrt{x + 1}}\,dx$ (f) No, substitution

63. $2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + C$

65. $\frac{1}{2}(x^4 e^{x^2} - 2x^2 e^{x^2} + 2e^{x^2}) + C$

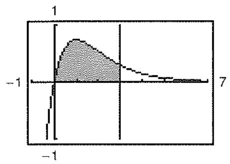
67. (a) and (b) $\frac{1}{3}\sqrt{4 + x^2}(x^2 - 8) + C$

69. $n = 0$: $x(\ln x - 1) + C$
$n = 1$: $\frac{1}{4}x^2(2\ln x - 1) + C$
$n = 2$: $\frac{1}{9}x^3(3\ln x - 1) + C$
$n = 3$: $\frac{1}{16}x^4(4\ln x - 1) + C$
$n = 4$: $\frac{1}{25}x^5(5\ln x - 1) + C$
$\displaystyle\int x^n \ln x\,dx = \frac{x^{n+1}}{(n + 1)^2}[(n + 1)\ln x - 1] + C$

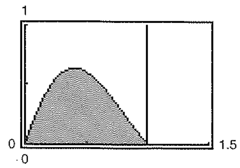
71–75. Proofs **77.** $-x^2\cos x + 2x\sin x + 2\cos x + C$

79. $\frac{1}{36}x^6(6\ln x - 1) + C$

81. $\dfrac{e^{-3x}(-3\sin 4x - 4\cos 4x)}{25} + C$

83.


$2 - \frac{8}{e^3} \approx 1.602$

85.


$\frac{\pi}{1+\pi^2}\left(\frac{1}{e}+1\right) \approx 0.395$

87. (a) 1 (b) $\pi(e-2) \approx 2.257$ (c) $\frac{1}{2}\pi(e^2+1) \approx 13.177$
(d) $\left(\frac{e^2+1}{4}, \frac{e-2}{2}\right) \approx (2.097, 0.359)$

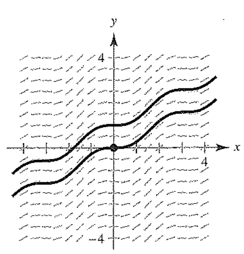
89. In Example 6, we showed that the centroid of an equivalent region was $\left(1, \frac{\pi}{8}\right)$. By symmetry, the centroid of this region is $\left(\frac{\pi}{8}, 1\right)$.

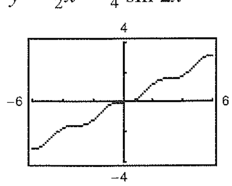
91. $\frac{7}{10\pi}(1-e^{-4\pi}) \approx 0.223$ **93.** $931,265

95. Proof **97.** $b_n = \frac{8h}{(n\pi)^2}\sin\frac{n\pi}{2}$

99. For any integrable function, $\int f(x)\,dx = C + \int f(x)\,dx$, but this cannot be used to imply that $C = 0$.

Section 8.3 *(page 538)*

1. $\int \sin^8 x\,dx$; The other integral can be found using u-substitution.
3. $-\frac{1}{6}\cos^6 x + C$ **5.** $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$
7. $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$
9. $-\frac{1}{3}(\cos 2\theta)^{3/2} + \frac{1}{7}(\cos 2\theta)^{7/2} + C$
11. $\frac{1}{12}(6x + \sin 6x) + C$ **13.** $2x^2 + 2x\sin 2x + \cos 2x + C$
15. $\frac{2}{3}$ **17.** $\frac{\pi}{4}$ **19.** $\frac{63\pi}{512}$ **21.** $\frac{1}{4}\ln|\sec 4x + \tan 4x| + C$
23. $\frac{\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|}{2\pi} + C$
25. $\frac{1}{2}\tan^4\frac{x}{2} - \tan^2\frac{x}{2} - 2\ln\left|\cos\frac{x}{2}\right| + C$
27. $\frac{1}{2}\left[\frac{\sec^5 2t}{5} - \frac{\sec^3 2t}{3}\right] + C$ **29.** $\frac{1}{24}\sec^6 4x + C$
31. $\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$
33. $\ln|\sec x + \tan x| - \sin x + C$
35. $\frac{12\pi\theta - 8\sin 2\pi\theta + \sin 4\pi\theta}{32\pi} + C$
37. $y = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$
39. (a)


(b) $y = \frac{1}{2}x - \frac{1}{4}\sin 2x$


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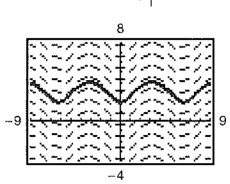
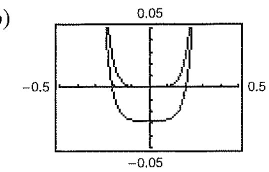


43. $\frac{1}{16}(2\sin 4x + \sin 8x) + C$

45. $\frac{1}{14}\cos 7t - \frac{1}{22}\cos 11t + C$
47. $\frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$
49. $\frac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$
51. $-\frac{1}{3}\cot 3x - \frac{1}{9}\cot^3 3x + C$
53. $\ln|\csc t - \cot t| + \cos t + C$
55. $\ln|\csc x - \cot x| + \cos x + C$ **57.** $t - 2\tan t + C$
59. π **61.** $3(1 - \ln 2)$ **63.** $\ln 2$ **65.** 4
67. (a) $\frac{1}{18}\tan^6 3x + \frac{1}{12}\tan^4 3x + C_1$, $\frac{1}{18}\sec^6 3x - \frac{1}{12}\sec^4 3x + C_2$
(b)


(c) Proof

69. (a) $\frac{1}{2}\sin^2 x + C$ (b) $-\frac{1}{2}\cos^2 x + C$
(c) $\frac{1}{2}\sin^2 x + C$ (d) $-\frac{1}{4}\cos 2x + C$
The answers are all the same, but they are written in different forms. Using trigonometric identities, you can rewrite each answer in the same form.

71. $\frac{1}{3}$ **73.** 1 **75.** $2\pi\left(1 - \frac{\pi}{4}\right) \approx 1.348$
77. (a) $\frac{\pi^2}{2}$ (b) $(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$ **79–81.** Proofs
83. $-\frac{1}{15}\cos x(3\sin^4 x + 4\sin^2 x + 8) + C$
85. $-\frac{1}{48}(8\cos^3 x\sin^3 x + 6\cos^3 x\sin x - 3\cos x\sin x - 3x) + C$
87. (a) and (b) Proofs
89. (a) Proof (b) $a_1 = 2, a_2 = -1, a_3 = \frac{2}{3}$

Section 8.4 *(page 547)*

1. (a) $x = 3\tan\theta$ (b) $x = 2\sin\theta$
(c) $x = 5\sin\theta$ (d) $x = 5\sec\theta$
3. $\frac{x}{16\sqrt{16-x^2}} + C$
5. $4\ln\left|\frac{4-\sqrt{16-x^2}}{x}\right| + \sqrt{16-x^2} + C$
7. $\ln\left|x + \sqrt{x^2-25}\right| + C$
9. $\frac{1}{15}(x^2-25)^{3/2}(3x^2+50) + C$
11. $\frac{(4+x^2)^{3/2}}{6} + C$ **13.** $\frac{1}{4}\left(\arctan\frac{x}{2} + \frac{2x}{4+x^2}\right) + C$
15. $\frac{1}{2}x\sqrt{49-16x^2} + \frac{49}{8}\arcsin\frac{4x}{7} + C$
17. $\frac{1}{2\sqrt{5}}\left(\sqrt{5}x\sqrt{36-5x^2} + 36\arcsin\frac{\sqrt{5}x}{6}\right) + C$
19. $4\arcsin\frac{x}{2} + x\sqrt{4-x^2} + C$ **21.** $-\frac{(1-x^2)^{3/2}}{3x^3} + C$
23. $-\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C$ **25.** $-\frac{x}{\sqrt{x^2+3}} + C$
27. $\frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C$
29. $\frac{1}{4}\left(\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right) + C$
31. $x\,\mathrm{arcsec}\,2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right| + C$
33. $2\arcsin\frac{x-2}{2} - \sqrt{4x-x^2} + C$
35. $\sqrt{x^2+6x+12} - 3\ln\left|\sqrt{x^2+6x+12} + (x+3)\right| + C$

37. (a) and (b) $\sqrt{3} - \frac{\pi}{3} \approx 0.685$

39. (a) and (b) $9(2 - \sqrt{2}) \approx 5.272$

41. (a) and (b) $-\frac{9}{2}\ln\left(\frac{2\sqrt{7}}{3} - \frac{4\sqrt{3}}{3} - \frac{\sqrt{21}}{3} + \frac{8}{3}\right) + 9\sqrt{3} - 2\sqrt{7} \approx 12.644$

43. Substitution: $u = x^2 + 1$, $du = 2x\,dx$

45. (a) $-\sqrt{1 - x^2} + C$; The answers are equivalent.

(b) $x - 3\arctan\frac{x}{3} + C$; The answers are equivalent.

47. True **49.** False. $\int_0^{\sqrt{3}} \frac{dx}{(1 + x^2)^{3/2}} = \int_0^{\pi/3} \cos\theta\, d\theta$

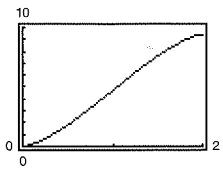
51. πab

53. $\ln\frac{5(\sqrt{2} + 1)}{\sqrt{26} + 1} + \sqrt{26} - \sqrt{2} \approx 4.367$ **55.** $6\pi^2$

57. $(0, 0.422)$

59. (a) $V = \frac{3\pi}{2} + 3\arcsin(d - 1) + 3(d - 1)\sqrt{2d - d^2}$

(b)



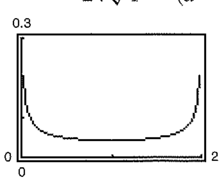
(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

$$y = \frac{3\pi}{4},\ y = \frac{3\pi}{2},\ y = \frac{9\pi}{4}$$

intersect the curve at $d = 0.596, 1.0, 1.404$. The dipstick would have these markings on it.

(d) $d'(t) = \frac{1}{24\sqrt{1 - (d - 1)^2}}$

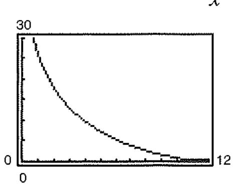
(e)



The minimum occurs at $d = 1$, which is the widest part of the tank.

61. (a) Proof

(b) $y = -12\ln\frac{12 - \sqrt{144 - x^2}}{x} - \sqrt{144 - x^2}$



(c) Vertical asymptote: $x = 0$ (d) About 5.2 m

63. (a) 187.2π lb (b) $62.4\pi d$ lb **65.** Proof

67. $12 + \frac{9\pi}{2} - 25\arcsin\frac{3}{5} \approx 10.050$

69. Putnum Problem A5, 2005

Section 8.5 *(page 557)*

1. (a) $\frac{A}{x} + \frac{B}{x - 8}$ (b) $\frac{A}{x - 3} + \frac{B}{(x - 3)^2} + \frac{C}{(x - 3)^3}$

(c) $\frac{A}{x} + \frac{Bx + C}{x^2 + 10}$ (d) $\frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$

3. $\frac{1}{6}\ln\left|\frac{x - 3}{x + 3}\right| + C$ **5.** $\ln\left|\frac{x - 1}{x + 4}\right| + C$

7. $5\ln|x - 2| - \ln|x + 2| - 3\ln|x| + C$

9. $x^2 + \frac{3}{2}\ln|x - 4| - \frac{1}{2}\ln|x + 2| + C$

11. $\frac{1}{x} + \ln|x^4 + x^3| + C$

13. $\frac{9}{x + 1} + 2\ln|x| - \ln|x + 1| + C$

15. $9\ln|x| - \frac{32}{7}\ln(7x^2 + 1) + C$

17. $\frac{1}{6}\left(\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2}\arctan\frac{x}{\sqrt{2}}\right) + C$

19. $\ln|x + 1| + \sqrt{2}\arctan\frac{x - 1}{\sqrt{2}} + C$ **21.** $\ln 3$

23. $\frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2 \approx 0.557$ **25.** $\ln|1 + \sec x| + C$

27. $\ln\left|\frac{\tan x + 2}{\tan x + 3}\right| + C$ **29.** $\frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C$

31. $2\sqrt{x} + 2\ln\left|\frac{\sqrt{x} - 2}{\sqrt{x} + 2}\right| + C$ **33–35.** Proofs

37. Substitution: $u = x^2 + 2x - 8$

39. Trigonometric substitution (tan) or inverse tangent rule

41. $12\ln\frac{9}{7}$ **43.** $\frac{5}{2}\ln 5$ **45.** 4.90, or \$490,000

47. (a) $V = 2\pi\left(\arctan 3 - \frac{3}{10}\right) \approx 5.963$

(b) $(\bar{x}, \bar{y}) \approx (1.521, 0.412)$

49. $x = \frac{n[e^{(n+1)kt} - 1]}{n + e^{(n+1)kt}}$ **51.** $\frac{\pi}{8}$

53. Putnam Problem B4, 1992

Section 8.6 *(page 564)*

1. No. The integral can be easily evaluated using basic integration rules.

	Trapezoidal	Simpson's	Exact
3.	2.7500	2.6667	2.6667
5.	0.6970	0.6933	0.6931
7.	20.2222	20.0000	20.0000
9.	12.6640	12.6667	12.6667
11.	0.3352	0.3334	0.3333
13.	0.5706	0.5930	0.5940

	Trapezoidal	Simpson's	Graphing Utility
15.	3.2833	3.2396	3.2413
17.	0.7828	0.7854	0.7854
19.	102.5553	93.3752	92.7437
21.	0.5495	0.5483	0.5493
23.	0.1940	0.1860	0.1858

25. (a) $\frac{1}{12}$ (b) 0 **27.** (a) $\frac{1}{4}$ (b) $\frac{1}{12}$

29. (a) $n = 366$ (b) $n = 26$

31. (a) $n = 77$ (b) $n = 8$

33. (a) $n = 643$ (b) $n = 48$

35. Trapezoidal Rule: 24.5
Simpson's Rule: 25.67

37. 0.701 **39.** $T_n = \frac{1}{2}(L_n + R_n)$ **41.** 89,250 m^2

43. 10,233.58 ft-lb **45.** 2.477 **47.** Proof

Section 8.7 *(page 570)*

1. Formula 40 **3.** $-\dfrac{1}{2}x(10 - x) + 25 \ln|5 + x| + C$

5. $-\dfrac{\sqrt{1 - x^2}}{x} + C$

7. $\frac{1}{24}(3x + \sin 3x \cos 3x + 2\cos^3 3x \sin 3x) + C$

9. $-2(\cot\sqrt{x} + \csc\sqrt{x}) + C$ **11.** $x - \frac{1}{2}\ln(1 + e^{2x}) + C$

13. $\dfrac{x^7}{49}(7 \ln x - 1) + C$ **15.** (a) and (b) $x\left(\ln\dfrac{x}{3} - 1\right) + C$

17. (a) and (b) $\ln\left|\dfrac{x - 1}{x}\right| + \dfrac{1}{x} + C$

19. $\frac{1}{2}\left[(x^2 + 1)\operatorname{arccsc}(x^2 + 1) + \ln\left(x^2 + 1 + \sqrt{x^4 + 2x^2}\right)\right] + C$

21. $\dfrac{\sqrt{x^4 - 1}}{x^2} + C$ **23.** $\dfrac{1}{36}\left(\dfrac{7}{7 - 6x} + \ln|7 - 6x|\right) + C$

25. $e^x \arccos(e^x) - \sqrt{1 - e^{2x}} + C$

27. $\frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$

29. $\dfrac{\sqrt{2}}{2}\arctan\dfrac{1 + \sin\theta}{\sqrt{2}} + C$ **31.** $-\dfrac{\sqrt{2 + 9x^2}}{2x} + C$

33. $\frac{1}{4}\left(2\ln|x| - 3\ln|3 + 2\ln|x||\right) + C$

35. $\dfrac{3x - 10}{2(x^2 - 6x + 10)} + \dfrac{3}{2}\arctan(x - 3) + C$

37. $\frac{1}{2}\ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$

39. $\dfrac{2}{1 + e^x} - \dfrac{1}{2(1 + e^x)^2} + \ln(1 + e^x) + C$

41. $\frac{2}{3}(2 - \sqrt{2}) \approx 0.3905$ **43.** $\frac{32}{5}\ln 2 - \frac{31}{25} \approx 3.1961$

45. $\dfrac{\pi}{2}$ **47.** $\dfrac{\pi^3}{8} - 3\pi + 6 \approx 0.4510$ **49–53.** Proofs

55. $\dfrac{1}{\sqrt{5}}\ln\left|\dfrac{2\tan(\theta/2) - 3 - \sqrt{5}}{2\tan(\theta/2) - 3 + \sqrt{5}}\right| + C$ **57.** $\ln 2$

59. $\frac{1}{2}\ln(3 - 2\cos\theta) + C$ **61.** $-2\cos\sqrt{\theta} + C$

63. $4\sqrt{3}$

65. (a) $\displaystyle\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

$\displaystyle\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

$\displaystyle\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

(b) $\displaystyle\int x^n \ln x\,dx = \frac{x^{n+1}}{n + 1}\ln x - \frac{x^{n+1}}{(n + 1)^2} + C$

(c) Proof

67. 1919.145 ft-lb **69.** About 401.4 **71.** $32\pi^2$

73. Putnam Problem A3, 1980

Section 8.8 *(page 579)*

1. One or both of the limits of integration are infinite, or the function has a finite number of infinite discontinuities on the interval you are considering.

3. To evaluate the improper integral $\int_a^\infty f(x)\,dx$, find the limit as $b \to \infty$ when f is continuous on $[a, \infty)$ or find the limit as $a \to -\infty$ when f is continuous on $(-\infty, b]$.

