Stat 254 - Chapter 5 Practice Problems - Solutions

1. d

2. (a) X = pile heigh of one rug in mm. Distribution; $X \sim Unif[6, 10]$
   (b) The distribution has height $1/(10-6) = 0.25$
   (c) (i) $P(X < 7.5) = (7.5 - 6) \times 0.25 = 0.375$
   (ii) $P(6.4 < X < 8.1) = (8.1 - 6.4) \times 0.25 = 0.425$
   (iii) $P(X > 8.5) = (10 - 8.5) \times 0.25 = 0.375$
   (d) $P(X < h) = 0.8 = (h = 6) \times 0.25$. Solving for h gives $h = 9.2$mm

3. This is an exponential distribution with $\lambda = 2$. So the mean is $mu = 1/\lambda = 1/2$ and the variance is $\sigma^2 = 1/\lambda^2 = 1/4$.

   The median is the value a such that $P(X < a) = 0.5$

   $\int_0^a 2e^{-2x}dx = -e^{-2x}|_0^a = -e^{-2a} + e^{-2(0)} = 1 - e^{-2a}$

   Then $0.5 = 1 - e^{-2a}$ means $e^{-2a} = 0.5$ so $a = \ln(0.5)/-2 = 0.347$

4. There is an outlier, suggesting that the data are not from a normally distributed population.

5. (a) $P(Z > 1.22)$ is between about 2.5% and 16%: Answer is (a) 0.1401
   (b) $P(Z \leq 1.45)$ is between about 84% and 97.5%: Answer is (d) 0.9265
   (c) $P(Z = -0.68) = 0$
   (d) $P(-2.22 < Z < -0.03)$ is between the regions of -2 to -1 and -3 to 0, therefore the area between should be between $16 - 2.5 = 13.5\%$ and $50 - 0.15 = 49.75\%$: Answer is (a) 0.4748
   (e) 20% is between 16% and 50% so z should be between -1 and 0: Answer is (d) -0.81
   (f) Highest 18% is between 16% and 50% so z should be between 0 and 1: Answer is (a) 0.92

6. (a) X = the length of one movie trailer in minutes
   $X \sim N(2.5, 0.25)$
   (b) $P(X > 3.2) = P(Z > \dfrac{3.2 - 2.5}{0.5}) = P(Z > 1.4)$: Answer between 2.5% and 16% is (c) 0.0808
   (c) $P(X < 1.1) = P(Z < \dfrac{1.1 - 2.5}{0.5}) = P(Z < -2.80)$: Answer between 0.15% and 2.5% is (b) = 0.0026
   (d) $P(2.6 < X < 3.7) = P(\dfrac{2.6 - 2.5}{0.5} < Z < \dfrac{3.7 - 2.5}{0.5}) = P(0.20 < Z < 2.40)$: Answer between $97.5 - 84 = 13.5\%$ and $99.7 - 50 = 49.7\%$ is (b) 0.4125
   (e) 90^{th} percentile should yield a z-score between +1 and +2, so the X value should be between $2.5 + 0.5 = 3$ and $2.5 + 2(0.5) = 3.5$ minutes. Answer is (c) 3.14 minutes.

7. X = service time for one person. Since $\mu = 4$, $\lambda = 1/4$, therefore $X \sim Exp(1/4)$ so $f(x) = \dfrac{1}{4}e^{-\frac{1}{4}x}$ for $x \geq 0$

$$P(X<3)=\int_0^3 \frac{1}{4}e^{-\frac{1}{4}x}dx = -e^{-\frac{1}{4}x}\Big|_0^3 = -e^{-\frac{1}{4}(3)} - -e^{-\frac{1}{4}(0)} = -0.4724-(-1)=0.5276$$

To determine the likelihood of this happening at least 4 out of 6 days, we define a new random variable

$Y =$ number of days with less than 3 minute service time out of 6 days, then $Y \sim Binom(6, 0.5276)$ using the probability calculated from the exponential distribution.

$$P(X\geq 4) = \binom{6}{4}0.5276^4 0.4724^2 + \binom{6}{5}0.5276^5 0.4724^1 + \binom{6}{6}0.5276^6 0.4724^1 = 0.5006$$

8. $X \sim Exp(1/3)$

   (a) $P(X>5)=\int_5^\infty \frac{1}{3}e^{-\frac{1}{3}x}dx = 1-\int_0^5 \frac{1}{3}e^{-\frac{1}{3}x}dx = 1--e^{-\frac{1}{3}x}|_0^5 = 1-(-e^{-\frac{1}{4}(5)}--e^{-\frac{1}{4}(0)}) = 1-(-0.1889-(-1))=0.1889$

   (b) $P(X>10)=\int_{10}^\infty \frac{1}{3}e^{-\frac{1}{3}x}dx = 1-\int_0^{10} \frac{1}{3}e^{-\frac{1}{3}x}dx = 1--e^{-\frac{1}{3}x}\Big|_0^{10} = 1-(-e^{-\frac{1}{4}(10)} - -e^{-\frac{1}{4}(0)}) = 1-(-0.03567-(-1))=0.03567$

