Geometric Vacuum Selection from Constrained Spacetime Foliations

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Abstract

We investigate a scalar-field framework for vacuum structure in General Relativity, in which the foliation defined by the gradient of a time-ordering field Θ is subject to a non-local geometric constraint. The level sets $\Theta = \text{const}$ define spacelike hypersurfaces interpreted as physical "present" leaves. The constraint fixes the asymptotic normalized average of the squared extrinsic curvature in the infrared limit to a constant scale K_0^2, enforced by a Lagrange multiplier that is constant on each leaf.

We show that this non-local condition acts as a vacuum selection principle: in the infrared limit it generically excludes the static Minkowski vacuum, compelling the ground state to be spontaneously curved. De Sitter spacetime emerges as a consistent asymptotic vacuum solution, with its expansion rate fixed by the geometric selection rule $H^2 = K_0^2/3$.

We further demonstrate that the infrared dominance of the constraint ensures compatibility with local gravitational physics. Standard Schwarzschild and post-Newtonian limits are recovered for local sources, as the constraint does not introduce unsuppressed local modifications of the gravitational dynamics.

These results suggest that the observed late-time cosmic acceleration may reflect a global geometric property of the spacetime foliation—a resistance to staticity—rather than the presence of an additional dynamical dark energy degree of freedom. While the implications for the early matter-dominated era remain open for future investigation, this framework offers a novel geometric perspective on the smallness and non-vanishing nature of the vacuum energy.

Conventions

We work in natural units $c = \hbar = 1$ adopting metric signature $(+,-,-,-)$. Our curvature conventions follow

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \qquad R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}. \tag{1}$$

For the nonlocal functional we set the smoothing/window parameter to its default value $W = 1$ throughout (no variation with respect to W is performed).

1 Motivation and scope

General Relativity provides an accurate local description of gravitational phenomena, yet it leaves the global structure of time and the selection of the vacuum largely unconstrained. In particular, both Minkowski and de Sitter spacetimes are admissible vacuum solutions, despite their radically different global properties. The observed accelerated expansion is therefore introduced phenomenologically through a cosmological constant, rather than emerging from a geometric principle.

In this work we explore a minimal geometric extension of General Relativity based on a scalar field Θ whose timelike gradient defines a preferred foliation of spacetime. Preferred-foliation frameworks closely related in spirit include Einstein–Æther and khronometric theories, where a timelike vector (or scalar-defined) structure selects a preferred slicing while remaining compatible with standard local tests in appropriate limits [1, 2, 3, 4]. The level sets $\Theta = \text{const}$ are interpreted as physical "present" hypersurfaces, providing a global time-ordering structure without affecting local causal relations or observer-dependent simultaneity.

The central ingredient is a non-local geometric constraint acting on the foliation itself: the normalized average of the squared extrinsic curvature K^2 over a causally defined domain on each hypersurface is fixed to a constant value. This leafwise constraint selects the global vacuum geometry without introducing unsuppressed local modifications of the gravitational dynamics.

We show that asymptotically flat Minkowski spacetime is not generically selected as an infrared vacuum, while de Sitter spacetime is admitted as a consistent asymptotic solution, with its expansion rate fixed by the constraint. Local Schwarzschild and post-Newtonian physics are recovered, as the constraint is dominated by infrared scales.

The purpose of this work is not to present a complete modified-gravity theory valid at all scales, nor a full cosmological model including perturbations and matter evolution. Rather, the framework is intended as a proof-of-principle demonstrating how a non-local, foliation-based geometric constraint can act as an infrared vacuum-selection mechanism within General Relativity. Issues such as perturbative stability, detailed cosmological histories, and observational constraints lie beyond the present scope and are left for future investigation.

2 Geometric setup

We consider a four-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$ endowed with a scalar field Θ. When the gradient $\nabla_\mu\Theta$ is timelike, the level sets $\Theta = \text{const}$ define a foliation of spacetime by spacelike hypersurfaces Σ_Θ. Scalar fields used to define preferred time functions and associated foliations in GR have been considered in earlier contexts; see e.g. [5]. Throughout this work, we restrict attention to configurations satisfying this condition.