5. Improper; $0 \le \frac{3}{5} \le 1$

7. Not improper; continuous on $[0, 1]$

9. Not improper; continuous on $[0, 2]$

11. Improper; infinite limits of integration

13. Infinite discontinuity at $x = 0$; 4

15. Infinite discontinuity at $x = 1$; diverges

17. $\frac{1}{8}$ **19.** Diverges **21.** Diverges **23.** 2

25. $\dfrac{1}{2(\ln 4)^2}$ **27.** π **29.** $\dfrac{\pi}{4}$ **31.** Diverges

33. Diverges **35.** 0 **37.** $-\dfrac{1}{4}$ **39.** Diverges

41. $\dfrac{\pi}{3}$ **43.** $\ln 3$ **45.** $\dfrac{\pi}{6}$ **47.** $\dfrac{2\pi\sqrt{6}}{3}$ **49.** $p > 1$

51. Proof **53.** Converges **55.** Converges

57. Converges **59.** Converges

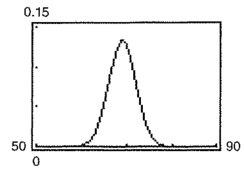
61. The improper integral diverges. **63.** $\frac{7}{8}$ **65.** π

67. (a) 1 (b) $\dfrac{\pi}{2}$ (c) 2π **69.** 2π

71. (a) $W = 20{,}000$ mile-tons (b) 4000 mi

73. (a) Proof (b) 48.66%

75. (a)



(b) About 0.1587 (c) 0.1587; same by symmetry

77. (a) \$807,992.41 (b) \$887,995.15 (c) \$1,116,666.67

79. $P = \dfrac{2\pi NI\left(\sqrt{r^2 + c^2} - c\right)}{kr\sqrt{r^2 + c^2}}$

81. False. Let $f(x) = \dfrac{1}{x + 1}$. **83.** True **85.** True

87. (a) and (b) Proofs

(c) The definition of the improper integral $\displaystyle\int_{-\infty}^{\infty}$ is not $\displaystyle\lim_{a\to\infty}\int_{-a}^{a}$ but rather that if you rewrite the integral that diverges, you can find that the integral converges.

89. Proof

91. $\dfrac{1}{s}, s > 0$ **93.** $\dfrac{2}{s^3}, s > 0$ **95.** $\dfrac{s}{s^2 + a^2}, s > 0$

97. $\dfrac{s}{s^2 - a^2}, s > |a|$

99. (a) $\Gamma(1) = 1, \Gamma(2) = 1, \Gamma(3) = 2$ (b) Proof
(c) $\Gamma(n) = (n - 1)!$

101. $c = 1$; $\ln 2$ **103.** $8\pi\left[\dfrac{(\ln 2)^2}{3} - \dfrac{\ln 4}{9} + \dfrac{2}{27}\right] \approx 2.01545$

105. $\displaystyle\int_0^1 2\sin u^2\,du$; 0.6278 **107.** Proof

Review Exercises for Chapter 8 *(page 583)*

1. $\dfrac{2}{9}(x^3 - 27)^{3/2} + C$ **3.** $-4\cot\dfrac{x + 8}{4} + C$

5. $\dfrac{1}{2} + \ln 2 \approx 1.1931$ **7.** $100 \arcsin\dfrac{x}{10} + C$

9. $-xe^{1-x} - e^{1-x} + C$

11. $\frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$

13. $x\tan x + \ln|\cos x| + C$

15. $\frac{1}{16}\left[(8x^2 - 1)\arcsin 2x + 2x\sqrt{1 - 4x^2}\right] + C$

17. $-\dfrac{\cos^5 x}{5} + C$ **19.** $\dfrac{\sin(\pi x - 1)[\cos^2(\pi x - 1) + 2]}{3\pi} + C$

21. $\dfrac{2}{3}\left(\tan^3 \dfrac{x}{2} + 3 \tan \dfrac{x}{2}\right) + C$

23. $\dfrac{\tan^3 x^2}{6} - \dfrac{\tan x^2}{2} + \dfrac{x^2}{2} + C$ **25.** $\tan\theta + \sec\theta + C$

27. $\dfrac{3\pi}{16} + \dfrac{1}{2} \approx 1.0890$ **29.** $\dfrac{3\sqrt{4 - x^2}}{x} + C$

31. $\frac{1}{3}(x^2 + 4)^{1/2}(x^2 - 8) + C$ **33.** $256 - 62\sqrt{17} \approx 0.3675$

35. (a), (b), and (c) $\frac{1}{3}\sqrt{4 + x^2}(x^2 - 8) + C$

37. $2 \ln|x + 2| - \ln|x - 3| + C$

39. $\frac{1}{4}[6 \ln|x - 1| - \ln(x^2 + 1) + 6 \arctan x] + C$

41. $x + \dfrac{1}{1 - x} + 2 \ln|x - 1| + C$

43. $-\ln|e^x + 1| + \frac{1}{2}\ln|e^x + 3| + \frac{1}{2}\ln|e^x - 1| + C$

	Trapezoidal	Simpson's	Graphing Utility
45.	0.2848	0.2838	0.2838
47.	0.6366	0.6847	0.7041

49. $\dfrac{1}{25}\left(\dfrac{4}{4 + 5x} + \ln|4 + 5x|\right) + C$ **51.** $1 - \dfrac{\sqrt{2}}{2}$

53. $\dfrac{1}{2}\ln|x^2 + 4x + 8| - \arctan\dfrac{x + 2}{2} + C$

55. $\dfrac{\ln|\tan \pi x|}{\pi} + C$ **57.** $\frac{1}{8}(\sin 2\theta - 2\theta \cos 2\theta) + C$

59. $\frac{4}{3}(x^{3/4} - 3x^{1/4} + 3 \arctan x^{1/4}) + C$

61. $2\sqrt{1 - \cos x} + C$ **63.** $\sin x \ln(\sin x) - \sin x + C$

65. $\dfrac{5}{2}\ln\left|\dfrac{x - 5}{x + 5}\right| + C$

67. $y = x \ln|x^2 + x| - 2x + \ln|x + 1| + C$ **69.** $\frac{1}{5}$

71. $\frac{1}{2}(\ln 4)^2 \approx 0.961$

73. $\pi^2 - 4 \sin 2 - 2 \cos 2 - 6 \approx 1.0647$ **75.** $\dfrac{\sqrt{27}}{5} \approx 1.0392$

77. $(\bar{x}, \bar{y}) = \left(0, \dfrac{4}{3\pi}\right)$ **79.** $\dfrac{32}{3}$ **81.** Diverges **83.** 1

85. $\dfrac{\pi}{4}$ **87.** (a) \$6,321,205.59 (b) \$10,000,000

89. (a) 0.4581 (b) 0.0135

P.S. Problem Solving *(page 585)*

1. (a) $\frac{4}{3}, \frac{16}{15}$ (b) Proof

3. (a) $R(n), I, T(n), L(n)$

(b) $S(4) = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx 5.42$

5. $\dfrac{\pi\sqrt{3}}{9} \approx 0.6046$ **7.** $(\bar{x}, \bar{y}) = \left(0, \dfrac{\sqrt{2}}{4}\right)$

9. (a) Proof (b) $x \arcsin x + \sqrt{1 - x^2} + C$ (c) 1

11. Proof **13.** (a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{4}$

15. $s(t) = -16t^2 + 12{,}000t\left(1 + \ln\dfrac{50{,}000}{50{,}000 - 400t}\right) + 1{,}500{,}000 \ln\dfrac{50{,}000 - 400t}{50{,}000}$; 557,168.626 ft

17. Proof **19.** (a) $\dfrac{2}{\pi}$ (b) 0

Chapter 9

Section 9.1 *(page 596)*

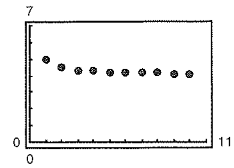
1. You need to be given one or more of the first few terms of a sequence, and then all other terms are defined using previous terms.

3. g; Factorial functions grow faster than exponential functions.

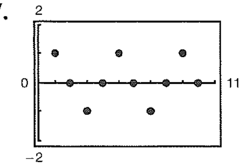
5. 3, 9, 27, 81, 243 **7.** 1, 0, −1, 0, 1 **9.** $2, -1, \frac{2}{3}, -\frac{1}{2}, \frac{2}{5}$

11. 3, 4, 6, 10, 18 **13.** c **14.** a **15.** d **16.** b

17. $n^2 + n$ **19.** $n(n - 1)(n - 2)$ **21.** 1 **23.** 2

25. 

Converges to 4

27. 

Diverges

29. Converges to 0 **31.** Diverges **33.** Converges to $\frac{3}{4}$

35. Converges to 0 **37.** Diverges **39.** Converges to 0

41. Converges to 1 **43.** Converges to 0

45. $6n - 4$; diverges **47.** $n^2 - 3$; diverges

49. $\dfrac{n + 1}{n + 2}$; converges **51.** $\dfrac{n + 1}{n}$; converges

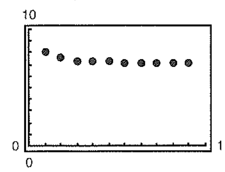
53. Monotonic, bounded **55.** Not monotonic, bounded

57. Monotonic, bounded **59.** Not monotonic, bounded

61. (a) $\left|7 + \dfrac{1}{n}\right| \geq 7 \Rightarrow$ bounded

$a_n > a_{n+1} \Rightarrow$ monotonic

So, $\{a_n\}$ converges.

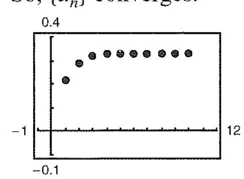
(b) 

Limit = 7

63. (a) $\left|\dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)\right| < \dfrac{1}{3} \Rightarrow$ bounded

$a_n < a_{n+1} \Rightarrow$ monotonic

So, $\{a_n\}$ converges.

(b) 

Limit = $\frac{1}{3}$

65. (a) No. $\lim_{n\to\infty} A_n$ does not exist.

(b)

n	1	2	3	4
A_n	\$10,045.83	\$10,091.88	\$10,138.13	\$10,184.60

n	5	6	7
A_n	\$10,231.28	\$10,278.17	\$10,325.28

n	8	9	10
A_n	\$10,372.60	\$10,420.14	\$10,467.90

67. \$26,125.00, \$27,300.63, \$28,529.15, \$29,812.97, \$31,154.55

69. Answers will vary. Sample answers:

(a) $a_n = 10 - \dfrac{1}{n}$ (b) $a_n = \dfrac{3n}{4n + 1}$

71. The sequence $\{a_n\}$ could converge or diverge. If $\{a_n\}$ is increasing, then it converges to a limit less than or equal to 1. If $\{a_n\}$ is decreasing, then it could converge (example: $a_n = 1/n$) or diverge (example: $a_n = -n$).

73. 1, 1.4142, 1.4422, 1.4142, 1.3797, 1.3480; Converges to 1

75. Proof

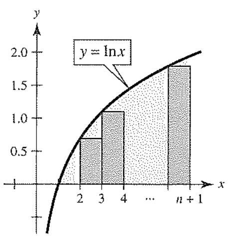
77. False. The sequence could also alternate between two values.

79. True

81. (a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144
(b) 1, 2, 1.5, 1.6667, 1.6, 1.6250, 1.6154, 1.6190, 1.6176, 1.6182 (c) Proof
(d) $\rho = \dfrac{1 + \sqrt{5}}{2} \approx 1.6180$

83. (a) 1.4142, 1.8478, 1.9616, 1.9904, 1.9976
(b) $a_n = \sqrt{2 + a_{n-1}}$ (c) $\lim_{n\to\infty} a_n = 2$

85. (a) Proof
(b)



(c) and (d) Proofs
(e) $\dfrac{\sqrt[20]{20!}}{20} \approx 0.4152$
$\dfrac{\sqrt[50]{50!}}{50} \approx 0.3897$
$\dfrac{\sqrt[100]{100!}}{100} \approx 0.3799$

87–89. Proofs **91.** Putnam Problem A1, 1990

Section 9.2 *(page 605)*

1. $\lim_{n\to\infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.
$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

3. You cannot make a conclusion. The series may either converge or diverge.

5. 1, 1.25, 1.361, 1.424, 1.464

7. 3, −1.5, 5.25, −4.875, 10.3125

9. 3, 4.5, 5.25, 5.625, 5.8125

11. Geometric series: $r = \frac{5}{2} > 1$ **13.** $\lim_{n\to\infty} a_n = 1 \neq 0$

15. $\lim_{n\to\infty} a_n = 1 \neq 0$ **17.** $\lim_{n\to\infty} a_n = \frac{1}{4} \neq 0$

19. Geometric series: $r = \frac{5}{6} < 1$

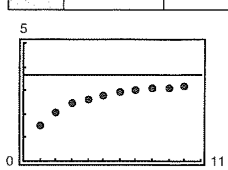
21. Geometric series: $r = 0.9 < 1$

23. Telescoping series: $a_n = 1/n - 1/(n + 1)$; Converges to 1.

25. (a) $\frac{11}{3}$
(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

(c)

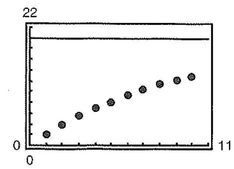


(d) The terms of the series decrease in magnitude relatively slowly, and the sequence of partial sums approaches the sum of the series relatively slowly.

27. (a) 20
(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

(c)



(d) The terms of the series decrease in magnitude relatively slowly, and the sequence of partial sums approaches the sum of the series relatively slowly.

29. 15 **31.** 3 **33.** 32 **35.** $\dfrac{1}{2}$ **37.** $\dfrac{\sin 1}{1 - \sin 1}$

39. (a) $\sum_{n=0}^{\infty} \dfrac{4}{10}(0.1)^n$ (b) $\dfrac{4}{9}$ **41.** (a) $\sum_{n=0}^{\infty} \dfrac{12}{100}\left(\dfrac{1}{100}\right)^n$ (b) $\dfrac{4}{33}$

43. (a) $\sum_{n=0}^{\infty} \dfrac{3}{40}(0.01)^n$ (b) $\dfrac{5}{66}$

45. Diverges **47.** Diverges **49.** Converges

51. Diverges **53.** Diverges **55.** Diverges

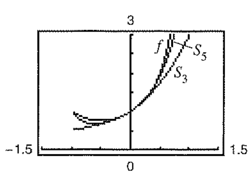
57. Diverges

59. Yes. If you remove a finite number of terms, the sum of the sequence of partial sums still diverges.

61. $|x| < \dfrac{1}{3}$; $\dfrac{3x}{1 - 3x}$ **63.** $0 < x < 2$; $\dfrac{x - 1}{2 - x}$

65. $-1 < x < 1$; $\dfrac{1}{1 + x}$

67. (a) x (b) $f(x) = \dfrac{1}{1 - x}$, $|x| < 1$
(c)



Answers will vary.

69. The required terms for the two series are $n = 100$ and $n = 5$, respectively. The second series converges at a higher rate.

71. $160{,}000(1 - 0.95^n)$ units

73. $\sum_{i=0}^{\infty} 200(0.75)^i$; Sum = \$800 million **75.** 152.42 ft

77. $\dfrac{1}{8}$; $\sum_{n=0}^{\infty} \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n = \dfrac{1/2}{1 - 1/2} = 1$

79. (a) $-1 + \sum_{n=0}^{\infty} \left(\dfrac{1}{2}\right)^n = -1 + \dfrac{a}{1 - r} = -1 + \dfrac{1}{1 - 1/2} = 1$
(b) No (c) 2

81. (a) 126 in.^{2} (b) 128 in.^{2}

83. The \$2,000,000 sweepstakes has a present value of \$1,146,992.12. After accruing interest over the 20-year period, it attains its full value.

85. (a) \$5,368,709.11 (b) \$10,737,418.23 (c) \$21,474,836.47

87. (a) \$14,739.84 (b) \$14,742.45

89. (a) \$518,136.56 (b) \$518,168.67

91. False. $\lim_{n\to\infty} \dfrac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

93. False. $\sum_{n=1}^{\infty} ar^n = \dfrac{a}{1-r} - a$; The formula requires that the geometric series begins with $n = 0$.

95. True

97. Answers will vary. Sample answer: $\sum_{n=0}^{\infty} 1, \; \sum_{n=0}^{\infty} (-1)$

99–101. Proofs **103.** Putnam Problem A2, 1984

Section 9.3 *(page 613)*

1. f must be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$.

3. Diverges **5.** Converges **7.** Converges

9. Diverges **11.** Diverges **13.** Converges

15. Converges **17.** Converges **19.** Diverges

21. Converges **23.** Diverges

25. $f(x)$ is not positive for $x \geq 1$.

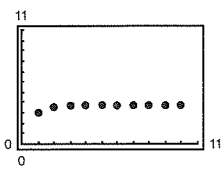
27. $f(x)$ is not always decreasing. **29.** Converges

31. Diverges **33.** Diverges **35.** Converges

37. Converges

39. (a)

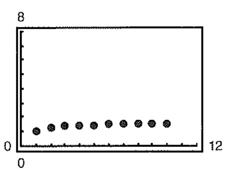
n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75



The partial sums approach the sum 3.75 very quickly.

(b)

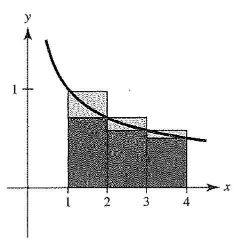
n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635



The partial sums approach the sum $\dfrac{\pi^2}{6} \approx 1.6449$ more slowly than the series in part (a).

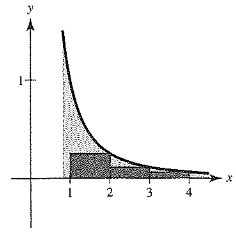
41. No. Because $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, $\sum_{n=10{,}000}^{\infty} \dfrac{1}{n}$ also diverges. The convergence or divergence of a series is not determined by the first finite number of terms of the series.

43. (a)



The area under the rectangles is greater than the area under the curve. Because $\int_1^{\infty} \dfrac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^{\infty} = \infty$ diverges, $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges.

(b)



The area under the rectangles is less than the area under the curve. Because $\int_1^{\infty} \dfrac{1}{x^2}\,dx = \left[-\dfrac{1}{x}\right]_1^{\infty} = 1$ converges, $\sum_{n=2}^{\infty} \dfrac{1}{n^2}$ converges $\left(\text{and so does } \sum_{n=1}^{\infty} \dfrac{1}{n^2}\right)$.

45. $p > 1$ **47.** $p > 1$ **49.** $p > 3$ **51.** Proof

53. $S_3 \approx 1.0748$, $R_3 \approx 0.0123$ **55.** $S_8 \approx 0.9597$, $R_8 \approx 0.1244$

57. $S_4 \approx 0.4049$, $R_4 \approx 5.6 \times 10^{-8}$

59. $N \geq 7$ **61.** $N \geq 16$

63. (a) $\sum_{n=2}^{\infty} \dfrac{1}{n^{1.1}}$ converges by the p-Series Test because $1.1 > 1$.

