MATH 251

Suggested Homework and Answers

From *Linear Algebra: A Modern Introduction, Poole, 3^{rd} Ed*

Repeated multiplication can be handled similarly. The idea is to use the addition and multiplication tables to reduce the result of each calculation to 0, 1, or 2.

Extending these ideas to vectors is straightforward.

Example 1.14

In $\mathbb{Z}_3^5$, let $\mathbf{u} = [2, 2, 0, 1, 2]$ and $\mathbf{v} = [1, 2, 2, 2, 1]$. Then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 2\cdot 1 + 2\cdot 2 + 0\cdot 2 + 1\cdot 2 + 2\cdot 1\\ &= 2 + 1 + 0 + 2 + 2\\ &= 1\end{aligned}$$

Vectors in $\mathbb{Z}_3^5$ are referred to as ***ternary vectors of length*** **5.**

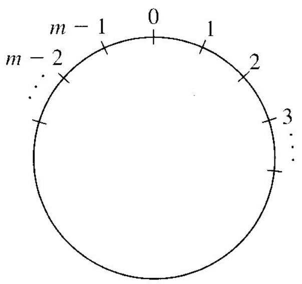

Figure 1.22
Arithmetic modulo m

In general, we have the set $\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}$ of ***integers modulo m*** (corresponding to an m-hour clock, as shown in Figure 1.22). A vector of length n whose entries are in $\mathbb{Z}_m$ is called an ***m-ary vector of length n.*** The set of all m-ary vectors of length n is denoted by $\mathbb{Z}_m^n$.

Exercises 1.1

1. Draw the following vectors in standard position in $\mathbb{R}^2$:

(a) $\mathbf{a} = \begin{bmatrix}3\\0\end{bmatrix}$ **(b)** $\mathbf{b} = \begin{bmatrix}2\\3\end{bmatrix}$

(c) $\mathbf{c} = \begin{bmatrix}-2\\3\end{bmatrix}$ **(d)** $\mathbf{d} = \begin{bmatrix}3\\-2\end{bmatrix}$

2. Draw the vectors in Exercise 1 with their tails at the point $(1, -3)$.

3. Draw the following vectors in standard position in $\mathbb{R}^3$:

(a) $\mathbf{a} = [0, 2, 0]$ **(b)** $\mathbf{b} = [3, 2, 1]$
(c) $\mathbf{c} = [1, -2, 1]$ **(d)** $\mathbf{d} = [-1, -1, -2]$

4. If the vectors in Exercise 3 are translated so that their heads are at the point $(1, 2, 3)$, find the points that correspond to their tails.

5. For each of the following pairs of points, draw the vector $\overrightarrow{AB}$. Then compute and redraw $\overrightarrow{AB}$ as a vector in standard position.

(a) $A = (1, -1), B = (4, 2)$
(b) $A = (0, -2), B = (2, -1)$
(c) $A = (2, \frac{3}{2}), B = (\frac{1}{2}, 3)$
(d) $A = (\frac{1}{3}, \frac{1}{3}), B = (\frac{1}{6}, \frac{1}{2})$

6. A hiker walks 4 km north and then 5 km northeast. Draw displacement vectors representing the hiker's trip and draw a vector that represents the hiker's net displacement from the starting point.

Exercises 7–10 refer to the vectors in Exercise 1. Compute the indicated vectors and also show how the results can be obtained geometrically.

7. $\mathbf{a} + \mathbf{b}$ **8.** $\mathbf{b} + \mathbf{c}$

9. $\mathbf{d} - \mathbf{c}$ **10.** $\mathbf{a} - \mathbf{d}$

Exercises 11 and 12 refer to the vectors in Exercise 3. Compute the indicated vectors.

11. $2\mathbf{a} + 3\mathbf{c}$ **12.** $2\mathbf{c} - 3\mathbf{b} - \mathbf{d}$

13. Find the components of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are as shown in Figure 1.23.

14. In Figure 1.24, A, B, C, D, E, and F are the vertices of a regular hexagon centered at the origin.

Express each of the following vectors in terms of $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$:

(a) $\overrightarrow{AB}$ **(b)** $\overrightarrow{BC}$

(c) $\overrightarrow{AD}$ **(d)** $\overrightarrow{CF}$

(e) $\overrightarrow{AC}$ **(f)** $\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA}$

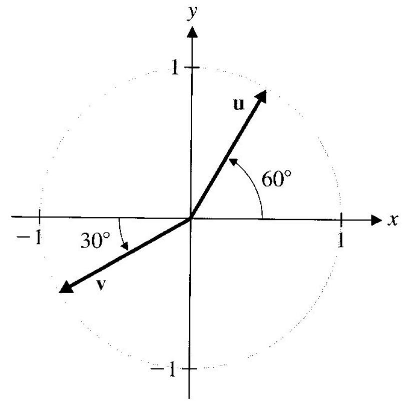

Figure 1.23

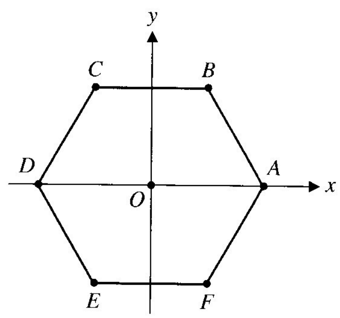

Figure 1.24

In Exercises 15 and 16, simplify the given vector expression. Indicate which properties in Theorem 1.1 you use.

15. $2(\mathbf{a} - 3\mathbf{b}) + 3(2\mathbf{b} + \mathbf{a})$

16. $-3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b})$

In Exercises 17 and 18, solve for the vector **x** *in terms of the vectors* **a** *and* **b**.

17. $\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a})$

18. $\mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b})$

In Exercises 19 and 20, draw the coordinate axes relative to **u** *and* **v** *and locate* **w**.

19. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$

20. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = -\mathbf{u} - 2\mathbf{v}$

In Exercises 21 and 22, draw the standard coordinate axes on the same diagram as the axes relative to **u** *and* **v**. *Use these to find* **w** *as a linear combination of* **u** *and* **v**.

21. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

23. Draw diagrams to illustrate properties (d) and (e) of Theorem 1.1.

24. Give algebraic proofs of properties (d) through (g) of Theorem 1.1.

In Exercises 25–28, **u** *and* **v** *are binary vectors. Find* $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{u} \cdot \mathbf{v}$ *in each case.*

25. $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **26.** $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

27. $\mathbf{u} = [1, 0, 1, 1], \mathbf{v} = [1, 1, 1, 1]$

28. $\mathbf{u} = [1, 1, 0, 1, 0], \mathbf{v} = [0, 1, 1, 1, 0]$

29. Write out the addition and multiplication tables for $\mathbb{Z}_4$.

30. Write out the addition and multiplication tables for $\mathbb{Z}_5$.

In Exercises 31–43, perform the indicated calculations.

31. $2 + 2 + 2$ in $\mathbb{Z}_3$ **32.** $2 \cdot 2 \cdot 2$ in $\mathbb{Z}_3$

33. $2(2 + 1 + 2)$ in $\mathbb{Z}_3$ **34.** $3 + 1 + 2 + 3$ in $\mathbb{Z}_4$

35. $2 \cdot 3 \cdot 2$ in $\mathbb{Z}_4$ **36.** $3(3 + 3 + 2)$ in $\mathbb{Z}_4$

37. $2 + 1 + 2 + 2 + 1$ in $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$

38. $(3 + 4)(3 + 2 + 4 + 2)$ in $\mathbb{Z}_5$

39. $8(6 + 4 + 3)$ in $\mathbb{Z}_9$ **40.** 2^{100} in $\mathbb{Z}_{11}$

41. $[2, 1, 2] + [2, 0, 1]$ in $\mathbb{Z}_3^3$ **42.** $[2, 1, 2] \cdot [2, 2, 1]$ in $\mathbb{Z}_3^3$

43. $[2, 0, 3, 2] \cdot ([3, 1, 1, 2] + [3, 3, 2, 1])$ in $\mathbb{Z}_4^4$ and in $\mathbb{Z}_5^4$

In Exercises 44–55, solve the given equation or indicate that there is no solution.

44. $x + 3 = 2$ in $\mathbb{Z}_5$ **45.** $x + 5 = 1$ in $\mathbb{Z}_6$

46. $2x = 1$ in $\mathbb{Z}_3$ **47.** $2x = 1$ in $\mathbb{Z}_4$

48. $2x = 1$ in $\mathbb{Z}_5$ **49.** $3x = 4$ in $\mathbb{Z}_5$

50. $3x = 4$ in $\mathbb{Z}_6$ **51.** $6x = 5$ in $\mathbb{Z}_8$

52. $8x = 9$ in $\mathbb{Z}_{11}$ **53.** $2x + 3 = 2$ in $\mathbb{Z}_5$

54. $4x + 5 = 2$ in $\mathbb{Z}_6$ **55.** $6x + 3 = 1$ in $\mathbb{Z}_8$

56. **(a)** For which values of a does $x + a = 0$ have a solution in $\mathbb{Z}_5$?
(b) For which values of a and b does $x + a = b$ have a solution in $\mathbb{Z}_6$?
(c) For which values of a, b, and m does $x + a = b$ have a solution in $\mathbb{Z}_m$?

57. **(a)** For which values of a does $ax = 1$ have a solution in $\mathbb{Z}_5$?
(b) For which values of a does $ax = 1$ have a solution in $\mathbb{Z}_6$?
(c) For which values of a and m does $ax = 1$ have a solution in $\mathbb{Z}_m$?

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Exercises 1.2

In Exercises 1–6, find $\mathbf{u}\cdot\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ **2.** $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ CAS **4.** $\mathbf{u} = \begin{bmatrix} 1.5 \\ 0.4 \\ -2.1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3.0 \\ 5.2 \\ -0.6 \end{bmatrix}$

5. $\mathbf{u} = [1, \sqrt{2}, \sqrt{3}, 0], \mathbf{v} = [4, -\sqrt{2}, 0, -5]$

CAS **6.** $\mathbf{u} = [1.12, -3.25, 2.07, -1.83]$,
$\mathbf{v} = [-2.29, 1.72, 4.33, -1.54]$

In Exercises 7–12, find $\|\mathbf{u}\|$ *for the given exercise, and give a unit vector in the direction of* $\mathbf{u}$.

7. Exercise 1 **8.** Exercise 2 **9.** Exercise 3

CAS **10.** Exercise 4 **11.** Exercise 5 CAS **12.** Exercise 6

In Exercises 13–16, find the distance $d(\mathbf{u}, \mathbf{v})$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *in the given exercise.*

13. Exercise 1 **14.** Exercise 2

15. Exercise 3 CAS **16.** Exercise 4

17. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, $n \geq 2$, and c is a scalar, explain why the following expressions make no sense:

(a) $\|\mathbf{u}\cdot\mathbf{v}\|$ (b) $\mathbf{u}\cdot\mathbf{v} + \mathbf{w}$
(c) $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ (d) $c\cdot(\mathbf{u} + \mathbf{w})$

In Exercises 18–23, determine whether the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is acute, obtuse, or a right angle.*

18. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **19.** $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

20. $\mathbf{u} = [5, 4, -3], \mathbf{v} = [1, -2, -1]$

CAS **21.** $\mathbf{u} = [0.9, 2.1, 1.2], \mathbf{v} = [-4.5, 2.6, -0.8]$

22. $\mathbf{u} = [1, 2, 3, 4], \mathbf{v} = [-3, 1, 2, -2]$

23. $\mathbf{u} = [1, 2, 3, 4], \mathbf{v} = [5, 6, 7, 8]$

In Exercises 24–29, find the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *in the given exercise.*

24. Exercise 18 **25.** Exercise 19 **26.** Exercise 20

CAS **27.** Exercise 21 CAS **28.** Exercise 22 CAS **29.** Exercise 23

30. Let $A = (-3, 2)$, $B = (1, 0)$, and $C = (4, 6)$. Prove that ΔABC is a right-angled triangle.

31. Let $A = (1, 1, -1)$, $B = (-3, 2, -2)$, and $C = (2, 2, -4)$. Prove that ΔABC is a right-angled triangle.

CAS **32.** Find the angle between a diagonal of a cube and an adjacent edge.

33. A cube has four diagonals. Show that no two of them are perpendicular.

In Exercises 34–39, find the projection of $\mathbf{v}$ *onto* $\mathbf{u}$. *Draw a sketch in Exercises 34 and 35.*

34. A parallelogram has diagonals determined by the vectors

$$\mathbf{d}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{d}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Show that the parallelogram is a rhombus (all sides of equal length) and determine the side length.

35. The rectangle $ABCD$ has vertices at A = $(1, 2, 3)$, B = $(3, 6, -2)$, and C = $(0, 5, -4)$. Determine the coordinates of vertex D.

36. An airplane heading due east has a velocity of 200 miles per hour. A wind is blowing from the north at 40 miles per hour. What is the resultant velocity of the airplane?

37. A boat heads north across a river at a rate of 4 miles per hour. If the current is flowing east at a rate of 3 miles per hour, find the resultant velocity of the boat.

38. Ann is driving a motorboat across a river that is 2 km wide. The boat has a speed of 20 km/h in still water, and the current in the river is flowing at 5 km/h. Ann heads out from one bank of the river for a dock directly across from her on the opposite bank. She drives the boat in a direction perpendicular to the current.

(a) How far downstream from the dock will Ann land?
(b) How long will it take Ann to cross the river?

39. Bert can swim at a rate of 2 miles per hour in still water. The current in a river is flowing at a rate of 1 mile per hour. If Bert wants to swim across the river to a point directly opposite, at what angle to the bank of the river must he swim?

In Exercises 40–45, find the projection of **v** *onto* **u**. *Draw a sketch in Exercises 40 and 41.*

40. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **41.** $\mathbf{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

42. $\mathbf{u} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ **43.** $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ -1 \\ -2 \end{bmatrix}$

CAS **44.** $\mathbf{u} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2.1 \\ 1.2 \end{bmatrix}$

CAS **45.** $\mathbf{u} = \begin{bmatrix} 3.01 \\ -0.33 \\ 2.52 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1.34 \\ 4.25 \\ -1.66 \end{bmatrix}$

Figure 1.39 suggests two ways in which vectors may be used to compute the area of a triangle. The area $\mathcal{A}$ of

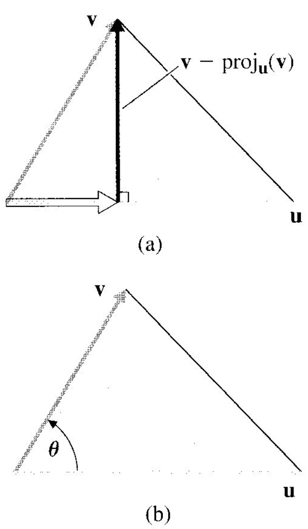


Figure 1.39

the triangle in part (a) is given by $\frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\|$, and part (b) suggests the trigonometric form of the area of a triangle: $\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$ (We can use the identity $\sin\theta = \sqrt{1 - \cos^2\theta}$ to find $\sin\theta$.)

In Exercises 46 and 47, compute the area of the triangle with the given vertices using both methods.

46. $A = (1, -1), B = (2, 2), C = (4, 0)$

47. $A = (3, -1, 4), B = (4, -2, 6), C = (5, 0, 2)$

In Exercises 48 and 49, find all values of the scalar k for which the two vectors are orthogonal.

48. $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} k+1 \\ k-1 \end{bmatrix}$ **49.** $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$

50. Describe all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

51. Describe all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.

52. Under what conditions are the following true for vectors **u** and **v** in $\mathbb{R}^2$ or $\mathbb{R}^3$?

(a) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ **(b)** $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$

53. Prove Theorem 1.2(b).

54. Prove Theorem 1.2(d).

In Exercises 55–57, prove the stated property of distance between vectors.

55. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ for all vectors **u** and **v**

56. $d(\mathbf{u}, \mathbf{w}) \le d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$ for all vectors **u**, **v**, and **w**

57. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

58. Prove that $\mathbf{u} \cdot c\mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ for all vectors **u** and **v** in $\mathbb{R}^n$ and all scalars c.

59. Prove that $\|\mathbf{u} - \mathbf{v}\| \ge \|\mathbf{u}\| - \|\mathbf{v}\|$ for all vectors **u** and **v** in $\mathbb{R}^n$. [*Hint:* Replace **u** by $\mathbf{u} - \mathbf{v}$ in the Triangle Inequality.]

60. Suppose we know that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$. Does it follow that $\mathbf{v} = \mathbf{w}$? If it does, give a proof that is valid in $\mathbb{R}^n$; otherwise, give a *counterexample* (that is, a *specific* set of vectors **u**, **v**, and **w** for which $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ but $\mathbf{v} \ne \mathbf{w}$).

61. Prove that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

62. **(a)** Prove that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

(b) Draw a diagram showing **u**, **v**, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in $\mathbb{R}^2$ and use (a) to deduce a result about parallelograms.

63. Prove that $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

Step 4: The distance $d(B, \mathcal{P})$ from B to $\mathcal{P}$ is

$$\begin{aligned}\|\mathrm{proj}_{\mathbf{n}}(\mathbf{v})\| &= \left|-\tfrac{2}{3}\right| \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| \\ &= \tfrac{2}{3} \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| \\ &= \tfrac{2}{3}\sqrt{3}\end{aligned}$$

In general, the distance $d(B, \mathcal{P})$ from the point $B = (x_0, y_0, z_0)$ to the plane whose general equation is $ax + by + cz = d$ is given by the formula

$$d(B, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (4)$$

You will be asked to derive this formula in Exercise 40.

Exercises 1.3

In Exercises 1 and 2, write the equation of the line passing through P with normal vector **n** *in (a) normal form and (b) general form.*

1. $P = (0, 0), \mathbf{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **2.** $P = (2, 1), \mathbf{n} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

In Exercises 3–6, write the equation of the line passing through P with direction vector **d** *in (a) vector form and (b) parametric form.*

3. $P = (1, 0), \mathbf{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ **4.** $P = (3, -3), \mathbf{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

5. $P = (0, 0, 0), \mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ **6.** $P = (-3, 1, 2), \mathbf{d} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

In Exercises 7 and 8, write the equation of the plane passing through P with normal vector **n** *in (a) normal form and (b) general form.*

7. $P = (0, 1, 0), \mathbf{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ **8.** $P = (-3, 1, 2), \mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

In Exercises 9 and 10, write the equation of the plane passing through P with direction vectors **u** *and* **v** *in (a) vector form and (b) parametric form.*

9. $P = (0, 0, 0), \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

10. $P = (4, -1, 3), \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 11 and 12, give the vector equation of the line passing through P and Q.

11. $P = (1, -2), Q = (3, 0)$

12. $P = (4, -1, 3), Q = (2, 1, 3)$

In Exercises 13 and 14, give the vector equation of the plane passing through P, Q, and R.

13. $P = (1, 1, 1), Q = (4, 0, 2), R = (0, 1, -1)$

14. $P = (1, 0, 0), Q = (0, 1, 0), R = (0, 0, 1)$

15. Find parametric equations and an equation in vector form for the lines in $\mathbb{R}^2$ with the following equations:

(a) $y = 3x - 1$ **(b)** $3x + 2y = 5$

16. Consider the vector equation $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$, where $\mathbf{p}$ and $\mathbf{q}$ correspond to distinct points P and Q in $\mathbb{R}^2$ or $\mathbb{R}^3$.

(a) Show that this equation describes the line segment $\overline{PQ}$ as t varies from 0 to 1.
(b) For which value of t is $\mathbf{x}$ the midpoint of $\overline{PQ}$, and what is $\mathbf{x}$ in this case?
(c) Find the midpoint of $\overline{PQ}$ when $P = (2, -3)$ and $Q = (0, 1)$.
(d) Find the midpoint of $\overline{PQ}$ when $P = (1, 0, 1)$ and $Q = (4, 1, -2)$.
(e) Find the two points that divide $\overline{PQ}$ in part (c) into three equal parts.
(f) Find the two points that divide $\overline{PQ}$ in part (d) into three equal parts.

17. Suggest a "vector proof" of the fact that, in $\mathbb{R}^2$, two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

18. The line ℓ passes through the point $P = (1, -1, 1)$ and has direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. For each of the following planes $\mathcal{P}$, determine whether ℓ and $\mathcal{P}$ are parallel, perpendicular, or neither:

(a) $2x + 3y - z = 1$ (b) $4x - y + 5z = 0$
(c) $x - y - z = 3$ (d) $4x + 6y - 2z = 0$

19. The plane $\mathcal{P}_1$ has the equation $4x - y + 5z = 2$. For each of the planes $\mathcal{P}$ in Exercise 18, determine whether $\mathcal{P}_1$ and $\mathcal{P}$ are parallel, perpendicular, or neither.

20. Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is perpendicular to the line with general equation $2x - 3y = 1$.

21. Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is parallel to the line with general equation $2x - 3y = 1$.

22. Find the vector form of the equation of the line in $\mathbb{R}^3$ that passes through $P = (-1, 0, 3)$ and is perpendicular to the plane with general equation $x - 3y + 2z = 5$.

23. Find the vector form of the equation of the line in $\mathbb{R}^3$ that passes through $P = (-1, 0, 3)$ and is parallel to the line with parametric equations

$$\begin{aligned} x &= 1 - t \\ y &= 2 + 3t \\ z &= -2 - t \end{aligned}$$

24. Find the normal form of the equation of the plane that passes through $P = (0, -2, 5)$ and is parallel to the plane with general equation $6x - y + 2z = 3$.

25. A cube has vertices at the eight points (x, y, z), where each of x, y, and z is either 0 or 1. (See Figure 1.34.)

(a) Find the general equations of the planes that determine the six faces (sides) of the cube.
(b) Find the general equation of the plane that contains the diagonal from the origin to $(1, 1, 1)$ and is perpendicular to the xy-plane.
(c) Find the general equation of the plane that contains the side diagonals referred to in Example 1.22.

26. Find the equation of the set of all points that are equidistant from the points $P = (1, 0, -2)$ and $Q = (5, 2, 4)$.

In Exercises 27 and 28, find the distance from the point Q to the line ℓ.

27. $Q = (2, 2)$, ℓ with equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

28. $Q = (0, 1, 0)$, ℓ with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

In Exercises 29 and 30, find the distance from the point Q to the plane $\mathcal{P}$.

29. $Q = (2, 2, 2)$, $\mathcal{P}$ with equation $x + y - z = 0$

30. $Q = (0, 0, 0)$, $\mathcal{P}$ with equation $x - 2y + 2z = 1$

Figure 1.66 suggests a way to use vectors to locate the point R on ℓ that is closest to Q.

31. Find the point R on ℓ that is closest to Q in Exercise 27.

32. Find the point R on ℓ that is closest to Q in Exercise 28.

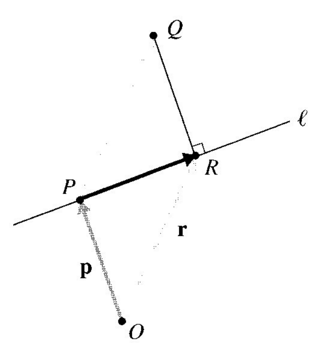


Figure 1.66
$\mathbf{r} = \mathbf{p} + \overrightarrow{PR}$

Figure 1.67 suggests a way to use vectors to locate the point R on $\mathcal{P}$ that is closest to Q.

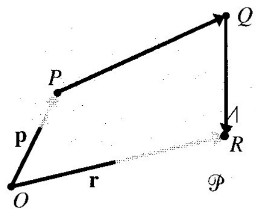


Figure 1.67
$\mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{QR}$

33. Find the point R on $\mathcal{P}$ that is closest to Q in Exercise 29.

34. Find the point R on $\mathcal{P}$ that is closest to Q in Exercise 30.

In Exercises 35 and 36, find the distance between the parallel lines.

35. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

36. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 37 and 38, find the distance between the parallel planes.

37. $2x + y - 2z = 0$ and $2x + y - 2z = 5$

38. $x + y + z = 1$ and $x + y + z = 3$

39. Prove equation (3) on page 43.

40. Prove equation (4) on page 44.

41. Prove that, in $\mathbb{R}^2$, the distance between parallel lines with equations $\mathbf{n}\cdot\mathbf{x} = c_1$ and $\mathbf{n}\cdot\mathbf{x} = c_2$ is given by $\dfrac{|c_1 - c_2|}{\|\mathbf{n}\|}$.

42. Prove that the distance between parallel planes with equations $\mathbf{n}\cdot\mathbf{x} = d_1$ and $\mathbf{n}\cdot\mathbf{x} = d_2$ is given by $\dfrac{|d_1 - d_2|}{\|\mathbf{n}\|}$.

If two nonparallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ and θ is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, then we define the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ to be either θ or $180° - \theta$, whichever is an acute angle. (Figure 1.68)

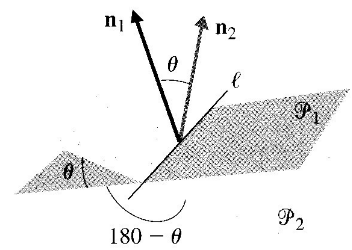


Figure 1.68

In Exercises 43–44, find the acute angle between the planes with the given equations.

