Graph Theory

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1 Definitions

Simple Graph A simple graph $G(V, E)$ has a vertex set V (denote V(G) if the graph is not clear), and an edge set $E \subset \binom{V}{2}$. [$E \subset \binom{V}{2}$ is the set of 2-element subsets of V] Remark, for an edge i, h we write it as ij or ji.

Compelete Graph Let $n \in \mathbb{N}$. The complete graph on n verticies, denoted as k_n, has vertex set $V = [n]$, and edge set $\binom{E=[n]}{2}$.

Path The path on n verticies, denoted P_n, has certe set V=[n], and edge set; $E = \{\{i, i+1\}|i \in [n-1]\}$.

Null Graph $G(V, E)$ has $V = E = \emptyset$, $G = \emptyset$.

Empty Graph The empty graph on n verticies $G(V E)$ has vertex set $V = [n]$, and $E = \emptyset$.

Cycle Graph For $n \in \mathbb{N}$, $n \geq 3$, the cycle graph on verticies, denoted as C_n, has vertex $E = i, i+1|i \in [n-1]$.

Wheel Graph Let $n \geq 4$. The wheel graph on n vertices, denoted as W_n is the graph $G(V, E)$, with vertex set $V = [n]$ and edge set.

$$E = \{\{i, i+1\}|i \in [n-2]\} \cup \{\{1, n-1\}\} \cup \{\{i, n\}|i \in [n-1]\}$$

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Bipartite graph A barpartite graph $G(V, E)$ is a graph s.t. V can be partitoned into two sets X, Y (i.e., $V = XUY$, $X \cap Y = \emptyset$ and $E \subset \{xy|x \in X, y \in Y\}$)

Hypercube The hypercube of degree d, denoted Q_d, has vertex set $V = 0, 1^d$ (i.e., the set of binary strings of length d). Now $(w_1, w_2, ..., w_d)$ and $(\tau_1, \tau_2, ...\tau_d)$ are adjacent in Q_d if and only if there exists exactly one $i \in [d]$ s.t. $w_i \neq \tau_i$.

Connected A graph is said to be connected if for every pair of vertices u, v there exists a path from u to v (i.e., $u - v$ path). A graph is disconnected if it is not connected. The connected subgraphs are called componenets.

Tree A tree is a connected, a cyclic graph.

Vertex Degree Let $G(V, E)$ be a graph. The degree of a vertex $v \in V$ is $deg(v) = |\{uv|uv \in E\}|$. A graph is d-regular if every vertex has degree d.

Walk Let $G(V, E)$ be a graph. A walk in G is a sequence of verticies. $(v_0, v_1, ..., v_k)$ s.t. for all $i \in \{0, ..., k-1\}$, $v_i v_{i+1} \in E(G)$.

Adjacency Matrix Let $G(V, E)$ be a graph. The adjacency matrix of G is a $|V| \times |V|$ matrix where A_{ij}=$\{1 : ij \in E(G), 0 : \text{Otherwise}\}$

Closed Walk Let $G(V, E)$ and let $v_0, v_1, ...v_k$ be a walk. We say that the walk is closed if $v_0 = v_k$.

Independent Set An independent set of a graph $G(V, E)$ is a set $S \subset V$ such that for every $i, j \in S$, $ij \notin E(G)$. Denote $\alpha(G)$ as the size of the largest independent set in G.

Graph Vertex Coloring A vertex coloring of a graph $G(V, E)$ is a function $\varphi : V(G) \to [n]$ s.t. whenever $uv \in E(G), \varphi(u) \neq \varphi(v)$. The chromatic number of G denoted $\chi(G)$, is the smallest $n \in \mathbb{N}$ s.t. there exists a coloring

$$\varphi : V(G) \to [n]$$

.

2 Lemma (Handshake Lemma)

Proposition Let $G(V, E)$ be a simple graph. Then

$$\sum_{v \in V} deg(v) = 2|E|$$

.

Proof By double counting. $2|E|$ counts twice the number of edges.

$$\sum_{v \in V} deg(v)$$

we note that $deg(v)$ counts the number of edges incident to v. Each edge has 2 endpoints, u and v. So uv is counted twice once in $deg(v)$ and once in deg(u). So

$$\sum_{v \in V} deg(v) = 2|E|$$

Q.E.D.

3 Lemma 1

Proposition Let $G(V, E)$ be a graph. Every closed walk of odd length at least 3, contains an odd cycle.