$\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges by the Integral Test because $\int_2^{\infty} \dfrac{1}{x \ln x}\,dx$ diverges.

(b) $\sum_{n=2}^{\infty} \dfrac{1}{n^{1.1}} = 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393 + \cdots$

$\sum_{n=2}^{\infty} \dfrac{1}{n \ln n} = 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930 + \cdots$

(c) $n \geq 3.431 \times 10^{15}$

65. (a) Let $f(x) = \dfrac{1}{x}$. f is positive, continuous, and decreasing on $[1, \infty)$.

$$S_n - 1 \leq \int_1^n \frac{1}{x}\,dx = \ln n$$

$$S_n \geq \int_1^{n+1} \frac{1}{x}\,dx \ln(n+1)$$

So, $\ln(n+1) \leq S_n \leq 1 + \ln n$.

(b) $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$

Also, $\ln(n+1) - \ln n > 0$ for $n \geq 1$. So, $0 \leq S_n - \ln n \leq 1$, and the sequence $\{a_n\}$ is bounded.

(c) $a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)]$

$$= \int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \geq 0$$

So, $a_n \geq a_{n+1}$.

(d) Because the sequence is bounded and monotonic, it converges to a limit, γ.

(e) 0.5822

67. (a) Diverges (b) Diverges

(c) $\sum_{n=2}^{\infty} x^{\ln n}$ converges for $x < \dfrac{1}{e}$.

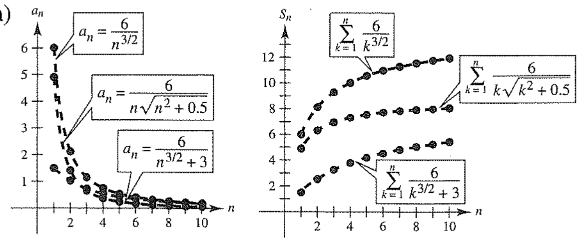
69. Diverges **71.** Converges **73.** Converges

75. Diverges **77.** Diverges **79.** Converges

Section 9.4 (page 620)

1. Yes. The test requires that $0 \le a_n \le b_n$ for all n greater than some integer N. The beginning terms do not affect the convergence or divergence of a series.

3. (a)



(b) $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$; Converges

(c) The magnitudes of the terms are less than the magnitudes of the terms of the p-series. Therefore, the series converges.

(d) The smaller the magnitudes of the terms, the smaller the magnitudes of the terms of the sequence of partial sums.

5. Diverges **7.** Diverges **9.** Diverges
11. Converges **13.** Converges **15.** Converges
17. Diverges **19.** Diverges **21.** Converges
23. Converges **25.** Diverges
27. Diverges; p-Series Test
29. Converges; Direct Comparison Test with $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$
31. Diverges; nth-Term Test **33.** Converges; Integral Test
35. $\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} na_n$; $\lim_{n\to\infty} na_n \ne 0$ but is finite.
The series diverges by the Limit Comparison Test.
37. Diverges **39.** Converges
41. $\lim_{n\to\infty} n\left(\frac{n^3}{5n^4+3}\right) = \frac{1}{5} \ne 0$
43. Diverges **45.** Converges
47. Convergence or divergence is dependent on the form of the general term for the series and not necessarily on the magnitudes of the terms.
49. (a) Proof
(b)

n	5	10	20	50	100
S_n	1.1839	1.2087	1.2212	1.2287	1.2312

(c) 0.1226 (d) 0.0277

51. Proof **53.** False. Let $a_n = \frac{1}{n^3}$ and $b_n = \frac{1}{n^2}$.
55. True **57.** True **59.** Proof **61.** $\sum_{n=1}^{\infty} \frac{1}{n^2}, \sum_{n=1}^{\infty} \frac{1}{n^3}$
63–69. Proofs
71. Putnam Problem 1, afternoon session, 1953

Section 9.5 (page 629)

1. The series diverges because of the nth-Term Test for Divergence.

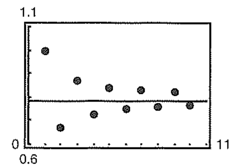
3. Σa_n is absolutely convergent if $\Sigma|a_n|$ converges. Σa_n is conditionally convergent if $\Sigma|a_n|$ diverges, but Σa_n converges.

5. (a)

n	1	2	3	4	5
S_n	1.0000	0.6667	0.8667	0.7238	0.8349

n	6	7	8	9	10
S_n	0.7440	0.8209	0.7543	0.8131	0.7605

(b)



(c) The points alternate sides of the horizontal line $y = \frac{\pi}{4}$ that represents the sum of the series. The distances between the successive points and the line decrease.

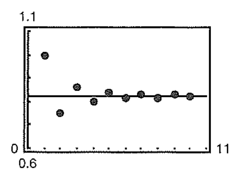
(d) The distance in part (c) is always less than the magnitude of the next term of the series.

7. (a)

n	1	2	3	4	5
S_n	1.0000	0.7500	0.8611	0.7986	0.8386

n	6	7	8	9	10
S_n	0.8108	0.8312	0.8156	0.8280	0.8180

(b)



(c) The points alternate sides of the horizontal line $y = \frac{\pi^2}{12}$ that represents the sum of the series. The distances between the successive points and the line decrease.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

9. Converges **11.** Converges **13.** Diverges
15. Diverges **17.** Converges **19.** Diverges
21. Diverges **23.** Converges **25.** Converges
27. Converges **29.** Converges **31.** $1.8264 \le S \le 1.8403$
33. $1.7938 \le S \le 1.8054$ **35.** 10 **37.** 7
39. 7 terms (Note that the sum begins with $n = 0$.)
41. Converges absolutely **43.** Converges absolutely
45. Converges conditionally **47.** Diverges
49. Converges conditionally **51.** Converges absolutely
53. Converges absolutely **55.** Converges conditionally
57. Converges absolutely
59. Overestimate; The next term is negative.

61. (a) False. For example, let $a_n = \frac{(-1)^n}{n}$.

Then $\sum a_n = \sum \frac{(-1)^n}{n}$ converges

and $\sum (-a_n) = \sum \frac{(-1)^{n+1}}{n}$ converges.

But, $\sum |a_n| = \sum \frac{1}{n}$ diverges.

(b) True. For if $\sum |a_n|$ converged, then so would $\sum a_n$ by Theorem 9.16.

63. $p > 0$

65. Proof; The converse is false. For example: Let $a_n = \frac{1}{n}$.

67. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

69. (a) No; $a_{n+1} \le a_n$ is not satisfied for all n. For example, $\frac{1}{9} < \frac{1}{8}$.
(b) Yes; 0.5

71. Diverges; p-Series Test **73.** Diverges; nth-Term Test

75. Diverges; Geometric Series Test

77. Converges; Integral Test

79. Converges; Alternating Series Test

81. You cannot arbitrarily change 0 to $1 - 1$.

Section 9.6 *(page 637)*

1. Converges **3.** Diverges **5.** Inconclusive **7.** Proof

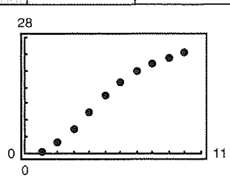
9. d **10.** c **11.** f **12.** b **13.** a **14.** e

15. (a) Proof

(b)

n	5	10	15	20	25
S_n	13.7813	24.2363	25.8468	25.9897	25.9994

(c)



(d) 26

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of partial sums approaches the sum of the series.

17. Converges **19.** Diverges **21.** Diverges

23. Diverges **25.** Converges **27.** Converges

29. Diverges **31.** Converges **33.** Converges

35. Diverges **37.** Converges **39.** Converges

41. Diverges **43.** Converges **45.** Diverges

47. Converges **49.** Converges **51.** Converges

53. Converges; Alternating Series Test

55. Converges; p-Series Test **57.** Diverges; nth-Term Test

59. Diverges; Geometric Series Test

61. Converges; Limit Comparison Test with $b_n = \frac{1}{2^n}$

63. Converges; Direct Comparison Test with $b_n = \frac{1}{3^n}$

65. Diverges; Ratio Test **67.** Converges; Ratio Test

69. Converges; Ratio Test **71.** a and c **73.** a and b

75. $\sum_{n=0}^{\infty} \frac{n+1}{7^{n+1}}$ **77.** Diverges; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

79. Converges; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ **81.** Diverges; $\lim a_n \ne 0$

83. Converges **85.** Converges **87.** $(-3, 3)$

89. $(-2, 0]$ **91.** $x = 0$ **93.** The test is inconclusive.

95. No; The series $\sum_{n=1}^{\infty} \frac{1}{n + 10{,}000}$ diverges.

97–103. Proofs

105. (a) Diverges (b) Converges (c) Converges
(d) Converges for all integers $x \ge 2$

107. Putnam Problem 7, morning session, 1951

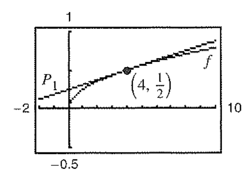
Section 9.7 *(page 648)*

1. The graphs of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$, and the slope of the graph of P is the same as the slope of the graph of f at the point $(c, f(c))$. If P is of degree n, then the first n derivatives of f and P agree at c. This allows the graph of P to resemble the graph of f near the point $(c, f(c))$.

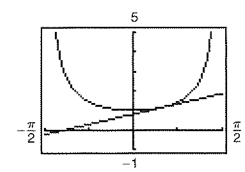
3. The accuracy is represented by the remainder of the Taylor polynomial. The remainder is $R_n(x) = \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$.

5. d **6.** c **7.** a **8.** b

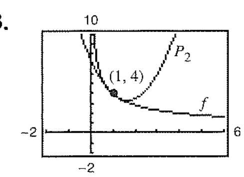
9. $P_1 = \frac{1}{16}x + \frac{1}{4}$



11. $P_1 = \frac{2\sqrt{3}}{3} + \frac{2}{3}\left(x - \frac{\pi}{6}\right)$



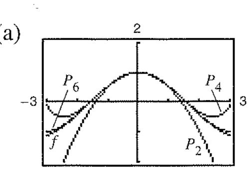
13.



x	0	0.8	0.9	1	1.1
$f(x)$	Error	4.4721	4.2164	4.0000	3.8139
$P_2(x)$	7.5000	4.4600	4.2150	4.0000	3.8150

x	1.2	2
$f(x)$	3.6515	2.8284
$P_2(x)$	3.6600	3.5000

15. (a)



(b) $f^{(2)}(0) = -1$ $\quad P_2^{(2)}(0) = -1$
$f^{(4)}(0) = 1$ $\quad P_4^{(4)}(0) = 1$
$f^{(6)}(0) = -1$ $\quad P_6^{(6)}(0) = -1$

(c) $f^{(n)}(0) = P_n^{(n)}(0)$

17. $1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4$ **19.** $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

21. $x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$ **23.** $1 + x + x^2 + x^3 + x^4 + x^5$

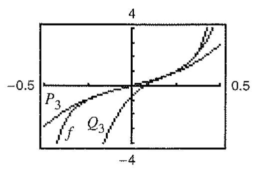
25. $1 + \frac{1}{2}x^2$ **27.** $2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$

29. $2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

31. $\ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4$

33. (a) $P_3(x) = \pi x + \dfrac{\pi^3}{3}x^3$

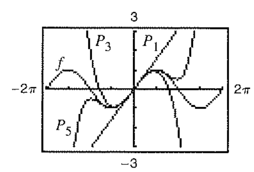
(b) $Q_3(x) = 1 + 2\pi\left(x - \frac{1}{4}\right) + 2\pi^2\left(x - \frac{1}{4}\right)^2 + \frac{8\pi^3}{3}\left(x - \frac{1}{4}\right)^3$



35. (a)

x	0	0.25	0.50	0.75	1.00
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0	0.25	0.50	0.75	1.00
$P_3(x)$	0	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0	0.2474	0.4794	0.6817	0.8417

(b)



(c) As the distance increases, the polynomial approximation becomes less accurate.

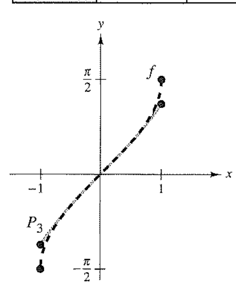
37. (a) $P_3(x) = x + \frac{1}{6}x^3$

(b)

x	-0.75	-0.50	-0.25	0	0.25
$f(x)$	-0.848	-0.524	-0.253	0	0.253
$P_3(x)$	-0.820	-0.521	-0.253	0	0.253

x	0.50	0.75
$f(x)$	0.524	0.848
$P_3(x)$	0.521	0.820

(c)



39. 2.7083 **41.** 0.227 **43.** 0.7419

45. $R_4 \le 2.03 \times 10^{-5}$; 0.000001

47. $R_5 \le 1.8 \times 10^{-8}$; 2.5×10^{-9}

49. $R_3 \le 7.82 \times 10^{-3}$; 0.00085 **51.** 3 **53.** 5 **55.** 2

57. $n = 9$; $\ln(1.5) \approx 0.4055$ **59.** $-0.3936 < x < 0$

61. $-0.9467 < x < 0.9467$

63. The tangent line to a function at a point is the first Taylor polynomial for the function at the point.

65. Substitute $2x$ into the polynomial for $f(x) = e^x$ to obtain the polynomial for $g(x) = e^{2x}$.

67. (a) $f(x) \approx P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

$g(x) \approx Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$

$Q_5(x) = xP_4(x)$

(b) $g(x) \approx P_6(x) = x^2 - \dfrac{x^4}{3!} + \dfrac{x^6}{5!}$

(c) $g(x) \approx P_4(x) = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!}$

69. (a) $Q_2(x) = -1 + \left(\dfrac{\pi^2}{32}\right)(x+2)^2$

(b) $R_2(x) = -1 + \left(\dfrac{\pi^2}{32}\right)(x-6)^2$

(c) No. Horizontal translations of the result in part (a) are possible only at $x = -2 + 8n$ (where n is an integer) because the period of f is 8.

71–73. Proofs

Section 9.8 *(page 658)*

1. A Maclaurin polynomial approximates a function, whereas a power series exactly represents a function. The Maclaurin polynomial has a finite number of terms and the power series has an infinite number of terms.

3. $R = 5$ **5.** 0 **7.** 2 **9.** $R = 1$ **11.** $R = \frac{1}{4}$

13. $R = \infty$ **15.** $(-4, 4)$ **17.** $(-1, 1]$ **19.** $(-\infty, \infty)$

21. $x = 0$ **23.** $(-6, 6)$ **25.** $(-5, 13]$ **27.** $(0, 2]$

29. $(0, 6)$ **31.** $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **33.** $(-\infty, \infty)$ **35.** $(-1, 1)$

37. $x = 3$ **39.** $R = c$ **41.** $(-k, k)$ **43.** $(-1, 1)$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ **47.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{(7n+6)!}$

49. (a) $(-3, 3)$ (b) $(-3, 3)$ (c) $(-3, 3)$ (d) $[-3, 3)$

51. (a) $(0, 2]$ (b) $(0, 2)$ (c) $(0, 2)$ (d) $[0, 2]$

53. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$; Answers will vary.

55. Answers will vary. Sample answer:

$\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for $-1 \le x < 1$. At $x = -1$, the convergence is conditional because $\sum \frac{1}{n}$ diverges.

$\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges for $-1 \le x \le 1$. At $x = \pm 1$, the convergence is absolute.

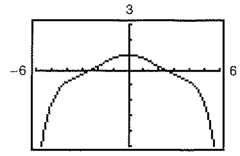
57. (a) For $f(x)$: $(-\infty, \infty)$; For $g(x)$: $(-\infty, \infty)$

(b) Proofs (c) $f(x) = \sin x$, $g(x) = \cos x$

59–63. Proofs

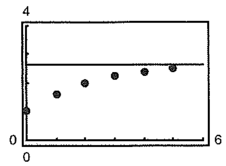
65. (a) and (b) Proofs

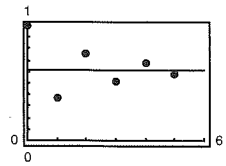
(c)



(d) 0.92

67. (a) $\frac{8}{3}$ (b) $\frac{8}{13}$



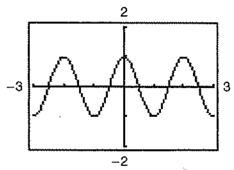


(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d)

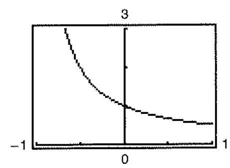
M	10	100	1000	10,000
N	5	14	24	35

69.



$f(x) = \cos \pi x$

71.



$f(x) = \dfrac{1}{1 + x}$

73. False. Let $a_n = \dfrac{(-1)^n}{n2^n}$. **75.** True **77.** Proof

79. (a) $(-1, 1)$ (b) $f(x) = \dfrac{c_0 + c_1x + c_2x^2}{1 - x^3}$

81. Proof

Section 9.9 *(page 666)*

1. You need to algebraically manipulate $\dfrac{b}{c - x}$ so that it resembles the form $\dfrac{a}{1 - r}$.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$ **5.** $\displaystyle\sum_{n=0}^{\infty} \frac{4}{3}\left(\frac{-x}{3}\right)^n$

7. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)\left(\frac{x-1}{5}\right)^n$ **9.** $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

$(-4, 6)$ $\left(-\frac{1}{3}, \frac{1}{3}\right)$

11. $\displaystyle -\frac{5}{9}\sum_{n=0}^{\infty}\left[\frac{2}{9}(x+3)\right]^n$ **13.** $\displaystyle -2\sum_{n=0}^{\infty}[5(x+1)]^n$

$\left(-\frac{15}{2}, \frac{3}{2}\right)$ $\left(-\frac{6}{5}, -\frac{4}{5}\right)$

15. $\displaystyle\sum_{n=0}^{\infty}\left[\frac{1}{(-3)^n} - 1\right]x^n$ **17.** $\displaystyle\sum_{n=0}^{\infty} x^n[1 + (-1)^n] = 2\sum_{n=0}^{\infty} x^{2n}$

$(-1, 1)$ $(-1, 1)$

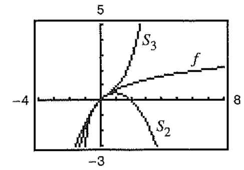
19. $\displaystyle 2\sum_{n=0}^{\infty} x^{2n}$ **21.** $\displaystyle\sum_{n=1}^{\infty} n(-1)^n x^{n-1}$ **23.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

$(-1, 1)$ $(-1, 1)$ $(-1, 1]$

25. $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ **27.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n (2x)^{2n}$

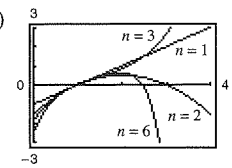
$(-1, 1)$ $\left(-\frac{1}{2}, \frac{1}{2}\right)$

29.



x	0.0	0.2	0.4	0.6	0.8	1.0
S_2	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.182	0.336	0.470	0.588	0.693
S_3	0.000	0.183	0.341	0.492	0.651	0.833

31. (a)



(b) $\ln x$, $0 < x \le 2$, $R = 1$
(c) -0.6931
(d) $\ln(0.5)$; The error is approximately 0.