43. $x + y + z = 0$ and $2x + y - 2z = 0$

44. $3x - y + 2z = 5$ and $x + 4y - z = 2$

In Exercises 45–46, show that the plane and line with the given equations intersect, and then find the acute angle of intersection between them.

45. The plane given by $x + y + 2z = 0$ and the line given by
$$\begin{aligned} x &= 2 + t \\ y &= 1 - 2t \\ z &= 3 + t \end{aligned}$$

46. The plane given by $4x - y - z = 6$ and the line given by
$$\begin{aligned} x &= t \\ y &= 1 + 2t \\ z &= 2 + 3t \end{aligned}$$

Exercises 47–48 explore one approach to the problem of finding the projection of a vector onto a plane. As Figure 1.69 shows, if $\mathcal{P}$ is a plane through the origin in $\mathbb{R}^3$ with normal vector $\mathbf{n}$, and $\mathbf{v}$ is a vector in $\mathbb{R}^3$, then $\mathbf{p} = \mathrm{proj}_{\mathcal{P}}(\mathbf{v})$ is a vector in $\mathcal{P}$ such that $\mathbf{v} - c\mathbf{n} = \mathbf{p}$ for some scalar c.

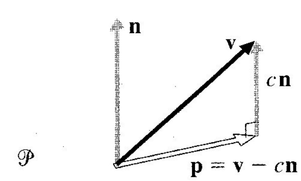


Figure 1.69
Projection onto a plane

Cross Product Part I

1. Find $\mathbf{a} \times \mathbf{b}$:
a) $\mathbf{a} = [0, 1, 1], \mathbf{b} = [3, -1, 2]$
b) $\mathbf{a} = [1, 1, 1], \mathbf{b} = [1, 2, 3]$

2. Find a normal vector for the plane:
a) parallel to $\mathbf{u} = [0, 1, 2]$ and $\mathbf{v} = [1, 1, 4]$
b) through the points $P = (0, -1, 1), Q = (2, 0, 2)$ and $R = (1, 2, -1)$

3. Compute the area of the triangle with vertices $A = (1, 2, 1), B = (2, 1, 0)$ and $C = (5, -1, 3)$.

Answers:

1. a) $[3, 3, -3]$ b) $[1, -2, 1]$

2. a) any nonzero multiple of $[2, 2, -1]$
b) any nonzero multiple of $[-5, 5, 5]$

3. $\frac{\sqrt{62}}{2}$

Cross Product Part II

1. Calculate $\mathbf{a} \times \mathbf{b}$ using cofactor expansion:
$\mathbf{a} = [0, 1, 1], \mathbf{b} = [3, -1, 2]$

2. Find the volume of the parallelepiped (slanted box) determined by $\mathbf{u} = [1, 4, 9], \mathbf{v} = [2, -6, 3]$ and $\mathbf{w} = [-1, 1, 1]$.

3. Determine whether the following vectors lie in a plane: $\mathbf{u} = [1, 7, -2]$, $\mathbf{v} = [2, 3, -2]$ and $\mathbf{w} = [-2, 5, -2]$.

4. Find the area of the parallelogram determined by $\mathbf{u} = [-2, 3]$ and $\mathbf{v} = [4, 9]$.

Answers:

1. $[3, 3, -3]$

2. 65

3. No. The volume of the parallelepiped is nonzero. (The exact volume is 28.)

4. 30

Exercises 2.1

In Exercises 1–6, determine which equations are linear equations in the variables x, y, and z. If any equation is not linear, explain why not.

1. $x - \pi y + \sqrt[3]{5}z = 0$

2. $x^2 + y^2 + z^2 = 1$

3. $x^{-1} + 7y + z = \sin\left(\frac{\pi}{9}\right)$

4. $2x - xy - 5z = 0$

5. $3\cos x - 4y + z = \sqrt{3}$

6. $(\cos 3)x - 4y + z = \sqrt{3}$

In Exercises 7–10, find a linear equation that has the same solution set as the given equation (possibly with some restrictions on the variables).

7. $2x + y = 7 - 3y$

8. $\dfrac{x^2 - y^2}{x - y} = 1$

9. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{4}{xy}$

10. $\log_{10} x - \log_{10} y = 2$

In Exercises 11–14, find the solution set of each equation.

11. $3x - 6y = 0$

12. $2x_1 + 3x_2 = 5$

13. $x + 2y + 3z = 4$

14. $4x_1 + 3x_2 + 2x_3 = 1$

In Exercises 15–18, draw graphs corresponding to the given linear systems. Determine geometrically whether each system has a unique solution, infinitely many solutions, or no solution. Then solve each system algebraically to confirm your answer.

15. $\begin{aligned} x + y &= 0 \\ 2x + y &= 3 \end{aligned}$

16. $\begin{aligned} x - 2y &= 7 \\ 3x + y &= 7 \end{aligned}$

17. $\begin{aligned} 3x - 6y &= 3 \\ -x + 2y &= 1 \end{aligned}$

18. $\begin{aligned} 0.10x - 0.05y &= 0.20 \\ -0.06x + 0.03y &= -0.12 \end{aligned}$

In Exercises 19–24, solve the given system by back substitution.

19. $\begin{aligned} x - 2y &= 1 \\ y &= 3 \end{aligned}$

20. $\begin{aligned} 2u - 3v &= 5 \\ 2v &= 6 \end{aligned}$

21. $\begin{aligned} x - y + z &= 0 \\ 2y - z &= 1 \\ 3z &= -1 \end{aligned}$

22. $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -5x_2 + 2x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$

23. $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 1 \\ x_2 + x_3 + x_4 &= 0 \\ x_3 - x_4 &= 0 \\ x_4 &= 1 \end{aligned}$

24. $\begin{aligned} x - 3y + z &= 5 \\ y - 2z &= -1 \end{aligned}$

The systems in Exercises 25 and 26 exhibit a "lower triangular" pattern that makes them easy to solve by forward substitution. (We will encounter forward substitution again in Chapter 3.) Solve these systems.

25. $\begin{aligned} x &= 2 \\ 2x + y &= -3 \\ -3x - 4y + z &= -10 \end{aligned}$

26. $\begin{aligned} x_1 &= -1 \\ -\tfrac{1}{2}x_1 + x_2 &= 5 \\ \tfrac{3}{2}x_1 + 2x_2 + x_3 &= 7 \end{aligned}$

Find the augmented matrices of the linear systems in Exercises 27–30.

27. $\begin{aligned} x - y &= 0 \\ 2x + y &= 3 \end{aligned}$

28. $\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + x_3 &= 0 \\ -x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}$

29. $\begin{aligned} x + 5y &= -1 \\ -x + y &= -5 \\ 2x + 4y &= 4 \end{aligned}$

30. $\begin{aligned} a - 2b + d &= 2 \\ -a + b - c - 3d &= 1 \end{aligned}$

In Exercises 31 and 32, find a system of linear equations that has the given matrix as its augmented matrix.

31. $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right]$

32. $\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right]$

For Exercises 33–38, solve the linear systems in the given exercises.

33. Exercise 27

34. Exercise 28

35. Exercise 29

36. Exercise 30

37. Exercise 31

38. Exercise 32

39. **(a)** Find a system of two linear equations in the variables x and y whose solution set is given by the parametric equations $x = t$ and $y = 3 - 2t$.

(b) Find another parametric solution to the system in part (a) in which the parameter is s and $y = s$.

40. **(a)** Find a system of two linear equations in the variables x_1, x_2, and x_3 whose solution set is given by the parametric equations $x_1 = t$, $x_2 = 1 + t$, and $x_3 = 2 - t$.

(b) Find another parametric solution to the system in part (a) in which the parameter is s and $x_3 = s$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1+t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Since t can take on the two values 0 and 1, there are exactly two solutions:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Remark For linear systems over $\mathbb{Z}_p$, there can never be infinitely many solutions. (Why not?) Rather, when there is more than one solution, the number of solutions is finite and is a function of the number of free variables and p. (See Exercise 59.)

Exercises 2.2

In Exercises 1–8, determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 9–14, use elementary row operations to reduce the given matrix to (a) row echelon form and (b) reduced row echelon form.

9. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 5 \\ 5 & -2 \\ 2 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -4 & -2 & 6 \\ 3 & -6 & 2 & 6 \end{bmatrix}$

13. $\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix}$

14. $\begin{bmatrix} -2 & 6 & -7 \\ 3 & -9 & 10 \\ 1 & -3 & 3 \end{bmatrix}$

15. Reverse the elementary row operations used in Example 2.9 to show that we can convert

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \quad \text{into}$$

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

16. In general, what is the elementary row operation that "undoes" each of the three elementary row operations $R_i \leftrightarrow R_j$, kR_i, and $R_i + kR_j$?

In Exercises 17 and 18, show that the given matrices are row equivalent and find a sequence of elementary row operations that will convert A into B.

17. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$

18. $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{bmatrix}$

19. What is wrong with the following "proof" that every matrix with at least two rows is row equivalent to a matrix with a zero row?

Perform $R_2 + R_1$ and $R_1 + R_2$. Now rows 1 and 2 are identical. Now perform $R_2 - R_1$ to obtain a row of zeros in the second row.

20. What is the net effect of performing the following sequence of elementary row operations on a matrix (with at least two rows)?

$$R_2 + R_1, R_1 - R_2, R_2 + R_1, -R_1$$

21. Students frequently perform the following type of calculation to introduce a zero into a matrix:

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{3R_2 - 2R_1} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}$$

However, $3R_2 - 2R_1$ is *not* an elementary row operation. Why not? Show how to achieve the same result using elementary row operations.

22. Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. Show that any of the three types of elementary row operations can be used to create a leading 1 at the top of the first column. Which do you prefer and why?

23. What is the rank of each of the matrices in Exercises 1–8?

24. What are the possible reduced row echelon forms of 3×3 matrices?

In Exercises 25–34, solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

25.
$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 4 \end{aligned}$$

26.
$$\begin{aligned} x + 2y &= -1 \\ 2x + y + z &= 1 \\ -x + y - z &= -1 \end{aligned}$$

27.
$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 4x_2 + 6x_3 &= 0 \end{aligned}$$

28.
$$\begin{aligned} 3w + 3x + y &= 1 \\ 2w + x + y + z &= 1 \\ 2w + 3x + y - z &= 2 \end{aligned}$$

29.
$$\begin{aligned} 2r + s &= 3 \\ 4r + s &= 7 \\ 2r + 5s &= -1 \end{aligned}$$

30.
$$\begin{aligned} -x_1 + 3x_2 - 2x_3 + 4x_4 &= 2 \\ 2x_1 - 6x_2 + x_3 - 2x_4 &= -1 \\ x_1 - 3x_2 + 4x_3 - 8x_4 &= -4 \end{aligned}$$

31.
$$\begin{aligned} \tfrac{1}{2}x_1 + x_2 - x_3 - 6x_4 &= 2 \\ \tfrac{1}{6}x_1 + \tfrac{1}{2}x_2 - 3x_4 + x_5 &= -1 \\ \tfrac{1}{3}x_1 - 2x_3 - 4x_5 &= 8 \end{aligned}$$

32.
$$\begin{aligned} \sqrt{2}x + y + 2z &= 1 \\ \sqrt{2}y - 3z &= -\sqrt{2} \\ -y + \sqrt{2}z &= 1 \end{aligned}$$

33.
$$\begin{aligned} w + x + 2y + z &= 1 \\ w - x - y + z &= 0 \\ x + y &= -1 \\ w + x + z &= 2 \end{aligned}$$

34.
$$\begin{aligned} a + b + c + d &= 10 \\ a + 2b + 3c + 4d &= 30 \\ a + 3b + 6c + 10d &= 65 \\ a + 4b + 8c + 15d &= 93 \end{aligned}$$

In Exercises 35–38, determine by inspection (i.e., without performing any calculations) whether a linear system with the given augmented matrix has a unique solution, infinitely many solutions, or no solution. Justify your answers.

35. $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$

36. $\left[\begin{array}{cccc|c} 3 & -2 & 0 & 1 & 1 \\ 1 & 2 & -3 & 1 & -1 \\ 2 & 4 & -6 & 2 & 0 \end{array}\right]$

37. $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \\ 9 & 10 & 11 & 12 & 0 \end{array}\right]$

38. $\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 7 & 7 & 7 & 7 & 7 \end{array}\right]$

39. Show that if $ad - bc \neq 0$, then the system

$$\begin{aligned} ax + by &= r \\ cx + dy &= s \end{aligned}$$

has a unique solution.

In Exercises 40–43, for what value(s) of k, if any, will the systems have (a) no solution, (b) a unique solution, and (c) infinitely many solutions?

40. $\begin{aligned} kx + y &= -2 \\ 2x - 2y &= 4 \end{aligned}$

41. $\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$

42. $\begin{aligned} x + y + z &= 2 \\ x + 4y - z &= k \\ 2x - y + 4z &= k^2 \end{aligned}$

43. $\begin{aligned} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= -2 \end{aligned}$

44. Give examples of homogeneous systems of m linear equations in n variables with $m = n$ and with $m > n$ that have (a) infinitely many solutions and (b) a unique solution.

In Exercises 45 and 46, find the line of intersection of the given planes.

45. $3x + 2y + z = -1$ and $2x - y + 4z = 5$

46. $4x + y - z = 0$ and $2x - y + 3z = 4$

47. **(a)** Give an example of three planes that have a common line of intersection (Figure 2.4).

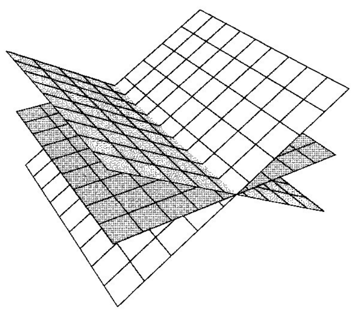


Figure 2.4

(b) Give an example of three planes that intersect in pairs but have no common point of intersection (Figure 2.5).

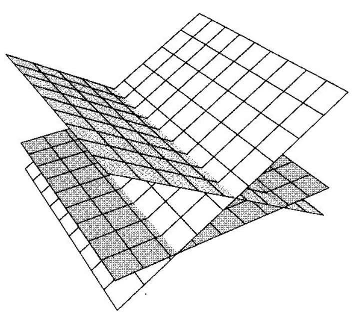


Figure 2.5

(c) Give an example of three planes, exactly two of which are parallel (Figure 2.6).

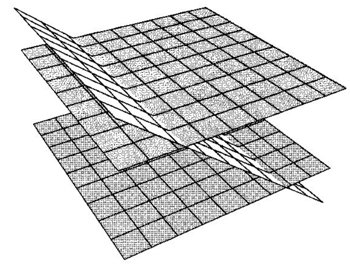


Figure 2.6

(d) Give an example of three planes that intersect in a single point (Figure 2.7).

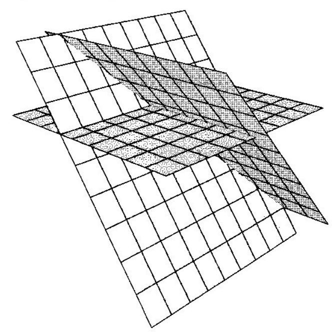


Figure 2.7

In Exercises 48 and 49, determine whether the lines $\mathbf{x} = \mathbf{p} + s\mathbf{u}$ *and* $\mathbf{x} = \mathbf{q} + t\mathbf{v}$ *intersect and, if they do, find their point of intersection.*

48. $\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

49. $\mathbf{p} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

50. Let $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Describe all points $Q = (a, b, c)$ such that the line through Q with direction vector $\mathbf{v}$ intersects the line with equation $\mathbf{x} = \mathbf{p} + s\mathbf{u}$.

51. Recall that the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector $\mathbf{u} \times \mathbf{v}$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. (See Exploration: The Cross Product in Chapter 1.) If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

relabel the vectors, if necessary, so that we can write $\mathbf{v}_m = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{m-1}\mathbf{v}_{m-1}$. Then the elementary row operations $R_m - c_1R_1$, $R_m - c_2R_2, \ldots, R_m - c_{m-1}R_{m-1}$ applied to A will create a zero row in row m. Thus, $\operatorname{rank}(A) < m$.

Conversely, assume that $\operatorname{rank}(A) < m$. Then there is some sequence of row operations that will create a zero row. A successive substitution argument analogous to that used in Example 2.25 can be used to show that $\mathbf{0}$ is a nontrivial linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Thus, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent.

In some situations, we can deduce that a set of vectors is linearly dependent without doing any work. One such situation is when the zero vector is in the set (as in Example 2.22). Another is when there are "too many" vectors to be independent. The following theorem summarizes this case. (We will see a sharper version of this result in Chapter 6.)

Theorem 2.8 Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.

Proof Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (column) vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_m]$ with these vectors as its columns. By Theorem 2.6, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A \mid \mathbf{0}]$ has a nontrivial solution. But, according to Theorem 2.6, this will always be the case if A has more columns than rows; it is the case here, since number of columns m is greater than number of rows n.

Example 2.26

The vectors $\begin{bmatrix}1\\3\end{bmatrix}$, $\begin{bmatrix}2\\4\end{bmatrix}$, and $\begin{bmatrix}3\\1\end{bmatrix}$ are linearly dependent, since there cannot be more than two linearly independent vectors in $\mathbb{R}^2$. (Note that if we want to find the actual dependence relation among these three vectors, we must solve the homogeneous system whose coefficient matrix has the given vectors as columns. Do this!)

Exercises 2.3

In Exercises 1–6, determine if the vector $\mathbf{v}$ *is a linear combination of the remaining vectors.*

1. $\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\-1\end{bmatrix}$

2. $\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}-1\\3\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\-6\end{bmatrix}$

3. $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

4. $\mathbf{v} = \begin{bmatrix}3\\1\\-2\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

5. $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix},$
$\mathbf{u}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$

CAS **6.** $\mathbf{v} = \begin{bmatrix}2.2\\4.0\\-2.2\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1.0\\0.5\\-0.5\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2.4\\1.2\\3.1\end{bmatrix},$
$\mathbf{u}_3 = \begin{bmatrix}1.2\\-2.3\\4.8\end{bmatrix}$

In Exercises 7 and 8, determine if the vector **b** *is in the span of the columns of the matrix A.*

7. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$

9. Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

10. Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$.

11. Show that $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

12. Show that $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$.

In Exercises 13–16, describe the span of the given vectors (a) geometrically and (b) algebraically.

13. $\begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

14. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

15. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

17. The general equation of the plane that contains the points $(1, 0, 3)$, $(-1, 1, -3)$, and the origin is of the form $ax + by + cz = 0$. Solve for a, b, and c.

18. Prove that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all in $\text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

19. Prove that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all in $\text{span}(\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w})$.

20. **(a)** Prove that if $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are vectors in $\mathbb{R}^n$, $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$, and $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$, then $\text{span}(S) \subseteq \text{span}(T)$. [*Hint:* Rephrase this question in terms of linear combinations.]

(b) Deduce that if $\mathbb{R}^n = \text{span}(S)$, then $\mathbb{R}^n = \text{span}(T)$ also.

21. **(a)** Suppose that vector $\mathbf{w}$ is a linear combination of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and that each $\mathbf{u}_i$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Prove that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ and therefore $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) \subseteq \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

(b) In part (a), suppose in addition that each $\mathbf{v}_j$ is also a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Prove that $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

(c) Use the result of part (b) to prove that

$$\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

[*Hint:* We know that $\mathbb{R}^3 = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.]

Use the method of Example 2.23 and Theorem 2.6 to determine if the sets of vectors in Exercises 22–31 are linearly independent. If, for any of these, the answer can be determined by inspection (i.e., without calculation), state why. For any sets that are linearly dependent, find a dependence relationship among the vectors.

22. $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

23. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

24. $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$

25. $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

26. $\begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

27. $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

28. $\begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$

29. $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

30. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

31. $\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

In Exercises 32–41, determine if the sets of vectors in the given exercise are linearly independent by converting the *vectors to row vectors and using the method of Example 2.25 and Theorem 2.7. For any sets that are linearly dependent, find a dependence relationship among the vectors.*

32. Exercise 22 **33.** Exercise 23

34. Exercise 24 **35.** Exercise 25

36. Exercise 26 **37.** Exercise 27

38. Exercise 28 **39.** Exercise 29

40. Exercise 30 **41.** Exercise 31

42. **(a)** If the columns of an $n \times n$ matrix A are linearly independent as vectors in $\mathbb{R}^n$, what is the rank of A? Explain.

(b) If the rows of an $n \times n$ matrix A are linearly independent as vectors in $\mathbb{R}^n$, what is the rank of A? Explain.

43. **(a)** If vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, will $\mathbf{u} + \mathbf{v}$, $\mathbf{v} + \mathbf{w}$, and $\mathbf{u} + \mathbf{w}$ also be linearly independent? Justify your answer.

(b) If vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, will $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{u} - \mathbf{w}$ also be linearly independent? Justify your answer.

44. Prove that two vectors are linearly dependent if and only if one is a scalar multiple of the other. [*Hint:* Separately consider the case where one of the vectors is $\mathbf{0}$.]

45. Give a "row vector proof" of Theorem 2.8.

46. Prove that every subset of a linearly independent set is linearly independent.

47. Suppose that $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}\}$ is a set of vectors in some $\mathbb{R}^n$ and that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. If $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, prove that $\operatorname{span}(S) = \operatorname{span}(S')$. [*Hint:* Exercise 21(b) is helpful here.]

48. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$, and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Suppose that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ with $c_1 \neq 0$. Prove that $\{\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

2.4 Applications

There are too many applications of systems of linear equations to do them justice in a single section. This section will introduce a few applications, to illustrate the diverse settings in which they arise.

Allocation of Resources

A great many applications of systems of linear equations involve allocating limited resources subject to a set of constraints.

Example 2.27 A biologist has placed three strains of bacteria (denoted I, II, and III) in a test tube, where they will feed on three different food sources (A, B, and C). Each day 2300 units of A, 800 units of B, and 1500 units of C are placed in the test tube, and each bacterium consumes a certain number of units of each food per day, as shown in Table 2.2. How many bacteria of each strain can coexist in the test tube and consume all of the food?

Table 2.2

	Bacteria Strain I	Bacteria Strain II	Bacteria Strain III
Food A	2	2	4
Food B	1	2	0
Food C	1	3	1

Table 2.7

	Bacteria Strain I	Bacteria Strain II	Bacteria Strain III
Food A	1	2	0
Food B	2	1	3
Food C	1	1	1

strain can coexist in the test tube and consume all of the food?

3. A florist offers three sizes of flower arrangements containing roses, daisies, and chrysanthemums. Each small arrangement contains one rose, three daisies, and three chrysanthemums. Each medium arrangement contains two roses, four daisies, and six chrysanthemums. Each large arrangement contains four roses, eight daisies, and six chrysanthemums. One day, the florist noted that she used a total of 24 roses, 50 daisies, and 48 chrysanthemums in filling orders for these three types of arrangements. How many arrangements of each type did she make?

4. **(a)** In your pocket you have some nickels, dimes, and quarters. There are 20 coins altogether and exactly twice as many dimes as nickels. The total value of the coins is \$3.00. Find the number of coins of each type.
(b) Find *all* possible combinations of 20 coins (nickels, dimes, and quarters) that will make exactly \$3.00.

5. A coffee merchant sells three blends of coffee. A bag of the house blend contains 300 grams of Colombian beans and 200 grams of French roast beans. A bag of the special blend contains 200 grams of Colombian beans, 200 grams of Kenyan beans, and 100 grams of French roast beans. A bag of the gourmet blend contains 100 grams of Colombian beans, 200 grams of Kenyan beans, and 200 grams of French roast beans. The merchant has on hand 30 kilograms of Colombian beans, 15 kilograms of Kenyan beans, and 25 kilograms of French roast beans. If he wishes to use up all of the beans, how many bags of each type of blend can be made?

6. Redo Exercise 5, assuming that the house blend contains 300 grams of Colombian beans, 50 grams of Kenyan beans, and 150 grams of French roast beans and the gourmet blend contains 100 grams of Colombian beans, 350 grams of Kenyan beans, and 50 grams of French roast beans. This time the merchant has on hand 30 kilograms of Colombian beans, 15 kilograms of Kenyan beans, and 15 kilograms of French roast beans. Suppose one bag of the house blend produces a profit of \$0.50, one bag of the special blend produces a profit of \$1.50, and one bag of the gourmet blend produces a profit of \$2.00. How many bags of each type should the merchant prepare if he wants to use up all of the beans *and* maximize his profit? What is the maximum profit?

Balancing Chemical Equations

In Exercises 7–14, balance the chemical equation for each reaction.

7. $FeS_2 + O_2 \longrightarrow Fe_2O_3 + SO_2$

8. $CO_2 + H_2O \longrightarrow C_6H_{12}O_6 + O_2$ (This reaction takes place when a green plant converts carbon dioxide and water to glucose and oxygen during photosynthesis.)