The scalar invariant

$$X \equiv g^{\mu\nu}\partial_\mu\Theta\,\partial_\nu\Theta \tag{2}$$

is assumed to be positive, ensuring that $\nabla_\mu\Theta$ is timelike. We then define the unit normal vector field to the foliation as

$$u_\mu \equiv \frac{\partial_\mu\Theta}{\sqrt{X}}\,. \tag{3}$$

The induced metric on each hypersurface Σ_Θ is given by

$$h_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu, \tag{4}$$

which projects tensors onto directions tangent to the foliation. We denote by γ_{ij} the (positive-definite) induced three-metric on each leaf Σ_Θ. Its determinant is $\gamma \equiv \det(\gamma_{ij})$, and the intrinsic volume element is $\sqrt{\gamma}\,d^3x$. The extrinsic curvature of the hypersurfaces is defined in the standard way as

$$K_{\mu\nu} \equiv h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \nabla_\alpha u_\beta. \tag{5}$$

A key scalar quantity in what follows is the squared extrinsic curvature,

$$K^2 \equiv K_{\mu\nu}K^{\mu\nu}, \tag{6}$$

which measures the curvature of the $\Theta = \text{const}$ hypersurfaces within spacetime. Together with the Ricci scalar R of the spacetime metric, these invariants will enter the action and the geometric constraint introduced in later sections.

3 Spherically symmetric sector

In order to analyze the geometric properties of the Θ-foliation and to construct explicit examples, we consider a static, spherically symmetric spacetime. The metric is written in the form

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\right), \tag{7}$$

where $A(r)$ and $B(r)$ are positive functions of the radial coordinate.

For the scalar field we adopt the ansatz

$$\Theta(t,r) = t + \psi(r), \tag{8}$$

which preserves spherical symmetry while allowing for nontrivial foliations tilted with respect to the static time coordinate. This form captures the most general Θ compatible with the symmetries of the metric.

3.1 Kinematic invariant

For the above ansatz, the kinetic scalar X takes the form

$$X = \frac{1}{A(r)} - \frac{\psi'(r)^2}{B(r)}. \tag{9}$$

The requirement that the foliation be spacelike implies $X > 0$, which constrains the admissible profiles $\psi(r)$.

3.2 Curvature invariants

The Ricci scalar associated with the metric is given by

$$R = R[A(r), B(r)], \tag{10}$$

where the explicit expression, though lengthy, depends only on the radial functions and their derivatives.

The squared extrinsic curvature of the $\Theta = \text{const}$ hypersurfaces can be written as

$$K^2 = K_{\mu\nu}K^{\mu\nu} = K^2\big(A(r), B(r), \psi(r), \psi'(r), \psi''(r)\big). \tag{11}$$

The explicit form is algebraically involved and will not be reproduced in full here. What is important for our purposes is its dependence on the radial coordinate through the functions $A(r)$, $B(r)$ and the foliation profile $\psi(r)$.

These local expressions will enter the construction of the action and the leafwise geometric constraint discussed in the following sections.

4 Constrained action

4.1 Local gravitational dynamics

We start from a local action consisting of the Einstein–Hilbert term and a minimal kinetic term for the scalar field Θ,

$$S_{\text{loc}} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}(R - 2\Lambda) - \frac{\alpha}{2}X\right], \tag{12}$$

where $\kappa = 8\pi G$, Λ is a cosmological constant and $X = g^{\mu\nu}\partial_\mu\Theta\,\partial_\nu\Theta$. We take the scalar field Θ to have dimensions of length (time in units $c = 1$), so that $\partial_\mu\Theta$ is dimensionless and the kinetic scalar X is dimensionless. Accordingly, the coupling constant α has dimensions $[L^{-4}]$. Although the local action includes a standard kinetic term for the scalar field Θ, this term is not introduced to model an additional physical scalar degree of freedom. Rather, Θ plays the role of a geometric time-ordering field whose gradient defines the spacetime foliation on which the non-local constraint acts. The kinetic term should therefore be understood as a regulator enforcing the timelike character of $\nabla_\mu\Theta$ and the regularity of the foliation. In the infrared regime of interest, the dynamics of Θ is dominated by the leafwise constraint, and the theory approaches a non-propagating, cuscuton-like limit. A detailed analysis of the scalar mode content is left for future investigation. This local action is fully diffeomorphism invariant and does not introduce any preferred frame by itself.

On the bare cosmological constant. We keep Λ explicit for generality. In the vacuum-selection discussion below we set the bare value to $\Lambda = 0$, so that the de Sitter curvature scale arises from the leafwise constraint as a selection rule rather than being put in by hand.

4.2 Leafwise geometric constraint

The defining ingredient of the theory is a non-local geometric constraint acting on the foliation defined by Θ. For each hypersurface Σ_Θ we consider a causally defined domain $D_\Theta \subset \Sigma_\Theta$, determined by the spacetime geometry and the foliation itself.

