Myhill-Nerode and DFA-Minimization

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1 Definitions

Definition 1.1. Let L be a language over Σ.We say that $x, y \in \Sigma^*$ are distinguishable with relation to L, $\exists z \in \Sigma^*$ s.t. $xz \in L$ and $yz \notin L$ (or vice-versa)

Definition 1.2. Distinguishable Set of Strings] A set of strings $\{x, ..., x_k\}$ is a distinguishable set of strings if forall distinct $i, j \in [x]$, x_i and x_j are distinguishable.

2 Proofs

Lemma 2.1. Let L be a regular language, and let M be a DFA such that $L(M) = L$. Let $x, y \in \Sigma^*$ be dustunguishable with relation to L. Then $M(x)$ and $M(y)$ halt on different states.

Proof. Suppose to the contrary that $M(x)$ and $M(y)$ halts on the same state q_i. Let $z \in \Sigma^*$ such that without laws of generality $xz \in L$ and $yz \notin L$. Observe that $M(xz)$ and $M(yz)$ transition M from q_0 to q_1 first. Then on string z, M transitions from q_1 to same state q_j.As $xz \in L$, $q_j \in F$. But the M accepts $yz \notin L$ by assumption. This contridicts the assumption that z distinguishes x, y. QED

Lemma 2.2. Suppose L is a lannguage with a set of k distinguishable strings. Then any DFA accepting L requires atleast k states.

Proof. If L is not regular, then for any DFA accepting L, $D(x_i)$ and $D(x_j)$ halt different states whenever $i \neq j$. So $|Q(D)| \geq k$. QED

Theorem 2.1. The DFA M constructed bt the Myhill-Nerode Theorem is minimum and unique to relableing.

Proof. We first show that M is minimum. Let D be another DFA accepting L. Let $q \in Q(D)$ Define:

$$S_q = \{w \in D(w) halts on q\}$$

Also DFA halts on seperate states when run on distinguishable strings, $S_q \subset [x]_l$ (where $[x]_l$ is an equivelence class under $\equiv_L$). Thus, $|Q(M)| \leq |Q(D)|$ So M is minimum We now show that M is unique. Suppose that $|Q(D) = |Q(M)|$, but for some strings $x, y, x \equiv_d y$, but $x \not\equiv_m y$. QED