9. $C_4H_{10} + O_2 \longrightarrow CO_2 + H_2O$ (This reaction occurs when butane, C_4H_{10}, burns in the presence of oxygen to form carbon dioxide and water.)

10. $C_7H_6O_2 + O_2 \longrightarrow H_2O + CO_2$

11. $C_5H_{11}OH + O_2 \longrightarrow H_2O + CO_2$ (This equation represents the combustion of amyl alcohol.)

12. $HClO_4 + P_4O_{10} \longrightarrow H_3PO_4 + Cl_2O_7$

13. $Na_2CO_3 + C + N_2 \longrightarrow NaCN + CO$

CAS **14.** $C_2H_2Cl_4 + Ca(OH)_2 \longrightarrow C_2HCl_3 + CaCl_2 + H_2O$

Network Analysis

15. Figure 2.18 shows a network of water pipes with flows measured in liters per minute.
(a) Set up and solve a system of linear equations to find the possible flows.
(b) If the flow through AB is restricted to 5 L/min, what will the flows through the other two branches be?
(c) What are the minimum and maximum possible flows through each branch?
(d) We have been assuming that flow is always *positive.* What would *negative* flow mean, assuming we allowed it? Give an illustration for this example.

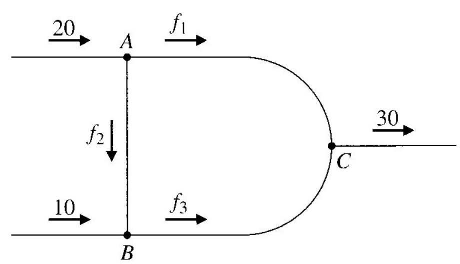


Figure 2.18

39. There are two fields whose total area is 1800 square yards. One field produces grain at the rate of $\frac{2}{3}$ bushel per square yard; the other field produces grain at the rate of $\frac{1}{2}$ bushel per square yard. If the total yield is 1100 bushels, what is the size of each field?

Over 2000 years ago, the Chinese developed methods for solving systems of linear equations, including a version of Gaussian elimination that did not become well known in Europe until the 19th century. (There is no evidence that Gauss was aware of the Chinese methods when he developed what we now call Gaussian elimination. However, it is clear that the Chinese knew the essence of the method, even though they did not justify its use.) The following problem is taken from the Chinese text Jiuzhang suanshu (Nine Chapters in the Mathematical Art)*, written during the early Han Dynasty, about 200* B.C.

40. There are three types of corn. Three bundles of the first type, two of the second, and one of the third make 39 measures. Two bundles of the first type, three of the second, and one of the third make 34 measures. And one bundle of the first type, two of the second, and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

41. Describe all possible values of a, b, c, and d that will make each of the following a valid addition table. [Problems 41–44 are based on the article "An Application of Matrix Theory" by Paul Glaister in *The Mathematics Teacher,* 85 (1992), pp. 220–223.]

(a)

+	a	b
c	2	3
d	4	5

(b)

+	a	b
c	3	6
d	4	5

42. What conditions on w, x, y, and z will guarantee that we can find a, b, c, and d so that the following is a valid addition table?

+	a	b
c	w	x
d	y	z

43. Describe all possible values of a, b, c, d, e, and f that will make each of the following a valid addition table.

(a)

+	a	b	c
d	3	2	1
e	5	4	3
f	4	3	1

(b)

+	a	b	c
d	1	2	3
e	3	4	5
f	4	5	6

44. Generalizing Exercise 42, find conditions on the entries of a 3×3 addition table that will guarantee that we can solve for a, b, c, d, e, and f as previously.

45. From elementary geometry we know that there is a unique straight line through any two points in a plane. Less well known is the fact that there is a unique parabola through any *three* noncollinear points in a plane. For each set of points below, find a parabola with an equation of the form $y = ax^2 + bx + c$ that passes through the given points. (Sketch the resulting parabola to check the validity of your answer.)

(a) $(0, 1), (-1, 4)$, and $(2, 1)$
(b) $(-3, 1), (-2, 2)$, and $(-1, 5)$

46. Through any three noncollinear points there also passes a unique circle. Find the circles (whose general equations are of the form $x^2 + y^2 + ax + by + c = 0$) that pass through the sets of points in Exercise 45. (To check the validity of your answer, find the center and radius of each circle and draw a sketch.)

The process of adding rational functions (ratios of polynomials) by placing them over a common denominator is the analogue of adding rational numbers. The reverse process of taking a rational function apart by writing it as a sum of simpler rational functions is useful in several areas of mathematics; for example, it arises in calculus when we need to integrate a rational function and in discrete mathematics when we use generating functions to solve recurrence relations. The decomposition of a rational function as a sum of partial fractions leads to a system of linear equations. In Exercises 47–50, find the partial fraction decomposition of the given form. (The capital letters denote constants.)

47. $\dfrac{3x + 1}{x^2 + 2x - 3} = \dfrac{A}{x - 1} + \dfrac{B}{x + 3}$

48. $\dfrac{x^2 - 3x + 3}{x^3 + 2x^2 + x} = \dfrac{A}{x} + \dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2}$

CAS **49.** $\dfrac{x - 1}{(x + 1)(x^2 + 1)(x^2 + 4)}$
$= \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 + 1} + \dfrac{Dx + E}{x^2 + 4}$

CAS **50.** $\dfrac{x^3 + x + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3} = \dfrac{A}{x} + \dfrac{B}{x - 1}$
$+ \dfrac{Cx + D}{x^2 + x + 1} + \dfrac{Ex + F}{x^2 + 1} + \dfrac{Gx + H}{(x^2 + 1)^2} + \dfrac{Ix + J}{(x^2 + 1)^3}$

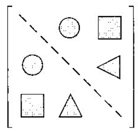


Figure 3.1
A symmetric matrix

Then A is symmetric, since $A^T = A$; but B is not symmetric, since $B^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \neq B$.

A symmetric matrix has the property that it is its own "mirror image" across its main diagonal. Figure 3.1 illustrates this property for a 3×3 matrix. The corresponding shapes represent equal entries; the diagonal entries (those on the dashed line) are arbitrary.

A componentwise definition of a symmetric matrix is also useful. It is simply the algebraic description of the "reflection" property.

A square matrix A is symmetric if and only if $A_{ij} = A_{ji}$ for all i and j.

Exercises 3.1

Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}, \quad E = [4 \quad 2], \quad F = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

In Exercises 1–16, compute the indicated matrices (if possible).

1. $A + 2D$ **2.** $2D - 5A$

3. $B - C$ **4.** $B - C^T$

5. AB **6.** B^2

7. $D + BC$ **8.** B^TB

9. $E(AF)$ **10.** $F(AF)$

11. FE **12.** EF

13. $B^TC^T - (CB)^T$ **14.** $DA - AD$

15. A^3 **16.** $(I_2 - A)^2$

17. Give an example of a nonzero 2×2 matrix A such that $A^2 = O$.

18. Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$. Find 2×2 matrices B and C such that $AB = AC$ but $B \neq C$.

19. A factory manufactures three products (doohickies, gizmos, and widgets) and ships them to two warehouses for storage. The number of units of each product shipped to each warehouse is given by the matrix

$$A = \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix}$$

(where a_{ij} is the number of units of product i sent to warehouse j and the products are taken in alphabetical order). The cost of shipping one unit of each product by truck is \$1.50 per doohickey, \$1.00 per gizmo, and \$2.00 per widget. The corresponding unit costs to ship by train are \$1.75, \$1.50, and \$1.00. Organize these costs into a matrix B and then use matrix multiplication to show how the factory can compare the cost of shipping its products to each of the two warehouses by truck and by train.

20. Referring to Exercise 19, suppose that the unit cost of distributing the products to stores is the same for each product but varies by warehouse because of the distances involved. It costs \$0.75 to distribute one unit from warehouse 1 and \$1.00 to distribute one unit from warehouse 2. Organize these costs into a matrix C and then use matrix multiplication to compute the total cost of distributing each product.

In Exercises 21–22, write the given system of linear equations as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

21. $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 - 5x_3 &= 4 \end{aligned}$

22. $\begin{aligned} -x_1 \quad\quad + 2x_3 &= 1 \\ x_1 - x_2 \quad\quad &= -2 \\ x_2 + x_3 &= -1 \end{aligned}$

In Exercises 23–28, let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix}$$

23. Use the matrix-column representation of the product to write each column of AB as a linear combination of the columns of A.

24. Use the row-matrix representation of the product to write each row of AB as a linear combination of the rows of B.

25. Compute the outer product expansion of AB.

26. Use the matrix-column representation of the product to write each column of BA as a linear combination of the columns of B.

27. Use the row-matrix representation of the product to write each row of BA as a linear combination of the rows of A.

28. Compute the outer product expansion of BA.

In Exercises 29 and 30, assume that the product AB makes sense.

29. Prove that if the columns of B are linearly dependent, then so are the columns of AB.

30. Prove that if the rows of A are linearly dependent, then so are the rows of AB.

In Exercises 31–34, compute AB by block multiplication, using the indicated partitioning.

31. $A = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 3 \end{array}\right]$, $B = \left[\begin{array}{cc|c} 2 & 3 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$

32. $A = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$, $B = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 5 & 4 \\ -2 & 3 & 2 \end{array}\right]$

33. $A = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$, $B = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

34. $A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 4 \end{array}\right]$, $B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & -1 \end{array}\right]$

35. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Compute $A^2, A^3, \ldots, A^7$.

(b) What is A^{2001}? Why?

36. Let $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Find, with justification, B^{2011}.

37. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find a formula for A^n $(n \geq 1)$ and verify your formula using mathematical induction.

38. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) Show that $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$.

(b) Prove, by mathematical induction, that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \text{ for } n \geq 1$$

39. In each of the following, find the 4×4 matrix $A = [a_{ij}]$ that satisfies the given condition:

(a) $a_{ij} = (-1)^{i+j}$ **(b)** $a_{ij} = j - i$

(c) $a_{ij} = (i-1)^j$ **(d)** $a_{ij} = \sin\left(\dfrac{(i+j-1)\pi}{4}\right)$

40. In each of the following, find the 6×6 matrix $A = [a_{ij}]$ that satisfies the given condition:

(a) $a_{ij} = \begin{cases} i+j & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$ **(b)** $a_{ij} = \begin{cases} 1 & \text{if } |i-j| \leq 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$

(c) $a_{ij} = \begin{cases} 1 & \text{if } 6 \leq i+j \leq 8 \\ 0 & \text{otherwise} \end{cases}$

41. Prove Theorem 3.1(a).

The next theorem says that the results of Example 3.21 are true in general.

Theorem 3.5 a. If A is a square matrix, then $A + A^T$ is a symmetric matrix.
b. For any matrix A, AA^T and A^TA are symmetric matrices.

Proof We prove (a) and leave proving (b) as Exercise 34. We simply check that

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

(using properties of the transpose and the commutativity of matrix addition). Thus, $A + A^T$ is equal to its own transpose and so, by definition, is symmetric.

Exercises 3.2

In Exercises 1–4, solve the equation for X, given that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$.

1. $X - 2A + 3B = O$

2. $3X = A - 2B$

3. $2(A + 2B) = 3X$

4. $2(A - B + 2X) = 3(X - B)$

In Exercises 5–8, write B as a linear combination of the other matrices, if possible.

5. $B = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

6. $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

7. $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

8. $B = \begin{bmatrix} 6 & -2 & 5 \\ -2 & 8 & 6 \\ 5 & 6 & 6 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

$A_4 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

In Exercises 9–12, find the general form of the span of the indicated matrices, as in Example 3.17.

9. $\text{span}(A_1, A_2)$ in Exercise 5

10. $\text{span}(A_1, A_2, A_3)$ in Exercise 6

11. $\text{span}(A_1, A_2, A_3)$ in Exercise 7

12. $\text{span}(A_1, A_2, A_3, A_4)$ in Exercise 8

In Exercises 13–16, determine whether the given matrices are linearly independent.

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 0 & 1 \\ 5 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 1 & 9 \\ 4 & 5 \end{bmatrix}$

16. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 9 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

17. Prove Theorem 3.2(a)–(d).

18. Prove Theorem 3.2(e)–(h).

19. Prove Theorem 3.3(c).

20. Prove Theorem 3.3(d).

21. Prove the half of Theorem 3.3(e) that was not proved in the text.

22. Prove that, for square matrices A and B, $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$.

In Exercises 23–25, if $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *find conditions on a, b, c, and d such that* $AB = BA$.

23. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **24.** $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ **25.** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

26. Find conditions on a, b, c, and d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

27. Find conditions on a, b, c, and d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with every 2×2 matrix.

28. Prove that if AB and BA are both defined, then AB and BA are both square matrices.

A square matrix is called ***upper triangular*** *if all of the entries below the main diagonal are zero. Thus, the form of an upper triangular matrix is*

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

where the entries marked $*$ are arbitrary. A more formal definition of such a matrix $A = [a_{ij}]$ *is that* $a_{ij} = 0$ *if* $i > j$.

29. Prove that the product of two upper triangular $n \times n$ matrices is upper triangular.

30. Prove Theorem 3.4(a)–(c).

31. Prove Theorem 3.4(e).

32. Using induction, prove that for all $n \geq 1$, $(A_1 + A_2 + \cdots + A_n)^T = A_1^T + A_2^T + \cdots + A_n^T$.

33. Using induction, prove that for all $n \geq 1$, $(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$.

34. Prove Theorem 3.5(b).

35. **(a)** Prove that if A and B are symmetric $n \times n$ matrices, then so is $A + B$.
(b) Prove that if A is a symmetric $n \times n$ matrix, then so is kA for any scalar k.

36. **(a)** Give an example to show that if A and B are symmetric $n \times n$ matrices, then AB need not be symmetric.
(b) Prove that if A and B are symmetric $n \times n$ matrices, then AB is symmetric if and only if $AB = BA$.

A square matrix is called ***skew-symmetric*** *if* $A^T = -A$.

37. Which of the following matrices are skew-symmetric?

(a) $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix}$

38. Give a componentwise definition of a skew-symmetric matrix.

39. Prove that the main diagonal of a skew-symmetric matrix must consist entirely of zeros.

40. Prove that if A and B are skew-symmetric $n \times n$ matrices, then so is $A + B$.

41. If A and B are skew-symmetric 2×2 matrices, under what conditions is AB skew-symmetric?

42. Prove that if A is an $n \times n$ matrix, then $A - A^T$ is skew-symmetric.

43. **(a)** Prove that any square matrix A can be written as the sum of a symmetric matrix and a skew-symmetric matrix. [*Hint:* Consider Theorem 3.5 and Exercise 42.]
(b) Illustrate part (a) for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

The ***trace*** *of an* $n \times n$ *matrix* $A = [a_{ij}]$ *is the sum of the entries on its main diagonal and is denoted by* $\mathrm{tr}(A)$. *That is,*

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

44. If A and B are $n \times n$ matrices, prove the following properties of the trace:
(a) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
(b) $\mathrm{tr}(kA) = k\mathrm{tr}(A)$, where k is a scalar

45. Prove that if A and B are $n \times n$ matrices, then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

46. If A is any matrix, to what is $\mathrm{tr}(AA^T)$ equal?

47. Show that there are no 2×2 matrices A and B such that $AB - BA = I_2$.

and there are no fractions in $\mathbb{Z}_3$. We must use multiplicative inverses rather than division.

Instead of $1/\det A = 1/2$, we use 2^{-1}; that is, we find the number x that satisfies the equation $2x = 1$ in $\mathbb{Z}_3$. It is easy to see that $x = 2$ is the solution we want: In $\mathbb{Z}_3$, $2^{-1} = 2$, since $2(2) = 1$. The formula for A^{-1} now becomes

$$A^{-1} = 2^{-1}\begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix} = 2\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$$

which agrees with our previous solution.

Exercises 3.3

In Exercises 1–10, find the inverse of the given matrix (if it exists) using Theorem 3.8.

1. $\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} \frac{3}{4} & \frac{3}{5} \\ \frac{5}{6} & \frac{2}{3} \end{bmatrix}$

6. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1\sqrt{2} \end{bmatrix}$

7. $\begin{bmatrix} -1.5 & -4.2 \\ 0.5 & 2.4 \end{bmatrix}$

8. $\begin{bmatrix} 3.55 & 0.25 \\ 8.52 & 0.60 \end{bmatrix}$

9. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

10. $\begin{bmatrix} 1/a & 1/b \\ 1/c & 1/d \end{bmatrix}$, where neither a, b, c, nor d is 0

In Exercises 11 and 12, solve the given system using the method of Example 3.25.

11. $2x + y = -1$
 $5x + 3y = 2$

12. $x_1 - x_2 = 2$
 $x_1 + 2x_2 = 5$

13. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.
 (a) Find A^{-1} and use it to solve the three systems $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, and $A\mathbf{x} = \mathbf{b}_3$.
 (b) Solve all three systems at the same time by row reducing the augmented matrix $[A \mid \mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$ using Gauss-Jordan elimination.
 (c) Carefully count the total number of individual multiplications that you performed in (a) and in (b). You should discover that, even for this 2×2 example, one method uses fewer operations. For larger systems, the difference is even more pronounced, and this explains why computer systems do not use one of these methods to solve linear systems.

14. Prove Theorem 3.9(b).

15. Prove Theorem 3.9(d).

16. Prove that the $n \times n$ identity matrix I_n is invertible and that $I_n^{-1} = I_n$.

17. (a) Give a counterexample to show that $(AB)^{-1} \neq A^{-1}B^{-1}$ in general.
 (b) Under what conditions on A and B is $(AB)^{-1} = A^{-1}B^{-1}$? Prove your assertion.

18. By induction, prove that if $A_1, A_2, \ldots, A_n$ are invertible matrices of the same size, then the product $A_1A_2\cdots A_n$ is invertible and $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}$.

19. Give a counterexample to show that $(A+B)^{-1} \neq A^{-1} + B^{-1}$ in general.

In Exercises 20–23, solve the given matrix equation for X. Simplify your answers as much as possible. (In the words of Albert Einstein, "Everything should be made as simple as possible, but not simpler.") Assume that all matrices are invertible.

20. $XA^{-1} = A^3$

21. $AXB = (BA)^2$

22. $(A^{-1}X)^{-1} = (AB^{-1})^{-1}(AB^2)$

23. $ABXA^{-1}B^{-1} = I + A$

In Exercises 24–30, let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix}$$

In each case, find an elementary matrix E that satisfies the given equation.

24. $EA = B$ **25.** $EB = A$ **26.** $EA = C$

27. $EC = A$ **28.** $EC = D$ **29.** $ED = C$

30. Is there an elementary matrix E such that $EA = D$? Why or why not?

In Exercises 31–38, find the inverse of the given elementary matrix.

31. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **32.** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ **36.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, c \neq 0$ **38.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, c \neq 0$

In Exercises 39 and 40, find a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I$. Use this sequence to write both A and A^{-1} as products of elementary matrices.

39. $A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$ **40.** $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

41. Prove Theorem 3.13 for the case of $AB = I$.

42. **(a)** Prove that if A is invertible and $AB = O$, then $B = O$.
(b) Give a counterexample to show that the result in part (a) may fail if A is not invertible.

43. **(a)** Prove that if A is invertible and $BA = CA$, then $B = C$.
(b) Give a counterexample to show that the result in part (a) may fail if A is not invertible.

44. A square matrix A is called ***idempotent*** if $A^2 = A$. (The word *idempotent* comes from the Latin *idem*, meaning "same," and *potere*, meaning "to have power." Thus, something that is idempotent has the "same power" when squared.)
(a) Find three idempotent 2×2 matrices.
(b) Prove that the only invertible idempotent $n \times n$ matrix is the identity matrix.

45. Show that if A is a square matrix that satisfies the equation $A^2 - 2A + I = O$, then $A^{-1} = 2I - A$.

46. Prove that if a symmetric matrix is invertible, then its inverse is symmetric also.

47. Prove that if A and B are square matrices and AB is invertible, then both A and B are invertible.

In Exercises 48–63, use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

48. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **49.** $\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

50. $\begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix}$ **51.** $\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$

52. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 5 & 1 \\ 2 & 3 & 0 \end{bmatrix}$ **53.** $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$

54. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ **55.** $\begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$

56. $\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$ **57.** $\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

58. $\begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

59. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$ **60.** $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ over $\mathbb{Z}_2$

61. $\begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix}$ over $\mathbb{Z}_5$ **62.** $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ over $\mathbb{Z}_3$

63. $\begin{bmatrix} 1 & 5 & 0 \\ 1 & 2 & 4 \\ 3 & 6 & 1 \end{bmatrix}$ over $\mathbb{Z}_7$

Partitioning large square matrices can sometimes make their inverses easier to compute, particularly if the blocks have a nice form. In Exercises 64–68, verify by block multiplication that the inverse of a matrix, if partitioned as shown, is as claimed. (Assume that all inverses exist as needed.)

64. $\begin{bmatrix} A & B \\ O & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix}$

with the entries placed in the order (1,1), (1,2), (1,3), (2,1), (3,1), (2,2), (2,3), (3,2), (3,3). In other words, the subdiagonal entries of A are replaced by the corresponding multipliers. (Check that this works!)

- Once an LU factorization of A has been computed, it can be used to solve as many linear systems of the form $A\mathbf{x} = \mathbf{b}$ as we like. We just need to apply the method of Example 3.34, varying the vector $\mathbf{b}$ each time.
- For matrices with certain special forms, especially those with a large number of zeros (so-called "sparse" matrices) concentrated off the diagonal, there are methods that will simplify the computation of an LU factorization. In these cases, this method is faster than Gaussian elimination in solving $A\mathbf{x} = \mathbf{b}$.
- For an invertible matrix A, an LU factorization of A can be used to find A^{-1}, if necessary. Moreover, this can be done in such a way that it simultaneously yields a factorization of A^{-1}. (See Exercises 15–18.)

Remark If you have a CAS (such as MATLAB) that has the LU factorization built in, you may notice some differences between your hand calculations and the computer output. This is because most CAS's will automatically try to perform partial pivoting to reduce round-off errors. (See the Exploration "Partial Pivoting," in Chapter 2.) Turing's paper is an extended discussion of such errors in the context of matrix factorizations.

This section has served to introduce one of the most useful matrix factorizations. In subsequent chapters, we will encounter other equally useful factorizations.

Exercises 3.4

In Exercises 1–6, solve the system $A\mathbf{x} = \mathbf{b}$ using the given LU factorization of A.

1. $A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 3 & -4 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{5}{4} & 1 \end{bmatrix}$
$\times \begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$
$\times \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 6 & -4 & 5 & -3 \\ 8 & -4 & 1 & 0 \\ 4 & -1 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 2 & -1 & 5 & 1 \end{bmatrix}$
$\times \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & -1 & 5 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 4 & 3 & 0 \\ -2 & -5 & -1 & 2 \\ 3 & 6 & -3 & -4 \\ -5 & -8 & 9 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix}$
$\times \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}$

In Exercises 7–12, find an LU factorization of the given matrix.

7. $\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$ **8.** $\begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 7 & 9 \end{bmatrix}$ **10.** $\begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & -6 & 7 \\ -1 & -2 & -9 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 2 & 2 & 1 \\ -2 & 4 & -1 & 2 \\ 4 & 4 & 7 & 3 \\ 6 & 9 & 5 & 8 \end{bmatrix}$

Generalize the definition of LU factorization to nonsquare matrices by simply requiring U to be a matrix in row echelon form. With this modification, find an LU factorization of the matrices in Exercises 13 and 14.

13. $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -7 & 3 & 8 & -2 \\ 1 & 1 & 3 & 5 & 2 \\ 0 & 3 & -3 & -6 & 0 \end{bmatrix}$

For an invertible matrix with an LU factorization $A = LU$, both L and U will be invertible and $A^{-1} = U^{-1}L^{-1}$. In Exercises 15 and 16, find L^{-1}, U^{-1}, and A^{-1} for the given matrix.

15. A in Exercise 1 **16.** A in Exercise 4

The inverse of a matrix can also be computed by solving several systems of equations using the method of Example 3.34. For an $n \times n$ matrix A, to find its inverse we need to solve $AX = I_n$ for the $n \times n$ matrix X. Writing this equation as $A[\mathbf{x}_1 \;\; \mathbf{x}_2 \cdots \mathbf{x}_n] = [\mathbf{e}_1 \;\; \mathbf{e}_2 \cdots \mathbf{e}_n]$, using the matrix-column form of AX, we see that we need to solve n systems of linear equations: $A\mathbf{x}_1 = \mathbf{e}_1, A\mathbf{x}_2 = \mathbf{e}_2, \ldots, A\mathbf{x}_n = \mathbf{e}_n$. Moreover, we can use the factorization $A = LU$ to solve each one of these systems.

In Exercises 17 and 18, use the approach just outlined to find A^{-1} for the given matrix. Compare with the method of Exercises 15 and 16.

17. A in Exercise 1 **18.** A in Exercise 4

In Exercises 19–22, write the given permutation matrix as a product of elementary (row interchange) matrices.

19. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **20.** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **22.** $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 23–25, find a P^TLU factorization of the given matrix A.