Throughout this work, the causal domain D_Θ is defined as the maximal causally accessible region of the hypersurface Σ_Θ, i.e. the largest connected subset of Σ_Θ whose points are mutually related by causal curves in the ambient spacetime geometry. This definition is intrinsic to the foliation and does not depend on the choice of any fiducial observer. In asymptotically flat spacetimes, D_Θ extends to spatial infinity, while in spacetimes with a cosmological horizon the causal domain is bounded by the horizon scale, providing a finite and geometrically natural infrared cutoff.

On each leaf we define the functionals

$$I_0[\Theta] \equiv \int_{D_\Theta} W\,\sqrt{\gamma}\,d^3x, \qquad I_1[\Theta] \equiv \int_{D_\Theta} W\,K^2\sqrt{\gamma}\,d^3x, \tag{13}$$

where γ is the determinant of the induced three-metric γ_{ij} on Σ_Θ, and $W \geq 0$ is an optional smoothing/window weight (set to $W = 1$ in this work). Then

$$\langle K^2\rangle_{D_\Theta} \equiv \frac{I_1[\Theta]}{I_0[\Theta]}. \tag{14}$$

The geometric constraint is imposed by requiring

$$\lim_{R\to\infty} \frac{\int_{D_\Theta(R)} K^2 \sqrt{\gamma}\, d^3x}{\int_{D_\Theta(R)} \sqrt{\gamma}\, d^3x} = K_0^2, \tag{15}$$

where K_0 is a fixed parameter setting the curvature scale of the foliation. Here, $D_\Theta(R)$ denotes a family of nested subdomains monotonically exhausting D_Θ as $R \to \infty$. In spacetimes where D_Θ is itself bounded (e.g. by a cosmological horizon), the limit is saturated at finite R and the constraint is automatically well-defined without external regulators. Conceptually, the use of global (or nonlocal) constraints to control vacuum properties has precedents in unimodular formulations and vacuum-energy sequestering mechanisms[1]. The present constraint, however, acts on a foliation-dependent extrinsic-curvature functional rather than on the metric determinant or on matter-sector vacuum contributions.s.

4.3 Full action

The constraint is enforced through a Lagrange multiplier $\lambda(\Theta)$ that depends only on the value of the scalar field and is therefore constant on each hypersurface Σ_Θ. The full action reads

$$S = S_{\text{loc}} + \int d\Theta\, \lambda(\Theta) \left[I_1[\Theta] - K_0^2 I_0[\Theta] \right]. \tag{16}$$

With $[\Theta] = L$, the leafwise Lagrange multiplier $\lambda(\Theta)$ has dimensions $[L^{-2}]$, ensuring that the non-local contribution to the action is dimensionless. Because λ is not a local field but a function of Θ only, the constraint does not generate a pointwise equation $K^2(x) = K_0^2$. Instead, it acts globally on the geometry of each leaf, selecting admissible foliations without introducing unsuppressed local modifications of the gravitational dynamics. Boundary terms required for a well-defined variational principle are assumed implicitly.

4.4 Variational structure and infrared dominance

A complete derivation of the Euler–Lagrange equations associated with the non-local action (16) is beyond the scope of this work. Nevertheless, some qualitative features of the variational structure can be stated. The variation of the leafwise functional

$$\langle K^2 \rangle_{D_\Theta} = \frac{I_1}{I_0}$$

with respect to the metric generically produces local contributions proportional to functional derivatives of $K_{\mu\nu}K^{\mu\nu}$ and $\sqrt{\gamma}$, multiplied by the leafwise Lagrange multiplier $\lambda(\Theta)$. Crucially, these contributions are weighted

[1] ...

by inverse powers of the domain volume I_0 and are therefore dominated by infrared scales. As a result, for localized sources and weak-field configurations, the induced corrections to the Einstein equations are parametrically suppressed by the ratio between local curvature scales and the infrared size of the causal domain D_Θ. In the limit where D_Θ extends to cosmological scales, these corrections vanish locally, ensuring recovery of standard Schwarzschild and post-Newtonian physics at leading order. This mechanism is conceptually analogous to other global-constraint frameworks, where non-local terms affect the infrared vacuum structure while leaving local gravitational dynamics effectively unchanged.