33. 0.245 **35.** 0.125 **37.** $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$, $-1 < x < 1$

39. $\displaystyle\sum_{n=0}^{\infty} (2n+1)x^n$, $-1 < x < 1$

41. $E(n) = 2$; Yes. Because the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

43. Proof **45.** (a) Proof (b) 3.14

47. $\ln \frac{3}{2} \approx 0.4055$; See Exercise 23.

49. $\ln \frac{7}{5} \approx 0.3365$; See Exercise 47.

51. $\arctan \frac{1}{2} \approx 0.4636$; See Exercise 50.

53. The series in Exercise 50 converges to its sum at a lower rate because its terms approach 0 at a much lower rate.

55. The series converges on the interval $(-5, 3)$ and perhaps also at one or both endpoints.

57. $S_1 = 0.3183098862$, $\dfrac{1}{\pi} \approx 0.3183098862$

Section 9.10 *(page 677)*

1. The Taylor series converges to $f(x)$ if and only if $R_n(x) \to 0$ as $n \to \infty$.

3. Multiply and divide as you would polynomials.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$ **7.** $\displaystyle\frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!}\left(x - \frac{\pi}{4}\right)^n$

9. $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x-1)^n$ **11.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$ **15.** $\displaystyle 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots$

17–19. Proofs **21.** $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$

23. $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!}$

25. $\displaystyle 1 + \frac{1}{4}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-5)x^n}{4^n n!}$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$ **29.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ **31.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$

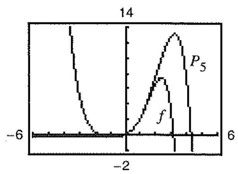
33. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{2n+1}$ **35.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$

37. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ **39.** $\displaystyle\frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n}}{(2n)!}\right]$

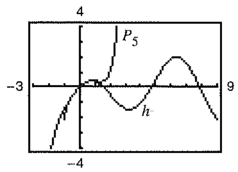
41. Proof **43.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

45. $\begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

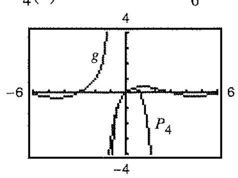
47. $P_5(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5$



49. $P_5(x) = x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{3}{40}x^5$



51. $P_4(x) = x - x^2 + \frac{5}{6}x^3 - \frac{5}{6}x^4$



53. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{(n+1)}x^{2n+3}}{(2n+3)(n+1)!}$ **55.** 0.6931 **57.** 7.3891

59. 0 **61.** 1 **63.** 0.8075 **65.** 0.9461 **67.** 0.4872

69. 0.2010 **71.** 0.7040 **73.** 0.3412

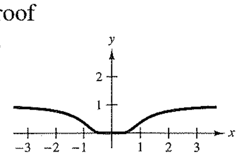
75. Square the series for $\cos x$, use a half-angle identity, or compute the coefficients using the definition.

First three terms: $1, \frac{x^2}{2}, \frac{x^4}{3}$

77. $f(x) = \dfrac{\sin(x+3)}{4}$; Answers will vary.

79. Proof

81. (a)



(b) Proof

(c) $\displaystyle\sum_{n=0}^{\infty} 0x^n = 0 \neq f(x)$; The series converges to f at $x = 0$ only.

83. Proof **85.** 20 **87.** -0.612864 **89.** $\displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$

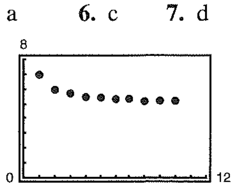
91. Proof **93.** Putnam Problem 4, morning session, 1962

Review Exercises for Chapter 9 *(page 680)*

1. 4, 34, 214, 1294, 7774 **3.** $-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, -\frac{1}{1024}$

5. a **6.** c **7.** d **8.** b

9.



Converges to 5

11. Converges to 0 **13.** Converges to 5 **15.** Diverges

17. Diverges **19.** $a_n = 5n - 2$; diverges

21. $a_n = \dfrac{1}{(n!+1)}$; converges **23.** Monotonic, bounded

25. (a)

n	1	2	3	4
A_n	\$8100.00	\$8201.25	\$8303.77	\$8407.56

n	5	6	7	8
A_n	\$8512.66	\$8619.07	\$8726.80	\$8835.89

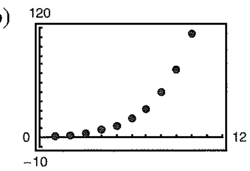
(b) \$13,148.96

27. 3, 4.5, 5.5, 6.25, 6.85

29. (a)

n	5	10	15	20	25
S_n	13.2	113.3	873.8	6648.5	50,500.3

(b)



31. $\frac{5}{3}$ **33.** $\frac{35}{3}$ **35.** (a) $\displaystyle\sum_{n=0}^{\infty} (0.09)(0.01)^n$ (b) $\frac{1}{11}$

37. Diverges **39.** Diverges

41. $120{,}000[1 - 0.92^n]$, $n > 0$ **43.** Diverges

45. Converges **47.** Diverges **49.** Diverges

51. Converges **53.** Diverges **55.** Converges

57. Converges **59.** Diverges **61.** 10 **63.** Diverges

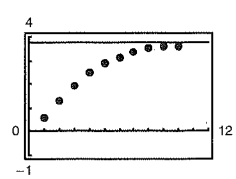
65. Diverges **67.** Converges

69. (a) Proof

(b)

n	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499

(c)



(d) 3.75

71. Converges; p-Series Test **73.** Diverges; nth-Term Test

75. Diverges; Limit Comparison Test

77. $P_3(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3$

79. $P_3(x) = 1 - 3(x-1) + 6(x-1)^2 - 10(x-1)^3$

81. 3 **83.** $(-10, 10)$ **85.** $[1, 3]$

87. Converges only at $x = 2$

89. (a) $(-5, 5)$ (b) $(-5, 5)$ (c) $(-5, 5)$ (d) $[-5, 5)$

91. Proof **93.** $\displaystyle\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n$ **95.** $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{x-1}{3}\right)^n$; $(-2, 4)$

97. $\ln\frac{5}{4} \approx 0.2231$ **99.** $e^{1/2} \approx 1.6487$

101. $\cos\frac{2}{3} \approx 0.7859$ **103.** $\displaystyle\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!}\left(x - \frac{3\pi}{4}\right)^n$

105. $\displaystyle\sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!}$ **107.** $\displaystyle -\sum_{n=0}^{\infty} (x+1)^n$

109. $1 + \frac{x}{5} - \frac{2x^2}{25} + \frac{6x^3}{125} - \frac{21x^4}{625} + \cdots$

111. (a)–(c) $1 + 2x + 2x^2 + \frac{4}{3}x^3$ **113.** $\sum_{n=0}^{\infty} \frac{(6x)^n}{n!}$

115. $\sum_{n=0}^{\infty} \frac{(-1)^n(5x)^{2n+1}}{(2n+1)!}$ **117.** 0.5

P.S. Problem Solving *(page 683)*

1. (a) 1 (b) Answers will vary. Sample answer: $0, \frac{1}{3}, \frac{2}{3}$
(c) 0

3. Proof **5.** (a) Proof (b) Yes (c) Any distance

7. (a) $\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}; \frac{1}{2}$ (b) $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$; 5.4366

9. For $a = b$, the series converges conditionally. For no values of a and b does the series converge absolutely.

11. Proof **13.** (a) and (b) Proofs

15. (a) The height is infinite. (b) The surface area is infinite.
(c) Proof

Chapter 10

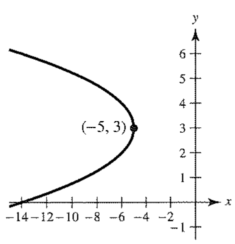
Section 10.1 *(page 696)*

1. A parabola is the set of all points that are equidistant from a fixed line, the directrix, and a fixed point, the focus, not on the line. An ellipse is the set of all points the sum of whose distances from two distinct fixed points called foci is constant. A hyperbola is the set of all points whose absolute value of the difference between the distances from two distinct fixed points called foci is constant.

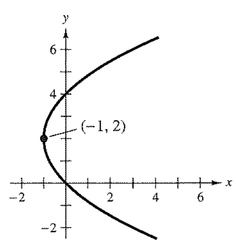
3. (a) $0 < e < 1$
(b) As e gets closer to 1, the graph of the ellipse flattens.

5. a **6.** e **7.** c **8.** b **9.** f **10.** d

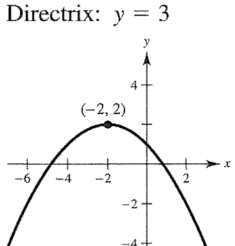
11. Vertex: $(-5, 3)$
Focus: $\left(-\frac{21}{4}, 3\right)$
Directrix: $x = -\frac{19}{4}$



13. Vertex: $(-1, 2)$
Focus: $(0, 2)$
Directrix: $x = -2$



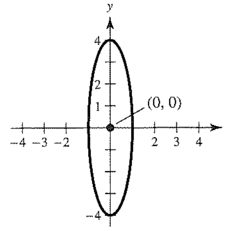
15. Vertex: $(-2, 2)$
Focus: $(-2, 1)$
Directrix: $y = 3$



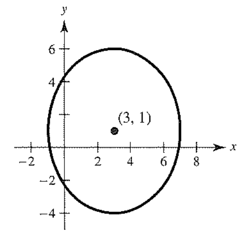
17. $(y - 4)^2 = 4(-2)(x - 5)$ **19.** $(x - 0)^2 = 4(8)(y - 5)$

21. $(x - 1)^2 = 4\left(-\frac{1}{3}\right)(y + 1)$ **23.** $\left(x - \frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y + \frac{4}{15}\right)$

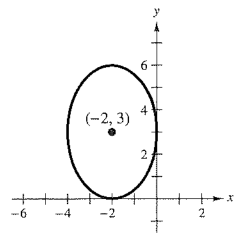
25. Center: $(0, 0)$
Foci: $\left(0, \pm\sqrt{15}\right)$
Vertices: $(0, \pm 4)$
$e = \frac{\sqrt{15}}{4}$



27. Center: $(3, 1)$
Foci: $(3, 4), (3, -2)$
Vertices: $(3, 6), (3, -4)$
$e = \frac{3}{5}$



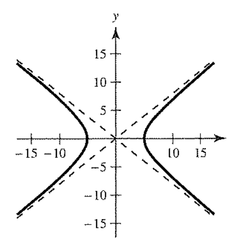
29. Center: $(-2, 3)$
Foci: $\left(-2, 3 \pm \sqrt{5}\right)$
Vertices: $(-2, 6), (-2, 0)$
$e = \frac{\sqrt{5}}{3}$



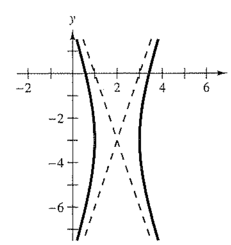
31. $\frac{x^2}{36} + \frac{y^2}{11} = 1$ **33.** $\frac{(x-3)^2}{9} + \frac{(y-5)^2}{16} = 1$

35. $\frac{x^2}{16} + \frac{7y^2}{16} = 1$

37. Center: $(0, 0)$
Foci: $\left(\pm\sqrt{41}, 0\right)$
Vertices: $(\pm 5, 0)$
$e = \frac{\sqrt{41}}{5}$
Asymptotes: $y = \pm\frac{4}{5}x$



39. Center: $(2, -3)$
Foci: $\left(2 \pm \sqrt{10}, -3\right)$
Vertices: $(1, -3), (3, -3)$
$e = \sqrt{10}$
Asymptotes: $y = -3 \pm 3(x - 2)$



41. $\frac{x^2}{1} - \frac{y^2}{25} = 1$ **43.** $\frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1$

45. $\frac{y^2}{4} - \frac{x^2}{12} = 1$ **47.** $\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1$

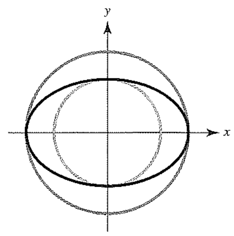
49. (a) $\left(6, \sqrt{3}\right)$: $2x - 3\sqrt{3}y - 3 = 0$
$\left(6, -\sqrt{3}\right)$: $2x + 3\sqrt{3}y - 3 = 0$
(b) $\left(6, \sqrt{3}\right)$: $9x + 2\sqrt{3}y - 60 = 0$
$\left(6, -\sqrt{3}\right)$: $9x - 2\sqrt{3}y - 60 = 0$

51. Parabola **53.** Hyperbola **55.** Circle

57. (a) Ellipse (b) Hyperbola (c) Circle
(d) Answers will vary. Sample answer: Eliminate the y^2-term.

59. Recall that $0 \le \sin^2\theta \le 1$. The circumference is given by

$$C = 4\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\sin^2\theta}\,d\theta$$



61. $\frac{9}{4}$ m **63.** (a) Proof (b) $(3, -3)$ **65.** $y = \frac{1}{180}x^2$

67. $\dfrac{16(4 + 3\sqrt{3} - 2\pi)}{3} \approx 15.536 \text{ ft}^2$

69. Minimum distance: 147,099,713.4 km
Maximum distance: 152,096,286.6 km

71. $e \approx 0.1373$ **73.** $e \approx 0.9671$

75. (a) Area $= 2\pi$

(b) Volume $= \dfrac{8\pi}{3}$

Surface area $= \dfrac{2\pi(9 + 4\sqrt{3}\pi)}{9} \approx 21.48$

(c) Volume $= \dfrac{16\pi}{3}$

Surface area $= \dfrac{4\pi[6 + \sqrt{3}\ln(2 + \sqrt{3})]}{3} \approx 34.69$

77. 37.96 **79.** 40 **81.** $\dfrac{(x-6)^2}{9} - \dfrac{(y-2)^2}{7} = 1$

83. 110.3 mi **85.** Proof

87. False. See the definition of a parabola. **89.** True

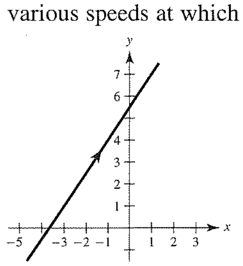
91. True **93.** Putnam Problem B4, 1976

Section 10.2 *(page 707)*

1. The position, direction, and speed at a given time

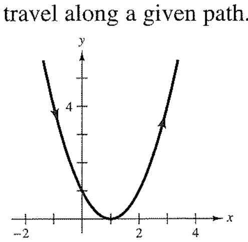
3. Different parametric representations can be used to represent various speeds at which objects travel along a given path.

5.



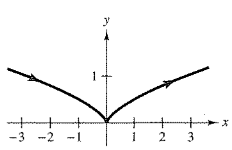
$3x - 2y + 11 = 0$

7.



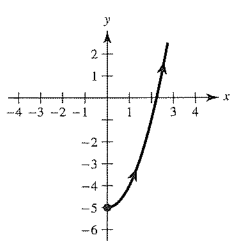
$y = (x - 1)^2$

9.



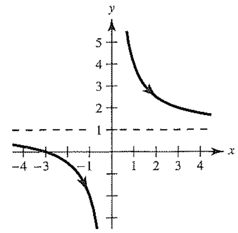
$y = \frac{1}{2}x^{2/3}$

11.



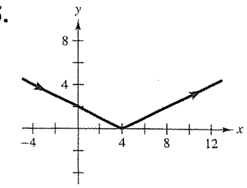
$y = x^2 - 5, \quad x \ge 0$

13.



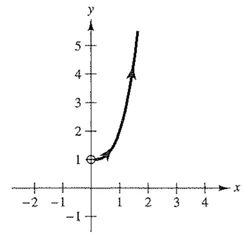
$y = \dfrac{x + 3}{x}$

15.



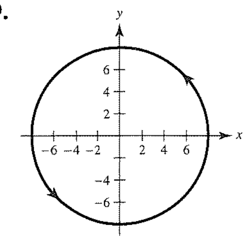
$y = \dfrac{|x - 4|}{2}$

17.



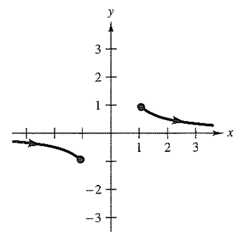
$y = x^3 + 1, \quad x > 0$

19.



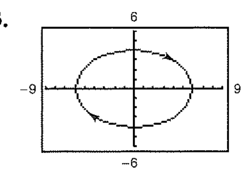
$x^2 + y^2 = 64$

21.



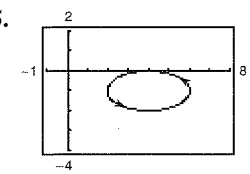
$y = \dfrac{1}{x}, \quad |x| \ge 1$

23.



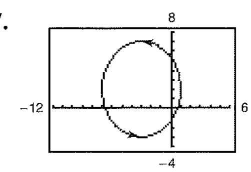
$\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$

25.



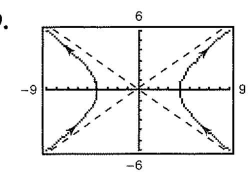
$\dfrac{(x-4)^2}{4} + \dfrac{(y+1)^2}{1} = 1$

27.



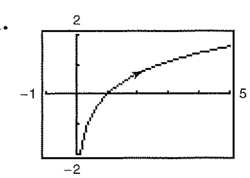
$\dfrac{(x+3)^2}{16} + \dfrac{(y-2)^2}{25} = 1$

29.



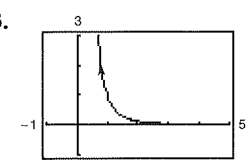
$\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

31.