23. $A = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$ **24.** $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

26. Prove that there are exactly $n!$ $n \times n$ permutation matrices.

In Exercises 27–28, solve the system $A\mathbf{x} = \mathbf{b}$ using the given factorization $A = P^TLU$. Because $PP^T = I$, $P^TLU\mathbf{x} = \mathbf{b}$ can be rewritten as $LU\mathbf{x} = P\mathbf{b}$. This system can then be solved using the method of Example 3.34.

27. $$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -\frac{5}{2} \end{bmatrix} = P^TLU, \; \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

28. $$A = \begin{bmatrix} 8 & 3 & 5 \\ 4 & 1 & 2 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = P^TLU, \; \mathbf{b} = \begin{bmatrix} 16 \\ -4 \\ 4 \end{bmatrix}$$

Solution Since $\mathbf{v} = 2\mathbf{e}_1 + 7\mathbf{e}_2 + 4\mathbf{e}_3$,

$$[\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

It should be clear that the coordinate vector of every (column) vector in $\mathbb{R}^n$ with respect to the standard basis is just the vector itself.

Example 3.54

In Example 3.44, we saw that $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$ are three vectors in the same subspace (plane through the origin) S of $\mathbb{R}^3$ and that $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ is a basis for S. Since $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$, we have

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

See Figure 3.3.

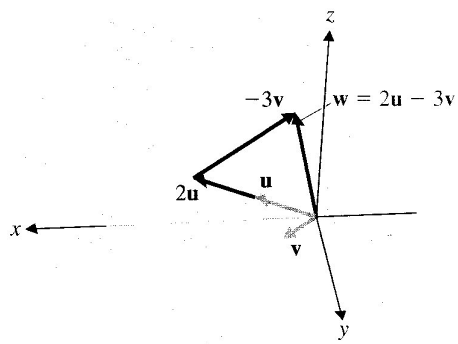


Figure 3.3
The coordinates of a vector with respect to a basis

Exercises 3.5

In Exercises 1–4, let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ *in* $\mathbb{R}^2$ *that satisfy the given property. In each case, either prove that S forms a subspace of* $\mathbb{R}^2$ *or give a counterexample to show that it does not.*

1. $x = 0$

2. $x \geq 0, y \geq 0$

3. $y = 2x$

4. $xy \geq 0$

In Exercises 5–8, let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *in* $\mathbb{R}^3$ *that satisfy the given property. In each case, either prove that S forms a subspace of* $\mathbb{R}^3$ *or give a counterexample to show that it does not.*

5. $x = y = z$

6. $z = 2x, y = 0$

7. $x - y + z = 1$

8. $|x - y| = |y - z|$

9. Prove that every line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

10. Suppose S consists of all points in $\mathbb{R}^2$ that are on the x-axis or the y-axis (or both). (S is called the *union* of the two axes.) Is S a subspace of $\mathbb{R}^2$? Why or why not?

In Exercises 11 and 12, determine whether **b** *is in col(A) and whether* **w** *is in row(A), as in Example 3.41.*

11. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 0 \\ 3 & -1 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 & -3 & -3 \end{bmatrix}$

13. In Exercise 11, determine whether $\mathbf{w}$ is in row(A), using the method described in the Remark following Example 3.41.

14. In Exercise 12, determine whether $\mathbf{w}$ is in row(A) using the method described in the Remark following Example 3.41.

15. If A is the matrix in Exercise 11, is $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ in null(A)?

16. If A is the matrix in Exercise 12, is $\mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$ in null(A)?

In Exercises 17–20, give bases for row(A), col(A), and null(A).

17. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ **18.** $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

20. $A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$

In Exercises 21–24, find bases for row(A) and col(A) in the given exercises using A^T.

21. Exercise 17 **22.** Exercise 18

23. Exercise 19 **24.** Exercise 20

25. Explain carefully why your answers to Exercises 17 and 21 are both correct even though there appear to be differences.

26. Explain carefully why your answers to Exercises 18 and 22 are both correct even though there appear to be differences.

In Exercises 27–30, find a basis for the span of the given vectors.

27. $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ **28.** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

29. $[2 \;\; -3 \;\; 1], [1 \;\; -1 \;\; 0], [4 \;\; -4 \;\; 1]$

30. $[3 \;\; 1 \;\; -1 \;\; 0], [0 \;\; -1 \;\; 2 \;\; -1], [4 \;\; 3 \;\; 8 \;\; 3]$

For Exercises 31 and 32, find bases for the spans of the vectors in the given exercises from among the vectors themselves.

31. Exercise 29 **32.** Exercise 30

33. Prove that if R is a matrix in echelon form, then a basis for row(R) consists of the nonzero rows of R.

34. Prove that if the columns of A are linearly independent, then they must form a basis for col(A).

For Exercises 35–38, give the rank and the nullity of the matrices in the given exercises.

35. Exercise 17

36. Exercise 18

37. Exercise 19

38. Exercise 20

39. If A is a 3×5 matrix, explain why the columns of A must be linearly dependent.

40. If A is a 4×2 matrix, explain why the rows of A must be linearly dependent.

41. If A is a 3×5 matrix, what are the possible values of nullity(A)?

42. If A is a 4×2 matrix, what are the possible values of nullity(A)?

In Exercises 43 and 44, find all possible values of rank(A) as a varies.

43. $A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}$ **44.** $A = \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix}$

Answer Exercises 45–48 by considering the matrix with the given vectors as its columns.

45. Do $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

46. Do $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

47. Do $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^4$?

48. Do $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$ form a basis for $\mathbb{R}^4$?

49. Do $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ form a basis for $\mathbb{Z}_2^3$?

50. Do $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ form a basis for $\mathbb{Z}_3^3$?

In Exercises 51 and 52, show that $\mathbf{w}$ *is in span($\mathcal{B}$) and find the coordinate vector* $[\mathbf{w}]_{\mathcal{B}}$.

51. $\mathcal{B} = \left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}, \mathbf{w} = \begin{bmatrix}1\\6\\2\end{bmatrix}$

52. $\mathcal{B} = \left\{\begin{bmatrix}3\\1\\4\end{bmatrix}, \begin{bmatrix}5\\1\\6\end{bmatrix}\right\}, \mathbf{w} = \begin{bmatrix}1\\3\\4\end{bmatrix}$

In Exercises 53–56, compute the rank and nullity of the given matrices over the indicated $\mathbb{Z}_p$.

53. $\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}$ over $\mathbb{Z}_2$

54. $\begin{bmatrix}1&1&2\\2&1&2\\2&0&0\end{bmatrix}$ over $\mathbb{Z}_3$

55. $\begin{bmatrix}1&3&1&4\\2&3&0&1\\1&0&4&0\end{bmatrix}$ over $\mathbb{Z}_5$

56. $\begin{bmatrix}2&4&0&0&1\\6&3&5&1&0\\1&0&2&2&5\\1&1&1&1&1\end{bmatrix}$ over $\mathbb{Z}_7$

57. If A is $m \times n$, prove that every vector in null(A) is orthogonal to every vector in row(A).

58. If A and B are $n \times n$ matrices of rank n, prove that AB has rank n.

59. **(a)** Prove that rank(AB) $\le$ rank(B). [*Hint:* Review Exercise 29 in Section 3.1.]
(b) Give an example in which rank(AB) $<$ rank(B).

60. **(a)** Prove that rank(AB) $\le$ rank(A). [*Hint:* Review Exercise 30 in Section 3.1 or use transposes and Exercise 59(a).]
(b) Give an example in which rank(AB) $<$ rank(A).

61. **(a)** Prove that if U is invertible, then rank(UA) = rank(A). [*Hint:* $A = U^{-1}(UA)$.]
(b) Prove that if V is invertible, then rank(AV) = rank(A).

62. Prove that an $m \times n$ matrix A has rank 1 if and only if A can be written as the outer product $\mathbf{u}\mathbf{v}^T$ of a vector $\mathbf{u}$ in $\mathbb{R}^m$ and $\mathbf{v}$ in $\mathbb{R}^n$.

63. If an $m \times n$ matrix A has rank r, prove that A can be written as the sum of r matrices, each of which has rank 1. [*Hint:* Find a way to use Exercise 62.]

64. Prove that, for $m \times n$ matrices A and B, rank $(A + B) \le$ rank(A) + rank(B).

65. Let A be an $n \times n$ matrix such that $A^2 = O$. Prove that rank(A) $\le n/2$. [*Hint:* Show that col(A) $\subseteq$ null(A) and use the Rank Theorem.]

66. Let A be a skew-symmetric $n \times n$ matrix. (See page 168)
(a) Prove that $\mathbf{x}^T A\mathbf{x} = 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
(b) Prove that $I + A$ is invertible. [*Hint:* Show that null($I + A$) = $\{\mathbf{0}\}$.]

3.6 Introduction to Linear Transformations

In this section, we begin to explore one of the themes from the introduction to this chapter. There we saw that matrices can be used to transform vectors, acting as a type of "function" of the form $\mathbf{w} = T(\mathbf{v})$, where the independent variable $\mathbf{v}$ and the dependent variable $\mathbf{w}$ are vectors. We will make this notion more precise now and look at several examples of such matrix transformations, leading to the concept of a *linear transformation*—a powerful idea that we will encounter repeatedly from here on.

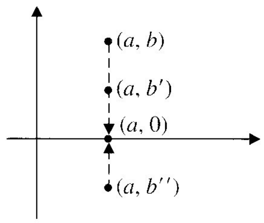


Figure 3.13
Projections are not invertible

Remark Figure 3.13 gives some idea why P in Example 3.63 is not invertible. The projection "collapses" $\mathbb{R}^2$ onto the x-axis. For P to be invertible, we would have to have a way of "undoing" it, to recover the point (a, b) we started with. However, there are infinitely many candidates for the image of $(a, 0)$ under such a hypothetical "inverse." Which one should we use? We cannot simply say that P^{-1} must send $(a, 0)$ to (a, b), since this cannot be a *definition* when we have no way of knowing what b should be. (See Exercise 42.)

Associativity

Theorem 3.3(a) in Section 3.2 stated the associativity property for matrix multiplication: $A(BC) = (AB)C$. (If you didn't try to prove it then, do so now. Even with all matrices restricted 2×2, you will get some feeling for the notational complexity involved in an "elementwise" proof, which should make you appreciate the proof we are about to give.)

Our approach to the proof is via linear transformations. We have seen that every $m \times n$ matrix A gives rise to a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$; conversely, every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has a corresponding $m \times n$ matrix $[T]$. The two correspondences are inversely related; that is, given A, $[T_A] = A$, and given T, $T_{[T]} = T$.

Let $R = T_A$, $S = T_B$, and $T = T_C$. Then, by Theorem 3.32,

$$A(BC) = (AB)C \quad \text{if and only if} \quad R \circ (S \circ T) = (R \circ S) \circ T$$

We now prove the latter identity. Let $\mathbf{x}$ be in the domain of T (and hence in the domain of both $R \circ (S \circ T)$ and $(R \circ S) \circ T$—why?). To prove that $R \circ (S \circ T) = (R \circ S) \circ T$, it is enough to prove that they have the same effect on $\mathbf{x}$. By repeated application of the definition of composition, we have

$$\begin{aligned}(R \circ (S \circ T))(\mathbf{x}) &= R((S \circ T)(\mathbf{x})) \\ &= R(S(T(\mathbf{x}))) \\ &= (R \circ S)(T(\mathbf{x})) = ((R \circ S) \circ T)(\mathbf{x})\end{aligned}$$

as required. (Carefully check how the definition of composition has been used four times.)

This section has served as an introduction to linear transformations. In Chapter 6, we will take a more detailed and more general look at these transformations. The exercises that follow also contain some additional explorations of this important concept.

Exercises 3.6

1. Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation corresponding to $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$. Find $T_A(\mathbf{u})$ and $T_A(\mathbf{v})$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

2. Let $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ be the matrix transformation corresponding to $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$. Find $T_A(\mathbf{u})$ and $T_A(\mathbf{v})$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

In Exercises 3–6, prove that the given transformation is a linear transformation, using the definition (or the Remark following Example 3.55).

3. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

4. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ -x \\ 3x-7y \end{bmatrix}$

5. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y+z \\ 2x+y-3z \end{bmatrix}$

6. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix}$

In Exercises 7–10, give a counterexample to show that the given transformation is not a linear transformation.

7. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x^2 \end{bmatrix}$

8. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| \\ |y| \end{bmatrix}$

9. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$

10. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$

In Exercises 11–14, find the standard matrix of the linear transformation in the given exercise.

11. Exercise 3

12. Exercise 4

13. Exercise 5

14. Exercise 6

In Exercises 15–18, show that the given transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is linear by showing that it is a matrix transformation.

15. F reflects a vector in the y-axis.

16. R rotates a vector 45° counterclockwise about the origin.

17. D stretches a vector by a factor of 2 in the x-component and a factor of 3 in the y-component.

18. P projects a vector onto the line $y = x$.

19. The three types of elementary matrices give rise to five types of 2×2 matrices with one of the following forms:

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Each of these elementary matrices corresponds to a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Draw pictures to illustrate the effect of each one on the unit square with vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$.

In Exercises 20–25, find the standard matrix of the given linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

20. Counterclockwise rotation through 120° about the origin

21. Clockwise rotation through 30° about the origin

22. Projection onto the line $y = 2x$

23. Projection onto the line $y = -x$

24. Reflection in the line $y = x$

25. Reflection in the line $y = -x$

26. Let ℓ be a line through the origin in $\mathbb{R}^2$, P_ℓ the linear transformation that projects a vector onto ℓ, and F_ℓ the transformation that reflects a vector in ℓ.

(a) Draw diagrams to show that F_ℓ is linear.

(b) Figure 3.14 suggests a way to find the matrix of F_ℓ, using the fact that the diagonals of a parallelogram bisect each other. Prove that $F_\ell(\mathbf{x}) = 2P_\ell(\mathbf{x}) - \mathbf{x}$, and use this result to show that the standard matrix of F_ℓ is

$$\frac{1}{d_1^2 + d_2^2}\begin{bmatrix} d_1^2 - d_2^2 & 2d_1d_2 \\ 2d_1d_2 & -d_1^2 + d_2^2 \end{bmatrix}$$

(where the direction vector of ℓ is $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$).

(c) If the angle between ℓ and the positive x-axis is θ, show that the matrix of F_ℓ is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

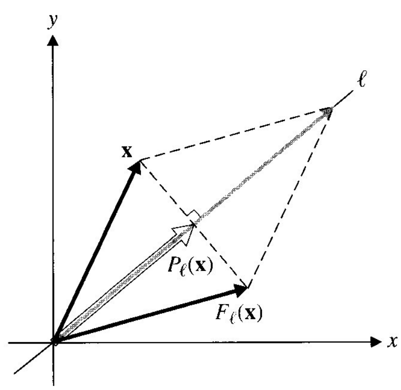


Figure 3.14

In Exercises 27 and 28, apply part (b) or (c) of Exercise 26 to find the standard matrix of the transformation.

27. Reflection in the line $y = 2x$

28. Reflection in the line $y = \sqrt{3}x$

29. Check the formula for $S \circ T$ in Example 3.60, by performing the suggested direct substitution.

In Exercises 30–35, verify Theorem 3.32 by finding the matrix of $S \circ T$ (a) by direct substitution and (b) by matrix multiplication of $[S][T]$.

30. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}, S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 \\ -y_2 \end{bmatrix}$

31. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -3x_1 + x_2 \end{bmatrix}, S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + 3y_2 \\ y_1 - y_2 \end{bmatrix}$

32. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + 3y_2 \\ 2y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$

33. $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 - x_2 + x_3 \end{bmatrix}, S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4y_1 - 2y_2 \\ -y_1 + y_2 \end{bmatrix}$

34. $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_2 - x_3 \end{bmatrix}, S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix}$

35. $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + x_3 \end{bmatrix}, S\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ y_2 - y_3 \\ -y_1 + y_3 \end{bmatrix}$

In Exercises 36–39, find the standard matrix of the composite transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

36. Counterclockwise rotation through 60°, followed by reflection in the line $y = x$

37. Reflection in the y-axis, followed by clockwise rotation through 30°

38. Clockwise rotation through 45°, followed by projection onto the y-axis, followed by clockwise rotation through 45°

39. Reflection in the line $y = x$, followed by counterclockwise rotation through 30°, followed by reflection in the line $y = -x$

In Exercises 40–43, use matrices to prove the given statements about transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

40. If R_θ denotes a rotation (about the origin) through the angle θ, then $R_\alpha \circ R_\beta = R_{\alpha+\beta}$.

41. If θ is the angle between lines ℓ and m (through the origin), then $F_m \circ F_\ell = R_{+2\theta}$. (See Exercise 26.)

42. **(a)** If P is a projection, then $P \circ P = P$.

(b) The matrix of a projection can never be invertible.

43. If ℓ, m, and n are three lines through the origin, then $F_n \circ F_m \circ F_\ell$ is also a reflection in a line through the origin.

44. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ (or from $\mathbb{R}^3$ to $\mathbb{R}^3$). Prove that T maps a straight line to a straight line or a point. [*Hint:* Use the vector form of the equation of a line.]

45. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ (or from $\mathbb{R}^3$ to $\mathbb{R}^3$). Prove that T maps parallel lines to parallel lines, a single line, a pair of points, or a single point.

In Exercises 46–51, let ABCD be the square with vertices $(-1, 1)$, $(1, 1)$, $(1, -1)$, and $(-1, -1)$. Use the results in Exercises 44 and 45 to find and draw the image of ABCD under the given transformation.

46. T in Exercise 3

47. D in Exercise 17

48. P in Exercise 18

49. The projection in Exercise 22

50. T in Exercise 31

51. The transformation in Exercise 37

52. Prove that $P_\ell(c\mathbf{v}) = cP_\ell(\mathbf{v})$ for any scalar c [Example 3.59(b)].

53. Prove that $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$

for all $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{R}^n$ and scalars c_1, c_2.

54. Prove that (as noted at the beginning of this section) the range of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is the column space of its matrix $[T]$.

55. If A is an invertible 2×2 matrix, what does the Fundamental Theorem of Invertible Matrices assert about the corresponding linear transformation T_A in light of Exercise 19?

Remark You will recall that a polynomial equation with real coefficients (such as the quadratic equation in Example 4.5) need not have real roots; it may have *complex* roots. (See Appendix C.) It is also possible to compute eigenvalues and eigenvectors when the entries of a matrix come from $\mathbb{Z}_p$, where p is prime. Thus, it is important to specify the setting we intend to work in before we set out to compute the eigenvalues of a matrix. However, unless otherwise specified, the eigenvalues of a matrix whose entries are *real* numbers will be assumed to be real as well.

Example 4.6

Interpret the matrix in Example 4.5 as a matrix over $\mathbb{Z}_3$ and find its eigenvalues in that field.

Solution The solution proceeds exactly as above, except we work modulo 3. Hence, the quadratic equation $\lambda^2 - 6\lambda + 8 = 0$ becomes $\lambda^2 + 2 = 0$. This equation is the same as $\lambda^2 = -2 = 1$, giving $\lambda = 1$ and $\lambda = -1 = 2$ as the eigenvalues in $\mathbb{Z}_3$. (Check that the same answer would be obtained by *first* reducing A modulo 3 to obtain $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and then working with this matrix.)

Example 4.7

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a) over $\mathbb{R}$ and (b) over the complex numbers $\mathbb{C}$.

Solution We must solve the equation

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

a + bi

(a) Over $\mathbb{R}$, there are no solutions, so A has no real eigenvalues.

(b) Over $\mathbb{C}$, the solutions are $\lambda = i$ and $\lambda = -i$. (See Appendix C.)

In the next section, we will extend the notion of determinant from 2×2 to $n \times n$ matrices, which in turn will allow us to find the eigenvalues of arbitrary square matrices. (In fact, this isn't quite true—but we will at least be able to find a polynomial equation that the eigenvalues of a given matrix must satisfy.)

Exercises 4.1

In Exercises 1–6, show that **v** *is an eigenvector of A and find the corresponding eigenvalue.*

1. $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 7–12, show that λ is an eigenvalue of A and find one eigenvector corresponding to this eigenvalue.

7. $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}, \lambda = 3$

8. $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}, \lambda = -2$

9. $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}, \lambda = 1$

10. $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}, \lambda = 4$

11. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \lambda = -1$

12. $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \\ 4 & 2 & 0 \end{bmatrix}, \lambda = 3$

In Exercises 13–18, find the eigenvalues and eigenvectors of A geometrically.

13. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (reflection in the y-axis)

14. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (reflection in the line $y = x$)

15. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (projection onto the x-axis)

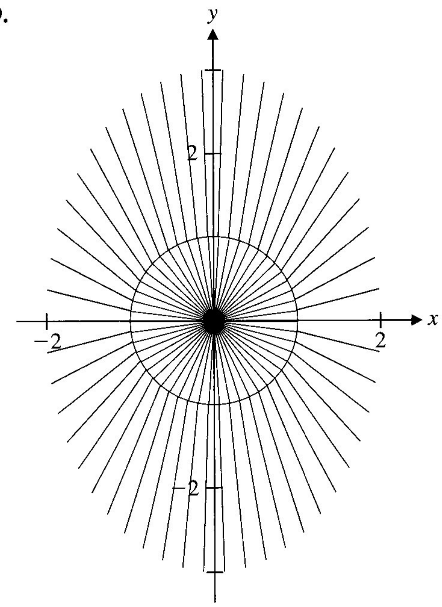
16. $A = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$ (projection onto the line through the origin with direction vector $\begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$)

17. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (stretching by a factor of 2 horizontally and a factor of 3 vertically)

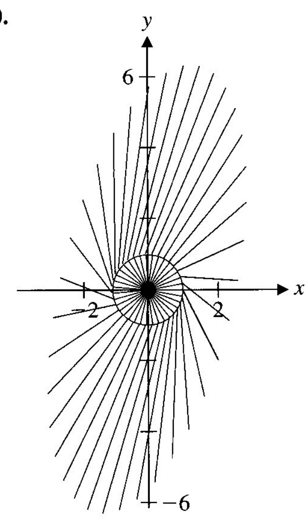
18. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (counterclockwise rotation of 90° about the origin)

In Exercises 19–22, the unit vectors **x** *in* $\mathbb{R}^2$ *and their images* $A\mathbf{x}$ *under the action of a* 2×2 *matrix A are drawn head-to-tail, as in Figure 4.7. Estimate the eigenvectors and eigenvalues of A from each "eigenpicture."*

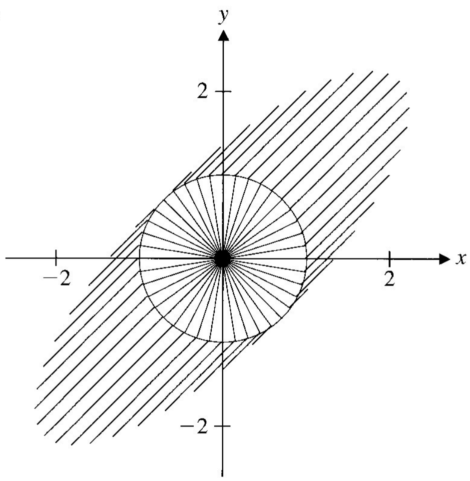
19.



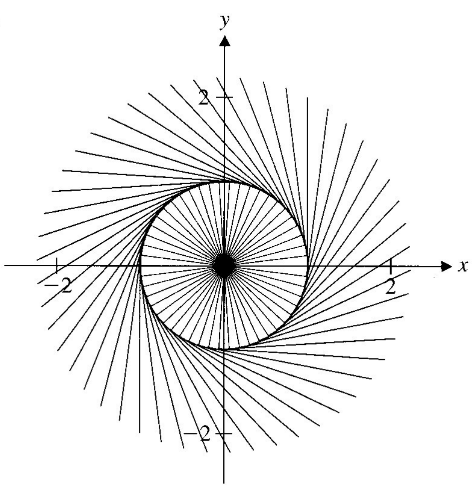
20.



21.



22.



In Exercises 23–26, use the method of Example 4.5 to find all of the eigenvalues of the matrix A. Give bases for each of the corresponding eigenspaces. Illustrate the eigenspaces and the effect of multiplying eigenvectors by A as in Figure 4.8.

23. $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ **24.** $A = \begin{bmatrix} 0 & 2 \\ 8 & 6 \end{bmatrix}$

25. $A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ **26.** $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

a + bi

In Exercises 27–30, find all of the eigenvalues of the matrix A over the complex numbers $\mathbb{C}$. Give bases for each of the corresponding eigenspaces.

27. $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ **28.** $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

29. $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ **30.** $A = \begin{bmatrix} 4 & 1-2i \\ 1+2i & 0 \end{bmatrix}$

In Exercises 31–34, find all of the eigenvalues of the matrix A over the indicated $\mathbb{Z}_p$.