5 Sanity checks and local GR recovery

5.1 Schwarzschild test

As a first consistency check, we evaluate the curvature invariants for the Schwarzschild solution,

$$A(r) = 1 - \frac{2M}{r}, \qquad B(r) = \left(1 - \frac{2M}{r}\right)^{-1}. \tag{17}$$

The Ricci scalar vanishes identically,

$$R\big|_{\text{Schw}} = 0, \tag{18}$$

as expected for a vacuum solution of General Relativity. Crucially, the existence of this solution is not obstructed by the leafwise constraint. Since the geometric condition (15) is defined as an asymptotic limit $R \to \infty$, the contributions from local matter distributions or compact objects like black holes are suppressed by the infinite volume of the integration domain. Consequently, the theory does not introduce unsuppressed local modifications of the gravitational dynamics. The recovery of standard weak-field and post-Newtonian behavior in the presence of preferred-structure fields follows from this infrared dominance, analogous to established results in Einstein–Æther theory; see e.g. [2] for explicit equivalence proofs

5.2 Static foliation limit

When the foliation coincides with constant-t hypersurfaces ($\psi = \text{const}$), the scalar invariants reduce to

$$X\big|_{\psi'=0} = \frac{1}{A(r)}, \tag{19}$$

$$K^2\big|_{\psi'=0} = 0. \tag{20}$$

In this static limit, the extrinsic curvature of the Θ-hypersurfaces vanishes identically. However, compatibility with General Relativity holds more generally. For generic asymptotically flat foliations where K^2 may be non-vanishing near the source, the contribution to the global average scales as the ratio of a finite strong-field volume to the infinite integration volume. In the limit $R \to \infty$, local sources effectively represent measure-zero corrections to the vacuum selection rule. This confirms that the global constraint acts purely as an infrared boundary condition, leaving local astrophysical tests of gravity unaffected.

6 Vacuum selection from the leafwise constraint

6.1 Minkowski spacetime

We now evaluate the leafwise geometric functional for Minkowski spacetime, written in static spherical coordinates,

$$A(r) = 1, \qquad B(r) = 1. \tag{21}$$

For foliations of the form $\Theta = t + \psi(r)$ the kinetic invariant becomes

$$X = 1 - \psi'(r)^2, \tag{22}$$

so that the timelike condition requires $|\psi'(r)| < 1$.

For a linear tilt $\psi(r) = vr$, one finds

$$K^2(r) = \frac{2v^2}{r^2(1 - v^2)}, \tag{23}$$

which is strictly positive but decays as r^{-2} at large radius.

6.2 Infrared behaviour of the leafwise average

The leafwise constraint does not involve the pointwise value of $K^2(r)$, but its normalized average over a causally defined domain D_Θ,

$$\langle K^2 \rangle_{D_\Theta} = \frac{\int_{D_\Theta} W\, K^2 \sqrt{\gamma}\, d^3x}{\int_{D_\Theta} W\, \sqrt{\gamma}\, d^3x}. \tag{24}$$

In Minkowski spacetime, the induced volume element grows as $\sqrt{\gamma} \sim r^2$, while $K^2 \sim r^{-2}$. As a consequence, the numerator scales linearly with the infrared cutoff, whereas the denominator grows cubically. In the infrared limit one therefore finds

$$\langle K^2 \rangle_{D_\Theta} \xrightarrow{\text{IR}} 0. \tag{25}$$

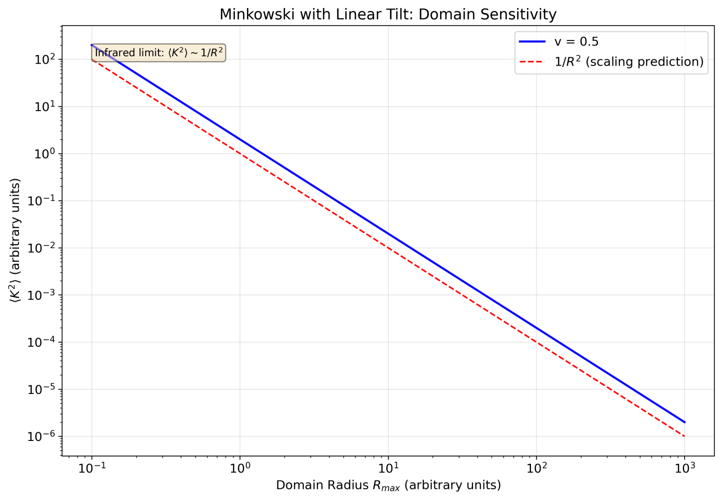

Figure 1: Infrared behaviour of the leafwise average $\langle K^2 \rangle$ in Minkowski spacetime with linear tilt. The numerical scaling confirms $\langle K^2 \rangle \propto R_{\max}^{-2}$, implying a vanishing average extrinsic-curvature invariant in the infrared limit $R_{\max} \to \infty$.