$y = \ln x$

33.



$y = \dfrac{1}{x^3}, \quad x > 0$

35. Both curves represent the parabola $y = x^2$.

	Domain	Orientation	Smooth
(a)	$-\infty < x < \infty$	Left to right	Yes
(b)	$-\infty < x < \infty$	Right to left	Yes

37. Each curve represents a portion of the line $y = 2x + 1$.

	Domain	Orientation	Smooth
(a)	$-\infty < x < \infty$	Up	Yes
(b)	$-1 \le x \le 1$	Oscillates	No, $\dfrac{dx}{d\theta} = \dfrac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$
(c)	$0 < x < \infty$	Down	Yes
(d)	$0 < x < \infty$	Up	Yes

39. $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ **41.** $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

43. $x = 4t$
$y = -7t$
(Solution is not unique.)

45. $x = 1 + 2\cos\theta$
$y = 1 + 2\sin\theta$
(Solution is not unique.)

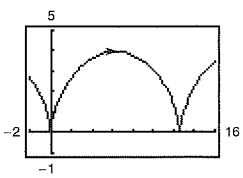
47. $x = 2 + 5\cos\theta$
$y = 4\sin\theta$
(Solution is not unique.)

49. $x = 2\tan\theta$
$y = \sec\theta$
(Solution is not unique.)

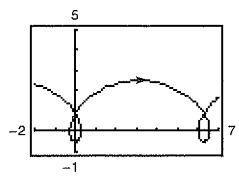
51. $x = t$
$y = 6t - 5$;
$x = t + 1$
$y = 6t + 1$
(Solution is not unique.)

53. $x = t$
$y = t^3$;
$x = \tan t$
$y = \tan^3 t$
(Solution is not unique.)

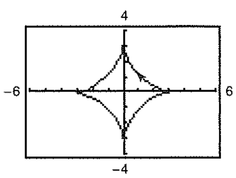
55. $x = t + 3,\ y = 2t + 1$ **57.** $x = t,\ y = t^2$

59.


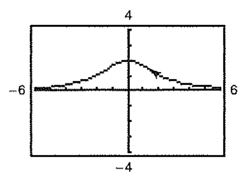
Not smooth at $\theta = 2n\pi$

61.


Smooth everywhere

63.


Not smooth at $\theta = \frac{1}{2}n\pi$

65.


Smooth everywhere

67. The orientation moves right to left on $[-1, 0]$ and left to right on $[0, 1]$, failing to determine a definite direction.

69. No. In the interval $0 < \theta < \pi$, $\cos\theta = \cos(-\theta)$ and $\sin^2\theta = \sin^2(-\theta)$. So, the parameter was not changed.

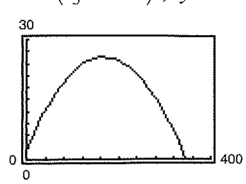
71. d; $(4, 0)$ is on the graph. **73.** b; $(1, 0)$ is on the graph.

75. $x = a\theta - b\sin\theta,\ y = a - b\cos\theta$

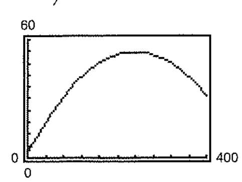
77. False. The graph of the parametric equations is the portion of the line $y = x$ when $x \ge 0$.

79. True

81. (a) $x = \left(\frac{440}{3}\cos\theta\right)t,\ y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2$

(b)


Not a home run

(c)


Home run

(d) 19.4°

Section 10.3 *(page 715)*

1. The slope of the curve at (x, y)

3. Horizontal tangent lines when $dy/dt = 0$ and $dx/dt \ne 0$ for some value of t; vertical tangent lines when $dx/dt = 0$ and $dy/dt \ne 0$ for some value of t

5. $-\dfrac{3}{t}$ **7.** -1

9. $\dfrac{dy}{dx} = \dfrac{3}{4},\ \dfrac{d^2y}{dx^2} = 0$; Neither concave upward nor concave downward

11. $\dfrac{dy}{dx} = 2t + 3,\ \dfrac{d^2y}{dx^2} = 2$

At $t = -2,\ \dfrac{dy}{dx} = -1,\ \dfrac{d^2y}{dx^2} = 2$; Concave upward

13. $\dfrac{dy}{dx} = -\cot\theta,\ \dfrac{d^2y}{dx^2} = -\dfrac{(\csc\theta)^3}{4}$

At $\theta = \dfrac{\pi}{4},\ \dfrac{dy}{dx} = -1,\ \dfrac{d^2y}{dx^2} = -\dfrac{\sqrt{2}}{2}$; Concave downward

15. $\dfrac{dy}{dx} = 2\csc\theta,\ \dfrac{d^2y}{dx^2} = -2\cot^3\theta$

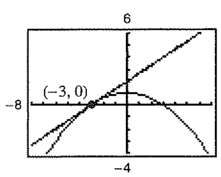
At $\theta = -\dfrac{\pi}{3},\ \dfrac{dy}{dx} = -\dfrac{4\sqrt{3}}{3},\ \dfrac{d^2y}{dx^2} = \dfrac{2\sqrt{3}}{9}$;
Concave upward

17. $\dfrac{dy}{dx} = -\tan\theta,\ \dfrac{d^2y}{dx^2} = \sec^4\theta\csc\dfrac{\theta}{3}$

At $\theta = \dfrac{\pi}{4},\ \dfrac{dy}{dx} = -1,\ \dfrac{d^2y}{dx^2} = \dfrac{4\sqrt{2}}{3}$; Concave upward

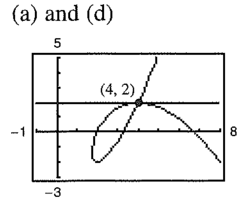
19. $\left(-\dfrac{2}{\sqrt{3}}, \dfrac{3}{2}\right)$: $3\sqrt{3}x - 8y + 18 = 0$
$(0, 2)$: $y - 2 = 0$
$\left(2\sqrt{3}, \dfrac{1}{2}\right)$: $\sqrt{3}x + 8y - 10 = 0$

21. $(0, 0)$: $2y - x = 0$
$(-3, -1)$: $y + 1 = 0$
$(-3, 3)$: $2x - y + 9 = 0$

23. (a) and (d)


(b) At $t = -\dfrac{1}{2},\ \dfrac{dx}{dt} = 6,\ \dfrac{dy}{dt} = 4$, and $\dfrac{dy}{dx} = \dfrac{2}{3}$.

(c) $y = \dfrac{2}{3}x + 2$

25. (a) and (d)


(b) At $t = -1,\ \dfrac{dx}{dt} = -3$, $\dfrac{dy}{dt} = 0$, and $\dfrac{dy}{dx} = 0$.

(c) $y = 2$

27. $y = \pm\frac{3}{4}x$ **29.** $y = 3x - 5$ and $y = 1$

31. Horizontal: $(-1, -\pi), (-1, \pi), (1, 2\pi), (1, -2\pi)$

Vertical: $\left(\dfrac{\pi}{2}, 1\right), \left(\dfrac{\pi}{2}, -1\right), \left(-\dfrac{3\pi}{2}, 1\right), \left(-\dfrac{3\pi}{2}, -1\right)$

33. Horizontal: $(9, 0)$
Vertical: None

35. Horizontal: $(2, 22), (6, -10)$
Vertical: None

37. Horizontal: $(0, 7), (0, -7)$
Vertical: $(7, 0), (-7, 0)$

39. Horizontal: $(5, -1), (5, -3)$
Vertical: $(8, -2), (2, -2)$

41. Horizontal: None
Vertical: $(4, 0)$

43. Concave downward: $-\infty < t < 0$
Concave upward: $0 < t < \infty$

45. Concave upward: $t > 0$

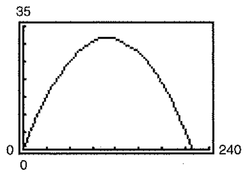
47. Concave downward: $0 < t < \dfrac{\pi}{2}$

Concave upward: $\dfrac{\pi}{2} < t < \pi$

49. $4\sqrt{13} \approx 14.422$ **51.** $\sqrt{2}(1 - e^{-\pi/2}) \approx 1.12$

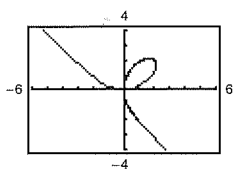
53. $\frac{1}{12}\left[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\right] \approx 3.249$ **55.** $6a$ **57.** $8a$

59. (a)



(b) 219.2 ft (c) 230.8 ft

61. (a)



(b) $(0, 0), \left(\dfrac{4\sqrt[3]{2}}{3}, \dfrac{4\sqrt[3]{4}}{3}\right)$

(c) About 6.557

63. (a) $27\pi\sqrt{13}$ (b) $18\pi\sqrt{13}$ **65.** 50π **67.** $\dfrac{12\pi a^2}{5}$

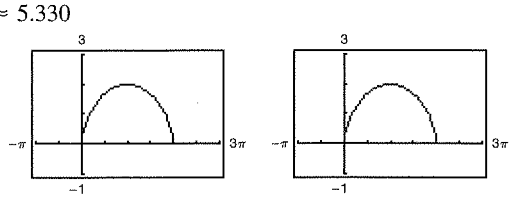
69. $S = 2\pi \displaystyle\int_0^2 (t + 2)\sqrt{9t^4 + 1}\, dt \approx 185.78$

71. $S = 2\pi \displaystyle\int_0^{\pi/2} \left(\sin\theta \cos\theta \sqrt{4\cos^2\theta + 1}\right) d\theta$

$= \dfrac{(5\sqrt{5} - 1)\pi}{6}$

≈ 5.330

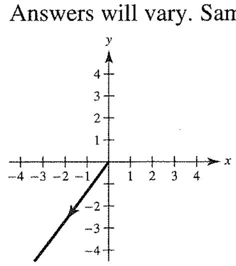
73. (a)



(b) The average speed of the particle on the second path is twice the average speed of the particle on the first path.

(c) 4π

75. Answers will vary. Sample answer: Let $x = -3t$, $y = -4t$.



77. Proof **79.** $\dfrac{3\pi}{2}$ **81.** d **82.** b **83.** f **84.** c

85. a **86.** e **87.** $\left(\frac{3}{4}, \frac{8}{5}\right)$ **89.** 288π

91. (a) $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1 - \cos\theta}; \dfrac{d^2y}{dx^2} = -\dfrac{1}{a(\cos\theta - 1)^2}$

(b) $y = \left(2 + \sqrt{3}\right)\left[x - a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right)\right] + a\left(1 - \dfrac{\sqrt{3}}{2}\right)$

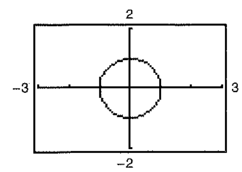
(c) $(a(2n + 1)\pi, 2a)$

(d) Concave downward on $(0, 2\pi)$, $(2\pi, 4\pi)$, etc.

(e) $s = 8a$

93. Proof

95. (a)



(b) Circle of radius 1 and center at $(0, 0)$ except the point $(-1, 0)$

(c) As t increases from -20 to 0, the speed increases, and as t increases from 0 to 20, the speed decreases.

97. False. $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{f'(t)} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}.$

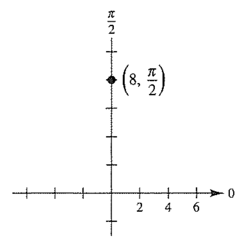
99. False. The resulting rectangular equation is a line.

Section 10.4 *(page 726)*

1. r is the directed distance from the origin to the point in the plane. θ is the directed angle, counterclockwise from the polar axis to the segment from the origin to the point in the plane.

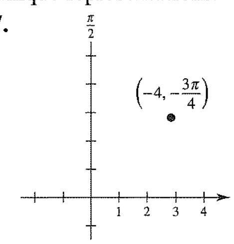
3. The rectangular coordinate system is a collection of points of the form (x, y), where x is the directed distance from the y-axis to the point and y is the directed distance from the x-axis to the point. Every point has a unique representation.
The polar coordinate system is a collection of points of the form (r, θ), where r is the directed distance from the origin O to a point P and θ is the directed angle, measured counterclockwise, from the polar axis to the segment $\overline{OP}$. Polar coordinates do not have unique representations.

5.



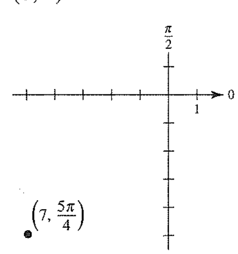
$(0, 8)$

7.



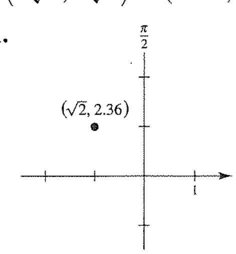
$\left(2\sqrt{2}, 2\sqrt{2}\right) \approx (2.828, 2.828)$

9.



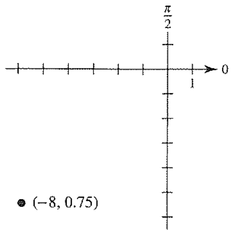
$(-4.95, -4.95)$

11.



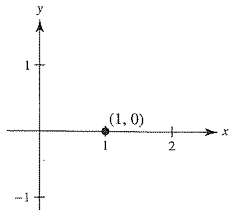
$(-1.004, 0.996)$

13.



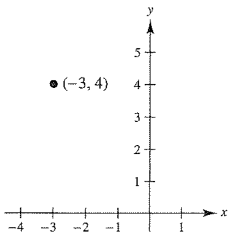
$(-5.854, -5.453)$

15.



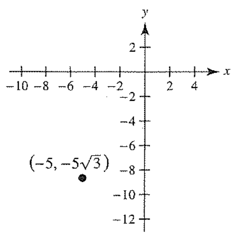
$(1, 0), (-1, \pi)$

17.



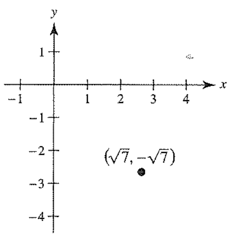
$(5, 2.214), (-5, 5.356)$

19.



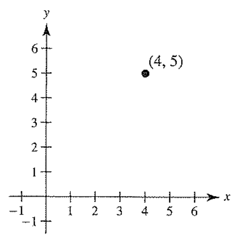
$\left(10, \frac{4\pi}{3}\right), \left(-10, \frac{\pi}{3}\right)$

21.



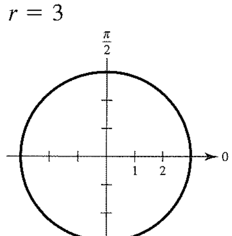
$\left(\sqrt{14}, \frac{7\pi}{4}\right), \left(-\sqrt{14}, \frac{3\pi}{4}\right)$

23.

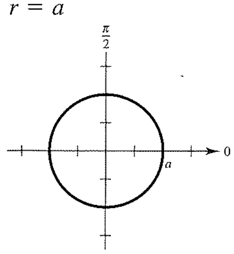


$(\sqrt{41}, 0.8961)$,
$(-\sqrt{41}, 4.0376)$

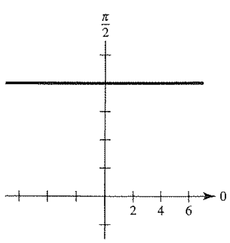
25. $r = 3$



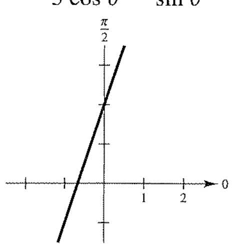
27. $r = a$



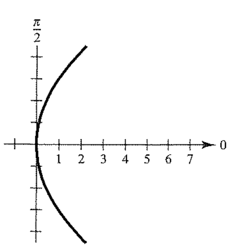
29. $r = 8 \csc \theta$



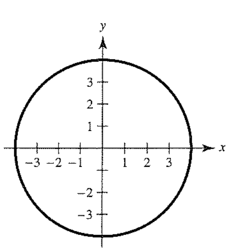
31. $r = \dfrac{-2}{3 \cos \theta - \sin \theta}$



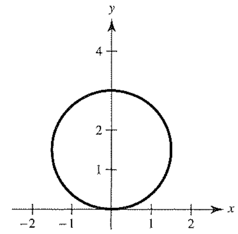
33. $r = 9 \csc^2 \theta \cos \theta$



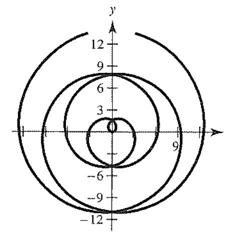
35. $x^2 + y^2 = 16$



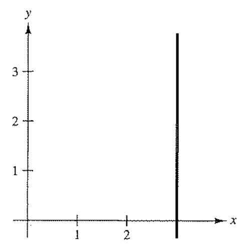
37. $x^2 + y^2 - 3y = 0$



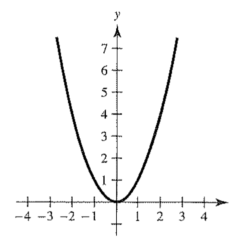
39. $\sqrt{x^2 + y^2} = \arctan \dfrac{y}{x}$



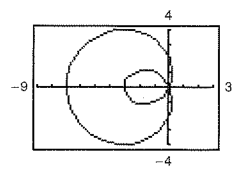
41. $x - 3 = 0$



43. $x^2 - y = 0$

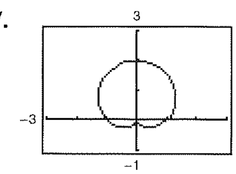


45.



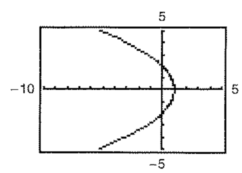
$0 \le \theta < 2\pi$

47.



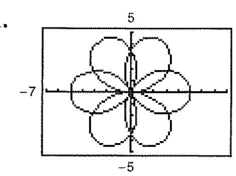
$0 \le \theta < 2\pi$

49.



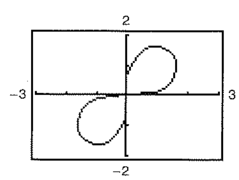
$-\pi < \theta < \pi$

51.



$0 \le \theta < 4\pi$

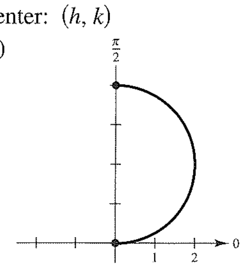
53.