31. $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ over $\mathbb{Z}_3$ **32.** $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ over $\mathbb{Z}_3$

33. $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$ over $\mathbb{Z}_5$ **34.** $A = \begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix}$ over $\mathbb{Z}_5$

35. **(a)** Show that the eigenvalues of the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are the solutions of the quadratic equation $\lambda^2 - \operatorname{tr}(A)\lambda + \det A = 0$, where $\operatorname{tr}(A)$ is the trace of A. (See page 168.)

(b) Show that the eigenvalues of the matrix A in part (a) are

$$\lambda = \tfrac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right)$$

(c) Show that the trace and determinant of the matrix A in part (a) are given by

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 \quad \text{and} \quad \det A = \lambda_1\lambda_2$$

where λ_1 and λ_2 are the eigenvalues of A.

36. Consider again the matrix A in Exercise 35. Give conditions on a, b, c, and d such that A has

(a) two distinct real eigenvalues,
(b) one real eigenvalue, and
(c) no real eigenvalues.

37. Show that the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

are $\lambda = a$ and $\lambda = d$, and find the corresponding eigenspaces.

a + bi **38.** Let a and b be real numbers. Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

over the complex numbers.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Gottfried Wilhelm von Leibniz (1646–1716) was born in Leipzig and studied law, theology, philosophy, and mathematics. He is probably best known for developing (with Newton, independently) the main ideas of differential and integral calculus. However, his contributions to other branches of mathematics are also impressive. He developed the notion of a determinant, knew versions of Cramer's Rule and the Laplace Expansion Theorem before others were given credit for them, and laid the foundation for matrix theory through work he did on quadratic forms. Leibniz also was the first to develop the binary system of arithmetic. He believed in the importance of good notation and, along with the familiar notation for derivatives and integrals, introduced a form of subscript notation for the coefficients of a linear system that is essentially the notation we use today.

By the late 19th century, the theory of determinants had developed to the stage that entire books were devoted to it, including Dodgson's *An Elementary Treatise on Determinants* in 1867 and Thomas Muir's monumental five-volume work, which appeared in the early 20th century. While their history is fascinating, today determinants are of theoretical more than practical interest. Cramer's Rule is a hopelessly inefficient method for solving a system of linear equations, and numerical methods have replaced any use of determinants in the computation of eigenvalues. Determinants are used, however, to give students an initial understanding of the characteristic polynomial (as in Sections 4.1 and 4.3).

Exercises 4.2

Compute the determinants in Exercises 1–6 using cofactor expansion along the first row and along the first column.

1. $\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 0 & -2 \\ 3 & 3 & 2 \\ 0 & -1 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Compute the determinants in Exercises 7–15 using cofactor expansion along any row or column that seems convenient.

7. $\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 0 & 4 \\ -3 & -2 & -1 \end{vmatrix}$

9. $\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix}$

10. $\begin{vmatrix} 0 & 0 & \sin\theta \\ \sin\theta & \cos\theta & \tan\theta \\ -\cos\theta & \sin\theta & -\cos\theta \end{vmatrix}$

11. $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ a & 0 & b \end{vmatrix}$

12. $\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix}$

13. $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix}$

14. $\begin{vmatrix} 3 & -2 & 0 & 1 \\ 1 & 3 & 0 & -1 \\ 0 & 2 & 2 & 4 \\ 3 & 1 & 0 & 0 \end{vmatrix}$

15. $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{vmatrix}$

In Exercises 16–18, compute the indicated 3×3 determinants using the method of Example 4.9.

16. The determinant in Exercise 6

17. The determinant in Exercise 8

18. The determinant in Exercise 11

19. Verify that the method indicated in (2) agrees with equation (1) for a 3×3 determinant.

20. Verify that definition (4) agrees with the definition of a 2×2 determinant when $n = 2$.

21. Prove Theorem 4.2. [*Hint:* A proof by induction would be appropriate here.]

In Exercises 22–25, evaluate the given determinant using elementary row and/or column operations and Theorem 4.3 to reduce the matrix to row echelon form.

22. The determinant in Exercise 1

23. The determinant in Exercise 9

24. The determinant in Exercise 13

25. The determinant in Exercise 14

In Exercises 26–34, use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

26. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -2 \\ 2 & 2 & 2 \end{vmatrix}$ **27.** $\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix}$

28. $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 5 & 2 \\ 3 & -1 & 4 \end{vmatrix}$ **29.** $\begin{vmatrix} 2 & 3 & -4 \\ 1 & -3 & -2 \\ -1 & 5 & 2 \end{vmatrix}$

30. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{vmatrix}$ **31.** $\begin{vmatrix} 4 & 1 & 3 \\ -2 & 0 & -2 \\ 5 & 4 & 1 \end{vmatrix}$

32. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ **33.** $\begin{vmatrix} 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

34. $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$

Find the determinants in Exercises 35–40, assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

35. $\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix}$ **36.** $\begin{vmatrix} 3a & -b & 2c \\ 3d & -e & 2f \\ 3g & -h & 2i \end{vmatrix}$

37. $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$ **38.** $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$

39. $\begin{vmatrix} 2c & b & a \\ 2f & e & d \\ 2i & h & g \end{vmatrix}$

40. $\begin{vmatrix} a & b & c \\ 2d-3g & 2e-3h & 2f-3i \\ g & h & i \end{vmatrix}$

41. Prove Theorem 4.3(a). **42.** Prove Theorem 4.3(f).

43. Prove Lemma 4.5. **44.** Prove Theorem 4.7.

In Exercises 45 and 46, use Theorem 4.6 to find all values of k for which A is invertible.

45. $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$

46. $A = \begin{bmatrix} k & k & 0 \\ k^2 & 4 & k^2 \\ 0 & k & k \end{bmatrix}$

In Exercises 47–52, assume that A and B are $n \times n$ matrices with $\det A = 3$ and $\det B = -2$. Find the indicated determinants.

47. $\det(AB)$ **48.** $\det(A^2)$ **49.** $\det(B^{-1}A)$

50. $\det(2A)$ **51.** $\det(3B^T)$ **52.** $\det(AA^T)$

In Exercises 53–56, A and B are $n \times n$ matrices.

53. Prove that $\det(AB) = \det(BA)$.

54. If B is invertible, prove that $\det(B^{-1}AB) = \det(A)$.

55. If A is idempotent (that is, $A^2 = A$), find all possible values of $\det(A)$.

56. A square matrix A is called ***nilpotent*** if $A^m = O$ for some $m > 1$. (The word *nilpotent* comes from the Latin *nil,* meaning "nothing," and *potere,* meaning "to have power." A nilpotent matrix is thus one that becomes "nothing"—that is, the zero matrix—when raised to some power.) Find all possible values of $\det(A)$ if A is nilpotent.

In Exercises 57–60, use Cramer's Rule to solve the given linear system.

57. $x + y = 1$
$x - y = 2$

58. $2x - y = 5$
$x + 3y = -1$

59. $2x + y + 3z = 1$
$y + z = 1$
$z = 1$

60. $x + y - z = 1$
$x + y + z = 2$
$x - y = 3$

In Exercises 61–64, use Theorem 4.12 to compute the inverse of the coefficient matrix for the given exercise.

61. Exercise 57

62. Exercise 58

63. Exercise 59

64. Exercise 60

65. If A is an invertible $n \times n$ matrix, show that adj A is also invertible and that

$$(\operatorname{adj} A)^{-1} = \frac{1}{\det A} A = \operatorname{adj}(A^{-1})$$

66. If A is an $n \times n$ matrix, prove that

$$\det(\operatorname{adj} A) = (\det A)^{n-1}$$

67. Verify that if $r < s$, then rows r and s of a matrix can be interchanged by performing $2(s - r) - 1$ interchanges of adjacent rows.

68. Prove that the Laplace Expansion Theorem holds for column expansion along the jth column.

69. Let A be a square matrix that can be partitioned as

$$A = \left[\begin{array}{c|c} P & Q \\ \hline O & S \end{array}\right]$$

where P and S are square matrices. Such a matrix is said to be in ***block (upper) triangular form.*** Prove that

$$\det A = (\det P)(\det S)$$

[*Hint:* Try a proof by induction on the number of rows of P.]

70. **(a)** Give an example to show that if A can be partitioned as

$$A = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right]$$

where P, Q, R, and S are all square, then it is not necessarily true that

$$\det A = (\det P)(\det S) - (\det Q)(\det R)$$

(b) Assume that A is partitioned as in part (a) and that P is invertible. Let

$$B = \left[\begin{array}{c|c} P^{-1} & O \\ \hline -RP^{-1} & I \end{array}\right]$$

Compute $\det(BA)$ using Exercise 69 and use the result to show that

$$\det A = \det P \det(S - RP^{-1}Q)$$

[The matrix $S - RP^{-1}Q$ is called the ***Schur complement*** of P in A, after Issai Schur (1875–1941), who was born in Belarus but spent most of his life in Germany. He is known mainly for his fundamental work on the representation theory of groups, but he also worked in number theory, analysis, and other areas.]

(c) Assume that A is partitioned as in part (a), that P is invertible, and that $PR = RP$. Prove that

$$\det A = \det(PS - RQ)$$

Proof The proof is indirect. We will assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly *dependent* and show that this assumption leads to a contradiction.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent, then one of these vectors must be expressible as a linear combination of the previous ones. Let $\mathbf{v}_{k+1}$ be the first of the vectors $\mathbf{v}_i$ that can be so expressed. In other words, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, but there are scalars $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v}_{k+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{1}$$

Multiplying both sides of equation (1) by A from the left and using the fact that $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for each i, we have

$$\begin{aligned}\lambda_{k+1}\mathbf{v}_{k+1} = A\mathbf{v}_{k+1} &= A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k)\\ &= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_kA\mathbf{v}_k \qquad (2)\\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_k\lambda_k\mathbf{v}_k\end{aligned}$$

Now we multiply both sides of equation (1) by λ_{k+1} to get

$$\lambda_{k+1}\mathbf{v}_{k+1} = c_1\lambda_{k+1}\mathbf{v}_1 + c_2\lambda_{k+1}\mathbf{v}_2 + \cdots + c_k\lambda_{k+1}\mathbf{v}_k \tag{3}$$

When we subtract equation (3) from equation (2), we obtain

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{k+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{k+1})\mathbf{v}_2 + \cdots + c_k(\lambda_k - \lambda_{k+1})\mathbf{v}_k$$

The linear independence of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ implies that

$$c_1(\lambda_1 - \lambda_{k+1}) = c_2(\lambda_2 - \lambda_{k+1}) = \cdots = c_k(\lambda_k - \lambda_{k+1}) = 0$$

Since the eigenvalues λ_i are all distinct, the terms in parentheses $(\lambda_i - \lambda_{k+1})$, $i = 1, \ldots, k$, are all nonzero. Hence, $c_1 = c_2 = \cdots = c_k = 0$. This implies that

$$\mathbf{v}_{k+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$$

which is impossible, since the eigenvector $\mathbf{v}_{k+1}$ cannot be zero. Thus, we have a contradiction, which means that our assumption that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent is false. It follows that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ must be linearly independent.

Exercises 4.3

In Exercises 1–12, compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, and (d) the algebraic and geometric multiplicity of each eigenvalue.

1. $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -9 \\ 1 & -5 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 2 \\ 3 & 0 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

10. $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -2 & 1 & 2 & -1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

13. Prove Theorem 4.18(b).

14. Prove Theorem 4.18(c). [*Hint:* Combine the proofs of parts (a) and (b) and see the fourth Remark following Theorem 3.9 (page 175).]

In Exercises 15 and 16, A is a 2 × 2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *corresponding to eigenvalues* $\lambda_1 = \frac{1}{2}$ *and* $\lambda_2 = 2$, *respectively, and* $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

15. Find $A^{10}\mathbf{x}$.

16. Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty$)?

In Exercises 17 and 18, A is a 3 × 3 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, *and* $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *corresponding to eigenvalues* $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = \frac{1}{3}$, *and* $\lambda_3 = 1$, *respectively, and* $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

17. Find $A^{20}\mathbf{x}$.

18. Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty$)?

19. **(a)** Show that, for any square matrix A, A^T and A have the same characteristic polynomial and hence the same eigenvalues.
(b) Give an example of a 2 × 2 matrix A for which A^T and A have different eigenspaces.

20. Let A be a nilpotent matrix (that is, $A^m = O$ for some $m > 1$). Show that $\lambda = 0$ is the only eigenvalue of A.

21. Let A be an idempotent matrix (that is, $A^2 = A$). Show that $\lambda = 0$ and $\lambda = 1$ are the only possible eigenvalues of A.

22. If $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ and c is a scalar, show that $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

23. **(a)** Find the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$$

(b) Using Theorem 4.18 and Exercise 22, find the eigenvalues and eigenspaces of A^{-1}, $A - 2I$, and $A + 2I$.

24. Let A and B be $n \times n$ matrices with eigenvalues λ and μ, respectively.
(a) Give an example to show that $\lambda + \mu$ need not be an eigenvalue of $A + B$.
(b) Give an example to show that $\lambda\mu$ need not be an eigenvalue of AB.
(c) Suppose λ and μ correspond to the *same* eigenvector $\mathbf{x}$. Show that, in this case, $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigenvalue of AB.

25. If A and B are two row equivalent matrices, do they necessarily have the same eigenvalues? Either prove that they do or give a counterexample.

Let p(x) be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

The ***companion matrix*** *of p(x) is the n × n matrix*

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (4)$$

26. Find the companion matrix of $p(x) = x^2 - 7x + 12$ and then find the characteristic polynomial of $C(p)$.

27. Find the companion matrix of $p(x) = x^3 + 3x^2 - 4x + 12$ and then find the characteristic polynomial of $C(p)$.

28. **(a)** Show that the companion matrix $C(p)$ of $p(x) = x^2 + ax + b$ has characteristic polynomial $\lambda^2 + a\lambda + b$.
(b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in part (a), then $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $C(p)$ corresponding to λ.

29. **(a)** Show that the companion matrix $C(p)$ of $p(x) = x^3 + ax^2 + bx + c$ has characteristic polynomial $-(\lambda^3 + a\lambda^2 + b\lambda + c)$.
(b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in part (a), then $\begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $C(p)$ corresponding to λ.

Solving for A, we have $A = PDP^{-1}$, which makes it easy to find powers of A. We compute

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

and, generally, $A^n = PD^nP^{-1}$ for all $n \geq 1$. (You should verify this by induction. Observe that this fact will be true for *any* diagonalizable matrix, not just the one in this example.)

Since

$$D^n = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

we have

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{2(-1)^n + 2^n}{3} & \dfrac{(-1)^{n+1} + 2^n}{3} \\ \dfrac{2(-1)^{n+1} + 2^{n+1}}{3} & \dfrac{(-1)^{n+2} + 2^{n+1}}{3} \end{bmatrix} \end{aligned}$$

Since we were only asked for A^{10}, this is more than we needed. But now we can simply set $n = 10$ to find

$$A^{10} = \begin{bmatrix} \dfrac{2(-1)^{10} + 2^{10}}{3} & \dfrac{(-1)^{11} + 2^{10}}{3} \\ \dfrac{2(-1)^{11} + 2^{11}}{3} & \dfrac{(-1)^{12} + 2^{11}}{3} \end{bmatrix} = \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$$

Exercises 4.4

In Exercises 1–4, show that A and B are not similar matrices.

1. $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 \\ -2 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

In Exercises 5–7, a diagonalization of the matrix A is given in the form $P^{-1}AP = D$. List the eigenvalues of A and bases for the corresponding eigenspaces.

5. $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{5}{8} & -\frac{3}{8} & -\frac{3}{8} \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

In Exercises 8–15, determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

8. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ **9.** $A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ **11.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 3 & 0 & 1 \end{bmatrix}$ **13.** $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **15.** $A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

In Exercises 16–23, use the method of Example 4.29 to compute the indicated power of the matrix.

16. $\begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix}^5$ **17.** $\begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}^{10}$

18. $\begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}^{-5}$ **19.** $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}^k$

20. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}^8$ **21.** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2002}$

22. $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}^k$ **23.** $\begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}^k$

In Exercises 24–29, find all (real) values of k for which A is diagonalizable.

24. $A = \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}$ **25.** $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

26. $A = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$ **27.** $A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **29.** $A = \begin{bmatrix} 1 & 1 & k \\ 1 & 1 & k \\ 1 & 1 & k \end{bmatrix}$

30. Prove Theorem 4.21(c).

31. Prove Theorem 4.22(b).

32. Prove Theorem 4.22(c).

33. Prove Theorem 4.22(e).

34. If A and B are invertible matrices, show that AB and BA are similar.

35. Prove that if A and B are similar matrices, then $\operatorname{tr}(A) = \operatorname{tr}(B)$. [*Hint:* Find a way to use Exercise 45 from Section 3.2.]

In general, it is difficult to show that two matrices are similar. However, if two similar matrices are diagonalizable, the task becomes easier. In Exercises 36–39, show that A and B are similar by showing that they are similar to the same diagonal matrix. Then find an invertible matrix P such that $P^{-1}AP = B$.

36. $A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

37. $A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ -6 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 2 & -1 \\ 2 & 2 & -4 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 & 0 \\ 6 & 5 & 0 \\ 4 & 4 & -1 \end{bmatrix}$

40. Prove that if A is similar to B, then A^T is similar to B^T.

41. Prove that if A is diagonalizable, so is A^T.

42. Let A be an invertible matrix. Prove that if A is diagonalizable, so is A^{-1}.

43. Prove that if A is a diagonalizable matrix with only one eigenvalue λ, then A is of the form $A = \lambda I$. (Such a matrix is called a ***scalar matrix.***)

44. Let A and B be $n \times n$ matrices, each with n distinct eigenvalues. Prove that A and B have the same eigenvectors if and only if $AB = BA$.

45. Let A and B be similar matrices. Prove that the algebraic multiplicities of the eigenvalues of A and B are the same.

Looking at the orthogonal matrices A and B in Example 5.7, you may notice that not only do their columns form orthonormal sets—so do their *rows*. In fact, every orthogonal matrix has this property, as the next theorem shows.

Theorem 5.7 If Q is an orthogonal matrix, then its rows form an orthonormal set.

Proof From Theorem 5.5, we know that $Q^{-1} = Q^T$. Therefore,

$$(Q^T)^{-1} = (Q^{-1})^{-1} = Q = (Q^T)^T$$

so Q^T is an orthogonal matrix. Thus, the columns of Q^T—which are just the rows of Q—form an orthonormal set.

The final theorem in this section lists some other properties of orthogonal matrices.

Theorem 5.8 Let Q be an orthogonal matrix.

a. Q^{-1} is orthogonal.
b. $\det Q = \pm 1$
c. If λ is an eigenvalue of Q, then $|\lambda| = 1$.
d. If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is Q_1Q_2.

Proof We will prove property (c) and leave the proofs of the remaining properties as exercises.

(c) Let λ be an eigenvalue of Q with corresponding eigenvector $\mathbf{v}$. Then $Q\mathbf{v} = \lambda\mathbf{v}$, and, using Theorem 5.6(b), we have

$$\|\mathbf{v}\| = \|Q\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$$

Since $\|\mathbf{v}\| \neq 0$, this implies that $|\lambda| = 1$.

a + bi

Remark Property (c) holds even for complex eigenvalues. The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is orthogonal with eigenvalues i and $-i$, both of which have absolute value 1.

Exercises 5.1

In Exercises 1–6, determine which sets of vectors are orthogonal.

1. $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -1 \end{bmatrix}$

5. $\begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ 2 \\ 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 7–10, show that the given vectors form an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$. Then use Theorem 5.2 to express **w** *as a linear combination of these basis vectors. Give the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$ of* **w** *with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$ or $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$.*

7. $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

8. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

9. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

10. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

In Exercises 11–15, determine whether the given orthogonal set of vectors is orthonormal. If it is not, normalize the vectors to form an orthonormal set.

11. $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

12. $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -\frac{5}{2} \end{bmatrix}$

14. $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix}$

15. $\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{6}/3 \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ -\sqrt{3}/6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

In Exercises 16–21, determine whether the given matrix is orthogonal. If it is, find its inverse.

16. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

18. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{2} & -\frac{1}{3} & \frac{2}{5} \\ -\frac{1}{2} & 0 & \frac{4}{5} \end{bmatrix}$

19. $\begin{bmatrix} \cos\theta\sin\theta & -\cos\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta & -\cos\theta\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

20. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & 0 & 1/\sqrt{6} \\ 0 & 2/3 & 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/3 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/3 & 0 & 1/\sqrt{2} \end{bmatrix}$

22. Prove Theorem 5.8(a).

23. Prove Theorem 5.8(b).

24. Prove Theorem 5.8(d).

25. Prove that every permutation matrix is orthogonal.

26. If Q is an orthogonal matrix, prove that any matrix obtained by rearranging the rows of Q is also orthogonal.

27. Let Q be an orthogonal 2×2 matrix and let **x** and **y** be vectors in $\mathbb{R}^2$. If θ is the angle between **x** and **y**, prove that the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ is also θ. (This proves that the linear transformations defined by orthogonal matrices are *angle-preserving* in $\mathbb{R}^2$, a fact that is true in general.)

28. **(a)** Prove that an orthogonal 2×2 matrix must have the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector.

(b) Using part (a), show that every orthogonal 2×2 matrix is of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $0 \le \theta < 2\pi$.

(c) Show that every orthogonal 2×2 matrix corresponds to either a rotation or a reflection in $\mathbb{R}^2$.

(d) Show that an orthogonal 2×2 matrix Q corresponds to a rotation in $\mathbb{R}^2$ if $\det Q = 1$ and a reflection in $\mathbb{R}^2$ if $\det Q = -1$.

In Exercises 29–32, use Exercise 28 to determine whether the given orthogonal matrix represents a rotation or a reflection. If it is a rotation, give the angle of rotation; if it is a reflection, give the line of reflection.

29. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

30. $\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$

31. $\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

32. $\begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$

Sections 5.1 and 5.2 have illustrated some of the advantages of working with orthogonal bases. However, we have not established that every subspace *has* an orthogonal basis, nor have we given a method for constructing such a basis (except in particular examples, such as Example 5.3). These issues are the subject of the next section.

Exercises 5.2

In Exercises 1–6, find the orthogonal complement $W^\perp$ of W and give a basis for $W^\perp$.

1. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x - y = 0 \right\}$

2. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 3x + 2y = 0 \right\}$

3. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$

4. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : -x + 3y - 5z = 0 \right\}$

5. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = -t, z = 3t \right\}$

6. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 2t, y = -t, z = \frac{1}{2}t \right\}$

In Exercises 7 and 8, find bases for the row space and null space of A. Verify that every vector in row(A) is orthogonal to every vector in null(A).

7. $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 \\ -2 & 0 & 4 & 0 & 2 \\ 2 & 2 & -2 & 3 & 1 \\ 2 & 4 & 2 & 5 & 1 \end{bmatrix}$

In Exercises 9 and 10, find bases for the column space of A and the null space of A^T for the given exercise. Verify that every vector in col(A) is orthogonal to every vector in null(A^T).

9. Exercise 7

10. Exercise 8

In Exercises 11–14, let W be the subspace spanned by the given vectors. Find a basis for $W^\perp$.

11. $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

12. $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

13. $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 6 \\ 3 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ -3 \\ -2 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 1 \end{bmatrix}$

14. $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 3 \\ -2 \\ 6 \\ -2 \\ 5 \end{bmatrix}$

In Exercises 15–18, find the orthogonal projection of $\mathbf{v}$ *onto the subspace W spanned by the vectors* $\mathbf{u}_i$. *(You may assume that the vectors* $\mathbf{u}_i$ *are orthogonal.)*

15. $\mathbf{v} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

16. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

17. $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

18. $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ -3 \\ 2 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 19–22, find the orthogonal decomposition of $\mathbf{v}$ *with respect to W.*

19. $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$

20. $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$

21. $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$

22. $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 3 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$

23. Prove Theorem 5.9(c).

24. Prove Theorem 5.9(d).

25. Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ a vector in $\mathbb{R}^n$. Suppose that $\mathbf{w}$ and $\mathbf{w}'$ are orthogonal vectors with $\mathbf{w}$ in W and that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$. Is it necessarily true that $\mathbf{w}'$ is in $W^\perp$? Either prove that it is true or find a counterexample.

26. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$ and let $W = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Is it necessarily true that $W^\perp = \operatorname{span}(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$? Either prove that it is true or find a counterexample.

In Exercises 27–29, let W be a subspace of $\mathbb{R}^n$*, and let* $\mathbf{x}$ *be a vector in* $\mathbb{R}^n$.

27. Prove that $\mathbf{x}$ is in W if and only if $\operatorname{proj}_W(\mathbf{x}) = \mathbf{x}$.

28. Prove that $\mathbf{x}$ is orthogonal to W if and only if $\operatorname{proj}_W(\mathbf{x}) = \mathbf{0}$.

29. Prove that $\operatorname{proj}_W(\operatorname{proj}_W(\mathbf{x})) = \operatorname{proj}_W(\mathbf{x})$.

30. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthonormal set in $\mathbb{R}^n$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$.