This result holds independently of the tilt parameter v and reflects a purely geometric scaling property of asymptotically flat spacetime.

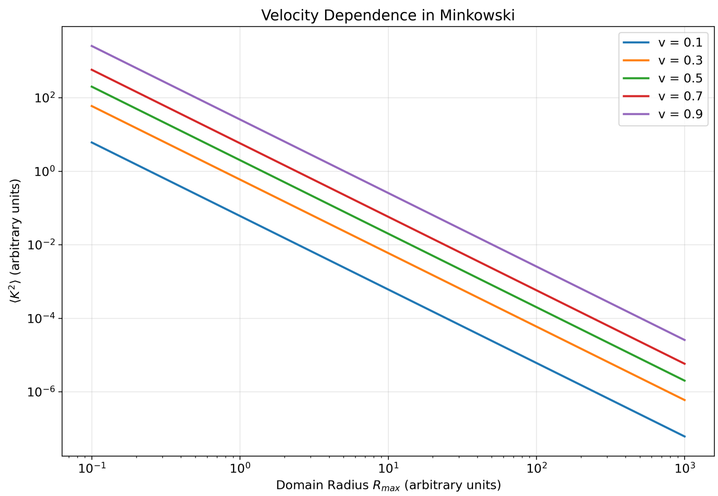

Figure 2: Dependence of $\langle K^2 \rangle$ on the foliation tilt parameter v in Minkowski spacetime (evaluated at finite infrared cutoffs). While the overall amplitude depends on v, the infrared decay with increasing $R_{\max}$ and the limit $\langle K^2 \rangle \to 0$ are universal, supporting the generic exclusion of Minkowski as an infrared vacuum for $K_0^2 > 0$.

This infrared scaling mechanism is qualitatively reminiscent of other approaches where large-scale (IR) effects can drive late-time acceleration without introducing a new local dark-energy field, albeit implemented here as a geometric selection rule rather than as a dynamical nonlocal modification of the field equations [6]. Therefore, for generic asymptotically flat foliations whose extrinsic curvature decays at large radius, Minkowski spacetime cannot satisfy the leafwise constraint $\langle K^2\rangle_{D_\Theta} = K_0^2$ for any $K_0^2 > 0$. This result follows from the infrared scaling of the normalized average and is independent of the specific tilt parameters of the foliation. While special constant-mean-curvature or hyperboloidal slicings of flat spacetime may evade this argument by enforcing a non-vanishing extrinsic curvature through global boundary conditions, such configurations require fine-tuned foliation choices and do not arise generically from the infrared-dominated causal domains considered here. In this sense, Minkowski spacetime is not selected as a natural infrared vacuum by the leafwise geometric constraint.

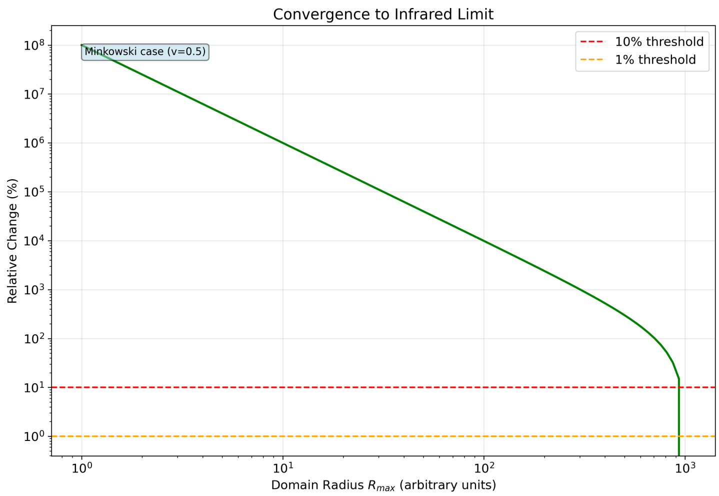

Figure 3: Convergence of the Minkowski leafwise average $\langle K^2\rangle(R_{\max})$ toward its infrared limit (example shown for $v = 0.5$). The plot illustrates that a genuinely infrared domain is required for robust convergence, motivating the interpretation of the constraint as an infrared boundary condition defined by horizon-scale causal domains.