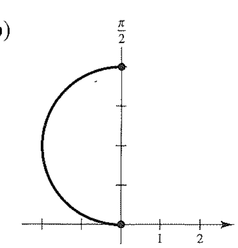
$0 \le \theta < \dfrac{\pi}{2}$

55. $(x - h)^2 + (y - k)^2 = h^2 + k^2$
Radius: $\sqrt{h^2 + k^2}$
Center: (h, k)

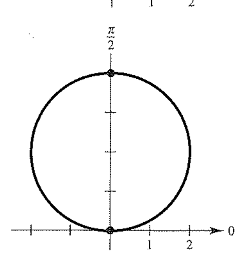
57. (a)



(b)



(c)



59. $\sqrt{17}$ **61.** About 5.6

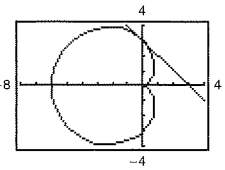
63. $\dfrac{dy}{dx} = \dfrac{-2\cos\theta\sin\theta + 2\cos\theta(1-\sin\theta)}{-2\cos\theta\cos\theta - 2\sin\theta(1-\sin\theta)}$

$(2, 0)$: $\dfrac{dy}{dx} = -1$

$\left(3, \dfrac{7\pi}{6}\right)$: $\dfrac{dy}{dx}$ is undefined.

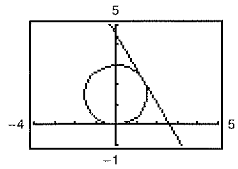
$\left(4, \dfrac{3\pi}{2}\right)$: $\dfrac{dy}{dx} = 0$

65. (a) and (b)



(c) $\dfrac{dy}{dx} = -1$

67. (a) and (b)



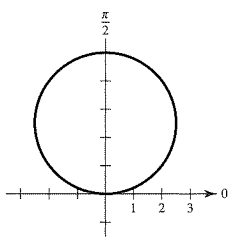
(c) $\dfrac{dy}{dx} = -\sqrt{3}$

69. Horizontal: $\left(2, \dfrac{3\pi}{2}\right), \left(\dfrac{1}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{1}{2}, \dfrac{5\pi}{6}\right)$

Vertical: $\left(\dfrac{3}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3}{2}, \dfrac{11\pi}{6}\right)$

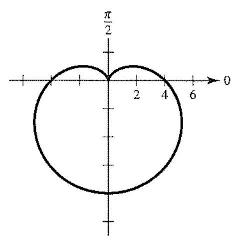
71. $\left(5, \dfrac{\pi}{2}\right), \left(1, \dfrac{3\pi}{2}\right)$

73.

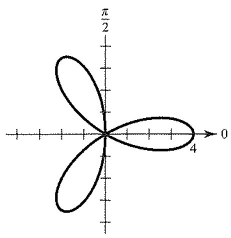


$\theta = 0$

75.

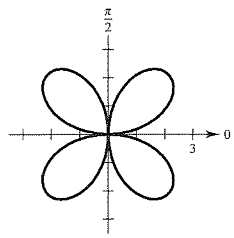


77.



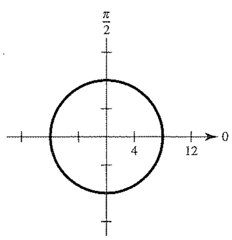
$\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

79.

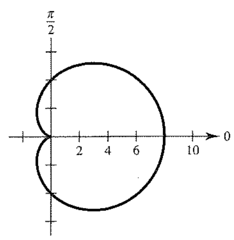


$\theta = 0, \dfrac{\pi}{2}$

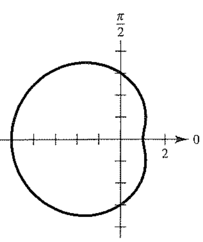
81.



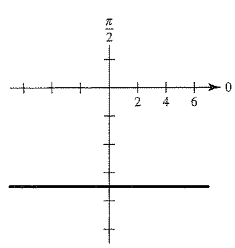
83.



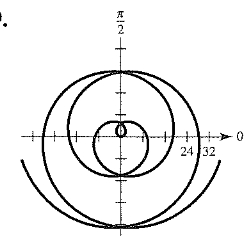
85.



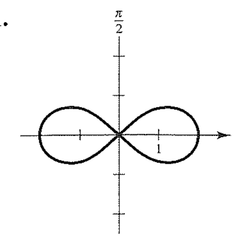
87.



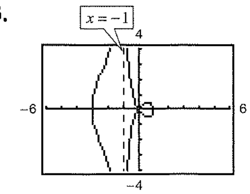
89.



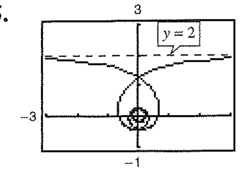
91.



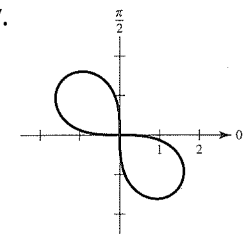
93.



95.



97.



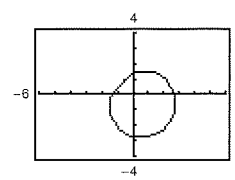
99. (a) To test for symmetry about the x-axis, replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$. If the substitution yields an equivalent equation, then the graph is symmetric about the x-axis.

(b) To test for symmetry about the y-axis, replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$. If the substitution yields an equivalent equation, then the graph is symmetric about the y-axis.

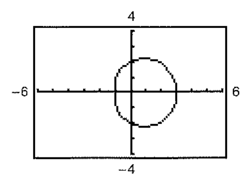
101. Proof

103. (a) $r = 2 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

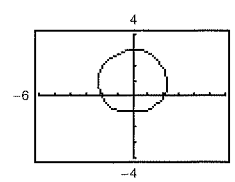
$= 2 - \dfrac{\sqrt{2}(\sin\theta - \cos\theta)}{2}$



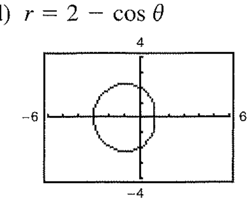
(b) $r = 2 + \cos\theta$



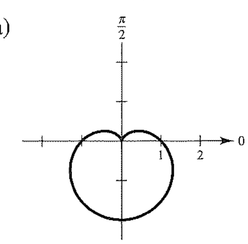
(c) $r = 2 + \sin\theta$



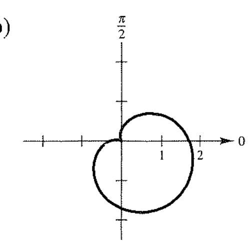
(d) $r = 2 - \cos\theta$

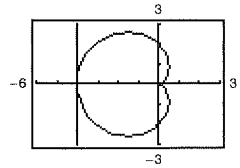


105. (a)

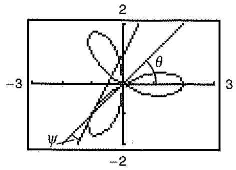


(b)

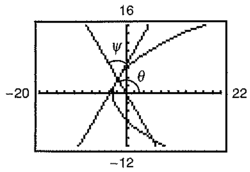


107. 

$\psi = \dfrac{\pi}{2}$

109. 

$\psi = \arctan \frac{1}{3} \approx 18.4°$

111. 

$\psi = \dfrac{\pi}{3}, 60°$

113. True **115.** True

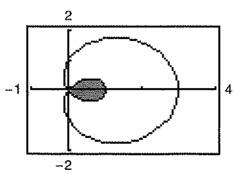
Section 10.5 *(page 735)*

1. Check that f is continuous and either nonnegative or nonpositive on the interval of consideration.

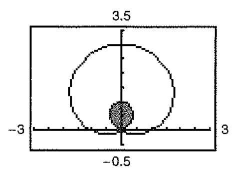
3. $8\displaystyle\int_0^{\pi/2} \sin^2\theta \, d\theta$ **5.** $\dfrac{1}{2}\displaystyle\int_{\pi/2}^{3\pi/2} (3 - 2\sin\theta)^2 \, d\theta$ **7.** 9π

9. $\dfrac{\pi}{3}$ **11.** $\dfrac{\pi}{16}$ **13.** $\dfrac{97\pi}{4} - 60 \approx 16.184$ **15.** $\dfrac{33\pi}{2}$

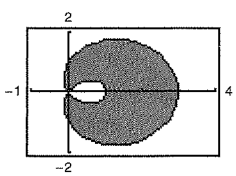
17. 4

19. 

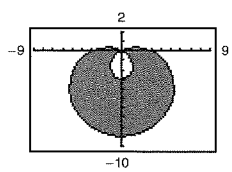
$\dfrac{2\pi - 3\sqrt{3}}{2}$

21. 

$\dfrac{2\pi - 3\sqrt{3}}{2}$

23. 

$\pi + 3\sqrt{3}$

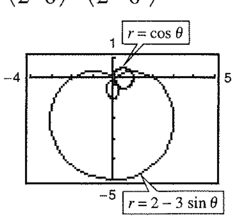
25. 

$9\pi + 27\sqrt{3}$

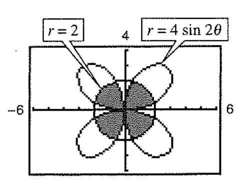
27. $\left(1, \dfrac{\pi}{2}\right), \left(1, \dfrac{3\pi}{2}\right), (0, 0)$

29. $\left(\dfrac{2 - \sqrt{2}}{2}, \dfrac{3\pi}{4}\right), \left(\dfrac{2 + \sqrt{2}}{2}, \dfrac{7\pi}{4}\right), (0, 0)$

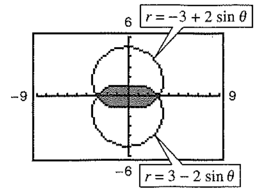
31. $\left(\dfrac{3}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{3}{2}, \dfrac{5\pi}{6}\right), (0, 0)$ **33.** $(2, 4), (-2, -4)$

35. 

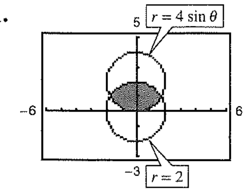
$(0, 0), (0.935, 0.363),$
$(0.535, -1.006)$
The graphs reach the pole at different times (θ-values).

37. 

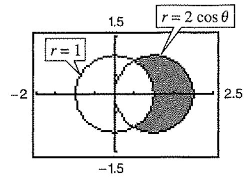
$\frac{4}{3}(4\pi - 3\sqrt{3})$

39. 

$11\pi - 24$

41. 

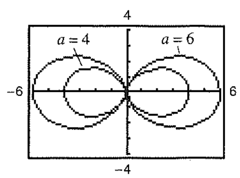
$\frac{2}{3}(4\pi - 3\sqrt{3})$

43. 

$\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$

45. $\dfrac{5\pi a^2}{4}$ **47.** $\dfrac{a^2}{2}(\pi - 2)$

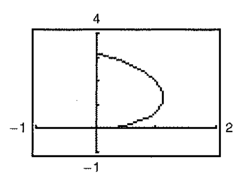
49. (a) $(x^2 + y^2)^{3/2} = ax^2$

(b) 

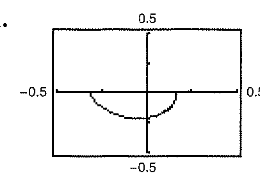
(c) $\dfrac{15\pi}{2}$

51. The area enclosed by the function is $\dfrac{\pi a^2}{4}$ if n is odd and is $\dfrac{\pi a^2}{2}$ if n is even.

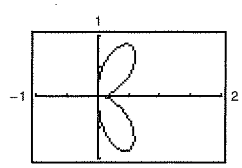
53. $\dfrac{4\pi}{3}$ **55.** 4π **57.** 8

59. 

About 4.16

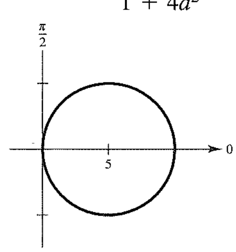
61. 

About 0.71

63. 

About 4.39

65. 36π **67.** $\dfrac{2\pi\sqrt{1 + a^2}}{1 + 4a^2}(e^{\pi a} - 2a)$ **69.** 21.87

71. (a) 

(b) $0 \le \theta < \pi$
(c) and (d) 25π

73. Answers will vary. Sample answer: $f(\theta) = \cos^2\theta + 1$, $g(\theta) = -\frac{3}{2}$

75. $40\pi^2$

77. (a) 16π

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

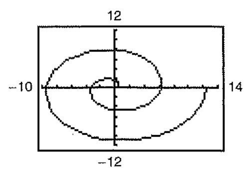
(c) and (d) For $\frac{1}{4}$ of area ($4\pi \approx 12.57$): 0.42

For $\frac{1}{2}$ of area ($8\pi \approx 25.13$): $1.57\left(\frac{\pi}{2}\right)$

For $\frac{3}{4}$ of area ($12\pi \approx 37.70$): 2.73

(e) No; Answers will vary.

79. (a)



The graph becomes larger and more spread out. The graph is reflected over the y-axis.

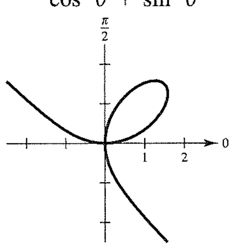
(b) $(an\pi, n\pi)$, where $n = 1, 2, 3, \ldots$

(c) About 21.26 (d) $\dfrac{4}{3\pi^3}$

81. $r = \sqrt{2}\cos\theta$

83. (a) $r = \dfrac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$

(b)

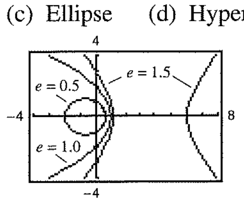


(c) $\dfrac{3}{2}$

Section 10.6 *(page 743)*

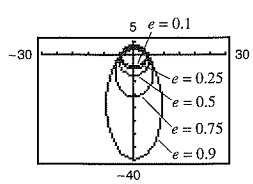
1. (a) Hyperbola (b) Parabola
(c) Ellipse (d) Hyperbola

3.



(a) Parabola
(b) Ellipse
(c) Hyperbola

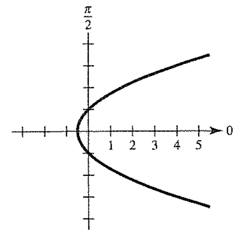
5.



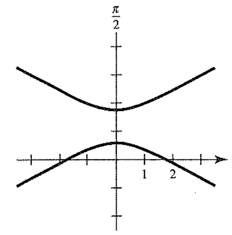
Ellipse
As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.

7. c **8.** f **9.** a **10.** e **11.** b **12.** d

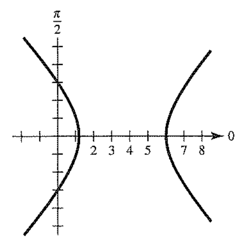
13. $e = 1$
Distance $= 1$
Parabola



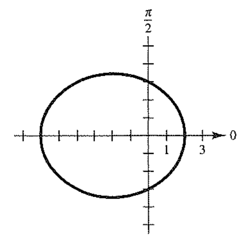
15. $e = 2$
Distance $= \frac{7}{8}$
Hyperbola



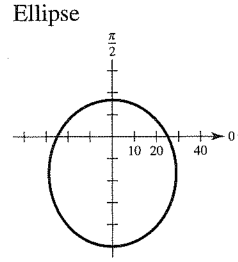
17. $e = \frac{3}{2}$
Distance $= 2$
Hyperbola



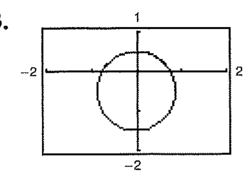
19. $e = \frac{1}{2}$
Distance $= 6$
Ellipse



21. $e = \frac{1}{2}$
Distance $= 50$
Ellipse

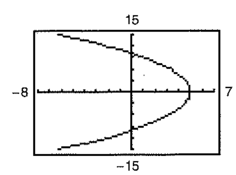


23.



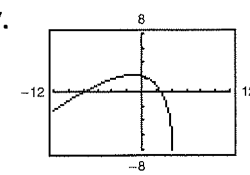
Ellipse
$e = \frac{1}{2}$

25.



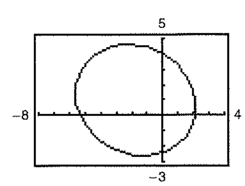
Parabola
$e = 1$

27.



Rotated $\frac{\pi}{3}$ radian counterclockwise.

29.



Rotated $\frac{\pi}{6}$ radian clockwise.