(a) Prove that

$$\|\mathbf{x}\|^2 \geq |\mathbf{x}\cdot\mathbf{v}_1|^2 + |\mathbf{x}\cdot\mathbf{v}_2|^2 + \cdots + |\mathbf{x}\cdot\mathbf{v}_k|^2$$

(This inequality is called ***Bessel's Inequality.***)

(b) Prove that Bessel's Inequality is an equality if and only if $\mathbf{x}$ is in $\operatorname{span}(S)$.

5.3 The Gram-Schmidt Process and the QR Factorization

In this section, we present a simple method for constructing an orthogonal (or orthonormal) basis for any subspace of $\mathbb{R}^n$. This method will then lead us to one of the most useful of all matrix factorizations.

The Gram-Schmidt Process

We would like to be able to find an orthogonal basis for a subspace W of $\mathbb{R}^n$. The idea is to begin with an arbitrary basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for W and to "orthogonalize" it one vector at a time. We will illustrate the basic construction with the subspace W from Example 5.3.

Example 5.12 Let $W = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Construct an orthogonal basis for W.

Solution Starting with $\mathbf{x}_1$, we get a second vector that is orthogonal to it by taking the component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$ (Figure 5.10).

- The QR factorization can be extended to arbitrary matrices in a slightly modified form. If A is $m \times n$, it is possible to find a sequence of orthogonal matrices $Q_1, \ldots, Q_{m-1}$ such that $Q_{m-1} \cdots Q_2 Q_1 A$ is an upper triangular $m \times n$ matrix R. Then $A = QR$, where $Q = (Q_{m-1} \cdots Q_2 Q_1)^{-1}$ is an orthogonal matrix. We will examine this approach in Exploration: The Modified QR Factorization.

Example 5.15 Find a QR factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution The columns of A are just the vectors from Example 5.13. The orthonormal basis for col(A) produced by the Gram-Schmidt Process was

$$\mathbf{q}_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

so

$$Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3] = \begin{bmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{bmatrix}$$

From Theorem 5.16, $A = QR$ for some upper triangular matrix R. To find R, we use the fact that Q has orthonormal columns and, hence, $Q^TQ = I$. Therefore,

$$Q^TA = Q^TQR = IR = R$$

We compute

$$R = Q^TA = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

Exercises 5.3

In Exercises 1–4, the given vectors form a basis for $\mathbb{R}^2$ or $\mathbb{R}^3$. Apply the Gram-Schmidt Process to obtain an orthogonal basis. Then normalize this basis to obtain an orthonormal basis.

1. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

3. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$

4. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 5 and 6, the given vectors form a basis for a subspace W of $\mathbb{R}^3$ or $\mathbb{R}^4$. Apply the Gram-Schmidt Process to obtain an orthogonal basis for W.

5. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$

6. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 0 \end{bmatrix}$

In Exercises 7 and 8, find the orthogonal decomposition of $\mathbf{v}$ *with respect to the subspace W.*

7. $\mathbf{v} = \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix}$, W as in Exercise 5

8. $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}$, W as in Exercise 6

Use the Gram-Schmidt Process to find an orthogonal basis for the column spaces of the matrices in Exercises 9 and 10.

9. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

11. Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$.

12. Find an orthogonal basis for $\mathbb{R}^4$ that contains the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$

In Exercises 13 and 14, fill in the missing entries of Q to make Q an orthogonal matrix.

13. $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & * \\ 0 & 1/\sqrt{3} & * \\ -1/\sqrt{2} & 1/\sqrt{3} & * \end{bmatrix}$

14. $Q = \begin{bmatrix} 1/2 & 2/\sqrt{14} & * & * \\ 1/2 & 1/\sqrt{14} & * & * \\ 1/2 & 0 & * & * \\ 1/2 & -3/\sqrt{14} & * & * \end{bmatrix}$

In Exercises 15 and 16, find a QR factorization of the matrix in the given exercise.

15. Exercise 9

16. Exercise 10

In Exercises 17 and 18, the columns of Q were obtained by applying the Gram-Schmidt Process to the columns of A. Find the upper triangular matrix R such that $A = QR$.

17. $A = \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix}, Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix}$

19. If A is an orthogonal matrix, find a QR factorization of A.

20. Prove that A is invertible if and only if $A = QR$, where Q is orthogonal and R is upper triangular with nonzero entries on its diagonal.

In Exercises 21 and 22, use the method suggested by Exercise 20 to compute A^{-1} for the matrix A in the given exercise.

21. Exercise 9

22. Exercise 15

23. Let A be an $m \times n$ matrix with linearly independent columns. Give an alternative proof that the upper triangular matrix R in a QR factorization of A must be invertible, using property (c) of the Fundamental Theorem.

24. Let A be an $m \times n$ matrix with linearly independent columns and let $A = QR$ be a QR factorization of A. Show that A and Q have the same column space.

Solution We begin by normalizing the vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$, with

$$\mathbf{q}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

Now, we compute the matrix A whose spectral decomposition is

$$\begin{aligned} A &= \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T \\ &= 3\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} - 2\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \\ &= 3\begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix} - 2\begin{bmatrix} \frac{16}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{6}{5} \end{bmatrix} \end{aligned}$$

It is easy to check that A has the desired properties. (Do this.)

Exercises 5.4

Orthogonally diagonalize the matrices in Exercises 1–10 by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

1. $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$

5. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

9. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

11. If $b \neq 0$, orthogonally diagonalize $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

12. If $b \neq 0$, orthogonally diagonalize $A = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix}$.

13. Let A and B be orthogonally diagonalizable $n \times n$ matrices and let c be a scalar. Use the Spectral Theorem to prove that the following matrices are orthogonally diagonalizable:

(a) $A + B$ **(b)** cA **(c)** A^2

14. If A is an invertible matrix that is orthogonally diagonalizable, show that A^{-1} is orthogonally diagonalizable.

15. If A and B are orthogonally diagonalizable and $AB = BA$, show that AB is orthogonally diagonalizable.

16. If A is a symmetric matrix, show that every eigenvalue of A is nonnegative if and only if $A = B^2$ for some symmetric matrix B.

In Exercises 17–20, find a spectral decomposition of the matrix in the given exercise.

17. Exercise 1 **18.** Exercise 2

19. Exercise 5 **20.** Exercise 8

In Exercises 21 and 22, find a symmetric 2×2 matrix with eigenvalues λ_1 and λ_2 and corresponding orthogonal eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

21. $\lambda_1 = -1, \lambda_2 = 2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

22. $\lambda_1 = 2, \lambda_2 = -2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

In Exercises 23 and 24, find a symmetric 3×3 matrix with eigenvalues λ_1, λ_2, and λ_3 and corresponding orthogonal eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

23. $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$
$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

24. $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$
$\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

25. Let $\mathbf{q}$ be a unit vector in $\mathbb{R}^n$ and let W be the subspace spanned by $\mathbf{q}$. Show that the orthogonal projection of a vector $\mathbf{v}$ onto W (as defined in Sections 1.2 and 5.2) is given by

$$\mathrm{proj}_W(\mathbf{v}) = (\mathbf{q}\mathbf{q}^T)\mathbf{v}$$

and that the matrix of this projection is thus $\mathbf{q}\mathbf{q}^T$. [*Hint:* Remember that, for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$.]

26. Let $\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$ and let W be the subspace spanned by this set.

(a) Show that the matrix of the orthogonal projection onto W is given by

$$P = \mathbf{q}_1\mathbf{q}_1^T + \cdots + \mathbf{q}_k\mathbf{q}_k^T$$

(b) Show that the projection matrix P in part (a) is symmetric and satisfies $P^2 = P$.

(c) Let $Q = [\mathbf{q}_1 \ \cdots \ \mathbf{q}_k]$ be the $n \times k$ matrix whose columns are the orthonormal basis vectors of W. Show that $P = QQ^T$ and deduce that $\mathrm{rank}(P) = k$.

27. Let A be an $n \times n$ real matrix, all of whose eigenvalues are real. Prove that there exist an orthogonal matrix Q and an upper triangular matrix T such that $Q^TAQ = T$. This very useful result is known as ***Schur's Triangularization Theorem.*** [*Hint:* Adapt the proof of the Spectral Theorem.]

28. Let A be a nilpotent matrix (see Exercise 56 in Section 4.2). Prove that there is an orthogonal matrix Q such that $Q^T AQ$ is upper triangular with zeros on its diagonal. [*Hint:* Use Exercise 27.]

5.5 Applications

Dual Codes

There are many ways of constructing new codes from old ones. In this section, we consider one of the most important of these.

First, we need to generalize the concepts of a generator and a parity check matrix for a code. Recall from Section 3.7 that a standard generator matrix for a code is an $n \times m$ matrix of the form

$$G = \begin{bmatrix} I_m \\ \hline A \end{bmatrix}$$

and a standard parity check matrix is an $(n - m) \times n$ matrix of the form

$$P = [B \mid I_{n-m}]$$

Observe that the form of these matrices guarantees that the columns of G are linearly independent and the rows of P are linearly independent. (Why?) In proving Theorem 3.37, we showed that G and P are associated with the *same* code if and only

The projection matrix becomes $[P] = AA^{+} = AA^{-1} = I$. (What is the geometric interpretation of this equality?)

Theorem 7.12 summarizes the key properties of the pseudoinverse of a matrix. (Before reading the proof of this theorem, verify these properties for the matrix in Example 7.32.)

Theorem 7.12 Let A be a matrix with linearly independent columns. Then the pseudoinverse A^{+} of A satisfies the following properties, called the ***Penrose conditions*** for A:

a. $AA^{+}A = A$
b. $A^{+}AA^{+} = A^{+}$
c. AA^{+} and $A^{+}A$ are symmetric.

Proof We prove condition (a) and half of condition (c) and leave the proofs of the remaining conditions as Exercises 54 and 55.

(a) We compute

$$\begin{aligned} AA^{+}A &= A((A^TA)^{-1}A^T)A \\ &= A(A^TA)^{-1}(A^TA) \\ &= AI = A \end{aligned}$$

(c) By Theorem 3.4, A^TA is symmetric. Therefore, $(A^TA)^{-1}$ is also symmetric, by Exercise 46 in Section 3.3. Taking the transpose of AA^{+}, we have

$$\begin{aligned} (AA^{+})^T &= (A(A^TA)^{-1}A^T)^T \\ &= (A^T)^T((A^TA)^{-1})^TA^T \\ &= A(A^TA)^{-1}A^T \\ &= AA^{+} \end{aligned}$$

Exercise 56 explores further properties of the pseudoinverse. In the next section, we will see how to extend the definition of A^{+} to handle *all* matrices, whether or not the columns of A are linearly independent.

Exercises 7.3

CAS

In Exercises 1–3, consider the data points $(1, 0)$, $(2, 1)$, and $(3, 5)$. Compute the least squares error for the given line. In each case, plot the points and the line.

1. $y = -2 + 2x$ **2.** $y = -3 + 2x$ **3.** $y = -3 + \frac{5}{2}x$

In Exercises 4–6, consider the data points $(-5, 3)$, $(0, 3)$, $(5, 2)$, and $(10, 0)$. Compute the least squares error for the given line. In each case, plot the points and the line.

4. $y = 2 - x$ **5.** $y = \frac{5}{2}$ **6.** $y = 2 - \frac{1}{5}x$

In Exercises 7–14, find the least squares approximating line for the given points and compute the corresponding least squares error.

7. $(1, 0), (2, 1), (3, 5)$

8. $(1, 5), (2, 3), (3, 2)$

9. $(0, 4), (1, 1), (2, 0)$

10. $(0, 2), (1, 2), (2, 5)$

11. $(-5, -1), (0, 1), (5, 2), (10, 4)$

12. $(-5, 3), (0, 3), (5, 2), (10, 0)$

13. $(1, 1), (2, 3), (3, 4), (4, 5), (5, 7)$

14. $(1, 10), (2, 8), (3, 5), (4, 3), (5, 0)$

In Exercises 15–18, find the least squares approximating parabola for the given points.

15. $(1, 1), (2, -2), (3, 3), (4, 4)$

16. $(1, 8), (2, 7), (3, 5), (4, 2)$

17. $(-2, 4), (-1, 7), (0, 3), (1, 0), (2, -1)$

18. $(-2, 0), (-1, -11), (0, -10), (1, -9), (2, 8)$

In Exercises 19–22, find a least squares solution of $A\mathbf{x} = \mathbf{b}$ *by constructing and solving the normal equations.*

19. $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

20. $A = \begin{bmatrix} 3 & -2 \\ 1 & -2 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

21. $A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 4 \end{bmatrix}$

22. $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

In Exercises 23 and 24, show that the least squares solution of $A\mathbf{x} = \mathbf{b}$ *is not unique and solve the normal equations to find all the least squares solutions.*

23. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

In Exercises 25 and 26, find the best approximation to a solution of the given system of equations.

25.
$$\begin{aligned} x + y - z &= 2 \\ -y + 2z &= 6 \\ 3x + 2y - z &= 11 \\ -x + z &= 0 \end{aligned}$$

26.
$$\begin{aligned} 2x + 3y + z &= 21 \\ x + y + z &= 7 \\ -x + y - z &= 14 \\ 2y + z &= 0 \end{aligned}$$

In Exercises 27 and 28, a QR factorization of A is given. Use it to find a least squares solution of $A\mathbf{x} = \mathbf{b}$.

27. $A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, R = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix},$

$R = \begin{bmatrix} \sqrt{6} & -\sqrt{6}/2 \\ 0 & 1/\sqrt{2} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

29. A tennis ball is dropped from various heights, and the height of the ball on the first bounce is measured. Use the data in Table 7.3 to find the least squares approximating line for bounce height b as a linear function of initial height h.

Table 7.3

h (cm)	20	40	48	60	80	100
b (cm)	14.5	31	36	45.5	59	73.5

30. Hooke's Law states that the length L of a spring is a linear function of the force F applied to it. (See Figure 7.17 and Example 6.92.) Accordingly, there are constants a and b such that

$$L = a + bF$$

Table 7.4 shows the results of attaching various weights to a spring.

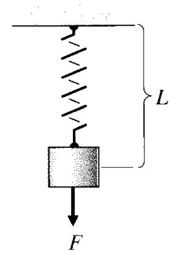


Figure 7.17

Table 7.4

F (oz)	2	4	6	8
L (in.)	7.4	9.6	11.5	13.6

Complex Numbers Exercises

Answers appear on the next page.

1. Express $z = (1 + 2i)(4 - 6i)^2$ in rectangular form.

2. Find $\overline{z}$ and $|z|$ for:
a) $z = 2 + 7i$
b) $z = -3 - 5i$

3. Find $\frac{1}{z}$ for the following. Give your answers in rectangular form.
a) $z = 1 - 5i$
b) $z = 1 + i$

4. Find $\frac{z_1}{z_2}$ for $z_1 = 2 + i$ and $z_2 = -7 + 5i$. Give your answer in rectangular form.

5. Let $z_1 = 2 + 2\sqrt{3}i$ and $z_2 = \frac{\sqrt{3}}{3} + \frac{1}{3}i$.
a) Convert z_1 and z_2 to polar form.
b) Compute $z_1 z_2$
c) Compute $\frac{z_1}{z_2}$.

6. Compute $(i + 1)^8$ by converting $i + 1$ to polar form.

7. Solve $z^6 - 64 = 0$.

8. Let $w = a + bi$ and $z = c + di$. Show that $|wz| = |w||z|$.

ANSWERS

1. $z = 76 - 88i$

2. a) $2 - 7i$, $\sqrt{53}$
b) $-3 + 5i$, $\sqrt{34}$

3. a) $\frac{1}{26} + \frac{5}{26}i$
b) $\frac{1}{2} - \frac{1}{2}i$

4. $\frac{-9}{74} - \frac{17}{74}i$.

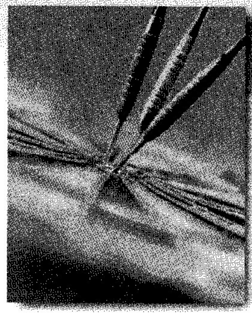
5. a) $z_1 = 4[\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})]$ and $z_2 = \frac{2}{3}[\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})]$.
b) $\frac{8}{3}[\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})] = \frac{8}{3}i$.
c) $6[\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})]$.

6. 16

7. $z = \pm 2, 1 \pm \sqrt{3}i, -1 \pm \sqrt{3}i$.

8.

$$\begin{aligned}
|wz| &= |(ac-bd)+(ad+bc)i| \\
&= \sqrt{(ac-bd)^2+(ad+bc)^2} \\
&= \sqrt{[(ac)^2-2acbd+(bd)^2]+[(ad)^2+2adbc+(bc)^2]} \\
&= \sqrt{(ac)^2+(bd)^2+(ad)^2+(bc)^2} \\
&= \sqrt{(a^2+b^2)c^2+(a^2+b^2)d^2} \\
&= \sqrt{(a^2+b^2)(c^2+d^2)} \\
&= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\
&= |w||z|
\end{aligned}$$



Answers to Selected Odd-Numbered Exercises

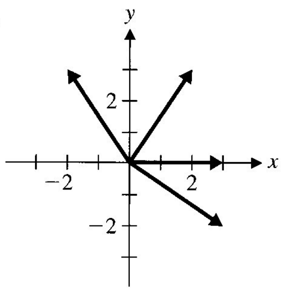
Answers are easy. It's asking the right questions [that's] hard.

—Doctor Who
"The Face of Evil,"
By Chris Boucher
BBC, 1977

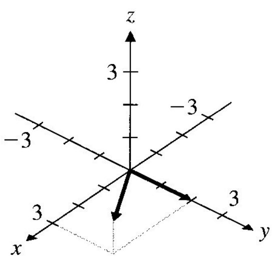
Chapter 1

Exercises 1.1

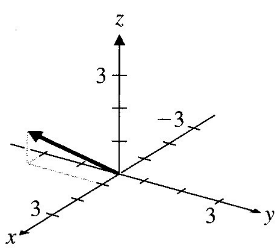
1.



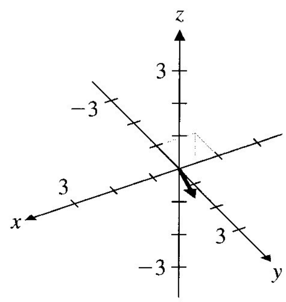
3. (a), (b)



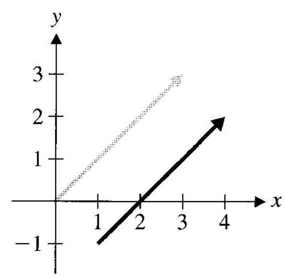
(c)



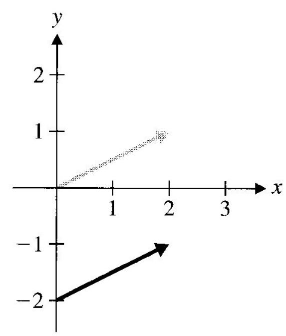
(d)



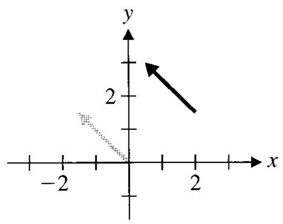
5. (a)



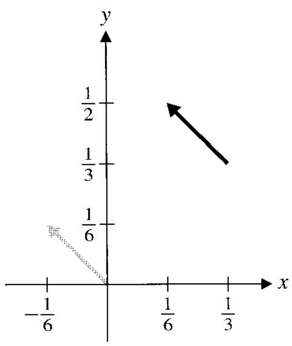
(b)



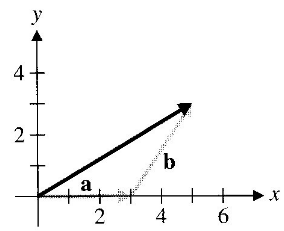
(c)



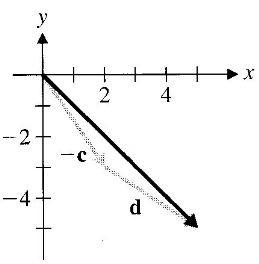
(d)



7. $\mathbf{a} + \mathbf{b} = [5, 3]$



9. $\mathbf{d} - \mathbf{c} = [5, -5]$



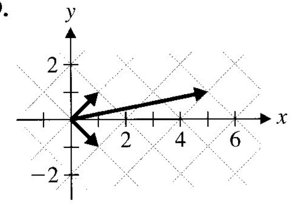
11. $[3, -2, 3]$

13. $\mathbf{u} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \mathbf{u} + \mathbf{v} = \begin{bmatrix} (1 - \sqrt{3})/2 \\ (\sqrt{3} - 1)/2 \end{bmatrix}, \mathbf{u} - \mathbf{v} = \begin{bmatrix} (1 + \sqrt{3})/2 \\ (1 + \sqrt{3})/2 \end{bmatrix}$

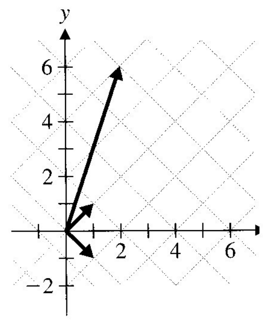
15. $5\mathbf{a}$

17. $\mathbf{x} = 3\mathbf{a}$

19.



21. $\mathbf{w} = -2\mathbf{u} + 4\mathbf{v}$



25. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u} \cdot \mathbf{v} = 1$

27. $\mathbf{u} + \mathbf{v} = [0, 1, 0, 0], \mathbf{u} \cdot \mathbf{v} = 1$

29.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

31. 0

33. 1

35. 0

37. 2, 0, 3

39. 5

41. $[1, 1, 0]$

43. 3, 2

45. $x = 2$

47. No solution

49. $x = 3$

51. No solution

53. $x = 2$

55. $x = 1$, or $x = 5$

57. **(a)** All $a \neq 0$ **(b)** $a = 1, 5$
(c) a and m can have no common factors other than 1 [i.e., the greatest common divisor (gcd) of a and m is 1].

Exercises 1.2

1. −1

3. 11

5. 2

7. $\sqrt{5}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

9. $\sqrt{14}, \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$

11. $\sqrt{6}, [1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}, 0]$

13. $\sqrt{17}$ **15.** $\sqrt{6}$

17. **(a)** $\mathbf{u} \cdot \mathbf{v}$ is a scalar, not a vector.

(c) $\mathbf{v} \cdot \mathbf{w}$ is a scalar and $\mathbf{u}$ is a vector.

19. Acute **21.** Acute **23.** Acute

25. 60° **27.** ≈88.10° **29.** ≈14.34°

31. Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = 0$, $\angle BAC$ is a right angle.

33. If we take the cube to be a unit cube (as in Figure 1.34), the four diagonals are given by the vectors

$$\mathbf{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Since $\mathbf{d}_i \cdot \mathbf{d}_j \neq 0$ for all $i \neq j$ (six possibilities), no two diagonals are perpendicular.

35. $D = (-2, 1, 1)$

37. 5 mi/h at an angle of ≈ 53.13° to the bank

39. 60°

41. $\begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ **43.** $\begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$ **45.** $\begin{bmatrix} -0.301 \\ 0.033 \\ -0.252 \end{bmatrix}$

47. $\mathcal{A} = \sqrt{45}/2$ **49.** $k = -2, 3$

51. $\mathbf{v}$ is of the form $k\begin{bmatrix} b \\ -a \end{bmatrix}$, where k is a scalar.

53. The Cauchy-Schwarz Inequality would be violated.

Exercises 1.3

1. **(a)** $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$ **(b)** $3x + 2y = 0$

3. **(a)** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) $x = 1 - t$
$y = 3t$

5. **(a)** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ **(b)** $x = t$, $y = -t$, $z = 4t$

7. **(a)** $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2$ **(b)** $3x + 2y + z = 2$

9. **(a)** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

(b) $x = 2s - 3t$
$y = s + 2t$
$z = 2s + t$

11. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

13. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$

15. **(a)** $x = t$, $y = -1 + 3t$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

17. Direction vectors for the two lines are given by $\mathbf{d}_1 = \begin{bmatrix} 1 \\ m_1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ m_2 \end{bmatrix}$. The lines are perpendicular if and only if $\mathbf{d}_1$ and $\mathbf{d}_2$ are orthogonal. But $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ if and only if $1 + m_1 m_2 = 0$ or, equivalently, $m_1 m_2 = -1$.