6.3 de Sitter spacetime

We next consider de Sitter spacetime in the static patch,

$$A(r) = 1 - H^2 r^2, \qquad B(r) = \left(1 - H^2 r^2\right)^{-1}. \tag{26}$$

For the Painlevé–Gullstrand slicing,

$$\psi'(r) = \frac{Hr}{1 - H^2 r^2}, \tag{27}$$

the kinetic invariant satisfies $X = 1$, and the squared extrinsic curvature of the Θ-hypersurfaces is constant,

$$K^2 = 3H^2. \tag{28}$$

Since K^2 is constant on the entire causal domain of the static patch, its normalized leafwise average coincides with the local value,

$$\langle K^2 \rangle_{D_\Theta} = 3H^2. \tag{29}$$

It is worth noting that the causal domain of the de Sitter foliation is bounded by the cosmological horizon at $r = H^{-1}$, so that D_Θ has finite volume. The infrared limit defining the constraint is therefore saturated at a finite, geometrically determined scale. This is in sharp contrast with the Minkowski case, where D_Θ extends to spatial infinity and the unbounded growth of the integration volume drives $\langle K^2 \rangle$ to zero. The vacuum selection mechanism can thus be traced to this structural asymmetry: de Sitter spacetime possesses a natural causal scale at which the constraint is intrinsically satisfied, whereas Minkowski spacetime lacks any such scale.

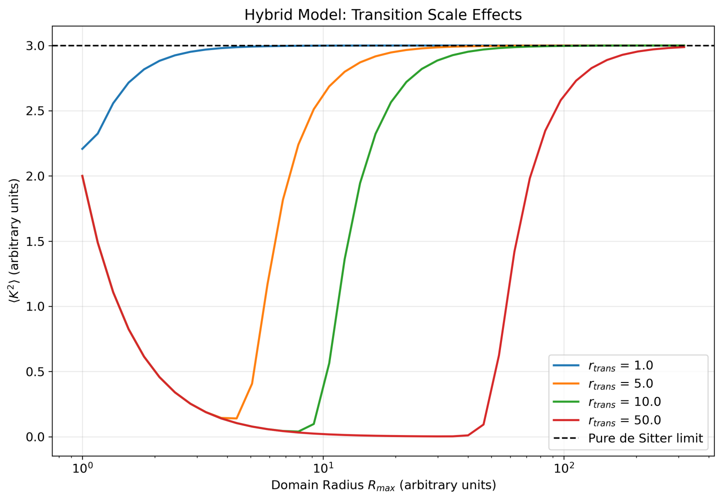

Figure 4: Hybrid toy model sensitivity: a Minkowski-like interior matched to a de Sitter exterior with transition radius r_{trans}. The leafwise average $\langle K^2 \rangle$ converges to the de Sitter value once the infrared domain extends well beyond the transition scale ($R_{\max} \gg r_{\text{trans}}$), illustrating infrared dominance of the vacuum selection rule in inhomogeneous settings.

The leafwise constraint therefore selects de Sitter spacetime as an admissible vacuum, with the expansion rate fixed by

$$H^2 = \frac{K_0^2}{3}. \tag{30}$$

This relation fixes the curvature scale of the *asymptotic* vacuum geometry selected by the leafwise constraint. Since the constraint is formally defined in the infinite-volume limit ($R \to \infty$), it does not impose a constant Hubble rate during the dynamical history at finite causal volume, where standard General Relativity is recovered. In the present case, the constancy of K^2 over the maximal causal domain ensures that the asymptotic average coincides with the pointwise value, providing a robust realization of the selected vacuum. More generally, however, the constraint requires only the global infrared average and does not enforce pointwise constancy of the extrinsic curvature.

6.4 Infrared dominance of the leafwise constraint

At this stage a potential concern may arise. Since the leafwise constraint fixes a non-zero value of the normalized average $\langle K^2 \rangle_{D_\Theta}$, one might worry that local Schwarzschild or post-Newtonian physics could be incompatible with the theory. This concern, however, relies on an incorrect interpretation of the constraint as a local condition.

The geometric constraint does not act pointwise on K^2, but only through its leafwise average over a causally defined domain. As a result, its effect is controlled by the large-scale structure of the foliation rather than by local curvature near compact sources. In the following we show explicitly that, for physically relevant foliations, the induced geometry on the $\Theta = \text{const}$ hypersurfaces remains locally Euclidean, ensuring full compatibility with standard Schwarzschild and post-Newtonian physics.