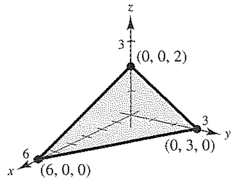
31. $r = \dfrac{8}{8 + 5\cos\left(\theta + \frac{\pi}{6}\right)}$

33. $r = \dfrac{3}{1 - \cos\theta}$ **35.** $r = \dfrac{1}{4 + \sin\theta}$

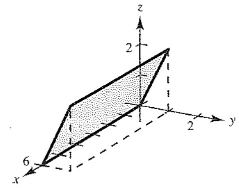
37. $r = \dfrac{8}{3 + 4\cos\theta}$ **39.** $r = \dfrac{2}{1 - \sin\theta}$

41. $r = \dfrac{16}{5 + 3\cos\theta}$ **43.** $r = \dfrac{9}{4 - 5\sin\theta}$

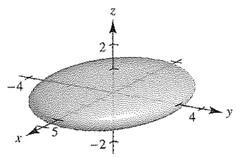
49. Plane



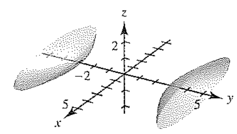
51. Plane



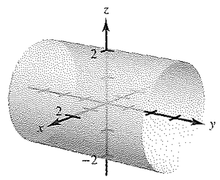
53. Ellipsoid



55. Hyperboloid of two sheets



57. Cylinder



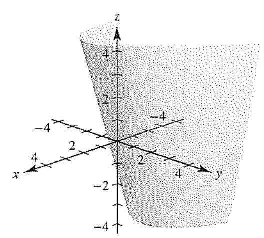
59. $x^2 + z^2 = 2y$

61. (a) $\left(2\sqrt{3}, -\frac{\pi}{3}, -5\right)$ (b) $\left(\sqrt{37}, -\frac{\pi}{3}, \arccos\left(-\frac{5\sqrt{37}}{37}\right)\right)$

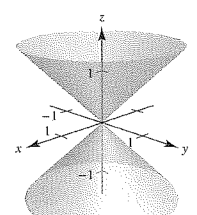
63. $(-5, 0, 1)$ **65.** $\left(-2\sqrt{2}, 0, 2\sqrt{2}\right)$

67. (a) $r^2 \cos 2\theta = 2z$ (b) $\rho = 2 \sec 2\theta \cos \phi \csc^2 \phi$

69. $z = y^2 + 3x$



71. $x^2 + y^2 - z^2 = 0$



P.S. Problem Solving *(page 817)*

1–3. Proofs **5.** (a) $\frac{3\sqrt{2}}{2} \approx 2.12$ (b) $\sqrt{5} \approx 2.24$

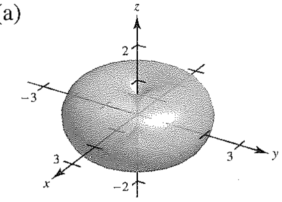
7. (a) $\frac{\pi}{2}$ (b) $\frac{1}{2}(\pi abk)k$

(c) $V = \frac{1}{2}(\pi ab)k^2$

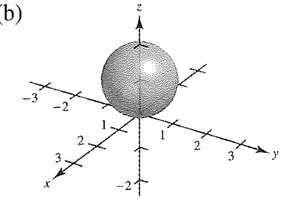
$V = \frac{1}{2}$(area of base)height

9. Proof

11. (a)



(b)



13. (a) Tension: $\frac{2\sqrt{3}}{3} \approx 1.1547$ lb

Magnitude of $\mathbf{u}$: $\frac{\sqrt{3}}{3} \approx 0.5774$ lb

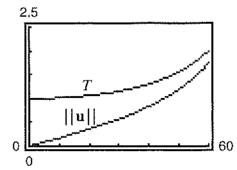
(b) $T = \sec \theta$, $\|\mathbf{u}\| = \tan \theta$; Domain: $0° \le \theta \le 90°$

(c)

θ	0°	10°	20°	30°
T	1	1.0154	1.0642	1.1547
$\|\mathbf{u}\|$	0	0.1763	0.3640	0.5774

θ	40°	50°	60°
T	1.3054	1.5557	2
$\|\mathbf{u}\|$	0.8391	1.1918	1.7321

(d)



(e) Both are increasing functions.

(f) $\lim_{\theta \to \pi/2^-} T = \infty$ and $\lim_{\theta \to \pi/2^-} \|\mathbf{u}\| = \infty$

Yes. As θ increases, both T and $\|\mathbf{u}\|$ increase.

15. $\langle 0, 0, \cos \alpha \sin \beta - \cos \beta \sin \alpha \rangle$; Proof

17. $D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$

$= \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{u} \times \mathbf{v}\|}$

19. Proof

Chapter 12

Section 12.1 *(page 825)*

1. You can use a vector-valued function to trace the graph of a curve. Recall that the terminal point of the position vector $\mathbf{r}(t)$ coincides with a point on the curve.

3. $(-\infty, -1) \cup (-1, \infty)$ **5.** $(0, \infty)$

7. $[0, \infty)$ **9.** $(-\infty, \infty)$

11. (a) $\frac{1}{2}\mathbf{i}$ (b) $\mathbf{j}$ (c) $\frac{1}{2}(s + 1)^2\mathbf{i} - s\mathbf{j}$

(d) $\frac{1}{2}\Delta t(\Delta t + 4)\mathbf{i} - \Delta t\mathbf{j}$

13. $\mathbf{r}(t) = 5t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$, $0 \le t \le 1$

$x = 5t$, $y = 2t$, $z = 2t$, $0 \le t \le 1$

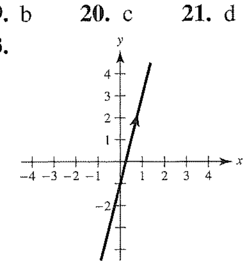
15. $\mathbf{r}(t) = (-3 + 2t)\mathbf{i} + (-6 - 3t)\mathbf{j} + (-1 - 7t)\mathbf{k}$, $0 \le t \le 1$

$x = -3 + 2t$, $y = -6 - 3t$, $z = -1 - 7t$, $0 \le t \le 1$

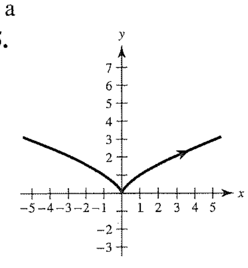
17. $t^2(5t - 1)$; No, the dot product is a scalar.

19. b **20.** c **21.** d **22.** a

23.



25.



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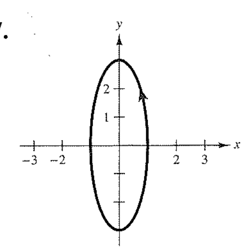

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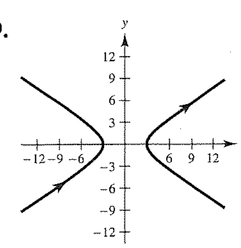
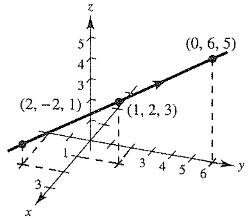
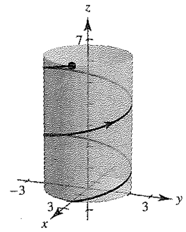
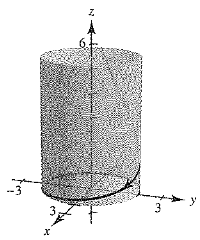
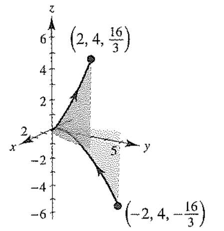
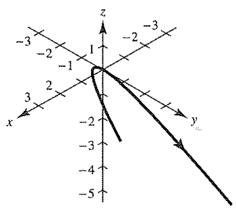


31.


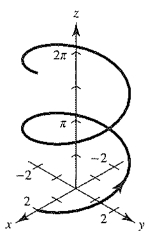
33.


35.


37.


39.


Parabola

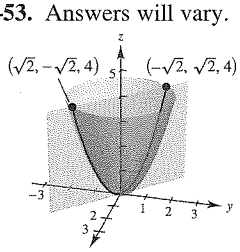
41.


(a) The helix is translated two units back on the x-axis.
(b) The height of the helix increases at a greater rate.
(c) The orientation of the graph is reversed.
(d) The radius of the helix is increased from 2 to 6.

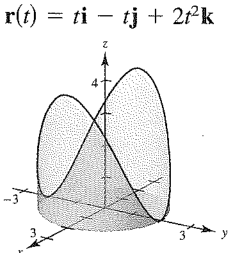
43. $\mathbf{u}(t) = 3t^2\mathbf{i} + (t - 1)\mathbf{j} + (t + 2)\mathbf{k}$

45. $\mathbf{u}(t) = 3t^2\mathbf{i} + 2(t - 1)\mathbf{j} + t\mathbf{k}$

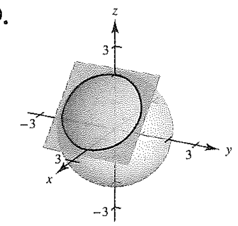
47–53. Answers will vary.

55.


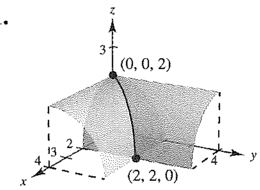
$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

57.


$\mathbf{r}(t) = 2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + 4 \sin^2 t\mathbf{k}$

59.


$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2} \cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$ and
$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2} \cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

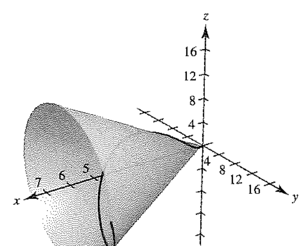
61.


$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

63. Let $x = t$, $y = 2t \cos t$, and $z = 2t \sin t$. Then

$$\begin{aligned} y^2 + z^2 &= (2t \cos t)^2 + (2t \sin t)^2 \\ &= 4t^2 \cos^2 t + 4t^2 \sin^2 t \\ &= 4t^2(\cos^2 t + \sin^2 t) \\ &= 4t^2. \end{aligned}$$

Because $x = t$, $y^2 + z^2 = 4x^2$.



65. $\pi\mathbf{i} - \mathbf{j}$ **67.** $\mathbf{0}$ **69.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

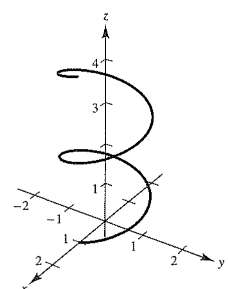
71. $\left(-\infty, -\frac{1}{2}\right), \left(-\frac{1}{2}, 0\right), (0, \infty)$ **73.** $[-1, 1]$

75. $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$, n is an integer.

77. It is a line; Answers will vary.

79. $\mathbf{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j}, & t \geq 3 \\ -\mathbf{i} + \mathbf{j}, & t < 3 \end{cases}$

81. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \frac{1}{\pi}t\mathbf{k}$, $0 \leq t \leq 4\pi$

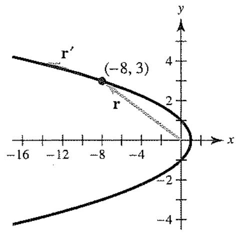


83–85. Proofs **87.** Not necessarily **89.** Yes; yes

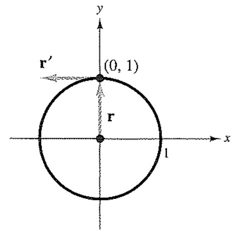
Section 12.2 *(page 834)*

1. $\mathbf{r}'(t_0)$ represents the vector that is tangent to the curve represented by $\mathbf{r}(t)$ at the point t_0.

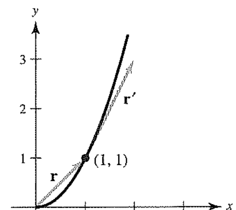
3. $\mathbf{r}'(t) = -2t\mathbf{i} + \mathbf{j}$
$\mathbf{r}(3) = -8\mathbf{i} + 3\mathbf{j}$
$\mathbf{r}'(3) = -6\mathbf{i} + \mathbf{j}$



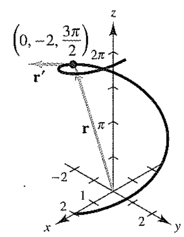
5. $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$
$\mathbf{r}\left(\dfrac{\pi}{2}\right) = \mathbf{j}$
$\mathbf{r}'\left(\dfrac{\pi}{2}\right) = -\mathbf{i}$



7. $\mathbf{r}'(t) = \langle e^t, 2e^{2t}\rangle$
$\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$
$\mathbf{r}'(0) = \mathbf{i} + 2\mathbf{j}$



9. $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$
$\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \left(\dfrac{3\pi}{2}\right)\mathbf{k}$
$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



11. $4t^3\mathbf{i} - 5\mathbf{j}$ **13.** $-9\sin t\cos^2 t\mathbf{i} + 6\sin^2 t\cos t\mathbf{j}$

15. $-e^{-t}\mathbf{i} + (5te^t + 5e^t)\mathbf{k}$

17. $\langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$

19. (a) $3t^2\mathbf{i} + t\mathbf{j}$ (b) $6t\mathbf{i} + \mathbf{j}$ (c) $18t^3 + t$

21. (a) $-4\sin t\mathbf{i} + 4\cos t\mathbf{j}$ (b) $-4\cos t\mathbf{i} - 4\sin t\mathbf{j}$ (c) 0

23. (a) $t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$ (b) $\mathbf{i} + t\mathbf{k}$ (c) $\dfrac{t^3}{2} + t$
(d) $-t\mathbf{i} - \frac{1}{2}t^2\mathbf{j} + \mathbf{k}$

25. (a) $\langle t\cos t, t\sin t, 1\rangle$
(b) $\langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$ (c) t
(d) $\langle -\sin t - t\cos t, \cos t - t\sin t, t^2\rangle$

27. $(-\infty, 0), (0, \infty)$ **29.** $\left(\dfrac{\pi}{2}, 2\pi\right)$

31. $(-\infty, -2), (-2, \infty)$

33. $\left(-\dfrac{\pi}{2} + n\pi, \dfrac{\pi}{2} + n\pi\right)$, n is an integer

35. (a) $\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$ (b) $-\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$
(c) $40t\mathbf{i} + 15t^2\mathbf{j} + 20t^3\mathbf{k}$ (d) $8t + 9t^2 + 5t^4$
(e) $8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$
(f) $2\mathbf{i} + 6\mathbf{j} + 8t\mathbf{k}$

37. (a) $7t^6$ (b) $12t^5\mathbf{i} - 5t^4\mathbf{j}$ **39.** $t^2\mathbf{i} + t\mathbf{j} + 9t\mathbf{k} + \mathbf{C}$

41. $\ln|t|\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$ **43.** $t\mathbf{i} + t^4\mathbf{j} + \dfrac{5^t}{\ln 5}\mathbf{k} + \mathbf{C}$

45. $e^t\mathbf{i} + t\mathbf{j} + (t\sin t + \cos t)\mathbf{k} + \mathbf{C}$

47. $4\mathbf{i} + \dfrac{1}{2}\mathbf{j} - \mathbf{k}$ **49.** $5\mathbf{i} + 6\mathbf{j} + \dfrac{\pi}{2}\mathbf{k}$

51. $2\mathbf{i} + (e^2 - 1)\mathbf{j} - (e^2 + 1)\mathbf{k}$

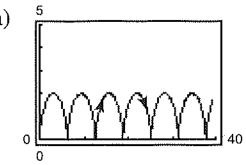
53. $2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$ **55.** $600\sqrt{3}t\mathbf{i} + (-16t^2 + 600t)\mathbf{j}$

57. $\dfrac{2 - e^{-t^2}}{2}\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k}$

59. The three components of $\mathbf{u}$ are increasing functions of t at $t = t_0$.

61–67. Proofs

69. (a)



The curve is a cycloid.

(b) The maximum of $\|\mathbf{r}'\|$ is 2 and the minimum of $\|\mathbf{r}'\|$ is 0. The maximum and the minimum of $\|\mathbf{r}'\|$ are 1.

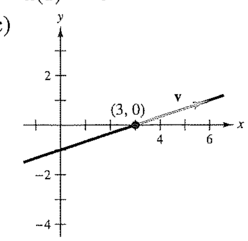
71. Proof **73.** True

75. False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$, then $\dfrac{d}{dt}[\|\mathbf{r}(t)\|] = 0$, but $\|\mathbf{r}'(t)\| = 1$.

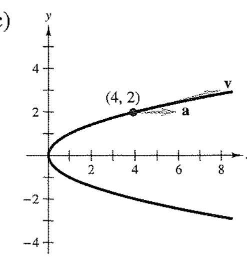
Section 12.3 *(page 842)*

1. The direction of the velocity vector provides the direction of motion at time t and the magnitude of the velocity vector provides the speed of the object.

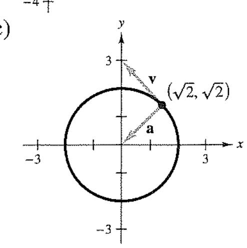
3. (a) $\mathbf{v}(t) = 3\mathbf{i} + \mathbf{j}$
$\|\mathbf{v}(t)\| = \sqrt{10}$
$\mathbf{a}(t) = \mathbf{0}$
(b) $\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}$
$\mathbf{a}(1) = \mathbf{0}$
(c)



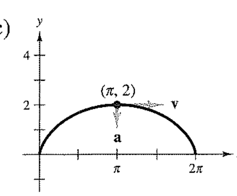
5. (a) $\mathbf{v}(t) = 2t\mathbf{i} + \mathbf{j}$
$\|\mathbf{v}(t)\| = \sqrt{4t^2 + 1}$
$\mathbf{a}(t) = 2\mathbf{i}$
(b) $\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$
$\mathbf{a}(2) = 2\mathbf{i}$
(c)



7. (a) $\mathbf{v}(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$
$\|\mathbf{v}(t)\| = 2$
$\mathbf{a}(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$
(b) $\mathbf{v}\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$
$\mathbf{a}\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$
(c)



9. (a) $\mathbf{v}(t) = \langle 1 - \cos t, \sin t\rangle$
$\|\mathbf{v}(t)\| = \sqrt{2 - 2\cos t}$
$\mathbf{a}(t) = \langle \sin t, \cos t\rangle$
(b) $\mathbf{v}(\pi) = \langle 2, 0\rangle$
$\mathbf{a}(\pi) = \langle 0, -1\rangle$
(c)



11. (a) $\mathbf{v}(t) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{35}$
$\mathbf{a}(t) = \mathbf{0}$
(b) $\mathbf{v}(1) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$
$\mathbf{a}(1) = \mathbf{0}$

13. (a) $\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{1 + 5t^2}$
$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$
(b) $\mathbf{v}(4) = \mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$
$\mathbf{a}(4) = 2\mathbf{j} + \mathbf{k}$

15. (a) $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} - \dfrac{t}{\sqrt{9 - t^2}}\mathbf{k}$

$\|\mathbf{v}(t)\| = \sqrt{\dfrac{18 - t^2}{9 - t^2}}$

$\mathbf{a}(t) = -\dfrac{9}{(9 - t^2)^{3/2}}\mathbf{k}$

(b) $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(0) = -\frac{1}{3}\mathbf{k}$

17. (a) $\mathbf{v}(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = -3\cos t\mathbf{j} - 3\sin t\mathbf{k}$

(b) $\mathbf{v}(\pi) = \langle 4, 0, -3\rangle$

$\mathbf{a}(\pi) = \langle 0, 3, 0\rangle$

19. (a) $\mathbf{v}(t) = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j} + e^t\mathbf{k}$

$\|\mathbf{v}(t)\| = e^t\sqrt{3}$

$\mathbf{a}(t) = -2e^t\sin t\mathbf{i} + 2e^t\cos t\mathbf{j} + e^t\mathbf{k}$

(b) $\mathbf{v}(0) = \langle 1, 1, 1\rangle$

$\mathbf{a}(0) = \langle 0, 2, 1\rangle$

21. $\mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(t) = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})$

23. $\mathbf{v}(t) = \left(\dfrac{t^2}{2} + \dfrac{9}{2}\right)\mathbf{j} + \left(\dfrac{t^2}{2} - \dfrac{1}{2}\right)\mathbf{k}$

$\mathbf{r}(t) = \left(\dfrac{t^3}{6} + \dfrac{9}{2}t - \dfrac{14}{3}\right)\mathbf{j} + \left(\dfrac{t^3}{6} - \dfrac{1}{2}t + \dfrac{1}{3}\right)\mathbf{k}$

$\mathbf{r}(2) = \dfrac{17}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

25. $\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

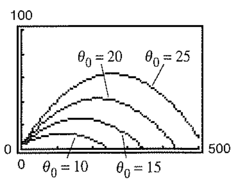
$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$

27. 45.5 ft; The ball will clear the fence.