19. **(a)** Perpendicular **(b)** Parallel
(c) Perpendicular **(d)** Perpendicular

21. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

23. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

25. **(a)** $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$
(b) $x - y = 0$ **(c)** $x + y - z = 0$

27. $3\sqrt{2}/2$ **29.** $2\sqrt{3}/3$ **31.** $(\frac{1}{2}, \frac{1}{2})$

33. $(\frac{4}{3}, \frac{4}{3}, \frac{8}{3})$ **35.** $18\sqrt{13}/13$ **37.** $\frac{5}{3}$

43. ≈78.9° **45.** ≈80.4°

Exercises 1.4

1. 13 N at approx N 67.38 E

3. $8\sqrt{3}$ N at an angle of 30° to $\mathbf{f}_1$

5. 4 N at an angle of 60° to $\mathbf{f}_2$

7. 5 N at an angle of 60° to the given force, $5\sqrt{3}$ N perpendicular to the 5 N force

9. $750\sqrt{2}$ N

11. 980 N

13. $\approx$ 117.6 N in the 15 cm wire, $\approx$ 88.2 N in the 20 cm wire

15. $[1, 0, 1, 1, 1]$

17. No error

19. No error

21. $d = 2$

23. $d = 0$

27. $d = 0$

31. $d = X$

33. (b) $[0, 4, 4, 9, 9, 0, 8, 3, 5, 6]$

35. (b) $[0, 3, 8, 7, 0, 9, 9, 0, 2, 6]$

Review Questions

1. (a) T (c) F (e) T (g) F (i) T

3. $\mathbf{x} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ **5.** 120° **7.** $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

9. $2x + 3y - z = 7$ **11.** $\sqrt{6}/2$

13. The Cauchy-Schwarz Inequality would be violated.

15. $2\sqrt{6}/3$ **17.** $x = 2$ **19.** 3

Chapter 2

Exercises 2.1

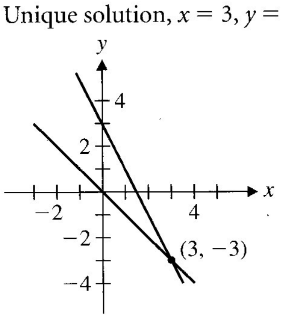
1. Linear **3.** Not linear because of the x^{-1} term

5. Linear **7.** $2x + 4y = 7$

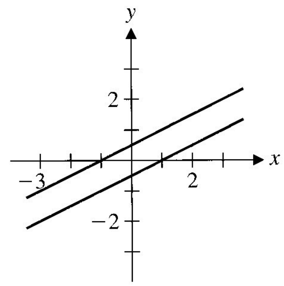
9. $x + y = 4(x, y \neq 0)$

11. $\left\{\begin{bmatrix} 2t \\ t \end{bmatrix}\right\}$ **13.** $\left\{\begin{bmatrix} 4 - 2s - 3t \\ s \\ t \end{bmatrix}\right\}$

15. Unique solution, $x = 3, y = -3$



17. No solution



19. $[7, 3]$ **21.** $\left[\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right]$

23. $[5, -2, 1, 1]$ **25.** $[2, -7, -32]$

27. $\left[\begin{array}{rr|r} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right]$ **29.** $\left[\begin{array}{rr|r} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right]$

31. $\begin{aligned} y + z &= 1 \\ x - y &= 1 \\ 2x - y + z &= 1 \end{aligned}$ **33.** $[1, 1]$

35. $[4, -1]$ **37.** No solution

39. (a) $\begin{aligned} 2x + y &= 3 \\ 4x + 2y &= 6 \end{aligned}$ (b) $\begin{aligned} x &= \tfrac{3}{2} - \tfrac{1}{2}s \\ y &= s \end{aligned}$

41. Let $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$. The solution is $x = \frac{1}{3}, y = -\frac{1}{2}$.

43. Let $u = \tan x$, $v = \sin y$, $w = \cos z$. One solution is $x = \pi/4$, $y = -\pi/6$, $z = \pi/3$. (There are infinitely many solutions.)

Exercises 2.2

1. No **3.** Reduced row echelon form

5. No **7.** No

9. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **11.** (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

13. (b) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

15. Perform elementary row operations in the order $R_4 + 29R_3$, $8R_3$, $R_4 - 3R_2$, $R_2 \leftrightarrow R_3$, $R_4 - R_1$, $R_3 + 2R_1$, and, finally, $R_2 + 2R_1$.

17. One possibility is to perform elementary row operations on A in the order $R_2 - 3R_1$, $\frac{1}{2}R_2$, $R_1 + 2R_2$, $R_2 + 3R_1$, $R_1 \leftrightarrow R_2$.

19. *Hint:* Pick a random 2×2 matrix and try this—carefully!

21. This is really two elementary row operations combined: $3R_2$ and $R_2 - 2R_1$.

23. Exercise 1: 3; Exercise 3: 2; Exercise 5: 2; Exercise 7: 3

25. $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ **27.** $t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **29.** $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

31. $\begin{bmatrix} 24 \\ -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

33. No solution

35. Unique solution

37. Infinitely many solutions

39. *Hint:* Show that if $ad - bc \neq 0$, the rank of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 2. (There are two cases: $a = 0$ and $a \neq 0$.) Use the Rank Theorem to deduce that the given system must have a unique solution.

41. **(a)** No solution if $k = -1$
(b) A unique solution if $k \neq \pm 1$
(c) Infinitely many solutions if $k = 1$

43. **(a)** No solution if $k = 1$
(b) A unique solution if $k \neq -2, 1$
(c) Infinitely many solutions if $k = -2$

45. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 9 \\ -10 \\ -7 \end{bmatrix}$

49. No intersection

51. The required vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are the solutions of the homogeneous system with augmented matrix

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right]$$

By Theorem 3, there are infinitely many solutions. If $u_1 \neq 0$ and $u_1v_2 - u_2v_1 \neq 0$, the solutions are given by

$$t\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

But a direct check shows that these are still solutions even if $u_1 = 0$ and/or $u_1v_2 - u_2v_1 = 0$.

53. $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ **55.** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **57.** $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Exercises 2.3

1. Yes **3.** No **5.** Yes **7.** Yes

9. We need to show that the vector equation $x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has a solution for all values of a and b.
This vector equation is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right]$. Row reduction yields $\left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b - a \end{array}\right]$, from which we can see that there is a (unique) solution.
[Further row operations yield $x = (a + b)/2$, $y = (a - b)/2$.] Hence, $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

11. We need to show that the vector equation $x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a solution for all values of a, b, and c. This vector equation is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right]$. Row reduction yields $\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & b + c - a \end{array}\right]$, from which we can see that there is a (unique) solution. [Further row operations yield $x = (a - b + c)/2$, $y = (a + b - c)/2$, $z = (-a + b + c)/2$.]

Hence, $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

13. **(a)** The line through the origin with direction vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
(b) The line with general equation $2x + y = 0$

15. **(a)** The plane through the origin with direction vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$
(b) The plane with general equation $2x - y + 4z = 0$

17. Substitution yields the linear system

$$\begin{aligned} a \quad\;\; + 3c &= 0 \\ -a + b - 3c &= 0 \end{aligned}$$

whose solution is $t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. It follows that there are infinitely many solutions, the simplest perhaps being $a = -3, b = 0, c = 1$.

19. $\mathbf{u} = \mathbf{u} + 0(\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$
$\mathbf{v} = (-1)\mathbf{u} + (\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$
$\mathbf{w} = 0\mathbf{u} + (-1)(\mathbf{u} + \mathbf{v}) + (\mathbf{u} + \mathbf{v} + \mathbf{w})$

21. **(c)** We must show that span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) = span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$). We know that span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$) $\subseteq \mathbb{R}^3$ = span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$). From Exercise 19, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ all belong to span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$). Therefore, by Exercise 21(b), span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) = span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$).

23. Linearly independent

25. Linearly dependent, $-\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

27. Linearly dependent, since the set contains the zero vector

29. Linearly independent

31. Linearly dependent, $\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$

43. **(a)** Yes **(b)** No

Exercises 2.4

1. $x_1 = 160, x_2 = 120, x_3 = 160$

3. two small, three medium, four large

5. 65 bags of house blend, 30 bags of special blend, 45 bags of gourmet blend

7. $4FeS_2 + 11O_2 \longrightarrow 2Fe_2O_3 + 8SO_2$

9. $2C_4H_{10} + 13O_2 \longrightarrow 8CO_2 + 10H_2O$

11. $2C_5H_{11}OH + 15O_2 \longrightarrow 12H_2O + 10CO_2$

13. $Na_2CO_3 + 4C + N_2 \longrightarrow 2NaCN + 3CO$

15. **(a)** $\begin{aligned} f_1 &= 30 - t \\ f_2 &= -10 + t \\ f_3 &= t \end{aligned}$ **(b)** $f_1 = 15, f_3 = 15$

(c) $\begin{aligned} 0 &\le f_1 \le 20 \\ 0 &\le f_2 \le 20 \\ 10 &\le f_3 \le 30 \end{aligned}$

(d) Negative flow would mean that water was flowing backward, against the direction of the arrow.

17. **(a)** $\begin{aligned} f_1 &= -200 + s + t \\ f_2 &= 300 - s - t \\ f_3 &= s \\ f_4 &= 150 - t \\ f_5 &= t \end{aligned}$ **(b)** $200 \le f_3 \le 300$

(c) If $f_3 = s = 0$, then $f_5 = t \ge 200$ (from the f_1 equation), but $f_5 = t \le 150$ (from the f_4 equation). This is a contradiction.

(d) $50 \le f_3 \le 300$

19. $I_1 = 3$ amps, $I_2 = 5$ amps, $I_3 = 2$ amps

21. **(a)** $I = 10$ amps, $I_1 = I_5 = 6$ amps, $I_2 = I_4 = 4$ amps, $I_3 = 2$ amps

(b) $R_{eff} = \frac{7}{5}$ ohms

(c) Yes; change it to 4 ohms.

23. Farming : Manufacturing = 2 : 3

25. The painter charges \$39/hr, the plumber \$42/hr, the electrician \$54/hr.

27. **(a)** Coal should produce \$100 million and steel \$160 million.

(b) Coal should reduce production by ≈ \$4.2 million and steel should increase production by ≈ \$5.7 million.

29. **(a)** Yes; push switches 1, 2, and 3 or switches 3, 4, and 5.

(b) No

31. The states that can be obtained are represented by those vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

in $\mathbb{Z}_2^5$ for which $x_1 + x_2 + x_4 + x_5 = 0$. (There are 16 such possibilities.)

33. If 0 = off, 1 = light blue, and 2 = dark blue, then the linear system that arises has augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

which reduces over $\mathbb{Z}_3$ to

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This yields the solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

where t is in $\mathbb{Z}_3$. Hence, there are exactly three solutions:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

where each entry indicates the number of times the corresponding switch should be pushed.

35. **(a)** Push squares 3 and 7.
(b) The 9×9 coefficient matrix A is row equivalent to $\mathbb{Z}_2$, so for any $\mathbf{b}$ in $\mathbb{Z}_2^9$, $A\mathbf{x} = \mathbf{b}$ has a unique solution.

37. Grace is 15, and Hans is 5.

39. 1200 and 600 square yards

41. **(a)** $a = 4 - d, b = 5 - d, c = -2 + d, d$ is arbitrary
(b) No solution

43. **(a)** No solution
(b) $[a, b, c, d, e, f] = [4, 5, 6, -3, -1, 0] + f[-1, -1, -1, 1, 1, 1]$

45. **(a)** $y = x^2 - 2x + 1$ **(b)** $y = x^2 + 6x + 10$

47. $A = 1, B = 2$

49. $A = -\frac{1}{5}, B = \frac{1}{3}, C = 0, D = -\frac{2}{15}, E = -\frac{1}{5}$

51. $a = \frac{1}{2}, b = \frac{1}{2}, c = 0$

Exercises 2.5

1.

n	0	1	2	3	4	5
x_1	0	0.8571	0.9714	0.9959	0.9991	0.9998
x_2	0	0.8000	0.9714	0.9943	0.9992	0.9998

Exact solution: $x_1 = 1, x_2 = 1$

3.

n	0	1	2	3	4	5	6
x_1	0	0.2222	0.2539	0.2610	0.2620	0.2622	0.2623
x_2	0	0.2857	0.3492	0.3582	0.3603	0.3606	0.3606

Exact solution (to four decimal places): $x_1 = 0.2623$, $x_2 = 0.3606$

5.

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997
x_2	0	0.2500	0.0834	0.1250	0.0972	0.1042	0.0996	0.1008	0.1000
x_3	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997

Exact solution: $x_1 = 0.3, x_2 = 0.1, x_3 = 0.3$

7.

n	0	1	2	3	4
x_1	0	0.8571	0.9959	0.9998	1.0000
x_2	0	0.9714	0.9992	1.0000	1.0000

After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

9.

n	0	1	2	3	4
x_1	0	0.2222	0.2610	0.2622	0.2623
x_2	0	0.3492	0.3603	0.3606	0.3606

After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

(b) $2x_1 + x_2 = 1$
$x_1 + 2x_2 = 1$

(c)

n	0	1	2	3	4	5	6	7
x_1	0	0	0.25	0.3125	0.3281	0.3320	0.3330	0.3332
x_2	1	0.5	0.375	0.3438	0.3360	0.3340	0.3335	0.3334

[Columns 1, 2, and 3 of this table are the *odd-numbered* columns 1, 3, and 5 from the table in part (a).] The iterates are converging to $x_1 = x_2 = 0.3333$.

(d) $x_1 = x_2 = \frac{1}{3}$

Review Questions

1. (a) F (c) F (e) T (g) T (i) F

3. $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ **5.** $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **7.** $k = -1$ **9.** $(0, 3, 1)$

11. $x - 2y + z = 0$ **13.** (a) Yes **15.** 1 or 2

17. If $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, then $(c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} = \mathbf{0}$. Linear independence of $\mathbf{u}$ and $\mathbf{v}$ implies $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Solving this system, we get $c_1 = c_2 = 0$. Hence $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.

19. Their ranks must be equal.

Chapter 3

Exercises 3.1

1. $\begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}$ **3.** Not possible

5. $\begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}$ **7.** $\begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}$

9. $[10]$ **11.** $\begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **15.** $\begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

19. $B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$, $BA = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix}$

Column i corresponds to warehouse i, row 1 contains the costs of shipping by truck, and row 2 contains the costs of shipping by train.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

23. $AB = [2\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 \quad 3\mathbf{a}_1 - \mathbf{a}_2 + 6\mathbf{a}_3 \quad \mathbf{a}_2 + 4\mathbf{a}_3]$ (where $\mathbf{a}_i$ is the ith column of A)

25. $\begin{bmatrix} 2 & 3 & 0 \\ -6 & -9 & 0 \\ 4 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -12 & -8 \\ -1 & 6 & 4 \\ 1 & -6 & -4 \end{bmatrix}$

27. $BA = \begin{bmatrix} 2\mathbf{A}_1 + 3\mathbf{A}_2 \\ \mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 \\ -\mathbf{A}_1 + 6\mathbf{A}_2 + 4\mathbf{A}_3 \end{bmatrix}$ (where $\mathbf{A}_i$ is the ith row of A)

29. If $\mathbf{b}_i$ is the ith column of B, then $A\mathbf{b}_i$ is the ith column of AB. If the columns of B are linearly dependent, then there are scalars $c_1, \ldots, c_n$ (not all zero) such that $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$. But then $c_1(A\mathbf{b}_1) + \cdots + c_n(A\mathbf{b}_n) = A(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) = A\mathbf{0} = \mathbf{0}$, so the columns of AB are linearly dependent.

31. $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 5 \end{array}\right]$ **33.** $\left[\begin{array}{cc|cc} 1 & 2 & 2 & 0 \\ 3 & 4 & 5 & 3 \\ \hline 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{array}\right]$

35. (a) $A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

(b) $A^{2001} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

37. $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

39. (a) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \end{bmatrix}$

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Exercises 3.2

1. $X = \begin{bmatrix} 5 & 4 \\ 3 & 5 \end{bmatrix}$ **3.** $X = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} \\ \frac{10}{3} & 4 \end{bmatrix}$

5. $B = 2A_1 + A_2$ **7.** Not possible

9. $\text{span}(A_1, A_2) = \left\{ \begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} w & x \\ 2x - 5w & x - w \end{bmatrix} \right\}$

11. $\text{span}(A_1, A_2, A_3) = \left\{ \begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -3b + 4c + 5e & b & c \\ 0 & e & 0 \end{bmatrix} \right\}$

13. Linearly independent **15.** Linearly independent

23. $a = d, c = 0$ **25.** $3b = 2c, a = d - c$

27. $a = d, b = c = 0$

29. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices and let $i > j$. Then, by the definition of an upper triangular matrix,

$a_{i1} = a_{i2} = \cdots = a_{i,\,i-1} = 0$ and
$b_{ij} = b_{i+1,\,j} = \cdots = b_{nj} = 0$

Now let $C = AB$. Then

$$\begin{aligned} c_{ij} = {} & a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{i,\,i-1}b_{i-1,\,j} + a_{ii}b_{ij} \\ & + a_{i,\,i+1}b_{i+1,\,j} + \cdots + a_{in}b_{nj} \\ = {} & 0 \cdot b_{1j} + 0 \cdot b_{2j} + \cdots 0 \cdot b_{i-1,\,j} + a_{ii} \cdot 0 \\ & + a_{i,\,i+1} \cdot 0 + \cdots + a_{in} \cdot 0 = 0 \end{aligned}$$

from which it follows that C is upper triangular.

35. (a) A, B symmetric $\Rightarrow (A + B)^T = A^T + B^T = A + B \Rightarrow A + B$ is symmetric

37. Matrices (b) and (c) are skew-symmetric.

41. Either A or B (or both) must be the zero matrix.

43. (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

47. *Hint:* Use the trace.

Exercises 3.3

1. $\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$ **3.** Not invertible

5. Not invertible **7.** $\begin{bmatrix} -1.6 & -2.8 \\ 0.\overline{3} & 1 \end{bmatrix}$

9. $\begin{bmatrix} a/(a^2 + b^2) & b/(a^2 + b^2) \\ -b/(a^2 + b^2) & a/(a^2 + b^2) \end{bmatrix}$ **11.** $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$

13. (a) $\mathbf{x}_1 = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(c) The method in part (b) uses fewer multiplications.

17. (b) $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$

21. $X = A^{-1}(BA)^2B^{-1}$

23. $X = (AB)^{-1}BA + A$

25. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **27.** $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

29. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **31.** $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **35.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

43. (a) If A is invertible, then $BA = CA \Rightarrow (BA)A^{-1} = (CA)A^{-1} \Rightarrow B(AA^{-1}) = C(AA^{-1}) \Rightarrow BI = CI \Rightarrow B = C$.

45. *Hint:* Rewrite $A^2 - 2A + I = O$ as $A(2I - A) = I$.

47. If AB is invertible, then there exists a matrix X such that $(AB)X = I$. But then $A(BX) = I$ too, so A is invertible (with inverse BX).

49. $\begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$ **51.** $\begin{bmatrix} 1/(a^2 + 1) & -a/(a^2 + 1) \\ a/(a^2 + 1) & 1/(a^2 + 1) \end{bmatrix}$

53. Not invertible

55. $\begin{bmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{bmatrix}, a \neq 0$

57. $\begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix}$

59. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a/d & -b/d & -c/d & 1/d \end{bmatrix}, d \neq 0$

61. Not invertible **63.** $\begin{bmatrix} 4 & 6 & 4 \\ 5 & 3 & 2 \\ 0 & 6 & 5 \end{bmatrix}$

69. $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right]$ **71.** $\left[\begin{array}{cc|cc} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$

Exercises 3.4

1. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **3.** $\begin{bmatrix} -3/2 \\ -2 \\ -1 \end{bmatrix}$ **5.** $\begin{bmatrix} -7 \\ -15 \\ -2 \\ 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

15. $L^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, U^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{bmatrix}, A^{-1} = \begin{bmatrix} -5/12 & 1/12 \\ 1/6 & 1/6 \end{bmatrix}$

19. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 5 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -16 \end{bmatrix}$

25. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

27. $\begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$ **31.** $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$

Exercises 3.5

1. Subspace **3.** Subspace

5. Subspace **7.** Not a subspace

11. **b** is in col(A), **w** is not in row(A).

15. No

17. $\{[1\ \ 0\ \ -1], [0\ \ 1\ \ 2]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A); $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for null(A).

19. $\{[1\ \ 0\ \ 1\ \ 0], [0\ \ 1\ \ -1\ \ 0], [0\ \ 0\ \ 0\ \ 1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a basis for col(A); $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for null(A).

21. $\{[1\ \ 0\ \ -1], [1\ \ 1\ \ 1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A)

23. $\{[1\ \ 1\ \ 0\ \ 1], [0\ \ 1\ \ -1\ \ 1], [0\ \ 1\ \ -1\ \ -1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A)

25. Both $\{[1\ \ 0\ \ -1], [0\ \ 1\ \ 2]\}$ and $\{[1\ \ 0\ \ -1], [1\ \ 1\ \ 1]\}$ are linearly independent spanning sets for row(A) $= \{[a\ \ b\ \ -a + 2b]\}$. Both $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ are linearly independent spanning sets for col(A) $= \mathbb{R}^2$.

27. $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

29. $\{[1\ \ 0\ \ 0], [0\ \ 1\ \ 0], [0\ \ 0\ \ 1]\}$

31. $\{[2\ \ -3\ \ 1], [1\ \ -1\ \ 0], [4\ \ -4\ \ 1]\}$

35. rank(A) $= 2$, nullity(A) $= 1$

37. rank(A) $= 3$, nullity(A) $= 1$

39. If A is 3×5, then rank(A) ≤ 3, so there cannot be more than three linearly independent columns.

41. nullity(A) $= 2, 3, 4,$ or 5

43. If $a = -1$, then $\text{rank}(A) = 1$; if $a = 2$, then $\text{rank}(A) = 2$; for $a \neq -1, 2$, $\text{rank}(A) = 3$.

45. Yes **47.** Yes **49.** No

51. $\mathbf{w}$ is in $\text{span}(\mathcal{B})$ if and only if the linear system with augmented matrix $[\mathcal{B} \mid \mathbf{w}]$ is consistent, which is true in this case, since

$$[\mathcal{B} \mid \mathbf{w}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

From this reduced row echelon form, it is also clear that $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

53. $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$

55. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$

57. Let $\mathbf{A}_1, \ldots, \mathbf{A}_m$ be the row vectors of A so that $\text{row}(A) = \text{span}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$. If $\mathbf{x}$ is in $\text{null}(A)$, then, since $A\mathbf{x} = \mathbf{0}$, we also have $\mathbf{A}_i \cdot \mathbf{x} = 0$ for $i = 1, \ldots, m$, by the row-column definition of matrix multiplication. If $\mathbf{r}$ is in $\text{row}(A)$, then $\mathbf{r}$ is of the form $\mathbf{r} = c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m$. Therefore,

$$\begin{aligned} \mathbf{r} \cdot \mathbf{x} &= (c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m) \cdot \mathbf{x} \\ &= c_1(\mathbf{A}_1 \cdot \mathbf{x}) + \cdots + c_m(\mathbf{A}_m \cdot \mathbf{x}) = 0 \end{aligned}$$

59. (a) If a set of columns of AB is linearly independent, then the corresponding columns of B are linearly independent (by an argument similar to that needed to prove Exercise 29 in Section 3.1). It follows that the *maximum* number k of linearly independent columns of AB [i.e., $k = \text{rank}(AB)$] is not more than the *maximum* number r of linearly independent columns of B [i.e., $r = \text{rank}(B)$]. In other words, $\text{rank}(AB) \leq \text{rank}(B)$.

61. (a) From Exercise 59(a), $\text{rank}(UA) \leq \text{rank}(A)$ and $\text{rank}(A) = \text{rank}((U^{-1}U)A) = \text{rank}(U^{-1}(UA)) \leq \text{rank}(UA)$. Hence, $\text{rank}(UA) = \text{rank}(A)$.