Lemma 1 (Euclidean induced metric on the SdS–PG foliation)**.** *Consider the static Schwarzschild–de Sitter line element*

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2 d\Omega^2, \qquad A(r) = 1 - \frac{2M}{r} - H^2 r^2, B(r) = \frac{1}{A(r)}. \tag{31}$$

Let the foliation be defined by the scalar field

$$\Theta(t, r) = t + \psi(r), \tag{32}$$

and choose the Painlevé–Gullstrand–type slicing

$$\psi'(r) = \frac{\sqrt{\frac{2M}{r} + H^2 r^2}}{A(r)}. \tag{33}$$

Then the metric induced on the hypersurfaces $\Sigma_\Theta : \ \Theta = \text{const}$ *is exactly Euclidean:*

$$\gamma_{ij}\,dx^i dx^j = dr^2 + r^2 d\Omega^2, \qquad \textit{hence} \qquad \sqrt{\gamma} = r^2 \sin\theta. \tag{34}$$

Proof. On a hypersurface $\Theta = \text{const}$ we have $d\Theta = dt + \psi'(r)\,dr = 0$, hence

$$dt = -\psi'(r)\,dr. \tag{35}$$

Substituting (35) into (31), the induced line element on Σ_Θ becomes

$$ds^2\big|_{\Sigma_\Theta} = \left(B(r) - A(r)\psi'(r)^2\right) dr^2 + r^2 d\Omega^2. \tag{36}$$

Using (33) and $B = 1/A$, we compute

$$B(r) - A(r)\psi'(r)^2 = \frac{1}{A(r)} - A(r)\,\frac{\frac{2M}{r} + H^2r^2}{A(r)^2} = \frac{1 - \left(\frac{2M}{r} + H^2r^2\right)}{A(r)}. \tag{37}$$

Finally, since $A(r) = 1 - \frac{2M}{r} - H^2r^2$, the numerator in (37) equals $A(r)$ and therefore

$$B(r) - A(r)\psi'(r)^2 = \frac{A(r)}{A(r)} = 1. \tag{38}$$

Substituting (38) into (36) yields (34). $\square$

As a consequence, the leafwise geometric constraint is dominated by the large-scale structure of the foliation, while local Schwarzschild and post-Newtonian physics remain unaffected. The apparent tension between the constraint and local gravity is therefore resolved at the geometric level.

7 Covariance and preferred foliation backgrounds

The formalism is generally covariant at the level of the action, but backgrounds for which $\nabla_\mu\Theta$ is timelike naturally select a preferred foliation of spacetime. This mechanism is conceptually related to those appearing in Einstein–Æther theories [1], Hořava–Lifshitz gravity [7], and khronometric models [3].

Although the foliation is defined by the gradient of a scalar field, the present framework differs from khronometric and scalar Einstein–Æther theories in a crucial way. The field Θ is not introduced as an independent dynamical degree of freedom with its own kinetic couplings, but serves a purely geometric role in defining the hypersurfaces on which the global constraint acts. Related scalar-aether and khronometric constructions, used for comparison, are discussed for instance in [4, 8]. The scalar field Θ is not introduced as an independent propagating degree of freedom describing new local physics. Instead, it serves a geometric role in defining the preferred foliation on which the global constraint acts. While a kinetic term is included in the local action, its role is to ensure the timelike character and regularity of the foliation, rather than to introduce an additional dynamical scalar mode in the infrared regime considered here.

The breaking of local Lorentz symmetry is therefore spontaneous rather than fundamental. The action remains fully diffeomorphism invariant, while specific solutions dynamically single out a preferred time direction through the gradient of Θ. In this sense, the preferred foliation does not represent a violation of covariance at the level of physical laws, but emerges as a property of the vacuum geometry selected by the leafwise constraint.

8 Computational details

Symbolic computations underlying the results of this work were performed using *Wolfram Mathematica* v. 14.3. Additional numerical analyses and consistency checks of the infrared scaling behaviour were carried out using Python-based routines. The calculations are organized in a sequence of notebooks implementing independent checks of the geometric invariants, the local action, and the evaluation of the leafwise constraint in symmetric settings:

1. `00_Setup_Assumptions.nb` - Global assumptions and symbol definitions.
2. `01_Invariants_X_K2_R.nb` - Computation of X, K^2, and R from the metric ansatz.
3. `02_Action_Lagrangian.nb` - Construction of the local four-dimensional Lagrangian and of the leafwise constraint functional.
4. `03_Leff_Radial_Reduction.nb` - Angular integration and construction of the radial kernels entering the leafwise averages.
5. `04_SanityChecks.nb` - Verification of limiting cases (Schwarzschild, $\psi' = 0$).
6. `05_Test_Minkowski.nb` - Evaluation of the leafwise average in Minkowski spacetime and analysis of its infrared behaviour.
7. `06_Test_deSitter.nb` - Verification that de Sitter spacetime satisfies the leafwise constraint through a cosmological (Painlevé–Gullstrand) slicing.
8. `07_Domain_sensitivity_calculations.py` - In addition to symbolic checks, quantitative sensitivity analyses were performed (using Python/SciPy) to verify the asymptotic scaling behavior of $\langle K^2 \rangle$ and the convergence rate of the integrals in the limit $R \to \infty$.