29. $v_0 = 40\sqrt{6}$ ft/sec; 78 ft **31.** Proof

33. (a) $\mathbf{r}(t) = \left(\frac{440}{3}\cos\theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3}\sin\theta_0\right)t - 16t^2\right]\mathbf{j}$

(b)



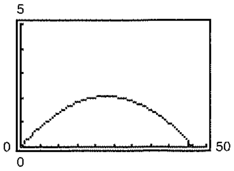
The minimum angle appears to be $\theta_0 = 20°$.

(c) $\theta_0 \approx 19.38°$

35. (a) $v_0 = 28.78$ ft/sec, $\theta = 58.28°$ (b) $v_0 \approx 32$ ft/sec

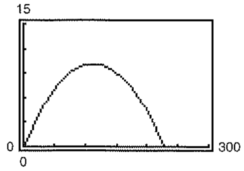
37. 1.91°

39. (a)



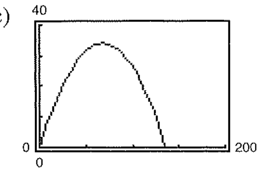
Maximum height: 2.1 ft
Range: 46.6 ft

(b)



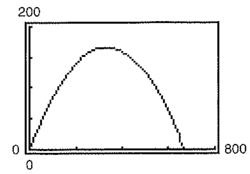
Maximum height: 10.0 ft
Range: 227.8 ft

(c)



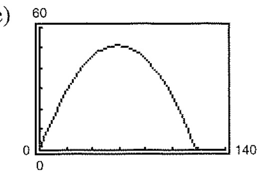
Maximum height: 34.0 ft
Range: 136.1 ft

(d)



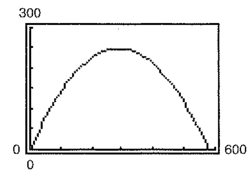
Maximum height: 166.5 ft
Range: 666.1 ft

(e)



Maximum height: 51.0 ft
Range: 117.9 ft

(f)



Maximum height: 249.8 ft
Range: 576.9 ft

41. Maximum height: 129.1 m; Range: 886.3 m **43.** Proof

45. $\mathbf{v}(t) = b\omega[(1 - \cos\omega t)\mathbf{i} + \sin\omega t\mathbf{j}]$

$\mathbf{a}(t) = b\omega^2(\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j})$

(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$.

(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots$.

47. $\mathbf{v}(t) = -b\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$

$\mathbf{v}(t) \cdot \mathbf{r}(t) = 0$

49. $\mathbf{a}(t) = -b\omega^2(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}) = -\omega^2\mathbf{r}(t)$; $\mathbf{a}(t)$ is a negative multiple of a unit vector from (0, 0) to $(\cos\omega t, \sin\omega t)$, so $\mathbf{a}(t)$ is directed toward the origin.

51. $8\sqrt{2}$ ft/sec

53. The particle could be changing direction.

55. This is true for uniform circular motion but not true for non-uniform circular motion.

57–59. Proofs **61.** True

63. False. Consider $\mathbf{r}(t) = \langle t^2, -t^2\rangle$. Then $\mathbf{v}(t) = \langle 2t, -2t\rangle$ and $\|\mathbf{v}(t)\| = \sqrt{8t^2}$.

Section 12.4 *(page 852)*

1. The unit tangent vector points in the direction of motion.

3. $\mathbf{T}(1) = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$ **5.** $\mathbf{T}\left(\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

7. $\mathbf{T}(e) = \dfrac{3e\mathbf{i} - \mathbf{j}}{\sqrt{9e^2 + 1}} \approx 0.9926\mathbf{i} - 0.1217\mathbf{j}$

9. $\mathbf{T}(0) = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$

$x = t$
$y = 0$
$z = t$

11. $\mathbf{T}(0) = \dfrac{\sqrt{2}}{2}(\mathbf{j} + \mathbf{k})$

$x = 1$
$y = 3t$
$z = -4 + 3t$

13. $\mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}\left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$

$x = \sqrt{2} - \sqrt{2}t$
$y = \sqrt{2} + \sqrt{2}t$
$z = 4$

15. $\mathbf{N}(2) = \dfrac{\sqrt{5}}{5}(-2\mathbf{i} + \mathbf{j})$

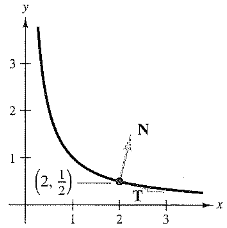
17. $\mathbf{N}(1) = -\dfrac{\sqrt{14}}{14}(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

19. $\mathbf{N}\left(\dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

21. $\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\mathbf{T}(2) = \dfrac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$

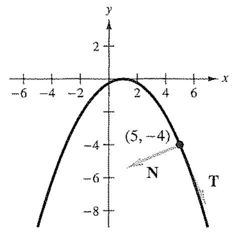
$\mathbf{N}(2) = \dfrac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$



23. $\mathbf{r}(2) = 5\mathbf{i} - 4\mathbf{j}$

$\mathbf{T}(2) = \dfrac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$

$\mathbf{N}(2) = \dfrac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$, perpendicular to $\mathbf{T}(2)$



25. $a_\mathbf{T} = -\sqrt{2}$, $a_\mathbf{N} = \sqrt{2}$ **27.** $a_\mathbf{T} = -\dfrac{7\sqrt{5}}{5}$, $a_\mathbf{N} = \dfrac{6\sqrt{5}}{5}$ **29.** $a_\mathbf{T} = \sqrt{2}e^{\pi/2}$, $a_\mathbf{N} = \sqrt{2}e^{\pi/2}$

31. $\mathbf{T}(t) = -\sin \omega t\mathbf{i} + \cos \omega t\mathbf{j}$
$\mathbf{N}(t) = -\cos \omega t\mathbf{i} - \sin \omega t\mathbf{j}$
$a_\mathbf{T} = 0$
$a_\mathbf{N} = a\omega^2$

33. $\|\mathbf{v}(t)\| = a\omega$; The speed is constant because $a_\mathbf{T} = 0$.

35. $a_\mathbf{T}$ is undefined.
$a_\mathbf{N}$ is undefined.

37. $a_\mathbf{T} = \dfrac{5\sqrt{6}}{6}$, $a_\mathbf{N} = \dfrac{\sqrt{30}}{6}$

39. $a_\mathbf{T} = \sqrt{3}$
$a_\mathbf{N} = \sqrt{2}$

41. The particle's motion is in a straight line.

43. $\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$
$\|\mathbf{v}(t)\| = \sqrt{9 + 16} = 5$
$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{0}$
$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$
$\mathbf{T}'(t) = 0 \implies \mathbf{N}(t)$ does not exist.
The path is a line. The speed is constant (5).

45. (a) $t = \frac{1}{2}$: $a_\mathbf{T} = \dfrac{\sqrt{2}\pi^2}{2}$, $a_\mathbf{N} = \dfrac{\sqrt{2}\pi^2}{2}$
$t = 1$: $a_\mathbf{T} = 0$, $a_\mathbf{N} = \pi^2$
$t = \frac{3}{2}$: $a_\mathbf{T} = -\dfrac{\sqrt{2}\pi^2}{2}$, $a_\mathbf{N} = \dfrac{\sqrt{2}\pi^2}{2}$
(b) $t = \frac{1}{2}$: Increasing because $a_\mathbf{T} > 0$.
$t = 1$: Maximum because $a_\mathbf{T} = 0$.
$t = \frac{3}{2}$: Decreasing because $a_\mathbf{T} < 0$.

47. $\mathbf{T}\left(\dfrac{\pi}{2}\right) = \dfrac{\sqrt{17}}{17}(-4\mathbf{i} + \mathbf{k})$
$\mathbf{N}\left(\dfrac{\pi}{2}\right) = -\mathbf{j}$
$\mathbf{B}\left(\dfrac{\pi}{2}\right) = \dfrac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{k})$

49. $\mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}(\mathbf{j} - \mathbf{k})$
$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}(\mathbf{j} + \mathbf{k})$
$\mathbf{B}\left(\dfrac{\pi}{4}\right) = -\mathbf{i}$

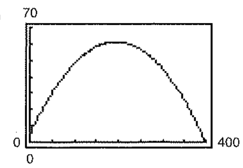
51. $\mathbf{T}\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{5}}{5}\left(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}\right)$
$\mathbf{N}\left(\dfrac{\pi}{3}\right) = -\dfrac{1}{2}\left(\sqrt{3}\mathbf{i} + \mathbf{j}\right)$
$\mathbf{B}\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{5}}{10}\left(\mathbf{i} - \sqrt{3}\mathbf{j} - 4\mathbf{k}\right)$

53. $\mathbf{N}(t) = \dfrac{1}{\sqrt{16t^2 + 9}}(-4t\mathbf{i} + 3\mathbf{j})$

55. $\mathbf{N}(t) = \dfrac{1}{\sqrt{5t^2 + 25}}(-t\mathbf{i} - 2t\mathbf{j} + 5\mathbf{k})$

57. $a_\mathbf{T} = \dfrac{-32(v_0 \sin \theta - 32t)}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$
$a_\mathbf{N} = \dfrac{32v_0 \cos \theta}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$
At maximum height, $a_\mathbf{T} = 0$ and $a_\mathbf{N} = 32$.

59. (a) $\mathbf{r}(t) = 60\sqrt{3}t\mathbf{i} + (5 + 60t - 16t^2)\mathbf{j}$
(b)

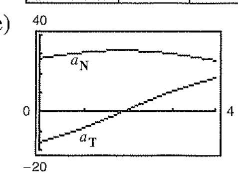


Maximum height $\approx$ 61.245 ft
Range $\approx$ 398.186 ft
(c) $\mathbf{v}(t) = 60\sqrt{3}\mathbf{i} + (60 - 32t)\mathbf{j}$
$\|\mathbf{v}(t)\| = 8\sqrt{16t^2 - 60t + 225}$
$\mathbf{a}(t) = -32\mathbf{j}$
(d)

t	0.5	1.0	1.5
Speed	112.85	107.63	104.61

t	2.0	2.5	3.0
Speed	104	105.83	109.98

(e)



The speed is decreasing when $a_\mathbf{T}$ and $a_\mathbf{N}$ have opposite signs.

61. (a) $4\sqrt{625\pi^2 + 1} \approx 314$ mi/h
(b) $a_\mathbf{T} = 0$, $a_\mathbf{N} = 1000\pi^2$
$a_\mathbf{T} = 0$ because the speed is constant.

63. (a) The centripetal component is quadrupled.
(b) The centripetal component is halved.

65. 4.74 mi/sec **67.** 4.67 mi/sec

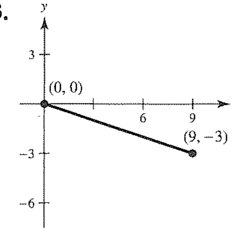
69. False. These vectors are perpendicular for an object traveling at a constant speed but not for an object traveling at a variable speed.

71. (a) and (b) Proofs **73–75.** Proofs

Section 12.5 *(page 864)*

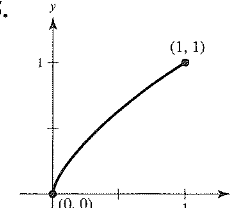
1. The curve bends more sharply at Q than at P.

3.



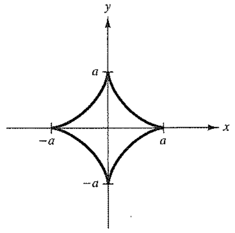
$3\sqrt{10}$

5.



$\dfrac{13\sqrt{13} - 8}{27}$

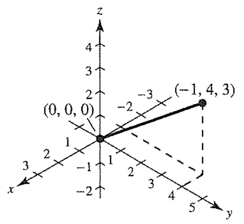
7.



$6a$

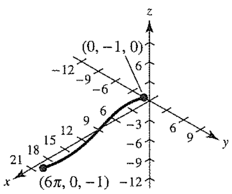
9. 362.9 ft

11.



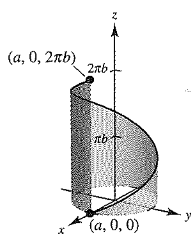
$\sqrt{26}$

13.



$\dfrac{3\sqrt{17}\pi}{2}$

15.



$2\pi\sqrt{a^2 + b^2}$

17. (a) $2\sqrt{21} \approx 9.165$ (b) 9.529
(c) Increase the number of line segments. (d) 9.571

19. 0 **21.** $\dfrac{1}{4}$ **23.** 0 **25.** $\dfrac{\sqrt{2}}{2}$ **27.** 1 **29.** $\dfrac{1}{4}$

31. $\dfrac{1}{a}$ **33.** $\dfrac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$ **35.** $\dfrac{3}{25}$ **37.** $\dfrac{12}{125}$

39. $\dfrac{7\sqrt{26}}{676}$ **41.** $K = 0$, $\dfrac{1}{K}$ is undefined.

43. $K = \dfrac{10}{101^{3/2}}, \dfrac{1}{K} = \dfrac{101^{3/2}}{10}$ **45.** $K = 4, \dfrac{1}{K} = \dfrac{1}{4}$

47. $K = \dfrac{12}{145^{3/2}}, \dfrac{1}{K} = \dfrac{145^{3/2}}{12}$ **49.** (a) $(1, 3)$ (b) 0

51. (a) $K \to \infty$ as $x \to 0$ (No maximum) (b) 0

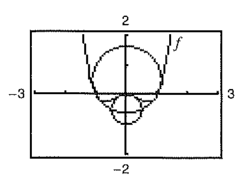
53. (a) $\left(\dfrac{1}{\sqrt{2}}, -\dfrac{\ln 2}{2}\right)$ (b) 0 **55.** $(0, 1)$

57. $(\pi + 2n\pi, 0)$ **59.** $c = \pm\sqrt{2}$

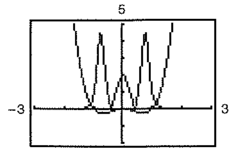
61. (a) $K = \dfrac{2|6x^2 - 1|}{(16x^6 - 16x^4 + 4x^2 + 1)^{3/2}}$

(b) $x = 0$: $x^2 + \left(y + \dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$x = 1$: $x^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{5}{4}$



(c)

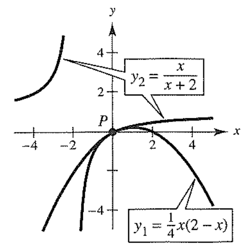


The curvature tends to be greatest near the extrema of the function and decreases as $x \to \pm\infty$. However, f and K do not have the same critical numbers.

Critical numbers of f: $x = 0, \pm\dfrac{\sqrt{2}}{2} \approx \pm 0.7071$

Critical numbers of K: $x = 0, \pm 0.7647, \pm 0.4082$

63. $a = \frac{1}{4}, b = 2$



65. (a) 12.25 units (b) $\frac{1}{2}$ **67–69.** Proofs

71. (a) 0 (b) 0 **73.** $\frac{1}{4}$ **75.** Proof

77. $K = \dfrac{1}{4a}\left|\csc \dfrac{\theta}{2}\right|$

Minimum: $K = \dfrac{1}{4a}$

There is no maximum.

79. 3327.5 lb **81.** Proof

83. False. See Exploration on page 855.

85. True **87–93.** Proofs

Review Exercises for Chapter 12 *(page 867)*

1. (a) All reals except $\dfrac{\pi}{2} + n\pi$, n is an integer.

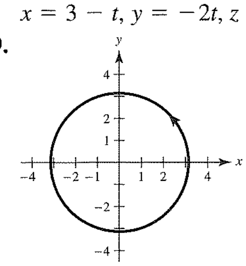
(b) Continuous except at $t = \dfrac{\pi}{2} + n\pi$, n is an integer.

3. (a) $[3, \infty)$ (b) Continuous for all $t \geq 3$

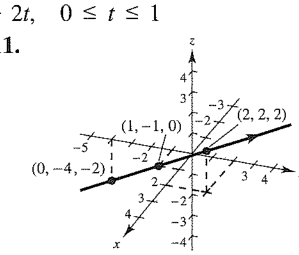
5. (a) $\mathbf{i} - \sqrt{2}\mathbf{k}$ (b) $-3\mathbf{i} + 4\mathbf{j}$
(c) $(2c - 1)\mathbf{i} + (c - 1)^2\mathbf{j} - \sqrt{c + 1}\mathbf{k}$
(d) $2\Delta t\mathbf{i} + \Delta t(\Delta t + 2)\mathbf{j} - \left(\sqrt{\Delta t + 3} - \sqrt{3}\right)\mathbf{k}$

7. $\mathbf{r}(t) = (3 - t)\mathbf{i} - 2t\mathbf{j} + (5 - 2t)\mathbf{k}$, $0 \leq t \leq 1$
$x = 3 - t, y = -2t, z = 5 - 2t$, $0 \leq t \leq 1$

9.

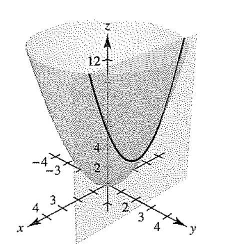


11.



13. $\mathbf{r}(t) = t\mathbf{i} + \left(-\frac{3}{4}t + 3\right)\mathbf{j}$

15.



$x = t, y = 2, z = t^2 + 4$

17. $\ln 3\mathbf{j} - \frac{1}{3}\mathbf{k}$

19. (a) $(2t + 4)\mathbf{i} - 6t\mathbf{j}$
(b) $2\mathbf{i} - 6\mathbf{j}$
(c) $40t + 8$

21. (a) $6t^2\mathbf{i} + 4\mathbf{j} - 2t\mathbf{k}$
(b) $12t\mathbf{i} - 2\mathbf{k}$
(c) $72t^3 + 4t$
(d) $-8\mathbf{i} - 12t^2\mathbf{j} - 48t\mathbf{k}$

23. $(-\infty, 1), (1, \infty)$