9. (a) $\int_0^1 xdx + \int_1^2 (2-x)dx = \frac{x^2}{2}\Big|_0^1 + (2x-\frac{x^2}{2})\Big|_1^2 = \frac{1^2}{2}-\frac{0^2}{2}+2(2)-\frac{2^2}{2}-(2(1)-\frac{1^2}{2})=1$

   (b) $P(X<1.3)=\int_0^1 xdx + \int_1^{1.3}(2-x)dx = \frac{x^2}{2}\Big|_0^1 + (2x-\frac{x^2}{2})\Big|_1^{1.3}$
   $= \frac{1^2}{2}-\frac{0^2}{2}+2(1.3)-\frac{1.3^2}{2}-(2(1)-\frac{1^2}{2}) = 0.755$
   OR: $P(X<1.3)=1-\int_{1.3}^2(2-x)dx = 1-(2x-\frac{x^2}{2})\Big|_{1.3}^2 = 1-\left(2(2)-\frac{2^2}{2}-(2(1.3)-\frac{1.3^2}{2})\right)$

   (c) $P(0.3<X<1.7)=\int_{0.3}^1 xdx + \int_1^{1.7}(2-x)dx = \frac{x^2}{2}\Big|_{0.3}^1 + (2x-\frac{x^2}{2})\Big|_1^{1.7} = 0.91$

   (d) Mean: $\mu = \int_{-\infty}^\infty xf(x)dx = \int_0^1 x^2dx + \int_1^2(2x-x^2)dx = \frac{x^3}{3}\Big|_0^1 + (x^2-\frac{x^3}{3})\Big|_1^2$
   $= (\frac{1}{3}-0)+(4-\frac{8}{3}-(1-\frac{1}{3}))=1$
   Variance: $\sigma^2 = \int_{-\infty}^\infty x^2f(x)dx-\mu^2 = \int_0^1 x^3dx + \int_1^2(2x^2-x^3)dx - 1^2$
   $= \frac{x^4}{4}\Big|_0^1 + (\frac{2x^3}{3}-\frac{x^4}{4})\Big|_1^2 - 1 = (\frac{1}{4}-0)+(\frac{2(8)}{3}-\frac{16}{4}-(\frac{2}{3}-\frac{1}{4}))-1 = 0.1667$

10. (a) $P(X>13)=P\left(Z>\frac{13-10}{2}\right)=P(Z>1.5)$: Answer between 2.5% and 16% is (a) 0.0668

    (b) $P(X<a)=0.03$ occurs between 1 and 2 standard deviations below the mean. When $\mu=10$ and $\sigma=2$ that is between $10-2(2)=6$ and $10=2=8$ years. Answer is (c) 6.2 years.

11. $f(x) = \frac{1}{10-7} = \frac{1}{3}$ on $[7, 10]$

(a) $\mu = \displaystyle\int_7^{10} \frac{x}{3} dx = \left.\frac{x^2}{6}\right|_7^{10} = 8.5$

$\sigma^2 = \displaystyle\int_7^{10} \frac{x^2}{3} dx - \mu^2 = \left.\frac{x^3}{9}\right|_7^{10} - 8.5^2 = 0.75$

(b) $P(X < 8.8) = \displaystyle\int_7^{8.8} \frac{1}{3} dx = \left.\frac{x}{3}\right|_7^{8.8} = 0.6$

(c) $P(X < a) = 0.95 = \displaystyle\int_7^{a} \frac{1}{3} dx = \left.\frac{x}{3}\right|_7^{a}$. Therefore $\dfrac{a-7}{3} = 0.95$ requires $a = 9.85$

(d) $P(7.4 < X < 9.5) = \displaystyle\int_{7.4}^{9.5} \frac{1}{3} dx = \left.\frac{x}{3}\right|_{7.4}^{9.5} = 0.7$

12. (a) $\displaystyle\int_0^1 k\sqrt{x} dx = \left.\frac{k}{3/2} x^{3/2}\right|_0^1 = k\left(\frac{2}{3}(1) - \frac{2}{3}(0)\right) = 1$. Therefore $k = \dfrac{3}{2}$

(b) $P(0.3 < X < 0.6) = \displaystyle\int_{0.3}^{0.6} \frac{3}{2}\sqrt{x} dx = \left. x^{3/2}\right|_{0.3}^{0.6} = \left(0.6^{3/2} - 0.3^{3/2}\right) = 0.3005$

(c) $\mu = \displaystyle\int_0^1 \frac{3}{2} x^{3/2} dx = \left.\frac{3}{2} x^{5/2}\right|_0^1 = 0.6$

$\sigma^2 = \displaystyle\int_0^1 \frac{3}{2} x^{5/2} dx - \mu^2 = \left.\frac{3}{2} x^{5/2}\right|_0^1 - 0.6^2 = 0.0686$

(d) $F(x) = \displaystyle\int_0^x \frac{3}{2}\sqrt{x} dx = \left. x^{3/2}\right|_0^x = x^{3/2}$ for $0 < x < 1$