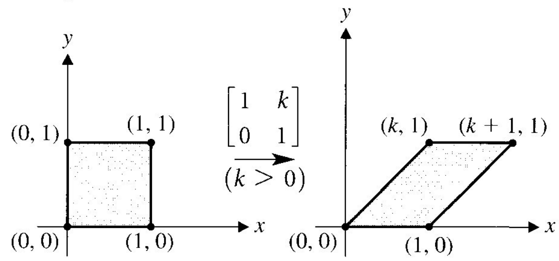
Exercises 3.6

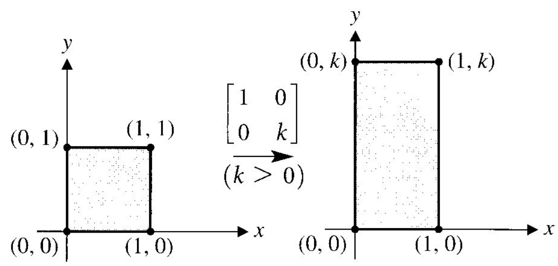
1. $T(\mathbf{u}) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ **13.** $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$

15. $[F] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **17.** $[D] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

19. $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ stretches or contracts in the x-direction (combined with a reflection in the y-axis if $k < 0$); $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ stretches or contracts in the y-direction (combined with a reflection in the x-axis if $k < 0$); $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection in the line $y = x$; $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ is a *shear* in the x-direction; $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ is a *shear* in the y-direction. For example,





21. $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$ **23.** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

25. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **27.** $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

31. $[S \circ T] = \begin{bmatrix} -8 & 5 \\ 4 & 1 \end{bmatrix}$

33. $[S \circ T] = \begin{bmatrix} 0 & 6 & -6 \\ 1 & -2 & 2 \end{bmatrix}$

35. $[S \circ T] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

37. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ **39.** $\begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

45. In vector form, let the parallel lines be given by $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and $\mathbf{x}' = \mathbf{p}' + t\mathbf{d}$. Their images are $T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{d}) = T(\mathbf{p}) + tT(\mathbf{d})$ and $T(\mathbf{x}') = T(\mathbf{p}' + t\mathbf{d}) = T(\mathbf{p}') + tT(\mathbf{d})$. Suppose $T(\mathbf{d}) \neq \mathbf{0}$. If $T(\mathbf{p}') - T(\mathbf{p})$ is parallel to $T(\mathbf{d})$, then the images represent the same line; otherwise the images represent distinct parallel lines. On the other hand, if $T(\mathbf{d}) = \mathbf{0}$,

91. One set of candidates for P and G is

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Review Questions

1. (a) T **(c)** F **(e)** T **(g)** T **(i)** T

3. Impossible

5. $\begin{bmatrix} \frac{17}{83} & -\frac{1}{83} \\ -\frac{1}{83} & \frac{5}{166} \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix}$

9. $\begin{bmatrix} 0 & -9 \\ 2 & 4 \\ 1 & -6 \end{bmatrix}$

11. Because $(I - A)(I + A + A^2) = I - A^3 = I - O = I$, $(I - A)^{-1} = I + A + A^2$.

13. A basis for row(A) is $\{[1, -2, 0, -1, 0], [0, 0, 1, 2, 0], [0, 0, 0, 0, 1]\}$; a basis for col($A$) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\}$ (or the standard basis for $\mathbf{R}^3$); and a basis for null(A) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

15. An invertible matrix has a trivial (zero) null space. If A is invertible, then so is A^T, and so both A and A^T have trivial null spaces. If A is not invertible, then A and A^T need not have the same null space. For example, take
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

17. Because A has n linearly independent columns, rank$(A) = n$. Hence rank$(A^TA) = n$ by Theorem 3.28. Because A^TA is $n \times n$, this implies that A^TA is invertible, by the Fundamental Theorem of Invertible Matrices. AA^T need not be invertible. For example, take $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

19. $\begin{bmatrix} -1/5\sqrt{2} & -3/5\sqrt{2} \\ 2/5\sqrt{2} & 6/5\sqrt{2} \end{bmatrix}$

Chapter 4

Exercises 4.1

1. $A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}, \lambda = 3$

3. $A\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\mathbf{v}, \lambda = -3$

5. $A\mathbf{v} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\mathbf{v}, \lambda = 3$

7. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **9.** $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

13. $\lambda = 1, E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right); \lambda = -1, E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

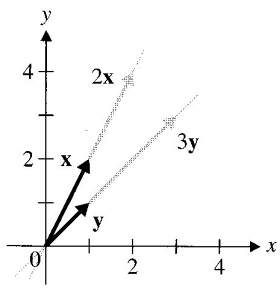
15. $\lambda = 0, E_0 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right); \lambda = 1, E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

17. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right); \lambda = 3, E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

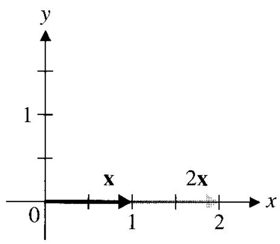
19. $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = 1; \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda = 2$

21. $\mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \lambda = 2; \mathbf{v} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \lambda = 0$

23. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right); \lambda = 3, E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



25. $\lambda = 2$, $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$



27. $\lambda = 1 + i$, $E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$; $\lambda = 1 - i$, $E_{1-i} =$ $\text{span}\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$

29. $\lambda = 1 + i$, $E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$; $\lambda = 1 - i$, $E_{1-i} =$ $\text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

31. $\lambda = 1, 2$ **33.** $\lambda = 4$

Exercises 4.2

1. 16 **3.** 0 **5.** -18 **7.** 6
9. -12 **11.** $a^2b + ab^2$ **13.** 4 **15.** $abdg$
17. 0 **25.** 2 **27.** -24 **29.** 0
31. 0 **33.** -24 **35.** 8 **37.** -4
39. -8 **45.** $k \neq 0, 2$ **47.** -6 **49.** $-\frac{3}{2}$
51. $(-2)3^n$
53. $\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$
55. 0, 1 **57.** $x = \frac{3}{2}, y = -\frac{1}{2}$
59. $x = -1, y = 0, z = 1$ **61.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$
63. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Exercises 4.3

1. (a) $\lambda^2 - 7\lambda + 12$ **(b)** $\lambda = 3, 4$
(c) $E_3 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$; $E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$
(d) The algebraic and geometric multiplicities are all 1.

3. (a) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$
(b) $\lambda = -2, 1, 3$
(c) $E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\right)$; $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$; $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right)$
(d) The algebraic and geometric multiplicities are all 1.

5. (a) $-\lambda^3 + \lambda^2$ **(b)** $\lambda = 0, 1$
(c) $E_0 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$; $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$
(d) $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 1.

7. (a) $-\lambda^3 + 9\lambda^2 - 27\lambda + 27$
(b) $\lambda = 3$
(c) $E_3 = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$
(d) $\lambda = 3$ has algebraic multiplicity 3 and geometric multiplicity 2.

9. (a) $\lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12$
(b) $\lambda = -1, 2, 3$
(c) $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$; $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right)$; $E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$
(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

11. **(a)** $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 4\lambda - 3$

(b) $\lambda = -1, 1, 3$

(c) $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$;

$E_1 = \text{span}\left(\begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \end{bmatrix}\right)$;

$E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$

(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 2.

15. $\begin{bmatrix} 2^{-9} + 3 \cdot 2^{10} \\ -2^{-9} + 3 \cdot 2^{10} \end{bmatrix}$ **17.** $\begin{bmatrix} 2 \\ (2 \cdot 3^{20} - 1)/3^{20} \\ 2 \end{bmatrix}$

23. **(a)** $\lambda = -2$, $E_{-2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$; $\lambda = 5$, $E_5 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(b) **(i)** $\lambda = -\frac{1}{2}$, $E_{-1/2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$; $\lambda = \frac{1}{5}$, $E_{1/5} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(iii) $\lambda = 0$, $E_0 = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$; $\lambda = 7$, $E_7 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

27. $\begin{bmatrix} -3 & 4 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $-\lambda^3 - 3\lambda^2 + 4\lambda - 12$

35. $A^2 = 4A - 5I$, $A^3 = 11A - 20I$

$A^4 = 24A - 55I$

37. $A^{-1} = -\frac{1}{5}A + \frac{4}{5}I$, $A^{-2} = -\frac{4}{25}A + \frac{11}{25}I$

Exercises 4.4

1. The characteristic polynomial of A is $\lambda^2 - 5\lambda + 1$, but that of B is $\lambda^2 - 2\lambda + 1$.

3. The eigenvalues of A are $\lambda = 2$ and $\lambda = 4$, but those of B are $\lambda = 1$ and $\lambda = 4$.

5. $\lambda_1 = 4$, $E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$; $\lambda_2 = 3$, $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

7. $\lambda_1 = 6$, $E_6 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}\right)$; $\lambda_2 = -2$, $E_{-2} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

9. Not diagonalizable

11. $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. Not diagonalizable

15. $P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 35839 & -69630 \\ -11605 & 24234 \end{bmatrix}$

19. $\begin{bmatrix} (3^k + 3(-1)^k)/4 & (3^{k+1} - 3(-1)^k)/4 \\ (3^k - (-1)^k)/4 & (3^{k+1} + (-1)^k)/4 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

23.

$$\begin{bmatrix} (5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (-5 + 2^{k+2} + (-3)^k)/10 \\ (2^{k+1} - 2(-3)^k)/5 & (2^k + 4(-3)^k)/5 & (2^{k+1} - 2(-3)^k)/5 \\ (-5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (5 + 2^{k+2} + (-3)^k)/10 \end{bmatrix}$$

25. $k = 0$ **27.** $k = 0$

29. All real values of k

35. If $A \sim B$, then there is an invertible matrix P such that $B = P^{-1}AP$. Therefore, we have

$$\begin{aligned} \text{tr}(B) &= \text{tr}(P^{-1}AP) = \text{tr}(P^{-1}(AP)) = \text{tr}((AP)P^{-1}) \\ &= \text{tr}(APP^{-1}) = \text{tr}(AI) = \text{tr}(A) \end{aligned}$$

using Exercise 45 in Section 3.2.

37. $P = \begin{bmatrix} 7 & -2 \\ 10 & -3 \end{bmatrix}$

39. $P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$

49. (b) $\dim E_{-1} = 1, \dim E_1 = 2, \dim E_2 = 3$

Exercises 4.5

1. (a) $\begin{bmatrix} 1 \\ 2.5 \end{bmatrix}, 6.000$

(b) $\lambda_1 = 6$

3. (a) $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}, 2.618$

(b) $\lambda_1 = (3 + \sqrt{5})/2 \approx 2.618$

5. (a) $m_5 = 11.001, \mathbf{y}_5 = \begin{bmatrix} -0.333 \\ 1.000 \end{bmatrix}$

7. (a) $m_8 = 10.000, \mathbf{y}_8 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

9.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 26 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 17.692 \\ 5.923 \end{bmatrix}$	$\begin{bmatrix} 18.018 \\ 6.004 \end{bmatrix}$	$\begin{bmatrix} 17.999 \\ 6.000 \end{bmatrix}$	$\begin{bmatrix} 18.000 \\ 6.000 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.308 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.335 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$
m_k	1	26	17.692	18.018	17.999	18.000

Therefore, $\lambda_1 \approx 18, \mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$.

11.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7.571 \\ 2.857 \end{bmatrix}$	$\begin{bmatrix} 7.755 \\ 3.132 \end{bmatrix}$	$\begin{bmatrix} 7.808 \\ 3.212 \end{bmatrix}$	$\begin{bmatrix} 7.823 \\ 3.234 \end{bmatrix}$	$\begin{bmatrix} 7.827 \\ 3.240 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.286 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.377 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.404 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.411 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.413 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$
m_k	1	7	7.571	7.755	7.808	7.823	7.827

Therefore, $\lambda_1 \approx 7.827, \mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$.

13.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 21 \\ 15 \\ 13 \end{bmatrix}$	$\begin{bmatrix} 16.809 \\ 12.238 \\ 10.714 \end{bmatrix}$	$\begin{bmatrix} 17.011 \\ 12.371 \\ 10.824 \end{bmatrix}$	$\begin{bmatrix} 16.999 \\ 12.363 \\ 10.818 \end{bmatrix}$	$\begin{bmatrix} 17.000 \\ 12.363 \\ 10.818 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.714 \\ 0.619 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.728 \\ 0.637 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$
m_k	1	21	16.809	17.011	16.999	17.000

Therefore, $\lambda_1 \approx 17, \mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$.

57. **(a)** $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 5, d_5 = 8$
(b) $d_n = d_{n-1} + d_{n-2}$
(c) $d_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

59. The general solution is $x(t) = -3C_1e^{-t} + C_2e^{4t}$, $y(t) = 2C_1e^{-t} + C_2e^{4t}$. The specific solution is $x(t) = -3e^{-t} + 3e^{4t}$, $y(t) = 2e^{-t} + 3e^{4t}$.

61. The general solution is $x_1(t) = (1+\sqrt{2})C_1e^{\sqrt{2}t} + (1-\sqrt{2})C_2e^{-\sqrt{2}t}$, $x_2(t) = C_1e^{\sqrt{2}t} + C_2e^{-\sqrt{2}t}$. The specific solution is $x_1(t) = (2+\sqrt{2})e^{\sqrt{2}t}/4 + (2-\sqrt{2})e^{-\sqrt{2}t}/4$, $x_2(t) = \sqrt{2}e^{\sqrt{2}t}/4 - \sqrt{2}e^{-\sqrt{2}t}/4$.

63. The general solution is $x(t) = -C_1 + C_3e^{-t}$, $y(t) = C_1 + C_2e^{t} - C_3e^{-t}$, $z(t) = C_1 + C_2e^{t}$. The specific solution is $x(t) = 2 - e^{-t}$, $y(t) = -2 + e^{t} + e^{-t}$, $z(t) = -2 + e^{t}$.

65. **(a)** $x(t) = -120e^{8t/5} + 520e^{11t/10}$, $y(t) = 240e^{8t/5} + 260^{11t/10}$. Strain X dies out after approximately 2.93 days; strain Y continues to grow.

67. $a = 10$, $b = 20$; $x(t) = 10e^{t}(\cos t + \sin t) + 10$, $y(t) = 10e^{t}(\cos t - \sin t) + 20$. Species Y dies out when $t \approx 1.22$.

71. $x(t) = C_1e^{2t} + C_2e^{3t}$

77. **(a)** $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}, \begin{bmatrix}9\\9\end{bmatrix}, \begin{bmatrix}27\\27\end{bmatrix}$ **(c)** Repeller

79. **(a)** $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ **(c)** Neither

81. **(a)** $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0.5\\-1\end{bmatrix}, \begin{bmatrix}1.75\\-0.5\end{bmatrix}, \begin{bmatrix}3.125\\-1.75\end{bmatrix}$ **(c)** Saddle point

83. **(a)** $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0.6\\0.6\end{bmatrix}, \begin{bmatrix}0.36\\0.36\end{bmatrix}, \begin{bmatrix}0.216\\0.216\end{bmatrix}$ **(c)** Attractor

85. $r = \sqrt{2}$, $\theta = 45°$, spiral repeller

87. $r = 2$, $\theta = -60°$, spiral repeller

89. $P = \begin{bmatrix}-1 & -1\\1 & 0\end{bmatrix}$, $C = \begin{bmatrix}0.2 & -0.1\\0.1 & 0.2\end{bmatrix}$, spiral attractor

91. $P = \begin{bmatrix}1/2 & -\sqrt{3}/2\\1 & 0\end{bmatrix}$, $C = \begin{bmatrix}1/2 & -\sqrt{3}/2\\\sqrt{3}/2 & 1/2\end{bmatrix}$, orbital center

Review Questions

1. **(a)** F **(c)** F **(e)** F **(g)** T **(i)** F

3. -18

5. Since $A^T = -A$, we have $\det A = \det(A^T) = \det(-A) = (-1)^n \det A = -\det A$ by Theorem 4.7 and the fact that n is odd. It follows that $\det A = 0$.

7. $A\mathbf{x} = \begin{bmatrix}5\\10\end{bmatrix} = 5\mathbf{x}$, $\lambda = 5$

9. **(a)** $4 - 3\lambda^2 - \lambda^3$

(c) $E_1 = \operatorname{span}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right)$, $E_{-2} = \operatorname{span}\left(\begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$

11. $\begin{bmatrix}162\\158\end{bmatrix}$ **13.** Not similar **15.** Not similar

17. 0, 1, or -1

19. If $A\mathbf{x} = \lambda\mathbf{x}$, then $(A^2 - 5A + 2I)\mathbf{x} = A^2\mathbf{x} - 5A\mathbf{x} + 2\mathbf{x} = 3^2\mathbf{x} - 5(3\mathbf{x}) + 2\mathbf{x} = -4\mathbf{x}$.

Chapter 5

Exercises 5.1

1. Orthogonal **3.** Not orthogonal **5.** Orthogonal

7. $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix}\frac{1}{2}\\-1\end{bmatrix}$ **9.** $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix}0\\\frac{2}{3}\\\frac{1}{3}\end{bmatrix}$ **11.** Orthonormal

13. $\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}, \begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{bmatrix}, \begin{bmatrix}2/3\sqrt{5}\\4/3\sqrt{5}\\-5/3\sqrt{5}\end{bmatrix}$

15. Orthonormal

17. Orthogonal, $\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

19. Orthogonal, $\begin{bmatrix}\cos\theta\sin\theta & \cos^2\theta & \sin\theta\\-\cos\theta & \sin\theta & 0\\-\sin^2\theta & -\cos\theta\sin\theta & \cos\theta\end{bmatrix}$

21. Not orthogonal

27.
$$\begin{aligned}\cos(\angle(Q\mathbf{x}, Q\mathbf{y})) &= \frac{(Q\mathbf{x})\cdot(Q\mathbf{y})}{\|Q\mathbf{x}\|\|Q\mathbf{y}\|}\\ &= \frac{(Q\mathbf{x})^TQ\mathbf{y}}{\sqrt{(Q\mathbf{x})^TQ\mathbf{x}}\sqrt{(Q\mathbf{y})^TQ\mathbf{y}}}\\ &= \frac{\mathbf{x}^TQ^TQ\mathbf{y}}{\sqrt{\mathbf{x}^TQ^TQ\mathbf{x}}\sqrt{\mathbf{y}^TQ^TQ\mathbf{y}}}\\ &= \frac{\mathbf{x}^T\mathbf{y}}{\sqrt{\mathbf{x}^T\mathbf{x}}\sqrt{\mathbf{y}^T\mathbf{y}}} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\\ &= \cos(\angle(\mathbf{x}, \mathbf{y}))\end{aligned}$$

29. Rotation, $\theta = 45°$ **31.** Reflection, $y = \sqrt{3}x$

33. (a) $A(A^T + B^T)B = AA^TB + AB^TB = IB + AI = B + A = A + B$

(b) From part (a),

$$\begin{aligned}\det(A+B) &= \det(A(A^T+B^T)B)\\ &= \det A \det(A^T+B^T)\det B\\ &= \det A \det((A+B)^T)\det B\\ &= \det A \det(A+B)\det B\end{aligned}$$

Assume that $\det A + \det B = 0$ (so that $\det B = -\det A$) but that $A + B$ is invertible. Then $\det(A+B) \neq 0$, so $1 = \det A \det B = \det A(-\det A) = -(\det A)^2$. This is impossible, so we conclude that $A + B$ cannot be invertible.

Exercises 5.2

1. $W^\perp = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0\right\}, \mathcal{B}^\perp = \left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

3. $W^\perp = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = t, z = -t\right\}, \mathcal{B}^\perp = \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$

5. $W^\perp = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 3z = 0\right\}, \mathcal{B}^\perp = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}\right\}$

7. $\text{row}(A)$: $\{[1\ \ 0\ \ 1], [0\ \ 1\ \ -2]\}$, $\text{null}(A)$: $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$

9. $\text{col}(A)$: $\left\{\begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix}\right\}$, $\text{null}(A^T)$: $\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ -7 \\ 0 \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix}\right\}$ **13.** $\left\{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

15. $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$ **17.** $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$

19. $\mathbf{v} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ -\frac{4}{5} \end{bmatrix}$ **21.** $\mathbf{v} = \begin{bmatrix} \frac{7}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

25. No

Exercises 5.3

1. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix},$

$\mathbf{q}_2 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

5. $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}\right\}$ **7.** $\mathbf{v} = \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{9} \end{bmatrix} + \begin{bmatrix} \frac{38}{9} \\ -\frac{38}{9} \\ \frac{19}{9} \end{bmatrix}$

9. $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -\frac{3}{35} \\ \frac{34}{35} \\ -\frac{1}{7} \end{bmatrix}, \begin{bmatrix} -\frac{15}{34} \\ 0 \\ \frac{9}{34} \end{bmatrix}\right\}$

13. $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

15. $\begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$

17. $R = \begin{bmatrix} 3 & 9 & \frac{1}{3} \\ 0 & 6 & \frac{2}{3} \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$

19. $A = AI$

21. $A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T =$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/2\sqrt{3} \\ 0 & 0 & 3/2\sqrt{3} \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

23. Let $R\mathbf{x} = \mathbf{0}$. Then $A\mathbf{x} = QR\mathbf{x} = Q\mathbf{0} = \mathbf{0}$. Since $A\mathbf{x}$ represents a linear combination of the columns of A (which are linearly independent), we must have $\mathbf{x} = \mathbf{0}$. Hence, R is invertible, by the Fundamental Theorem.

Exercises 5.4

1. $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

3. $Q = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

5. $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

7. $Q = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

9. $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix},$

$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

11. $Q^TAQ = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} \cdot$

$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} = D$

13. (a) If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Therefore, $A + B$ is symmetric, by Exercise 35 in Section 3.2, and so is orthogonally diagonalizable, by the Spectral Theorem.

15. If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Since $AB = BA$, AB is also symmetric, by Exercise 36 in Section 3.2. Hence, AB is orthogonally diagonalizable, by the Spectral Theorem.

17. $A = \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix}$

19. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

21. $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ **23.** $\begin{bmatrix} \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{5}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{8}{3} \end{bmatrix}$

Exercises 5.5

1. $G' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, C' = C$

3. $G' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, C' is equivalent to C but $C' \neq C$

5. $P' = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, C' is equivalent to C but $C' \neq C$

7. $P' = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$, C' is equivalent to C but $C' \neq C$

9. $C^{\perp} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

11. $C^{\perp} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

13. $G^{\perp} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, P^{\perp} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

15. $G^{\perp} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, P^{\perp} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

17. $G^{\perp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, P^{\perp} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

23. $2x^2 + 6xy + 4y^2$ **25.** 123

27. -5 **29.** $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

31. $\begin{bmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix}$ **33.** $\begin{bmatrix} 5 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$

35. $Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}, y_1^2 + 6y_2^2$

37. $Q = \begin{bmatrix} 2/\sqrt{5} & 2/3\sqrt{5} & -1/3 \\ 0 & 5/3\sqrt{5} & 2/3 \\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3 \end{bmatrix}, 9y_1^2 + 9y_2^2 - 9y_3^2$

Exercises 7.3

1. $\|\mathbf{e}\| = \sqrt{2} \approx 1.414$ **3.** $\|\mathbf{e}\| = \sqrt{6}/2 \approx 1.225$

5. $\|\mathbf{e}\| = \sqrt{7} \approx 2.646$

7. $y = -3 + \frac{5}{2}x$, $\|\mathbf{e}\| \approx 1.225$

9. $y = \frac{11}{3} - 2x$, $\|\mathbf{e}\| \approx 0.816$

11. $y = \frac{7}{10} + \frac{8}{25}x$, $\|\mathbf{e}\| \approx 0.447$

13. $y = -\frac{1}{5} + \frac{7}{5}x$, $\|\mathbf{e}\| \approx 0.632$

15. $y = 3 - \frac{18}{5}x + x^2$ **17.** $y = \frac{18}{5} - \frac{17}{10}x - \frac{1}{2}x^2$

19. $\bar{\mathbf{x}} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{15} \end{bmatrix}$ **21.** $\bar{\mathbf{x}} = \begin{bmatrix} \frac{4}{3} \\ -\frac{5}{6} \end{bmatrix}$

23. $\bar{\mathbf{x}} = \begin{bmatrix} 4+t \\ -5-t \\ -5-2t \\ t \end{bmatrix}$ **25.** $\begin{bmatrix} \frac{42}{11} \\ \frac{19}{11} \\ \frac{42}{11} \end{bmatrix}$

27. $\bar{\mathbf{x}} = \begin{bmatrix} \frac{5}{3} \\ -2 \end{bmatrix}$ **29.** $y = 0.92 + 0.73x$

31. (a) If we let the year 1920 correspond to $t = 0$, then $y = 56.6 + 2.9t$; 79.9 years

33. (a) $p(t) = 150e^{0.131t}$

35. 139 days

37. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}$ **39.** $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

41. $\begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}, \begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}$ **45.** $A^+ = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \end{bmatrix}$

47. $A^+ = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$ **49.** $A^+ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

51. $A^+ = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

53. (a) If A is invertible, so is A^T, and we have $A^+ = (A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}$.

Exercises 7.4

1. 2, 3 **3.** $\sqrt{2}, 0$ **5.** 5 **7.** 2, 3

9. $\sqrt{5}, 2, 0$

11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

15. $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} [1]$

17. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$

21. $A = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1/\sqrt{2} \quad 1/\sqrt{2}] + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $[-1/\sqrt{2} \quad 1/\sqrt{2}]$ (Exercise 3)

23. (Exercise 7) $A = 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [0 \quad 1] + 2 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [1 \quad 0]$

33. The line segment $[-1, 1]$

35. The solid ellipse $\dfrac{y_1^2}{5} + \dfrac{y_2^2}{4} \le 1$

37. (a) $\|A\|_2 = \sqrt{2}$ **(b)** $\text{cond}_2(A) = \infty$

39. (a) $\|A\|_2 = 1.95$ **(b)** $\text{cond}_2(A) = 38.11$

41. $A^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$ **43.** $A^+ = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{5} & 0 \end{bmatrix}$

45. $A^+ = \begin{bmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix} 0.52 \\ 1.04 \end{bmatrix}$

47. $A^+ = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

61. $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

63. $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Exercises 7.5

1. $g(x) = \frac{1}{3}$ **3.** $g(x) = \frac{3}{5}x$

5. $g(x) = \frac{3}{16} + \frac{15}{16}x^2$ **7.** $\{1, x - \frac{1}{2}\}$

9. $g(x) = x - \frac{1}{6}$

11. $g(x) = (4e - 10) + (18 - 6e)x \approx 0.87 + 1.69x$

13. $g(x) = \frac{1}{20} - \frac{3}{5}x + \frac{3}{2}x^2$