Proof By induction on odd $k \in \mathbb{Z}^+$, $k \geq 3$. Base Case: Any closed walk of length 3 includes an odd cycle so the lemma holds. Inductive Hypothesis: Fix $k \in \mathbb{Z}^+$ odd, $k \geq 3.$, and suppose the lemma holds. Inductive Step: Conside a closed walk of length $k+2$, $v_0, v_1, ..., v_k + 2$. If $v_0 = v_{k+2}$ are the only repeated verticies then the walk unduces an odd cycle, and we are done. Suppose instead there are only other repeated verticies in the walk. Let $0 \leq i < j \leq k+2$, where we don't have both $i = 0$ and $j = k+2$. Suppose $v_i = v_j$, then $v_i, v_{i+1}, ..., v_k$ has odd length, then $v_i, ..., v_j$ contains an odd cycle by the inductive hypothesis. Suppose instead $v_i, ..., v_j$ has even length. Observe that $v_0, ..., v_i, v_{j+1}, ..., v_k + 2$ is a closed walk of odd length at most k. So by the Inductive Hypothesis, $v_0, ..., v_i, v_j + 1, ..., v_{k+2}$ has an odd cycle. Thus $v_0, ..., v_{k+2}$(The Original Walk) has an odd cycle. Q.E.D

4 Theorm 1

Proposition Let $G(V, E)$ be a graph, and let A be its adjcency matrix. For all $n \in \mathbb{Z}^+$, $(A^n)ij$ counts the number of $i - j$ walks of length n.

Proof By induction on $n \in \mathbb{Z}^+$. Base case: n=1. So $A^1 = A$. Now there exists a walk of length 1 from $i - j$ if and only if $ij \in ij \in E(G)$. This is counted by A^{ij}. Inductive Hypothesis: Fix $k \geq 1$, and supposse that $(A^k)ij$ counts the number of $i - j$ walks of length k. Inductive Step: Consider $A^{k+1} = A^K \times A$. By the Inductive Hypothesis, $(A^k)ij$ counts the number of $i - j$ walks of length k. Similarly Aij counts the number of $i - j$ walks of length k. Observe that:

$$(A^{K+1})ij = \sum_{x=1}^{n=|V|} ((A^k)_{ix} \times Axj)$$

Now $(A^k)ix$ counts the number of k-length walks from $i - x$. Now $Axj = 1$ if and only if $xj \in E(G)$. So we may extend a walk of length k from $i - x$, to walk of length $k+1$ from $i - j$ if and only if $xj \in E(G)$. By the rule of sum, we add up over all the $x \in V(G)$. Q.E.D.

5 Theorm 2

Proposition A graph $G(V, E)$ is bipartite if and only if G contains no cycles of odd length.

Proof Suppose G is bipartite with parts from X and Y.

$$V(G) = X \cup Y, X \cap Y = 0$$

Consider a walk of length$n$. As no two vertices in a fixed part are adjacent, only walks of even length can be closed. A cycle is a closed walk where only the endpoints are repeated. So G contains no odd cycles. Conversely, suppose G has no odd cycles. We construct a bipartition of G. Without laws of generality, suppose G is connected. For if G is not connected we apply the following construction to each connected component. Fix $v \in V(G)$. Let: $X = \{x \in v(G)|dist(v, x)$ is even$\}$, $Y = \{y \in V(G)|dist(v, y)$ is odd$\}$. So $V(G) = X \cup Y$, and $X \cap Y = \emptyset$. We show that as two verticies in the same part are adjacent. Suppose to the contrary that there exists a closed odd walk $(v_1, ...y_1, y_2, ...v)$ By Lemma 1, $(v_1, ..., y_1, y_2, ...v)$ contains an odd cycle, contradicting the assumption that G has no odd cycles .Similarly, no two vertices in X are adjacent. So G is bipartite. Q.E.D.

6 Theorm 3

Proposition $2^{\mathbb{N}}$ is uncountable.

Proof Suppose to the contrary that $2^{\mathbb{N}}$ is countable. Let $h : \mathbb{N} \to 2^{\mathbb{N}}$ be a bijection. We obtain a contradiction (contradicting the surjectivity of h). We construct a set: $S \in 2^{\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}$, $h(n) + S$. Def: $S = \{i \in \mathbb{N}|i \notin h(i)\}$. We show that S is not in the range of h. Suppose $i \in h(i)$. Then $i \notin S$ Thus, $h(i) \neq S$. Similarly, if $i \notin h(i)$, then $i \in S$. So $h(i) \neq S$. Thus, S ins not in the range of h, contridicting the assumption that h was a bijection. Q.E.D.

7 Corralary to Theorm 3

Proposition $\mathbb{R}$ is uncountable.

Proof We show $[0, 1]$ is uncountable. We represent $S \in 2^{\mathbb{N}}$ as a binary string w, where $w_i = \{1 : i \in S, 0 : i \notin S\}$ we map $w \mapsto 0.$ w, which is an injection. So $[0, 1]$ is uncountable. Q.E.D.