An additional notebook, `GR_Spherical_Foliation_K2_R.nb`, provides an independent derivation of the geometric invariants through explicit construction of the Christoffel symbols, Riemann tensor, and extrinsic curvature tensor $K_{\mu\nu}$.

The notebooks are provided as supplementary material.

8.1 Scope of formal verification (explicit boundaries)

For precision, we state explicitly what is and is not currently certified in the Lean development associated with this work.

Starting point for K^2. The Lean proofs certify the full analytic chain *starting from* the explicit coordinate expression

$$K^2_{\text{general}} = \frac{\text{NumK2}}{\text{DenK2}},$$

and all substitutions/limits built on top of it (Minkowski IR scaling, de Sitter selection, abstract averaging lemmas, and IR-dominance statements). At present, the Lean code does *not* yet include a complete differential-geometric derivation of this coordinate formula from first principles (u^μ, projectors, connection, and tensorial construction of $K_{\mu\nu}K^{\mu\nu}$).

Causal domain D_Θ. In Lean, the causal domains are formalized via nested proxy families suitable for infrared analysis (balls B_R, and in de Sitter $B_{\min(R,H^{-1})}$), for which well-posedness, positivity of the normalization denominator, and finite-radius saturation are proved. What is not yet formalized is the full Lorentzian definition of mutual causal reachability and a general theorem proving equivalence between that definition and the proxy domains in all backgrounds.

Accordingly, statements of formal certification should be read with the following canonical scope: *The Lean artifact verifies the analytic chain of results starting from the explicit coordinate expression $K^2_{general}$ and the proxy domains used in the manuscript; it does not yet certify the derivation of $K^2_{general}$ from the full geometric definition $K_{\mu\nu}K^{\mu\nu}$, nor the equivalence between the proxy domains and the general Lorentzian causal-domain definition.*

Reproducibility note. Build verified at commit `<hash>` with Lean `v4.27.0` / mathlib `a3a10db0e9d66acbebf76c5e6a135066525ac900`; CI reproduces `lake build` on every push. Reference artifact release: `v2.0`.

9 Conclusion

We have presented a scalar-field framework in which the foliation defined by the gradient of Θ is subject to a non-local geometric constraint acting leafwise on spacetime hypersurfaces. The constraint fixes the asymptotic value of the normalized average of the squared extrinsic curvature in the infrared limit, rather than imposing a pointwise condition on the geometry.

While the theory locally reproduces the field equations of General Relativity when the foliation aligns with static time coordinates, the asymptotic constraint leads to a non-trivial selection of the vacuum structure. Asymptotically flat Minkowski spacetime is excluded as a vacuum solution because its leafwise average vanishes in the infrared limit (scaling as R^{-2}), whereas de Sitter spacetime emerges as a consistent asymptotic solution with curvature scale fixed by the geometric selection rule

$$H^2 = \frac{K_0^2}{3}. \tag{39}$$

Because the constraint is defined as a limit at infinity ($R \to \infty$), it acts as a boundary condition on the foliation. Local matter distributions and compact objects contribute only at measure-zero to the infinite-volume average, ensuring that standard Schwarzschild and post-Newtonian physics are fully recovered without introducing unsuppressed local modifications or additional forces.

The present framework is intended as a proof-of-principle for vacuum selection in the strict infrared regime. A complete cosmological implementation requires a finite-domain prescription for the causal regions D_Θ. We anticipate that the resulting finite-volume corrections—suppressed in the early universe but relevant at late times—will provide the dynamical mechanism driving the evolution toward the selected de Sitter vacuum. Developing such a finite-domain formulation and its implications for the radiation- and matter-dominated eras is left for future work.

These results suggest that the observed late-time cosmic acceleration may reflect a global geometric property of the spacetime foliation defining the present—a resistance to staticity enforced at the cosmological horizon—rather than requiring the introduction of an additional dynamical dark energy degree of freedom.

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